Title	Mass Renormalization in the Nelson Model
Author(s)	大澤, 進
Degree Grantor	北海道大学
Degree Name	博士(理学)
Dissertation Number	甲第12684号
Issue Date	2017-03-23
DOI	https://doi.org/10.14943/doctoral.k12684
Doc URL	https://hdl.handle.net/2115/65333
Type	doctoral thesis
File Information	Susumu_Osawa.pdf

Mass Renormalization in the Nelson Model

(ネルソンモデルにおける質量のくりこみ)

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March 2017

Abstract

The asymptotic behavior of the effective mass $m_{\rm eff}(\Lambda)$ of the so-called Nelson model in quantum field theory is considered, where Λ is an ultraviolet cutoff parameter of the model. Let m be the bare mass of the model. It is shown that for sufficiently small coupling constant $|\alpha|$ of the model, $m_{\rm eff}(\Lambda)/m$ can be expanded as $m_{\rm eff}(\Lambda)/m = 1 + \sum_{n=1}^\infty a_n(\Lambda)\alpha^{2n}$. A physical folklore is that $a_n(\Lambda) \sim [\log\Lambda]^{(n-1)}$ as $\Lambda\to\infty$. It is rigorously shown that

$$0 < \lim_{\Lambda\to\infty} a_1(\Lambda) < C, \quad C_1 \le \lim_{\Lambda\to\infty} a_2(\Lambda)/\log\Lambda \le C_2$$

with some constants C, C_1 and C_2.

Contents

1 Introduction and main results

This thesis is based on joint work with F. Hiroshima (F. Hiroshima and S. Osawa, Mass renormalization in the Nelson model, accepted for publication in International Journal of Mathematics and Mathematical Sciences).

The model considered in this paper is the so-called Nelson model [7], which describes a non-relativistic nucleon with bare mass $m > 0$ interacting with a quantized scalar field with mass $\nu > 0$. The nucleon is governed by a Schrödinger operator. Let us first define the Nelson Hamiltonian. We use relativistic unit and employ the total momentum representation. Then the Hilbert space of states is the boson Fock space over $L^2(\mathbb{R}^3)$ which is given by

$$\mathcal{F} = \oplus_{n=0}^{\infty}[\otimes_s^n L^2(\mathbb{R}^3)],$$

where $\otimes_s^n$ denotes the n-fold symmetric tensor product and $\otimes_s^0 L^2(\mathbb{R}^3) = \mathbb{C}$. Then $\Phi \in \mathcal{F}$ can be written as $\Phi = \{\Phi^{(0)}, \Phi^{(1)}, \Phi^{(2)}, \dots\}$, where $\Phi^{(n)} \in \otimes_s^n L^2(\mathbb{R}^3)$. The Fock vacuum $\Omega \in \mathcal{F}$ is defined by $\Omega = \{1, 0, 0, \dots\}$. Let $a(f)$, $f \in L^2(\mathbb{R}^3)$, be the annihilation operator and $a(f)^*$, $f \in L^2(\mathbb{R}^3)$, the creation operator on $\mathcal{F}$, which are defined by

$$\begin{aligned}
&D(a(f)^*) = \{\Psi \in \mathcal{F} | \sum_{n=0}^{\infty}(n+1)\|S_{n+1}(f \otimes \Psi^{(n)})\|^2_{\otimes^n L^2(\mathbb{R}^3)} < \infty\},\\
&(a(f)^*\Psi)^{(0)} = 0,\\
&(a(f)^*\Psi)^{(n+1)} = \sqrt{n+1}S_{n+1}(f \otimes \Psi^{(n)}),
\end{aligned}$$

and $a(f) = (a(f)^*)^*$, where S_n is the symmetrizer, $D(X)$ the domain of operator X, and $\|\cdot\|_{\mathcal{K}}$ the norm on $\mathcal{K}$. They satisfy canonical commutation relations:

$$[a(f), a(g)^*] = (f, g), \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0$$

on a suitable dense domain, where $[X, Y] = XY - YX$ and $(\cdot, \cdot)$ is the inner product on $\mathcal{K}$ (linear in the second variable). Let T be a self-adjoint operator on $L^2(\mathbb{R}^3)$. Then we define the self-adjoint operator $\mathrm{d}\Gamma(T)$ on $\mathcal{F}$ by $\mathrm{d}\Gamma(T) = \oplus_{n=0}^{\infty} T^{(n)}$, where

$$T^{(n)} = \overline{\left(\sum_{j=1}^{n} I \otimes \cdots \otimes I \otimes \overset{j\text{th}}{\breve{T}} \otimes I \otimes \cdots \otimes I\right) \lceil \hat{\otimes}^n D(T)} \qquad (n \geq 1)$$

with $T^{(0)} = 0$. Here, for a closable operator T, $\overline{T}$ denotes the closure of T. The operator $\mathrm{d}\Gamma(T)$ is called the second quantization of T. The free energy of the scalar field is given by $H_{\mathrm{f}} = \mathrm{d}\Gamma(\omega)$, where $\omega(k) = \sqrt{|k|^2 + \nu^2}$ ($k = (k_1, k_2, k_3) \in \mathbb{R}^3$, $\nu > 0$) is considered as a multiplication operator on $L^2(\mathbb{R}^3)$. Similarly the momentum of the scalar field is given by $P_{\mathrm{f}\mu} = \mathrm{d}\Gamma(k_\mu)$ ($\mu = 1, 2, 3$). The coupling of the nucleon and a scalar field is mediated through the Segal field operator $\Phi_s(g)$ defined by

$$\Phi_s(g) = \frac{1}{\sqrt{2}}(a(g) + a(g)^*),$$

where g is a cutoff function given by $g(k) = \frac{\hat{\phi}(k)}{\sqrt{\omega(k)}}$. Here $\hat{\phi}$ is the form factor with infrared cutoff $\kappa > 0$ and ultraviolet cutoff $\Lambda > 0$, which are defined by

$$\hat{\phi}(k) = \begin{cases} 0 & |k| < \kappa, \\ (2\pi)^{-\frac{3}{2}} & \kappa \leq |k| \leq \Lambda, \\ 0 & |k| > \Lambda. \end{cases}$$

The Nelson Hamiltonian with total momentum $p \in \mathbb{R}^3$ is given by a self-adjoint operator on $\mathcal{F}$:

$$H(p) = \frac{1}{2m}(p - P_{\mathrm{f}})^2 + H_{\mathrm{f}} + \alpha\Phi_s(g),$$

where $\alpha \in \mathbb{R}$ is a coupling constant. Let $E(p, \alpha)$ be the energy-momentum relation (the infimum of the spectrum $\sigma(H(p))$) defined by

$$E(p, \alpha) = \inf \sigma(H(p)).$$

Then the effective mass $m_{\text{eff}} = m_{\text{eff}}(\Lambda)$ is defined by

$$\frac{1}{m_{\text{eff}}} = \frac{1}{3} \Delta_p E(p, \alpha) \restriction_{p=0} .$$

Here Δ_p denotes the three dimensional Laplacian in the variable p. We are concerned with the asymptotic behavior of m_{eff} as the ultraviolet cutoff goes to infinity. It is however a subtle problem. Removal of the ultraviolet cutoff Λ through mass renormalization means to find sequences $\{m\}$ and $\{\Lambda\}$ such that $m \to 0$, $\Lambda \to \infty$, and m_{eff} converges to the experimentally determined mass. To achieve this, we want to find constants $0 < \gamma < 1$ and $0 < b_0 < \infty$ such that

$$\lim_{\Lambda \to \infty} \frac{m_{\text{eff}}/m}{(\Lambda/m)^{\gamma}} = b_0. \tag{1.1}$$

If we succeed in finding constants γ and b_0 such as in (1.1), taking $m = \Lambda^{-\gamma/(1-\gamma)} (m^*/b_0)^{1/(1-\gamma)}$, we have

$$\lim_{\Lambda \to \infty} m_{\text{eff}}(\Lambda) = m^*,$$

where m^* is the experimentally determined mass of the nucleon, see [10, p.316]. The mass renormalization is, however, a subtle problem, and unfortunately, we cannot yet find constants γ and b_0 such as in (1.1). For that reason we turn to perturbartive renormalization, by which we try to guess the proper value of γ. Main results obtained in this paper are summarized as follows:

Theorem 3.1 Let $\kappa > 0$. Then m_{eff} is an analytic function of α^2 and can be expanded in the following power series for sufficiently small $|\alpha|$

$$\frac{m_{\text{eff}}}{m} = 1 + \sum_{n=1}^{\infty} a_n(\Lambda) \alpha^{2n}.$$

Theorem 3.2 There exists a strictly positive constant C such that

$$\lim_{\Lambda \to \infty} a_1(\Lambda) = C.$$

Theorem 3.8 There exist some constants C_1 and C_2 such that

$$C_1 \le \lim_{\Lambda \to \infty} \frac{a_2(\Lambda)}{\log \Lambda} \le C_2.$$

From Theorems 3.2 and 3.8, if $D = \lim_{\Lambda \to \infty} a_2(\Lambda)/\log \Lambda > 0$, it is suggested that $\gamma = D\alpha^2/C$. So, the mass of the Nelson model is renormalizeble for sufficiently small $|\alpha|$.

The effective mass and energy-momentum relation have been studied mainly in non-relativistic electrodynamics. Spohn [9] investigates the upper and lower bound of the effective mass of the polaron model from a functional integral point of view. Hiroshima and Spohn [6] study a perturbative mass renormalization including fourth order in the coupling constant in the case of a spinless electron. Hiroshima and Ito [4, 5] study it in the case of an electron with spin 1/2. Bach, Chen, Fröhlich and Sigal [1] show that the energy-momentum relation is equal to the infimum of the essential spectrum of the Hamiltonian for $\kappa \geq 0$. Fröhlich and Pizzo [2] investigate energy-momentum relation when infrared cutoff goes to 0.

2 Analytic properties

In order to investigate the effective mass in a perturbation theory we have to check the analytic properties of $E(p, \alpha)$.

2.1 Analytic family in the sense of Kato

Lemma 2.1. *$H(p)$ is an analytic family in the sense of Kato.*

Proof. We prove $H(p)$ is an analytic family of type (A). We see that

$$H(p) = H_0 + \sum_{\mu=1}^{3} p_\mu \frac{1}{2m}(p_\mu - 2P_{\mathrm{f}\mu}) + \alpha H_{\mathrm{I}},$$

where $H_0 = \frac{1}{2m}P_{\mathrm{f}}^2 + H_{\mathrm{f}}$ and $H_{\mathrm{I}} = \Phi_s(g)$. Hence all we have to do is to prove the following facts.

(a) $D(H_0) \subset \cap_{\mu=1}^{3} D(P_{\mathrm{f}\mu}) \cap D(H_{\mathrm{I}})$.

(b) There exist real constants a_μ, b_μ $(\mu = 1,2,3)$, c and d such that for any $\Psi \in D(H_0)$

$$\begin{aligned} &\|\frac{1}{2m}(p_\mu - 2P_{\mathrm{f}\mu})\Psi\|_{\mathcal{F}} \le a_\mu \|H_0\Psi\|_{\mathcal{F}} + b_\mu\|\Psi\|_{\mathcal{F}} \quad (\mu = 1,2,3), \\ &\|H_{\mathrm{I}}\Psi\|_{\mathcal{F}} \le c\|H_0\Psi\|_{\mathcal{F}} + d\|\Psi\|_{\mathcal{F}}. \end{aligned}$$

We prove (a) at first. Since $\cap_{\mu=1}^{3} D(P_{\mathrm{f}\mu}^2) \subset D(P_{\mathrm{f}\mu})$, we have $D(H_0) = \cap_{\mu=1}^{3} D(P_{\mathrm{f}\mu}^2) \cap D(H_{\mathrm{f}}) \subset \cap_{\mu=1}^{3} D(P_{\mathrm{f}\mu})$. Additionally, since $\|\omega^{-1/2}g\|_{L^2(\mathbb{R}^3)} < \infty$, we have $g \in D(\omega^{-1/2})$. Furthermore, since ω is a nonnegative and injective self-adjoint operator on $L^2(\mathbb{R}^3)$, it follows that $D(\mathrm{d}\Gamma(\omega)^{1/2}) \subset D(a(g)) \cap D(a(g)^*) = D(H_{\mathrm{I}})$. Hence we have $D(H_0) \subset D(\mathrm{d}\Gamma(\omega)) \subset D(\mathrm{d}\Gamma(\omega)^{1/2})$. Together with them, (a) is proven . Next we prove (b). Let Ψ be an arbitrary vector in $D(H_0)$. Then we have

$$\|\frac{1}{2m}(p_\mu - 2P_{\mathrm{f}\mu})\Psi\|_{\mathcal{F}} \le \frac{|p_\mu|}{2m}\|\Psi\|_{\mathcal{F}} + \frac{1}{m}\|P_{\mathrm{f}\mu}\Psi\|_{\mathcal{F}}.$$

Since $\|P_{\mathrm{f}\mu}\Psi\|_{\mathcal{F}}^2 \le 2m\|H_0^{1/2}\Psi\|_{\mathcal{F}}^2$, we have $\|H_0^{1/2}\Psi\|_{\mathcal{F}}^2 \le \|(H_0+1)\Psi\|_{\mathcal{F}}^2$. Hence

$$\|\frac{1}{2m}(p_\mu - 2P_{\mathrm{f}\mu})\Psi\|_{\mathcal{F}} \le \sqrt{\frac{2}{m}}\|H_0\Psi\|_{\mathcal{F}} + (\frac{|p_\mu|}{2m} + \sqrt{\frac{2}{m}})\|\Psi\|_{\mathcal{F}}. \tag{2.1}$$

Since $D(H_0) \subset D(\mathrm{d}\Gamma(\omega)^{1/2})$,

$$\begin{aligned} &\|a(g)\Psi\|_{\mathcal{F}} \le \|\omega^{-1/2}g\|_{L^2(\mathbb{R}^3)}\|H_{\mathrm{f}}^{1/2}\Psi\|_{\mathcal{F}}, \\ &\|a(g)^*\Psi\|_{\mathcal{F}} \le \|\omega^{-1/2}g\|_{L^2(\mathbb{R}^3)}\|H_{\mathrm{f}}^{1/2}\Psi\|_{\mathcal{F}} + \|g\|_{L^2(\mathbb{R}^3)}\|\Psi\|_{\mathcal{F}} \end{aligned}$$

hold. Hence

$$\|H_{\mathrm{I}}\Psi\|_{\mathcal{F}} \le \sqrt{2}\|\omega^{-1/2}g\|_{L^2(\mathbb{R}^3)}\|H_{\mathrm{f}}^{1/2}\Psi\|_{\mathcal{F}} + \frac{1}{\sqrt{2}}\|g\|_{L^2(\mathbb{R}^3)}\|\Psi\|_{\mathcal{F}}.$$

From triangle inequality, we have $\|H_{\mathrm{f}}^{1/2}\Psi\|_{\mathcal{F}} \le \|H_{\mathrm{f}}\Psi\|_{\mathcal{F}} + \|\Psi\|_{\mathcal{F}}$. In addition,

$$\|H_0\Psi\|_{\mathcal{F}}^2 - \|H_{\mathrm{f}}\Psi\|_{\mathcal{F}}^2 = \|\frac{1}{2m}P_{\mathrm{f}}^2\Psi\|_{\mathcal{F}}^2 + \frac{1}{m}\Re(P_{\mathrm{f}}^2\Psi, H_{\mathrm{f}}\Psi).$$

Since P_{f}^2 and H_{f} are strongly commutative and nonnegative self-adjoint operators on $\mathcal{F}$, $(P_{\mathrm{f}}^2\Psi, H_{\mathrm{f}}\Psi) \ge 0$ holds. Hence $\|H_{\mathrm{f}}\Psi\|_{\mathcal{F}} \le \|H_0\Psi\|_{\mathcal{F}}$. Then we have

$$\|H_{\mathrm{I}}\Psi\|_{\mathcal{F}} \le \sqrt{2}\|\omega^{-1/2}g\|_{L^2(\mathbb{R}^3)}\|H_0\Psi\|_{\mathcal{F}} + (\sqrt{2}\|\omega^{-1/2}g\|_{L^2(\mathbb{R}^3)} + \frac{1}{\sqrt{2}}\|g\|_{L^2(\mathbb{R}^3)})\|\Psi\|_{\mathcal{F}}. \tag{2.2}$$

From (2.1) and (2.2), (b) is proven . Hence $H(p)$ is an analytic family of type (A). Since every analytic family of type (A) is an analytic family of in the sense of Kato, it is an analytic family in the sense of Kato. □

We denote the ground state of $H(p)$ by $\psi_g(p)$.

Lemma 2.2. *(1) $E(p,\alpha)$ is analytic in p and α if $|p|$ and $|\alpha|$ are sufficiently small. (2) $\psi_g(p)$ is strongly analytic in p and α if $|p|$ and $|\alpha|$ are sufficiently small.*

Proof. From [8, Theoren XII.9], (1) follows, and from [8, Theoren XII.8], (2) follows. □

2.2 Formulae

In this section we expand $m/m_{\rm eff}$ with respect to α.

Lemma 2.3. *The ratio $m/m_{\rm eff}$ can be expressed as*

$$\frac{m}{m_{\rm eff}}=1-\frac{2}{3}\sum_{\mu=1}^{3}\frac{(P_{{\rm f}\mu}\psi_g(0),{\psi_g}_\mu'(0))}{(\psi_g(0),\psi_g(0))},$$

where ${\psi_g}_\mu'(0)=s-\partial p_\mu\psi_g(p)\lceil_{p=0}$.

Proof. Since $E(p,\alpha)$ is symmetry, $E(p,\alpha)=E(-p,\alpha)$, we have

$$\partial p_\mu E(p,\alpha)\lceil_{p=0}=0,\quad \mu=1,2,3. \tag{2.3}$$

Since $H(p)\psi_g(p)=E(p,\alpha)\psi_g(p)$, for any $\Psi\in D(H(p))$,

$$(H(p)\Psi,\psi_g(p))=E(p,\alpha)(\Psi,\psi_g(p))$$

holds. Taking a derivative with respect to p_μ on the both sides above, we have

$$\begin{aligned}&(H_\mu'(p)\Psi,\psi_g(p))+(H(p)\Psi,\psi_{g_\mu}'(p))=E_\mu'(p,\alpha)(\Psi,\psi_g(p))+E(p,\alpha)(\Psi,\psi_{g_\mu}'(p)), \qquad (2.4)\\ &(H_\mu''(p)\Psi,\psi_g(p))+2(H_\mu'(p)\Psi,\psi_{g_\mu}'(p))+(H(p)\Psi,\psi_{g_\mu}''(p))\\ &=E_\mu''(p,\alpha)(\Psi,\psi_g(p))+2E_\mu'(p,\alpha)(\Psi,\psi_{g_\mu}'(p))+E(p,\alpha)(\Psi,{\psi_g}_\mu\prime\prime(p)).\end{aligned}$$

Here $'$ denotes the derivative or strong derivative with respect to p_μ, and $H_\mu'(p)=\frac{1}{m}(p_\mu-P_{{\rm f}\mu})$, $H_\mu''(p)=\frac{1}{m}$. Setting $\Psi=\psi_g(0)$ and $p=0$, we have

$$E_\mu''(0,\alpha)=\frac{1}{m}\frac{(\psi_g(0),\psi_g(0))-2(P_{{\rm f}\mu}\psi_g(0),\psi_{g\mu}\prime(0))}{(\psi_g(0),\psi_g(0))}.$$

This expression and the definition of the effective mass prove the lemma.

□

2.3 Perturbative expansions

We define operators A^+ and A^- by $A^+=\frac{1}{\sqrt{2}}a(g)^*$ and $A^-=\frac{1}{\sqrt{2}}a(g)$. Then $H_{\rm I}=A^++A^-$. Moreover, let $\mathcal{F}^{(n)}=\otimes_s^nL^2(\mathbb{R}^3)$ and

$$\psi_g(0)=\sum_{n=0}^\infty\frac{\alpha^n}{n!}\varphi_n. \tag{2.5}$$

Since $E(p,\alpha)$ is symmetry $E(p,-\alpha)=E(p,\alpha)$, we have

$$E(0,\alpha)=\sum_{n=0}^\infty\frac{\alpha^{2n}}{(2n)!}E_{2n}. \tag{2.6}$$

Since $\ker H_0\neq\{0\}$, H_0 is not injective. However, we define the operator $\frac{1}{H_0}$ (notational simplicity we write $\frac{1}{H_0}$ for H_0^{-1} in what follows) on $\mathcal{F}$ as follows.

$$\begin{aligned}&D(\frac{1}{H_0})=\{\Psi\in\mathcal{F}|\sum_{n=1}^\infty\|(\frac{1}{2m}|k_1+\cdots+k_n|^2+\sum_{i=1}^n\omega(k_i))^{-1}\Psi^{(n)}(k_1,\cdots,k_n)\|^2<\infty\},\\ &(\frac{1}{H_0}\Psi)^{(0)}=0,\\ &(\frac{1}{H_0}\Psi)^{(n)}=(\frac{1}{2m}|k_1+\cdots+k_n|^2+\sum_{i=1}^n\omega(k_i))^{-1}\Psi^{(n)}(k_1,\cdots,k_n)\quad(n\geq1).\end{aligned}$$

We define the subspace $\mathcal{F}_{\rm fin}$ of $\mathcal{F}$ as $\mathcal{F}_{\rm fin}=\{\{\Psi^{(n)}\}_{n=0}^\infty\in\mathcal{F}|\Psi^{(l)}=0 \text{ for } l\geq q \text{ with some } q\}$.

Lemma 2.4. *It holds that* $\mathcal{F}_{\rm fin} \subset D(\frac{1}{H_0})$.

Proof. Let $\Psi \in \mathcal{F}_{\rm fin}$. Then $\|\frac{1}{H_0}\Psi\|^2 = \sum_{n=1}^{\infty} \|(\frac{1}{H_0}\Psi)^{(n)}\|^2 < \infty$. Hence the lemma follows. □

Lemma 2.5. *Let* $\psi_g(0) = \sum_{n=0}^{\infty}(\alpha^n/n!)\varphi_n$. *Then* $\varphi_0 = \Omega$, $\varphi_1 = -\frac{1}{H_0}H_{\rm I}\Omega$ *and the recurrence formulas*

$$\varphi_{2l} = \frac{1}{H_0}\{-2lH_{\rm I}\varphi_{2l-1} + \sum_{j=1}^{l}\binom{2l}{2j}E_{2j}\varphi_{2l-2j}\} \quad (l \geq 1), \tag{2.7}$$

$$\varphi_{2l+1} = \frac{1}{H_0}\{-(2l+1)H_{\rm I}\varphi_{2l} + \sum_{j=1}^{l}\binom{2l+1}{2j}E_{2j}\varphi_{2l+1-2j}\} \quad (l \geq 0) \tag{2.8}$$

follow, with

$$\varphi_{2l} \in \mathcal{F}^{(2)} \oplus \mathcal{F}^{(4)} \oplus \cdots \oplus \mathcal{F}^{(2l)} \quad (l \geq 1), \tag{2.9}$$

$$\varphi_{2l+1} \in \mathcal{F}^{(1)} \oplus \mathcal{F}^{(3)} \oplus \cdots \oplus \mathcal{F}^{(2l+1)} \quad (l \geq 0), \tag{2.10}$$

and E_{2l} *is given by*

$$E_{2l} = 2l(\Omega, H_{\rm I}\varphi_{2l-1}) \quad (l \geq 1). \tag{2.11}$$

Proof. We have $E(0,0) = E_0$ by substituting $\alpha = 0$ in (2.6). Since $E(0,0)$ is the ground state energy of H_0, $E(0,0) = 0$. Hence $E_0 = 0$. Since φ_0 is the ground state of H_0, φ_0 can be Ω. We can find that $(\varphi_n, \Omega) = \delta_{0n}$ for $n = 0, 1, \cdots$ holds in the same way as [6]. From now we set $H = H(p)$, $\psi_g = \psi_g(0)$, $E = E(p,\alpha)$ and $'$ means (strong)derivative with respect to α.

$$(H\Psi, \psi_g) = E(\Psi, \psi_g) \tag{2.12}$$

holds for $\Psi \in D(H)$. Differentiating (2.12) with respect to α, we have

$$(H_{\rm I}\Psi, \psi_g) + (H\Psi, \psi_g') = E'(\Psi, \psi_g) + E(\Psi, \psi_g').$$

Hence $\psi_g' \in D(H)$ and we have

$$H_{\rm I}\psi_g + H\psi_g' = E'\psi_g + E\psi_g'. \tag{2.13}$$

Substituting $p = 0$ and $\alpha = 0$ into (2.13) and taking into account $(\varphi_n, \Omega) = \delta_{0n}$, we have $\varphi_1 = -\frac{1}{H_0}H_{\rm I}\Omega$. Differentiating (2.12) n times with respect to α, we also have

$$(H\Psi, \psi_g^{(n)}) + n(\Psi, H_{\rm I}\psi_g^{(n-1)}) = \sum_{j=1}^{n}\binom{n}{j}E^{(j)}(\Psi, \psi_g^{(n-j)}).$$

By the induction on n, we have $\psi_g^{(n)} \in D(H)$ and

$$H\psi_g^{(n)} + nH_{\rm I}\psi_g^{(n-1)} = \sum_{j=1}^{n}\binom{n}{j}E^{(j)}\psi_g^{(n-j)}.$$

Substituting $p = 0$ and $\alpha = 0$ into both sides above, we have

$$H_0\varphi_{2l} + 2lH_{\rm I}\varphi_{2l-1} = \sum_{j=1}^{l}\binom{2l}{2j}E_{2j}\varphi_{2l-2j} \quad (l \geq 1),$$

$$H_0\varphi_{2l+1} + (2l+1)H_{\rm I}\varphi_{2l} = \sum_{j=1}^{l}\binom{2l+1}{2j}E_{2j}\varphi_{2l+1-2j} \quad (l \geq 0).$$

From now on, we shall prove

$$
\begin{aligned}
&\varphi_n^{(i)} = 0, \quad (i > n,\ i = 0),\\
&\varphi_{2l}^{(2i+1)} = 0, \quad (l \geq 1,\ 0 \leq i \leq l-1),\\
&\mathrm{supp}_{k\in\mathbb{R}^{3\cdot 2i}}\varphi_{2l}^{(2i)}(k) = S_{2i} \text{ or } \emptyset, \quad (l \geq 1,\ 1 \leq i \leq l),\\
&\varphi_{2l+1}^{(2i)} = 0, \quad (l \geq 0,\ 0 \leq i \leq l),\\
&\mathrm{supp}_{k\in\mathbb{R}^{3(2i+1)}}\varphi_{2l+1}^{(2i+1)}(k) = S_{2i+1} \text{ or } \emptyset, \quad (l \geq 0,\ 0 \leq i \leq l),
\end{aligned}
$$

where we set $\varphi_n = \{\varphi_n^{(i)}\}_{i=0}^{\infty}$ by induction for $n \geq 1$, and

$$
S_i = \{(k_1, \cdots, k_i) \in \mathbb{R}^{3i} | \kappa \leq |k_1| \leq \Lambda, \cdots, \kappa \leq |k_i| \leq \Lambda\}.
$$

Since $\varphi_1 = -\frac{1}{H_0}H_{\mathrm{I}}\Omega \in \mathcal{F}^{(1)}$, $\varphi_1^{(i)} = 0$, $(i > 1,\ i = 0)$. Moreover, since

$$
\varphi_1^{(1)}(k_1) = -\frac{1}{\sqrt{2}(2\pi)^{3/2}\sqrt{\omega(k_1)}E(k_1)}\chi_{[\kappa,\Lambda]}(|k_1|),
$$

we have $\mathrm{supp}_{k_1\in\mathbb{R}^3}\varphi_1^{(1)}(k_1) = S_1$, where $E(k) = |k|^2/2m + \omega(k)$. Assume that the assumption of the induction holds when $n \leq 2l+1$, $(l \geq 0)$. Then

$$
H_0\varphi_{2l+2} + (2l+2)H_{\mathrm{I}}\varphi_{2l+1} = \sum_{j=1}^{l+1}\binom{2l+2}{2j}E_{2j}\varphi_{2l+2-2j}. \tag{2.14}
$$

It is derived that $\varphi_{2l+2}^{(i)} = 0$, $(i > 2l+2,\ i = 0)$ by $(\varphi_n, \Omega) = \delta_{0n}$ and (2.14). By the assumption of the induction, $(H_{\mathrm{I}}\varphi_{2l+1})^{(2i+1)} = 0$, $(0 \leq i \leq l)$ holds. When $1 \leq q \leq l$, it holds that

$$
\begin{aligned}
(H_0\varphi_{2l+2})^{(2q)} &= -2(l+1)(H_{\mathrm{I}}\varphi_{2l+1})^{(2q)} + \sum_{j=1}^{l+1}\binom{2l+2}{2j}E_{2j}\varphi_{2l+2-2j}^{(2q)}\\
&= -\sqrt{2}(l+1)\{\frac{1}{\sqrt{2q}}\sum_{i=1}^{2q}\frac{1}{(2\pi)^{\frac{3}{2}}}\frac{\chi_{[\kappa,\Lambda]}(|k_i|)}{\sqrt{2}\sqrt{\omega(k_i)}}\varphi_{2l+1}^{(2q-1)}(k_1, \cdots, \hat{k_i}, \cdots, k_{2q})\\
&\quad + \sqrt{2q+1}\int dk\frac{1}{(2\pi)^{\frac{3}{2}}}\frac{\chi_{[\kappa,\Lambda]}(|k|)}{\sqrt{2}\sqrt{\omega(k)}}\varphi_{2l+1}^{(2q+1)}(k, k_1, \cdots, k_{2q})\} + \sum_{j=1}^{l+1}\binom{2l+2}{2j}E_{2j}\varphi_{2l+1-2j}^{(2q)},
\end{aligned}
$$

where $\hat{k_i}$ means that k_i is omitted. By the assumption of the induction, the supports of the functions

$$
\frac{1}{(2\pi)^{\frac{3}{2}}}\frac{\chi_{[\kappa,\Lambda]}(|k_i|)}{\sqrt{2}\sqrt{\omega(k_i)}}\varphi_{2l+1}^{(2q-1)}(k_1, \cdots, \hat{k_i}, \cdots, k_{2q}), \qquad \int dk\frac{1}{(2\pi)^{\frac{3}{2}}}\frac{\chi_{[\kappa,\Lambda]}(|k|)}{\sqrt{2}\sqrt{\omega(k)}}\varphi_{2l+1}^{(2q+1)}(k, k_1, \cdots, k_{2q})
$$

and $\varphi_{2l+1-2j}^{(2q)}$ are S_{2q} or $\emptyset$. Furthermore,

$$
\begin{aligned}
(H_0\varphi_{2l+2})^{(2l+2)}(k_1, \cdots, k_{2l+2}) &= -2(l+1)(A^+\varphi_{2l+1})^{(2l+2)}(k_1, \cdots, k_{2l+2})\\
&= -\sqrt{2(l+1)}\sum_{i=1}^{2l+2}\frac{1}{(2\pi)^{\frac{3}{2}}}\frac{\chi_{[\kappa,\Lambda]}(|k_i|)}{\sqrt{2}\sqrt{\omega(k_i)}}\varphi_{2l+1}^{(2l+1)}(k_1, \cdots, \hat{k_i}, \cdots, k_{2l+2})
\end{aligned}
$$

holds. By the assumption of the induction, the support of the right hand side is S_{2l+2} or $\emptyset$. Hence we have $\mathrm{supp}_{k\in\mathbb{R}^{3\cdot 2i}}\varphi_{2l+2}^{(2i)}(k) = S_{2i}$ or $\emptyset, (1 \leq i \leq l+1)$. We can prove $\varphi_{2l+3}^{(i)} = 0, (i > 2l+3,\ i = 0)$,

$\varphi^{(2i)}_{2l+3} = 0, (1 \leq i \leq l+1)$, $\mathrm{supp}_{k\in\mathbb{R}^{3(2i+1)}}\varphi^{(2i+1)}_{2l+3}(k) = S_{2i+1}$ or $\emptyset, (0 \leq i \leq l+1)$ in a similar way. From the discussion so far, we have

$$-2lH_{\mathrm{I}}\varphi_{2l-1} + \sum_{j=1}^{l}\binom{2l}{2j}E_{2j}\varphi_{2l-2j} \in \mathcal{F}_{\mathrm{fin}} \quad (l \geq 1),$$

$$-(2l+1)H_{\mathrm{I}}\varphi_{2l} + \sum_{j=1}^{l}\binom{2l+1}{2j}E_{2j}\varphi_{2l+1-2j} \in \mathcal{F}_{\mathrm{fin}} \quad (l \geq 0).$$

Hence we have

$$\varphi_{2l} = \frac{1}{H_0}\{-2lH_{\mathrm{I}}\varphi_{2l-1} + \sum_{j=1}^{l}\binom{2l}{2j}E_{2j}\varphi_{2l-2j}\} + b_{2l}\Omega \quad (l \geq 1),$$

$$\varphi_{2l+1} = \frac{1}{H_0}\{-(2l+1)H_{\mathrm{I}}\varphi_l + \sum_{j=1}^{l}\binom{2l+1}{2j}E_{2j}\varphi_{2l+1-2j}\} + b_{2l+1}\Omega \quad (l \geq 0),$$

where b_{2l} and b_{2l+1} are some constants. Since $(\varphi_{2l}, \Omega) = 0$, $(l \geq 1)$ and $(\varphi_{2l+1}, \Omega) = 0$, $(l \geq 0)$, $b_{2l} = b_{2l+1} = 0$. Hence (2.7) and (2.8) are proven . By the discussion so far, (2.9) and (2.10) are also proven . We can derive (2.11) by (2.7) and $(\varphi_n, \Omega) = \delta_{0n}$. □

3 Main theorems

For notational simplicity we set $\hat{\phi}_j = \hat{\phi}(k_j)$ and $\omega_j = \omega(k_j)$ for $k_j \in \mathbb{R}^3, j = 1, 2$. Let

$$E_j = \frac{|k_j|^2}{2m} + \omega_j, \quad j = 1, 2, \quad E_{12} = \frac{|k_1 + k_2|^2}{2m} + \omega_1 + \omega_2, \quad \omega(r) = \sqrt{r^2 + \nu^2}, \quad F(r) = \frac{r^2}{2m} + \omega(r).$$

Theorem 3.1. *Let $\kappa > 0$. Then m_{eff} is an analytic function of α^2 and can be expanded in the following power series for sufficiently small $|\alpha|$:*

$$\frac{m_{\mathrm{eff}}}{m} = 1 + \sum_{n=1}^{\infty} a_n(\Lambda)\alpha^{2n}.$$

Proof. By the power series (2.5), we have

$$(\psi_g, \psi_g) = (\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\varphi_n, \sum_{m=0}^{\infty}\frac{\alpha^m}{m!}\varphi_m) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{n+m}}{n!m!}(\varphi_n, \varphi_m).$$

By Lemma 2.5, $(\varphi_n, \varphi_m) \neq 0$ if and only if both n and m are even or odd. Then we have

$$(\psi_g, \psi_g) = 1 + \sum_{n=1}^{\infty} b_n(\Lambda)\alpha^{2n}.$$

From the fact that both m_{eff}^{-1} and (ψ_g, ψ_g) are analytic functions of α^2 and Lemma 2.3, we have the following power series

$$-\frac{2}{3}\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}\psi_g, \psi_{g_\mu}\prime(0)) = \sum_{n=0}^{\infty} c_n(\Lambda)\alpha^{2n}.$$

Since $\psi'_{g_\mu}(0)$ is an analytic function of α, we can write

$$\psi'_{g_\mu}(0) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \Phi_n^\mu.$$

We note that

$$c_0(\Lambda) = -\frac{2}{3} \sum_{\mu=1}^{3} (P_{\mathrm{f}\mu}\varphi_0, \Phi_0^\mu) = -\frac{2}{3} \sum_{\mu=1}^{3} (\mathrm{d}\Gamma(k_\mu)\varphi_0, \Phi_0^\mu) = 0.$$

Hence if $|\alpha|$ is sufficiently small, then we have the following power series.

$$\begin{aligned}
\frac{m_{\mathrm{eff}}}{m} &= \frac{(\psi_g, \psi_g)}{(\psi_g, \psi_g) - \frac{2}{3}\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}\psi_g(0), \psi_{g_\mu}\prime(0))} = \frac{1 + \sum_{n=1}^{\infty} b_n(\Lambda)\alpha^{2n}}{1 + \sum_{n=1}^{\infty}(b_n(\Lambda) + c_n(\Lambda))\alpha^{2n}} \\
&= (1 + \sum_{n=1}^{\infty} b_n(\Lambda)\alpha^{2n}) \sum_{n=0}^{\infty} (-\sum_{l=1}^{\infty} (b_l(\Lambda) + c_l(\Lambda))\alpha^{2l})^n. \qquad (3.1)
\end{aligned}$$

This proves the theorem. □

Theorem 3.2. *There exists strictly positive constant C such that $\lim_{\Lambda\to\infty} a_1(\Lambda) = C$.*

Proof. From (3.1), we have

$$\frac{m_{\mathrm{eff}}}{m} = \{1 + b_1(\Lambda)\alpha^2 + O(\alpha^4)\}[1 - \{b_1(\Lambda) + c_1(\Lambda)\}\alpha^2 + O(\alpha^4)] \ = 1 - c_1(\Lambda)\alpha^2 + O(\alpha^4).$$

Therefore $a_1(\Lambda) = -c_1(\Lambda)$. Since

$$\sum_{n=1}^{\infty} c_n(\Lambda)\alpha^{2n} = -\frac{2}{3}\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}\psi_g, \psi'_{g_\mu}(0)) = -\frac{2}{3}\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\varphi_n, \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\Phi_n^\mu),$$

we have

$$\begin{aligned}
c_1(\Lambda) &= -\frac{2}{3}\sum_{\mu=1}^{3}\{\frac{1}{0!2!}(P_{\mathrm{f}\mu}\varphi_0, \Phi_2^\mu) + \frac{1}{1!1!}(P_{\mathrm{f}\mu}\varphi_1, \Phi_1^\mu) + \frac{1}{2!0!}(P_{\mathrm{f}\mu}\varphi_2, \Phi_0^\mu)\} \\
&= -\frac{2}{3}\sum_{\mu=1}^{3}\{(P_{\mathrm{f}\mu}\varphi_1, \Phi_1^\mu) + \frac{1}{2}(P_{\mathrm{f}\mu}\varphi_2, \Phi_0^\mu)\}. \qquad (3.2)
\end{aligned}$$

Substituting $p = 0$ into (2.4) and using (2.3), we have

$$-\frac{1}{m}(P_{\mathrm{f}\mu}\Psi, \psi_g) + ((H_0 + \alpha H_{\mathrm{I}})\Psi, \psi'_{g_\mu}(0)) = E(0,\alpha)(\Psi, \psi'_{g_\mu}(0)). \qquad (3.3)$$

In addition, by setting $\alpha = 0$, we have $-\frac{P_{\mathrm{f}\mu}}{m}\varphi_0 + H_0\Phi_0^\mu = 0$. Since $P_{\mathrm{f}\mu}\varphi_0 = \mathrm{d}\Gamma(k_\mu)\Omega = 0$, $H_0\Phi_0^\mu = 0$ holds. Hence we have

$$\Phi_0^\mu = c_0\Omega, \quad (c_0 \text{ is some constant.}) \qquad (3.4)$$

Differentiating both sides of (3.3) with respect to α, we have

$$\begin{aligned}
&-\frac{1}{m}(P_{\mathrm{f}\mu}\Psi, s-\frac{d}{d\alpha}\psi_g) + (H(0)\Psi, s-\frac{d}{d\alpha}\psi'_{g_\mu}(0)) + (H_{\mathrm{I}}\Psi, \psi'_{g_\mu}(0)) \\
&= E(0,\alpha)(\Psi, s-\frac{d}{d\alpha}\psi'_{g_\mu}(0)) + \frac{d}{d\alpha}E(0,\alpha)(\Psi, \psi'_{g_\mu}(0)).
\end{aligned}$$

Substituting $\alpha = 0$ into both sides, we have

$$(H_0\Psi, \Phi_1^\mu) = \frac{1}{m}(\Psi, P_{\mathrm{f}\mu}\varphi_1) - (\Psi, H_{\mathrm{I}}\Phi_0^\mu).$$

Therefore $\Phi_1^\mu \in D(H_0)$ and

$$H_0\Phi_1^\mu = \frac{1}{m}P_{\mathrm{f}\mu}\varphi_1 - H_{\mathrm{I}}\Phi_0^\mu = -\frac{1}{m}P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega - c_0 H_{\mathrm{I}}\Omega.$$

Since $-\frac{1}{m}P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega - c_0H_{\mathrm{I}}\Omega \in \mathcal{F}_{\mathrm{fin}}$, we have

$$\Phi_1^\mu = -\frac{1}{m}\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega - c_0\frac{1}{H_0}H_{\mathrm{I}}\Omega + c_1\Omega, \tag{3.5}$$

where c_1 is some constant. By $a_1(\Lambda) = -c_1(\Lambda)$, (3.2), (3.4) and (3.5), we have

$$\begin{aligned} a_1(\Lambda) &= \frac{2}{3}\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}\varphi_1, \Phi_1^\mu) \\ &= \frac{2}{3m}\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega, \frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega) - \frac{2c_0}{3}\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega, \frac{1}{H_0}H_{\mathrm{I}}\Omega) - \frac{2c_1}{3}\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega, \Omega). \end{aligned}$$

It is also seen that

$$\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega, \frac{1}{H_0}H_{\mathrm{I}}\Omega) = \sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega, \Omega) = 0.$$

Thus we have

$$\begin{aligned} a_1(\Lambda) &= \frac{2}{3m}\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega, \frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega) \\ &= \frac{2}{3m}\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}\frac{1}{H_0}A^+\Omega, \frac{1}{H_0}P_{f_\mu}\frac{1}{H_0}A^+\Omega) = \frac{2}{3m}\int\frac{|\hat{\phi}(k)|^2}{\omega(k)}\frac{|k|^2}{E(k)^3}dk. \end{aligned}$$

Changing variables into polar coordinate, we have

$$\frac{2}{3m}\int\frac{|\hat{\phi}(k)|^2}{\omega(k)}\frac{|k|^2}{E(k)^3}dk = \frac{8\pi}{3m(2\pi)^3}\int_\kappa^\Lambda\frac{r^4}{\omega(r)F(r)^3}dr.$$

Since $\frac{r^4}{\omega(r)F(r)^3} = O(r^{-3})$ $(r \to \infty)$, the improper integral $\int_\kappa^\infty \frac{r^4}{\omega(r)F(r)^3}dr$ converges. It is trivial to see that $\lim_{\Lambda\to\infty} a_1(\Lambda) > 0$. Thus the theorem follows.

□

Lemma 3.3. *It follows that $\Phi_n^\mu \in \mathcal{F}_{\mathrm{fin}}$ for $n \in \mathbb{N}\cup\{0\}$.*

Proof. By (3.4), we have $\Phi_0^\mu \in \mathcal{F}_{\mathrm{fin}}$. Assume that $\Phi_n^\mu \in \mathcal{F}_{\mathrm{fin}}$ holds when $n \le k-1$. Differentiating both sides of (3.3) k times with respect to α and substituting $\alpha = 0$, we have

$$-\frac{1}{m}(P_{\mathrm{f}\mu}\Psi, \varphi_k) + (H_0\Psi, \Phi_k^\mu) + k(H_{\mathrm{I}}\Psi, \Phi_{k-1}^\mu) = \sum_{j=1}^{k}\binom{k}{j}E_j(\Psi, \Phi_{k-j}^\mu),$$

however, $E_j = 0$ when j is odd. Since $\Phi_{k-1}^\mu \in \mathcal{F}_{\mathrm{fin}}$, $\Phi_{k-1}^\mu \in D(H_{\mathrm{I}})$ and

$$(H_0\Psi, \Phi_k^\mu) = \frac{1}{m}(\Psi, P_{\mathrm{f}\mu}\varphi_k) - k(\Psi, H_{\mathrm{I}}\Phi_{k-1}^\mu) + (\Psi, \sum_{j=1}^{k}\binom{k}{j}E_j\Phi_{k-j}^\mu).$$

Thus $\Phi_k^\mu \in D(H_0)$ and

$$H_0 \Phi_k^\mu = \frac{1}{m} P_{\mathrm{f}\mu} \varphi_k - k H_{\mathrm{I}} \Phi_{k-1}^\mu + \sum_{j=1}^{k} \binom{k}{j} E_j \Phi_{k-j}^\mu. \tag{3.6}$$

Since $\varphi_k \in \mathcal{F}_{\mathrm{fin}}$, $H_{\mathrm{I}} \Phi_{k-1}^\mu \in \mathcal{F}_{\mathrm{fin}}$ and $\Phi_{k-j}^\mu \in \mathcal{F}_{\mathrm{fin}}$ $(j = 1, \cdots, k)$, by the assumption of induction, $H_0 \Phi_k^\mu \in \mathcal{F}_{\mathrm{fin}}$. Hence $\Phi_n^\mu \in \mathcal{F}_{\mathrm{fin}}$ holds when $n = k$. □

Lemma 3.4. *It holds that* $\Phi_0^\mu = c_0 \Omega$, $\Phi_1^\mu = -\frac{1}{m} \frac{1}{H_0} P_{\mathrm{f}\mu} \frac{1}{H_0} H_{\mathrm{I}} \Omega - c_0 \frac{1}{H_0} H_{\mathrm{I}} \Omega + c_1 \Omega$ *and the recurrence formulas:*

$$\Phi_{2l}^\mu = \frac{1}{H_0} \{ \frac{1}{m} P_{\mathrm{f}\mu} \varphi_{2l} - 2l H_{\mathrm{I}} \Phi_{2l-1}^\mu + \sum_{j=1}^{l} \binom{2l}{2j} E_{2j} \Phi_{2l-2j}^\mu \} + c_{2l} \Omega$$
$$(l \geq 1, c_{2l} \textit{ is some constant.}) \tag{3.7}$$

$$\Phi_{2l+1}^\mu = \frac{1}{H_0} \{ \frac{1}{m} P_{\mathrm{f}\mu} \varphi_{2l+1} - (2l+1) H_{\mathrm{I}} \Phi_{2l}^\mu + \sum_{j=1}^{l} \binom{2l+1}{2j} E_{2j} \Phi_{2l+1-2j}^\mu \} + c_{2l+1} \Omega,$$
$$(l \geq 0, c_{2l+1} \textit{ is some constant.}) \tag{3.8}$$

Proof. The first and second expressions are proven in Theorem 3.2. From (3.6), it follows that

$$H_0 \Phi_{2l}^\mu = \frac{1}{m} P_{\mathrm{f}\mu} \varphi_{2l} - 2l H_{\mathrm{I}} \Phi_{2l-1}^\mu + \sum_{j=1}^{l} \binom{2l}{2j} E_{2j} \Phi_{2l-2j}^\mu \quad (l \geq 1)$$
$$H_0 \Phi_{2l+1}^\mu = \frac{1}{m} P_{\mathrm{f}\mu} \varphi_{2l+1} - (2l+1) H_{\mathrm{I}} \Phi_{2l}^\mu + \sum_{j=1}^{l} \binom{2l+1}{2j} E_{2j} \Phi_{2l+1-2j}^\mu \quad (l \geq 0).$$

These prove the lemma. □

Lemma 3.5. *It is proven that* $a_2(\Lambda)$ *can be expanded as*

$$a_2(\Lambda) = \frac{2}{3m} \sum_{j=1}^{8} \mathrm{I}_j(\Lambda) + \frac{E_2(\Lambda)}{m} \mathrm{I}_9(\Lambda) - a_1(\Lambda) \mathrm{I}_{10}(\Lambda) + a_1(\Lambda)^2,$$

where I_j *are given by*

$$\mathrm{I}_1(\Lambda) = \frac{1}{4}\iint dk_1 dk_2 \frac{|\hat{\phi}_1|^2|\hat{\phi}_2|^2}{\omega_1\omega_2}(\frac{|k_1|^2}{E_1^3} + \frac{|k_2|^2}{E_2^3})(\frac{1}{E_1} + \frac{1}{E_2})\frac{1}{E_{12}},$$

$$\mathrm{I}_2(\Lambda) = \frac{1}{8}\iint dk_1 dk_2 \frac{|\hat{\phi}_1|^2|\hat{\phi}_2|^2}{\omega_1\omega_2}(\frac{|k_1|^2}{E_1^4} + \frac{|k_2|^2}{E_2^4})\frac{1}{E_{12}},$$

$$\mathrm{I}_3(\Lambda) = \frac{1}{8}\iint dk_1 dk_2 \frac{|\hat{\phi}_1|^2|\hat{\phi}_2|^2}{\omega_1\omega_2}(\frac{1}{E_1^2} + \frac{1}{E_2^2})(\frac{1}{E_1} + \frac{1}{E_2})\frac{(k_1,k_2)}{E_{12}^2},$$

$$\mathrm{I}_4(\Lambda) = \frac{1}{4}\iint dk_1 dk_2 \frac{|\hat{\phi}_1|^2|\hat{\phi}_2|^2}{\omega_1\omega_2}(\frac{|k_1|^2}{E_1^2} + \frac{|k_2|^2}{E_2^2})(\frac{1}{E_1} + \frac{1}{E_2})\frac{1}{E_{12}^2},$$

$$\mathrm{I}_5(\Lambda) = \frac{1}{4}\iint dk_1 dk_2 \frac{|\hat{\phi}_1|^2|\hat{\phi}_2|^2}{\omega_1\omega_2 E_1^2 E_2^2}\frac{(k_1,k_2)}{E_{12}},$$

$$\mathrm{I}_6(\Lambda) = \frac{1}{8}\iint dk_1 dk_2 \frac{|\hat{\phi}_1|^2|\hat{\phi}_2|^2}{\omega_1\omega_2}(\frac{1}{E_1} + \frac{1}{E_2})^2\frac{|k_1|^2+|k_2|^2}{E_{12}^3},$$

$$\mathrm{I}_7(\Lambda) = \frac{1}{4}\iint dk_1 dk_2 \frac{|\hat{\phi}_1|^2|\hat{\phi}_2|^2}{\omega_1\omega_2}(\frac{1}{E_1} + \frac{1}{E_2})^2\frac{(k_1,k_2)}{E_{12}^3},$$

$$\mathrm{I}_8(\Lambda) = \frac{1}{4}\iint dk_1 dk_2 \frac{|\hat{\phi}_1|^2|\hat{\phi}_2|^2}{\omega_1\omega_2}(\frac{1}{E_1} + \frac{1}{E_2})\frac{(k_1,k_2)}{E_{12}^4},$$

$$\mathrm{I}_9(\Lambda) = \frac{1}{2}\int \frac{|\hat{\phi}(k)|^2|k|^2}{\omega(k)E(k)^4}dk,$$

$$\mathrm{I}_{10}(\Lambda) = \frac{1}{2}\int \frac{|\hat{\phi}(k)|^2}{\omega(k)E(k)^2}dk.$$

The proof of Lemma 3.5 is given in the next section. The asymptotic behaviors of terms $\mathrm{I}_j(\Lambda)$ as $\Lambda \to \infty$ is given in the lemma below. Only two terms $\mathrm{I}_1(\Lambda)$ and $\mathrm{I}_2(\Lambda)$ logarithmically diverge, and other terms converge as $\Lambda \to \infty$.

Lemma 3.6. *(1)-(3) follow:*

(1) There exist some constants C_3 and C_4 such that $C_3 \leq \lim_{\Lambda\to\infty} \frac{\mathrm{I}_1(\Lambda)}{\log\Lambda} \leq C_4$.

(2) There exist some constants C_5 and C_6 such that $C_5 \leq \lim_{\Lambda\to\infty} \frac{\mathrm{I}_2(\Lambda)}{\log\Lambda} \leq C_6$.

(3) For $j = 3,4,5,6,7,8$ $\lim_{\Lambda\to\infty} |\mathrm{I}_j(\Lambda)| < \infty$.

The proof of Lemma 3.6 is technical and also given in the next section.

Lemma 3.7. *It holds that*

$$\lim_{\Lambda\to\infty} \frac{E_2(\Lambda)}{\log\Lambda} = -\frac{m}{\pi^2}. \tag{3.9}$$

Proof. From (2.11), we have $E_2(\Lambda) = -\frac{1}{2\pi^2}\int_\kappa^\Lambda dr \frac{r^2}{\omega(r)F(r)}$ and $\lim_{r\to\infty} \frac{\frac{r^2}{\omega(r)F(r)}}{\frac{1}{r}} = 2m$. It implies (3.9). □

Now we are in the position to state the main theorem in this paper.

Theorem 3.8. *There exist some constants C_1 and C_2 such that*

$$C_1 \leq \lim_{\Lambda\to\infty} \frac{a_2(\Lambda)}{\log\Lambda} \leq C_2.$$

Proof. We have $\mathrm{I}_9(\Lambda) = \frac{1}{4\pi^2}\int_\kappa^\Lambda \frac{r^4}{\omega(r)F(r)^4}dr$. Since $\frac{r^4}{\omega(r)F(r)^4} = O(r^{-5})$ $(r \to \infty)$, we have

$$\lim_{\Lambda\to\infty} |\mathrm{I}_9(\Lambda)| < \infty. \tag{3.10}$$

We also have $\mathrm{I}_{10}(\Lambda) = \frac{1}{4\pi^2}\int_\kappa^\Lambda \frac{r^2}{\omega(r)F(r)^2}dr$. Then $\frac{r^2}{\omega(r)F(r)^2} = O(r^{-3})$ $(r \to \infty)$ and we also have

$$\lim_{\Lambda\to\infty} |\mathrm{I}_{10}(\Lambda)| < \infty. \tag{3.11}$$

By (3.10) and (3.11), Theorem 3.2 and Lemmas 3.5 and 3.6 we can conclude the theorem. □

4 Proof of Lemmas 3.5 and 3.6

In this section we prove Lemmas 3.5 and 3.6.

4.1 Proof of Lemma 3.5

From (3.1) and $a_1(\Lambda) = -c_1(\Lambda)$, we have $a_2(\Lambda) = -c_2(\Lambda) - b_1(\Lambda)a_1(\Lambda) + a_1(\Lambda)^2$. Here

$$\begin{aligned}
b_1(\Lambda) &= (\varphi_1, \varphi_1) = \frac{1}{2}\int \frac{|\phi(\hat{k})|^2}{\omega(k)E(k)^2}dk, \\
c_2(\Lambda) &= -\frac{2}{3}\{\frac{1}{0!4!}\sum_{\mu=1}^3 (P_{\mathrm{f}\mu}\varphi_0, \Phi_4^\mu) + \frac{1}{1!3!}\sum_{\mu=1}^3 (P_{\mathrm{f}\mu}\varphi_1, \Phi_3^\mu) \\
&\quad + \frac{1}{2!2!}\sum_{\mu=1}^3 (P_{\mathrm{f}\mu}\varphi_2, \Phi_2^\mu) + \frac{1}{3!1!}\sum_{\mu=1}^3 (P_{\mathrm{f}\mu}\varphi_3, \Phi_1^\mu) + \frac{1}{4!0!}\sum_{\mu=1}^3 (P_{f_\mu}\varphi_4, \Phi_0^\mu)\} \\
&= -\frac{1}{9}\sum_{\mu=1}^3 (P_{\mathrm{f}\mu}\varphi_1, \Phi_3^\mu) - \frac{1}{6}\sum_{\mu=1}^3 (P_{\mathrm{f}\mu}\varphi_2, \Phi_2^\mu) - \frac{1}{9}\sum_{\mu=1}^3 (P_{f_\mu}\varphi_3, \Phi_1^\mu).
\end{aligned} \tag{4.1}$$

Using recurrence formulas (2.7), (2.8), (3.7) and (3.8), we have

$$\begin{aligned}
\varphi_2 &= 2(\frac{1}{H_0}H_\mathrm{I})^2\Omega, \\
\varphi_3 &= -6(\frac{1}{H_0}H_\mathrm{I})^3\Omega - 3E_2(\frac{1}{H_0})^2 H_\mathrm{I}\Omega, \\
\Phi_2^\mu &= \frac{2}{m}\frac{1}{H_0}P_{\mathrm{f}\mu}(\frac{1}{H_0}H_\mathrm{I})^2\Omega + \frac{2}{m}\frac{1}{H_0}H_\mathrm{I}\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega \\
&\quad + 2c_0(\frac{1}{H_0}H_\mathrm{I})^2\Omega - 2c_1\frac{1}{H_0}H_\mathrm{I}\Omega + c_2\Omega
\end{aligned}$$

and

$$\begin{aligned}
\Phi_3^\mu = &-\frac{6}{m}\frac{1}{H_0}P_{\mathrm{f}\mu}(\frac{1}{H_0}H_\mathrm{I})^3\Omega - \frac{6}{m}\frac{1}{H_0}H_\mathrm{I}\frac{1}{H_0}P_{\mathrm{f}\mu}(\frac{1}{H_0}H_\mathrm{I})^2\Omega - \frac{6}{m}(\frac{1}{H_0}H_\mathrm{I})^2\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega \\
&- \frac{3}{m}E_2\frac{1}{H_0}P_{\mathrm{f}\mu}(\frac{1}{H_0})^2 H_\mathrm{I}\Omega - \frac{3}{m}E_2(\frac{1}{H_0})^2 P_{f_\mu}\frac{1}{H_0}H_\mathrm{I}\Omega \\
&- 6c_0(\frac{1}{H_0}H_\mathrm{I})^3\Omega + 6c_1(\frac{1}{H_0}H_\mathrm{I})^2\Omega - 3c_2\frac{1}{H_0}H_\mathrm{I}\Omega - 3c_0E_2(\frac{1}{H_0})^2 H_\mathrm{I}\Omega + c_3\Omega.
\end{aligned}$$

Substituting them into (4.1), we have

$$c_2(\Lambda) = -\frac{2}{3m}\{\sum_{\mu=1}^3 (P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega, \frac{1}{H_0}P_{f_\mu}(\frac{1}{H_0}H_\mathrm{I})^3\Omega) + \sum_{\mu=1}^3 (P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega, \frac{1}{H_0}H_\mathrm{I}\frac{1}{H_0}P_{\mathrm{f}\mu}(\frac{1}{H_0}H_\mathrm{I})^2\Omega)$$

$$
\begin{aligned}
&+\sum_{\mu=1}^3(P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega,(\frac{1}{H_0}H_\mathrm{I})^2\frac{1}{H_0}P_{f_\mu}\frac{1}{H_0}H_\mathrm{I}\Omega)+\sum_{\mu=1}^3(P_{\mathrm{f}\mu}(\frac{1}{H_0}H_\mathrm{I})^2\Omega,\frac{1}{H_0}P_{f_\mu}(\frac{1}{H_0}H_\mathrm{I})^2\Omega)\\
&+\sum_{\mu=1}^3(P_{f_\mu}(\frac{1}{H_0}H_\mathrm{I})^2\Omega,\frac{1}{H_0}H_\mathrm{I}\frac{1}{H_0}P_{f_\mu}\frac{1}{H_0}H_\mathrm{I}\Omega)+\sum_{\mu=1}^3(P_{f_\mu}(\frac{1}{H_0}H_\mathrm{I})^3\Omega,\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega)\}\\
&-\frac{E_2}{3m}\{\sum_{\mu=1}^3(P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega,\frac{1}{H_0}P_{\mathrm{f}\mu}(\frac{1}{H_0})^2H_\mathrm{I}\Omega)+\sum_{\mu=1}^3(P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega,(\frac{1}{H_0})^2P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega)\\
&+\sum_{\mu=1}^3(P_{\mathrm{f}\mu}(\frac{1}{H_0})^2H_\mathrm{I}\Omega,\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega)\}-\frac{2c_0}{3}\sum_{\mu=1}^3(P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega,(\frac{1}{H_0}H_\mathrm{I})^3\Omega)\\
&+\frac{2c_1}{3}\sum_{\mu=1}^3(P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega,(\frac{1}{H_0}H_\mathrm{I})^2\Omega)-\frac{c_2}{3}\sum_{\mu=1}^3(P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega,\frac{1}{H_0}H_\mathrm{I}\Omega)\\
&-\frac{c_0E_2}{3}\sum_{\mu=1}^3(P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega,(\frac{1}{H_0})^2H_\mathrm{I}\Omega)+\frac{c_3}{9}\sum_{\mu=1}^3(P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega,\Omega)\\
&-\frac{2c_0}{3}\sum_{\mu=1}^3(P_{\mathrm{f}\mu}(\frac{1}{H_0}H_\mathrm{I})^2\Omega,(\frac{1}{H_0}H_\mathrm{I})^2\Omega)+\frac{2c_1}{3}\sum_{\mu=1}^3(P_{\mathrm{f}\mu}(\frac{1}{H_0}H_\mathrm{I})^2\Omega,\frac{1}{H_0}H_\mathrm{I}\Omega)\\
&-\frac{c_2}{3}\sum_{\mu=1}^3(P_{\mathrm{f}\mu}(\frac{1}{H_0}H_\mathrm{I})^2\Omega,\Omega)-\frac{2c_0}{3}\sum_{\mu=1}^3(P_{\mathrm{f}\mu}(\frac{1}{H_0}H_\mathrm{I})^3\Omega,\frac{1}{H_0}H_\mathrm{I}\Omega)\\
&+\frac{2c_1}{3}\sum_{\mu=1}^3(P_{\mathrm{f}\mu}(\frac{1}{H_0}H_\mathrm{I})^3\Omega,\Omega)-\frac{c_0E_2}{3}\sum_{\mu=1}^3(P_{\mathrm{f}\mu}(\frac{1}{H_0})^2H_\mathrm{I}\Omega,\frac{1}{H_0}H_\mathrm{I}\Omega)\\
&+\frac{c_1E_2}{3}\sum_{\mu=1}^3(P_{\mathrm{f}\mu}(\frac{1}{H_0})^2H_\mathrm{I}\Omega,\Omega)=\sum_{j=1}^{21}(j).
\end{aligned}
$$

We estimate 21 terms $(1)-(21)$ above. We can however directly see that $0=(10)=\cdots=(21)$:

$$
\begin{aligned}
&\sum_{\mu=1}^3(P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega,(\frac{1}{H_0}H_\mathrm{I})^3\Omega)=\sum_{\mu=1}^3(P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega,(\frac{1}{H_0}H_\mathrm{I})^2\Omega)=\sum_{\mu=1}^3(P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega,\frac{1}{H_0}H_\mathrm{I}\Omega)\\
&=\sum_{\mu=1}^3(P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega,(\frac{1}{H_0})^2H_\mathrm{I}\Omega)=\sum_{\mu=1}^3(P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega,\Omega)=\sum_{\mu=1}^3(P_{\mathrm{f}\mu}(\frac{1}{H_0}H_\mathrm{I})^2\Omega,(\frac{1}{H_0}H_\mathrm{I})^2\Omega)\\
&=\sum_{\mu=1}^3(P_{\mathrm{f}\mu}(\frac{1}{H_0}H_\mathrm{I})^2\Omega,\frac{1}{H_0}H_\mathrm{I}\Omega)=\sum_{\mu=1}^3(P_{\mathrm{f}\mu}(\frac{1}{H_0}H_\mathrm{I})^2\Omega,\Omega)=\sum_{\mu=1}^3(P_{\mathrm{f}\mu}(\frac{1}{H_0}H_\mathrm{I})^3\Omega,\frac{1}{H_0}H_\mathrm{I}\Omega)\\
&=\sum_{\mu=1}^3(P_{\mathrm{f}\mu}(\frac{1}{H_0}H_\mathrm{I})^3\Omega,\Omega)=\sum_{\mu=1}^3(P_{\mathrm{f}\mu}(\frac{1}{H_0})^2H_\mathrm{I}\Omega,\frac{1}{H_0}H_\mathrm{I}\Omega)=\sum_{\mu=1}^3(P_{\mathrm{f}\mu}(\frac{1}{H_0})^2H_\mathrm{I}\Omega,\Omega)=0.
\end{aligned}
$$

We can compute remaining terms $(1)-(9)$ as

$$
\begin{aligned}
&(1)\ \sum_{\mu=1}^3(P_{\mathrm{f}\mu}\frac{1}{H_0}H_\mathrm{I}\Omega,\frac{1}{H_0}P_{\mathrm{f}\mu}(\frac{1}{H_0}H_\mathrm{I})^3\Omega)=\sum_{\mu=1}^3(A^+\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}A^+\Omega,(\frac{1}{H_0}A^+)^2\Omega)\\
&=\frac{1}{8}\iint dk_1dk_2\frac{|\hat\phi_1|^2|\hat\phi_2|^2}{\omega_1\omega_2}(\frac{|k_1|^2}{E_1^3}+\frac{|k_2|^2}{E_2^3})(\frac{1}{E_1}+\frac{1}{E_2})\frac{1}{E_{12}},
\end{aligned}
$$

(2) $\displaystyle\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega, \frac{1}{H_0}H_{\mathrm{I}}\frac{1}{H_0}P_{\mathrm{f}\mu}(\frac{1}{H_0}H_{\mathrm{I}})^2\Omega) = \sum_{\mu=1}^{3}(A^+\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}A^+\Omega, \frac{1}{H_0}P_{\mathrm{f}\mu}(\frac{1}{H_0}A^+)^2\Omega)$

$$= \frac{1}{8}\iint dk_1dk_2\frac{|\hat{\phi}_1|^2|\hat{\phi}_2|^2}{\omega_1\omega_2}(\frac{|k_1|^2}{E_1^2}+\frac{|k_2|^2}{E_2^2})(\frac{1}{E_1}+\frac{1}{E_2})\frac{1}{E_{12}^2}$$
$$+\frac{1}{8}\iint dk_1dk_2\frac{|\hat{\phi}_1|^2|\hat{\phi}_2|^2}{\omega_1\omega_2}(\frac{1}{E_1^2}+\frac{1}{E_2^2})(\frac{1}{E_1}+\frac{1}{E_2})\frac{(k_1,k_2)}{E_{12}^2}.$$

(3) $\displaystyle\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega, (\frac{1}{H_0}H_{\mathrm{I}})^2\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega) = \sum_{\mu=1}^{3}(A^+\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}A^+\Omega, \frac{1}{H_0}A^+\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}A^+\Omega)$

$$= \frac{1}{8}\iint dk_1dk_2\frac{|\hat{\phi}_1|^2|\hat{\phi}_2|^2}{\omega_1\omega_2}(\frac{|k_1|^2}{E_1^4}+\frac{|k_2|^2}{E_2^4})\frac{1}{E_{12}}+\frac{1}{4}\iint dk_1dk_2\frac{|\hat{\phi}_1|^2|\hat{\phi}_2|^2}{\omega_1\omega_2E_1^2E_2^2}\frac{(k_1,k_2)}{E_{12}}.$$

(4) $\displaystyle\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}(\frac{1}{H_0}H_{\mathrm{I}})^2\Omega, \frac{1}{H_0}P_{\mathrm{f}\mu}(\frac{1}{H_0}H_{\mathrm{I}})^2\Omega) = \sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}(\frac{1}{H_0}A^+)^2\Omega, \frac{1}{H_0}P_{\mathrm{f}\mu}(\frac{1}{H_0}A^+)^2\Omega)$

$$= \frac{1}{8}\iint dk_1dk_2\frac{|\hat{\phi}_1|^2|\hat{\phi}_2|^2}{\omega_1\omega_2}(\frac{1}{E_1}+\frac{1}{E_2})^2\frac{|k_1|^2+|k_2|^2}{E_{12}^3}+\frac{1}{4}\iint dk_1dk_2\frac{|\hat{\phi}_1|^2|\hat{\phi}_2|^2}{\omega_1\omega_2}(\frac{1}{E_1}+\frac{1}{E_2})^2\frac{(k_1,k_2)}{E_{12}^3}.$$

(5) $\displaystyle\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}(\frac{1}{H_0}H_{\mathrm{I}})^2\Omega, \frac{1}{H_0}H_{\mathrm{I}}\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega) = \sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}(\frac{1}{H_0}A^+)^2\Omega, \frac{1}{H_0}A^+\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}A^+\Omega)$

$$= \frac{1}{8}\iint dk_1dk_2\frac{|\hat{\phi}_1|^2|\hat{\phi}_2|^2}{\omega_1\omega_2}(\frac{|k_1|^2}{E_1^2}+\frac{|k_2|^2}{E_2^2})(\frac{1}{E_1}+\frac{1}{E_2})\frac{1}{E_{12}^2}+\frac{1}{4}\iint dk_1dk_2\frac{|\hat{\phi}_1|^2|\hat{\phi}_2|^2}{\omega_1\omega_2}(\frac{1}{E_1}+\frac{1}{E_2})\frac{(k_1,k_2)}{E_{12}^4}.$$

(6) $\displaystyle\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}(\frac{1}{H_0}H_{\mathrm{I}})^3\Omega, \frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega) = \sum_{\mu=1}^{3}((\frac{1}{H_0}A^+)^2\Omega, A^+\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}A^+\Omega)$

$$= \frac{1}{8}\iint dk_1dk_2\frac{|\hat{\phi}_1|^2|\hat{\phi}_2|^2}{\omega_1\omega_2}(\frac{|k_1|^2}{E_1^3}+\frac{|k_2|^2}{E_2^3})(\frac{1}{E_1}+\frac{1}{E_2})\frac{1}{E_{12}}.$$

(7) $\displaystyle\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega, \frac{1}{H_0}P_{\mathrm{f}\mu}(\frac{1}{H_0})^2H_{\mathrm{I}}\Omega) = \frac{1}{2}\int\frac{|\hat{\phi}(k)|^2|k|^2}{\omega(k)E(k)^4}dk.$

(8) $\displaystyle\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega, (\frac{1}{H_0})^2P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega) = \frac{1}{2}\int\frac{|\hat{\phi}(k)|^2|k|^2}{\omega(k)E(k)^4}dk.$

(9) $\displaystyle\sum_{\mu=1}^{3}(P_{\mathrm{f}\mu}(\frac{1}{H_0})^2H_{\mathrm{I}}\Omega, \frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}H_{\mathrm{I}}\Omega) = \frac{1}{2}\int\frac{|\hat{\phi}(k)|^2|k|^2}{\omega(k)E(k)^4}dk.$

Thus the lemma follows. □

4.2 Proof of Lemma 3.6

4.2.1 Proof of $C_3 \le \lim_{\Lambda\to\infty}\frac{\mathrm{I}_1(\Lambda)}{\log\Lambda}$ and $C_5 \le \lim_{\Lambda\to\infty}\frac{\mathrm{I}_2(\Lambda)}{\log\Lambda}$

Changing variables to polar coordinates, we have

$$\mathrm{I}_1(\Lambda) = \frac{2\pi^2}{(2\pi)^6}\int_{-1}^{1}dz\int_{\kappa}^{\Lambda}dr_1\int_{\kappa}^{\Lambda}\frac{r_1^2r_2^2}{\omega(r_1)\omega(r_2)}(\frac{r_1^2}{F(r_1)^3}+\frac{r_2^2}{F(r_2)^3})(\frac{1}{F(r_1)}+\frac{1}{F(r_2)})\frac{1}{L(r_1,r_2,z)}dr_2,$$

where

$$L(r_1,r_2,z) = \frac{r_1^2+r_2^2+2r_1r_2z}{2m}+\omega(r_1)+\omega(r_2).$$

We define $h(r_1,r_2)$, $h_1(r_1,r_2)$, $h_2(r_1,r_2)$, $S(\Lambda)$, $S_1(\Lambda)$ and $S_2(\Lambda)$ as

$$\begin{aligned}
&h(r_1,r_2)=\frac{r_1^2r_2^2}{\omega(r_1)\omega(r_2)}\Big(\frac{r_1^2}{F(r_1)^3}+\frac{r_2^2}{F(r_2)^3}\Big)\Big(\frac{1}{F(r_1)}+\frac{1}{F(r_2)}\Big)\frac{1}{L(r_1,r_2,1)},\\
&h_1(r_1,r_2)=\frac{r_1^2r_2^4}{\omega(r_1)\omega(r_2)F(r_2)^4L(r_1,r_2,1)},\\
&h_2(r_1,r_2)=h(r_1,r_2)-h_1(r_1,r_2),\\
&S(\Lambda)=\int_\kappa^\Lambda\int_\kappa^\Lambda h(r_1,r_2)dr_1dr_2,\\
&S_1(\Lambda)=\int_\kappa^{\kappa+1}dr_2\int_{\kappa+1+\nu+m}^\Lambda h_1(r_1,r_2)dr_1,\\
&S_2(\Lambda)=\int_\kappa^{\kappa+1}dr_2\int_\kappa^{\kappa+1+\nu+m}h_1(r_1,r_2)dr_1+\int_{\kappa+1}^\Lambda dr_1\int_\kappa^\Lambda h_1(r_1,r_2)dr_2+\int_\kappa^\Lambda\int_\kappa^\Lambda h_2(r_1,r_2)dr_1dr_2.
\end{aligned}$$

Then

$$\frac{4\pi^2}{(2\pi)^6}S(\Lambda)\le \mathrm{I}_1(\Lambda). \tag{4.2}$$

In addition, $S(\Lambda)=S_1(\Lambda)+S_2(\Lambda)$ follows. Since $h_1(r_1,r_2)>0$ and $h_2(r_1,r_2)>0$, $S_2(\Lambda)>0$. Hence

$$S(\Lambda)>S_1(\Lambda). \tag{4.3}$$

Let r_2 satisfy $\kappa\le r_2\le\kappa+1$. Suppose that $\kappa+1+\nu+m\le r_1\le\Lambda$. Since $\nu<r_1$, $r_1^2+\nu^2<2r_1^2$ holds. Therefore we have $\omega(r_1)<\sqrt{2}r_1$. Since $r_2<r_1$, we have $r_1r_2<r_1^2$ and $r_2^2<r_1^2$. Thus $L(r_1,r_2,1)<2(\frac{1}{m}+\sqrt{2})r_1^2$. So,

$$\int_{\kappa+1+\nu+m}^\Lambda\frac{r_1^2}{\omega(r_1)L(r_1,r_2,1)}dr_1>\frac{1}{2\sqrt{2}(\frac{1}{m}+\sqrt{2})}\int_{\kappa+1+\nu+m}^\Lambda r_1^{-1}dr_1=\frac{1}{2\sqrt{2}(\frac{1}{m}+2)}(\log\Lambda-\log(\kappa+1+\nu+m))$$

follows. When $\kappa\le r_2\le\kappa+1$, we have

$$\omega(r_2)\le\sqrt{(\kappa+1)^2+\nu^2},\quad F(r_2)\le\frac{(\kappa+1)^2}{2m}+\sqrt{(\kappa+1)^2+\nu^2}.$$

Then

$$\begin{aligned}
S_1(\Lambda)&=\int_\kappa^{\kappa+1}dr_2\frac{r_2^4}{\omega(r_2)F(r_2)^4}\int_{\kappa+1+\nu+m}^\Lambda\frac{r_1^2}{\omega(r_1)L(r_1,r_2,1)}dr_1\\
&>\frac{K}{2\sqrt{2}(\frac{1}{m}+\sqrt{2})}(\log\Lambda-\log(\kappa+1+\nu+m))\int_\kappa^{\kappa+1}r_2^4dr_2\\
&=\frac{K\{(\kappa+1)^5-\kappa^5\}}{10\sqrt{2}(\frac{1}{m}+\sqrt{2})}(\log\Lambda-\log(\kappa+1+\nu+m)),
\end{aligned}\tag{4.4}$$

where

$$K=\frac{1}{\sqrt{(\kappa+1)^2+\nu^2}(\frac{(\kappa+1)^2}{2m}+\sqrt{(\kappa+1)^2+\nu^2})^4}.$$

From (4.2), (4.3), and (4.4), $C_3\le\lim_{\Lambda\to\infty}\frac{\mathrm{I}_1(\Lambda)}{\log\Lambda}$ follows.

The proof of $C_5\le\lim_{\Lambda\to\infty}\frac{\mathrm{I}_2(\Lambda)}{\log\Lambda}$ is similar to that of $C_3\le\lim_{\Lambda\to\infty}\frac{\mathrm{I}_1(\Lambda)}{\log\Lambda}$. Then we omit it. □

4.2.2 Proof of $\lim_{\Lambda\to\infty}\frac{\mathrm{I}_1(\Lambda)}{\log\Lambda}\le C_4$

We redefine $h(r_1,r_2)$, $h_1(r_1,r_2)$, $h_2(r_1,r_2)$, $S_1(\Lambda)$, $S_2(\Lambda)$ and $S(\Lambda)$ as

$$
\begin{aligned}
h(r_1,r_2) &= \frac{r_1^2 r_2^2}{\omega(r_1)\omega(r_2)}\Big(\frac{r_1^2}{F(r_1)^3}+\frac{r_2^2}{F(r_2)^3}\Big)\Big(\frac{1}{F(r_1)}+\frac{1}{F(r_2)}\Big)\frac{1}{L(r_1,r_2,-1)},\\
h_1(r_1,r_2) &= \frac{r_1^2 r_2^4}{\omega(r_1)\omega(r_2)F(r_2)^4 L(r_1,r_2,-1)},\\
h_2(r_1,r_2) &= \frac{r_1^2 r_2^4}{\omega(r_1)\omega(r_2)F(r_1)F(r_2)^3 L(r_1,r_2,-1)},\\
S_1(\Lambda) &= \int_\kappa^\Lambda\int_\kappa^\Lambda h_1(r_1,r_2)dr_1dr_2,\\
S_2(\Lambda) &= \int_\kappa^\Lambda\int_\kappa^\Lambda h_2(r_1,r_2)dr_1dr_2,\\
S(\Lambda) &= \int_\kappa^\Lambda\int_\kappa^\Lambda h(r_1,r_2)dr_1dr_2.
\end{aligned}
$$

Then we have

$$\mathrm{I}_1(\Lambda)\le\frac{4\pi^2}{(2\pi)^6}S(\Lambda). \tag{4.5}$$

Since $h(r_1,r_2)=h_1(r_1,r_2)+h_1(r_2,r_1)+h_2(r_1,r_2)+h_2(r_2,r_1)$, we have

$$S(\Lambda)=2(S_1(\Lambda)+S_2(\Lambda)). \tag{4.6}$$

Let B be $B=\frac{(\kappa+1)^3-\kappa^3}{6\nu^2}$. Since $\nu<\omega(r_1)$ and $2\nu<L(r_1,r_2,-1)$,

$$\int_\kappa^{\kappa+1}\frac{r_1^2}{\omega(r_1)L(r_1,r_2,-1)}dr_1<B \tag{4.7}$$

follows. Let Y be $Y=2r_2+\kappa+1$. Since $r_1<\omega(r_1)$ and $r_1<L(r_1,r_2,-1)$,

$$\int_{\kappa+1}^{Y}\frac{r_1^2}{\omega(r_1)L(r_1,r_2,-1)}dr_1<2r_2 \tag{4.8}$$

holds. When $Y\le r_1$, since $2r_2<r_1$, we have $r_1-r_2>r_1/2$. Then $L(r_1,r_2,-1)>\frac{(r_1-r_2)^2}{2m}>\frac{r_1^2}{8m}$. So,

$$\int_Y^\Lambda\frac{r_1^2}{\omega(r_1)L(r_1,r_2,-1)}dr_1<8m\int_Y^\Lambda\frac{dr_1}{r_1}<8m\log\Lambda \tag{4.9}$$

follows. From (4.7), (4.8) and (4.9), we have

$$\int_\kappa^\Lambda\frac{r_1^2}{\omega(r_1)L(r_1,r_2,-1)}dr_1<B+2r_2+8m\log\Lambda.$$

Using this, we see that $S_1(\Lambda)<\int_\kappa^\Lambda(B+2r_2+8m\log\Lambda)\frac{r_2^4}{\omega(r_2)F(r_2)^4}dr_2$. Since $r_2<\omega(r_2)$ and $r_2^2/2m<F(r_2)$, we have

$$S_1(\Lambda)<\int_\kappa^\Lambda(2r_2+B+8m\log\Lambda)\frac{16m^4}{r_2^5}dr_2=\frac{32m^4}{3}\Big(\frac{1}{\kappa^3}-\frac{1}{\Lambda^3}\Big)+4m^4(B+8m\log\Lambda)\Big(\frac{1}{\kappa^4}-\frac{1}{\Lambda^4}\Big)$$

$$< \frac{32m^5}{\kappa^4}\log\Lambda + \frac{32m^4}{3\kappa^3} + \frac{4m^4B}{\kappa^4}. \tag{4.10}$$

Since $r_1 < L(r_1, r_2, -1)$,

$$\int_\kappa^\Lambda \frac{r_1^2}{\omega(r_1)F(r_1)L(r_1,r_2,-1)}dr_1 < 2m\int_\kappa^\Lambda \frac{dr_1}{r_1^2} < \frac{2m}{\kappa}.$$

Then we have

$$\begin{aligned} S_2(\Lambda) &= \int_\kappa^\Lambda dr_2 \frac{r_2^4}{\omega(r_2)F(r_2)^3}\int_\kappa^\Lambda \frac{r_1^2}{\omega(r_1)F(r_1)L(r_1,r_2,-1)}dr_1 < \frac{2m}{\kappa}\int_\kappa^\Lambda \frac{r_2^4}{\omega(r_2)F(r_2)^3}dr_2 \\ &< \frac{16m^4}{\kappa}\int_\kappa^\Lambda \frac{dr_2}{r_2^3} < \frac{8m^4}{\kappa^3}. \end{aligned} \tag{4.11}$$

From (4.6), (4.10) and (4.11), it follows that

$$S(\Lambda) < \frac{64m^5}{\kappa^4}\log\Lambda + \frac{112m^4}{3\kappa^3} + \frac{8m^2B}{\kappa^4}. \tag{4.12}$$

From (4.5) and (4.12), the lemma follows. □

4.2.3 Proof of $\lim_{\Lambda\to\infty}\frac{\mathrm{I}_2(\Lambda)}{\log\Lambda} \le C_6$

We redefine $h(r_1, r_2)$, $h_1(r_1, r_2)$, $S(\Lambda)$, and $S_1(\Lambda)$ as

$$\begin{aligned} h(r_1,r_2) &= \frac{r_1^2r_2^2}{\omega(r_1)\omega(r_2)}\left(\frac{r_1^2}{F(r_1)^4} + \frac{r_2^2}{F(r_2)^4}\right)\frac{1}{L(r_1,r_2,-1)}, \\ h_1(r_1,r_2) &= \frac{r_1^2r_2^4}{\omega(r_1)\omega(r_2)F(r_2)^4L(r_1,r_2,-1)}, \\ S(\Lambda) &= \int_\kappa^\Lambda\int_\kappa^\Lambda h(r_1,r_2)dr_1dr_2, \\ S_1(\Lambda) &= \int_\kappa^\Lambda\int_\kappa^\Lambda h_1(r_1,r_2)dr_1dr_2. \end{aligned}$$

We have

$$\mathrm{I}_2(\Lambda) = \frac{\pi^2}{(2\pi)^6}\int_{-1}^1 dz\int_\kappa^\Lambda dr_1\int_\kappa^\Lambda \frac{r_1^2r_2^2}{\omega(r_1)\omega(r_2)}\left(\frac{r_1^2}{F(r_1)^4} + \frac{r_2^2}{F(r_2)^4}\right)\frac{1}{L(r_1,r_2,z)}dr_2.$$

Then

$$\mathrm{I}_2(\Lambda) \le \frac{2\pi^2}{(2\pi)^6}S(\Lambda) \tag{4.13}$$

holds. Since $h(r_1, r_2) = h_1(r_1, r_2) + h_1(r_2, r_1)$, we have

$$S(\Lambda) = 2S_1(\Lambda). \tag{4.14}$$

We have

$$\int_\kappa^\Lambda \frac{r_1^2}{\omega(r_1)L(r_1,r_2,-1)}dr_1 < B + 2r_2 + 8m\log\Lambda$$

in the same way as the proof of $\lim_{\Lambda\to\infty}\frac{\mathrm{I}_1(\Lambda)}{\log\Lambda} \le C_4$. Since $r_2 < \omega(r_2)$ and $r_2^2/2m < F(r_2)$, we have

$$\begin{aligned} S_1(\Lambda) &< \int_\kappa^\Lambda (B + 2r_2 + 8m\log\Lambda)\frac{r_2^4}{\omega(r_2)F(r_2)^4}dr_2 \\ &< 32m^4\int_\kappa^\Lambda \frac{dr_2}{r_2^4} + 16m^4(B + 8m\log\Lambda)\int_\kappa^\Lambda \frac{dr_2}{r_2^5} < \frac{32m^5}{\kappa^4}\log\Lambda + \frac{32m^4}{3\kappa^3} + \frac{4m^4B}{\kappa^4}. \end{aligned} \tag{4.15}$$

From (4.13) , (4.14), and (4.15), the lemma follows. □

4.2.4 Proof of $\lim_{\Lambda\to\infty}\frac{\mathrm{I}_3(\Lambda)}{\log\Lambda}=0$

We define $h(r_1,r_2,z)$ as

$$h(r_1,r_2,z)=\frac{zr_1^3r_2^3}{\omega(r_1)\omega(r_2)}\left(\frac{1}{F(r_1)^2}+\frac{1}{F(r_2)^2}\right)\left(\frac{1}{F(r_1)}+\frac{1}{F(r_2)}\right)\frac{1}{L(r_1,r_2,z)^2}$$

and redefine $S(\Lambda)$ as

$$S(\Lambda)=\int_{-1}^{1}dz\int_{\kappa}^{\Lambda}dr_2\int_{\kappa}^{\Lambda}h(r_1,r_2,z)dr_1.$$

Then we have $\mathrm{I}_3(\Lambda)=\frac{\pi^2}{(2\pi)^6}S(\Lambda)$. We divide $S(\Lambda)$ in the following way.

$$\begin{aligned}
S(\Lambda)&=\int_{-1}^{0}dz\int_{\kappa}^{\Lambda}dr_2\int_{\kappa}^{\Lambda}h(r_1,r_2,z)dr_1+\int_{0}^{1}dz\int_{\kappa}^{\Lambda}dr_2\int_{\kappa}^{\Lambda}h(r_1,r_2,z)dr_1\\
&=-\int_{1}^{0}dz\int_{\kappa}^{\Lambda}dr_2\int_{\kappa}^{\Lambda}h(r_1,r_2,-z)dr_1+\int_{0}^{1}dz\int_{\kappa}^{\Lambda}dr_2\int_{\kappa}^{\Lambda}h(r_1,r_2,z)dr_1\\
&=\int_{0}^{1}dz\int_{\kappa}^{\Lambda}dr_2\int_{\kappa}^{\Lambda}(h(r_1,r_2,z)+h(r_1,r_2,-z))dr_1\\
&=\int_{0}^{1}dz\int_{\kappa}^{\Lambda}dr_2\int_{\kappa}^{\Lambda}g(r_1,r_2,z)dr_1, \qquad (4.16)
\end{aligned}$$

where

$$\begin{aligned}
g(r_1,r_2,z)&=h(r_1,r_2,z)+h(r_1,r_2,-z)\\
&=-\frac{2z^2r_1^4r_2^4}{m\omega(r_1)\omega(r_2)}\left(\frac{1}{F(r_1)^2}+\frac{1}{F(r_2)^2}\right)\left(\frac{1}{F(r_1)}+\frac{1}{F(r_2)}\right)\frac{\left(\frac{r_1^2+r_2^2}{m}+2\omega(r_1)+2\omega(r_2)\right)}{L(r_1,r_2,z)^2L(r_1,r_2,-z)^2}. \qquad (4.17)
\end{aligned}$$

Since $g(r_1,r_2,z)\le 0$, $S(\Lambda)$ is decreasing in Λ.

$$\begin{aligned}
S(\Lambda)&=\int_{0}^{1}dz\int_{\kappa}^{\Lambda}dr_2\int_{r_2}^{\Lambda}g(r_1,r_2,z)dr_1+\int_{0}^{1}dz\int_{\kappa}^{\Lambda}dr_1\int_{r_1}^{\Lambda}g(r_1,r_2,z)dr_2\\
&=2\int_{0}^{1}dz\int_{\kappa}^{\Lambda}dr_2\int_{r_2}^{\Lambda}g(r_1,r_2,z)dr_1\\
&=2\int_{0}^{1}dz\int_{\kappa}^{\Lambda}dr_2\int_{r_2}^{2r_2}g(r_1,r_2,z)dr_1+2\int_{0}^{1}dz\int_{\kappa}^{\Lambda}dr_2\int_{2r_2}^{\Lambda}g(r_1,r_2,z)dr_1. \qquad (4.18)
\end{aligned}$$

Since $\kappa\le r_1$, we have $1\le r_1/\kappa$. Hence $r_1^2+\nu^2\le\frac{\kappa^2+\nu^2}{\kappa^2}r_1^2$. Therefore we have $\omega(r_1)\le\frac{\sqrt{\kappa^2+\nu^2}}{\kappa}r_1$, and similarly $\omega(r_2)\le\frac{\sqrt{\kappa^2+\nu^2}}{\kappa}r_2$. When $0\le z\le 1$, we have

$$L(r_1,r_2,z)>\frac{r_1^2}{2m}. \qquad (4.19)$$

Then

$$L(r_1,r_2,-z)=\frac{(r_1-r_2)^2+2r_1r_2(1-z)}{2m}+\omega(r_1)+\omega(r_2)>\omega(r_1)>r_1. \qquad (4.20)$$

When $r_2\le r_1$, we have $\frac{r_2^2}{m}\le\frac{r_1^2}{m}$. Then it holds that

$$2\omega(r_1)\le\frac{2r_1^2}{\kappa^2}\sqrt{\kappa^2+\nu^2},\quad 2\omega(r_2)\le\frac{2r_2}{\kappa}\sqrt{\kappa^2+\nu^2}\le\frac{2r_1^2}{\kappa^2}\sqrt{\kappa^2+\nu^2}.$$

Thus we have

$$\frac{r_1^2}{m} + \frac{r_2^2}{m} + 2\omega(r_1) + 2\omega(r_2) \le 2(\frac{1}{m} + \frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})r_1^2. \tag{4.21}$$

When $r_2 \le r_1 \le 2r_2$, we have

$$\frac{1}{F(r_1)} + \frac{1}{F(r_2)} < \frac{10m}{r_1^2}, \quad \frac{1}{F(r_1)^2} + \frac{1}{F(r_2)^2} < \frac{68m^2}{r_1^4}. \tag{4.22}$$

From (4.17), (4.19), (4.20), (4.21) and (4.22), it follows that

$$\begin{aligned}-g(r_1,r_2,z) &\le \frac{2}{m}\frac{z^2r_1^4r_2^4}{r_1r_2}\frac{68m^2}{r_1^4}\frac{10m}{r_1^2}2(\frac{1}{m} + \frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})r_1^2(\frac{2m}{r_1^2})^2\frac{1}{r_1^2} \\ &= 68\cdot 10\cdot 16m^4(\frac{1}{m} + \frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})\frac{z^2r_2^3}{r_1^7}.\end{aligned}$$

Hence

$$-2\int_0^1 dz\int_\kappa^\Lambda dr_2\int_{r_2}^{2r_2} g(r_1,r_2,z)dr_1 \le 1190\frac{m^4}{\kappa^2}(\frac{1}{m} + \frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2}). \tag{4.23}$$

When $2r_2 \le r_1$, we have

$$\frac{1}{F(r_1)} + \frac{1}{F(r_2)} < \frac{5m}{2r_2^2}, \quad \frac{1}{F(r_1)^2} + \frac{1}{F(r_2)^2} < \frac{17m^2}{4r_2^4}. \tag{4.24}$$

Since $r_2 \le r_1/2$, we have $r_1/2 \le r_1 - r_2$. Then we have

$$L(r_1,r_2,\pm z) = \frac{(r_1-r_2)^2 + 2r_1r_2(1\pm z)}{2m} + \omega(r_1) + \omega(r_2) > \frac{(r_1-r_2)^2}{2m} \ge \frac{r_1^2}{8m}. \tag{4.25}$$

From (4.17), (4.21), (4.24) and (4.25), it follows that

$$\begin{aligned}-g(r_1,r_2,z) &\le \frac{2}{m}\frac{z^2r_1^4r_2^4}{r_1r_2}\frac{17m^2}{4r_2^4}\frac{5m}{2r_2^2}2(\frac{1}{m} + \frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})r_1^2(\frac{8m}{r_1^2})^4 \\ &= 2^{11}\cdot 5\cdot 17m^6(\frac{1}{m} + \frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})\frac{z^2}{r_1^3r_2^3}.\end{aligned}$$

Hence

$$-2\int_0^1 dz\int_\kappa^\Lambda dr_2\int_{2r_2}^{\Lambda} g(r_1,r_2,z)dr_1 \le 2^7\cdot 5\cdot 17\frac{m^6}{\kappa^4}(\frac{1}{m} + \frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2}). \tag{4.26}$$

Then by (4.18), (4.23) and (4.26), we have

$$-S(\Lambda) \le \frac{1190m^4}{\kappa^2}(\frac{1}{m} + \frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2}) + 2^7\cdot 5\cdot 17\frac{m^6}{\kappa^4}(\frac{1}{m} + \frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2}).$$

Since $S(\Lambda)$ is decreasing and bounded below, it converges as $\Lambda \to \infty$. This fact proves the lemma. □

4.2.5 Proof of $\lim_{\Lambda\to\infty} \frac{\mathrm{I}_4(\Lambda)}{\log\Lambda} = 0$

We redefine $h(r_1, r_2, z)$ as

$$h(r_1,r_2,z) = \frac{r_1^2 r_2^2}{\omega(r_1)\omega(r_2)}\left(\frac{r_1^2}{F(r_1)^2} + \frac{r_2^2}{F(r_2)^2}\right)\left(\frac{1}{F(r_1)} + \frac{1}{F(r_2)}\right)\frac{1}{L(r_1,r_2,z)^2}.$$

Then we have $\mathrm{I}_4(\Lambda) = \frac{2\pi^2}{(2\pi)^6}\int_{-1}^1 dz \int_\kappa^\Lambda dr_1 \int_\kappa^\Lambda h(r_1,r_2,z)dr_2$. We define $J(\Lambda)$ as

$$J(\Lambda) = \int_{-1}^1 dz \int_\kappa^\Lambda dr_1 \int_\kappa^\Lambda h(r_1,r_2,z)dr_2.$$

(Step 1) We define $S(\Lambda, z)$ as

$$S(\Lambda,z) = \int_\kappa^\Lambda dr_1 \int_\kappa^\Lambda h(r_1,r_2,z)dr_2.$$

Our first task is to prove that $\lim_{\Lambda\to\infty} S(\Lambda,z)$ exists for all $z \in \mathrm{I} = [-1,1]$. Since $h(r_1,r_2,z) > 0$, $S(\Lambda,z)$ is increasing in Λ. Let

$$\begin{aligned} S_1(\Lambda,z) &= 2\int_\kappa^\Lambda dr_2 \int_{r_2}^{2r_2} h(r_1,r_2,z)dr_1, \\ S_2(\Lambda,z) &= 2\int_\kappa^\Lambda dr_2 \int_{2r_2}^{\Lambda} h(r_1,r_2,z)dr_1. \end{aligned}$$

Then

$$\begin{aligned} S(\Lambda,z) &= \int_\kappa^\Lambda dr_2 \int_{r_2}^\Lambda h(r_1,r_2,z)dr_1 + \int_\kappa^\Lambda dr_1 \int_{r_1}^\Lambda h(r_1,r_2,z)dr_2 \\ &= 2\int_\kappa^\Lambda dr_2 \int_{r_2}^\Lambda h(r_1,r_2,z)dr_1 \\ &= 2\int_\kappa^\Lambda dr_2 \int_{r_2}^{2r_2} h(r_1,r_2,z)dr_1 + 2\int_\kappa^\Lambda dr_2 \int_{2r_2}^\Lambda h(r_1,r_2,z)dr_1 \\ &= S_1(\Lambda,z) + S_2(\Lambda,z) \end{aligned} \tag{4.27}$$

holds. We have

$$\frac{1}{F(r_1)} + \frac{1}{F(r_2)} < 2m\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right), \quad \frac{r_1^2}{F(r_1)^2} + \frac{r_2^2}{F(r_2)^2} < 4m^2\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) \tag{4.28}$$

and

$$r_1 < L(r_1,r_2,z). \tag{4.29}$$

Let $r_2 \le r_1 \le 2r_2$. Since $1/r_2^2 \le 4/r_1^2$, it holds that

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} \le 5/r_1^2. \tag{4.30}$$

Then from (4.28) and (4.30), it follows that

$$\frac{1}{F(r_1)} + \frac{1}{F(r_2)} < \frac{10m}{r_1^2}, \quad \frac{r_1^2}{F(r_1)^2} + \frac{r_2^2}{F(r_2)^2} < \frac{20m^2}{r_1^2}. \tag{4.31}$$

Hence from (4.29) and (4.31), it follows that $h(r_1, r_2, z) < \frac{200m^3 r_2}{r_1^5}$. Therefore we have

$$S_1(\Lambda, z) < 400m^3 \int_\kappa^\Lambda r_2 dr_2 \int_{r_2}^{2r_2} \frac{dr_1}{r_1^5} = \frac{375m^3}{8}(\frac{1}{\kappa^2} - \frac{1}{\Lambda^2}) < \frac{375m^3}{8\kappa^2}. \tag{4.32}$$

Let $2r_2 \le r_1$. Since $r_1/2 \le r_1 - r_2$, we have

$$\frac{r_1^2}{8m} \le \frac{(r_1 - r_2)^2}{2m} < L(r_1, r_2, z). \tag{4.33}$$

Since $r_2^2 \le r_1^2$, we have

$$\frac{1}{F(r_1)} + \frac{1}{F(r_2)} < \frac{4m}{r_2^2}, \quad \frac{r_1^2}{F(r_1)^2} + \frac{r_2^2}{F(r_2)^2} < \frac{8m^2}{r_2^2}. \tag{4.34}$$

Hence from (4.33) and (4.34), it follows that $h(r_1, r_2, z) < \frac{2048m^5}{r_1^3 r_2^3}$. Therefore we have

$$S_2(\Lambda, z) < 4096m^5 \int_\kappa^\Lambda \frac{dr_2}{r_2^3} \int_{2r_2}^\Lambda \frac{dr_1}{r_1^3} < 512m^5 \int_\kappa^\Lambda \frac{dr_2}{r_2^5} < \frac{128m^5}{\kappa^4}. \tag{4.35}$$

From (4.27), (4.32) and (4.35), it follows that

$$S(\Lambda, z) < \frac{375m^3}{8\kappa^2} + \frac{128m^5}{\kappa^4}.$$

Since $S(\Lambda, z)$ is increasing in Λ and bounded above for all $z \in \mathrm{I}$, it converges as Λ goes to infinity.

(Step 2) Our second task is to prove that $J(\Lambda)$ converges when Λ goes to infinity. Let $M(r_1, r_2)$ be

$$M(r_1, r_2) = \frac{r_1^2 r_2^2}{\omega(r_1)\omega(r_2)}(\frac{r_1^2}{F(r_1)^2} + \frac{r_2^2}{F(r_2)^2})(\frac{1}{F(r_1)} + \frac{1}{F(r_2)})\frac{1}{L(r_1, r_2, -1)^2}.$$

$|h(r_1, r_2, z)| \le M(r_1, r_2)$ holds for all $(r_1, r_2, z) \in [\kappa, \Lambda]^2 \times \mathrm{I}$, and by (Step 1), there exists

$$\lim_{\Lambda \to \infty} \int_\kappa^\Lambda \int_\kappa^\Lambda M(r_1, r_2) dr_1 dr_2.$$

Since

$$M_\Lambda = \int_\kappa^\Lambda \int_\kappa^\Lambda M(r_1, r_2) dr_1 dr_2 \to M_\infty = \lim_{\Lambda \to \infty} \int_\kappa^\Lambda \int_\kappa^\Lambda M(r_1, r_2) dr_1 dr_2,$$

from Cauchy convergence condition, for any $\epsilon > 0$, there exists $\Lambda_0 \in [\kappa, \infty)$ such that if $\Lambda_0 < \Lambda_1 \le \Lambda_2$, $|M_{\Lambda_2} - M_{\Lambda_1}| < \epsilon$. Then for $\Lambda_0 < \Lambda_1 \le \Lambda_2$ and all $z \in \mathrm{I}$,

$$\begin{aligned}
&|S(\Lambda_2, z) - S(\Lambda_1, z)| \\
&= |\int_\kappa^{\Lambda_1} dr_1 \int_{\Lambda_1}^{\Lambda_2} h(r_1, r_2, z) dr_2 + \int_\kappa^{\Lambda_1} dr_2 \int_{\Lambda_1}^{\Lambda_2} h(r_1, r_2, z) dr_1 + \int_{\Lambda_1}^{\Lambda_2} \int_{\Lambda_1}^{\Lambda_2} h(r_1, r_2, z) dr_1 dr_2| \\
&\le \int_\kappa^{\Lambda_1} dr_1 \int_{\Lambda_1}^{\Lambda_2} M(r_1, r_2) dr_2 + \int_\kappa^{\Lambda_1} dr_2 \int_{\Lambda_1}^{\Lambda_2} M(r_1, r_2) dr_1 + \int_{\Lambda_1}^{\Lambda_2} \int_{\Lambda_1}^{\Lambda_2} M(r_1, r_2) dr_1 dr_2 \\
&= |M_{\Lambda_2} - M_{\Lambda_1}| < \epsilon.
\end{aligned}$$

Therefore $\sup_{z \in I} |S(\Lambda_2, z) - S(\Lambda_1, z)| \le |M_{\Lambda_2} - M_{\Lambda_1}| < \epsilon$ holds. Since family of functions $(S(\Lambda, \cdot))_{\Lambda \in [\kappa, \infty)}$ on I satisfies uniform Cauchy conditions, it converges uniformly on I. Since $[\kappa, \Lambda]^2$ is a Jordan measurable bounded closed set of $\mathbb{R}^2$, the function $S(\Lambda, z)$ is continuous on I. Hence

$$S(\infty, z) = \lim_{\Lambda \to \infty} \int_\kappa^\Lambda \int_\kappa^\Lambda h(r_1, r_2, z) dr_1 dr_2$$

is continuous on I. Since both $S(\Lambda, z)$ and $S(\infty, z)$ are integrable on Jordan measurable set I, by uniform convergence theorem, we have

$$\lim_{\Lambda\to\infty}\int_{-1}^{1} S(\Lambda, z)dz = \int_{-1}^{1} S(\infty, z)dz.$$

It implies that $J(\Lambda)$ converges as $\Lambda \to \infty$. □

4.2.6 Proof of $\lim_{\Lambda\to\infty}\frac{\mathrm{I}_5(\Lambda)}{\log\Lambda} = 0$

We redefine $h(r_1, r_2, z)$, $g(r_1, r_2, z)$ and $S(\Lambda)$ as

$$\begin{aligned}
h(r_1, r_2, z) &= \frac{zr_1^3r_2^3}{\omega(r_1)\omega(r_2)F(r_1)^2F(r_2)^2L(r_1, r_2, z)},\\
g(r_1, r_2, z) &= h(r_1, r_2, z) + h(r_1, r_2, -z),\\
S(\Lambda) &= \int_{-1}^{1} dz\int_{\kappa}^{\Lambda} dr_1\int_{\kappa}^{\Lambda} h(r_1, r_2, z)dr_2.
\end{aligned}$$

Then $\mathrm{I}_5(\Lambda) = \frac{2\pi^2}{(2\pi)^6}S(\Lambda)$. We have

$$S(\Lambda) = \int_0^1 dz\int_\kappa^\Lambda dr_2\int_\kappa^\Lambda g(r_1, r_2, z)dr_1$$

in the same way as (4.16). Since

$$g(r_1, r_2, z) = -\frac{2z^2r_1^4r_2^4}{m^2\omega(r_1)\omega(r_2)F(r_1)^2F(r_2)^2L(r_1, r_2, z)L(r_1, r_2, -z)} \le 0,$$

$S(\Lambda)$ is decreasing in Λ. Since $r_1 < L(r_1, r_2, z)$, we have

$$\frac{r_1^4}{\omega(r_1)F(r_1)^2L(r_1, r_2, z)} < \frac{4m^2}{r_1^2}.$$

Similarly, we have

$$\frac{r_2^4}{\omega(r_2)F(r_2)^2L(r_1, r_2, -z)} < \frac{4m^2}{r_2^2}.$$

Hence

$$\begin{aligned}
-S(\Lambda) &= \frac{2}{m^2}\int_0^1 dzz^2\int_\kappa^\Lambda dr_2\frac{r_2^4}{\omega(r_2)F(r_2)^2L(r_1, r_2, -z)}\int_\kappa^\Lambda\frac{r_1^4}{\omega(r_1)F(r_1)^2L(r_1, r_2, z)}dr_1\\
&< \frac{2}{3m^2}\int_\kappa^\Lambda dr_2\frac{4m^2}{r_2^2}\int_\kappa^\Lambda\frac{4m^2}{r_1^2}dr_1 < \frac{32m^2}{3\kappa^2}.
\end{aligned}$$

Since $S(\Lambda)$ is decreasing and bounded below, it converges as $\Lambda \to \infty$. □

4.2.7 Proof of $\lim_{\Lambda\to\infty}\frac{\mathrm{I}_6(\Lambda)}{\log\Lambda} = 0$

We redefine $h(r_1, r_2, z)$, $J(\Lambda)$, $S(\Lambda, z)$, $S_1(\Lambda, z)$ and $S_2(\Lambda, z)$ as

$$\begin{aligned}
h(r_1, r_2, z) &= \frac{r_1^2r_2^2}{\omega(r_1)\omega(r_2)}\left(\frac{1}{F(r_1)} + \frac{1}{F(r_2)}\right)^2\frac{r_1^2 + r_2^2}{L(r_1, r_2, z)^3},\\
J(\Lambda) &= \int_{-1}^{1} dz\int_\kappa^\Lambda dr_2\int_\kappa^\Lambda h(r_1, r_2, z)dr_1,
\end{aligned}$$

$$S(\Lambda, z) = \int_\kappa^\Lambda dr_1 \int_\kappa^\Lambda h(r_1, r_2, z) dr_2,$$
$$S_1(\Lambda, z) = 2\int_\kappa^\Lambda dr_2 \int_{r_2}^{2r_2} h(r_1, r_2, z) dr_1,$$
$$S_2(\Lambda, z) = 2\int_\kappa^\Lambda dr_2 \int_{2r_2}^{\Lambda} h(r_1, r_2, z) dr_1.$$

We have $\mathrm{I}_6(\Lambda) = \frac{\pi^2}{(2\pi)^6} J(\Lambda)$.

(Step 1) Our first task is to prove that $\lim_{\Lambda\to\infty} S(\Lambda, z)$ exists for all $z \in \mathrm{I}$. Since $h(r_1, r_2, z) > 0$, $S(\Lambda, z)$ is increasing in Λ. We have

$$S(\Lambda, z) = S_1(\Lambda, z) + S_2(\Lambda, z) \tag{4.36}$$

in the same way as (4.27). When $r_2 \le r_1$, it holds that

$$r_1^2 + r_2^2 \le 2r_1^2, \quad \frac{1}{F(r_1)} + \frac{1}{F(r_2)} < \frac{4m}{r_2^2}.$$

When $r_2 \le r_1 \le 2r_2$, it also holds that

$$\frac{1}{F(r_1)} + \frac{1}{F(r_2)} < \frac{10m}{r_1^2}.$$

Then we have

$$\int_{r_2}^{2r_2} h(r_1, r_2, z) dr_1 < \int_{r_2}^{2r_2} \frac{r_1^2 r_2^2}{r_1 r_2} (\frac{10m}{r_1^2})^2 \frac{2r_1^2}{r_1^3} dr_1 = \frac{175m^2}{3r_2^2}.$$

Hence

$$S_1(\Lambda, z) < \frac{350m^2}{3} \int_\kappa^\Lambda \frac{dr_2}{r_2^2} < \frac{350m^2}{3\kappa}. \tag{4.37}$$

Let $2r_2 \le r_1$. Since $r_1/2 \le r_1 - r_2$, we have

$$\frac{r_1^2}{8m} \le \frac{(r_1 - r_2)^2}{2m} < L(r_1, r_2, z).$$

Then

$$\int_{2r_2}^{\Lambda} h(r_1, r_2, z) dr_1 < \int_{2r_2}^{\Lambda} \frac{r_1^2 r_2^2}{r_1 r_2} (\frac{4m}{r_2^2})^2 (\frac{8m}{r_1^2})^3 2r_1^2 dr_1 = \frac{16384 m^5}{r_2^3} \int_{2r_2}^{\Lambda} \frac{dr_1}{r_1^3} < \frac{2048 m^5}{r_2^5}.$$

Therefore

$$S_2(\Lambda, z) < 4096 m^5 \int_\kappa^\Lambda \frac{dr_2}{r_2^5} < \frac{1024 m^5}{\kappa^4}. \tag{4.38}$$

From (4.36), (4.37) and (4.38), it follows that $S(\Lambda, z) < \frac{350m^2}{3\kappa} + \frac{1024m^5}{\kappa^4}$. Since $S(\Lambda, z)$ is increasing in Λ and bounded above, it converges as Λ goes to infinity.

(Step 2) Our second task is to prove $J(\Lambda)$ converges as Λ goes to infinity. This step is the same as that of $\lim_{\Lambda\to\infty} \frac{\mathrm{I}_4(\Lambda)}{\log \Lambda} = 0$. □

4.2.8 Proof of $\lim_{\Lambda\to\infty}\frac{\mathrm{I}_7(\Lambda)}{\log\Lambda}=0$

We redefine $h(r_1,r_2,z)$, $g(r_1,r_2,z)$ and $S(\Lambda)$ as

$$\begin{aligned}
h(r_1,r_2,z) &= \frac{zr_1^3r_2^3}{\omega(r_1)\omega(r_2)}(\frac{1}{F(r_1)}+\frac{1}{F(r_2)})^2\frac{1}{L(r_1,r_2,z)^3},\\
g(r_1,r_2,z) &= h(r_1,r_2,z)+h(r_1,r_2,-z),\\
S(\Lambda) &= \int_{-1}^1 dz\int_\kappa^\Lambda dr_1\int_\kappa^\Lambda h(r_1,r_2,z)dr_2.
\end{aligned}$$

Then we have $\mathrm{I}_7(\Lambda)=\frac{2\pi^2}{(2\pi)^6}S(\Lambda)$, and

$$S(\Lambda)=\int_0^1 dz\int_\kappa^\Lambda dr_1\int_\kappa^\Lambda g(r_1,r_2,z)dr_2$$

in the same way as $\lim_{\Lambda\to\infty}\frac{\mathrm{I}_3(\Lambda)}{\log\Lambda}=0$. We define $g_1(r_1,r_2,z)$ and $g_2(r_1,r_2,z)$ as

$$\begin{aligned}
g_1(r_1,r_2,z) &= -\frac{6z^2r_1^4r_2^4}{m\omega(r_1)\omega(r_2)}(\frac{1}{F(r_1)}+\frac{1}{F(r_2)})^2\frac{(\frac{r_1^2+r_2^2}{2m}+\omega(r_1)+\omega(r_2))^2}{L(r_1,r_2,z)^3L(r_1,r_2,-z)^3},\\
g_2(r_1,r_2,z) &= -\frac{2z^4r_1^6r_2^6}{m^3\omega(r_1)\omega(r_2)}(\frac{1}{F(r_1)}+\frac{1}{F(r_2)})^2\frac{1}{L(r_1,r_2,z)^3L(r_1,r_2,-z)^3}.
\end{aligned}$$

Then we have $g(r_1,r_2,z)=g_1(r_1,r_2,z)+g_2(r_1,r_2,z)$. We redefine $S_1(\Lambda)$ and $S_2(\Lambda)$ by

$$\begin{aligned}
S_1(\Lambda) &= \int_0^1 dz\int_\kappa^\Lambda dr_1\int_\kappa^\Lambda g_1(r_1,r_2,z)dr_2,\\
S_2(\Lambda) &= \int_0^1 dz\int_\kappa^\Lambda dr_1\int_\kappa^\Lambda g_2(r_1,r_2,z)dr_2.
\end{aligned}$$

Then

$$S(\Lambda)=S_1(\Lambda)+S_2(\Lambda).$$

Since $g_1(r_1,r_2,z)\le 0$, $S_1(\Lambda)$ is decreasing in Λ. We divide $S_1(\Lambda)$ in the following way:

$$\begin{aligned}
S_1(\Lambda) &= \int_0^1 dz\int_\kappa^\Lambda dr_1\int_{r_1}^\Lambda g_1(r_1,r_2,z)dr_2+\int_0^1 dz\int_\kappa^\Lambda dr_2\int_{r_2}^\Lambda g_1(r_1,r_2,z)dr_1\\
&= \int_0^1 dz\int_\kappa^\Lambda dr_1\int_{r_1}^\Lambda g_1(r_1,r_2,z)dr_2+\int_0^1 dz\int_\kappa^\Lambda dr_1\int_{r_1}^\Lambda g_1(r_2,r_1,z)dr_2\\
&= 2\int_0^1 dz\int_\kappa^\Lambda dr_1\int_{r_1}^\Lambda g_1(r_1,r_2,z)dr_2.
\end{aligned}$$

Let $\kappa\le r_1\le r_2$. Then we have

$$\frac{r_1^2+r_2^2}{2m}+\omega(r_1)+\omega(r_2)\le(\frac{1}{m}+\frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})r_2^2$$

in the same way as (4.21). Let $r_1\le r_2\le 2r_1$. Then we also have

$$\frac{1}{F(r_1)}+\frac{1}{F(r_2)}<\frac{10m}{r_2^2}.$$

Therefore

$$-g_1(r_1,r_2,z) \le \frac{6}{m}\frac{z^2r_1^4r_2^4}{r_1r_2}(\frac{10m}{r_2^2})^2\{(\frac{1}{m}+\frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})r_2^2\}^2(\frac{2m}{r_2^2})^3\frac{1}{r_2^3} = 4800m^4(\frac{1}{m}+\frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})^2\frac{z^2r_1^3}{r_2^6},$$

and

$$-\int_{r_1}^{2r_1} g_1(r_1,r_2,z)dr_2 \le 4800m^4(\frac{1}{m}+\frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})^2 z^2r_1^3\int_{r_1}^{2r_1}\frac{dr_2}{r_2^6} = 930m^4(\frac{1}{m}+\frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})^2\frac{z^2}{r_1^2}.$$

Hence

$$-2\int_0^1 dz\int_\kappa^\Lambda dr_1\int_{r_1}^{2r_1} g_1(r_1,r_2,z)dr_2 \le 1860m^4(\frac{1}{m}+\frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})^2\int_0^1 z^2dz\int_\kappa^\Lambda\frac{dr_2}{r_2^2} < \frac{620m^4}{\kappa}(\frac{1}{m}+\frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})^2.$$

Let $2r_1 \le r_2$. Then we have

$$\frac{1}{F(r_1)}+\frac{1}{F(r_2)} < \frac{5m}{2r_1^2}.$$

In addition, since $r_2/2 \le r_2 - r_1$, we can see that

$$\frac{r_2^2}{8m} \le \frac{(r_2-r_1)^2}{2m} < L(r_1,r_2,-z).$$

Therefore

$$\begin{aligned}-g_1(r_1,r_2,z) &\le \frac{6}{m}\frac{z^2r_1^4r_2^4}{r_1r_2}(\frac{5m^2}{2r_1^2})^2\{(\frac{1}{m}+\frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})r_2^2\}^2(\frac{2m}{r_2^2})^3(\frac{8m}{r_2^2})^3\\ &= 153600m^9(\frac{1}{m}+\frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})^2\frac{z^2}{r_1r_2^5}.\end{aligned}$$

Then we have

$$-\int_{2r_1}^{\Lambda} g_1(r_1,r_2,z)dr_2 \le \frac{153600m^9z^2}{r_1}(\frac{1}{m}+\frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})^2\int_{2r_1}^\Lambda\frac{dr_2}{r_2^5} \le 2400m^9(\frac{1}{m}+\frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})^2\frac{z^2}{r_1^5}.$$

Hence

$$\begin{aligned}-2\int_0^1 dz\int_\kappa^\Lambda dr_1\int_{2r_1}^\Lambda g_1(r_1,r_2,z)dr_2 &\le 4800m^9(\frac{1}{m}+\frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})^2\int_0^1 z^2dz\int_\kappa^\Lambda\frac{dr_1}{r_1^5}\\ &< \frac{400m^9}{\kappa^4}(\frac{1}{m}+\frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})^2.\end{aligned}$$

Then we have

$$-S_1(\Lambda) < \frac{620m^4}{\kappa}(\frac{1}{m}+\frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})^2 + \frac{400m^9}{\kappa^4}(\frac{1}{m}+\frac{2\sqrt{\kappa^2+\nu^2}}{\kappa^2})^2.$$

Since $S_1(\Lambda)$ is decreasing and bounded below, it converges as $\Lambda \to \infty$. Since $g_2(r_1,r_2,z) \le 0$, $S_2(\Lambda)$ is also decreasing in Λ. Let $r_1 \le r_2$. Then

$$\frac{1}{F(r_1)}+\frac{1}{F(r_2)} < \frac{4m}{r_1^2}.$$

Therefore

$$-g_2(r_1,r_2,z) \le \frac{2}{m^3}\frac{z^4r_1^6r_2^6}{r_1r_2}(\frac{4m}{r_1^2})^2\frac{8m^3}{r_2^6}\frac{1}{r_2^3} = \frac{256m^2z^4r_1}{r_2^4}.$$

Then

$$-\int_{r_1}^{\Lambda} g_2(r_1,r_2,z)dr_2 \le 256m^2z^4r_1\int_{r_1}^{\Lambda}\frac{dr_2}{r_2^4} \le \frac{256m^2z^4}{3r_1^2}.$$

Hence

$$-S_2(\Lambda) \le \frac{512m^2}{3}\int_0^1 z^4dz\int_{\kappa}^{\Lambda}\frac{dr_1}{r_1^2} < \frac{512m^2}{15\kappa}.$$

Since $S_2(\Lambda)$ is decreasing in Λ and bounded below, it converges. Since both $S_1(\Lambda)$ and $S_2(\Lambda)$ converge, $S(\Lambda)$ converges. □

4.2.9 Proof of $\lim_{\Lambda\to\infty}\frac{\mathrm{I}_8(\Lambda)}{\log\Lambda}=0$

We redefine $h(r_1,r_2,z)$, $g(r_1,r_2,z)$, $g_1(r_1,r_2,z)$, $g_2(r_1,r_2,z)$, $S(\Lambda)$, $S_1(\Lambda)$ and $S_2(\Lambda)$ as

$$\begin{aligned}
h(r_1,r_2,z) &= \frac{zr_1^3r_2^3}{\omega(r_1)\omega(r_2)}\left(\frac{1}{F(r_1)}+\frac{1}{F(r_2)}\right)\frac{1}{L(r_1,r_2,z)^4},\\
g(r_1,r_2,z) &= h(r_1,r_2,z)+h(r_1,r_2,-z),\\
g_1(r_1,r_2,z) &= -\frac{2z^2r_1^4r_2^4}{m\omega(r_1)\omega(r_2)}\left(\frac{1}{F(r_1)}+\frac{1}{F(r_2)}\right)\frac{1}{L(r_1,r_2,z)L(r_1,r_2,-z)^4}, && (4.39)\\
g_2(r_1,r_2,z) &= -\frac{2z^2r_1^4r_2^4}{m\omega(r_1)\omega(r_2)}\left(\frac{1}{F(r_1)}+\frac{1}{F(r_2)}\right)G(r_1,r_2,z), && (4.40)\\
S(\Lambda) &= \int_{-1}^{1}dz\int_{\kappa}^{\Lambda}dr_2\int_{\kappa}^{\Lambda}h(r_1,r_2,z)dr_1,\\
S_1(\Lambda) &= \int_{\kappa}^{\Lambda}dr_2\int_{r_2}^{\Lambda}dr_1\int_0^{1-\frac{1}{r_1^{1/4}r_2^{1/2}}}g_1(r_1,r_2,z)dz,\\
S_2(\Lambda) &= \int_{\kappa}^{\Lambda}dr_2\int_{r_2}^{\Lambda}dr_1\int_{1-\frac{1}{r_1^{1/4}r_2^{1/2}}}^{1}g_1(r_1,r_2,z)dz,
\end{aligned}$$

where

$$G(r_1,r_2,z)=\frac{1}{L(r_1,r_2,z)^2L(r_1,r_2,-z)^3}+\frac{1}{L(r_1,r_2,z)^3L(r_1,r_2,-z)^2}+\frac{1}{L(r_1,r_2,z)^4L(r_1,r_2,-z)}.$$

Furthermore, we define $S_3(\Lambda)$ as

$$S_3(\Lambda)=\int_0^1dz\int_{\kappa}^{\Lambda}dr_2\int_{r_2}^{\Lambda}g_2(r_1,r_2,z)dr_1.$$

Then we have $\mathrm{I}_8(\Lambda)=\frac{2\pi^2}{(2\pi)^6}S(\Lambda)$, and

$$S(\Lambda)=2\int_0^1dz\int_{\kappa}^{\Lambda}dr_2\int_{r_2}^{\Lambda}g(r_1,r_2,z)dr_1$$

in the same way as the proof of $\lim_{\Lambda\to\infty}\frac{\mathrm{I}_3(\Lambda)}{\log\Lambda}=0$. Since $g(r_1,r_2,z)=g_1(r_1,r_2,z)+g_2(r_1,r_2,z)$, it holds that

$$S(\Lambda)=2S_1(\Lambda)+2S_2(\Lambda)+2S_3(\Lambda). \tag{4.41}$$

Since $g_1(r_1,r_2,z)\le 0$ and $g_2(r_1,r_2,z)\le 0$, $S_i(\Lambda)$ $(i=1,2,3)$ are decreasing in Λ. Let $r_2\le r_1$. Then

$$\frac{1}{F(r_1)}+\frac{1}{F(r_2)}<\frac{4m}{r_2^2}. \tag{4.42}$$

Let $0\le z\le 1-\frac{1}{r_1^{1/4}r_2^{1/2}}$. Then we have

$$\frac{1}{(1-z)^4}\le r_1r_2^2. \tag{4.43}$$

We have

$$L(r_1,r_2,-z)=\frac{(r_1-r_2)^2+2r_1r_2(1-z)}{2m}+\omega(r_1)+\omega(r_2)>\frac{r_1r_2(1-z)}{m}. \tag{4.44}$$

Using (4.43) and (4.44), we have

$$\frac{1}{L(r_1,r_2,-z)^4}<\frac{m^4}{r_1^4r_2^4(1-z)^4}\le\frac{m^4}{r_1^3r_2^2}. \tag{4.45}$$

From (4.39), (4.42) and (4.45), it follows that

$$-g_1(r_1,r_2,z)\le\frac{2}{m}\frac{z^2r_1^4r_2^4}{r_1r_2}\frac{4m}{r_2^2}\frac{2m}{r_1^2}\frac{m^4}{r_1^3r_2^2}=\frac{16m^5}{r_1^2r_2}.$$

Hence we have

$$-S_1(\Lambda)\le 16m^5\int_\kappa^\Lambda\frac{dr_2}{r_2}\int_{r_2}^\Lambda\frac{1}{r_1^2}(1-\frac{1}{r_1^{1/4}r_2^{1/2}})dr_1<16m^5\int_\kappa^\Lambda\frac{dr_2}{r_2}\int_{r_2}^\Lambda\frac{dr_1}{r_1^2}<\frac{16m^5}{\kappa}.$$

Since $S_1(\Lambda)$ is decreasing in Λ and bounded below, it converges. When $r_2\le r_1$ and $1-\frac{1}{r_1^{1/4}r_2^{1/2}}\le z\le 1$, from (4.39) and (4.42), it holds that

$$-g_1(r_1,r_2,z)<\frac{2}{m}\frac{r_1^4r_2^4}{r_1r_2}\frac{4m}{r_2^2}\frac{2m}{r_1^2}\frac{1}{r_1^4}=\frac{16mr_2}{r_1^3}.$$

Hence we have

$$\begin{aligned}
&-S_2(\Lambda)<16m\int_\kappa^\Lambda dr_2\int_{r_2}^\Lambda dr_1\int_{1-\frac{1}{r_1^{1/4}r_2^{1/2}}}^1\frac{r_2}{r_1^3}dz=16m\int_\kappa^\Lambda dr_2\int_{r_2}^\Lambda\frac{r_2}{r_1^3}r_1^{-1/4}r_2^{-1/2}dr_1\\
&=16m\int_\kappa^\Lambda dr_2r_2^{1/2}\int_{r_2}^\Lambda r_1^{-13/4}dr_1=\frac{64m}{9}\int_\kappa^\Lambda r_2^{1/2}(r_2^{-9/4}-\Lambda^{-9/4})dr_2<\frac{64m}{9}\int_\kappa^\Lambda r_2^{-7/4}dr_2\\
&=\frac{256m}{27}(\kappa^{-3/4}-\Lambda^{-3/4})<\frac{256m}{27\kappa^{3/4}}.
\end{aligned}$$

Since $S_2(\Lambda)$ is decreasing in Λ and bounded below, it converges. We have

$$G(r_1,r_2,z)<\frac{4m^2}{r_1^7}+\frac{8m^3}{r_1^8}+\frac{16m^4}{r_1^9}. \tag{4.46}$$

From (4.40), (4.42) and (4.46), we have

$$-g_2(r_1,r_2,z)\le\frac{2}{m}\frac{z^2r_1^4r_2^4}{r_1r_2}\frac{4m}{r_2^2}(\frac{4m^2}{r_1^7}+\frac{8m^3}{r_1^8}+\frac{16m^4}{r_1^9})$$

$$< 8r_1^3 r_2 (\frac{4m^2}{r_1^7} + \frac{8m^3}{r_1^8} + \frac{16m^4}{r_1^9}) = \frac{32m^2 r_2}{r_1^4} + \frac{64m^3 r_2}{r_1^5} + \frac{128m^4 r_2}{r_1^6}.$$

Hence

$$\begin{aligned} -S_3(\Lambda) &< 32m^2 \int_\kappa^\Lambda dr_2 r_2 \int_{r_2}^\Lambda \frac{dr_1}{r_1^4} + 64m^3 \int_\kappa^\Lambda dr_2 r_2 \int_{r_2}^\Lambda \frac{dr_1}{r_1^5} + 128m^4 \int_\kappa^\Lambda dr_2 r_2 \int_{r_2}^\Lambda \frac{dr_1}{r_1^6} \\ &< \frac{32m^2}{3} \int_\kappa^\Lambda \frac{1}{r_2^2} dr_2 + 16m^3 \int_\kappa^\Lambda \frac{1}{r_2^3} dr_2 + \frac{128m^4}{5} \int_\kappa^\Lambda \frac{1}{r_2^4} dr_2 < \frac{32m^2}{3\kappa} + \frac{8m^3}{\kappa^2} + \frac{128m^4}{15\kappa^3}. \end{aligned}$$

Since $S_3(\Lambda)$ is decreasing in Λ and bounded below, it converges. Since $S_i(\Lambda)(i = 1, 2, 3)$ converge, $\mathrm{I}_8(\Lambda)$ converges by (4.41). □

5 Concluding remarks

(1) The Nelson model is defined as the self-adjoint operator

$$H_V = (-\frac{1}{2}\Delta + V) \otimes 1 + 1 \otimes H_\mathrm{f} + \alpha \int_{\mathbb{R}^3}^\oplus \phi(x) dx, \tag{5.1}$$

acting in the Hilbert space $L^2(\mathbb{R}) \otimes \mathcal{F} \cong \int_{\mathbb{R}^3}^\oplus \mathcal{F} dx$. Here $V : \mathbb{R}^3 \to \mathbb{R}$ is an external potential and

$$\phi(x) = \frac{1}{\sqrt{2}} \int \left\{ a^\dagger(k) e^{-ikx} \hat{\varphi}(k)/\sqrt{\omega(k)} + a(k) e^{ikx} \hat{\varphi}(-k)/\sqrt{\omega(k)} \right\} dk.$$

In the case of $V = 0$, $H_{V=0}$ is translation invariant and the relationship between H_V and $H(p)$ is given by

$$H_{V=0} = \int_{\mathbb{R}^3}^\oplus H(p) dp.$$

Furthermore the ground state energy of $H(p = 0)$ coincides with that of $H_{V=0}$.
(2) We show that $m_\mathrm{eff}(\Lambda)/m = 1 + \sum_{n=1}^\infty a_n(\Lambda)\alpha^{2n}$ and $\lim_{\Lambda\to\infty} a_2(\Lambda) = \pm\infty$. It is also expected that $\lim_{\Lambda\to\infty} a_n(\Lambda)$ diverges and the signatures are alternatively changed. Hence $\lim_{\Lambda\to\infty} m_\mathrm{eff}(\Lambda)/m$ may converge but it is not trivial to see it directly.
(3) The relativistic Nelson model is defined by replacing $-\frac{1}{2}\Delta + V$ with the semi-relativistic Schrödinger operator $\sqrt{-\Delta + 1} + V$ in (5.1), i.e.,

$$H_V^{rel} = (\sqrt{-\Delta + 1} + V) \otimes 1 + 1 \otimes H_\mathrm{f} + \int_{\mathbb{R}^3}^\oplus \phi(x) dx.$$

Then it follows that

$$H_{V=0}^{rel} = \int_{\mathbb{R}^3}^\oplus H^{rel}(p) dp,$$

where $H^{rel}(p) = \sqrt{(p - P_\mathrm{f})^2 + 1} + H_\mathrm{f} + \phi(0)$. Then the effective mass $m_\mathrm{eff}(\Lambda)$ of $H^{rel}(p)$ is defined in the same way as that of $H(p)$. We are also interested in seeing the asymptotic behavior of $m_\mathrm{eff}(\Lambda)$ as $\Lambda \to \infty$. However $\sqrt{(p - P_\mathrm{f})^2 + 1}$ is a non-local operator and then estimates are rather complicated.

Another interesting non-local model is the so-called semi-relativistic Pauli-Fierz model defined by

$$H_V^{PF} = \sqrt{\left(-i\nabla \otimes -\alpha \int_{\mathbb{R}^3}^\oplus A(x) dx\right)^2 + 1} + V \otimes 1 + 1 \otimes H_\mathrm{f},$$

where $A(x)$ is a quantized radiation field. See [3] for the detail. Then it follows that

$$H_{V=0}^{PF} = \int_{\mathbb{R}^3}^\oplus H^{PF}(p) dp,$$

where $H^{PF}(p) = \sqrt{(p - P_\mathrm{f} - \alpha A(0))^2 + 1} + H_\mathrm{f}$. It is also interesting to investigate the asymptotic behavior of the effective mass of the semi-relativistic Pauli-Fierz model.

Acknowledgments

I would like to express my gratitude to my advisor Asao Arai for helpful comments. I would also like to thank Fumio Hiroshima for his cooperation and persistent help. This work would not have been possible without his collaboration and support.

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