Free Subordination and Belinschi-Nica Semigroup

Octavio Arizmendi
Department of Probability and Statistics, CIMAT,
Guanajuato, Mexico
Email: octavius@cimat.mx

Takahiro Hasebe*
Department of Mathematics, Hokkaido University
Kita 10, Nishi 8, Kitaku, Sapporo 060-0810, Japan.
Email: thasebe@math.sci.hokudai.ac.jp

October 5, 2015

Abstract

We realize the Belinschi-Nica semigroup of homomorphisms as a free multiplicative subordination. This realization allows to define more general semigroups of homomorphisms with respect to free multiplicative convolution. For these semigroups we show that a differential equation holds, generalizing the complex Burgers equation. We give examples of free multiplicative subordination and find a relation to the Markov-Krein transform, Boolean stable laws and monotone stable laws. A similar idea works for additive subordination, and in particular we study the free additive subordination associated to the Cauchy distribution and show that it is a homomorphism with respect to monotone, Boolean and free additive convolutions.

Mathematics Subject Classification 2010: 46L54
Keywords: Free subordination, Boolean stable law, monotone stable law, Markov-Krein transform

Contents

*Supported by European Commision, Marie Curie Actions – International Incoming Fellowships (Project 328112 ICNCP) at University of Franche-Comté; supported also by JSPS, Global COE program "Fostering top leaders in mathematics – broadening the core and exploring new ground" at Kyoto university.

1 Introduction

In [BN08], Belinschi and Nica defined, for each $t > 0$, the map

$$\mathbb{B}_t(\mu) = \left(\mu^{\boxplus(1+t)}\right)^{\uplus\frac{1}{1+t}}, \qquad \mu \in \mathcal{P}(\mathbb{R}).$$

The family $\{\mathbb{B}_t\}_{t\geq 0}$ is a composition semigroup of homomorphisms in the sense that it satisfies

$$\mathbb{B}_t(\mu_1 \boxtimes \mu_2) = \mathbb{B}_t(\mu_1) \boxtimes \mathbb{B}_t(\mu_2), \qquad \mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}_+), \quad t \geq 0$$

and

$$\mathbb{B}_t \circ \mathbb{B}_s = \mathbb{B}_{t+s} \text{ on } \mathcal{P}(\mathbb{R}), \qquad s, t \geq 0.$$

In this paper, we realize $\mathbb{B}_t$ as a free multiplicative subordination measure, giving a new look at the semigroup $\{\mathbb{B}_t\}_{t\geq 0}$ and generalizing it. That is, we are able to show that $\mathbb{B}_t(\mu)$ satisfies the formula

$$\boldsymbol{\sigma}_t \boxtimes \mu = \boldsymbol{\sigma}_t \circlearrowright \mathbb{B}_t(\mu), \qquad \mu \in \mathcal{P}(\mathbb{R}_+),$$

where $\circlearrowright$ denotes monotone multiplicative convolution and $\boldsymbol{\sigma}_t$ is the Bernoulli law

$$\boldsymbol{\sigma}_t = \frac{t}{1+t}\delta_0 + \frac{1}{1+t}\delta_{1+t}.$$

We can generalize the map $\mathbb{B}_t$ by changing $\boldsymbol{\sigma}_t$ to any other measure σ on $\mathbb{R}_+$ ($\sigma \neq \delta_0$) or σ on $\mathbb{T}$, to obtain the map $\mathbb{B}_\sigma$ defined by

$$\sigma \boxtimes \mu = \sigma \circlearrowright \mathbb{B}_\sigma(\mu), \tag{1.1}$$

where $\sigma \in \mathcal{P}(\mathbb{R}_+) \setminus \{\delta_0\}, \mu \in \mathcal{P}(\mathbb{R}_+)$ or $\sigma, \mu \in \mathcal{P}(\mathbb{T})$. This relation means that $\mathbb{B}_\sigma(\mu)$ is exactly the *multiplicative subordination* for the convolution $\boxtimes$ (see [Bi98, BB07] for more details and [L08] for an operator model) and then $\mathbb{B}_t$ appears in the special case when $\sigma = \boldsymbol{\sigma}_t$. One may think of this generalization artificial, but important properties of $\{\mathbb{B}_t\}_{t\geq 0}$ remain true for $\mathbb{B}_\sigma$, as the following results show.

Theorem 1.1. *Let $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}_+)$, $\sigma, \sigma_1, \sigma_2 \in \mathcal{P}(\mathbb{R}_+)\setminus\{\delta_0\}$ or let $\mu_1, \mu_2, \sigma_1, \sigma_2, \sigma \in \mathcal{P}(\mathbb{T})$. Then*

$$\mathbb{B}_\sigma(\mu_1 \boxtimes \mu_2) = \mathbb{B}_\sigma(\mu_1) \boxtimes \mathbb{B}_\sigma(\mu_2) \tag{1.2}$$

and

$$\mathbb{B}_{\sigma_1} \circ \mathbb{B}_{\sigma_2} = \mathbb{B}_{\sigma_2 \circlearrowright \sigma_1} \ \text{on } \mathcal{P}(\mathbb{R}_+). \tag{1.3}$$

Unfortunately, we are not able to show (1.2) in the general case $\mu_2 \in \mathcal{P}(\mathbb{R})$, nor (1.3) on $\mathcal{P}(\mathbb{R})$, since the existence of the multiplicative subordination $\mathbb{B}_\sigma(\mu)$ is not known for the general case $\sigma \in \mathcal{P}(\mathbb{R}_+) \setminus \{\delta_0\}$ and $\mu \in \mathcal{P}(\mathbb{R})$.

The measure $\boldsymbol{\sigma}_t$ satisfies the relation

$$\boldsymbol{\sigma}_t \circlearrowright \boldsymbol{\sigma}_s = \boldsymbol{\sigma}_{t+s}, \quad s, t \geq 0.$$

Hence, in terms of Theorem 1.1, the property $\mathbb{B}_t \circ \mathbb{B}_s = \mathbb{B}_{t+s}$ is a consequence of the fact that $\{\boldsymbol{\sigma}_t\}_{t\geq 0}$ forms a $\circlearrowright$-convolution semigroup.

The transformation $\mathbb{B}_\sigma$ is versatile. For example, we may realize free and Boolean convolution powers (modulo some dilation) by choosing the right measures σ. For Boolean convolution powers we choose $\sigma = \delta_{1/t}$, $t > 0$ and for free convolution powers we choose the Bernoulli law $\sigma = (1 - 1/t)\delta_0 + (1/t)\delta_1$, $t \geq 1$. Then the relations

$$\mathbf{D}_{1/t}(\mu \boxtimes \nu)^{\boxplus t} = \mathbf{D}_{1/t}(\mu)^{\boxplus t} \boxtimes \mathbf{D}_{1/t}(\mu)^{\boxplus t}$$

and

$$\mathbf{D}_{1/t}(\mu \boxtimes \nu)^{\uplus t} = \mathbf{D}_{1/t}(\mu)^{\uplus t} \boxtimes \mathbf{D}_{1/t}(\mu)^{\uplus t},$$

which were proved in [BN08], are just special cases of (1.2). Note here that $\mathbf{D}_t$ is the dilation of a measure by t.

On the other hand, the maps $\mathbb{B}_t$ can describe the laws of a free Brownian motion. Let F_μ be the reciprocal Cauchy transform and η_μ be the η-transform. The domains F_μ, η_μ are respectively $\mathbb{C}^+, \mathbb{C}^-$ when $\mu \in \mathcal{P}(\mathbb{R})$ and $\mathbb{D}^c, \mathbb{D}$ when $\mu \in \mathcal{P}(\mathbb{T})$. It is shown in [BN08] that the function $h(t, z) := z - F_{\mathbb{B}_t(\mu)}(z)$ satisfies the complex Burgers equation

$$\frac{\partial h}{\partial t} = -h(t, z)\frac{\partial h}{\partial z}.$$

We can extend this differential equation in terms of $\mathbb{B}_\sigma$.

Theorem 1.2. *Let $\mu \in \mathcal{P}(\mathbb{R}_+)$ and let $\{\nu_t\}_{t\geq 0}$ be a weakly continuous $\circlearrowleft$-convolution semigroup of probability measures on $\mathbb{R}_+$ or $\mathbb{T}$ such that $\nu_0 = \delta_1$ and let K be its generator defined by*

$$zK(z) = \frac{d}{dt}\bigg|_0 \eta_{\nu_t}(z).$$

Then the function $H(t,z) := z - F_{\mathbb{B}_{\nu_t}}(z)$ satisfies

$$\frac{\partial H}{\partial t} = -zK\left(\frac{H}{z}\right)\frac{\partial H}{\partial z}.$$

In terms of $\eta(t,z) := 1 - F_{\mathbb{B}_{\nu_t}(\mu)}(z)/z$, we have

$$\frac{\partial \eta}{\partial t} = -K(\eta)\left(-z\frac{\partial \eta}{\partial z} + \eta\right).$$

We have not been able to find out a noncommutative stochastic process whose marginal distributions are described by this differential equation. Finding such an example may be an interesting question.

A similar idea works for additive free convolution. Let $\mathbb{A}_\nu(\mu)$ denote the probability measure (called the *additive subordination*) characterized by

$$\sigma \boxplus \mu = \sigma \rhd \mathbb{A}_\sigma(\mu), \quad \mu, \sigma \in \mathcal{P}(\mathbb{R}), \tag{1.4}$$

where $\rhd$ is monotone convolution. The additive subordination $\mathbb{A}_\sigma(\mu)$ exists for any $\mu, \sigma \in \mathcal{P}(\mathbb{R})$; see [BB07, Bi98] for more details and [L07] for an operator model. The additive subordination was originally introduced in [V93].

Theorem 1.3. *For $\mu_1, \mu_2, \sigma_1, \sigma_2, \sigma \in \mathcal{P}(\mathbb{R})$, we have*

$$\mathbb{A}_\sigma(\mu_1 \boxplus \mu_2) = \mathbb{A}_\sigma(\mu_1) \boxplus \mathbb{A}_\sigma(\mu_2)$$

and

$$\mathbb{A}_{\sigma_1} \circ \mathbb{A}_{\sigma_2} = \mathbb{A}_{\sigma_2 \rhd \sigma_1} \text{ on } \mathcal{P}(\mathbb{R}).$$

Theorem 1.4. *Let $\mu \in \mathcal{P}(\mathbb{R})$ and let $\{\nu_t\}_{t\geq 0}$ be a weakly continuous $\rhd$-convolution semigroup of measures on $\mathbb{R}$ and let J be its generator defined by*

$$J(z) = \frac{d}{dt}\bigg|_0 F_{\nu_t}(z).$$

Then the function $G(t,z) := z - F_{\mathbb{A}_{\nu_t}(\mu)}(z)$ satisfies

$$\frac{\partial G}{\partial t} = J(z - G)\frac{\partial G}{\partial z}.$$

The paper is organized as follows. We show Theorem 1.1, 1.2 in Section 3 and then give examples, in particular focusing on the Belinschi-Nica semigroup and free and Boolean convolutions powers. We then show Theorem 1.3, 1.4 and give some examples in Section 4.

We give more examples of the multiplicative subordination $\mathbb{B}_\sigma$ when σ is a beta distribution and Boolean stable law in Section 5. The former example turns out to be related to the Markov transform [K98] and scale mixtures of Boolean stable laws developed in [AH].

Finally we consider additive subordination $\mathbb{A}_\sigma$ when σ is a Cauchy distribution and study its properties in Section 6.

2 Preliminaries

2.1 Notations

We collect basic notations used in this paper.

(1) $\mathcal{P}(I)$ is the set of Borel probability measures on a set I. We will use $I = \mathbb{R}, \mathbb{R}_+ = [0, \infty), \mathbb{R}_- = (-\infty, 0], \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

(2) For $a \in \mathbb{R}$, we denote by $\mathbf{D}_a(\mu)$ the dilation of a probability measure μ, i.e. if a random variable X follows μ, then $\mathbf{D}_a(\mu)$ is the law of aX.

(3) For $p \geq 0$ and $\mu \in \mathcal{P}(\mathbb{R}_+)$, let μ^p be the push-forward of μ by the map $x \mapsto x^p$.

(4) $\mathbb{C}^+, \mathbb{C}^-$ denote the complex upper half-plane and the lower half-plane, respectively.

(5) $\mathbb{D}$ denotes the open unit disc of $\mathbb{C}$ with center 0 and $\mathbb{T}$ denotes the unit circle of $\mathbb{C}$.

(6) For $\lambda, M > 0$, $\Gamma_{\lambda,M}$ is the truncated cone $\{z \in \mathbb{C}^+ : |\mathrm{Re}(z)| < \lambda \,\mathrm{Im}(z), \mathrm{Im}(z) > M\}$.

(7) The logarithm $z \mapsto \log z$ is the principal value defined in $\mathbb{C} \setminus \mathbb{R}_-$.

(8) For $p \in \mathbb{R}$, the power $z \mapsto z^p$ denotes the principal value $e^{p \log z}$ in $\mathbb{C} \setminus \mathbb{R}_-$.

2.2 Additive convolutions

Additive free, monotone and Boolean convolutions are binary operations on the set $\mathcal{P}(\mathbb{R})$, denoted by $\boxplus, \rhd$ and $\uplus$ respectively. Each convolution arises as the sum of (free, monotonically or Boolean) independent random variables, and is characterized in terms of some transforms.

Let $\mu \in \mathcal{P}(\mathbb{R})$. Let G_μ be the *Cauchy transform* $\int_{\mathbb{R}} \frac{1}{z-x}\,\mu(dx)$ and let F_μ, η_μ be the *reciprocal Cauchy transform* and *η-transform*:

$$F_\mu(z) = \frac{1}{G_\mu(z)}, \qquad z \in \mathbb{C}^+, \tag{2.1}$$

$$\eta_\mu(z) = 1 - zF_\mu\left(\frac{1}{z}\right), \qquad z \in \mathbb{C}^-. \tag{2.2}$$

When $\mu \in \mathcal{P}(\mathbb{R}_+)$, we consider G_μ, F_μ, η_μ in $\mathbb{C} \setminus \mathbb{R}_+$. It is known in [BV93] that, for any $\lambda > 0$, there exist $\lambda', M, M' > 0$ such that F_μ is univalent in $\Gamma_{\lambda',M'}$ and $F_\mu(\Gamma_{\lambda',M'}) \supset \Gamma_{\lambda,M}$. Then the univalent compositional left inverse F_μ^{-1} exists in $\Gamma_{\lambda,M}$, and the *Voiculescu transform* of μ is defined by

$$\phi_\mu(z) := F_\mu^{-1}(z) - z, \quad z \in \Gamma_{\lambda,M}. \tag{2.3}$$

The three convolutions are then characterized as follows:

[BV93] $$\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z) \qquad \text{in some } \Gamma_{\alpha,\beta}, \tag{2.4}$$

[M00] $$F_{\mu_1 \rhd \mu_2}(z) = F_{\mu_1}(F_{\mu_2}(z)), \qquad z \in \mathbb{C}^+, \tag{2.5}$$

[SW97] $$\eta_{\mu_1 \uplus \mu_2}(z) = \eta_{\mu_1}(z) + \eta_{\mu_2}(z), \qquad z \in \mathbb{C}^-. \tag{2.6}$$

For any $\mu \in \mathcal{P}(\mathbb{R})$, $t \geq 1$ and $s \geq 0$, probability measures $\mu^{\boxplus t}$ and $\mu^{\uplus s}$ exist such that $\phi_{\mu^{\boxplus t}}(z) = t\phi_\mu(z)$ (in the common domain) and that $\eta_{\mu^{\uplus s}}(z) = s\eta_\mu(z)$ in $\mathbb{C}^-$. The proofs can be found in [BB05] and [SW97] respectively.

2.3 Multiplicative convolutions on $\mathbb{R}_+$

Multiplicative free [V87, BV93] and monotone [B05, F09a] convolutions are binary operations on $\mathcal{P}(\mathbb{R}_+)$, denoted by $\mu \boxtimes \nu$ and $\mu \circlearrowright \nu$, respectively. Let X, Y be positive random variables with distributions μ and ν respectively. Then $\mu \boxtimes \nu$ is the law of $X^{1/2}YX^{1/2}$ when X, Y are free, and $\mu \circlearrowright \nu$ is the law of $X^{1/2}YX^{1/2}$ when $X-1, Y-1$ are monotonically independent. We can extend the definition for $\mu \in \mathcal{P}(\mathbb{R}_+)$ and an arbitrary $\nu \in \mathcal{P}(\mathbb{R})$; see [BV93] and [AH].

It is always true that $\eta_\mu(-0) = 0$. If moreover $\delta_0 \neq \mu \in \mathcal{P}(\mathbb{R}_+)$, the function η_μ is strictly increasing in $(-\infty, 0)$, so that one can define $\eta_\mu^{-1}(z)$ and the Σ-*transform*

$$\Sigma_\mu(z) := \frac{\eta_\mu^{-1}(z)}{z} \tag{2.7}$$

in the interval $(\eta_\mu(-\infty), 0)$. Suppose that $\mu, \nu \in \mathcal{P}(\mathbb{R}_+)$ and $\mu \neq \delta_0 \neq \nu$. Then the multiplicative free and monotone convolutions are characterized by

[BV93] $$\Sigma_{\mu\boxtimes\nu}(z) = \Sigma_\mu(z)\Sigma_\nu(z), \qquad z \in (\alpha, 0), \tag{2.8}$$

[B05] $$\eta_{\mu\circlearrowright\nu}(z) = \eta_\mu(\eta_\nu(z)), \qquad z \in (-\infty, 0), \tag{2.9}$$

where $\alpha := \max\{\eta_{\mu_1}(-\infty), \eta_{\mu_2}(-\infty)\}$.

For $\delta_0 \neq \mu \in \mathcal{P}(\mathbb{R}_-)$, the formula (2.8) is still true if we define $\Sigma_\mu(z) := -\Sigma_{\mathbf{D}_{-1}(\mu)}(z)$ in $(\eta_{\mathbf{D}_{-1}(\mu)}(-\infty), 0)$. This definition of Σ_μ is compatible with the property $\Sigma_{\mathbf{D}_s(\tau)}(z) = \frac{1}{s}\Sigma_\tau(z)$ for $s > 0$, $\delta_0 \neq \tau \in \mathcal{P}(\mathbb{R}_+)$. Moreover, the formula (2.8) may be extended to the case where both $\mu \in \mathcal{P}(\mathbb{R}_+), \nu \in \mathcal{P}(\mathbb{R})$ have compact supports [V87, RS07] and the case where $\mu \in \mathcal{P}(\mathbb{R}_+)$ and ν is symmetric [APA09]. The formula (2.9) can be extended to the general case $\mu \in \mathcal{P}(\mathbb{R}_+), \nu \in \mathcal{P}(\mathbb{R})$ [AH].

If $\delta_0 \neq \mu \in \mathcal{P}(\mathbb{R}_+)$, then the convolution power $\mu^{\boxtimes t} \in \mathcal{P}(\mathbb{R}_+)$, satisfying $\Sigma_{\mu^{\boxtimes t}}(z) = (\Sigma_\mu(z))^t$, is well defined for any $t \geq 1$ [BB05].

2.4 Multiplicative convolutions on $\mathbb{T}$

Multiplicative free and monotone convolutions are also defined on $\mathcal{P}(\mathbb{T})$, denoted by the same symbols $\boxtimes$ and $\circlearrowright$ respectively (see [V87] and [B05]). Let U and V unitaries with distributions μ and ν, respectively. The measure $\mu \boxtimes \nu$ is the distribution of UV when U and V are free, and $\mu \circlearrowright \nu$ is the distribution of UV when $U-1$ and $V-1$ are monotonically independent.

Let $\mu \in \mathcal{P}(\mathbb{T})$. Now we consider $G_\mu(z)$ and $F_\mu(z)$ outside the unit disc $\mathbb{D}$, and $\eta_\mu(z) = 1 - zF_\mu\left(\frac{1}{z}\right)$ in the unit disc $\mathbb{D}$. Let $m_1(\mu) := \int_{\mathbb{T}} w\, d\mu(w) \neq 0$ be the first moment of μ. The function η_μ has a convergent series expansion $\eta_\mu(z) = m_1(\mu)z + o(z)$, and so one can define $\eta_\mu^{-1}(z)$ and

$$\Sigma_\mu(z) := \frac{\eta_\mu^{-1}(z)}{z} \tag{2.10}$$

in a neighborhood of 0. Suppose that $m_1(\mu), m_1(\nu) \neq 0$. Then the multiplicative free convolution is characterized by [BV93]

$$\Sigma_{\mu\boxtimes\nu}(z) = \Sigma_\mu(z)\Sigma_\nu(z) \text{ in a neighborhood of } 0. \tag{2.11}$$

The multiplicative monotone convolution of $\mu, \nu \in \mathcal{P}(\mathbb{T})$ is characterized by [B05]

$$\eta_{\mu\circlearrowright\nu}(z) = \eta_\mu(\eta_\nu(z)), \qquad z \in \mathbb{D}. \tag{2.12}$$

2.5 Positive stable, free Poisson and beta distributions

Let $\alpha \in (0,1]$. We denote by $\mathbf{b}_\alpha^+, \mathbf{f}_\alpha^+, \mathbf{m}_\alpha^+$ Boolean, free and monotone *positive stable distributions*, respectively. They are characterized as (2.14), (2.16), (2.20). Similarly, we denote by $\mathbf{b}_\alpha^-, \mathbf{f}_\alpha^-, \mathbf{m}_\alpha^-$ the negative stable distributions (i.e. the reflection of positive ones regarding the point 0).

Let $\boldsymbol{\pi}$ be the free Poisson distribution with density

$$\frac{1}{2\pi}\sqrt{\frac{4-x}{x}}1_{(0,4)}(x)\,dx.$$

Let $\boldsymbol{\beta}_{p,q}$ be the beta distribution with parameters $p,q>0$:

$$\boldsymbol{\beta}_{p,q}(dx) = \frac{1}{B(p,q)}x^{p-1}(1-x)^{q-1}1_{(0,1)}(x)\,dx,$$

where $B(p,q)$ is the beta function. In particular we put $\boldsymbol{\beta}_\alpha := \boldsymbol{\beta}_{\alpha,1-\alpha}$ for $\alpha \in (0,1)$. By the weak continuity, we define $\boldsymbol{\beta}_0 = \delta_0$ and $\boldsymbol{\beta}_1 = \delta_1$.

Useful relations are summarized here.

$$\begin{aligned}
&F_{\mathbf{b}_\alpha^+}(z) = z-(-z)^{1-\alpha}, && \text{(see [SW97])} && (2.13)\\
&\eta_{\mathbf{b}_\alpha^+}(z) = -(-z)^{\alpha}, && && (2.14)\\
&\Sigma_{\mathbf{b}_\alpha^+}(z) = (-z)^{\frac{1-\alpha}{\alpha}}, && && (2.15)\\
&\phi_{\mathbf{f}_\alpha^+}(z) = (-z)^{1-\alpha}, && \text{(see [BV93])} && (2.16)\\
&\Sigma_{\boldsymbol{\pi}}(z) = 1-z, && && (2.17)\\
&F_{\boldsymbol{\beta}_\alpha}(z) = z\left(1-\frac{1}{z}\right)^{\alpha}, && \text{(see [H14])} && (2.18)\\
&\eta_{\boldsymbol{\beta}_\alpha}(z) = 1-(1-z)^{\alpha}, && && (2.19)\\
&F_{\mathbf{m}_\alpha^+}(z) = -((-z)^\alpha+1)^{1/\alpha}, && \text{(see [H10])} && (2.20)\\
&\eta_{\mathbf{m}_\alpha^+}(z) = 1-((-z)^\alpha+1)^{1/\alpha}, && && (2.21)\\
&\Sigma_{\mathbf{m}_\alpha^+}(z) = \frac{((1-z)^\alpha-1)^{1/\alpha}}{-z}, && && (2.22)
\end{aligned}$$

for $z<0$. Moreover, for $\mu \in \mathcal{P}(\mathbb{R}_+)$, the following hold:

$$\begin{aligned}
&G_{\mathbf{D}_{-1}(\mu)}(z) = -G_\mu(-z), \quad z \in \mathbb{C}\setminus\mathbb{R}_-; && (2.23)\\
&\eta_{\mathbf{D}_{-1}(\mu)}(z) = \eta_\mu(-z), \quad z \in \mathbb{C}\setminus\mathbb{R}_-; && (2.24)\\
&\Sigma_{\mathbf{D}_{-1}(\mu)}(z) = -\Sigma_\mu(z), \quad z \in (\eta_\mu(\infty),0). && (2.25)
\end{aligned}$$

These formulas give us the characterization of negative stable distributions.

3 The main results: Multiplication case

3.1 Proof of Theorem 1.1

We start from the positive real line case. Let $\sigma \in \mathcal{P}(\mathbb{R}_+)\setminus\{\delta_0\}, \mu \in \mathcal{P}(\mathbb{R}_+)$. The multiplicative subordination $\mathbb{B}_\sigma(\mu)$ is characterized by

$$\Sigma_{\mathbb{B}_\sigma(\mu)}(z) = \Sigma_\mu(\eta_\sigma(z)), \quad z \in (-\alpha, 0) \qquad (3.1)$$

for some $\alpha > 0$, see [Bi98, BB07]. The homomorphism property then follows easily from Eq. (3.1):

$$\begin{aligned}\Sigma_{\mathbb{B}_\sigma(\mu_1\boxtimes\mu_2)}(z) &= \Sigma_{\mu_1\boxtimes\mu_2}(\eta_\sigma(z)) \\ &= \Sigma_{\mu_1}(\eta_\sigma(z))\Sigma_{\mu_2}(\eta_\sigma(z)) \\ &= \Sigma_{\mathbb{B}_\sigma(\mu_1)}(z)\Sigma_{\mathbb{B}_\sigma(\mu_2)}(z) \\ &= \Sigma_{\mathbb{B}_\sigma(\mu_1)\boxtimes\mathbb{B}_\sigma(\mu_2)}(z)\end{aligned}$$

in some interval $(-\beta, 0)$ and so $\mathbb{B}_\sigma(\mu_1\boxtimes\mu_2) = \mathbb{B}_\sigma(\mu_1)\boxtimes\mathbb{B}_\sigma(\mu_2)$. The compositional relation is proved as follows:

$$\begin{aligned}\Sigma_{\mathbb{B}_{\sigma_1}(\mathbb{B}_{\sigma_2}(\mu))}(z) &= \Sigma_{\mathbb{B}_{\sigma_2}(\mu)}(\eta_{\sigma_1}(z)) = \Sigma_\mu(\eta_{\sigma_2}(\eta_{\sigma_1}(z)) \\ &= \Sigma_\mu(\eta_{\sigma_2 \circlearrowright \sigma_1}(z)) = \Sigma_{\mathbb{B}_{\sigma_2\circlearrowright\sigma_1}(\mu)}(z)\end{aligned}$$

in some interval $(-\gamma, 0)$ and hence $\mathbb{B}_{\sigma_1}\circ\mathbb{B}_{\sigma_2} = \mathbb{B}_{\sigma_2\circlearrowright\sigma_1}$ on $\mathcal{P}(\mathbb{R}_+)$.

On the unit circle case, the same computation is valid in suitable neighborhoods of 0 of Σ-transforms when probability measures have non-vanishing means. If probability measures have vanishing means, then we may use the results of [RS07] or we may resort to an approximation argument by using moments.

3.2 Proof of Theorem 1.2

The measures ν_t satisfy $\eta_{\nu_t}\circ\eta_{\nu_s} = \eta_{\nu_{t+s}}$ and $\eta_{\nu_0}(z) = z$. By differentiating this compositional relation as regards s at 0, we obtain the partial differential equation

$$\frac{\partial\eta_{\nu_t}}{\partial t}(z) = zK(z)\frac{\partial\eta_{\nu_t}}{\partial z}(z). \tag{3.2}$$

Let $\eta = \eta(t,z) := \eta_{\mathbb{B}_{\nu_t}(\mu)}(z)$ for simplicity, and let η^{-1} denote the compositional inverse with respect to z. From (3.1) and (3.2), the partial derivatives $\frac{\partial\eta^{-1}}{\partial t}$ and $\frac{\partial\eta^{-1}}{\partial z}$ satisfy the following:

$$\frac{\frac{\partial\eta^{-1}}{\partial t}(t,z)}{\frac{\partial\eta^{-1}}{\partial z}(t,z) - \frac{\eta^{-1}(t,z)}{z}} = \frac{z\Sigma'_\mu(\eta_{\nu_t})\frac{\partial\eta_{\nu_t}}{\partial t}(t,z)}{z\Sigma'_\mu(\eta_{\nu_t})\frac{\partial\eta_{\nu_t}}{\partial z}(t,z)} = zK(z). \tag{3.3}$$

Differentiate $\eta(t,\eta^{-1}(t,z)) = z$ with respect to t and z, and then we have

$$\frac{\partial\eta^{-1}}{\partial t}(t,\eta) = -\frac{\frac{\partial\eta}{\partial t}(t,z)}{\frac{\partial\eta}{\partial z}(t,z)}, \qquad \frac{\partial\eta^{-1}}{\partial z}(t,\eta) = \frac{1}{\frac{\partial\eta}{\partial z}(t,z)}. \tag{3.4}$$

Replace z by $\eta(t,z)$ in Eq. (3.3) and then, thanks to (3.4), we have

$$\frac{\partial\eta}{\partial t} = -K(\eta)\left(\eta - z\frac{\partial\eta}{\partial z}\right). \tag{3.5}$$

By the way, from (2.2), the function H can be written as

$$H(t,z) = z - F_{\mathbb{B}_{\nu_t}}(z) = z\eta(t,1/z),$$

so that

$$\frac{\partial H}{\partial t}(t,1/z) = \frac{1}{z}\frac{\partial\eta}{\partial t}(t,z), \quad \frac{\partial H}{\partial z}(t,1/z) = \eta(t,z) - z\frac{\partial\eta}{\partial z}(t,z). \tag{3.6}$$

Combining (3.4) and (3.6), the desired equation follows.

3.3 Examples

We present three examples mentioned in the introduction including the Belinschi-Nica semigroup.

Proposition 3.1. *Recall that the measures $\boldsymbol{\rho}_t, \boldsymbol{\sigma}_t$ are defined by*

$$\boldsymbol{\rho}_t = (1-t)\delta_0 + t\delta_1, \quad \boldsymbol{\sigma}_t = \frac{t}{1+t}\delta_0 + \frac{1}{1+t}\delta_{1+t}.$$

Then

$$\begin{aligned}
\mathbb{B}_{\delta_{1/t}}(\mu) &= \mathbf{D}_{1/t}(\mu^{\uplus t}), && t > 0, && (3.7)\\
\mathbb{B}_{\boldsymbol{\rho}_{1/t}}(\mu) &= \mathbf{D}_{1/t}(\mu^{\boxplus t}), && t \geq 1, && (3.8)\\
\mathbb{B}_{\boldsymbol{\sigma}_t}(\mu) &= (\mu^{\boxplus(1+t)})^{\uplus \frac{1}{1+t}}, && t \geq 0, && (3.9)\\
\delta_s \circlearrowright \delta_t &= \delta_{st}, && s,t > 0, && (3.10)\\
\boldsymbol{\rho}_s \circlearrowright \boldsymbol{\rho}_t &= \boldsymbol{\rho}_{st}, && s,t > 0, && (3.11)\\
\boldsymbol{\sigma}_s \circlearrowright \boldsymbol{\sigma}_t &= \boldsymbol{\sigma}_{s+t}, && s,t \geq 0. && (3.12)
\end{aligned}$$

Proof. We can compute the following:

$$\begin{aligned}
\eta_{\delta_{1/t}}(z) &= \tfrac{z}{t}, && (3.13)\\
\eta_{\boldsymbol{\rho}_{1/t}}(z) &= \tfrac{z}{t+(1-t)z}, && (3.14)\\
\eta_{\boldsymbol{\sigma}_t}(z) &= \tfrac{z}{1-tz}. && (3.15)
\end{aligned}$$

Then (3.10) – (3.12) are easy to prove. Moreover, the following formulas are known for $t > 0$ and $\mu \in \mathcal{P}(\mathbb{R}_+)$ ([BN08]):

$$\begin{aligned}
\Sigma_{\mu^{\uplus t}}(z) &= \frac{1}{t}\Sigma_\mu\left(\frac{z}{t}\right), && (3.16)\\
\Sigma_{\mu^{\boxplus t}}(z) &= \frac{1}{t}\Sigma_\mu\left(\frac{z}{t+(1-t)z}\right), && (3.17)\\
\Sigma_{\mathbf{D}_t(\mu)}(z) &= \frac{1}{t}\Sigma_\mu(z). && (3.18)
\end{aligned}$$

From (3.1), (3.13), (3.16) and (3.18), it follows that

$$\Sigma_{\mathbb{B}_{\delta_{1/t}}(\mu)}(z) = \Sigma_\mu\left(\frac{z}{t}\right) = \Sigma_{\mathbf{D}_{1/t}(\mu^{\uplus t})}(z).$$

The proof of (3.8) is similar.

We can easily show that $\eta_{\boldsymbol{\sigma}_t} = \eta_{\boldsymbol{\rho}_{1/(1+t)}} \circ \eta_{\delta_{1+t}}$, and hence $\boldsymbol{\sigma}_t = \boldsymbol{\rho}_{1/(1+t)} \circlearrowright \delta_{1+t}$. Now Theorem 1.1 is applicable:

$$\mathbb{B}_{\boldsymbol{\sigma}_t}(\mu) = \mathbb{B}_{\delta_{1+t}}(\mathbb{B}_{\boldsymbol{\rho}_{1/(1+t)}}(\mu)) = (\mu^{\boxplus(1+t)})^{\uplus\frac{1}{1+t}}.$$

Note that the dilation operator $\mathbf{D}_s$ commutes with the free power and Boolean power. □

The following "commutation relation" was proved in [BN08] and was crucial in the proof of the semigroup property $\mathbb{B}_t \circ \mathbb{B}_s = \mathbb{B}_{t+s}$. We are able to give a different proof in terms of multiplicative subordination functions.

Proposition 3.2. *Let $\mu \in \mathcal{P}(\mathbb{R})$ and p, q be two real numbers such that $p \geq 1$ and $1 - \frac{1}{p} < q$. Then*

$$(\mu^{\boxplus p})^{\uplus q} = (\mu^{\uplus q'})^{\boxplus p'}, \tag{3.19}$$

where p', q' are defined by $p' := pq/(1 - p + pq)$ and $q' := 1 - p + pq$.

Proof. This is a straight consequence of the relation

$$\boldsymbol{\rho}_{1/p} \circlearrowright \delta_{1/q} = \delta_{1/q'} \circlearrowright \boldsymbol{\rho}_{1/p'}$$

and (1.3). □

When $\mu = \nu$, the subordination map has a special meaning.

Example 3.3. *The map $\mu \mapsto \mathbb{B}_\mu(\mu)$ is the multiplicative Boolean-to-free Bercovici-Pata map on $\mathcal{P}(\mathbb{R}_+) \setminus \{\delta_0\}$, since $\Sigma_{\mathbb{B}_\mu(\mu)}(z) = \Sigma_\mu(\eta_\mu(z)) = \frac{z}{\eta_\mu(z)}$ [AH13b]. The measure $\mathbb{B}_\mu(\mu)$ is $\boxtimes$-infinitely divisible for any $\mu \in \mathcal{P}(\mathbb{R}_+) \setminus \{\delta_0\}$, but this map is not surjective onto the set of $\boxtimes$-infinitely divisible measures.*

Example 3.4. *We specialize in $\mathbb{B}_t$ by taking $\nu_t := \boldsymbol{\sigma}_t$. In this case $\eta_{\nu_t} = \frac{z}{1-tz}$ and so*

$$K(z) = \frac{1}{z}\frac{d}{dt}\bigg|_0 \eta_{\nu_t}(z) = z.$$

Thus we recover the complex Burgers equation of Belinschi and Nica:

$$\frac{\partial H}{\partial t} = -H\frac{\partial H}{\partial z}.$$

Example 3.5. *Let $\nu_t = \delta_{e^{-t}}$ and consider $\mathbb{B}_{\delta_{e^{-t}}}(\mu) = \mathbf{D}_{e^{-t}}\mu^{\uplus e^t}$. Then $\eta_{\nu_t} = ze^{-t}$ and so*

$$K(z) = \frac{1}{z}\frac{d}{dt}\bigg|_0 \eta_{\nu_t}(z) = -1,$$

and so we obtain

$$\frac{\partial H}{\partial t} = z\frac{\partial H}{\partial z}.$$

Example 3.6. *Since $\{\boldsymbol{\rho}_t\}_{t\geq 0}$ satisfies $\boldsymbol{\rho}_s \circlearrowright \boldsymbol{\rho}_t = \boldsymbol{\rho}_{st}$ for $s, t \geq 0$, we put $\nu_t = \boldsymbol{\rho}_{e^{-t}}$. Consider the composition semigroup $\mathbb{B}_{\nu_t}(\mu) = \mathbf{D}_{e^{-t}}\mu^{\boxplus e^t}$. Then*

$$K(z) = \frac{1}{z}\frac{d}{dt}\bigg|_0 \frac{z}{e^t + (1 - e^t)z} = z - 1.$$

Thus the differential equation becomes

$$\frac{\partial H}{\partial t} = (z - H)\frac{\partial H}{\partial z}.$$

Example 3.7. *The Boolean stable law* $\mathbf{b}^+_\alpha$ *satisfies* $\mathbf{b}^+_\alpha \circlearrowleft \mathbf{b}^+_\beta = \mathbf{b}^+_{\alpha\beta}$ *for* $\alpha, \beta \in (0,1]$*; see (2.14). So we define* $\nu_t = \mathbf{b}^+_{e^{-t}}$ *to obtain a composition semigroup. The generator is computed as*

$$K(z) = -\frac{1}{z}\frac{d}{dt}\bigg|_0 (-z)^{e^{-t}} = -\log(-z).$$

The differential equation for $H(t,z)$ *is:*

$$\frac{\partial H}{\partial t} = z \log\left(\frac{H}{z}\right)\frac{\partial H}{\partial z}.$$

Example 3.8. *The beta distribution* $\boldsymbol{\beta}_\alpha$ *satisfies* $\boldsymbol{\beta}_\alpha \circlearrowleft \boldsymbol{\beta}_\gamma = \boldsymbol{\beta}_{\alpha\gamma}$ *for* $\alpha, \gamma \in [0,1]$ *and* $\boldsymbol{\beta}_1 = \delta_1$*; see (2.19). So we define* $\nu_t := \boldsymbol{\beta}_{e^{-t}}$ *and consider the composition semigroup* $\mathbb{B}_{\nu_t}$*. The generator* K *is given by*

$$K(z) = \frac{1}{z}\frac{d}{dt}\bigg|_0 \left(1-(1-z)^{e^{-t}}\right) = \frac{(1-z)\log(1-z)}{z},$$

and so the differential equation for $H(t,z)$ *is given by*

$$\frac{\partial H}{\partial t} = z\left(\frac{z}{H}-1\right)\log\left(1-\frac{H}{z}\right)\frac{\partial H}{\partial z}.$$

The maps $\mathbb{B}_{\mathbf{b}_\alpha}$ and $\mathbb{B}_{\boldsymbol{\beta}^+_\alpha}$ will be investigated more in Section 5.

4 The main results: Additive case

4.1 Proof of Theorem 1.3

The measure $\mathbb{A}_\nu(\mu)$ is characterized by

$$\phi_{\mathbb{A}_\nu(\mu)}(z) = \phi_\mu(F_\nu(z)), \quad z \in \Gamma_{\lambda,M} \tag{4.1}$$

for some $\lambda, M > 0$. The homomorphism property follows from (4.1):

$$\begin{aligned}\phi_{\mathbb{A}_\nu(\mu_1\boxplus\mu_2)}(z) &= \phi_{\mu_1\boxplus\mu_2}(F_\nu(z)) = \phi_{\mu_1}(F_\nu(z)) + \phi_{\mu_2}(F_\nu(z)) \\ &= \phi_{\mathbb{A}_\nu(\mu_1)}(z) + \phi_{\mathbb{A}_\nu(\mu_2)}(z) = \phi_{\mathbb{A}_\nu(\mu_1)\boxplus\mathbb{A}_\nu(\mu_2)}(z),\end{aligned}$$

and so $\mathbb{A}_\nu(\mu_1 \boxplus \mu_2) = \mathbb{A}_\nu(\mu_1) \boxplus \mathbb{A}_\nu(\mu_2)$. The compositional relation is proved as follows:

$$\begin{aligned}\phi_{\mathbb{A}_{\nu_1}(\mathbb{A}_{\nu_2}(\mu))}(z) &= \phi_{\mathbb{A}_{\nu_2}(\mu)}(F_{\nu_1}(z)) = \phi_\mu(F_{\nu_2}(F_{\nu_1}(z)) \\ &= \phi_\mu(F_{\nu_2\rhd\nu_1}(z)) = \phi_{\mathbb{A}_{\nu_2\rhd\nu_1}(\mu)}(z),\end{aligned}$$

and hence $\mathbb{A}_{\nu_1} \circ \mathbb{A}_{\nu_2} = \mathbb{A}_{\nu_2\rhd\nu_1}$.

4.2 Proof of Theorem 1.4

The measures ν_t satisfy $F_{\nu_t} \circ F_{\nu_s} = F_{\nu_{t+s}}$ and $F_{\nu_0}(z) = z$. By differentiating this compositional relation as regards s at 0, the following partial differential equation is obtained:

$$\frac{\partial F_{\nu_t}}{\partial t}(z) = J(z)\frac{\partial F_{\nu_t}}{\partial z}(z). \tag{4.2}$$

Let $F = F(t,z) := F_{\mathbb{A}_{\nu_t}(\mu)}(z)$, and let F^{-1} denote the compositional inverse with respect to z. From (4.1) and (4.2), the partial derivatives $\frac{\partial F^{-1}}{\partial t}$ and $\frac{\partial F^{-1}}{\partial z}$ satisfy the following:

$$\frac{\frac{\partial F^{-1}}{\partial t}(t,z)}{\frac{\partial F^{-1}}{\partial z}(t,z) - 1} = \frac{\phi'_\mu(F_{\nu_t})\frac{\partial F_{\nu_t}}{\partial t}(t,z)}{\phi'_\mu(F_{\nu_t})\frac{\partial F_{\nu_t}}{\partial z}(t,z)} = J(z). \tag{4.3}$$

Differentiating $F(t, F^{-1}(t,z)) = z$ with respect to t and z, we have

$$\frac{\partial F^{-1}}{\partial t}(t,F) = -\frac{\frac{\partial F}{\partial t}(t,z)}{\frac{\partial F}{\partial z}(t,z)}, \qquad \frac{\partial F^{-1}}{\partial z}(t,F) = \frac{1}{\frac{\partial F}{\partial z}(t,z)}. \tag{4.4}$$

Replace z by $F(z)$ in Eq. (4.3), thanks to (4.4), we have

$$\frac{\partial F}{\partial t} = J(F)\left(\frac{\partial F}{\partial z} - 1\right). \tag{4.5}$$

The conclusion therefore follows.

4.3 Examples

The following is the additive analog of Example 3.3.

Example 4.1. *The map $\mu \mapsto \mathbb{A}_\mu(\mu)$ is the additive Boolean-to-free Bercovici-Pata bijection on $\mathcal{P}(\mathbb{R})$, since $\phi_{\mathbb{A}_\mu(\mu)}(z) = \phi_\mu(F_\mu(z)) = z - F_\mu(z)$. A measure ν can be written as $\nu = \mathbb{A}_\mu(\mu)$ for some $\mu \in \mathcal{P}(\mathbb{R})$ if and only if ν is $\boxplus$-infinitely divisible.*

Example 4.2. *Let ν_t be the arcsine law with mean 0 and variance t:*

$$\nu_t(dx) = \frac{1}{\pi\sqrt{2t - x^2}} 1_{(-\sqrt{2t},\sqrt{2t})}(x)\,dx.$$

Then it holds that $\nu_{s+t} = \nu_s \rhd \nu_t$, $s,t \geq 0$, $\nu_0 = \delta_0$, because F_{ν_t} is given by $\sqrt{z^2 - 2t}$. The generator is given by $J(z) = -\frac{1}{z}$. So the differential equation of Theorem 1.4 becomes

$$\frac{\partial G}{\partial t} = \frac{1}{G - z}\frac{\partial G}{\partial z}.$$

Example 4.3. *Let ν_t be $\mathbf{c}_{at,bt}$, where $\mathbf{c}_{a,b}$ is the Cauchy distribution*

$$\mathbf{c}_{a,b}(dx) = \frac{b}{\pi[(x-a)^2 + b^2]} 1_{\mathbb{R}}(x)\,dx, \quad a \in \mathbb{R}, \quad b > 0.$$

We define $\mathbf{c}_{a,0} = \delta_a$ considering the weak continuity. Then $\nu_{s+t} = \nu_s \rhd \nu_t$, $s,t \geq 0$, $\nu_0 = \delta_0$. The generator J is the constant $-a + bi$, and so the differential equation is

$$\frac{\partial G}{\partial t} = (-a + bi)\frac{\partial G}{\partial z}.$$

5 Examples of multiplicative free subordination

We study the subordination map $\mathbb{B}_\nu$ when $\nu = \boldsymbol{\beta}_\alpha$ or $\mathbf{b}_\alpha^+$ and find its relation with the Markov transform. Many computations in this section may be extended to symmetric probability measures by using the S-transform for symmetric measures, but we restrict ourselves to the positive and negative cases.

5.1 Transforms $\mathbf{M}_\alpha^+$ and $\mathbf{U}_\alpha^+$

Definition 5.1. (1) *For $\mu \in \mathcal{P}(\mathbb{R}_+)$ and $\alpha \in (0,1]$, let $\mathbf{M}_\alpha^+ : \mathcal{P}(\mathbb{R}_+) \to \mathcal{P}(\mathbb{R}_+)$ be the map defined by*

$$\mathbf{M}_\alpha^+(\mu) = \mathbb{B}_{\boldsymbol{\beta}_\alpha}(\mu)^{\boxtimes 1/\alpha} \boxtimes \mathbf{m}_\alpha^+. \tag{5.1}$$

(2) *For $\mu \in \mathcal{P}(\mathbb{R}_+)$ and $\alpha \in (0,1]$, let $\mathbf{U}_\alpha^+ : \mathcal{P}(\mathbb{R}_+) \to \mathcal{P}(\mathbb{R}_+)$ be the map defined by*

$$\mathbf{U}_\alpha^+(\mu) = \mathbb{B}_{\mathbf{b}_\alpha^+}(\mu)^{\boxtimes 1/\alpha}. \tag{5.2}$$

Theorem 5.2. (1) *The probability measure $\mathbf{M}_\alpha^+(\mu)$ is characterized by*

$$F_{\mathbf{M}_\alpha^+(\mu)}(z) = -(-F_\mu(-(-z)^\alpha))^{1/\alpha}, \qquad z < 0, \quad \mu \in \mathcal{P}(\mathbb{R}_+). \tag{5.3}$$

The map $\mathbf{M}_\alpha^+$ satisfies a semigroup and a homomorphism properties:

$$\mathbf{M}_\alpha^+ \circ \mathbf{M}_\alpha^+ = \mathbf{M}_{\alpha\beta}^+; \tag{5.4}$$

$$\mathbf{M}_\alpha^+(\mu \rhd \nu) = \mathbf{M}_\alpha^+(\mu) \rhd \mathbf{M}_\alpha^+(\nu). \tag{5.5}$$

(2) *The probability measure $\mathbf{U}_\alpha^+(\mu)$ is characterized by*

$$\eta_{\mathbf{U}_\alpha^+(\mu)}(z) = -\left(-\eta_\mu(-(-z)^\alpha)\right)^{1/\alpha}, \qquad z < 0, \quad \mu \in \mathcal{P}(\mathbb{R}_+). \tag{5.6}$$

Moreover, we have a semigroup and homomorphism properties:

$$\mathbf{U}_\alpha^+ \circ \mathbf{U}_\beta^+ = \mathbf{U}_{\alpha\beta}^+; \tag{5.7}$$

$$\mathbf{U}_\alpha^+(\mu \circlearrowright \nu) = \mathbf{U}_\alpha^+(\mu) \circlearrowright \mathbf{U}_\alpha^+(\nu); \tag{5.8}$$

$$\mathbf{U}_\alpha^+(\mu \boxtimes \nu) = \mathbf{U}_\alpha^+(\mu) \boxtimes \mathbf{U}_\alpha^+(\nu). \tag{5.9}$$

Remark 5.3. (1) *The fact that $\mathbf{M}_\alpha^+$ is a homomorphism regarding $\rhd$ may be seen as the monotonic counterpart of [AH, Theorem 4.14], where the maps*

$$\mathbf{B}_\alpha^+(\mu) = \mu^{1/\alpha} \circledast \mathbf{b}_\alpha^+ = \mu^{\boxtimes 1/\alpha} \boxtimes \mathbf{b}_\alpha^+,$$
$$\mathbf{N}_\alpha^+(\mu) = \mu^{1/\alpha} \circledast \mathbf{n}_\alpha^+,$$
$$\mathbf{F}_\alpha^+(\mu) = \mu^{\boxtimes 1/\alpha} \boxtimes \mathbf{f}_\alpha^+$$

are shown to be homomorphisms with respect to $\uplus, *, \boxplus$, respectively (the measure $\mathbf{n}_\alpha^+$ is a classical positive α-stable law). They also become compositional semigroups: $\mathbf{B}_\alpha^+ \circ \mathbf{B}_\beta^+ = \mathbf{B}_{\alpha\beta}^+$, $\mathbf{N}_\alpha^+ \circ \mathbf{N}_\beta^+ = \mathbf{N}_{\alpha\beta}^+$, $\mathbf{F}_\alpha^+ \circ \mathbf{F}_\beta^+ = \mathbf{F}_{\alpha\beta}^+$.

(2) *The formula (5.3) becomes simpler on* $\mathbb{R}_-$. *We introduce the conjugated map* $\mathbf{M}_\alpha^- := \mathbf{D}_{-1} \circ \mathbf{M}_\alpha^+ \circ \mathbf{D}_{-1}$ *on* $\mathcal{P}(\mathbb{R}_-)$. *Then the formula (5.1) changes to*

$$\mathbf{M}_\alpha^-(\mu) = \mathbb{B}_{\boldsymbol{\beta}_\alpha}(\mathbf{D}_{-1}(\mu))^{\boxtimes 1/\alpha} \boxtimes \mathbf{m}_\alpha^-. \tag{5.10}$$

and (5.3) can be written as

$$F_{\mathbf{M}_\alpha^-(\mu)}(z) = (F_\mu(z^\alpha))^{1/\alpha}, \qquad z > 0. \tag{5.11}$$

We can also define $\mathbf{U}_\alpha^-$ *on* $\mathcal{P}(\mathbb{R}_-)$, *but this does not simplify (5.6) so much.*

Proof. We are going to show (5.11) instead of (5.3). Let $F(z) := (F_\mu(z^\alpha))^{1/\alpha}$ and $\eta(z) := 1 - zF(1/z)$. Using the relation (2.2) twice, we have

$$\eta(z) = 1 - z\left(F_\mu\left(\frac{1}{z^\alpha}\right)\right)^{1/\alpha} = 1 - (1 - \eta_\mu(z^\alpha))^{1/\alpha},\ z > 0,$$

so that η is strictly increasing on $(-\infty, 0)$ and it has the compositional inverse

$$\eta^{-1}(w) = [\eta_\mu^{-1}(1 - (1-w)^\alpha)]^{1/\alpha} = \left(\eta_\mu^{-1}(\eta_{\boldsymbol{\beta}_\alpha}(w))\right)^{1/\alpha},\ w \in (-\varepsilon, 0)$$

for some $\varepsilon > 0$. Hence

$$\begin{aligned}
\Sigma(w) := \frac{\eta^{-1}(w)}{w} &= -\left(\frac{\eta_\mu^{-1}(\eta_{\boldsymbol{\beta}_\alpha}(w))}{(-w)^\alpha}\right)^{1/\alpha} \\
&= -\left(\frac{(-\eta_{\boldsymbol{\beta}_\alpha}(w))(-\Sigma_\mu \circ \eta_{\boldsymbol{\beta}_\alpha}(w))}{(-w)^\alpha}\right)^{1/\alpha} \\
&= \frac{((1-w)^\alpha - 1)^{1/\alpha}}{-w}(\Sigma_{\mathbb{B}_{\boldsymbol{\beta}_\alpha}(\mathbf{D}_{-1}(\mu))}(w))^{1/\alpha} \\
&= \Sigma_{\mathbf{m}_\alpha^-}(w)\Sigma_{\mathbb{B}_{\boldsymbol{\beta}_\alpha}(\mathbf{D}_{-1}(\mu))^{\boxtimes 1/\alpha}}(w), \\
&= \Sigma_{\mathbf{M}_\alpha^-(\mu)}, \qquad w \in (-\varepsilon', 0)
\end{aligned}$$

for some $\varepsilon' > 0$, where the formula (2.22) was used on the fourth line. This computation shows (5.11). (5.4) and (5.5) are simple applications of (5.3).

The proof of (5.6) is following. We obtain

$$\begin{aligned}
\frac{1}{w}\eta_{\mathbf{U}_\alpha^+(\mu)}^{-1}(w) &= \Sigma_{\mathbf{U}_\alpha^+(\mu)}(w) = \Sigma_{\mathbb{B}_{\mathbf{b}_\alpha^+}(\mu)^{\boxtimes 1/\alpha}}(w) = \Sigma_{\mathbb{B}_{\mathbf{b}_\alpha^+}(\mu)}(w)^{1/\alpha} \\
&= (\Sigma_\mu(-(-w)^\alpha))^{1/\alpha} = \frac{1}{-w}(-\eta_\mu^{-1}(-(-w)^\alpha))^{1/\alpha}, \quad w \in (-\delta, 0)
\end{aligned}$$

for some $\delta > 0$, where (3.1) was used on the fourth equality. After some more computation we have (5.6). (5.7) and (5.8) are easy applications of (5.6). (5.9) follows from the relation $\Sigma_{\mathbf{U}_\alpha^+(\mu)}(w) = (\Sigma_\mu(-(-w)^\alpha))^{1/\alpha}$. $\square$

Example 5.4. *What is shown below is several computations involving* $\mathbf{M}_\alpha^+$, $\mathbf{U}_\alpha^+$, $\mathbb{B}_{\boldsymbol{\beta}_\alpha}$ *and* $\mathbb{B}_{\mathbf{b}_\alpha^+}$ $(0 < \alpha \leq 1)$.

(1) $\mathbf{M}_\alpha^+(\mathbf{m}_\beta^+) = \mathbf{m}_{\alpha\beta}^+$. *In particular,* $\mathbf{M}_\alpha^+(\delta_1) = \mathbf{m}_\alpha^+$.

(2) $\mathbf{M}_\alpha^+(\boldsymbol{\beta}_\alpha) = \mathbf{b}_\alpha^+$.

(3) $\mathbf{M}_\alpha^+(\boldsymbol{\rho}_\alpha) = \boldsymbol{\beta}_\alpha$.

(4) $\mathbb{B}_{\boldsymbol{\beta}_\alpha}(\boldsymbol{\pi}^{\boxtimes 1/\alpha}) = \boldsymbol{\pi}$, *since*

$$\Sigma_{\mathbb{B}_{\boldsymbol{\beta}_\alpha}(\boldsymbol{\pi}^{\boxtimes 1/\alpha})}(z) = \Sigma_{\boldsymbol{\pi}^{\boxtimes 1/\alpha}}(\eta_{\boldsymbol{\beta}_\alpha}(z)) = \Sigma_{\boldsymbol{\pi}}(\eta_{\boldsymbol{\beta}_\alpha}(z))^{1/\alpha} = 1 - z = \Sigma_{\boldsymbol{\pi}}(z).$$

(5) $\mathbf{M}_\alpha^+(\boldsymbol{\pi}) = \mathbf{m}_\alpha^+ \boxtimes \boldsymbol{\pi}$ *because of (4) above and (5.1).*

(6) *In [AH13a], a family of measures is introduced. Part of that family is $\mu_{\alpha,p}$ which is characterized by*

$$G_{\mu_{\alpha,p}}(z) = -\left(\frac{\left(1 + \left(-\frac{1}{z}\right)^\alpha\right)^p - 1}{p}\right)^{1/\alpha}, \quad z \in \mathbb{C}^+, \tag{5.12}$$

for $\alpha, p \in (0,1]$.[1] *The powers $z \mapsto z^\alpha$ and $z \mapsto z^{1/\alpha}$ are the principal value. We have the identity*

$$\mu_{\alpha,p} = \mathbf{M}_\alpha^+(\boldsymbol{\beta}_{1-p,1+p}), \quad \alpha, p \in (0,1], \tag{5.13}$$

which can be checked by using the fact that the beta distribution $\boldsymbol{\beta}_{1-p,1+p}$ coincides with $\mu_{1,p}$ [AH13a].

(7) $\mathbb{B}_{\mathbf{b}_\alpha^+}\left(\mathbf{b}^+_{\frac{\alpha\beta}{1-\beta+\alpha\beta}}\right) = \mathbf{b}_\beta^+$, $\alpha, \beta \in (0,1]$.

(8) $\mathbf{M}_\alpha^+(\mu) = \mathbf{m}_\alpha^+ \circlearrowright \mathbf{U}_\alpha^+(\mu)$ *for $\mu \in \mathcal{P}(\mathbb{R}_+)$. This is proved as follows. First we can show that*

$$\mathbf{b}_\alpha^+ = \boldsymbol{\beta}_\alpha \circlearrowright \mathbf{m}_\alpha^+, \quad \alpha \in (0,1]. \tag{5.14}$$

by computing η-transforms. Starting from the relations (5.1) and (1.1), we have

$$\mathbf{M}_\alpha^+(\mu) = \mathbf{m}_\alpha^+ \boxtimes \mathbb{B}_{\boldsymbol{\beta}_\alpha}(\mu^{\boxtimes 1/\alpha}) = \mathbf{m}_\alpha^+ \circlearrowright \mathbb{B}_{\mathbf{m}_\alpha^+}(\mathbb{B}_{\boldsymbol{\beta}_\alpha}(\mu^{\boxtimes 1/\alpha})).$$

Thanks to Theorem 1.1 and (5.14), we get $\mathbb{B}_{\mathbf{m}_\alpha^+} \circ \mathbb{B}_{\boldsymbol{\beta}_\alpha} = \mathbb{B}_{\boldsymbol{\beta}_\alpha \circlearrowright \mathbf{m}_\alpha^+} = \mathbb{B}_{\mathbf{b}_\alpha^+}$.

(9) $\mu^{1/\alpha} \circledast \mathbf{b}_\alpha^+ = \mu^{\boxtimes 1/\alpha} \boxtimes \mathbf{b}_\alpha^+ = \mu \circlearrowright \mathbf{b}_\alpha^+ = \boldsymbol{\beta}_\alpha \circlearrowright \mathbf{M}_\alpha^+(\mu)$ *for $\mu \in \mathcal{P}(\mathbb{R}_+)$. The first and second equalities were proved in [AH]. The last one is proved as follows:*

$$\begin{aligned}\eta_{\boldsymbol{\beta}_\alpha \circlearrowright \mathbf{M}_\alpha^+(\mu)}(z) &= 1 - (1 - \eta_{\mathbf{M}_\alpha^+(\mu)}(z))^\alpha = 1 - \left(zF_{\mathbf{M}_\alpha^+(\mu)}(1/z)\right)^\alpha \\ &= 1 + (-z)^\alpha F_\mu(-(-z)^{-\alpha}) = \eta_\mu(-(-z)^\alpha), \quad z < 0.\end{aligned}$$

The last expression is equal to $\eta_{\mathbf{b}_\alpha^+ \circledast \mu^{1/\alpha}}(z)$ as shown in [AH].

[1] The parametrization is different from that in the original paper [AH13a], and the dilation is omitted here.

5.2 Markov transform, mixture of Boolean stable laws and $\mathbf{M}_\alpha^+$

Definition 5.5. *Let $\mathcal{M}(\nu)$ be the Markov (or Markov-Krein) transform of ν defined by*

$$G_{\mathcal{M}(\nu)}(z) = \exp\left(\int_{\mathbb{R}} \log\left(\frac{1}{z-x}\right)\, \nu(dx)\right);$$

see [K98]. In general ν can be taken to be a Rayleigh measure, but now we consider a probability measure ν satisfying $\int_{\mathbb{R}} \log(2+|x|)\,\nu(dx) < \infty$. It is characterized by the differential equation

$$\frac{d}{dz} G_{\mathcal{M}(\nu)}(z) = -G_\nu(z) G_{\mathcal{M}(\nu)}(z). \tag{5.15}$$

We compute the Markov transforms of classical scale mixtures of Boolean stable laws $\mu^{1/\alpha} \circledast \mathbf{b}_\alpha^+$, $\mu \in \mathcal{P}(\mathbb{R}_+)$.

Proposition 5.6. *Let $\alpha \in (0,1]$ as usual. Then*

$$\mathcal{M}(\nu^{1/\alpha} \circledast \mathbf{b}_\alpha^+) = \mathbf{M}_\alpha^+(\mathcal{M}(\nu)), \quad \nu \in \mathcal{P}(\mathbb{R}_+). \tag{5.16}$$

Remark 5.7. (1) *In terms of $\mathbb{R}_-$, we have*

$$\mathcal{M}((\mathbf{D}_{-1}(\nu))^{1/\alpha} \circledast \mathbf{b}_\alpha^-) = \mathbf{M}_\alpha^-(\mathcal{M}(\nu)), \quad \nu \in \mathcal{P}(\mathbb{R}_-). \tag{5.17}$$

(2) *In terms of the map $\mathbf{B}_\alpha^+$ (see Remark 5.3), we may write (5.16) as*

$$\mathcal{M} \circ \mathbf{B}_\alpha^+ = \mathbf{M}_\alpha^+ \circ \mathcal{M}, \tag{5.18}$$

i.e. the Markov transform intertwines the compositional semigroups $(\mathbf{B}_\alpha^+)_{\alpha\in(0,1]}$ and $(\mathbf{M}_\alpha^+)_{\alpha\in(0,1]}$.

Proof. We consider the formula (5.17). Let $\mu = \mathcal{M}(\nu)$ for $\nu \in \mathcal{P}(\mathbb{R}_-)$. It follows from elementary calculus that

$$\begin{aligned} \frac{d}{dz} \log G_{\mathbf{M}_\alpha^-(\mu)}(z) &= \frac{1}{\alpha}\frac{d}{dz} \log\left(G_\mu(z^\alpha)\right) = z^{\alpha-1}\frac{G_\mu'(z^\alpha)}{G_\mu(z^\alpha)} \\ &= -z^{\alpha-1} G_\nu(z^\alpha) = z^{\alpha-1} G_{\mathbf{D}_{-1}(\nu)}(-z^\alpha) \\ &= -G_{(\mathbf{D}_{-1}(\nu))^{1/\alpha}\circledast \mathbf{b}_\alpha^-}(z), \quad z>0. \end{aligned}$$

The last line is a consequence of the formula (4.4) in [AH]. This shows (5.17). ☐

Example 5.8. (1) $\mathcal{M}(\mathbf{b}_\alpha^+) = \mathbf{M}_\alpha^+(\delta_1) = \mathbf{m}_\alpha^+$ *since* $\mathcal{M}(\delta_1) = \delta_1$.

(2) $\mathcal{M}((\boldsymbol{\rho}_\alpha)^{1/\alpha} \circledast \mathbf{b}_\alpha^+) = \mathbf{b}_\alpha^+$, *since* $\mathbf{M}_\alpha^+(\mathcal{M}(\boldsymbol{\rho}_\alpha)) = \mathbf{M}_\alpha^+(\boldsymbol{\beta}_\alpha) = \mathbf{b}_\alpha^+$ *from Example 5.4(2), (3).*

(3) $\mathcal{M}((\mathbf{f}_\alpha^+)^{\uplus\alpha}) = \mathbf{f}_\alpha^+$. *This is not directly related to Proposition 5.6, but seems interesting to compare with $\mathcal{M}(\mathbf{b}_\alpha^+) = \mathbf{m}_\alpha^+$. The proof is as follows. Note that (5.15) is equivalent to $(\log F_{\mathcal{M}(\nu)})' = 1/F_\nu$. Recalling the relation*

$$F_{\mathbf{f}_\alpha^+}^{-1}(z) = z + (-z)^{1-\alpha}, \qquad z \in \Gamma_{\lambda,M} \tag{5.19}$$

for some $\lambda, M > 0$, we get

$$(\log F_{\mathbf{f}_\alpha^+})' = \frac{1}{F_{\mathbf{f}_\alpha^+} + (1-\alpha)(-F_{\mathbf{f}_\alpha^+})^{1-\alpha}}. \tag{5.20}$$

From (5.19), we have $(-F_{\mathbf{f}_\alpha^+})^{1-\alpha} = z - F_{\mathbf{f}_\alpha^+}$. Combining this with (5.20), we obtain $(\log F_{\mathbf{f}_\alpha^+})' = 1/((1-\alpha)z + \alpha F_{\mathbf{f}_\alpha^+}) = 1/F_{(\mathbf{f}_\alpha^+)^{\uplus\alpha}}$, the conclusion.

(4) *We can show that the inverse Markov transform $\nu_p := \mathcal{M}^{-1}(\boldsymbol{\beta}_{1-p,1+p})$ has the Cauchy transform*

$$G_{\nu_p}(z) = \frac{p}{z} \cdot \frac{(z-1)^{p-1}}{z^p - (z-1)^p} = p\left(\frac{1}{z} + \frac{(z-1)^{p-1} - z^{p-1}}{z^p - (z-1)^p}\right),$$

and so

$$\nu_p = p\delta_0 + \frac{p \sin \pi p}{\pi} \frac{x^{p-1}(1-x)^{p-1}}{x^{2p} - 2x^p(1-x)^p \cos \pi p + (1-x)^{2p}} 1_{(0,1)}(x)\, dx.$$

Using this measure, we have $\mu_{\alpha,p} = \mathcal{M}((\nu_p)^{1/\alpha} \circledast \mathbf{b}_\alpha^+)$ for $0 < \alpha, p \le 1$. Notice the relation $\nu_p = \boldsymbol{\rho}_{1-p} \circledast \tau_p$ exists, where

$$\tau_p(dx) := \frac{p \sin \pi p}{(1-p)\pi} \frac{x^{p-1}(1-x)^{p-1}}{x^{2p} - 2x^p(1-x)^p \cos \pi p + (1-x)^{2p}} 1_{(0,1)}(x)\, dx$$

is the law of a random variable $\mathbb{G}_p$ studied in [BFRY06].

6 Additive subordination by Cauchy distribution

We are going to study $\mathbb{A}_\nu$ in details when ν is the Cauchy distribution $\mathbf{c}_{a,b}$. The corresponding map $\mathbb{A}_{\mathbf{c}_{a,b}}$ gives us freely infinitely divisible distributions with rational function densities. The map $\mathbb{A}_{\mathbf{c}_{a,b}}$ is denoted by $\mathbb{A}_{a,b}$ below.

Proposition 6.1. *For $a \in \mathbb{R}$, $b \ge 0$ and $\mu \in \mathcal{P}(\mathbb{R})$, $\mathbb{A}_{a,b}(\mu)$ is characterized by*

$$F_{\mathbb{A}_{a,b}(\mu)}(z) = F_\mu(z - a + ib) + a - ib, \quad z \in \mathbb{C}^+. \tag{6.1}$$

Proof. The Eq. (4.1) is equivalent to $F^{-1}_{\mathbb{A}_{a,b}(\mu)}(z) - a + ib = F_\mu^{-1}(z - a + ib)$ in a domain $\Gamma_{\lambda,M}$, and so the conclusion follows in a domain of $\mathbb{C}^+$, which in fact is valid in $\mathbb{C}^+$ by analyticity. □

We denote by $m_1(\mu)$, $m_2(\mu)$ and $\sigma^2(\mu)$ the first moment, second moment and variance of a probability measure μ, respectively. The notation $f \sim g$, $|x| \to \infty$ means that $\frac{f(x)}{g(x)} \to 1$ as $|x| \to \infty$.

Proposition 6.2. (1) *For any $\mu, \nu \in \mathcal{P}(\mathbb{R})$ and any $a \in \mathbb{R}, b \ge 0$, we have the homomorphism properties*

$$\mathbb{A}_{a,b}(\mu \rhd \nu) = \mathbb{A}_{a,b}(\mu) \rhd \mathbb{A}_{a,b}(\nu), \tag{6.2}$$

$$\mathbb{A}_{a,b}(\mu \boxplus \nu) = \mathbb{A}_{a,b}(\mu) \boxplus \mathbb{A}_{a,b}(\nu), \tag{6.3}$$

$$\mathbb{A}_{a,b}(\mu \uplus \nu) = \mathbb{A}_{a,b}(\mu) \uplus \mathbb{A}_{a,b}(\nu). \tag{6.4}$$

(2) *For any $a \in \mathbb{R}$, $b > 0$ and any μ which is not a point measure, $\mathbb{A}_{a,b}(\mu)$ has a real analytic density on $\mathbb{R}$. Moreover, if μ has a finite variance, then the density of $\mathbb{A}_{a,b}(\mu)$ $\sim \frac{b\sigma^2(\mu)}{\pi x^4}$ as $|x| \to \infty$.*

(3) *If μ has a finite variance, then $m_1(\mathbb{A}_{a,b}(\mu)) = m_1(\mu)$ and $m_2(\mathbb{A}_{a,b}(\mu)) = m_2(\mu)$.*

Proof. (1) The second equality was proved in Theorem 1.3. The others can be shown from the formula (6.1).

(2) If μ is not a point measure, $\mathrm{Im}(F_\mu(z)) > \mathrm{Im}(z)$ for $z \in \mathbb{C}^+$. Using the Stieltjes inversion formula, we have

$$\mathbb{A}_{a,b}(\mu)(dx) = \frac{\mathrm{Im}(F_\mu(x-a+ib)) - b}{\pi|F_\mu(x-a+ib)+a-ib|^2}dx.$$

The density function is real analytic and strictly positive on $\mathbb{R}$. The reciprocal Cauchy transform F_μ has the Nevanlinna representation as $F_\mu(z) = z - m_1(\mu) + \int_{\mathbb{R}} \frac{\rho_\mu(du)}{u-z}$ for a finite non-negative measure ρ_μ which satisfies $\rho_\mu(\mathbb{R}) = \sigma^2(\mu)$ [M92]. So $F_\mu(x-a+ib)+a-ib \sim x$ as $|x| \to \infty$ and

$$\mathrm{Im}(F_\mu(x-a+ib)+a-ib) = \int_{\mathbb{R}} \frac{b}{(u-x+a)^2+b^2}\rho_\mu(du) \sim \frac{b\sigma^2}{x^2}.$$

Therefore, the density of $\mathbb{A}_{a,b}(\mu)$ behaves as $\sim \frac{b\sigma^2}{\pi x^4}$ as $|x| \to \infty$.

(3) The first moment $m_1(\mu)$ is characterized by $F_\mu(iy) - iy + m_1(\mu) = o(1)$ $(y \to \infty)$, and so $m_1(\mathbb{A}_{a,b}(\mu)) = m_1(\mu)$. The variance $\sigma^2(\mu) = \rho_\mu(\mathbb{R})$ is equal to $\lim_{y\to\infty} |F_\mu(iy) - iy + m_1(\mu)|y$ [M92]. Applying this formula to $\mathbb{A}_{a,b}(\mu)$, we have $\sigma^2(\mu) = \sigma^2(\mathbb{A}_{a,b}(\mu))$ and hence $m_2(\mathbb{A}_{a,b}(\mu)) = m_2(\mu)$. □

We prove that $\mathbb{A}_{a,b}(\mu)$ is freely infinitely divisible for large b if μ has finite variance. We introduce a subclass of freely infinitely divisible distributions characterized by *univalent inverse* Cauchy transforms.

Definition 6.3. *A probability measure μ on $\mathbb{R}$ is said to be in class $\mathcal{UI}$ if F_μ^{-1}, originally defined in some $\Gamma_{\lambda,M}$, extends to a univalent function in $\mathbb{C}^+$.*

The class $\mathcal{UI}$ is a subset of all freely infinitely divisible distributions as proved in [AH13a].

Theorem 6.4. *Let μ be a probability measure with finite variance σ^2. Then $\mathbb{A}_{a,b}(\mu) \in \mathcal{UI}$ for $b \geq 2\sigma$.*

Remark 6.5. *Assume that μ is not $\boxplus$-infinitely divisible. If we use the free divisibility indicator $\phi(\mu)$, the assumption $b \geq 2\sigma$ can be weakened to $b \geq 2\sigma\sqrt{1-\phi(\mu)}$.*

Proof. Let $\mathbb{C}_s := \{z \in \mathbb{C} : \mathrm{Im}(z) > s\}$. Maassen proved that F_μ is univalent in $\mathbb{C}_\sigma$ and $F_\mu(\mathbb{C}_\sigma) \supset \mathbb{C}_{2\sigma}$ ([M92], Lemma 2.4). Therefore, $F_{\mathbb{A}_{a,b}(\mu)}(\mathbb{C}_{-b+\sigma}) = F_\mu(\mathbb{C}_\sigma) + a - ib \supset \mathbb{C}_{2\sigma-b} \supset \mathbb{C}^+$, so that $F^{-1}_{\mathbb{A}_{a,b}(\mu)}$ can be defined in $\mathbb{C}^+$ as a univalent function. □

Starting from atomic measures, we obtain many freely infinitely divisible measures whose densities are rational functions.

Example 6.6. *Let $b > 0, n > 1, \lambda_j > 0, \sum_{j=1}^n \lambda_j = 1$ and $a_j \in \mathbb{R}$. The measure $\mathbb{A}_{a,b}(\sum_{j=1}^n \lambda_j \delta_{a_j})$ has a rational function density. If $b > 0$ is large enough, it is freely infinitely divisible.*

Let $\mu_p := \frac{p}{2}(\delta_{-1} + \delta_1) + (1-p)\delta_0$, $0 \le p \le 1$. Then $G_{\mu_p}(z) = \frac{z^2-1+p}{z(z^2-1)}$, so that $G_{\mathbb{A}_{0,b}(\mu_p)}(z) = \frac{z^2-b^2-1+p+2ibz}{z^3-(1+b^2)z+2ib(z^2-p/2)}$. The measure $\mathbb{A}_{0,b}(\mu_p)$ is given by

$$\mathbb{A}_{0,b}(\mu_p)(dx) = \frac{bp(x^2+b^2+1-p)}{\pi[x^6+2(b^2-1)x^4+(b^4+2b^2(1-2p)+1)x^2+p^2b^2]}dx.$$

From Theorem 6.4, $\mathbb{A}_{0,b}(\mu_p) \in \mathcal{UI}$ for $b \ge 2\sqrt{p}$.

In the particular case $p = 1$ and $b = 2$, $\mathbb{A}_{0,2}(\mu_1)$ is a scaled t-distribution with 3 degrees of freedom:

$$\mathbb{A}_{0,2}(\mu_1)(dx) = \frac{2}{\pi(1+x^2)^2}dx, \quad x \in \mathbb{R}.$$

We can see $\mathbb{A}_{0,b}(\mu_1)$ is not freely infinitely divisible for $b < 2$ as follows. The Voiculescu transform of μ_1 is $\phi_{\mu_1}(z) = \frac{-z+\sqrt{z^2+4}}{2}$. This function cannot be analytic in $\mathbb{C}_b$. From [BV93, Theorem 5.10], $\mathbb{A}_{0,b}(\mu_1)$ is not freely infinitely divisible. Note that the Student distribution is freely infinitely divisible for more parameters [H14].

Remark 6.7 (Multiplicative analogue on $\mathbb{T}$)**.** *For $a \ge 0$ and $b \in \mathbb{R}$, let $\hat{\mathbf{c}} = \hat{\mathbf{c}}_{a,b}$ be a probability measure on $\mathbb{T}$ defined by*

$$\hat{\mathbf{c}}(d\theta) = \frac{1}{2\pi}\frac{1-e^{-2a}}{1+e^{-2a}-2e^{-a}\cos(\theta-b)}d\theta, \quad 0 \le \theta < 2\pi.$$

This is an analogue of the Cauchy distribution on $\mathbb{R}$ since both are the Poisson kernels. Let $c := e^{-a+ib}$, then $\eta_{\hat{\mathbf{c}}}(z) = cz$ and

$$\Sigma_{\hat{\mathbf{c}}}(z) = k_{\hat{\mathbf{c}}}(z) = c^{-1} = e^{-ib}\exp\left(a\int_{\mathbb{T}}\frac{1+\zeta z}{1-\zeta z}\omega(d\zeta)\right),$$

where ω is the normalized Haar measure. The map $\mathbb{B}_{\hat{\mathbf{c}}}$ is denoted by $\mathbb{B}_{a,b}$. It is not hard to see that $\mathbb{B}_{a,b}(\mu)$ is characterized by

$$\eta_{\mathbb{B}_{a,b}(\mu)}(z) = c^{-1}\eta_\mu(cz). \tag{6.5}$$

which is the analog of (6.1) for the multiplicative case. Moreover, for any pair of measures $\mu, \nu \in \mathcal{P}(\mathbb{T})$ and any $a \in \mathbb{R}, b \ge 0$, we have the homomorphism properties

$$\mathbb{B}_{a,b}(\mu \circlearrowright \nu) = \mathbb{B}_{a,b}(\mu) \circlearrowright \mathbb{B}_{a,b}(\nu), \tag{6.6}$$
$$\mathbb{B}_{a,b}(\mu \boxtimes \nu) = \mathbb{B}_{a,b}(\mu) \boxtimes (\mathbb{B}_{a,b}(\nu), \tag{6.7}$$
$$\mathbb{B}_{a,b}(\mu \,\rotatebox[origin=c]{90}{$\Join$}\, \nu) = \mathbb{B}_{a,b}(\mu) \,\rotatebox[origin=c]{90}{$\Join$}\, \mathbb{B}_{a,b}(\nu), \tag{6.8}$$

where $\rotatebox[origin=c]{90}{$\Join$}$ is the multiplicative Boolean convolution (see [F09a, F09b]).

References

[AH13a] O. Arizmendi and T. Hasebe, On a class of explicit Cauchy–Stieltjes transforms related to monotone stable and free Poisson laws, Bernoulli 19 (5B) (2013), 2750–2767.

[AH13b] O. Arizmendi and T. Hasebe, Semigroups related to additive and multiplicative, free and Boolean convolutions, Studia Math. 215 (2013), no. 2, 157–185.

[AH] O. Arizmendi and T. Hasebe, Classical scale mixtures of Boolean stable laws, Trans. Amer. Math. Soc., to appear. arXiv:1405.2162

[APA09] O. Arizmendi and V. Pérez-Abreu, The S-transform for symmetric probability measures with unbounded supports, Proc. Amer. Math. Soc. 137 (2009), 3057–3066.

[BB05] S.T. Belinschi and H. Bercovici, Partially defined semigroups relative to multiplicative free convolution, Int. Math. Res. Notices, No. 2 (2005), 65–101.

[BB07] S. Belinschi and H. Bercovici, A new approach to subordination results in free probability, Journal d'Analyse Math. 101 (2007), 357–365.

[BN08] S.T. Belinschi and A. Nica, On a remarkable semigroup of homomorphisms with respect to free multiplicative convolution, Indiana Univ. Math. J. 57, No. 4 (2008), 1679–1713.

[B05] H. Bercovici, Multiplicative monotonic convolution, Illinois J. Math. 49, No. 3 (2005), 929–951.

[BV93] H. Bercovici and D. Voiculescu, Free convolution of measures with unbounded support, Indiana Univ. Math. J. 42, No. 3 (1993), 733–773.

[BFRY06] J. Bertoin, T. Fujita, B. Roynette and M. Yor, On a particular class of self-decomposable random variables: the duration of a Bessel excursion straddling an independent exponential time, Prob. Math. Stat. 26 (2006), 315–366.

[Bi98] Ph. Biane, Processes with free increments, Math. Z. 227 (1) (1998), 143–174

[F09a] U. Franz, Monotone and boolean convolutions for non-compactly supported probability measures, Indiana Univ. Math. J. 58, No. 3 (2009), 1151–1186.

[F09b] U. Franz, Boolean convolution of probability measures on the unit circle, Analyse et probabilités, Séminaires et Congrès 16 (2009), 83–93.

[H14] T. Hasebe, Free infinite divisibility for beta distributions and related ones, Electron. J. Probab. 19 (2014), no. 81, 1–33.

[H10] T. Hasebe, Monotone convolution and monotone infinite divisibility from complex analytic viewpoint, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 13, No. 1 (2010), 111–131.

[K98] S. Kerov, Interlacing measures, Amer. Math. Soc. Transl. Ser. 2, 181, Amer. Math. Soc., Providence, RI, 1998, pp. 35–83.

[L07] R. Lenczewski, Decompositions of the free additive convolution, J. Funct. Anal. 246 (2007), 330–365.

[L08] R. Lenczewski, Operators related to subordination for free multiplicative convolutions, Indiana Univ. Math. J. 57 (2008), 1055–1103.

[M92] H. Maassen, Addition of freely independent random variables, J. Funct. Anal. 106 (1992), 409–438.

[M00] N. Muraki, Monotonic convolution and monotonic Lévy-Hinčin formula, preprint, 2000.

[RS07] N. Raj Rao and R. Speicher, Multiplication of free random variables and the S-transform: The case of vanishing mean, Elect. Comm. Probab. 12 (2007), 248–258.

[SW97] R. Speicher and R. Woroudi, Boolean convolution, in: Free Probability Theory, D. Voiculescu (ed.), Fields Inst. Comm. 12, Amer. Math. Soc., 1997, 267–280.

[V87] D. Voiculescu, Multiplication of certain noncommuting random variables, J. Operator Theory 18 (1987), 223–235.

[V93] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory I, Comm. Math. Phys. 155 (1993), 71–92.