Derivations on a C^∞-ring

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The main purpose of this paper is to provide several results on objects lying between differential geometry and algebraic geometry such as C^∞-rings and derivations on a C^∞-ring. A C^∞-ring is defined as a set with operations by C^∞-functions on Euclidean spaces. A derivation on a C^∞-ring is defined in two ways, an $\mathbb{R}$-derivation and a C^∞-derivation. The main result of this paper is to show that any $\mathbb{R}$-derivation is a C^∞-derivation for some classes of C^∞-rings (Theorem 3.1, 3.2).

1 Introduction

A "C^∞**-ring**" is an $\mathbb{R}$-algebra with operations by C^∞-functions $\mathbb{R}^n \to \mathbb{R}$([2, 5]). See also Definition 2.1 for the definition of C^∞-rings and homomorphisms of C^∞-rings.

Let us begin with a motivating example in C^∞-manifolds. Let M, N be C^∞-manifolds and $f : N \to M$ a C^∞-map between C^∞-manifolds. Write an $\mathbb{R}$-algebra $C^\infty(M)$ as a set of C^∞-functions $M \to \mathbb{R}$, and define the homomorphism $f^* : C^\infty(M) \to C^\infty(N)$ as $f^*(h) := h \circ f \in C^\infty(N)$ for $h \in C^\infty(M)$.

Then $C^\infty(M)$ is a kind of C^∞**-ring** with the following property;
for any $l \in \{0\} \cup \mathbb{N}$ and $g \in C^\infty(\mathbb{R}^l)$ (if $l = 0$, g is a constant value $c \in \mathbb{R}$), there exists an operation $\Phi_g : C^\infty(M)^l \to C^\infty(M)$ defined by $\Phi_g(h_1, \dots, h_l) := g \circ (h_1, \dots, h_l) \in C^\infty(M)$ where $\big(g \circ (h_1, \dots, h_l)\big)(x) := g\big(h_1(x), \dots, h_l(x)\big)$ for any $x \in M$ (if $l = 0$, Φ_g is a constant function $c \in \mathbb{R}$).

Recall that we can regard a C^∞-vector field $V : N \to TM$ on N along f as an $\mathbb{R}$**-derivation** $V : C^\infty(M) \to C^\infty(N)$ along f^*, that is, V is an $\mathbb{R}$-linear map which satisfies the following <u>product rule</u>;

$$V(h_1 h_2) = f^*(h_1) \cdot V(h_2) + f^*(h_2) \cdot V(h_1) \text{ for any } h_1, h_2 \in C^\infty(M).$$

Note that in this case, V turns to be a C^∞**-derivation**, that is, V satisfies the following chain rule;

$$V\big(g\circ(h_1,\dots,h_l)\big)=\sum_{i=1}^{l} f^*\Big(\frac{\partial g}{\partial x_i}\circ(h_1,\dots,h_l)\Big)\cdot V(h_i)$$

$$\text{for any } l\in\mathbb{N},\ h_1,\dots,h_l\in C^\infty(M), \text{and } g\in C^\infty(\mathbb{R}^l).$$

In general, an $\mathbb{R}$-derivation needs not to be a C^∞-derivation (Example 5.1).

Suppose that $\mathfrak{C}$, $\mathfrak{D}$ are C^∞-rings and $\phi:\mathfrak{C}\to\mathfrak{D}$ a homomorphism of C^∞-rings. The aim of this paper is to show that any $\mathbb{R}$-derivation $V:\mathfrak{C}\to\mathfrak{D}$ along ϕ becomes a C^∞-derivation under certain conditions of $\mathfrak{C}$, $\mathfrak{D}$ and ϕ.

In §2, we recall the notions of C^∞-rings and derivations of C^∞-rings. To prove the main results Theorem 3.1 and 3.2, we define k-jet determined C^∞-rings and prove that a Weil algebra is a C^∞-ring. One of our main results Theorem 3.1 is announced in [7].

In §3, we formulate the main results Theorem 3.1 and 3.2, and in §4, we prove them.

In §5, we introduce the two kinds of cotangent modules of C^∞-rings. We give an example of an $\mathbb{R}$-derivation which is not a C^∞-derivation, by using a cotangent module.

2 Differentiable rings and their derivations

In this section, we recall the notions of C^∞-rings and derivations of C^∞-rings.

Definition 2.1 ([2, 3]) *1. A C^∞**-ring** is a set $\mathfrak{C}$ which is endowed with the following data; for any $l\in\{0\}\cup\mathbb{N}$ and any C^∞-map $f:\mathbb{R}^l\to\mathbb{R}$ (if $l=0$, f is a constant value $c\in\mathbb{R}$), there are given an operation $\Phi_f:\mathfrak{C}^l\to\mathfrak{C}$ (if $l=0$, an operation Φ_f is a constant on $\mathfrak{C}$) such that*

- *for any $k\in\{0\}\cup\mathbb{N}$, any C^∞-maps $g:\mathbb{R}^k\to\mathbb{R}$ and $f_i:\mathbb{R}^l\to\mathbb{R}(i=1,\dots,k)$,*

$$\Phi_g(\Phi_{f_1}(c_1,\dots,c_l),\dots,\Phi_{f_k}(c_1,\dots,c_l))=\Phi_{g\circ(f_1,\dots,f_k)}(c_1,\dots,c_l) \text{ for any } c_1,\cdots,c_l\in\mathfrak{C}, \text{ and}$$

- *for all projections $\pi:\mathbb{R}^l\to\mathbb{R}$ defined as $\pi_i(x_1,\dots,x_l)=x_i(i=1,\cdots,l)$,*

$$\Phi_{\pi_i}(c_1,\dots,c_l)=c_i \text{ for any } c_1,\dots,c_l\in\mathfrak{C}.$$

2. Let $\mathfrak{C}$ and $\mathfrak{D}$ be C^∞-rings with operations $\Phi_f:\mathfrak{C}^l\to\mathfrak{C}$ and $\Psi_f:\mathfrak{D}^l\to\mathfrak{D}$ for any C^∞-function

$f : \mathbb{R}^l \to \mathbb{R}$. *A homomorphism between C^∞-rings is a map $\phi : \mathfrak{C} \to \mathfrak{D}$ which satisfies*

$$\phi\big(\Phi_f(c_1, \ldots, c_n)\big) = \Psi_f\big(\phi(c_1), \ldots, \phi(c_n)\big) \text{ for any } c_1, \cdots, c_l \in \mathfrak{C}.$$

Remark that any C^∞-ring has the natural $\mathbb{R}$-algebra structure. Define the addition on $\mathfrak{C}$ by $c + c' := \Phi_{(x,y)\mapsto x+y}(c, c')$, the multiplication on $\mathfrak{C}$ by $c \cdot c' := \Phi_{(x,y)\mapsto xy}(c, c')$, and define the scalar multiplication by $\lambda \in \mathbb{R}$ by $\lambda c := \Phi_{x\mapsto \lambda x}(c)$. We see that elements 0 and 1 in $\mathfrak{C}$ are given by $0_{\mathfrak{C}} := \Phi_{\emptyset\mapsto 0}(\emptyset)$ and $1_{\mathfrak{C}} := \Phi_{\emptyset\mapsto 1}(\emptyset)$.

Let $\mathfrak{C}$ be a C^∞-ring with operations $\Phi_f : \mathfrak{C}^l \to \mathfrak{C}$ for any C^∞-function $f : \mathbb{R}^l \to \mathbb{R}$. Let $I \subset \mathfrak{C}$ be an ideal of the $\mathbb{R}$-algebra. We define a C^∞-ring structure on the quotient $\mathbb{R}$-algebra $\mathfrak{C}/I$ as follows. For any natural number $l \in \mathbb{N}$ and a C^∞-function $f \in C^\infty(\mathbb{R}^l)$, there exists l C^∞-functions $g_1, \ldots, g_l \in C^\infty(\mathbb{R}^{2l})$ which satisfy

$$f(x_1 + y_1, \ldots, x_l + y_l) - f(x_1, \ldots, x_l) = \sum_{k=1}^{l} y_k g_k(x_1, \ldots, x_l, y_1, \ldots, y_l)$$

for any $(x_1, \ldots, x_l), (y_1, \ldots, y_l) \in \mathbb{R}^l$ by Hadamard's lemma in [6]. Then we have

$$\Phi_f(c_1 + i_1, \ldots, c_l + i_l) - \Phi_f(c_1, \ldots, c_l) = \sum_{k=1}^{l} i_k \cdot \Phi_{g_k}(c_1, \ldots, c_l, i_1, \ldots, i_l)$$

for any $c_1, \ldots, c_l \in \mathfrak{C}$ and $i_1, \ldots, i_l \in I$. Therefore the quotient $\mathbb{R}$-algebra $\mathfrak{C}/I$ has a C^∞-ring structure with a following operation $\Phi^I_f : (\mathfrak{C}/I)^l \to \mathfrak{C}/I$ for any $f \in C^\infty(\mathbb{R}^n)$ defined as

$$\Phi^I_f(c_1 + I, \ldots, c_l + I) := \Phi_f(c_1, \ldots, c_l) + I \text{ for any } c_1, \ldots, c_l \in \mathfrak{C}.$$

As is seen in Introduction, $C^\infty(M)$ is a C^∞-ring for a C^∞-manifold M. The set $C^\infty_p(M)$ of C^∞-function-germs on M at a point p is a C^∞-ring, which has the unique maximal ideal m_p consisting of germs with zero values at p.

The set $C^\infty_p(M)/{m_p}^{k+1}$ of k-jets of C^∞-functions on M at a point p has a C^∞-ring structure.

Example 2.1 *The set $\mathbb{R}$ of real numbers has a structure of C^∞-ring by the operation $\Phi_f : \mathbb{R}^l \to \mathbb{R}$ for $f \in C^\infty(\mathbb{R}^l)$ as $\Phi_f(r_1, \ldots, r_l) := f(r_1, \ldots, r_l)$ for any $r_1, \ldots, r_l \in \mathbb{R}$.*

Regarding operations by smooth functions on Euclidean spaces, which contain the addition, the multiplication and scalar multiplications by real values, we define two kinds of derivations on a C^∞-ring as follows.

Definition 2.2 ([3]) *Let $\mathfrak{C}$ be a C^∞-ring and $\mathfrak{M}$ be a $\mathfrak{C}$-module.*

1. *An* $\mathbb{R}$**-derivation** *is an* $\mathbb{R}$*-linear map* $d : \mathfrak{C} \to \mathfrak{M}$ *such that*

$$d(c_1 c_2) = c_2 \cdot d(c_1) + c_1 \cdot d(c_2) \text{ for any } c_1, c_2 \in \mathfrak{C}.$$

2. *A* C^∞**-derivation** *is an* $\mathbb{R}$*-linear map* $d : \mathfrak{C} \to \mathfrak{M}$ *such that*

$$d\big(\Phi_f(c_1, \ldots, c_n)\big) = \sum_{i=1}^{n} \big(\Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n)\big) \cdot d(c_i) \text{ for any } n \in \mathbb{N}, f \in C^\infty(\mathbb{R}^n) \text{ and } c_1, \ldots, c_n \in \mathfrak{C}.$$

By Definition 2.2, we have that any C^∞-derivation is an $\mathbb{R}$-derivation.

Example 2.2 *Let* M *be a* C^∞*-manifold and* $\Gamma(T^*M)$ *the set of* C^∞*-sections to the cotangent bundle* T^*M *on* M*.*

1. *For any* $f \in C^\infty(M)$*, define a* C^∞*-section* $df : M \to T^*M$ *by* $df(v) := v(f)$ *for any* $x \in M$ *and* $v \in T_xM$*.*

   Define an $\mathbb{R}$*-linear mapping* $d : C^\infty(M) \to \Gamma(T^*M)$ *by* $d(f) := df$ *for any* $f \in C^\infty(M)$*. This* $\mathbb{R}$*-mapping* d *is the* C^∞*-derivation.*

2. *Let* $V : M \to TM$ *be a* C^∞*-vector field of* M *as* V_x *is a tangent vector at a point* x *on* M*.*

   Define $V(f) \in C^\infty(M)$ *by* $\big(V(f)\big)(x) := V_x(f)$ *regarding the tangent vector* V_x *as a differential* $C^\infty(M) \to \mathbb{R}$ *at* x*. We regard* $V : C^\infty(M) \to C^\infty(M)$ *as an* $\mathbb{R}$*-derivation. Then* V *turns out to be a* C^∞*-derivation.*

We will define k-jet determined C^∞-rings. They will be used to prove our Theorem 3.1 and 3.2.

Definition 2.3 ([3]) *Let* $\mathfrak{C}$ *be a* C^∞*-ring.*

1. *An* $\mathbb{R}$**-point of** $\mathfrak{C}$ *is defined as a surjective homomorphism* $p : \mathfrak{C} \to \mathbb{R}$ *of* C^∞*-rings (see Definition 2.1 (2), Example 2.1).*

2. *Take any* $\mathbb{R}$*-point* $p : \mathfrak{C} \to \mathbb{R}$*. We call a* C^∞*-ring* $\mathfrak{C}_p$ *the* **localization of** $\mathfrak{C}$ **at** p *if the following universal conditions are satisfied:*

   (a) *There exists a homomorphism* $i_p : \mathfrak{C} \to \mathfrak{C}_p$ *of* C^∞*-rings such that* $i_p(c)$ *is invertible for any* $c \in p^{-1}(\mathbb{R}\backslash\{0\})$*.*

   (b) *For any homomorphism* $i : \mathfrak{C} \to \mathfrak{D}$ *of* C^∞*-rings which satisfies that* $i_p(c)$ *is invertible for any* $c \in p^{-1}(\mathbb{R}\backslash\{0\})$*, there exists a unique homomorphism* $\phi_i : \mathfrak{C}_p \to \mathfrak{D}$ *of* C^∞*-rings such that* $i = \phi_i \circ i_p$*.*

The localization $\mathfrak{C}_p$ always exists for any $\mathbb{R}$-point $p : \mathfrak{C} \to \mathbb{R}$. $\mathfrak{C}_p$ has the unique maximal ideal $m_p \subset \mathfrak{C}_p$ such that $\mathfrak{C}_p/m_p$ is isomorphic to $\mathbb{R}$ ([3]).

3. *Let $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$. For any $\mathbb{R}$-point $p : \mathfrak{C} \to \mathbb{R}$, define a natural homomorphism $j^k_p : \mathfrak{C} \to \mathfrak{C}_p/{m_p}^{k+1}$ by $j^k_p(c) := c + m_p^{k+1}$ for $c \in \mathfrak{C}$. Define a homomorphism j^k by*

$$j^k := (j^k_p)_{p:\mathfrak{C}\to\mathbb{R}} : \mathfrak{C} \to \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/m_p^{k+1}$$

(If $k = \infty$, we mean ${m_p}^{k+1}$ by ${m_p}^{\infty} := \cap_{k\in\mathbb{N}} {m_p}^k$).

Example 2.3 *Let M be a C^∞-manifold and $p \in M$. Define an $\mathbb{R}$-point $e_p : C^\infty(M) \to \mathbb{R}$ by $e_p(f) := f(p)$ for $f \in C^\infty(M)$.*

For the $\mathbb{R}$-point e_p, the localization $\left(C^\infty(M)\right)_{e_p}$ is isomorphic to the C^∞-ring $C^\infty_p(M)$ of C^∞-function-germs at p. Its unique maximal ideal is given by $m_{e_p} = \{[f,U]_p \in C^\infty_p(M) | f(p) = 0\}$.

Remark 2.1 *In I. Moerdijk and G.E. Reyes ([5] Definition 4.1), the point determined C^∞-rings, near-point determined C^∞-rings and germ determined C^∞-rings are introduced.*

Generalizing them, we define k-jet determined C^∞-rings by $\mathbb{R}$-points and their localizations.

Definition 2.4 *Let $\mathfrak{C}$ be a C^∞-ring and $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$. $\mathfrak{C}$ is a* **k-jet determined C^∞-ring** *if the homomorphism $j^k : \mathfrak{C} \to \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/m_p^{k+1}$ defined in Definition 2.3 (3) is injective. Especially, a 0-jet determined C^∞-ring is nothing but a point determined C^∞-ring introduced in [5].*

Example 2.4 *Suppose that M is a C^∞-manifold.*

1. *$C^\infty(M)$ is a point determined C^∞-ring.*

   For any $\mathbb{R}$-point $e_p : C^\infty(M) \to \mathbb{R}$, $j^0_p : C^\infty(M) \to C^\infty_p(M)/m_p$ is identified with e_p by the isomorphism $C^\infty_p(M) \tilde{=} \mathbb{R}$. Then, the homomorphism j^0 is identified with $e : C^\infty(M) \to \prod_{p\in M} \mathbb{R}$ defined by $e(f) := \{f(p)\}_{p\in M}$, which is clearly injective.

2. *Let $l, k \in \mathbb{N}$ and $0 \le l < k \le \infty$. Then $C^\infty_p(M)/{m_p}^{k+1}$ is not an l-jet determined C^∞-ring, but a k-jet determined C^∞-ring.*

   First, $C^\infty_p(M)/{m_p}^{k+1}$ has a unique $\mathbb{R}$-point $e_p : C^\infty_p(M)/{m_p}^{k+1} \to \mathbb{R}$ with $e_p(f + {m_p}^{k+1}) := f(p)$ for any $f + {m_p}^{k+1} \in C^\infty_p(M)/{m_p}^{k+1}$. For any $l' \le k$, we have a homomorphism

$$j^{k,l'} : C^\infty_p(M)/{m_p}^{k+1} \to C^\infty_p(M)/{m_p}^{l'+1}$$

by $j^{k,l'}(f + {m_p}^{k+1}) := f + {m_p}^{l'+1}$*. Then the homomorphism*

$$j^{l'} : C_p^\infty(M)/{m_p}^{k+1} \to \left(C_p^\infty(M)/{m_p}^{k+1}\right)/\left(m_p/{m_p}^{k+1}\right)^{l'+1}$$

is identified with $j^{k,l'}$ *by the isomorphism*

$$\left(C_p^\infty(M)/{m_p}^{k+1}\right)/\left(m_p/{m_p}^{k+1}\right)^{l'+1} \doteq C_p^\infty(M)/{m_p}^{l'+1}.$$

For $l < k$, $j^{k,l}$ *is not injective and* $C_p^\infty(M)/{m_p}^{k+1}$ *is not* l*-jet determined.*

$j^{k,k}$ *is an identity of* $C_p^\infty(M)/{m_p}^{k+1}$*, so* $C_p^\infty(M)/{m_p}^{k+1}$ *is* k*-jet determined.*

3. *The* C^∞*-ring* $C_p^\infty(M)$ *of* C^∞*-function-germs on* (M,p) *is not* k*-jet determined for any* $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$*.*

   Since there exists a nonzero C^∞*-function-germ* $f \in C_p^\infty(M)$ *such that* $f \in m_p^{k+1}$*, the homomorphism* $j^k : C_p^\infty(M) \to C_p^\infty(M)/m_p^{k+1}$ *is not injective.*

Then we have

Proposition 2.1 *Let* $\mathfrak{C}$ *be a* C^∞*-ring and* $k, l \in \{0\} \cup \mathbb{N} \cup \{\infty\} (k \leq l)$*.*

If $\mathfrak{C}$ *is a* k*-jet determined* C^∞*-ring, then* $\mathfrak{C}$ *is also an* l*-jet determined* C^∞*-ring.*

Proof) We have the homomorphism $j^{k,l} : \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/{m_p}^{l+1} \to \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/{m_p}^{k+1}$ such that $j^k = j^{k,l} \circ j^l$.

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{j^l} & \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/{m_p}^{l+1}. \\ & j^k \searrow & \downarrow j^{k,l} \\ & & \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/{m_p}^{k+1} \end{array}$$

If j^k is injective, then j^l is also injective. Therefore, we have Proposition 2.1. ■

Definition 2.5 ([5]) *A* **Weil algebra** *is a local* $\mathbb{R}$*-algebra* W *(a local ring with an* $\mathbb{R}$*-algebra structure) which is a finite dimensional* $\mathbb{R}$*-vector space and is written as* $W = \mathbb{R} \oplus m$ *(the first component is the* $\mathbb{R}$*-algebra structure and the second is the maximal ideal in* W*).*

Example 2.5 ([5]) *Let* $k \in \{0\} \cup \mathbb{N}$*,* $n \in \mathbb{N}$ *and* $m \subset C_0^\infty(\mathbb{R}^n)$ *be the maximal ideal of* $C_0^\infty(\mathbb{R}^n)$*.*

The ring

$$J_n^k := C_0^\infty(\mathbb{R}^n)/m^{k+1}$$

is called **the ring of jets of order k (in n-variables)**, *or* **the ring of k-jets**. *This J^k_n is a Weil algebra for any k.*

Lemma 2.1 ([5]) *Let W be a Weil algebra with a maximal ideal $m \subset W$ which satisfies $W = \mathbb{R} \oplus m$. Then $m^k = 0$ for some $k \in \mathbb{N}$.*

The reverse of Lemma 2.1 is false, i.e. there exists a local $\mathbb{R}$-algebra which is <u>an infinite dimensional $\mathbb{R}$-vector space</u> and has a maximal ideal m with $m^k = 0$ for some $k \in \mathbb{N}$.

From several results in [5], we have following results.

Lemma 2.2 *Let W be a Weil algebra. Then W is isomorphic to a ring $C^\infty_0(\mathbb{R}^n)/I$ for some ideal $I \subset C^\infty_0(\mathbb{R}^n)$ and W is finite dimensional as a real vector space.*

Proof) A Weil algebra W turns out to be isomorphic to a ring $C^\infty_0(\mathbb{R}^n)/I$ for some ideal $I \subset C^\infty_0(\mathbb{R}^n)$ from Theorem 3.17 of [5].

Moreover A is finite dimensional as a real vector space from Corollary 3.18 of [5]. ■

Lemma 2.3 ([5]) *Let W be a Weil algebra and $\mathfrak{C}$ a C^∞-ring.*

Suppose that $\phi : W \to \mathfrak{C}$ (resp. $\psi : \mathfrak{C} \to W$) is a homomorphism of $\mathbb{R}$-algebras. Then ϕ (resp. ψ) is a homomorphism of C^∞-rings.

Proof) The homomorphism $\phi : W \to \mathfrak{C}$ of $\mathbb{R}$-algebras turns out to be a homomorphism of C^∞-rings from Corollary 3.19 of [5].

The homomorphism $\psi : \mathfrak{C} \to W$ of $\mathbb{R}$-algebras turns out to be a homomorphism of C^∞-rings from Proposition 3.20 of [5]. ■

From Lemma 2.2 and 2.3, we can regard Weil algebras as C^∞-rings.

Definition 2.6 *An $\mathbb{R}$-algebra W is the* **weakly nilpotent local $\mathbb{R}$-algebra** *if W has a unique maximal ideal $m \subset W$ and any $y \in m$ is a nilpotent element of W, i.e. $y^k = 0$ for some $k \in \mathbb{N}$.*

Example 2.6 *Let W be a formal algebra $\mathbb{R}[x_i | i \in \mathbb{N}] = \lim_{i\to\infty} \mathbb{R}[x_1, \ldots, x_i]$, and an ideal $I := \langle x_i^i | i \in \mathbb{N}\rangle_{\mathbb{R}} = \lim_{i\to\infty}\langle x_1, x_2^2 \ldots, x_i^i\rangle_{\mathbb{R}}$ of W.*

A quotient $\mathbb{R}$-algebra

$$B := W/I = \lim_{i\to\infty} \mathbb{R}[x_1, \ldots, x_i]/\langle x_1, x_2^2 \ldots, x_i^i\rangle_{\mathbb{R}}$$

is a weakly nilpotent local $\mathbb{R}$-algebra with a unique maximal ideal

$$m = \lim_{i\to\infty} \langle x_1, \ldots, x_i\rangle_{\mathbb{R}}/\langle x_1, x_2^2, \ldots, x_i^i\rangle_{\mathbb{R}}.$$

Proposition 2.2 *A Weil algebra is the weakly nilpotent local $\mathbb{R}$-algebra.*

Proposition 2.3 *Let W is a local $\mathbb{R}$-algebra with a maximal ideal $m \subset W$. Suppose that the maximal ideal m is generated by finite elements $a_1, \dots, a_n \in m$ as the $\mathbb{R}$-algebra.*

If W is the weakly nilpotent local algebra, W is also the Weil algebra.

Proof) Clearly W is an $\mathbb{R}$-finite dimension vector space generated by $a_1, \dots, a_n, 1$.

Suppose $p = \sum_{i=1}^{n} c_i a_i \in m (c_1, \dots, c_n \in \mathbb{R})$. There exists $i_k \in \mathbb{N}$ such that ${a_k}^{i_k} = 0$ for each $k = 1, \dots, n$. Then for $c_1 a_1 + c_2 a_2 \in W (c_1, c_2 \in \mathbb{R})$, we have

$$(c_1 a_1 + c_2 a_2)^{i_1 + i_2} = \sum_{j=0}^{i_1+i_2} \begin{pmatrix} i_1 + i_2 \\ j \end{pmatrix} {c_1}^j {c_2}^{i_1+i_2-j} {a_1}^j {a_2}^{i_1+i_2-j} = 0$$

since $j \geq i_1$ or $i_1 + i_2 - j \geq i_2$, i.e. $a_1^j = 0$ or $a_2^{i_1+i_2-j} = 0$ for any $j = 0, 1, \dots, i_1 + i_2$. For $p \in m$, we have $p^{\sum_{k=1} i_k} = 0$. Therefore,

$$m^{\sum_{k=1} i_k} = 0.$$

We have that W is the Weil algebra. ■

Corollary 2.1 *Let W be a weakly nilpotent local $\mathbb{R}$-algebra. W is a C^∞-ring.*

Proof) Suppose that W is finitely generated. Then W is a Weil algebra, and is also a finitely presented C^∞-ring from Lemma 2.2

Suppose that W is arbitrary weakly nilpotent local $\mathbb{R}$-algebra. There exists an indexed set $\{W_\lambda\}_{\lambda \in \Lambda}$ of finitely generated $\mathbb{R}$-subalgebras W_λ of W with an ordered set Λ of $\lambda < \lambda'$. W_λ is a Weil algebra for each $\lambda \in \Lambda$, i.e. a local C^∞-ring. Then $W = \lim_{\lambda \in \Lambda} W_\lambda$ is also a C^∞-ring since the limit of C^∞-rings exists([5]).
■

3 Main Results

We give sufficient conditions that any $\mathbb{R}$-derivation becomes a C^∞-derivation.

Theorem 3.1 *Let $\mathfrak{C}, \mathfrak{D}$ be C^∞-rings, $\phi : \mathfrak{C} \to \mathfrak{D}$ a homomorphism of C^∞-rings and $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$. Suppose that $\mathfrak{D}$ is a k-jet determined C^∞-ring.*

Then any $\mathbb{R}$-derivation $V : \mathfrak{C} \to \mathfrak{D}$ over ϕ is a C^∞-derivation.

Corollary 3.1 *Let $\mathfrak{C}$ be a k-jet determined C^∞-ring with the form $C^\infty(\mathbb{R}^n)/I$ for some ideal I of $C^\infty(\mathbb{R}^n)$. Any $\mathbb{R}$-derivation $V : \mathfrak{C} \to \mathfrak{C}$ is a C^∞-derivation.*

Moreover, there exist smooth functions $a_1(x), \dots, a_n(x) \in C^\infty(\mathbb{R}^n)$ such that

$$V(f(x)+I) = \left(\sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}\right)(x) + I \text{ for any } f(x)+I \in C^\infty(\mathbb{R}^n)/I.$$

The C^∞-ring $C^\infty(M)$ for a C^∞-manifold M is a point determined C^∞-ring by Example 2.4 (1). Moreover, $C^\infty(M)$ is isomorphic to a quotient C^∞-ring $C^\infty(\mathbb{R}^N)/m_M^0$ by an embedding $M \hookrightarrow \mathbb{R}^N$ for some $N \in \mathbb{N}$ by Proposition 3.2 (1) in [3], when m_M^0 is the ideal of C^∞-functions on $\mathbb{R}^N$ vanishing on M.

Example 3.1 *Let $f : N \to M$ be a C^∞-mapping of C^∞-manifolds and $V : C^\infty(M) \to C^\infty(N)$ an $\mathbb{R}$-derivation along the pull-back $f^* : C^\infty(M) \to C^\infty(N)$.*

Since $C^\infty(N)$ is a point determined C^∞-ring, the $\mathbb{R}$-derivation V is a C^∞-derivation by Theorem 3.1.

Theorem 3.2 *Let W be a weakly nilpotent $\mathbb{R}$-algebra and $\mathfrak{C}$ a C^∞-ring.*

Suppose that $V : W \to \mathfrak{M}_{\mathfrak{C}}$ (resp. $V' : \mathfrak{C} \to \mathfrak{M}_W$) is an $\mathbb{R}$-derivation along a homomorphism $\phi : W \to \mathfrak{C}$ to a $\mathfrak{C}$-module $\mathfrak{M}_{\mathfrak{C}}$ (resp. a homomorphism $\psi : \mathfrak{C} \to W$ to a W-module $\mathfrak{M}_W$). Then V (resp. V') is a C^∞-derivation.

Example 3.2 *$C^\infty(\mathbb{R})/\langle x^{k+1}\rangle_{C^\infty(\mathbb{R})}$ is not a point determined C^∞-ring but a k-jet determined C^∞-ring for an ideal $\langle x^{k+1}\rangle_{C^\infty(\mathbb{R})}$ of $C^\infty(\mathbb{R})$ generated by x^{k+1}. Moreover, $C^\infty(\mathbb{R})/\langle x^{k+1}\rangle_{C^\infty(\mathbb{R})}$ is a Weil algebra with the maximal ideal $\langle x\rangle_{C^\infty(\mathbb{R})}/\langle x^{k+1}\rangle_{C^\infty(\mathbb{R})}$ and the maximal ideal is generated by finite elements $x^1 + \langle x^{k+1}\rangle_{C^\infty(\mathbb{R})}, \dots, x^k + \langle x^{k+1}\rangle_{C^\infty(\mathbb{R})}$ as the $\mathbb{R}$-vector space.*

Any $\mathbb{R}$-derivation $V : C^\infty(\mathbb{R})/\langle x^{k+1}\rangle_{C^\infty(\mathbb{R})} \to C^\infty(\mathbb{R})/\langle x^{k+1}\rangle_{C^\infty(\mathbb{R})}$ is a C^∞-derivation from Theorem 3.2. We can write V as

$$V\left(f(x) + \langle x^{k+1}\rangle_{C^\infty(\mathbb{R})}\right) = \left(v\frac{\partial f}{\partial x}\right)(x) + \langle x^{k+1}\rangle_{C^\infty(\mathbb{R})}$$

by $v(x) \in C^\infty(\mathbb{R})$ such that $v(x) + \langle x^{k+1}\rangle_{C^\infty(\mathbb{R})} = V(x + \langle x^{k+1}\rangle_{C^\infty(\mathbb{R})})$.

4 Proof of Main Results

For the proof of Theorem 3.1, we will prepare a lemma.

Suppose that $f(x)$ is a smooth function $\mathbb{R}^n \to \mathbb{R}$.

We write $f(x)$, $f(r)$ and $\Phi_f(c)$ as $f(x_1, \dots, x_n)$, $f(r_1, \dots, r_n)$ and $\Phi_f(c_1, \dots, c_n)$ for the coordinate $(x_1, \dots, x_n)$ of $\mathbb{R}^n$, $r := (r_1, \dots, r_n) \in \mathbb{R}^n$ and $c := (c_1, \dots, c_n)$ by $c_1, \dots, c_n \in \mathfrak{C}$.

For $\alpha = (\alpha_1, \dots, \alpha_n) \in (\{0\} \cup \mathbb{N})^n$, we define $e_i := (0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$, $|\alpha| := \sum_{i=1}^n \alpha_i \in \{0\} \cup \mathbb{N}$, $\alpha! := \prod_{i=1}^n \alpha_i!$, $c^\alpha := \prod_{i=1}^n c_i^{\alpha_i}$ for $c_1, \dots, c_n \in \mathfrak{C}$ and $\frac{\partial^\alpha f}{\partial x^\alpha} := \frac{\partial^{|\alpha|} f}{\partial {x_1}^{\alpha_1} \cdots \partial {x_n}^{\alpha_n}}$.

Lemma 4.1 *Let $\mathfrak{C}, \mathfrak{D}$ be C^∞-rings, $\phi : \mathfrak{C} \to \mathfrak{D}$ a homomorphism of C^∞-rings. Suppose that $\mathfrak{D}$ is a local C^∞-ring which is a weakly nilpotent local $\mathbb{R}$-algebra with a unique maximal ideal $m \subset \mathfrak{D}$.*

Then any $\mathbb{R}$-derivation $V : \mathfrak{C} \to \mathfrak{D}$ along ϕ is a C^∞-derivation.

Proof) Suppose $c_1, \ldots, c_n \in \mathfrak{C}$ and $f \in C^\infty(\mathbb{R}^n)$. We have to prove that

$$V(\Phi_f(c_1, \ldots, c_n)) = \sum_{i=1}^{n} \phi(\Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n))V(c_i). \tag{1}$$

From the locality of $\mathfrak{D}$, take a real number $p_i \in \mathbb{R}$ and a natural number $k_i \in \mathbb{N}$ such that $\phi(c_i) - p_i = \phi(c_i - p_i) \in m$ and $(\phi(c_i) - p_i)^{k_i} = 0$ for each $i = 1, \ldots, n$. Let $k \in \mathbb{N}$ be the sum of k_i. Then

$$\phi\big((c-p)^\alpha\big) = 0 \text{ for any } |\alpha| \geq k+1. \tag{2}$$

Using Hadamard's lemma by induction, there exists smooth functions $G_{\alpha,p}, G^i_{\beta,p} : \mathbb{R}^n \to \mathbb{R}(|\alpha| = k+2, |\beta| = k+1, i = 1, \ldots, n)$ such that

$$\begin{aligned} f(x) &= f(p) + \sum_{1\leq|\alpha|\leq k+1} \frac{(x-p)^\alpha}{\alpha!}\frac{\partial^\alpha f}{\partial x^\alpha}(p) + \sum_{|\alpha|=k+2}(x-p)^\alpha G_{\alpha,p}(x), \\ \frac{\partial f}{\partial x_i}(x) &= \sum_{0\leq|\beta|\leq k} \frac{(x-p)^\beta}{\beta!}\frac{\partial^{\beta+e_i} f}{\partial x^{\beta+e_i}}(p) + \sum_{|\beta|=k+1}(x-p)^\beta G^i_{\beta,p}(x). \end{aligned}$$

We have

$$\Phi_f(c) = f(p) + \sum_{1\leq|\alpha|\leq k+1} \frac{\partial^\alpha f}{\partial x^\alpha}(p)\frac{(c-p)^\alpha}{\alpha!} + \sum_{|\alpha|=k+2}(c-p)^\alpha \Phi_{G_{\alpha,p}}(c), \tag{3}$$

$$\Phi_{\frac{\partial f}{\partial x_i}}(c) = \sum_{0\leq|\beta|\leq k} \frac{\partial^{\beta+e_i} f}{\partial x^{\beta+e_i}}(p)\frac{(c-p)^\beta}{\beta!} + \sum_{|\beta|=k+1}(c-p)^\beta \Phi_{G^i_{\beta,p}}(c). \tag{4}$$

For each $i = 1, \ldots, n$, we can describe $\phi\big(\Phi_{\frac{\partial f}{\partial x_i}}(c)\big)$ as followings by the equations (2) and (4).

$$\begin{aligned} \phi\big(\Phi_{\frac{\partial f}{\partial x_i}}(c)\big) =& \phi\Big(\sum_{0\leq|\beta|\leq k} \frac{(c-p)^\beta}{\beta!}\frac{\partial^{\beta+e_i} f}{\partial x^{\beta+e_i}}(p) + \sum_{|\beta|=k+1}(c-p)^\beta \Phi_{G^i_{\beta,p}}(c)\Big) \\ =& \phi\Big(\sum_{0\leq|\beta|\leq k} \frac{(c-p)^\beta}{\beta!}\frac{\partial^{\beta+e_i} f}{\partial x^{\beta+e_i}}(p)\Big) \\ =& \phi\Big(\sum_{0\leq|\beta|\leq k} (\beta+e_i)_i \frac{(c-p)^\beta}{(\beta+e_i)!}\frac{\partial^{\beta+e_i} f}{\partial x^{\beta+e_i}}(p)\Big) = \phi\Big(\sum_{1\leq|\alpha|\leq k+1} \alpha_i \frac{(c-p)^{\alpha-e_i}}{\alpha!}\frac{\partial^\alpha f}{\partial x^\alpha}(p)\Big) \end{aligned} \tag{5}$$

($\{\beta \in (\{0\}\cup\mathbb{N})^n | 0 \leq |\beta| \leq k\}$ and $\{\alpha \in (\{0\}\cup\mathbb{N})^n | 1 \leq |\alpha| \leq k+1, \alpha_i \geq 1\}$ are one-to-one relation with $\alpha \leftrightarrow$

$\alpha + e_i)$. Then we can describe $V(\Phi_f(c))$ as followings by the equations (2) and (5).

$$\begin{aligned}V(\Phi_f(c)) = &\sum_{1\leq|\alpha|\leq k+1}\sum_{i=1}^{n}\frac{\partial^\alpha f}{\partial x^\alpha}(p)\phi\left(\frac{\alpha_i(c-p)^{\alpha-e_i}}{\alpha!}\right)V(c_i)\\ &+\sum_{|\alpha|=k+2}\left\{V((c-p)^\alpha)\phi(\Phi_{G_{\alpha,p}}(c))+\phi\left((c-p)^\alpha\right)V(\Phi_{G_{\alpha,p}}(c))\right\}\\ =&\sum_{i=1}^{n}\phi\left(\sum_{1\leq|\alpha|\leq k+1}\alpha_i\frac{(c-p)^{\alpha-e_i}}{\alpha!}\frac{\partial^\alpha f}{\partial x^\alpha}(p)\right)V(c_i)=\sum_{i=1}^{n}\phi\big(\Phi_{\frac{\partial f}{\partial x_i}}(c)\big)V(c_i).\end{aligned}$$

Therefore, we have the equation (1). ■

We can prove Theorem 3.2 in the same way as the proof of Lemma 4.1.

Proof of Theorem 3.1) Suppose $c_1, \ldots, c_n \in \mathfrak{C}$ and $f \in C^\infty(\mathbb{R}^n)$. We have to show that

$$V(\Phi_f(c_1,\ldots,c_n)) = \sum_{i=1}^{n}\phi(\Phi_{\frac{\partial f}{\partial x_i}}(c_1,\ldots,c_n))V(c_i). \tag{6}$$

Take any $\mathbb{R}$-point $q : \mathfrak{D} \to \mathbb{R}$ and $l \in \{0\} \cup \mathbb{N}$. $\mathfrak{D}_q/{m_q}^{l+1}$ is a local C^∞-ring with a unique maximal ideal $m_q/{m_q}^{l+1}$. $\mathfrak{D}_q/{m_q}^{l+1}$ satisfies that $(\mathfrak{D}_q/{m_q}^{l+1})/(m_q/{m_q}^{l+1})$ is isomorphic to $\mathfrak{D}_q/m_q$, or to $\mathbb{R}$ and $(m_q/{m_q}^{l+1})^{l+1} = 0$. Therefore, $\mathfrak{D}_q/{m_q}^{l+1}$ is a weakly nilpotent local $\mathbb{R}$-algebra. For $j^l_q : \mathfrak{D} \to \mathfrak{D}_q/{m_q}^{l+1}$, $j^l_q \circ V : \mathfrak{C} \to \mathfrak{D}_q/{m_q}^{l+1}$ is an $\mathbb{R}$-derivation along $j^l_q \circ \phi : \mathfrak{C} \to \mathfrak{D}_q/{m_q}^{l+1}$. From Lemma 4.1, $j^l_q \circ V$ is the C^∞-derivation. Therefore, we have

$$j^l_q\Big(\sum_{i=1}^{n}\phi\big(\Phi_{\frac{\partial f}{\partial x_i}}(c_1,\ldots,c_n)\big)V(c_i)\Big) = j^l_q\Big(V\big(\Phi_f(c_1,\ldots,c_n)\big)\Big). \tag{7}$$

Suppose that $\mathfrak{D}$ is k-jet determined for $k \in \{0\} \cup \mathbb{N}$. For any $\mathbb{R}$-point $q : \mathfrak{D} \to \mathbb{R}$, we have an equation (7) for $l = k$. From the assumption of k-jet determined, we have the equation (6). Therefore, we have that V is a C^∞-derivation.

Suppose that $\mathfrak{D}$ is ∞-jet determined. For any $\mathbb{R}$-point $q : \mathfrak{D} \to \mathbb{R}$ and $l \in \{0\} \cup \mathbb{N}$, we have an equation (7). From the assumption of ∞-jet determined, we have the equation (6). Therefore, we have that V is C^∞-derivation. ■

5 Cotangent modules

We explain the $\mathbb{R}$-cotangent module and the C^∞-cotangent module and construct them with naturality. In [1], an $\mathbb{R}$-derivation is called a Kähler derivation.

Definition 5.1 ([1, 3]) *Let $\mathfrak{C}$ be a C^∞-ring, $\mathfrak{M}$ a $\mathfrak{C}$-module and $d : \mathfrak{C} \to \mathfrak{M}$ an $\mathbb{R}$-derivation(resp. a C^∞-derivation). We call the pair $(\mathfrak{M}, d)$ an* $\mathbb{R}$**-cotangent module** *(resp. a* C^∞**-cotangent module***) for $\mathfrak{C}$ if $(\mathfrak{M}, d)$ satisfies that*

> *for any $\mathbb{R}$-derivation (resp. C^∞-derivation) $d' : \mathfrak{C} \to \mathfrak{M}'$ to a $\mathfrak{C}$-module $\mathfrak{M}'$, there exists a unique homomorphism $\phi : \mathfrak{M} \to \mathfrak{M}'$ of $\mathfrak{C}$-modules such that $\phi \circ d = d'$.*

We construct an $\mathbb{R}$-cotangent module and a C^∞-cotangent module for $\mathfrak{C}$ respectively. Let $\mathfrak{F}_\mathfrak{C}$ be a free $\mathfrak{C}$-module generated by $d(c)(c \in \mathfrak{C})$. Define elements of $\mathfrak{F}_\mathfrak{C}$ as

$$\begin{aligned} Sum_\mathfrak{C}(c, c') :=& d(c + c') - d(c) - d(c'), \\ Prod_\mathfrak{C}(c, c') :=& d(c \cdot c') - cd(c') - c'd(c), \\ Func_\mathfrak{C}(f, c_1, \dots, c_m) :=& d(\Phi_f(c_1, \dots, c_m)) - \sum_{i=1}^{m} \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \dots, c_m) d(c_i) \\ &\text{for any } m \in \{0\} \cup \mathbb{N}, c, c', c_1, \dots, c_m \in \mathfrak{C}, f \in C^\infty(\mathbb{R}^m) \end{aligned}$$

and $\mathfrak{C}$-submodules of $\mathfrak{F}_\mathfrak{C}$ as

$$\mathfrak{M}_{\mathfrak{C},\mathbb{R}} := \left\langle Sum_\mathfrak{C}(c, c'), Prod_\mathfrak{C}(c, c'), d(r1_\mathfrak{C}) \middle| c, c' \in \mathfrak{C}, r \in \mathbb{R} \right\rangle_\mathfrak{C} \text{ and}$$

$$\mathfrak{M}_{\mathfrak{C},C^\infty} := \left\langle Sum_\mathfrak{C}(c, c'), Func_\mathfrak{C}(f, c_1, \dots, c_m), d(r1_\mathfrak{C}) \middle| \begin{array}{c} m \in \{0\} \cup \mathbb{N}, c, c', c_1, \dots, c_m \in \mathfrak{C}, \\ f \in C^\infty(\mathbb{R}^m), r \in \mathbb{R} \end{array} \right\rangle_\mathfrak{C}.$$

Let $K = \mathbb{R}, C^\infty$. Define a $\mathfrak{C}$-module $\Omega_{\mathfrak{C},K}$ by $\mathfrak{F}_\mathfrak{C}/\mathfrak{M}_{\mathfrak{C},K}$ and a K-derivation $d_{\mathfrak{C},K} : \mathfrak{C} \to \Omega_{\mathfrak{C},K}$ of $\mathfrak{C}$ as $d_{\mathfrak{C},K}(c) := d(c) + \mathfrak{M}_{\mathfrak{C},K}$.

For any K-derivation $V : \mathfrak{C} \to \mathfrak{M}$, we can define a homomorphism $\phi : \Omega_{\mathfrak{C},K} \to \mathfrak{M}$ of $\mathfrak{C}$ as $\phi(d(c)+\mathfrak{M}_{\mathfrak{C},K}) := V(c)$. ϕ satisfies that $\phi \circ d_{\mathfrak{C},K} = V$. Suppose that there exists another homomorphism $\phi' : \Omega_{\mathfrak{C},K} \to \mathfrak{M}$ of $\mathfrak{C}$ which satisfies $\phi' \circ d_{\mathfrak{C},K} = V$. Since $\phi(d(c) + \mathfrak{M}_{\mathfrak{C},K}) = V(c) = \phi'(d(c) + \mathfrak{M}_{\mathfrak{C},K})$ for any $c \in \mathfrak{C}$, we have $\phi = \phi'$. Therefore, ϕ is a unique homomorphism which satisfies $\phi \circ d_{\mathfrak{C},K} = V$.

Espencially for another K-cotangent module $(\mathfrak{M}', d')$ of $\mathfrak{C}$, there exists a unique homomorphism $\phi : \Omega_{\mathfrak{C},K} \to \mathfrak{M}'$ of $\mathfrak{C}$ which satisfies that $\phi \circ d_{\mathfrak{C},K} = d'$. For the property of the K-cotangent module $(\mathfrak{M}', d')$ of $\mathfrak{C}$, there exists a unique homomorphism $\psi : \mathfrak{M}' \to \Omega_{\mathfrak{C},K}$ of $\mathfrak{C}$ which satisfies that $\psi \circ d' = d_{\mathfrak{C},K}$. Then, $\phi \circ \psi \circ d' = d'$ and $\psi \circ \phi \circ d_{\mathfrak{C},K} = d_{\mathfrak{C},K}$. $\phi \circ \psi = id_{\mathfrak{M}'}$ and $\psi \circ \phi = id_{\Omega_{\mathfrak{C},K}}$ from the properties of $\mathfrak{M}'$ and $\Omega_{\mathfrak{C},K}$. Then $\mathfrak{M}'$ is isomorphic to $\Omega_{\mathfrak{C},K}$ as $\mathfrak{C}$-modules.

Therefore for any C^∞-ring $\mathfrak{C}$, there exists a K-cotangent module $(\Omega_{\mathfrak{C},K}, d_{\mathfrak{C},K})$ which is unique up to unique isomorphism of $\mathfrak{C}$-modules.

From D. Joyce [3] Remark 5.12., we have the following example showing that the assumption in our Theorems is essential.

Example 5.1 *Let $\mathfrak{C}$ be a C^∞-ring $C^\infty(\mathbb{R})$ of C^∞-functions on $\mathbb{R}$. For the $\mathbb{R}$-cotangent module $(\Omega_{\mathfrak{C},\mathbb{R}}, d_{\mathfrak{C},\mathbb{R}})$, $d_{\mathfrak{C},\mathbb{R}} : \mathfrak{C} \to \Omega_{\mathfrak{C},\mathbb{R}}$ is an $\mathbb{R}$-derivation but not a C^∞-derivation.*

In fact, for the exponential $e^x \in C^\infty(\mathbb{R})$, $e^x d_{\mathfrak{C},\mathbb{R}}(x) - d_{\mathfrak{C},\mathbb{R}}(e^x) \neq 0$ in $\Omega_{\mathfrak{C},\mathbb{R}}$. Note that the assumptions of Theorem 3.1 and 3.2 are not satisfied in this example.

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