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Proceedings of the 42nd Sapporo Symposium on Partial Differential Equations

- In memory of Professor Taira Shirota -

Edited by

S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, T. Ozawa
T. Sakajo, Y. Tonegawa, and K. Tsutaya

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Sapporo, 2017

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PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 8 through August 10 in 2017 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Professor Taira Shirota started the symposium more than 40 years ago. Professor Kôji Kubota and late Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

It is our deep regret to learn that Professor Taira Shirota passed away on June 27, 2017 at the age of 94. We would like to dedicate this proceeding to late Professor Shirota for his significant contribution to the organization of the Sapporo Symposium on PDE.

S.-I. Ei (Hokkaido University)
Y. Giga (The University of Tokyo)
N. Hamamuki (Hokkaido University)
S. Jimbo (Hokkaido University)
H. Kubo (Hokkaido University)
T. Ozawa (Waseda Universtiy)
T. Sakajo (Kyoto University)
Y. Tonegawa (Tokyo Institute of Technology)
K. Tsutaya (Hirosaki University)

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On uniqueness for the supercritical harmonic map heat flow

T. Mine (Kyoto Institute of Technology)
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L. Fanelli (SAPIENZA Università di Roma)
Time decay of Schödinger evolutions: the role played by the angular Hamiltonian

The 42nd Sapporo Symposium on Partial Differential Equations

- In memory of Professor Taira Shirota -

第４２回偏微分方程式論札幌シンポジウム

- 白田平先生を追悼して -

Period	August 8, 2017 - August 10, 2017
Venue	7-310, 7-219/220, Faculty of Science Bld. No.7, Hokkaido University
Organizers	Hideo Kubo, Shuichi Jimbo
Program Committee	Shin-Ichiro Ei, Yoshikazu Giga, Nao Hamamuki, Shuichi Jimbo, Hideo Kubo, Tohru Ozawa, Takashi Sakajo, Yoshihiro Tonegawa, Kimitoshi Tsutaya
URL	http://www.math.sci.hokudai.ac.jp/sympo/sapporo/program170808.html

August 8, 2017 (Tuesday)

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On the uniqueness of least energy elastic curves

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14:10-14:40 *

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Numerical analysis of the Oseen-type Peterlin viscoelastic model

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Scattering by the ring magnetic field in the three dimensional space

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11:30-12:30 Luca Fanelli (SAPIENZA Università di Roma)
Time decay of Schrödinger evolutions: the role played by the angular Hamiltonian

* Free discussion with speakers

Concentration-compactness and finite-time singularities for Chen's flow

Yann Bernard * Glen Wheeler † Valentina-Mira Wheeler †

Abstract

Chen's flow is a fourth-order curvature flow motivated by the spectral decomposition of immersions, a program classically pushed by B.-Y. Chen since the 1970s. In curvature flow terms the flow sits at the critical level of scaling together with the most popular extrinsic fourth-order curvature flow, the Willmore and surface diffusion flows. Unlike them however the famous Chen conjecture indicates that there should be no stationary nonminimal data, and so in particular the flow should drive all closed submanifolds to singularities. We investigate this idea, proving that (1) closed data becomes extinct in finite time in all dimensions and for any codimension; (2) singularities are characterised by concentration of curvature in L^n for intrinsic dimension $n \in \{2, 3, 4\}$ and any codimension (a Lifespan Theorem); and (3) for $n = 2$ and in one codimension only, there exists an explicit ε_2 such that if the L^2 norm of the tracefree curvature is initially smaller than ε_2, the flow remains smooth until it shrinks to a point, and that the blowup of that point is an embedded smooth round sphere.

1 Introduction

Suppose $f : M^n \to \mathbb{R}^N$, $N > n$ is a smooth isometric immersion. We assume that M^n is closed and complete. Denote by $\vec{H}$ the mean curvature vector of f. Then

$$(\Delta f)(p) = \vec{H}(p)$$

for all $p \in M^n$, where Δ here refers to the rough Laplacian. The rough Laplacian is the induced connection on the pullback bundle $f^*(T\mathbb{R}^{n+1})$. Applying the operator again yields

$$(\Delta^2 f)(p) = (\Delta \vec{H})(p)\,.$$

If $\Delta^2 f \equiv 0$, we call f *biharmonic*. Chen's conjecture is the statement that $\Delta \vec{H} \equiv 0$ implies $\vec{H} \equiv 0$. This conjecture is motivated by Chen's work in the spectral decomposition of immersed submanifolds. There has been much

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activity on the conjecture (see as a sample the recent papers [2, 8, 19, 20, 26, 27, 28, 29, 30, 34, 38] and Chen's recent survey [6]), but still it remains open.

In this paper we study the heat flow for Δ^2: this is a one-parameter family of smooth isometric immersions $f : M^n \times [0,T) \to \mathbb{R}^N$ satisfying $f(p,0) = f_0(p)$ for a given smooth isometric immersion $f_0 : M^n \to \mathbb{R}^N$ and

$$(\partial_t f)(p,t) = -(\Delta^2 f)(p,t)\,, \tag{CF}$$

for all $(p,t) \in M^n \times (0,T)$. We call (CF) *Chen's flow* and f_0 the initial data. Since Δ^2 is a fourth-order quasilinear elliptic operator, local existence and uniqueness for (CF) is standard. Details can be found in [1, Chapter 3]. See also [9, Chapter 5], [33] and [18].

Theorem 1. *Let $f_0 : M^n \to \mathbb{R}^N$ be a smooth closed isometrically immersed submanifold. There exists a $T \in (0,\infty]$ and unique one-parameter family of smooth closed isometric immersions $f : M^n \times [0,T) \to \mathbb{R}^N$ such that* (CF) *is satisfied and T is maximal.*

Note that *maximal* above means that there does not exist another family $\hat{f} : M^n \times [0,\hat{T}) \to \mathbb{R}^N$ of smooth closed isometrically immersed hypersurfaces satisfying (CF), $\hat{f}(p,0) = f_0(p)$ with $\hat{T} > T$.

A simple consequence of the argument used by Jiang [10] is that there are no closed biharmonic submanifolds of Euclidean space. Therefore the following qualitative property of the flow is natural to expect.

Theorem 2. *Chen's flow $f : M^n \times [0,T) \to \mathbb{R}^N$ with smooth, closed initial data $f_0 : M^n \to \mathbb{R}^N$ has finite maximal time of existence, with the explicit estimate*

$$T \leq \frac{\mu(f_0)}{C_n}\,, \tag{1}$$

where for $n \in \{2,3,4\}$ we have $C_n = 4\omega_n^{\frac{4}{n}} n^2$, and for $n > 4$ we have $C_n = \frac{\omega_n^{\frac{4}{n}}}{n^2 4^{4n+3}}$. Here ω_n denotes the area of the unit n-sphere. Furthermore, if equality is achieved in (1)*, then $\mu(f_t) \searrow 0$ as $t \nearrow T$.*

Given Theorem 2, it is natural to ask for a classification of finite-time singularities. For higher-order curvature flow such as Chen's flow, such classifications are very difficult. For example, a classification of singular geometries remains well open for the two most popular extrinsic fourth-order curvature flow, that is, the Willmore flow and the surface diffusion flow (see for example [14, 15, 16, 21, 24, 35, 36, 37, 39]).

For both the surface diffusion and Willmore flows, the general principle of concentration or compactness from the classical theory of harmonic map heat flow remains valid. We are able to obtain a similar result here: We present the following characterisation of finite-time singularities, also called a concentration-compactness alternative or lifespan theorem.

Theorem 3. *Let $n \in \{2,3,4\}$. There exist constants $\varepsilon_1 > 0$ and $c < \infty$ depending only on n and N with the following property. Let $f : M^n \times [0,T) \to \mathbb{R}^N$ be a Chen flow with smooth initial data. If ρ is chosen with*

$$\int_{f^{-1}(B_\rho(x))} |A|^n d\mu\Big|_{t=0} = \varepsilon(x) \leq \varepsilon_1 \qquad \text{for all } x \in \mathbb{R}^N, \tag{2}$$

then the maximal time T of smooth existence satisfies

$$T \geq \frac{1}{c}\rho^4, \tag{3}$$

and we have the estimate

$$\int_{f^{-1}(B_\rho(x))} |A|^n d\mu \leq c\varepsilon_1 \qquad \text{for all } t \in \Big[0, \frac{1}{c}\rho^4\Big]. \tag{4}$$

Remark 1. *Our proof applies to a general class of flows, including the Willmore flow and the surface diffusion flow. This is new for the Willmore flow in three and four dimensions (the two dimensional case is the main result of [15]), and completely new for the surface diffusion flow in four dimensions. In three dimensions a lifespan theorem for the surface diffusion flow is known [36], however the constants (ε_1, c) there for $n = 3$ depend on the measure of the initial data. Here our constants are universal.*

The main result of [15] and the lifespan theorems from [21, 23, 35, 36, 37] (assuming the external force vanishes identically) are generalised by our work here. See Theorem **??** *for a precise statement.*

The concentration phenomenon that Theorem 3 guarantees can be seen as follows. If $\rho(t)$ denotes the largest radius such that (2) holds at time t, then $\rho(t) \leq \sqrt[4]{c(T-t)}$ and so at least ε_1 of the curvature concentrates in a ball $f^{-1}(B_{\rho(T)}(x))$. That is,

$$\lim_{t\to T} \int_{f^{-1}(B_{\rho(t)}(x))} |A|^n d\mu \geq \varepsilon_1 \,,$$

where $x = x(t)$ is understood to be the centre of a ball where the integral above is maximised.

Although Theorem 3 yields a characterisation of finite time singularities as space-time concentrations of curvature locally in L^2, it does not give any information at all about the asymptotic geometry of such a singularity. One of the simplest observations in this direction is that for spherical initial data with radius r_0, the flow shrinks homothetically to a point with maximal time

$$T = \frac{r_0}{n^2} \,.$$

As the evolution is homothetic, parabolic rescaling about the space-time singularity reveals a standard round sphere. This asymptotic behaviour is called *shrinking to a round point*.

One may therefore hope that this behaviour holds in a neighbourhood of a sphere. This is our final result of the paper, proved using blowup analysis.

Theorem 4. *There exists an absolute constant $\varepsilon_2 > 0$ such that if $f : M^2 \times [0,T) \to \mathbb{R}^3$ is Chen's flow*

$$\int_M |A^o|^2 d\mu \Big|_{t=0} \leq \varepsilon_2 < 8\pi \tag{5}$$

then $T < \infty$, and $f(M^2, t)$ shrinks to a round point as $t \to T$.

Our strategy for obtaining these results is as follows. First, we use standard techniques in curvature flow to describe the Chen flow in the normal bundle. Then, we calculate evolution equations for various geometric quantities. These are used, together with some inequalities from classical differential geometry, to prove Theorem 2. Our analysis for the remaining results relies on control obtained via localised integral estimates. The key tools that facilitate this are the Michael-Simon Sobolev inequality [25] and the divergence theorem. We derive several consequences of these that are particularly adapted to our purposes here, but hold more generally. Integral estimates valid along the flow follow as a consequence, including control on the local growth of the L^n norm of A. This allows us to use an argument due to Struwe to prove the lifespan theorem. Finally, for the global analysis for the flow, we need to first prove monotonicity for the L^2 norm of A, then conduct blowup analysis. The monotonicity allows us to conclude using a standard argument that the blowup must be a standard round sphere, proving Theorem 4.

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The energy-critical Zakharov system[1]

Kenji Nakanishi (Osaka University)

1. Introduction

The Zakharov system

$$\begin{cases} i\dot{u} - \Delta u = nu, & u(t,x) : \mathbb{R}^{1+d} \to \mathbb{C}, \\ \ddot{n}/\alpha^2 - \Delta n = -\Delta |u|^2, & n(t,x) : \mathbb{R}^{1+d} \to \mathbb{R} \end{cases} \quad (\alpha > 0 \text{: constant})$$

is a simple model for the Langmuir turbulence in plasma, generated by nonlinear interactions between electric and hydrodynamic oscillations. We are interested in the existence and global behavior of solutions in the special case $d = 4$.

For simplicity, put $D := \sqrt{-\Delta}$ and $N := n - iD^{-1}\dot{n}/\alpha$. Then the equation is transformed into the first order (in time) system

$$\begin{cases} i\dot{u} - \Delta u = nu, \quad n = \operatorname{Re} N, \quad u : \mathbb{R}^{1+d} \to \mathbb{C}, \\ i\dot{N} + \alpha D N = \alpha D |u|^2, \quad N : \mathbb{R}^{1+d} \to \mathbb{C}. \end{cases} \tag{1.1}$$

It is a Hamiltonian system with the energy and the mass conserved

$$E(u,N) := \int_{\mathbb{R}^d} |\nabla u|^2 + |N|^2/2 - n|u|^2 dx, \quad M(u) := \int_{\mathbb{R}^d} |u|^2 dx.$$

1.1. **Energy-critical dimension.** In view of the above conservation laws, it is natural to look for solutions in the energy space:

$$(u(t), N(t)) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d),$$

as we are interested in the global dynamics of (generally large) solutions. The nonlinear part of $E(u,N)$ is controlled through Sobolev by the linear part iff $d \le 4$, because

$$|\langle n, |u|^2 \rangle| \le \|N\|_{L^2(\mathbb{R}^d)} \|u\|^2_{L^4(\mathbb{R}^d)}, \quad H^1(\mathbb{R}^d) \subset L^4(\mathbb{R}^d),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of $L^2(\mathbb{R}^d)$:

$$\langle f, g \rangle := \operatorname{Re} \int_{\mathbb{R}^d} f(x) \overline{g(x)} dx.$$

Indeed, the subsonic limit $\alpha \to \infty$ is the nonlinear Schrödinger equation

$$i\dot{u} - \Delta u = |u|^2 u \quad (N = |u|^2),$$

which is energy-critical on $\mathbb{R}^4$. In terms of the wave component, $N \in L^2(\mathbb{R}^4)$ is energy-critical as the potentil:

$$\inf_{\|\varphi\|_{L^2(\mathbb{R}^4)}=1} \langle (-\Delta - n)\varphi | \varphi \rangle = \begin{cases} 0 & (\|n\|_{L^2(\mathbb{R}^4)} \le \|W^2\|_{L^2(\mathbb{R}^4)}) \\ -\infty & (\|n\|_{L^2(\mathbb{R}^4)} > \|W^2\|_{L^2(\mathbb{R}^4)}), \end{cases}$$

where $W(x) \in \dot{H}^1(\mathbb{R}^4) \setminus L^2(\mathbb{R}^4)$ is the unique positive radial maximizer for the Sobolev inequality $\|\varphi\|_{L^4(\mathbb{R}^4)} \le C\|\nabla\varphi\|_{L^2(\mathbb{R}^4)}$ or the ground state of the static NLS:

$$W(x) := (|x|^2 + 1/8)^{-1}, \quad -\Delta W = W^3, \tag{1.2}$$

[1]This talk is based on joint work with Ioan Bejenaru, Zihua Guo, and Sebastian Herr.

which is a resonance at 0 for the Schrödinger operator $-\Delta - W^2$. The 4D Zakharov system has the corresponding ground state solution $(u, N) = (W, W^2)$.

The energy space is also critical for the 4D Zakharov system because of the Strichartz estimate $L_t^2 L_x^4$ with the prohibited critical Sobolev embedding $H_4^1(\mathbb{R}^4) \not\subset L^\infty(\mathbb{R}^4)$:

$$\|e^{-it\Delta}\varphi\|_{L_t^2 H_4^1} \lesssim \|\varphi\|_{H^1(\mathbb{R}^4)},$$

where H_p^s denotes the inhomogeneous Sobolev space on $\mathbb{R}^4$ with the norm $\|\varphi\|_{H_p^s} = \|\sqrt{1-\Delta}^s \varphi\|_{L^p(\mathbb{R}^4)}$, and the space-time norm is abbreviated by

$$\|u\|_{L_t^p X}^p := \int \|u(t)\|_X^p dt,$$

for any Banach space X of functions on $\mathbb{R}^4$.

1.2. **Main difficulties for the Zakharov system.** The linear energy estimate for (1.1) in $H^s \times H^\sigma$ is

$$\|u\|_{L_t^\infty H^s} \lesssim \|u(0)\|_{H^s} + \|nu\|_{L_t^1 H^s}, \quad \|N\|_{L_t^\infty H^\sigma} \lesssim \|N(0)\|_{H^\sigma} + \|D|u|^2\|_{L_t^1 H^\sigma}.$$

In order to make this estimate closed in $H^s \times H^\sigma$, the regularity should be the same for u and nu, and the same for N and $D|u|^2$, so

$$s \le \min(\sigma, s), \quad \sigma \le s - 1,$$

which is impossible.

This derivative loss is avoided by exploiting the non-resonance of interactions. Using a bilinear Fourier multiplier $B(N, u)$, the equation for u is transformed into

$$(i\partial_t - \Delta)(u - B(N, u)) = (nu)_{LH} - B(\alpha D|u|^2, u) - B(N, nu),$$

where $(nu)_{LH}$ contains only the interactions where the frequency of N is lower than that of u. It means essentially that the derivative on $(nu)_{LH}$ acts only on u. Specifically we have, for any $s \ge 0$,

$$\|(nu)_{LH}\|_{L_t^2 H_{4/3}^s} \lesssim \|n\|_{L_t^\infty L^2} \|u\|_{L_t^2 H_4^s},$$

which is just the Hölder inequality if $s = 0$. Hence this term can be handled by the double endpoint Strichartz estimate:

$$\|u\|_{L_t^2 H_4^s} \lesssim \|u(0)\|_{H^s} + \|(i\partial_t - \Delta)u\|_{L_t^2 H_{4/3}^s}.$$

The bilinear form $B(N, u)$ contains the interactions between higher frequencies of N and lower ones of u, but it is much better than Nu and roughly similar to $\langle D\rangle^{-2}(Nu)$. However, when $s = 1$, the product estimate using $H^2 \subset H_4^1$

$$\|\langle D\rangle^{-2}(Nu)\|_{H_4^1} \lesssim \|Nu\|_{L^2} \lesssim \|N\|_{L^2} \|u\|_{L^\infty}$$

cannot be combined with the endpoint Strichartz estimate, because of $H_4^1 \not\subset L^\infty$. Essentially the same problem appears in the cubic term:

$$\|B(N, nu)\|_{H_{4/3}^1} \lesssim \|N\|_{L^2} \|nu\|_{L^2} \lesssim \|N\|_{L^2}^2 \|u\|_{L^\infty}.$$

2. Main results

First we have local wellposedness and small data scattering in general Sobolev spaces.

Theorem 1. *Let $s, \sigma \geq 0$ satisfy*

$$\begin{cases} 0 \leq \sigma \leq \min(2s-1, s+1), \quad (s,\sigma) \neq (2,3), \\ s < 4\sigma + 1, \quad s \leq \sigma + 2, \quad s \leq 2\sigma + 11/8, \end{cases} \tag{2.1}$$

or $(s,\sigma) = (1,0)$.

Then the Zakharov system (1.1) is time-locally wellposed in $(u,N) \in (H^s \times H^\sigma)(\mathbb{R}^4)$, and the maximal existence interval is common among all (s,σ) in (2.1). Moreover, if $\|(u(0),N(0))\|_{H^{1/2}\times H^0}$ is small enough, then the solution (u,N) is global, satisfying

$$\|u(t) - e^{-it\Delta}u_\pm\|_{H^s} + \|N(t) - e^{it\alpha D}N_\pm\|_{H^\sigma} \to 0 \quad (t \to \pm\infty)$$

for some scattering states $(u_\pm, N_\pm) \in (H^s \times H^\sigma)(\mathbb{R}^4)$. The smallness condition is independent of (s,σ).

$(s,\sigma) = (1/2, 0)$ is the lowest corner point in (2.1), and the other corner points are $(s,\sigma) = (2,3), (7/4, 3/16), (21/8, 5/8), (1,0)$, where the first two are not included, and the last one corresponding to the energy space. The latter four exponents are intermediate in the sense that for some $(s_0,\sigma_0), (s_1,\sigma_1)$ in the range (2.1)

$$H^{s_0} \times H^{\sigma_0} \subset H^s \times H^\sigma \subset H^{s_1} \times H^{\sigma_1}.$$

This does not guarantee local wellposedness at (s,σ). In fact we have the following illposedness at one of the corner $(s,\sigma) = (2,3)$.

Theorem 2. *There exists a radial function $\varphi \in H^2(\mathbb{R}^4)$ such that for any $\varepsilon > 0$, any $N(0) \in H^3(\mathbb{R}^4)$, and any $T > 0$, the Zakharov system (1.1) has no solution on $0 \leq t \leq T$ satisfying*

$$u(0) = \varepsilon\varphi, \quad (u,N) \in C([0,T]; \mathcal{S}' \times \mathcal{S}') \cap L^2((0,T); H^1 \times H^3).$$

On the other hand, the unique solution (u,N) in $H^2(\mathbb{R}^4) \times H^2(\mathbb{R}^4)$ given by the previous result satisfies $N(t) \notin H^3(\mathbb{R}^4)$ for all $t \neq 0$.

The above results except for the local wellposedness in the energy space for large data are in [3].

Global analysis of large solutions has an extra difficulty because the $\|N\|_{L_t^\infty L^2}$ norm does not decay either for large time or for shorter time interval, but the Strichartz estimate does not allow us to use other space-time norms for N.

Our idea to overcome this difficulty is to treat nu as a potential term rather than a nonlinear one, and to derive a Strichartz estimate with a potential solving a wave equation which is uniform with respect to the $L_t^\infty L^2$ norm of the potential. The Strichartz estimate with a wave potential is proven using the normal form argument and the profile decomposition, together with the improved free Strichartz for radial solutions. Thus we obtain global existence in the radial energy space under the ground state constraint:

Theorem 3. *Let $(u(0), N(0)) \in H^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$ be radial functions satisfying*

$$E(u(0),N(0)) < E_S(W) \quad \textit{and} \quad \|N(0)\|_{L^2(\mathbb{R}^4)} < \|W^2\|_{L^2(\mathbb{R}^4)},$$

where W is the ground state of NLS defined in (1.2)*, and E_S is the energy for NLS:*

$$E(\varphi) := \int_{\mathbb{R}^4} \frac{|\nabla\varphi|^2}{2} - \frac{|\varphi|^4}{4} dx.$$

Then the unique solution (u, N) given by Theorem (1) *is global and bounded in the energy norm, satisfying the above constraints for all $t \in \mathbb{R}$.*

We expect that those global solutions scatter, while the solutions in the other region

$$E(u(0), N(0)) < E_S(W) \quad \text{and} \quad \|N(0)\|_{L^2} \geq \|W^2\|_{L^2}$$

blow up.

3. Preceding and other results

3.1. **Local well-posedness.**

- For $d \leq 3$, Bourgain-Colliander [4] proved in the energy space.
- Ginibre-Tsutsumi-Velo [6] extended the range of (s, σ) for all d.
- Bejenaru-Herr-Holmer-Tataru [2] and Bejenaru-Herr [1] further extended the range for $d = 2, 3$.

3.2. **Final data problem.** $\exists!(u, N)$ for $t \gg 1$, asymptotic to a given free solution $(e^{-it\Delta}u_+, e^{it\alpha D}N_+)$

- Ozawa-Tsutsumi [23] $d = 3$ in weighted Sobolev spaces, either small data or $\operatorname{supp} \hat{u}_+ \cap \{|\xi| = \alpha\} = \emptyset$
- Shimomura [26] removed the condition on $\operatorname{supp} \hat{u}_+$, but $\hat{N}_+(\xi = 0) = 0$.
- Ginibre-Velo [7] removed the condition on $\hat{N}_+(\xi = 0)$, extending to Sobolev spaces on $L^2 \cap L^1$, and to $d = 2$.

3.3. **Blow-up.**

- Glangedas and Merle [8] constructed blow-up in finite time for $d = 2$.
- For $d = 2, 3$ and $E(u, N) < 0$, Merle [22] proved blow-up either in finite or in infinite time.

3.4. **Small data scattering.**

- For $d = 3$, Guo-Nakanishi [11] proved for radial data in the energy space.
- Hani-Pusateri-Shatah [14] proved for non-radial data in weighted Sobolev spaces with high regularity and decay.
- Guo-Lee-Nakanishi-Wang [10] improved it to non-radial data with one angular derivative in the energy space.
- For $d \geq 4$, Kato-Tsugawa [16] independently obtained the scattering for $\sigma = s - 1/2 \geq (d - 4)/2$. Their proof is based on bilinear estimates.

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On the uniqueness of least energy elastic curves

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1 Introduction

This talk addresses one of the most classical problems on elastic curves. The problem is formulated as the minimizing problem of the total squared curvature, so-called *bending energy*,

$$\mathcal{B}[\gamma] := \int_\gamma \kappa^2 ds,$$

where γ is a planar curve of fixed length and clamped endpoints, i.e., the positions and the tangential directions at the endpoints are fixed as in Figure 1. Here s denotes the arc length parameter and κ denotes the (signed) curvature.

The above problem is motivated to determine the shapes of inextensible and flexible elastic rods of clamped endpoints. The history goes back to at least the study of L. Euler in 1744 [2]. Thanks to the pioneering work by Euler and a great number of following studies by numerous authors, the above problem is now well-understood, at least, at the level of equation (see e.g. [3, 4, 7, 9]). Notwithstanding, even today, there remain many open

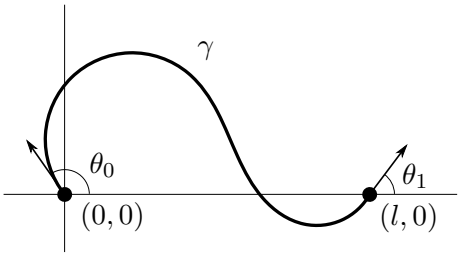

Figure 1: Clamped curve.

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problems on the above boundary value problem. In particular, although it is possible to exist multiple global minimizers for a given condition, it is expected that global minimizers are unique for generic constraints (cf. [8]); however, there is no general result to ensure the uniqueness only in terms of constraints. In this talk we give a uniqueness result under simple assumptions on the constraints.

2 Main results

2.1 Uniqueness for inextensible problem

We state our main result on the uniqueness. To this end we define the set of admissible curves more precisely.

Let $I = (0,1)$. Let $0 < l < L$ and $\theta_0, \theta_1 \in (-\pi, \pi)$. We say that $\gamma \in C^\infty(\bar{I}; \mathbb{R}^2)$ is *admissible* if γ is of constant speed L, i.e., $|\dot{\gamma}| \equiv L$, and moreover satisfies the clamped boundary condition:

$$\begin{aligned} &\gamma(0) = (0,0), \quad \dot{\gamma}(0) = L(\cos\theta_0, \sin\theta_0), \\ &\gamma(1) = (l,0), \quad \dot{\gamma}(1) = L(\cos\theta_1, \sin\theta_1). \end{aligned}$$

We denote the set of all admissible curves by $\mathcal{A}^L_{\theta_0,\theta_1,l} \subset C^\infty(\bar{I}; \mathbb{R}^2)$. Then our problem is formulated as

$$\min_{\gamma \in \mathcal{A}^L_{\theta_0,\theta_1,l}} \mathcal{B}[\gamma]. \tag{2.1}$$

We mention that the existence of minimizers follows by a direct method in the calculus of variations (and a bootstrap argument).

Our main result states that, if the angle condition $\theta_0\theta_1 < 0$ holds, then the uniqueness of global minimizers is guaranteed as the straightening limit $l \uparrow L$.

Theorem 2.1. *Let $L > 0$ and $\theta_0, \theta_1 \in (-\pi, \pi)$. Suppose that $\theta_0\theta_1 < 0$. Then there exists $\bar{l} \in (0, L)$ such that for any $l \in (\bar{l}, L)$ the energy $\mathcal{B}$ admits a unique global minimizer in $\mathcal{A}^L_{\theta_0,\theta_1,l}$.*

2.2 Uniqueness for extensible problem

The minimizing problem on the total squared curvature $\mathcal{B}$ is not easy to tackle directly since there are a number of constraints. In our study we first bypass the length constraint on admissible curves.

Let $l > 0$ and $\theta_0, \theta_1 \in (-\pi, \pi)$. Define

$$\mathcal{A}_{\theta_0,\theta_1,l} := \bigcup_{L \in (l,\infty)} \mathcal{A}^L_{\theta_0,\theta_1,l}.$$

For $\varepsilon > 0$, we consider the following minimizing problem:

$$\min_{\gamma \in \mathcal{A}_{\theta_0,\theta_1,l}} \mathcal{E}_\varepsilon[\gamma], \tag{2.2}$$

where $\mathcal{E}_\varepsilon$ denotes the modified total squared curvature, i.e.,

$$\mathcal{E}_\varepsilon[\gamma] := \varepsilon^2 \mathcal{B}[\gamma] + \mathcal{L}[\gamma], \quad \mathcal{L}[\gamma] := \int_\gamma ds. \tag{2.3}$$

The above problem is an "extensible" problem in the sense that there is no length constraint on admissible curves unlike the original "inextensible" problem. The extensible problem is easier to deal with at least in view of the number of constraints. For the proof of Theorem 2.1 we first obtain the following

Theorem 2.2. *Let $l > 0$ and $\theta_0, \theta_1 \in (-\pi, \pi)$. Suppose that $\theta_0\theta_1 < 0$. Then there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ the energy $\mathcal{E}_\varepsilon$ admits a unique global minimizer in $\mathcal{A}_{\theta_0,\theta_1,l}$.*

Then we investigate the precise connection between the inextensible and extensible problems. It turns out that the uniqueness in Theorem 2.2 particularly implies that the inextensible problem (2.1) can be rephrased in terms of the extensible problem (2.2).

Proposition 2.3. *Let $L > 0$ and $\theta_0, \theta_1 \in (-\pi, \pi)$. Suppose that $\theta_0\theta_1 < 0$. Then there are $\bar{l} \in (0, L)$ and a strictly decreasing function $\tilde{\varepsilon} : (\bar{l}, L) \to (0, \infty)$ such that, for any $l \in (\bar{l}, L)$ and any minimizer γ_l of $\mathcal{B}$ in $\mathcal{A}^L_{\theta_0,\theta_1,l}$, the dilated curve $\frac{L}{l}\gamma_l$ is a minimizer of $\mathcal{E}_{\tilde{\varepsilon}(l)}$ in $\mathcal{A}_{\theta_0,\theta_1,L}$.*

Note that the distance of the endpoints of the dilated curve $\frac{L}{l}\gamma$ is fixed as L (not l). Combining Theorem 2.2 and Proposition 2.3, we immediately obtain Theorem 2.1. In what follows we give a sketch of the proof of Theorem 2.2, which is the most important part.

3 Convexity and uniqueness

A key step in the proof of Theorem 2.2 is to obtain an a priori convexity of minimizers, which is our main contribution (Proposition 3.2). Once such a convexity is ensured, we easily obtain the uniqueness by utilizing a change of variables in terms of the radius of curvature parameterized by the tangential angle. The change of variables has been used in the stability analysis of M. Born in 1906 [1]. Surprisingly, this procedure totally "convexifies" our problem, and this convexification immediately implies the uniqueness of minimizers. In the rest of this section we first review Born's convexification, and then state our result on the convexity of minimizers.

We prepare some notations. For a smooth regular curve γ, we denote by $\tilde{\gamma}$ the arc length parameterization of γ. In addition, we denote by ϑ_γ the tangential angle function, i.e., a unique smooth function such that $\partial_s\tilde{\gamma}(s) = (\cos\vartheta_\gamma(s), \sin\vartheta_\gamma(s))$. Such a function is unique up to addition by a constant of $2\pi\mathbb{Z}$. Finally, we denote by κ the signed curvature, i.e., $\kappa(s) := \partial_s\vartheta_{\tilde{\gamma}}(s)$.

3.1 Born's convexification

For $l > 0$ and $\theta_0, \theta_1 \in \mathbb{R}$ with $\theta_0 \neq \theta_1$, we denote by $\tilde{\mathcal{A}}_{\theta_0,\theta_1,l}$ the set of all smooth constant speed curves joining $(0,0)$ to $(l,0)$ such that the tangential angles are strictly monotone functions from θ_0 to θ_1. Notice that $\tilde{\mathcal{A}}_{\theta_0,\theta_1,l} \subset \mathcal{A}_{\theta_0,\theta_1,l}$ if $\theta_0, \theta_1 \in (-\pi, \pi)$. Remark that the constraint of $\tilde{\mathcal{A}}_{\theta_0,\theta_1,l}$ completely fixes the variation of the tangential angle of a curve unlike our original clamped boundary condition. Concerning this space of convex curves, we have the following uniqueness.

Proposition 3.1. *Let $l > 0$ and $\theta_0, \theta_1 \in \mathbb{R}$ with $\theta_0 \neq \theta_1$. Then, for any $\varepsilon > 0$ the energy $\mathcal{E}_\varepsilon : \tilde{\mathcal{A}}_{\theta_0,\theta_1,l} \to (0,\infty)$ admits at most one minimizer in $\tilde{\mathcal{A}}_{\theta_0,\theta_1,l}$.*

Proposition 3.1 is proved by Born's convexification method. This method is simple and elegant, so we give a complete proof of this proposition.

Proof of Proposition 3.1. We may assume that $\theta_0 < \theta_1$ without loss of generality. For any $\gamma \in \tilde{\mathcal{A}}_{\theta_0,\theta_1,l}$, we can define the radius of curvature function $\rho : [\theta_0, \theta_1] \to (0,\infty)$ parameterized by the tangential angle as $\rho(\phi) := 1/\kappa(\vartheta_{\tilde{\gamma}}^{-1}(\phi))$. (Notice that in this case κ is positive.) Then, for any $\varepsilon > 0$ and $\gamma \in \tilde{\mathcal{A}}_{\theta_0,\theta_1,l}$, the energy $\mathcal{E}_\varepsilon$ is represented as

$$\mathcal{E}_\varepsilon[\gamma] = \int_0^{\mathcal{L}[\gamma]} \left(\varepsilon^2\kappa^2 + 1\right) ds = \int_{\theta_0}^{\theta_1} \left(\frac{\varepsilon^2}{\rho} + \rho\right) d\phi =: \tilde{\mathcal{E}}_\varepsilon[\rho].$$

In particular, for any fixed ε, the energy $\tilde{\mathcal{E}}_\varepsilon$ is strictly convex with respect to ρ since $\rho > 0$ and the integrand $f(\rho) = \varepsilon^2/\rho + \rho$ is strictly convex in $(0,\infty)$. Moreover, the constraints on the positions of γ at the endpoints

$$\int_0^{\mathcal{L}[\gamma]} \cos\vartheta_{\tilde{\gamma}} ds = l, \quad \int_0^{\mathcal{L}[\gamma]} \sin\vartheta_{\tilde{\gamma}} ds = 0,$$

are also expressed in terms of ρ as

$$\int_{\theta_0}^{\theta_1} \rho\cos\phi d\phi = l, \quad \int_{\theta_0}^{\theta_1} \rho\sin\phi d\phi = 0. \tag{3.1}$$

Conversely, if a smooth function $\rho : [\theta_0, \theta_1] \to (0,\infty)$ satisfying (3.1) is given, then we can restore a unique smooth curve $\gamma \in \tilde{\mathcal{A}}_{\theta_0,\theta_1,l}$ such that $\rho(\phi) = 1/\kappa(\vartheta_{\tilde{\gamma}}^{-1}(\phi))$.

We now denote by $\tilde{\mathcal{R}}_{\theta_0,\theta_1,l}$ the set of all functions $\rho \in C^\infty([\theta_0,\theta_1];(0,\infty))$ satisfying (3.1). Clearly, the set $\tilde{\mathcal{R}}_{\theta_0,\theta_1,l}$ is convex. Moreover, by the above arguments, we find that the minimizing problem of $\mathcal{E}_\varepsilon : \tilde{\mathcal{A}}_{\theta_0,\theta_1,l} \to (0,\infty)$ is equivalent to the minimizing problem of $\tilde{\mathcal{E}}_\varepsilon : \tilde{\mathcal{R}}_{\theta_0,\theta_1,l} \to (0,\infty)$. More explicitly, there is a bijection Φ from $\tilde{\mathcal{R}}_{\theta_0,\theta_1,l}$ to $\tilde{\mathcal{A}}_{\theta_0,\theta_1,l}$ such that the equality $\mathcal{E}_\varepsilon[\Phi(\rho)] = \tilde{\mathcal{E}}_\varepsilon[\rho]$ holds for any $\varepsilon > 0$ and $\rho \in \tilde{\mathcal{R}}_{\theta_0,\theta_1,l}$. In addition, we easily find that the energy $\tilde{\mathcal{E}}_\varepsilon : \tilde{\mathcal{R}}_{\theta_0,\theta_1,l} \to (0,\infty)$ admits at most one minimizer since $\tilde{\mathcal{E}}_\varepsilon$ is a strictly convex functional defined on a convex set. Therefore, we also find that the energy $\mathcal{E}_\varepsilon : \tilde{\mathcal{A}}_{\theta_0,\theta_1,l} \to (0,\infty)$ admits at most one minimizer. The proof is now complete. □

3.2 A priori convexity

As a direct corollary of Proposition 3.1, if we obtain an a priori convexity of minimizers, then the uniqueness is guaranteed. Our result gives such a guaranty for any small ε under the assumption that $\theta_0\theta_1 < 0$.

Proposition 3.2. *Let $l > 0$ and $\theta_0, \theta_1 \in (-\pi, \pi)$. Suppose that $\theta_0\theta_1 < 0$. Then there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ any minimizer γ of $\mathcal{E}_\varepsilon$ in $\mathcal{A}_{\theta_0,\theta_1,l}$ satisfies the following properties.*

(1) The signed curvature of γ has no sign change.

(2) The total absolute curvature of γ is equal to $|\theta_0| + |\theta_1|$.

This proposition is proved by our recent result on the convergence of minimizers of $\mathcal{E}_\varepsilon$ as $\varepsilon \to 0$ and the expressions of the curvatures of solution curves in terms of the Jacobi elliptic functions [6]. In the proof of convergence, a first order expansion of the energy (as in [5]) plays an important role.

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Strauss's radial compactness and its application to nonlinear elliptic problem with variable critical exponent

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1. Problem

Let $1 < p < N$. Assume $q = q(x) \in L^\infty(\mathbb{R}^N)$ and $q(x) \geq 1$ for a.e. $x \in \mathbb{R}^N$. In this talk, we consider on compactness and non-compactness for the embedding from radial Sobolev spaces $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to *variable exponent* Lebesgue spaces $L^{q(\cdot)}(\mathbb{R}^N)$. In particular, we point out that the behavior of $q(\cdot)$ at infinity and the origin plays an essential role on compactness and non-compactness. Variable exponent Lebesgue spaces $L^{q(\cdot)}(\mathbb{R}^N)$ which are given by

$$L^{q(\cdot)}(\mathbb{R}^N) = \left\{ u \text{ is a real measurable function on } \mathbb{R}^N \,\middle|\, \int_{\mathbb{R}^N} |u(x)|^{q(x)} dx < \infty \right\}$$

are Banach spaces with the following norm:

$$\|u\|_{q(\cdot)} = \inf \left\{ \lambda > 0 \,\middle|\, \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{q(x)} dx \leq 1 \right\}.$$

As an application we prove the existence of a solution of the following nonlinear elliptic equation (P) when $q(x) \searrow p$ as $|x| \to \infty$.

$$(P) \begin{cases} -\Delta_p u(x) + u(x)^{p-1} = u(x)^{q(x)-1}, \quad u(x) \geq 0, \quad x \in \mathbb{R}^N, \\ u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N), \end{cases}$$

where $\Delta_p u(x) = \mathrm{div}\,(|\nabla u(x)|^{p-2} \nabla u(x))$ is $p-$Laplacian.

2. Compactness and non-compactness of the embedding

Sobolev-type embeddings have been studied by many researchers so far. It is well-known that the embedding from $W^{1,p}(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ is continuous if $q \in [p, p^*]$, where $p^* := \frac{Np}{N-p}$. In addition, this embedding is not compact for any $q \in [p, p^*]$ due to the translation invariance on $\mathbb{R}^N$. However, the embedding from radial Sobolev spaces $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ is compact for $q \in (p, p^*)$ (see [15] ($p = 2$ case), [12] ($p \neq 2$ case)). Note that even radial Sobolev spaces $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, it is not compact if $q = p$ or p^*. Especially, in the case $q = p$, the cause of non-compactness is "vanishing" phenomenon. More precisely, if we consider the following scaling:

$$u_\lambda(x) = \lambda^{\frac{N}{p}} u(\lambda x) \quad \text{for } \lambda > 0 \text{ and } x \in \mathbb{R}^N,$$

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then we see that

$$\|u_\lambda\|_{L^p(\mathbb{R}^N)} = \|u\|_{L^p(\mathbb{R}^N)}, \quad \|\nabla u_\lambda\|_{L^p(\mathbb{R}^N)} = \lambda\|\nabla u\|_{L^p(\mathbb{R}^N)} \to 0 \text{ as } \lambda \to 0.$$

Due to $L^p(\mathbb{R}^N)$ norm preservation, the embedding $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$ is not compact.

The generalization of this embedding to variable exponent Lebesgue spaces is considered in [10]. They showed that if $p < \operatorname{ess\,inf}_{\mathbb{R}^N} q(x) \le \operatorname{ess\,sup}_{\mathbb{R}^N} q(x) < p^*$, then the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(\cdot)}(\mathbb{R}^N)$ is compact. Note that the critical case, that is $\operatorname{ess\,inf}_{\mathbb{R}^N} q(x) = p$ or $\operatorname{ess\,sup}_{\mathbb{R}^N} q(x) = p^*$, is not treated in [10]. However, in a bounded domain Ω, Kurata and Shioji [11] treat the critical case $\operatorname{ess\,sup}_{\Omega} q(x) = p^*$. More precisely, they showed that if there exist $x_0 \in \Omega, C_0 > 0, \eta > 0$, and $0 < \ell < 1$ such that $\operatorname{ess\,sup}_{\Omega\setminus B_\eta(x_0)} q(x) < p^*$ and

$$q(x) \le p^* - \frac{C_0}{|\log|x - x_0||^\ell} \quad \text{for a.e. } x \in \Omega \cap B_\eta(x_0),$$

then the embedding from $W^{1,p}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is compact. Conversely, if

$$q(x) \ge p^* - \frac{C_0}{|\log|x - x_0||} \quad \text{for a.e. } x \in \Omega \cap B_\eta(x_0),$$

then the embedding from $W^{1,p}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is not compact. For more generalized Sobolev spaces $W^{k,p(\cdot)}(\Omega)$, see [13].

In this talk, we shall treat the other critical case, that is $\operatorname{ess\,inf}_{\mathbb{R}^N} q(x) = p$. Our first purpose is to obtain a sufficiently condition of compactness and non-compactness of the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(\cdot)}(\mathbb{R}^N)$. We observe that the decay speed of $q(\cdot)$ at infinity decides whether the embedding is compact or not. Our results are as follows.

Theorem 1 (Non-compactness) If there exist positive constants R, C and a open set Γ in $\mathbb{S}^{N-1}$ such that

$$q(x) \le p + \frac{C}{|\log|x||} \quad \text{for } x \in (R, +\infty) \times \Gamma, \tag{2.1}$$

then the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(\cdot)}(\mathbb{R}^N)$ is not compact.

Theorem 2 (Compactness) If there exist positive constants R_1, R_2, C_1, C_2, and $\ell_1, \ell_2 \in (0, 1)$ such that

$$q(x) \le p^* - \frac{C_1}{|\log|x||^{\ell_1}} \quad \text{for } x \in B_{R_1}(0), \tag{2.2}$$

$$q(x) \ge p + \frac{C_2}{|\log|x||^{\ell_2}} \quad \text{for } x \in \mathbb{R}^N \setminus B_{R_2}(0), \tag{2.3}$$

then the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(\cdot)}(\mathbb{R}^N)$ is compact.

Remark 3 In Theorem 2, we do not need the constraint $p \le q(x) \le p^*$ for all $x \in \mathbb{R}^N$. Namely $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \subset L^{q(\cdot)}(\mathbb{R}^N)$ holds true whenever $q(x)$ satisfies $q(x) \le p^*$ in B_{R_1} and $q(x) \ge p$ in $\mathbb{R}^N \setminus B_{R_2}$.

3. Existence of a weak solution of (P)

As an application of Theorem 2, we discuss the existence of a weak solution of (P) under the hypotheses (2.2), (2.3) in Theorem 2.

$p(x)$-Laplace type elliptic equation on $\mathbb{R}^N$ is studied by many researchers so far in several subjects: multiplicity of solutions (see e.g. [1], [9]), existence of solutions of equations involving several nonlinearities (see e.g. [2], [8]) or equations under periodic assumptions (see e.g. [7], [17]) and so on. Especially, existence of solutions of (P) involving variable Sobolev critical exponent, that is $\operatorname{ess\,sup}_{\mathbb{R}^N} q(x) = p^* := \frac{Np}{N-p}$, is studied by [4] and [14]. On the other hand, the case where $\operatorname{ess\,inf}_{\mathbb{R}^N} q(x) = p$ is also critical case in (P). However even for p-Laplace equation there are no other results in the case where $\operatorname{ess\,inf}_{\mathbb{R}^N} q(x) = p$. The important setting in our research is $\operatorname{ess\,inf}_{\mathbb{R}^N} q(x) = p$.

First, set energy functional E from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $\mathbb{R}$ as

$$E(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) - \int_{\mathbb{R}^N} \frac{1}{q(x)} u_+^{q(x)} dx. \tag{3.1}$$

Then we see that $E \in C^1(W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N), \mathbb{R})$. And also we see that $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ is a solution of (P) in the sense of (3.2) if and only if u is a critical point of E, that is $E'(u) = 0$ in the dual space $\left(W^{1,p}_{rad}(\mathbb{R}^N)\right)^*$. Therefore we shall find a critical point of E.

Mountain pass method which has been introduced by Ambrosetti and Rabinowitz [3] is useful to show existence of a critical point of E (or existence of a weak solution of (P)). In order to use mountain pass method, we need to check the following two conditions (I), (II).

(I) Does E satisfy **the mountain pass geometry** ? Namely, does E satisfy (i)-(iii) ?

$$\begin{cases} (i)\ E(0) = 0. \\ (ii)\ \text{There exist}\, \rho > 0,\ \alpha > 0\, \text{such that}\, E(u) \geq \alpha\, \text{for any}\, u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)\, \text{with}\, \|u\|_{W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)} = \rho. \\ (iii)\ \text{There exists}\, u_1 \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)\, \text{such that}\, \|u_1\|_{W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)} \geq \rho\, \text{and}\, E(u_1) < \alpha. \end{cases}$$

(II) Does E satisfy **Palais-Smale condition** at some level $\beta \in \mathbb{R}$? (briefly, we say (P.-S.)$_\beta$ condition) Namely, does E satisfy that if $E(u_m) \to \beta$ and $E'(u_m) \to 0$ as $m \to \infty$ (briefly, we say (P.-S.)$_\beta$ sequence), then $\{u_m\}_{m\in\mathbb{N}}$ has a strongly convergent subsequence in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$?

However the critical case $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} q(x) = p$ causes some difficulties to apply mountain pass method to (P). More precisely, it is difficult to show the condition $(I)(ii)$ since we can not directly apply the fibering map method, which is known a standard method to show the mountain pass geometry, due to the variable critical exponent $q(\cdot)$. Furthermore, even though it is clear that bounded (P.-S.) sequence has a strongly convergence subsequence from Theorem 2, it is not trivial to show the boundedness of all (P.-S.) sequence.

To overcome these difficulties, we introduce the condition (C) (see Section 4) defined in [6] or [5] instead of the (P.-S.) condition in Section 4. Our result is as follows.

Theorem 4 Assume that $q(x)$ satisfies $\operatorname{ess\,inf}_{x\in B_R} q(x) > p$ and the hypotheses (2.2), (2.3) in Theorem 2. Then there exists a nontrivial weak solution $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ of (P) in the sense of

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u \nabla\phi + u^{p-1}\phi - u^{q(x)-1}\phi\right) dx = 0 \quad \text{for any } \phi \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \tag{3.2}$$

Remark 5 If $q(x)$ is radially symmetric satisfying the hypotheses of Theorem 4, the weak form (3.2) holds true even for non-radial solution $\phi \in W^{1,p}(\mathbb{R}^N)$. Indeed, since u and $q(x)$ are radially symmetric, it follows that for all $\phi \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$

$$\int_0^\infty \left(|u'(r)|^{p-2}u'(r)\phi'(r) + u^{p-1}\phi - u^{q(r)-1}\phi\right) r^{N-1}dr = 0,$$

where $r = |x|$. If for any $\psi \in C_c^\infty(\mathbb{R}^N)$ we consider the radial function $\Psi(r) = \int_{\omega\in\mathbb{S}^{N-1}} \psi(r\omega)\, dS_\omega$, then we have

$$\begin{aligned}&\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u\nabla\psi + u^{p-1}\psi - u^{q(x)-1}\psi\right) dx\\ &= \int_0^\infty \left(|u'(r)|^{p-2}u'(r)\Psi'(r) + u^{p-1}\Psi - u^{q(r)-1}\Psi\right) r^{N-1}dr = 0.\end{aligned}$$

Therefore we see that u satisfies (3.2) even for non-radial functions ϕ.

4. Sketch of the proof

In this section, we show Theorems in Section 2 and Section 3. First we give the proof of Theorem 1. After that, we state outline of the proofs of Theorem 2 and Theorem 4.

Proof of Theorem 1

We shall show Theorem 1 in the same way as [11]. Set $r(x) = q(x) - p$ for $x \in \mathbb{R}^N$. Let $\phi \in C_c^\infty(\mathbb{R}^N)$ be a radial function satisfying $\phi \equiv 1$ on $B_{\frac{1}{2}}$ and $\mathrm{supp}\phi \subset B_1$. For $m \in \mathbb{N}$, we define $\phi_m(x) = m^{-\frac{N}{p}}\phi(\frac{x}{m})$. Then for any $m \in \mathbb{N}$ we obtain

$$\|\phi_m\|_{L^p(\mathbb{R}^N)} = \|\phi\|_{L^p(B_1)}, \quad \|\nabla\phi_m\|_{L^p(\mathbb{R}^N)} = m^{-1}\|\nabla\phi\|_{L^p(B_1)}.$$

Since $\{\phi_m\}_{m=1}^\infty$ is a bounded sequence in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ and $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ is reflexive Banach space, there exist a weakly convergent subsequence $\{\phi_{m_j}\}_{j=1}^\infty$ and $\phi_\infty \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that $\phi_{m_j} \rightharpoonup \phi_\infty$ in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ as $j \to \infty$. By compactness of the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^r(\mathbb{R}^N)$ for $p < r < p^*$, we have $\phi_{m_j} \to \phi_\infty$ in $L^r(\mathbb{R}^N)$ and $\phi_{m_j} \to \phi_\infty$ a.e. in $\mathbb{R}^N$ which yields that $\phi_\infty \equiv 0$. On the other hand, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\phi_m(x)|^{q(x)}\, dx &= \int_{B_m} m^{-\frac{N}{p}(p+r(x))} \left|\phi\left(\frac{x}{m}\right)\right|^{q(x)} dx\\ &= \int_{B_1} m^{-\frac{N}{p}r(my)}|\phi(y)|^{q(my)}\, dy\\ &\geq \int_{B_{\frac{1}{2}}\setminus B_{\frac{1}{4}}} m^{-\frac{N}{p}r(my)}\, dy.\end{aligned}$$

Since Γ is open in $\mathbb{S}^{N-1}$, there exists a smooth domain $D \subset \mathbb{S}^{N-1}$ such that $D \subset \Gamma$. By using the polar coordinates as $y = s\omega$ $(s > 0, \omega \in \mathbb{S}^{N-1})$ we obtain

$$\int_{\mathbb{R}^N} |\phi_m(x)|^{q(x)}\, dx \geq \int_{s=\frac{1}{4}}^{\frac{1}{2}} \int_{\omega\in D} m^{-\frac{N}{p}r(ms\omega)}\, s^{N-1} ds dS_\omega.$$

By the assumption (2.1), we obtain $r(ms\omega) \le C_0 |\log ms|^{-1}$ for large m, $s \in (1/4, 1/2)$, and $\omega \in D \subset \Gamma$. Moreover for $s \in (1/4, 1/2)$ and large m, it holds $\log ms = \log m + \log s \ge \frac{1}{2}\log m$ which yields that

$$r(ms\omega) \le \frac{2C_0}{\log m}.$$

Therefore we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |\phi_m(x)|^{q(x)}\,dx &\ge \int_{s=\frac{1}{4}}^{\frac{1}{2}} \int_{\omega\in D} e^{-\frac{N}{p}\log m \frac{2C_0}{\log m}}\, s^{N-1} ds dS_\omega \\ &= \mathcal{H}^{N-1}(D)\, e^{-\frac{2C_0 N}{p}}\, \frac{2^{-N} - 4^{-N}}{N} > 0\end{aligned}$$

for large m, where $\mathcal{H}^d$ is the $d-$dimensional Hausdorff measure. Thus, if we assume the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ is compact, then we have $\int_{\mathbb{R}^N} |\phi_\infty|^{q(x)}\,dx > 0$ which contradicts $\phi_\infty \equiv 0$. Hence the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(\cdot)}(\mathbb{R}^N)$ is not compact. □

Outline of the proof of Theorem 2

It is enough to show that if $u_m \rightharpoonup 0$ in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ as $m \to \infty$, then

$$\int_{\mathbb{R}^N} |u_m(x)|^{q(x)}\,dx \to 0 \quad \text{as } m \to \infty. \tag{4.1}$$

We divide $\int_{\mathbb{R}^N} |u_m(x)|^{q(x)} dx$ into three terms as follows:

$$\begin{aligned}\int_{\mathbb{R}^N} |u_m(x)|^{q(x)} dx &= \int_{B_{R_1}} |u_m(x)|^{q(x)} dx + \int_{B_R\setminus B_{R_1}} |u_m(x)|^{q(x)} dx + \int_{\mathbb{R}^N\setminus B_R} |u_m(x)|^{q(x)} dx \\ &=: I_1(m) + I_2(m,R) + I_3(m,R),\end{aligned} \tag{4.2}$$

where R is sufficiently larger than R_2.

Firstly, by the Kurata-Shioji's result [11] we have

$$I_1(m) = o(1) \text{ as } m \to \infty. \tag{4.3}$$

Next, for $I_2(m,R)$ we have

$$I_2(m,R) = o(1) \text{ as } m \to \infty \text{ for fixed } R > 0. \tag{4.4}$$

Finally we can obtain the following estimate of $I_3(m,R)$.

Proposition 6

$$I_3(m,R) = o(1) \text{ uniformly in } m \text{ as } R \to \infty. \tag{4.5}$$

Proposition 6 is shown by the following pointwise estimate (4.6) by [15] ($p = 2$) and [12].

Lemma 7 ([15], [12]) For any $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$ we have

$$|u(x)| \le \left(\frac{p}{\omega_{N-1}}\right)^{\frac{1}{p}} |x|^{-\frac{N-1}{p}} \|u\|_{L^p(\mathbb{R}^N)}^{\frac{p-1}{p}} \|\nabla u\|_{L^p(\mathbb{R}^N)}^{\frac{1}{p}}. \tag{4.6}$$

From (4.2), (4.3), (4.4) and (4.5) we obtain (4.1). □

Outline of the proof of Theorem 4

Cerami [6] and Bartolo-Benci-Fortunato [5] have proposed a variant of (P.-S.) condition. In this paper, we use the condition (C) introduced by [6] and [5] and the mountain pass theorem under the condition (C) (Theorem 9). Let V be a real Banach space and $E \in C^1(V, \mathbb{R})$. First, we define the condition (C) based on [6] and [5].

Definition 8 ([6], [5] Definition 1.1.) We say that E satisfies the condition (C) in (c_1, c_2), $(-\infty \leq c_1 < c_2 \leq +\infty)$, if

(i) every bounded sequence $\{u_k\} \subset E^{-1}((c_1, c_2))$, for which $\{E(u_k)\}$ is bounded and $E'(u_k) \to 0$, possesses a convergent subsequence, and

(ii) for any $c \in (c_1, c_2)$ there exist $\sigma, \rho, \alpha > 0$ such that $[c - \sigma, c + \sigma] \subset (c_1, c_2)$ and for any $u \in E^{-1}([c - \sigma, c + \sigma])$ with $\|u\| \geq \rho, \|E'(u)\|_* \|u\| \geq \alpha$.

Theorem 9 (Mountain pass theorem under the condition (C)) Let E satisfy the condition (C) in $(0, +\infty)$. Assume that

(i) $E(0) = 0$

(ii) There exist $\rho > 0,\ \alpha > 0$ such that $E(u) \geq \alpha$ for any $u \in V$ with $\|u\| = \rho$.

(iii) There exists $u_1 \in V$ such that $\|u_1\| \geq \rho$ and $E(u_1) < \alpha$.

Define

$$P = \{\, p \in C([0, 1], V) \mid p(0) = 0,\ p(1) = u_1 \,\}.$$

Then

$$\beta = \inf_{p \in P} \sup_{0 \leq t \leq 1} E(p(t)) \geq \alpha$$

is a critical value, that is there exists a critical point $u_0 \in V$ such that $E(u_0) = \beta$.

Note that Theorem 9 can be shown in the same way as the proof of Theorem 6.1 in p.109 in [16] by substituting the deformation theorem under the condition (C) for Theorem 3.4 in p.83 in [16]. We omit detail. From now on, let $V = W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ and E be defined by (3.1). We show the following two propositions.

Proposition 10 Assume that $q(x)$ satisfies the hypotheses (2.2), (2.3) in Theorem 2 and $\operatorname{ess\,inf}_{x \in B_R} q(x) > p$. Then J satisfies the condition (C) on $\mathbb{R}$.

Proposition 11 Assume that $q(x)$ satisfies the hypotheses (2.2), (2.3) in Theorem 2 and $\operatorname{ess\,inf}_{x \in B_R} q(x) > p$. Then J has the mountain pass geometry, that is J satisfies (i), (ii) and (iii) in Theorem 9.

From Proposition 10, Proposition 11, and Theorem 9, we can show the existence of a nontrivial critical point $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ which is a weak solution of (P). □

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE WAVE EQUATION WITH SPACE-DEPENDENT DAMPING

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This talk is based on a joint work with Motohiro Sobajima (Tokyo University of Science).

1. Introduction

Let $N \geq 2$ and let Ω be an exterior domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. We assume that $0 \notin \bar{\Omega}$ without loss of generality. We study the initial-boundary value problem for the damped wave equation

$$\begin{cases} u_{tt} - \Delta u + a(x)u_t = 0, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here $u = u(x,t)$ is a real-valued unknown function. We assume that $a(x) \in C^2(\bar{\Omega})$, $a(x) > 0$ on $\bar{\Omega}$. The term $a(x)u_t$ describes the damping effect, which plays a role in reducing the energy of the wave. The initial data (u_0, u_1) is compactly supported, let us say $\operatorname{supp}(u_0, u_1) \subset \{x \in \Omega; |x| \leq R_0\}$ with some $R_0 > 0$, and satisfies the compatibility condition of order 1, namely,

$$(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega). \tag{1.2}$$

Then, it is well known that there exists a unique strong solution

$$u \in \bigcap_{i=0}^{2} C^i([0,\infty); H^{2-i}(\Omega)) \tag{1.3}$$

(see Ikawa [3, Theorem 2.25]). Also, the solution u enjoys the finite propagation property, namely, $\operatorname{supp} u(\cdot,t) \subset \{x \in \Omega; |x| \leq R_0 + t\}$.

Our purpose is to study how the damping term $a(x)u_t$ affects the behavior of the solution as time tends to infinity. As a typical case, we assume that there exist constants $\alpha \in \mathbb{R}$ and $a_0 > 0$ such that

$$\lim_{R\to\infty} \sup_{|x|\geq R} ||x|^\alpha a(x) - a_0| = 0. \tag{1.4}$$

Roughly speaking, this assumption means $a(x) \sim a_0|x|^{-\alpha}$ as $|x| \to \infty$.

The total energy of the solution u is given by

$$E(t;u) = \frac{1}{2}\int_\Omega (u_t^2 + |\nabla u|^2)\, dx.$$

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By differentiating it and using the equation (1.1), we easily obtain the energy identity

$$E(t;u) + \int_0^t \int_\Omega a(x) u_t(x,s)^2 \, dxds = E(0;u).$$

Since $a(x)$ is positive, we see that $E(t;u)$ is monotone decreasing with respect to t.

Then, it is natural to ask whether the total energy decays to zero. Mochizuki [13] considered the case $\Omega = \mathbb{R}^N$ with $N \neq 2$ and proved that if $\alpha > 1$, then there exists initial data (u_0, u_1) such that for the associated solution u, the total energy $E(t;u)$ does not decay to zero. Moreover, there exists a solution w of the free wave equation

$$w_{tt} - \Delta w = 0$$

such that $\lim_{t\to\infty} E(t; u - w) = 0$ (see [14] for further improvements).

On the other hand, Matsumura [12] studied the case $\alpha \leq 1$ and proved that the total energy decays to zero (see [15, 14] for further improvements). Therefore, for the energy decay, the threshold is given by $\alpha = 1$.

Concerning the asymptotic profile of the solution, the diffusion phenomena for the classical damped wave equation

$$u_{tt} - \Delta u + u_t = 0$$

has been studied by [11, 29, 7, 17, 10, 2, 16, 1, 26]. The diffusion phenomena means that the solution u is approximated by a solution of the heat equation

$$v_t - \Delta v = 0$$

as $t \to \infty$.

The main purpose of this talk is to show the diffusion phenomena when the damping depends on the space variables. In particular, we consider the case $\alpha < 1$ (recall that when $\alpha > 1$, the solution behaves like that of the wave equation by the Mochizuki's result mentioned above). To this end, we consider the corresponding parabolic problem

$$\begin{cases} v_t - a(x)^{-1}\Delta v = 0, & x \in \Omega, t > 0, \\ v(x,t) = 0, & x \in \partial\Omega, t > 0, \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{1.5}$$

This equation is formally obtained by dropping the term u_{tt} from the equation (1.1) and dividing it by $a(x)$.

Our main result reads as follows:

Theorem 1.1 ([22, 23, 24])**.** *We assume that a satisfies* (1.4) *with some $\alpha < 1$ and $a_0 > 0$. Let u be the solution to* (1.1) *and let v be the solution to* (1.5) *with $v_0(x) = u_0(x) + a(x)^{-1}u_1(x)$. Then, for any $\delta > 0$, there exists $C = C(N, \alpha, a_0, R_0, \delta) > 0$ such that we have for $t \geq 1$*

$$\left\| \sqrt{a(\cdot)}(u(\cdot,t) - v(\cdot,t)) \right\|_{L^2} \leq C(1+t)^{-\frac{N-\alpha}{2(2-\alpha)} - \frac{1-\alpha}{2-\alpha} + \delta} \left\| (u_0, u_1) \right\|_{H^2 \times H^1}. \tag{1.6}$$

As a byproduct of Theorem 1.1, we obtain the almost optimal L^2-estimate for the solution.

Corollary 1.2. *Under the assumption of Theorem 1.1, the solution u of* (1.1) *satisfies*

$$\left\| \sqrt{a(\cdot)} u(\cdot, t) \right\|_{L^2} \le C(1+t)^{-\frac{N-\alpha}{2(2-\alpha)}+\delta} \left\| (u_0, u_1) \right\|_{H^2 \times H^1},$$

where $\delta > 0$ is an arbitrary small number and $C = C(N, \alpha, a_0, R_0, \delta) > 0$. When $N = 2$ or $\alpha \in [0, 1)$, we may take $\delta = 0$.

Here, we give several remarks. Theorem 1.1 when $\Omega = \mathbb{R}^N$ and $a(x) = (1 + |x|^2)^{-\alpha/2}$ has been proved by [28]. Also, Corollary 1.2 when $\Omega = \mathbb{R}^N$ and for radially symmetric $a(x)$ has been proved by [27, 21, 8].

Our approach requires the compactness of the support of the initial data. Recently, in [25], for $a(x) = |x|^{-\alpha}$ with $\alpha \in [0, 1)$, we could remove the compactness assumption on the initial data and proved several sharp energy decay estimates for polynomially decaying initial data. For general unbounded damping coefficients, recently, Ikehata and Takeda [5] showed the existence of a unique solution without the compactness assumption on the initial data and obtained uniform energy decay estimates.

2. Strategy of the proof

Our strategy for the proof of Theorem 1.1 is the following. First, we treat the term u_{tt} as a perturbation and write the equation (1.1) as

$$u_t - a(x)^{-1}\Delta u = -a(x)^{-1} u_{tt}.$$

This is natural because we expect the diffusion phenomena, and for the solution v to the parabolic problem (1.5), the term v_{tt} decays faster than v_t and $a(x)^{-1}\Delta v$. Then, by Duhamel principle, the above equation can be formally transformed into the integral equation

$$u(t) = e^{ta(x)^{-1}\Delta} u_0 - \int_0^t e^{(t-s)a(x)^{-1}\Delta} [a(x)^{-1} u_{ss}(s)]\, ds.$$

We further apply the integration by parts to obtain

$$\begin{aligned} u(t) - e^{ta(x)^{-1}\Delta}[u_0 + a(x)^{-1}u_1] &= -\int_{t/2}^t e^{(t-s)a(x)^{-1}\Delta}[a(x)^{-1}u_{tt}(s)]\, ds \\ &\quad - e^{\frac{t}{2}a(x)^{-1}\Delta}[a(x)^{-1}u_t(t/2)] \\ &\quad - \int_0^{t/2} a(x)^{-1}\Delta e^{(t-s)a(x)^{-1}\Delta}[a(x)^{-1}u_t(s)]\, ds. \end{aligned} \tag{2.1}$$

By putting $v(t) = e^{ta(x)^{-1}\Delta}[u_0 + a(x)^{-1}u_1]$, it suffices to estimate the each term in the right-hand side.

To this end, we first investigate the heat semigroup $e^{ta(x)^{-1}\Delta}$. By the Beurling-Deny criterion and the Gagliardo-Nirenberg inequality, we derive the following optimal estimate for the semigroup $e^{ta(x)^{-1}\Delta}$. To state the results, we put $d\mu = a(x)dx$ and define the weighted Lebesgue spaces

$$L^q_{d\mu}(\Omega) := \left\{ f \in L^q_{\mathrm{loc}}(\Omega)\,;\, \|f\|_{L^q_{d\mu}} = \left(\int_\Omega |f(x)|^q a(x)\, dx \right)^{1/q} < \infty \right\}.$$

Proposition 2.1. *When $\alpha \in [0,1)$, we have*

$$\|e^{ta(x)^{-1}\Delta}v_0\|_{L^2_{d\mu}} \le Ct^{-\frac{N-\alpha}{2-\alpha}\left(\frac{1}{q}-\frac{1}{2}\right)}\|v_0\|_{L^q_{d\mu}},$$

where $q \in [1,2]$. When $\alpha < 0$, we have

$$\|e^{ta(x)^{-1}\Delta}v_0\|_{L^2_{d\mu}} \le Ct^{-(N/2)(1/q-1/2)}\left(\int_\Omega |v_0(x)|^q |x|^{-\alpha(N-2)(2-q)/4}\,d\mu\right)^{1/q},$$

where $q \in [1,2]$ $(N=2)$, $q \in (p'_\alpha, 2]$ $(N \ge 3)$ with $p_\alpha = 2N(2-\alpha)/(-\alpha(N-2))$.

Applying the above estimate to the right-hand side of (2.1), we see that it suffices to estimate the terms including $a(x)^{-1}u_{tt}(x,s), a(x)^{-1}u_t(x,t/2), a(x)^{-1}u_t(x,s)$. For this, we apply a weighted energy method developed by Lions and Masmoudi [9], Ikehata [4] and Todorova and Yordanov [27]. Namely, we consider the weighted energy

$$\int_\Omega (|\nabla u|^2 + u_t^2)\Phi(x,t)\,dx$$

with a weight function having a form

$$\Phi(x,t) = \exp\left(\beta\frac{A(x)}{1+t}\right),$$

where $\beta \in \mathbb{R}$ and $A(x)$ is an appropriate function.

Todorova and Yordanov [27] pointed out that to obtain optimal energy estimates, the function $A(x)$ should satisfy the Poisson equation

$$\Delta A(x) = a(x) \tag{2.2}$$

and the conditions

$$A_1|x|^{2-\alpha} \le A(x) \le A_2|x|^{2-\alpha}, \tag{2.3}$$

$$\frac{|\nabla A(x)|^2}{a(x)A(x)} \le \frac{2-\alpha}{N-\alpha} + \varepsilon, \tag{2.4}$$

and construct an explicit solution of (2.2) satisfying (2.3) and (2.4) when $a(x)$ is radially symmetric. However, when $a(x)$ is not radially symmetric, in general the solution $A(x)$ of (2.2) does not satisfy (2.3), (2.4) (a counter example is given in [22]). To overcome this difficulty, we weaken the above equation to the inequality

$$(1-\varepsilon)a(x) \le \Delta A(x) \le (1+\varepsilon)a(x) \tag{2.5}$$

with a small parameter $\varepsilon > 0$.

Proposition 2.2. *If $a(x) \in C^2(\bar\Omega)$ is positive and satisfies* (1.4)*, then, for any $\varepsilon \in (0,1)$, there exists $A_\varepsilon \in C^2(\bar\Omega)$ satisfying* (2.5)*,* (2.3)*,* (2.4)*.*

Finally, using the function $A_\varepsilon(x)$ constructed above, we define the weight function

$$\Phi_\varepsilon(x,t) = \exp\left(\frac{1}{h+2\varepsilon}\frac{A_\varepsilon(x)}{1+t}\right),$$

where $\varepsilon \in (0,1)$ and $h = \frac{2-\alpha}{N-\alpha}$, and we introduce the following energy.

$$E_{\partial x}(t;u) := \int_\Omega |\nabla u|^2 \Phi_\varepsilon \, dx, \quad E_{\partial t}(t;u) := \int_\Omega |u_t|^2 \Phi_\varepsilon \, dx, \tag{2.6}$$

$$E_a(t;u) := \int_\Omega a(x)|u|^2 \Phi_\varepsilon \, dx, \quad E_*(t;u) := 2\int_\Omega u u_t \Phi_\varepsilon \, dx, \tag{2.7}$$

and $E_1(t;u) := E_{\partial x}(t;u) + E_{\partial t}(t;u)$ and $E_2(t;u) := E_*(t;u) + E_a(t;u)$.

Computing the time derivatives of $E_1(t;u)$ and $E_2(t;u)$, we have the following weighted energy estimates.

Proposition 2.3. *Assume that* (u_0,u_1) *satisfies* $\operatorname{supp}(u_0,u_1) \subset \{x \in \mathbb{R}^N; |x| \le R_0\}$ *and the compatibility condition of order* 1. *Let* u *be a solution of the problem* (1.1). *For every* $\delta > 0$ *and* $k = 0, 1$, *there exist* $\varepsilon > 0$ *and* $M_{\delta,k,R_0} > 0$ *such that for every* $t \ge 0$,

$$(1+t)^{\frac{N-\alpha}{2-\alpha}+2k+1-\delta}\Big(E_{\partial x}(t;\partial_t^k u) + E_{\partial t}(t;\partial_t^k u)\Big) + (1+t)^{\frac{N-\alpha}{2-\alpha}+2k-\delta} E_a(t;\partial_t^k u)$$
$$\le M_{\delta,k,R_0} \|(u_0,u_1)\|^2_{H^{k+1}\times H^k(\Omega)}.$$

Applying these energy estimates to the remainder terms of the right-hand side of (2.1), we can obtain the desired estimate for $u(t) - v(t)$.

3. An open problem

Finally, we suggest an open problem. When $\alpha = 1$, Ikehata, Todorova and Yordanov [6] showed almost optimal energy estimate. More precisely, under the assumption that

$$a_1(1+|x|^2)^{-1/2} \le a(x) \le a_2(1+|x|^2)^{-1/2},$$

they showed

$$E(t;u) = \begin{cases} O(t^{-a_1}) & (1 < a_1 < N), \\ O(t^{-N+\delta}) & (a_1 \ge N), \end{cases}$$

where $\delta > 0$ is an arbitrary small number. This indicates that the asymptotic behavior of the solution may change depending on the value of a_1. We expect that if a_1 is sufficiently large, then the diffusion phenomena still holds. However, if a_1 is small, we do not have any idea for the asymptotic profile.

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Concentration Phenomena of a Least-Energy Solution to a Semilinear Elliptic Problem in Piecewise Smooth Domains

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1 Introduction

In this paper we consider a singular perturbation problem to a semilinear Neumann problem

$$\begin{cases} \epsilon^2 \Delta u - u + u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain, $p > 1$ and $(N-2)p < N+2$ are satisfied, $\epsilon > 0$ is a constant, and ν is the outer unit normal vector on $\partial\Omega$. We are interested in the asymptotic behavior of solutions to (1.1).

Such a singular perturbation problem is originally considered in the stationary Keller-Segel model or the shadow system of the Gierer-Meinhardt model, and (1.1) is obtained as a reduced problem of those problems. In the case that Ω has a smooth boundary, various researchers investigated (1.1). Lin, Ni and Takagi [16] are pioneers researchers of the problem, and they proved the existence and some properties of least-energy solutions to (1.1) for sufficiently small $\epsilon > 0$. Here a least-energy solution is a solution which attains the least positive critical value of the associated energy functional with (1.1):

$$J_\epsilon(u) = \frac{1}{2}\int_\Omega (\epsilon^2 |\nabla u|^2 + u^2)\, dx - \frac{1}{p+1}\int_\Omega u^{p+1}\, dx \quad \text{for } u \in H^1(\Omega).$$

Next Ni and Takagi [19, 20] investigated the asymptotic behavior of a least-energy solution as $\epsilon \to 0$. They proved that, by singular perturbation, the point concentration phenomena of a least-energy solution occurs, that is, the solution concentrates at its maximum point $P_\epsilon \in \partial\Omega$. Furthermore there holds

$$H(P_\epsilon) \to \max_{P \in \partial\Omega} H(P) \quad \text{as } \epsilon \to 0,$$

where $H(P)$ is the mean curvature of $\partial\Omega$ at $P \in \partial\Omega$. Other many mathematicians also investigated concentration phenomena; e.g., Byeon [5], del Pino and Felmer [6] (concentration

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at a single point), Gui [10], Gui and Wei [11], Gui, Wei and Winter [12] (concentration at many points), Malchiodi and Montenegro [17, 18] (concentration on boundary). Kabeya and Ni [13] investigated the concentration phenomena of semilinear elliptic problems with exponential nonlinearity instead of power nonlinearity. On the other hand, under the Dirichlet boundary condition, concentration phenomena was also investigated by, e.g., [6], the author [14], Ni and Wei [21]. Then the maximum point moves to the most distant point from the boundary. Moreover, under the mixed boundary condition, Azorero, et al. [1, 2] obtained the asymptotic behavior of the energy of solutions, which is affected by the Dirichlet and Neumann parts of $\partial\Omega$, respectively.

In precedents results above, we assumed that $\partial\Omega$ is smooth. On the other hand, we are interested in the case that $\partial\Omega$ has *piecewise smoothness*, and we know less results on that case than results on the smooth boundary case. When Ω is a rectangle $(0,a)\times(0,b)\subset\mathbb{R}^2$ with constants $a>0$ and $b>0$, Shi [23] investigated the following problem

$$\begin{cases} \Delta u+\lambda f(u)=0 & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \nu}=0 & \text{on } \partial\Omega \end{cases}$$

in the view of the bifurcation theory. On the other hand, Dipierro [7, 8] invesigated (1.1) with the piecewise smooth $\partial\Omega$ and $(N-2)$ dimensional subset $\partial\Omega_0\subset\partial\Omega$ of non-smooth points. Namely they assumed that $\partial\Omega$ has a smooth edge $\partial\Omega_0$, and $H_0(P)$ denotes the open angle at $P\in\partial\Omega_0$. Then they constructed a solution which concentrates at $P\in\partial\Omega_0$ where $H_0(P)$ attains its strict local maximum or minimum.

Dipierro's result implies that concentration phenomena occurs on edges, that is, $(N-2)$ dimensional subset, and $H_0(P)$ plays a similar role to the mean curvature $H(P)$. On the other hand, our aim in this paper is to prove that concentration phenomena occurs at the *vertex* as well as the edge. Especially we focus our attention on a least-energy solution u_ϵ to (1.1), and we show that u_ϵ concentrates at the point having the least *angle* (the least *solid angle* if $N=3$) less than π $(N=2)$ or 2π $(N=3)$. Moreover we obain the asymptotic profile of u_ϵ as $\epsilon\to 0$.

In order to investigate the behavior of least-energy solutions by our method, we are required to use the regularity properties of the solutions. Hence we only assume the case $N=2,3$. If we assume higher dimensional case $N\geq 4$, then it seems difficult to show the regularity of solutions.

2 Main Result

Now we will express our main result. In our arguments in this paper, we can assume more general nonlinear term than that of the equation (1.1). Namely let $\Omega\subset\mathbb{R}^N$ $(N\geq 2)$ be a Lipschitz domain. In arguments below we consider the following problem

$$\begin{cases} \epsilon^2\Delta u-u+f(u)=0 & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \nu}=0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\epsilon>0$, and ν is the outer unit normal vector on $\partial\Omega$. Here $f(t)$ satisfies the following conditions:

(fi) $f(t) \in C^{1,l}(\mathbb{R})$ with $0 < l < 1$, $f(t) > 0$ $(t > 0)$ and $f(t) \equiv 0$ $(t \le 0)$;

($f$ii) $\lim_{t\to 0} f(t)/t = 0$ and $f(t)/t$ is increasing for $t > 0$;

($f$iii) $f(t) = O(t^p)$ as $t \to \infty$ with $p > 1$ and $(N-2)p < N+2$;

($f$iv) there exists a constant $\theta \in (0, 1/2)$ such that $F(t) \le \theta t f(t)$ for $t \ge 0$.

Since Ω is not smooth in our problem, we discuss about a weak solution $u \in H^1(\Omega)$ to (2.1), which satisfies

$$\int_\Omega (\nabla u \nabla \varphi + u\varphi - f(u)\varphi)\, dx = 0 \quad \text{for any } \varphi \in H^1(\Omega).$$

Moreover we focus our attention on a least-energy solution to (2.1), which attains the least positive critical value of

$$J_\epsilon(u) = \frac{1}{2}\int_\Omega (\epsilon^2 |\nabla u|^2 + u^2)\, dx - \int_\Omega F(u)\, dx \quad \text{for } u \in H^1(\Omega)$$

with $F(t) = \int_0^t f(s)\, ds$.

In our arguments we will show the asymptotic profile of a solution to (2.1), and then a *ground state solution* $w = w_0$ to the following entire space problem plays an important role, that is,

$$\begin{cases} \Delta w - w + f(w) = 0 & \text{in } \mathbb{R}^N, \\ w > 0 & \text{in } \mathbb{R}^N, \\ w(z) \to 0 & \text{as } |z| \to \infty, \\ w(0) = \max\limits_{z\in\mathbb{R}^N} w(z). \end{cases} \tag{2.2}$$

In order to precisely define a ground state solution to (2.2), we introduce the following energy functional

$$I(w) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla w|^2 + w^2)\, dz - \int_{\mathbb{R}^N} F(w)\, dz \quad \text{for } w \in H^1(\mathbb{R}^N).$$

Then we obtain the next proposition:

Proposition 2.1 *Under conditions* (fi)–($f$iv)*, there exists a solution* w_0 *to* (2.2) *such that*

(a) $w_0 > 0$ *in* $\mathbb{R}^N$ *and* $w_0 \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$;

(b) *for any solutions* $w \in H^1(\mathbb{R}^N)$ *to* (2.2)*, there holds* $0 < I(w_0) \le I(w)$;

(c) $w_0(z) = w_0(r)$ *with* $r = |z|$ *and* $w_0'(r) < 0$ *for* $r > 0$;

(d) $w_0(r), w_0'(r) \le C r^{-\frac{N-1}{2}} e^{-r}$ *for* $r > 0$ *with some constant* $C > 0$.

The solution w_0 found in Proposition 2.1 is said to be a ground state solution to (2.2). The existence of w_0 to (2.2) satisfying (a) and (b) is proved by Berestycki, Gallouët, Kavian [3] ($N = 2$) or Berestycki, Lions [4] ($N \ge 3$). Moreover, by Gidas, Ni and Nirenberg's result (cf., Theorem 2 in [9]), it is known that w_0 satisfies (c) and (d). Through arguments in this paper, we also assume the next condition:

$(f\mathrm{v})$ a ground state solution w_0 to (2.2) is unique.

Since w_0 is radially symmetric, $(f\mathrm{v})$ is equivalent to the uniqueness of the following ODE

$$\begin{cases} w'' + \dfrac{N-1}{r} w' + f(w) = 0 & \text{for } r > 0, \\ w > 0 & \text{for } r > 0, \\ w(r) \to 0 & \text{as } r \to \infty. \end{cases} \tag{2.3}$$

For example a solution to (2.3) is unique if $f(t) = t^p$ holds with $p > 1$ and $(N-2)p < N+2$ (e.g., Kwong [15]). Concerning more general cases, e.g., Pucci and Serrin [22].

Next we introduce some notations and assume the geometrical condition of the Lipschitz domain Ω. Let $x_0 \in \partial\Omega$. In some neighborhood of x_0, the boundary $\partial\Omega$ is expressed by a graph. If the graph is of class $C^{2,\gamma}$, then x_0 is said to be a *smooth point*. On the other hand, if not, then x_0 is said to be a *non-smooth point*. For two kinds of those points, we assume the following geometrical condition:

> If $x_0 \in \partial\Omega$ is a smooth point, then some neighborhood of x_0 only consists of smooth points. On the other hand, if $x_0 \in \partial\Omega$ is a non-smooth point, then there exists $R_* > 0$ such that the intersection $B_{R_*}(x_0) \cap \Omega$ is expressed by
>
> $$B_{R_*}(x_0) \cap \Omega = \{x \in \mathbb{R}^N \mid x - x_0 = r\sigma, 0 < r < R_*, \sigma \in \mathcal{S}(x_0)\},$$
>
> where $\mathcal{S}(x_0) \subset \mathbb{S}^{N-1}$ is a Lipschitz domain. Moreover, for any $x_1, x_2 \in B_{R_*}(x_0) \cap \Omega$, it holds that
>
> $$ax_1 + bx_2 \in \{x \in \mathbb{R}^N \mid x - x_0 = r\sigma, 0 < r < \infty, \sigma \in \mathcal{S}(x_0)\}$$
>
> for any $a, b > 0$ (BD)

Condition (BD) implies that a neighborhood of any non-smooth point is a *convex cone*. For example a convex polyhedron satisfies the condition (BD).

Under $(f\mathrm{i})$–$(f\mathrm{v})$ and (BD), we obtain the following theorem. Here, for a set $\mathcal{U} \subset \mathbb{R}^k$ $(k \le N)$, the Hausdorff measure of $\mathcal{U}$ is denoted by $\mathcal{H}_k(\mathcal{U})$.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Assume $N = 2, 3$, $(f\mathrm{i})$–$(f\mathrm{v})$, (BD) and the existence of a non-smooth point $P_0 \in \partial\Omega$ such that*

$$\mathcal{H}_{N-1}(\mathcal{S}(P_0)) = \min_{P \in \partial\Omega} \mathcal{H}_{N-1}(\mathcal{S}(P)) < \frac{1}{2}\mathcal{H}_{N-1}(\mathbb{S}^{N-1}). \tag{2.4}$$

Then the following statements hold:

(i) *For sufficiently small $\epsilon > 0$, there exists a least-energy solution $u_\epsilon \in H^2(\Omega)$. Moreover $u_\epsilon \in C^2(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ to (1.1) with some $0 < \alpha < 1$, $u_\epsilon > 0$ in Ω and it holds that*

$$J_\epsilon(u_\epsilon) = \epsilon^N \left\{ \frac{\mathcal{H}_{N-1}(\mathcal{S}(P_0))}{\mathcal{H}_{N-1}(\mathbb{S}^{N-1})} I(w_0) + o(1) \right\} \quad \epsilon \to 0. \tag{2.5}$$

(ii) *Let x_ϵ be a maximum point of u_ϵ. Then, for any $\epsilon_j \to 0$ $(j \to \infty)$, there exists a subsequence $\{\epsilon_{j_k}\}$ and a point $P_1 \in \partial\Omega$ such that $\mathcal{H}_{N-1}(\mathcal{S}(P_1)) = \mathcal{H}_{N-1}(\mathcal{S}(P_0))$ and*

$x_{\epsilon_{j_k}} \to P_1$ *as* $k \to \infty$. *Moreover, for sufficiently small* $R > 0$, *it holds that*

$$\begin{aligned} &\frac{1}{2}\int_{B_R(P_1)\cap\Omega} (\epsilon_{j_k}^2 |\nabla u_{\epsilon_{j_k}}|^2 + u_{\epsilon_{j_k}}^2)\ dx - \int_{B_R(P_1)\cap\Omega} F(u_{\epsilon_{j_k}})\ dx \\ &= J_{\epsilon_{j_k}}(u_{\epsilon_{j_k}}) + o\left(\epsilon_{j_k}^N\right) \quad (k \to \infty), \end{aligned} \tag{2.6}$$

and, for any $R_0 > 0$,

$$\left\| u_{\epsilon_{j_k}}(x) - w_0\left(\frac{x - P_1}{\epsilon_{j_k}}\right)\right\|_{C^{0,\alpha}\left(B_{\epsilon_{j_k}R_0}(P_1)\cap\Omega\right)} \to 0 \quad (k \to \infty). \tag{2.7}$$

For a least-energy solution v_ϵ concentrating at a smooth point, it is known (cf. [19]) that v_ϵ satisfies

$$J_\epsilon(v_\epsilon) = \epsilon^2 \left\{ \frac{1}{2} I(w_0) + o(1) \right\} \quad \text{as } \epsilon \to 0. \tag{2.8}$$

Hence if $\mathcal{H}_{N-1}(\mathcal{S}(P_0)) < 2^{-1}\mathcal{H}_{N-1}(\mathbb{S}^{N-1})$, then the condition (2.4) guarantees that u_ϵ concentrates at a non-smooth point (compare (2.5) with (2.8)). On the other hand, if $\mathcal{H}_{N-1}(\mathcal{S}(P_0)) > 2^{-1}\mathcal{H}_{N-1}(\mathbb{S}^{N-1})$ for any non-smooth point P_0, then u_ϵ does not concentrates at any non-smooth point (the concentration phenomena of u_ϵ occurs at a smooth point on $\partial\Omega$). Moreover (2.6) implies that the energy concentration occurs, and (2.7) implies that the asymptotic profile of u_ϵ is characterized by the ground state solution w_0 to (2.2).

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Analysis of unidirectional diffusion equation and its gradient flow structure

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Abstract

We study a nonlinear diffusion equation with irreversibility condition: $u_t = (\Delta u + f)_+$ in a bounded domain of $\mathbf{R}^n$ with Dirichlet or mixed boundary condition. Under some suitable conditions, we prove the unique existence of a strong solution and show its gradient structure, comparison principle, and long time behaviour of the solution. The construction of the strong solution is done through the backward Euler time discretization by using a regularity estimate of the solution of the classical obstacle problem. This talk is based on a joint work with Goro Akagi (Tohoku University).

1 Problem and main results

In this talk, we consider a strong solution of the following irreversible diffusion equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\Gamma = \partial\Omega$. We fix $T > 0$ and define $Q := \Omega \times (0, T)$.

$$\begin{cases} u_t = \big(\Delta u + f(x,t)\big)_+ & (x,t) \in Q := \Omega \times (0,T), \\ u(x,t) = 0 & (x,t) \in \Gamma \times (0,T), \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases} \tag{1.1}$$

where $(a)_+ := \max(a, 0)$.

Definition 1.1 (strong solution)**.** Let $f \in L^2(Q)$, $u_0 \in L^2(\Omega)$. Then u is called a strong solution of (1.1) iff

1. $u \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$
2. $u_t = \big(\Delta u + f\big)_+$ $\mathcal{H}^{n+1}$-a.e. in Q
3. $u(0,\cdot) = u_0 \in L^2(\Omega)$

We remark that it is not easy to define a notion of the weak solution (i.e. H^1 solution) in a standard way due to the strong nonlinearity of the sign function $(\cdot)_+$. On the other hand, it is possible to consider a viscosity solution since it is a degenerate parabolic equation. However, it is difficult to connect the viscosity solution and the gradient structure as described in Theorem 1.7, due to the lack of differentiability of the viscosity solution in general.

Theorem 1.2 (complementarity form). *u is a strong solution of* (1.1) *iff*

1. $u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$,
2. $\partial_t u \geq 0$ *a.e. in* Q,
3. $\partial_t u - \Delta u - f \geq 0$ *a.e. in* Q,
4. $(\partial_t u - \Delta u - f)\,\partial_t u = 0$ *a.e. in* Q,
5. $u(0,\cdot) = u_0$.

Theorem 1.3 (uniqueness). *A strong solution of* (1.1) *is unique, if it exists.*

Theorem 1.4 (existence). *We suppose $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $f \in L^2(Q)$. If there exists $f^* \in L^2(\Omega)$ with $f(x,t) \leq f^*(x)$ a.e. in Q, then there is a strong solution of* (1.1).

Theorem 1.5 (comparison principle). *Let u^i $(i=1,2)$ be a strong solution of* (1.1) *with $u_0 = u_0^i \in H^2(\Omega)\cap H_0^1(\Omega)$, $f = f^i \in L^2(Q)$, respectively. We suppose that there exists $f^* \in L^2(\Omega)$ with $f^i(x,t) \leq f^*(x)$ a.e. in Q $(i=1,2)$. If $u_0^1 \leq u_0^2$ a.e. in Ω and $f^1 \leq f^2$ a.e. in Q, then $u^1 \leq u^2$ a.e. in Q holds.*

Theorem 1.6 (asymptotic behavior). *If $f \in L^2(\Omega)$, then there exists $\bar{u} \in H^2(\Omega)\cap H_0^1(\Omega)$ such that*

$$\lim_{t\to\infty} \|u(\cdot,t) - \bar{u}\|_{H^1(\Omega)} = 0,$$

where $\bar{u}$ is given as a unique solution of the following variational inequality:

$$\bar{u} \in K := \{v \in H_0^1(\Omega);\ v \geq u_0 \ \text{a.e. in } \Omega\},$$
$$\int_\Omega \nabla\bar{u}\cdot\nabla(v-\bar{u})\,dx \geq \langle f, v-\bar{u}\rangle \qquad ({}^\forall v \in K),$$

which is an obstacle problem with obstacle u_0.

Theorem 1.7 (gradient flow structure). *We suppose $f \in L^2(\Omega)$. We define*

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega fu\,dx.$$

Then $[t \mapsto E(u(\cdot,t))] \in W^{1,1}(0,T)$ and

$$\frac{d}{dt}E(u(\cdot,t)) = -\int_\Omega |u_t|^2\,dx \leq 0 \quad \text{a.e. } t \in (0,T)$$

holds.

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Extinction behavior of solutions of the logarithmic diffusion equation

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Abstract

This talk is concerned with the behavior of positive solutions to the logarithmic diffusion equation on the whole line. It is known that due to fast diffusion of the equation, the extinction of solutions may occur in finite time. In this talk, I will investigate precisely the behavior of solutions near the extinction time in the one-dimensional case. By using an intersection number argument, it is shown that the hot spot usually goes to $+\infty$ or $-\infty$, but in some cases, the hot spot remains bounded but does not converge to any particular point.

1 Introduction

The logarithmic diffusion equation

$$u_t = \Delta(\log u) \tag{1.1}$$

has been studied in various contexts. It appears as the central limit approximation of Carleman's model of the Boltzman equation [15], and the expansion into a vacuum of a thermalized electron cloud [14]. It also arises as a model for long Van-der-Waals interactions in thin films of a liquid spreading on a solid surface, if certain nonlinear fourth order terms are neglected. There is a relation with the Ricci flow on $\mathbb{R}^2$ in differential geometry [11], namely the evolution of a Riemannian metric $g_{ij}(\tau)$ given by $\partial_\tau g_{ij} = -2R_{ij}$. If the metric is conformal, there is a function w such that $g_{ij} = w\delta_{ij}$, where δ_{ij} denotes the standard Euclidean metric, and the problem is reduced to (1.1).

For the logarithmic diffusion equation (1.1), it is known that some positive solutions may vanish in finite time even if initial data are positive (see, e.g, [2, 3, 5, 6, 13, 19]. In this talk, I will discuss the behavior of solutions near the extinction time.

In the one-dimensional case, we consider the problem

$$\begin{cases} u_t = (\log u)_{xx}, & x \in \mathbb{R},\ t > 0, \\ \lim\limits_{x\to-\infty}(\log u)_x = \alpha(t), \quad \lim\limits_{x\to+\infty}(\log u)_x = -\beta(t), & t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $\alpha(t),\ \beta(t)$ are given positive continuous functions corresponding to the decay rate of u as $x \to \pm\infty$. The initial value $u_0 : \mathbb{R} \to (0,\infty)$ at $t = 0$ is assumed to be a positive continuous function incorporating prescribed asymptotic conditions at $x = \pm\infty$. Then it was shown by Rodríguez-Vázquez [16] that there exists a unique positive solution which satisfies (1.2) in the classical sense as long as the solution remains positive.

Integrating the equation in (1.2) over $\mathbb{R}$ and applying the asymptotic conditions at $x = \pm\infty$, we see that the total mass satisfies

$$\frac{d}{dt}\int_{\mathbb{R}} u(x,t)dx = -\{\alpha(t) + \beta(t)\}.$$

This implies that the solution must vanish at $t = T$ determined by

$$\int_0^T \{\alpha(t) + \beta(t)\}dt = \int_{\mathbb{R}} u_0(x)dx. \tag{1.3}$$

Concerning the behavior of solutions of (1.2) near the extinction time, Hsu [12] (see also Daskalopoulos-Kenig [10]) considered the case where $\alpha(t) \equiv \beta(t) \equiv \gamma$ (constant) and proved that any even symmetric solution satisfies

$$(T-t)^{-1}u(x,t) \to \varphi(x) \qquad (t \uparrow T),$$

where φ is a positive solution of

$$\begin{cases} (\log\varphi)_{xx} + \varphi = 0, & x \in \mathbb{R}, \\ (\log\varphi)_x \to \gamma > 0 & (x \to \pm\infty). \end{cases} \tag{1.4}$$

If both $\alpha(t) \equiv \alpha_0$ and $\beta(t) \equiv \beta_0$ are positive constant but they are not necessarily equal, then the method of Hsu [12] is not applicable. In fact In this case, the following result was recently established by Shimojo-Takáč-Yanagida [18].

Theorem 1.1 *Let u be a solution of (1.2) with $\alpha(t) \equiv \alpha_0 > 0$ and $\beta(t) \equiv \beta_0 > 0$. Then there exist a constant $c \in \mathbb{R}$ and a positive function φ such that*

$$v(x,t) = e^s u(x, T - e^{-s}) \to \varphi(x - cs) \qquad (s \to \infty)$$

uniformly in $x \in \mathbb{R}$*, where* c *and* $\varphi(z)$ *satisfy*

$$(\log \varphi)_{zz} + c\varphi_{zz} + \varphi = 0, \qquad z \in \mathbb{R},$$

with the asymptotic conditions

$$\lim_{z\to-\infty} (\log \varphi)_z = \alpha_0, \quad \lim_{z\to+\infty} (\log \varphi)_z = -\beta_0.$$

In particular, this theorem implies that as $t \uparrow T$, the hot spot (i.e., the maximal point of $u(x,t)$) tends to $-\infty$ if $\alpha_0 < \beta_0$ and tends to $-\infty$ if $\alpha_0 > \beta_0$, and converges to a point if $\alpha_0 = \beta_0$. This is in contract to the linear heat equation on $\mathbb{R}^N$ for which the hot spot of any positive L^1 solutions converges to a point. See [17] and the references therein about the behavior of hot spots for other problems.

The next natural question is what happens if $\alpha(t)$ and $\beta(t)$ are not necessarily constant. In this case, the following results shows that the motion of the hot spot $h(t)$ near the extinction time could be very complicated.

Theorem 1.2 *There exists* $t_0 \in (0,T)$ *such that the hot spot of* $u(x,t)$ *consists one point for* $t \in (t_0, T)$*. Moreover, the unique hot spot* $h(t)$ *satisfies the following:*
(i) *if* $\alpha(T) > \beta(T)$*, then* $h(t) \to +\infty$ *as* $t \uparrow T$*.*
(ii) *if* $\alpha(T) < \beta(T)$*, then* $h(t) \to -\infty$ *as* $t \uparrow T$*.*
(iii) *For any positive constants* T*,* γ*, and any positive initial data, there exist positive continuous functions* $\alpha(t)$ *and* $\beta(t)$ *with* $\alpha(T) = \beta(T) = \gamma$ *such that*

$$\limsup_{t\uparrow T} h(t) = +\infty, \qquad \liminf_{t\uparrow T} h(t) = -\infty.$$

We remark that this theorem holds true if we replace $h(t)$ with the center of gravity

$$g(t) := \frac{\displaystyle\int_{\mathbb{R}} xu(x,t)dx}{\displaystyle\int_{\mathbb{R}} u(x,t)dx}.$$

The rest of this note is organized as follows. In Section 2 we describe some fundamental properties of solutions of the logarithmic diffusion equation, and introduce a transformation which is useful in studying the extinction behavior. Then we consider the existence of traveling solutions of the transformed equation. In Section 3 we sketch the proof of Theorem 2.

2 Fundamental properties

In this section we describe fundamental properties of solutions to (1.2).

We can observe complicated behavior of solutions near the extinction point by choosing $\alpha(s)$ and $\beta(s)$ suitably.

Introducing the transformation of variables

$$u(x,t) = (T-t)v(x,s), \qquad t = T - e^{-s}, \tag{2.1}$$

we have

$$\begin{cases} v_s = (\log v)_{xx} + v, & x \in \mathbb{R}, \quad s \in (-\log T, \infty) \\ (\log v)_x \to +\alpha(T - e^{-s}) > 0 & (x \to -\infty) \\ (\log v)_x \to -\beta(T - e^{-s}) < 0 & (x \to +\infty) \\ v(x, -\log T) = v_0(x) := \dfrac{1}{T} u_0(x) \end{cases} \tag{2.2}$$

Thus, in order to study the behavior of u as $t \to T$, we need to study the behavior of v as $s \to \infty$.

We note

$$\begin{aligned} \int_{\mathbb{R}} v(x,s)ds &= \frac{1}{T-t} \int_{\mathbb{R}} u(x,t)dx \\ &= \frac{1}{T-t} \Big[\int_{\mathbb{R}} u_0(x)dx - \int_0^t \big\{\alpha(t) + \beta(t)\big\}dt \big\} \Big] \\ &= \frac{1}{T-t} \Big[\int_0^T \big\{\alpha(t) + \beta(t)\big\}dt - \int_0^t \big\{\alpha(t) + \beta(t)\big\}dt \big\} \Big] \\ &= \frac{1}{T-t} \int_{T-t}^T \big\{\alpha(t) + \beta(t)\big\}dt. \end{aligned}$$

Hence

$$\int_{\mathbb{R}} v(x,s)ds \to \alpha(T) + \beta(T) \quad \text{ as } s \to \infty.$$

In particular, if $\alpha(t) \equiv \alpha_0$ and $\beta_(t) \equiv \beta_0$, then

$$\int_{\mathbb{R}} v(x,s)dx \equiv \alpha_0 + \beta_0, \qquad s \in [-\log T, \infty).$$

Next we consider traveling solutions of

$$v_s = (\log v)_{xx} + v, \qquad x \in \mathbb{R}, \tag{2.3}$$

which plays an essential role in the proof of Theorem 1.2. We Introduce a traveling coordinate $z = x - cs$, and consider solutions of the form $v(x,s) = \varphi(z)$, $z = x - cs$,

where c denotes the propagation speed, $\varphi > 0$ is the profile of the traveling wave. Substituting $v = \varphi(z)$ in (2.3), we see that φ must satisfy

$$\begin{cases} (\log\varphi)_{zz} + c\varphi_z + \varphi = 0, \quad z \in R \\ (\log\varphi)_z(-\infty) = \alpha, \quad (\log\varphi)_z(-\infty) = -\beta. \end{cases} \tag{2.4}$$

If we introduce an auxiliary variable $\psi := (\log\varphi)_z$, then (2.4) is equivalent to the system

$$\begin{cases} \varphi_z = \varphi\psi, \\ \psi_z = -c\varphi\psi - \varphi. \end{cases} \tag{2.5}$$

By the phase plane analysis, we can show that for each $\alpha, \beta > 0$, there exists a unique $c = c(\alpha, \beta)$ such that (2.5) has an orbit connecting $(0, \alpha)$ and $(-\beta, 0)$, and the corresponding solution of (2.4) is positive and unimodal, that is, there exists $z_0 \in \mathbb{R}$ such that $\varphi_z(z) > 0$ for $z \in (-\infty, z_0)$ and $\varphi_z(z) < 0$ for $z \in (-\infty, z_0)$. Hence we may assume $\varphi_z(0) = 0$ to fix the phase, which leads to the uniqueness of the solution of (2.4). We will denote the unique solution by $\varphi(\cdot; \alpha, \beta)$. We can also show that the propagation speed of the traveling wave is monotone decreasing in $\alpha \in (0, \infty)$ and monotone increasing in $\beta \in (0, \infty)$, and

$$c \gtreqless 0 \Longleftrightarrow \alpha \gtreqless \beta.$$

We note that for $\alpha = \beta$, the traveling solution with $c = 0$ becomes a stationary solution.

3 Outline of the proof of Theorem 1.2

By (2.1), it suffices to consider the motion of the hot spot of $v(x, s)$ for (2.2) as $s \to \infty$. We first give some preliminary lemmas.

Lemma 3.1 *The intersection number principle holds, that is, the number of intersection points of two solutions of* (2.2) *with different initial values is non-increasing in* s.

The principle intersection number is well known for semilinear equations, but it is not trivial for the fast diffusion equation, because we must exclude the possibility of appearance of intersection points from $\pm\infty$. This was done in [18].

Lemma 3.2 *The solution* v *of* (2.2) *is uniformly bounded in* $s \in (-\log T, \infty)$.

This can be proved by the comparison argument. It should be noted that this property holds true only for some initial values with critical mass. In fact, if the mass is smaller than the critical mass, then the solution vanishes in finite time, whereas if the mass is larger, then the solution grows exponentially.

Lemma 3.3 *The solution v of (2.2) becomes unimodal in finite time, that is, there exists $s_0 \in (-\log T, \infty)$ such that $v_x > 0$ for $x \in (-\infty, h(t))$ and $v_x > 0$ for $x \in (h(t), \infty)$ for all $s > s_0$.*

To show this, we use the fact that $v = \exp(Ax + Bs + C)$, where A, B, C are arbitrary constants, satisfies (2.3). By the boundedness of the solution of (2.2) and the intersection number principle, the solution must become unimodal eventually.

Using the above lemmas, we can show Theorem 2 as follows. First, we consider the intersection of $v(x,s)$ and $v_1(x,s) := \varphi(x - c(\alpha_1, \beta_1)s + \gamma_1)$ and $v_2(x,s) := \varphi(x - c(\alpha_2, \beta_2)t - \gamma_2)$, where α_1, β_1, α_2, β_2 are constants satisfying

$$\alpha_1 < \alpha(T) < \alpha_2, \quad \beta_1 > \beta(T) > \beta_2,$$

and γ_1, γ_2 are positive constants. Note that

$$c(\alpha_1, \beta_1) < c(\alpha(T), \beta(T)) < c(\alpha_2, \beta_2).$$

If we take γ_1, γ_2 sufficiently large, then $v(x,t)$ intersects $v_1(x,s)$ and $v_2(x,s)$ only once. Then by using Lemma 3.3, we can show that $h(t)$ must move with the speed between $c(\alpha_1, \beta_1)$ and $c(\alpha_2, \beta_2)$. This proves (i) and (ii).

The proof of (iii) can be obtained again by the intersection number principle. We can control the position of the hot spot by choosing $\alpha(t)$ and $\beta(t)$ suitably and considering the intersection with $v_1(x,s)$ and $v_2(x,s)$ carefully. Then we can show that the behavior of the hot spot satisfies the desired properties.

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Numerical analysis of the Oseen-type Peterlin viscoelastic model

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The talk will be based on our paper [1] which is a joint work with M. Lukáčová-Medvid'ová, H. Mizerová and M. Tabata.

1 Introduction

In the talk, error estimates of a linear stabilized Lagrange–Galerkin scheme for the Oseen-type diffusive Peterlin viscoelastic model are presented. The model is a coupled nonlinear system of incompressible flow equations and a equation of the so-called conformation tensor, cf. [2]. As for the nonlinear scheme, see our another paper [3].

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain and T a positive constant. The problem to be dealt with is to find $(u, p, C) : \Omega \times (0, T) \to \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$ such that

$$\frac{Du}{Dt} - \nabla \cdot [2\nu D(u)] + \nabla p = \nabla \cdot [(\operatorname{tr} C)C] + f \quad \text{in } \Omega \times (0, T), \tag{1a}$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega \times (0, T), \tag{1b}$$

$$\frac{DC}{Dt} - \varepsilon \Delta C = (\nabla u)C + C(\nabla u)^T - (\operatorname{tr} C)^2 C + (\operatorname{tr} C)I + F \quad \text{in } \Omega \times (0, T), \tag{1c}$$

$$u = 0, \quad \varepsilon \frac{\partial C}{\partial n} = 0, \quad \text{on } \partial\Omega \times (0, T), \tag{1d}$$

$$u = u^0, \quad C = C^0, \quad \text{in } \Omega, \text{ at } t = 0, \tag{1e}$$

where u is the velocity, p is the pressure, C is the conformation tensor, $\nu > 0$ is a fluid viscosity, $\varepsilon > 0$ is an elastic stress viscosity, $(f, F) : \Omega \times (0, T) \to \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is a pair of given external forces, $D(u)$ is the strain-rate tensor defined by

$$D(u) \equiv \frac{1}{2}\Big[\nabla u + (\nabla u)^T\Big],$$

I is the identity matrix, $n : \Gamma \to \mathbb{R}^d$ is the outward unit normal, $(u^0, C^0) : \Omega \to \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is a pair of given initial functions, and D/Dt is the material derivative defined by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + w \cdot \nabla$$

for a given velocity satisfying the next hypothesis.

Hypothesis 1.1 *The function w satisfies $w \in C([0,T]; W_0^{1,\infty}(\Omega)^d)$.*

Let $(\cdot,\cdot)$ be the $L^2(\Omega)$ inner product, and the inner products in $L^2(\Omega)^d$ and $L^2(\Omega)^{d\times d}$ are denoted by the same notation. Let $V := H_0^1(\Omega)^d$, $Q := L_0^2(\Omega) := \{q \in L^2(\Omega);\ \int_\Omega q\ dx = 0\}$ and $W := \{D \in H^1(\Omega)^{d\times d};\ D = D^T\}$. We define the bilinear forms a_u on $V \times V$, b on $V \times Q$, $\mathcal{A}$ on $(V \times Q) \times (V \times Q)$ and a_c on $W \times W$ by

$$\begin{aligned} a_u(u,v) &:= 2\big(D(u), D(v)\big), \\ b(u,q) &:= -(\nabla \cdot u, q), \\ \mathcal{A}\big((u,p),(v,q)\big) &:= \nu a_u(u,v) + b(u,q) + b(v,p), \\ a_c(C,D) &:= (\nabla C, \nabla D), \end{aligned}$$

respectively. The weak formulation of the problem (1) is to find $(u,p,C) : (0,T) \to V \times Q \times W$ such that for $t \in (0,T)$

$$\left(\frac{Du}{Dt}(t), v\right) + \mathcal{A}\big((u,p)(t),(v,q)\big) = -\big([\operatorname{tr} C(t)]C(t), \nabla v\big) + \big(f(t), v\big), \tag{2a}$$

$$\begin{aligned} \left(\frac{DC}{Dt}(t), D\right) + \varepsilon a_c\big(C(t), D\big) &= 2\big([\nabla u(t)]C(t), D\big) - \big([\operatorname{tr} C(t)]^2 C(t), D\big) \\ &\quad + \big([\operatorname{tr} C(t)]I, D\big) + \big(F(t), D\big), \\ &\qquad \forall (v,q,D) \in V \times Q \times W, \end{aligned} \tag{2b}$$

with $(u(0), C(0)) = (u^0, C^0)$.

2 A linear stabilized Lagrange–Galerkin scheme

The stabilized Lagrange–Galerkin method proposed originally in [4,5] is a combination of the Brezzi–Pitkäranta's pressure-stabilization method [6] and the method of characteristics, and employs P1-element for the velocity and pressure, where the error estimates have been established for the Oseen equations [8] and for the Navier–Stokes equations [7]. For the model (1) we propose a stabilized Lagrange–Galerkin scheme which employs P1-element for all the unknowns u, p and C.

Let Δt be a time increment, $N_T := \lfloor T/\Delta t \rfloor$ the total number of time steps and $t^n := n\Delta t$ for $n = 0, \ldots, N_T$. Let g be a function defined in $\Omega \times (0,T)$ and $g^n := g(\cdot, t^n)$. For the approximation of the material derivative we employ the first-order characteristics method,

$$\frac{Dg}{Dt}(x, t^n) = \frac{g^n(x) - \big(g^{n-1} \circ X_1^n\big)(x)}{\Delta t} + O(\Delta t), \tag{3}$$

where $X_1^n : \Omega \to \mathbb{R}^d$ is a mapping defined by

$$X_1^n(x) := x - w^n(x)\Delta t,$$

and the symbol $\circ$ means the composition of functions,

$$(g^{n-1} \circ X_1^n)(x) := g^{n-1}(X_1^n(x)).$$

For the details on deriving the approximation (3) of Dg/Dt, see, e.g., [8]. The point $X_1^n(x)$ is called the upwind point of x with respect to w^n. The next proposition presents a sufficient condition to ensure that all upwind points defined by X_1^n are in Ω.

Proposition 2.1 ([9]) *Suppose Hypothesis 1.1 holds. Then, under the condition,*

$$\Delta t |w|_{C([0,T];W^{1,\infty})} < 1, \tag{4}$$

$X_1^n : \Omega \to \Omega$ is bijective for $n \in \{0, \ldots, N_T\}$.

For the sake of simplicity we suppose that Ω is a polygonal domain. Let $\mathcal{T}_h = \{K\}$ be a triangulation of $\bar{\Omega}$ $(= \bigcup_{K\in\mathcal{T}_h} K)$, h_K the diameter of $K \in \mathcal{T}_h$ and $h := \max_{K\in\mathcal{T}_h} h_K$ the maximum element size. We consider a regular family of subdivisions $\{\mathcal{T}_h\}_{h\downarrow 0}$ satisfying the inverse assumption. We define the discrete function spaces X_h, M_h, W_h, V_h and Q_h by

$$\begin{aligned}
X_h &:= \left\{v_h \in C(\bar{\Omega})^d;\ v_{h|K} \in P_1(K)^d, \forall K \in \mathcal{T}_h\right\},\\
M_h &:= \left\{q_h \in C(\bar{\Omega});\ q_{h|K} \in P_1(K), \forall K \in \mathcal{T}_h\right\},\\
W_h &:= \left\{D_h \in C(\bar{\Omega})^{d\times d} \cap W;\ D_{h|K} \in P_1(K)^{d\times d}, \forall K \in \mathcal{T}_h\right\},\\
V_h &:= X_h \cap V, \qquad Q_h := M_h \cap Q,
\end{aligned}$$

respectively, where $P_1(K)$ is the polynomial space of linear functions on $K \in \mathcal{T}_h$.

Let δ_0 be a small positive constant fixed arbitrarily and $(\cdot,\cdot)_K$ the $L^2(K)^d$ inner product. We define the bilinear forms $\mathcal{A}_h$ on $(V \times H^1(\Omega)) \times (V \times H^1(\Omega))$ and $\mathcal{S}_h$ on $H^1(\Omega) \times H^1(\Omega)$ by

$$\begin{aligned}
\mathcal{A}_h\left((u,p),(v,q)\right) &:= \nu a_u(u,v) + b(u,q) + b(v,p) - \mathcal{S}_h(p,q),\\
\mathcal{S}_h(p,q) &:= \delta_0 \sum_{K\in\mathcal{T}_h} h_K^2 (\nabla p, \nabla q)_K.
\end{aligned}$$

We note that $\mathcal{S}_h$ is the Brezzi–Pitkäranta's pressure-stabilization.

Let $(f_h, F_h) := (\{f_h^n\}_{n=1}^{N_T}, \{F_h^n\}_{n=1}^{N_T}) \subset L^2(\Omega)^d \times L^2(\Omega)^{d\times d}$ and $(u_h^0, C_h^0) \in V_h \times W_h$ be given. A linear stabilized Lagrange–Galerkin scheme for (1) is to find $(u_h, p_h, C_h) := \{(u_h^n, p_h^n, C_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h \times W_h$ such that, for $n = 1, \ldots, N_T$,

$$\begin{aligned}
&\left(\frac{u_h^n - u_h^{n-1}\circ X_1^n}{\Delta t}, v_h\right) + \mathcal{A}_h\big((u_h^n, p_h^n), (v_h, q_h)\big)\\
&\qquad = -\big([\mathrm{tr}C_h^n]C_h^{n-1}, \nabla v_h\big) + (f_h^n, v_h), \qquad (5a)\\
&\left(\frac{C_h^n - C_h^{n-1}\circ X_1^n}{\Delta t}, D_h\right) + \varepsilon a_c(C_h^n, D_h) = 2\big((\nabla u_h^n)C_h^{n-1}, D_h\big)\\
&\qquad - \big([\mathrm{tr}C_h^{n-1}]^2 C_h^n, D_h\big) + \big([\mathrm{tr}C_h^{n-1}]I, D_h\big) + (F_h^n, D_h), \qquad (5b)\\
&\qquad\qquad \forall (v_h, q_h, D_h) \in V_h \times Q_h \times W_h.
\end{aligned}$$

3 Main results

We use the following norms and seminorms,

$$\|u\|_{\ell^\infty(X)} := \max_{n=0,\dots,N_T} \|u^n\|_X, \qquad \|u\|_{\ell^2(X)} := \left\{ \Delta t \sum_{n=1}^{N_T} \|u^n\|_X^2 \right\}^{1/2},$$

$$|p|_h := \left\{ \sum_{K\in\mathcal{T}_h} h_K^2 (\nabla p, \nabla p)_K \right\}^{1/2}, \quad |p|_{\ell^2(|\cdot|_h)} := \left\{ \Delta t \sum_{n=1}^{N_T} |p^n|_h^2 \right\}^{1/2},$$

for $X = L^\infty(\Omega)$, $L^2(\Omega)$ and $H^1(\Omega)$. $\overline{D}_{\Delta t}$ is the backward difference operator defined by

$$\overline{D}_{\Delta t} u^n := \frac{u^n - u^{n-1}}{\Delta t}.$$

The existence and uniqueness of the solution of scheme (5) are ensured by the following proposition.

Proposition 3.1 *Suppose Hypothesis 1.1 holds. Then, for any h and Δt satisfying* (4) *there exists a unique solution $(u_h, p_h, C_h) \subset V_h \times Q_h \times W_h$ of scheme* (5).

We state the main results after preparing a projection.

Definition 3.2 *For $(u,p,C) \in V \times Q \times W$ we define the* Stokes–Poisson *projection $(\hat{u}_h, \hat{p}_h, \hat{C}_h) \in V_h \times Q_h \times W_h$ of (u,p,C) by*

$$\begin{aligned} &\mathcal{A}_h\left((\hat{u}_h,\hat{p}_h),(v_h,q_h)\right) + a_c(\hat{C}_h, D_h) + (\hat{C}_h, D_h) \\ &\qquad = \mathcal{A}\left((u,p),(v_h,q_h)\right) + a_c(C, D_h) + (C, D_h), \\ &\qquad\qquad \forall (v_h, q_h, D_h) \in V_h \times Q_h \times W_h. \end{aligned}$$

The Stokes–Poisson projection derives an operator $\Pi_h^{\mathrm{SP}} : V \times Q \times W \to V_h \times Q_h \times W_h$ defined by $\Pi_h^{\mathrm{SP}}(u,p,C) := (\hat{u}_h, \hat{p}_h, \hat{C}_h)$. The i-th component of $\Pi_h^{\mathrm{SP}}(u,p,C)$ is denoted by $[\Pi_h^{\mathrm{SP}}(u,p,C)]_i$ for $i = 1,2,3$. We now impose the conditions,

$$(u_h^0, C_h^0) = ([\Pi_h^{\mathrm{SP}}(u,p,C)]_1, [\Pi_h^{\mathrm{SP}}(u,p,C)]_3), \quad (f_h, F_h) = (f, F). \tag{6}$$

Theorem 3.3 (error estimates I) *Suppose that Hypothesis 1.1 holds and that the solution (u,p,C) of* (2) *is smooth enough. Then, there exist positive constants h_0, c_0 and $c_\dagger$ such that, for any pair $(h, \Delta t)$ satisfying*

$$h \in (0, h_0], \quad \Delta t \le \begin{cases} c_0 (1 + |\log h|)^{-1/2} & (d=2), \\ c_0 h^{1/2} & (d=3), \end{cases} \tag{7}$$

the solution (u_h, p_h, C_h) of scheme (5) *with* (6) *is estimated as follows.*

$$\|C_h\|_{\ell^\infty(L^\infty)} \le \|C\|_{C(L^\infty)} + 1, \tag{8}$$

$$\|u_h - u\|_{\ell^\infty(L^2)},\ \|u_h - u\|_{\ell^2(H^1)},\ |p_h - p|_{\ell^2(|\cdot|_h)},$$

$$\|C_h - C\|_{\ell^\infty(H^1)},\ \left\| \overline{D}_{\Delta t} C_h - \frac{\partial C}{\partial t} \right\|_{\ell^2(L^2)} \le c_\dagger (\Delta t + h). \tag{9}$$

Theorem 3.4 (error estimates II) *Suppose that Hypothesis 1.1 holds and that the solution* (u, p, C) *of* (2) *is smooth enough. Let* h_0 *and* c_0 *be the constants stated in Theorem 3.3. Then, there exists a positive constant* $c_\ddagger$ *such that, for any pair* $(h, \Delta t)$ *with* (7) *the solution* (u_h, p_h, C_h) *of scheme* (5) *with* (6) *satisfies the estimates,*

$$\left\| \overline{D}_{\Delta t} u_h - \frac{\partial u}{\partial t} \right\|_{\ell^2(L^2)}, \quad \|p_h - p\|_{\ell^2(L^2)} \le c_\ddagger (\Delta t + h). \tag{10}$$

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Large time behavior of solutions to the viscous conservation law with dispersion

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1 Introduction

In this talk, we consider asymptotic behavior of global solutions to the following equation:

$$\begin{aligned} u_t + (f(u))_x + ku_{xxx} = u_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1.1}$$

where $u_0 \in L^1(\mathbb{R})$, $f(u) = (b/2)u^2 + (c/3)u^3$ with $b, c \in \mathbb{R}$ and $k \in \mathbb{R}$. This is a nonlinear model for taking into account the dispersive (described by the term of the third order derivative u_{xxx}) and dissipative processes (like a heat equation) as well as the convection effects. This equation is called the equation of viscous conservation law with dispersion for the conserved quantity $u(x,t)$. It is also called the generalized Korteweg-de Vries-Burgers equation (we call it generalized KdV-Burgers equation for short). The goal of our study is to obtain an asymptotic profile of the solution $u(x,t)$ and to examine the optimality of the asymptotic rate, by constructing the second asymptotic profile of $u(x,t)$.

First of all, we recall known result concerning this problem. When $k = 0$, (1.1) becomes the generalized Burgers equation:

$$\begin{aligned} u_t + (f(u))_x = u_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{1.2}$$

It was shown in Matsumura and Nishihara [12] that the solution of (1.2) converges to a nonlinear diffusion wave defined by

$$\chi(x,t) \equiv \frac{1}{\sqrt{1+t}} \chi_* \left(\frac{x}{\sqrt{1+t}} \right), \quad t \geq 0, \quad x \in \mathbb{R}, \tag{1.3}$$

where

$$\chi_*(x) \equiv \frac{1}{b} \frac{(e^{b\delta/2} - 1)e^{-x^2/4}}{\sqrt{\pi} + (e^{b\delta/2} - 1) \int_{x/2}^{\infty} e^{-y^2} dy}, \quad \delta \equiv \int_{\mathbb{R}} u_0(x) dx, \quad b \neq 0. \tag{1.4}$$

Note that $\chi(x,t)$ is a solution of the Burgers equation

$$\chi_t + \left(\frac{b}{2} \chi^2 \right)_x = \chi_{xx}, \tag{1.5}$$

satisfying

$$\int_{\mathbb{R}} \chi(x,0) dx = \delta.$$

Moreover, if $u_0 \in L^1_1(\mathbb{R}) \cap H^1(\mathbb{R})$ and $\|u_0\|_{L^1_1} + \|u_0\|_{H^1}$ is sufficiently small, then the optimal asymptotic rate to the nonlinear diffusion wave are obtained by Kato [9] by constructing the second asymptotic profile $V_1(x,t)$ which is the leading term of $u - \chi$. Here $L^1_\beta(\mathbb{R})$ is a subset of $L^1(\mathbb{R})$ whose elements satisfy $\|u_0\|_{L^1_\beta} \equiv \int_{\mathbb{R}} |u_0(x)|(1+|x|)^\beta dx < \infty$. Indeed, the following decay estimate are established:

$$\|u(\cdot,t) - \chi(\cdot,t) - V_1(\cdot,t)\|_{L^\infty} \leq C(\|u_0\|_{L^1_1} + \|u_0\|_{H^1})(1+t)^{-1}, \quad t \geq 1, \tag{1.6}$$

where

$$V_1(x,t) \equiv -\frac{cd}{12\sqrt{\pi}} V_*\left(\frac{x}{\sqrt{1+t}}\right)(1+t)^{-1}\log(1+t), \quad t \geq 0, \quad x \in \mathbb{R}, \tag{1.7}$$

$$V_*(x) \equiv (b\chi_*(x) - x)e^{-x^2/4}\eta_*(x) = 2\frac{d}{dx}(\eta_*(x)e^{-x^2/4}), \tag{1.8}$$

$$\eta_*(x) \equiv \exp\left(\frac{b}{2}\int_{-\infty}^{x} \chi_*(y)dy\right), \quad d \equiv \int_{\mathbb{R}} (\eta_*(y))^{-1}(\chi_*(y))^3 dy. \tag{1.9}$$

From (1.6), the triangle inequality and (1.7), we see that the original solution $u(x,t)$ tends to the nonlinear diffusion wave $\chi(x,t)$ at the rate of $t^{-1}\log t$, and in addition, if $\delta \neq 0$ and $c \neq 0$, then this asymptotic rate is optimal with respect to the time decaying order. Also we see that $u - \chi$ tends to the second asymptotic profile $V_1(x,t)$ at the rate of t^{-1}.

Next, we consider the case where $b = k = 1$ and $c = 0$ in (1.1):

$$\begin{gathered} u_t + uu_x + u_{xxx} = u_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{gathered} \tag{1.10}$$

This equation is called the KdV-Burgers equation. It was shown in Kaikina and Ruiz-Paredes [5] that if $u_0 \in L_1^1(\mathbb{R}) \cap H^s(\mathbb{R})$ with $s > -1/2$, the following estimate

$$\|u(\cdot,t) - \chi(\cdot,t-1) - V_2(\cdot,t)\|_{L^\infty} \leq Ct^{-1}\sqrt{\log t} \tag{1.11}$$

holds for sufficiently large t, where

$$V_2(x,t) \equiv -\frac{d}{32\sqrt{\pi}} V_*\left(\frac{x}{\sqrt{t}}\right)t^{-1}\log t$$

with $V_*(x)$ being defined by (1.8). We see from this result that the solution of (1.10) also tends to the nonlinear diffusion wave $\chi(x,t)$ at the rate of $t^{-1}\log t$ and this rate is optimal. Here, we note that $V_1(x,t)$ and $V_2(x,t)$ are essentially the same functions. On the other hand, the asymptotic rate given by (1.11) is rougher than (1.6) by $\sqrt{\log t}$, although they mentioned in [5] that the term $\sqrt{\log t}$ in the estimate (1.11) could be removed by a more delicate consideration, without any proof.

We consider (1.1) in general and obtained the following result:

Theorem 1.1 (In Preparation)**.** *Assume that $u_0 \in L^1(\mathbb{R}) \cap H^3(\mathbb{R})$ and $\|u_0\|_{L^1} + \|u_0\|_{H^3}$ is sufficiently small. Then* (1.1) *has a unique global solution $u(x,t)$ satisfying $u \in C^0([0,\infty); H^3)$ and $\partial_x u \in L^2(0,\infty; H^3)$. Moreover if $u_0 \in L_1^1(\mathbb{R}) \cap H^3(\mathbb{R})$ and $\|u_0\|_{L_1^1} + \|u_0\|_{H^3}$ is sufficiently small, then we have*

$$\|u(\cdot,t) - \chi(\cdot,t) - V(\cdot,t)\|_{L^\infty} \leq C(\|u_0\|_{L_1^1} + \|u_0\|_{H^3})(1+t)^{-1}, \quad t \geq 1, \tag{1.12}$$

where $\chi(x,t)$ is defined by (1.3)*, while $V(x,t)$ is defined by*

$$V(x,t) \equiv -\frac{d}{4\sqrt{\pi}}\left(\frac{b^2k}{8} + \frac{c}{3}\right)V_*\left(\frac{x}{\sqrt{1+t}}\right)(1+t)^{-1}\log(1+t), \quad t \geq 0, \quad x \in \mathbb{R}, \tag{1.13}$$

with $V_*(x)$ being defined by (1.8)*.*

Remark 1.2. From (1.12), the triangle inequality and (1.13), if $\delta \neq 0$ and $(b^2k)/8 + c/3 \neq 0$, we see that the original solution $u(x,t)$ tends to the nonlinear diffusion wave $\chi(x,t)$ at the rate of $t^{-1}\log t$. Actually, we have

$$\tilde{C}(1+t)^{-1}\log(1+t) \leq \|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} \leq C(1+t)^{-1}\log(1+t) \tag{1.14}$$

holds for sufficiently large t. Therefore, this asymptotic rate $t^{-1}\log t$ is optimal with respect to the time decaying order. However, if $(b^2k)/8 + c/3 = 0$, then we find the asymptotic rate t^{-1} without involving $V(x,t)$.

Remark 1.3. We are able to improve the estimate (1.11), and unify the results due to Kato [9] and Kaikina and Ruiz-Paredes [5].

2 Decay estimates for solutions to (1.1)

We can prove the decay estimates for solutions to (1.1). First, we introduce the green function associated with the linearized equation for (1.1) and its estimates. Here and later, for $f, g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, we denote the Fourier transform of f and the inverse Fourier transform of g as follows:

$$\hat{f}(\xi) \equiv F[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx,$$
$$\check{g}(x) \equiv F^{-1}[g](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} g(\xi) d\xi.$$

We denote the green function

$$S(x,t) \equiv F^{-1}[e^{-t|\xi|^2 + itk\xi^3}](x). \tag{2.1}$$

By a direct calculation, we can show the estimates of derivatives of $S(x,t)$. For the proof, see Lemma A.1. and Lemma A.2. in [8].

Lemma 2.1. *Let l be a non-negative integer. Then, for $p \in [2, \infty]$, we have*

$$\|\partial_x^l S(\cdot,t)\|_{L^1} \le C t^{-l/2}(1 + t^{-1/4}), \quad t > 0, \tag{2.2}$$
$$\|\partial_x^l S(\cdot,t)\|_{L^p} \le C t^{-(1/2)(1-1/p)-l/2}, \quad t > 0. \tag{2.3}$$

Moreover, for the convolution $S(t) * f$, we obtain following estimate:

Lemma 2.2. *Let m be a positive integer. Suppose $f \in H^m(\mathbb{R}) \cap L^1(\mathbb{R})$. Then the estimate*

$$\|\partial_x^l (S(t) * f)\|_{L^2} \le C(1+t)^{-1/4-l/2} \|f\|_{L^1} + e^{-t} \|\partial_x^l f\|_{L^2}, \quad t \ge 0 \tag{2.4}$$

holds for any integer $0 \le l \le m$.

Now we turn back to (1.1). Local existence and uniqueness of the solution to (1.1) can be shown by the standard iterative method, and a priori estimates can be obtained by the energy method (see e.g. [8], [10], [12]). Therefore one can obtain global existence of the solution satisfying

$$\|u(\cdot,t)\|_{H^3}^2 + \int_0^t \|\partial_x u(\cdot,s)\|_{H^3}^2 ds \le C \|u_0\|_{H^3}^2, \quad t \ge 0. \tag{2.5}$$

Moreover, the solution satisfies the following decay estimates:

Lemma 2.3. *Assume that $u_0 \in L^1(\mathbb{R}) \cap H^3(\mathbb{R})$ and $\|u_0\|_{L^1} + \|u_0\|_{H^3}$ is sufficiently small. Then the solution $u(x,t)$ to (1.1) satisfies*

$$\|\partial_x^l u(\cdot,t)\|_{L^1} \le C(\|u_0\|_{L^1} + \|u_0\|_{H^3}) t^{-l/2}(1 + t^{-1/4}), \quad t > 0, \tag{2.6}$$
$$\|\partial_x^l u(\cdot,t)\|_{L^2} \le C(\|u_0\|_{L^1} + \|u_0\|_{H^3})(1+t)^{-1/4-l/2}, \quad t \ge 0 \tag{2.7}$$

for $l = 0, 1, 2, 3$.

3 Basic lemmas and auxiliary problem

In order to show basic estimates for auxiliary problems, we prepare a couple of lemmas. The first one is concerned with the decay estimates for the semigroup $e^{t\Delta}$, where $e^{t\Delta} f \equiv F^{-1}[e^{-t|\xi|^2} \hat{f}(\xi)]$ associated with the heat equation ($k = 0$ in (2.1)):

Lemma 3.1. *Let m be a positive integer. Suppose $f \in H^m(\mathbb{R}) \cap L^1(\mathbb{R})$. Then the estimate*

$$\|\partial_x^l e^{t\Delta} f\|_{L^2} \le C(1+t)^{-1/4-l/2} \|f\|_{L^1} + e^{-t} \|\partial_x^l f\|_{L^2}, \quad t \ge 0 \tag{3.1}$$

holds for any integer $0 \le l \le m$.

Next, we treat the nonlinear diffusion wave $\chi(x,t)$ defined by (1.3), and the heat kernel

$$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

A direct calculation yields

$$|\chi(x,t)| \le C|\delta|(1+t)^{-1/2} e^{-x^2/4(1+t)}, \quad t \ge 0, \ x \in \mathbb{R}. \tag{3.2}$$

Moreover, we can estimate derivatives of $\chi(x,t)$ and $G(x,t)$ (for the proof, see e.g. Lemma 4.1 of [12]).

Lemma 3.2. *Let α and β be non-negative integers. Then, for $p \in [1,\infty]$, we have*

$$\|\partial_x^\alpha \partial_t^\beta \chi(\cdot,t)\|_{L^p} \le C|\delta|(1+t)^{-(1/2)(1-1/p)-\alpha/2-\beta}, \quad t \ge 0, \tag{3.3}$$

$$\|\partial_x^\alpha \partial_t^\beta G(\cdot,t)\|_{L^p} \le Ct^{-(1/2)(1-1/p)-\alpha/2-\beta}, \quad t > 0. \tag{3.4}$$

Next, for the latter sake, we define

$$\eta_1(x,t) \equiv \eta_*\left(\frac{x}{\sqrt{1+t}}\right) = \exp\left(\frac{b}{2}\int_{-\infty}^{x} \chi(y,t)dy\right), \tag{3.5}$$

$$\eta_2(x,t) \equiv (\eta_1(x,t))^{-1}. \tag{3.6}$$

For these functions, we can easily show

$$\min\{1, e^{b\delta/2}\} \le \eta_1(x,t) \le \max\{1, e^{b\delta/2}\}, \tag{3.7}$$

$$\min\{1, e^{-b\delta/2}\} \le \eta_2(x,t) \le \max\{1, e^{-b\delta/2}\}. \tag{3.8}$$

Moreover, we have the following estimates by using Lemma 3.2 (for the proof, see Corollary 2.3 of [9]).

Lemma 3.3. *Let l be a positive integer and $p \in [1,\infty]$. For $i = 1,2$, if $|\delta| \le 1$, then we have*

$$\|\partial_x^l \eta_i(\cdot,t)\|_{L^p} \le C|\delta|(1+t)^{-(1/2)(1-1/p)-l/2+1/2}, \quad t \ge 0. \tag{3.9}$$

In order to prove the main theorem, we introduce an auxiliary problem. We set $\psi(x,t) = u(x,t) - \chi(x,t)$, where $u(x,t)$ is the original solution to (1.1) and $\chi(x,t)$ is the nonlinear diffusion wave defined by (1.3). Then, $\psi(x,t)$ satisfies following equation:

$$\psi_t + (b\chi\psi)_x + \left(\frac{b}{2}\psi^2\right)_x + \left(\frac{c}{3}(\psi+\chi)^3\right)_x + k\psi_{xxx} + k\chi_{xxx} - \psi_{xx} = 0. \tag{3.10}$$

Since the initial data is small, we can expect that the higher order terms are small. Moreover, by virtue of Lemma 2.3 and Lemma 3.2, we see that ψ_{xxx} decays faster than the other terms at $t \to \infty$. For this, one may guess that the main term of asymptotic expansion of ψ at $t \to \infty$ is governed by the solution to the following equation:

$$v_t + (b\chi v)_x + \left(\frac{c}{3}\chi^3\right)_x + k\chi_{xxx} - v_{xx} = 0. \tag{3.11}$$

This observation leads to the following auxiliary problem:

$$\begin{gathered} z_t + (b\chi z)_x - z_{xx} = \partial_x \lambda(x,t), \quad t > 0, \quad x \in \mathbb{R}, \\ z(x,0) = z_0(x), \quad x \in \mathbb{R}, \end{gathered} \tag{3.12}$$

where $\lambda(x,t)$ is a given regular function decaying fast enough at spatial infinity. The explicit representation formula (3.14) below plays an important roles in the proof of the main theorem, especially in the proofs of Proposition 4.2 and Proposition 4.3 below. If we set

$$U[h](x,t,s) \equiv \int_{\mathbb{R}} \partial_x (G(x-y,t-s)\eta_1(x,t))\eta_2(y,s)\left(\int_{-\infty}^{y} h(\xi)d\xi\right)dy, \quad 0 \le s < t, \quad x \in \mathbb{R}, \tag{3.13}$$

then we have:

Lemma 3.4. *The solution of* (3.12) *is given by*

$$z(x,t) = U[z_0](x,t,0) + \int_0^t U[\partial_x \lambda(s)](x,t,s)ds, \quad t > 0, \quad x \in \mathbb{R}. \tag{3.14}$$

For the first term and the second term of (3.14), the following estimates were given by Kato [9].

Lemma 3.5. *Let m be a positive integer. Assume that $|\delta| \le 1$, $z_0 \in H^m(\mathbb{R}) \cap L^1_1(\mathbb{R})$ and $\int_{\mathbb{R}} z_0(x)dx = 0$. Then the estimate*

$$\|\partial_x^l U[z_0](\cdot,t,0)\|_{L^2} \le C(\|z_0\|_{H^l} + \|z_0\|_{L^1_1})(1+t)^{-3/4-l/2}, \quad t>0 \tag{3.15}$$

holds for any integer $0 \le l \le m$.

Lemma 3.6. *Let m be a positive integer. Assume that $|\delta| \le 1$ and $\lambda \in C^0(0,\infty;H^m) \cap C^0(0,\infty;W^{m,1})$. Then the estimate*

$$\begin{aligned}\left\|\partial_x^l \int_0^t U[\partial_x \lambda(s)](\cdot,t,s)ds\right\|_{L^2} \le & C\int_0^{t/2} (1+t-s)^{-3/4-l/2}\|\lambda(\cdot,s)\|_{L^1}ds \\ & + C\sum_{n=0}^{l}\int_{t/2}^t (1+t-s)^{-3/4}(1+s)^{-(l-n)/2}\|\partial_x^n\lambda(\cdot,s)\|_{L^1}ds \\ & + C\sum_{n=0}^{l}\left(\int_0^t e^{-(t-s)}(1+s)^{-(l-n)}\|\partial_x^n\lambda(\cdot,s)\|_{L^2}^2 ds\right)^{1/2}\end{aligned} \tag{3.16}$$

holds for any integer $0 \le l \le m$.

4 Sketch of the proof of Theorem 1.1.

In this section, we will give the sketch of the proof of Theorem 1.1. Based on the observation given in the previous section, we consider

$$\begin{aligned} v_t + (b\chi v)_x + \left(\frac{c}{3}\chi^3\right)_x + k\chi_{xxx} - v_{xx} = 0, \quad t>0, \quad x\in\mathbb{R}, \\ v(x,0) = 0, \quad x\in\mathbb{R}. \end{aligned} \tag{4.1}$$

Applying Lemma 3.4, the solution $v(x,t)$ to (4.1) is given by

$$v(x,t) = \int_0^t U\left[\left(\partial_x\left(-\frac{c}{3}\chi^3\right) - k\chi_{xxx}\right)(s)\right](x,t,s)ds. \tag{4.2}$$

By using the lemmas derived in Section 3, we can estimate derivatives of $v(x,t)$ as follows:

Lemma 4.1. *Let l be a non-negative integer. Assume that $|\delta| \le 1$. Then we have*

$$\|\partial_x^l v(\cdot,t)\|_{L^2} \le C|\delta|(1+t)^{-3/4-l/2}\log(1+t), \quad t \ge 0. \tag{4.3}$$

In particular, we get

$$\|\partial_x^l v(\cdot,t)\|_{L^\infty} \le C|\delta|(1+t)^{-1-l/2}\log(1+t), \quad t \ge 0. \tag{4.4}$$

We set

$$w(x,t) = u(x,t) - \chi(x,t) - v(x,t). \tag{4.5}$$

Then $w(x,t)$ satisfies the following equation:

$$\begin{aligned} w_t + (b\chi w)_x - w_{xx} = g(w,\chi,v)_x - kw_{xxx} - kv_{xxx}, \quad t>0, \quad x\in\mathbb{R}, \\ w(x,0) = w_0, \end{aligned} \tag{4.6}$$

where we have set

$$g(w,\chi,v) = -\frac{b}{2}(w+v)^2 - \frac{c}{3}\left(w^3+v^3+3(w+v)(w+\chi)(\chi+v)\right),$$

$$w_0(x) = u_0(x) - \chi(x,0).$$

By the assumption on the initial data, (1.4) and (3.3), we get $w_0 \in L^1_1(\mathbb{R}) \cap H^3(\mathbb{R})$ and $\int_{\mathbb{R}} w_0(x)dx = 0$. From Lemma 3.4, we get

$$w(x,t) = U[w_0](x,t,0) + \int_0^t U[\partial_x g(w,\chi,v)(s)](x,t,s)ds - k\int_0^t U[(w_{xxx}+v_{xxx})(s)](x,t,s)ds. \tag{4.7}$$

By using the above Lemmas, we can show the following proposition.

Proposition 4.2. *If $u_0 \in L^1_1(\mathbb{R}) \cap H^3(\mathbb{R})$ and $\|u_0\|_{L^1_1} + \|u_0\|_{H^3}$ is sufficiently small, then the estimate*

$$\|\partial_x^l(u(\cdot,t) - \chi(\cdot,t) - v(\cdot,t))\|_{L^2} \le C(\|u_0\|_{L^1_1} + \|u_0\|_{H^3})(1+t)^{-3/4-l/2}, \quad t \ge 0 \tag{4.8}$$

holds for $l = 0, 1$, where $\chi(x,t)$ is defined by (1.3)*, while $v(x,t)$ is the solution to* (4.1)*. In particular, we get*

$$\|u(\cdot,t) - \chi(\cdot,t) - v(\cdot,t)\|_{L^\infty} \le C(\|u_0\|_{L^1_1} + \|u_0\|_{H^3})(1+t)^{-1}, \quad t \ge 0. \tag{4.9}$$

On the other hand, the main term of $v(x,t)$ is governed by $V(x,t)$ as follows.

Proposition 4.3. *Assume that $|\delta| \le 1$. Then the estimate*

$$\|v(\cdot,t) - V(\cdot,t)\|_{L^\infty} \le C|\delta|(1+t)^{-1}, \ t \ge 1 \tag{4.10}$$

holds. Here $v(x,t)$ is the solution to (4.1) *and $V(x,t)$ is defined by* (1.14)*.*

Combining Proposition 4.2 and Proposition 4.3, we prove Theorem 1.1.

5 Idea of the derivation of the second asymptotic profile

We already mentionned that the second asymptotic profile of the solution to (1.1) is given by (1.13) (Theorem 1.1). We would like to introduce the idea of the derivation of the second asymptotic profile by heuristic discussion.

First we consider the heat equation:

$$\begin{aligned} u_t - u_{xx} = 0, \quad t > 0, \ x \in \mathbb{R}, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R} \end{aligned} \tag{5.1}$$

where $u_0(x) \in C_0(\mathbb{R})$ and $\int_{\mathbb{R}} u_0(x)dx \equiv \delta$. It is well known that the solution to (5.1) tends to $\delta G(x,t)$. Formally, we denote it

$$u(x,t) \sim \delta G(x,t), \quad t \to \infty. \tag{5.2}$$

Next, for a parameter $s \in \mathbb{R}$, we consider the following equation:

$$\begin{aligned} z_t - (b\chi z)_x - z_{xx} = 0, \quad t > s, \ x \in \mathbb{R}, \\ z(x,s) = z_0(x). \end{aligned} \tag{5.3}$$

We set

$$r(x,t) = \int_{-\infty}^{x} z(y,t)dy.$$

Then, it follows that

$$\begin{aligned} r_t - b\chi r_x - r_{xx} = 0, \quad t > s, \ x \in \mathbb{R}, \\ r(x,s) = \int_{-\infty}^{x} z(y,s)dy, \end{aligned} \tag{5.4}$$

so that $\eta_2(x,t)r(x,t)$ satisfies the heat equation

$$(\eta_2(x,t)r(x,t))_t - (\eta_2(x,t)r(x,t))_{xx} = 0. \tag{5.5}$$

Therefore, we have

$$(\eta_2(x,t)r(x,t)) \sim M[z](s)G(x,t-s), \quad t \to \infty, \tag{5.6}$$

or

$$r(x,t) \sim M[z](s)G(x,t-s)\eta_1(x,t), \quad t \to \infty, \tag{5.7}$$

where η_1 and η_2 are defined by (3.5) and (3.6), and $M[z](s) = \int_{\mathbb{R}} \eta_2(y,s)(\int_{-\infty}^{y} z(\xi,s)d\xi)dy$. Lemma 3.4, (5.4) and (5.7) lead that

$$z(x,t) = U[z_0](x,t,s) \sim M[z](s)(G(x,t-s)\eta_1(x,t))_x, \quad t \to \infty. \tag{5.8}$$

Now, recalling that the solution to (4.1) is given by (4.2), we have

$$\begin{aligned} v(x,t) &= \int_0^t U\left[\left(\partial_x\left(-\frac{c}{3}\chi^3\right)-k\chi_{xxx}\right)(s)\right](x,t,s)ds \\ &\sim \int_0^t M\left[\partial_x\left(-\frac{c}{3}\chi^3\right)-k\chi_{xxx}\right](s)(G(x,t-s)\eta_1(x,t))_x ds \\ &= \int_0^t \partial_x(G(x,t-s)\eta_1(x,t))\int_{\mathbb{R}}\eta_2(y,s)\left(-\frac{c}{3}\chi^3-k\chi_{xx}\right)(y,s)dyds \\ &= -d\left(\frac{b^2k}{8}+\frac{c}{3}\right)\int_0^t \partial_x(G(x,t-s)\eta_1(x,t))(1+s)^{-1}ds, \end{aligned} \tag{5.9}$$

where we used $\int_{\mathbb{R}}\eta_2(y,s)\chi(y,s)^3dy = d(1+s)^{-1}$ and $\int_{\mathbb{R}}\eta_2(y,s)\chi_{xx}(y,s)dy = (b^2d/8)(1+s)^{-1}$. Now, we rewrite

$$\begin{aligned} &\int_0^t \partial_x(G(x,t-s)\eta_1(x,t))(1+s)^{-1}ds \\ =&\eta_1(x,t)\int_0^t \partial_x G(x,t-s)(1+s)^{-1}ds + \partial_x\eta_1(x,t)\int_0^t G(x,t-s)(1+s)^{-1}ds. \end{aligned} \tag{5.10}$$

To proceed further, let us take $x=0$. Then we have

$$\begin{aligned} I(t) &\equiv \int_0^t \partial_x(G(x,t-s)\eta_1(x,t))\mid_{x=0}(1+s)^{-1}ds \\ &= \frac{b}{2}\chi(0,t)\eta_1(0,t)\int_0^t G(0,t-s)(1+s)^{-1}ds. \end{aligned} \tag{5.11}$$

Since the factor $(1+s)^{-1}$ is irrelevant to the integral when s is close to 0, we may guess that the leading term of the above integral is

$$\int_0^t G(0,t)(1+s)^{-1}ds = \frac{1}{\sqrt{4\pi}}t^{-1/2}\log(1+t). \tag{5.12}$$

Actually, it follows that

$$\begin{aligned} &\left|\int_0^t G(0,t-s)(1+s)^{-1}ds - \int_0^t G(0,t)(1+s)^{-1}ds\right| \\ =&\frac{1}{\sqrt{4\pi}}\int_0^t\left(\frac{1}{\sqrt{t-s}}-\frac{1}{\sqrt{t}}\right)\frac{1}{1+s}ds \\ =&\frac{1}{\sqrt{4\pi}}\int_0^t\frac{\sqrt{t}-\sqrt{t-s}}{\sqrt{t}\sqrt{t-s}}\frac{1}{1+s}ds \\ =&\frac{1}{\sqrt{4\pi}}\int_0^t\frac{1}{\sqrt{t}\sqrt{t-s}(\sqrt{t}+\sqrt{t-s})}\frac{s}{1+s}ds \\ \le&\frac{1}{\sqrt{4\pi}}\int_0^t\frac{1}{t\sqrt{t-s}}ds = \frac{1}{\sqrt{\pi}}t^{-1/2}. \end{aligned} \tag{5.13}$$

Therefore, $I(t)$ can be well approximated by

$$\frac{b}{4\sqrt{\pi}}\chi(0,t)\eta_1(0,t)t^{-1/2}\log(1+t). \tag{5.14}$$

As for the general case, replacing $G(x,t-s)$ by $G(x,t+1)$ and restricting the interval of the s-integral to $(0,t/2)$, from (5.9), we see that the leading term would be

$$\begin{aligned} v(x,t) \sim& -d\left(\frac{b^2k}{8}+\frac{c}{3}\right)\int_0^{t/2}\partial_x(G(x,1+t)\eta_1(x,t))(1+s)^{-1}ds \\ \sim& -d\left(\frac{b^2k}{8}+\frac{c}{3}\right)\partial_x(G(x,1+t)\eta_1(x,t))\log(1+t) \\ =&V(x,t). \end{aligned}$$

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ON UNIQUENESS FOR THE SUPERCRITICAL HARMONIC MAP HEAT FLOW

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Abstract. We examine the question of uniqueness for the equivariant reduction of the harmonic map heat flow in the energy supercritical dimension $d \geq 3$. It is shown that, generically, singular data can give rise to two distinct solutions which are both stable, and satisfy the local energy inequality. We also discuss how uniqueness can be retrieved. This is a joint work with Pierre Germain and Tej-Eddine Ghoul.

1. Introduction

1.1. **The equation, its scaling and energy.** The harmonic map heat flow is the following evolution equation

$$\begin{cases} \partial_t u - \Delta u = A(u)(\nabla u, \nabla u) \\ u(t=0) = u_0. \end{cases}$$

where $u_0 = u_0(x)$ is a map from $\mathbb{R}^d$ to a Riemannian manifold $M \subset \mathbb{R}^k$ with second fundamental form A; and the solution $u = u(t,x)$ is a map from $[0,\infty) \times \mathbb{R}^d$ to M.

The set of solutions is invariant by the scaling transform $u(t,x) \mapsto u(\lambda^2 t, \lambda x)$ for $\lambda > 0$. This makes scale-invariant spaces such as L^∞ of particular relevance for the data - we will come back to them.

The harmonic map heat flow is the gradient flow for the Dirichlet energy $\int |\nabla u|^2$, and as such, it enjoys (formally) an energy inequality. It can be localized to yield (formally) the following local energy inequality, for a solution of (1.4) on $[0,T] \times \mathbb{R}^d$, with data u_0, locally in $H^1_{t,x}$: for any τ, $(\psi^\ell)_{\ell=1\dots d}$ in $\mathcal{C}^\infty_0([0,\infty) \times \mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \frac{1}{2}\tau(t=0)|\nabla u_0|^2\,dx$$
$$\geq \int_0^\infty \int_{\mathbb{R}^d} \left[\tau|\partial_t u|^2 - \frac{1}{2}\left(\partial_t\tau + \partial_\ell\psi^\ell\right)|\nabla u|^2 + \partial_i\tau\partial_i u^k\partial_t u^k + \psi^\ell\partial_t u^k\partial_\ell u^k + \partial_i\psi^\ell\partial_i u^k\partial_\ell u^k\right] dx\,dt. \tag{1.1}$$

Comparing the scaling of the energy with that of the equation gives rise to the notion of criticality. In the following, we will focus on the supercritical case $\mathbf{d \geq 3}$.

1.2. **The solutions of the harmonic map heat flow and their uniqueness.** The harmonic map heat flow was introduced by Eells and Sampson [13], who also showed that the solution is globally smooth for smooth data if the sectional curvature of M is negative. Without this assumption, global smooth solutions are also guaranteed if the image of u_0 is contained in a ball of radius $\frac{\pi}{2\sqrt{\kappa}}$, where κ is an upper bound for the sectional curvature of M (see Jost [21], Lin-Wang [26]). Notice that in the case where M is a sphere, this corresponds

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to u_0 being valued in a hemisphere, a condition which will also play a role in the present article. It was then discovered that, without these assumptions, singularities can form out of smooth solutions: see Coron-Ghidaglia [11] and Chen-Ding [8].

Weak solutions can be built up for less regular data, simply of finite Dirichlet energy. This was established by Chen [7] and Rubinstein-Sternberg-Keller [29] for the case where the target is a sphere, and extended by Chen-Struwe [9] to general manifolds, with the help of the monotonicity formula discovered by Struwe [31].

Under which conditions are these weak solutions unique? Which is the largest space for the data yielding unique solutions? Successive improvements gave $W^{1,d}$ (Lin-Wang [27]), and, under a smallness condition on the data and the solution, L^∞ (Koch-Lamm [22]), and BMO (Wang [34]).

These last results are optimal: indeed, for data which are large in L^∞, but not continuous, uniqueness is lost, and examples of non-unique solutions can be constructed, see Coron [10], and Bethuel-Coron-Ghidaglia-Soyeur [1]. For these examples however, uniqueness can be salvaged if one requires that solutions satisfy the monotonicity formula (itself essentially a consequence of the local energy inequality, see Moser [28]). This led Struwe [32] to ask whether this criterion could indeed imply uniqueness. Germain and Rupflin [17] showed that this could not be the case, at least for very specific data, having in particular infinite energy.

These developments lead to the following questions; **can one describe the loss of uniqueness (or well-posedness) for generic large initial data in L^∞? Can one find a criterion to restore uniqueness in a meaningful way?**

We try and answer this question in the framework of equivariant solutions, which will be presented in the following subsection.

1.3. **The equivariant reduction and self-similar solutions.** For the rest of this paper, the target manifold M is the d-sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$. The harmonic map heat flow equation becomes

$$\begin{cases} \partial_t u - \Delta u = |\nabla u|^2 u \\ u(t=0) = u_0. \end{cases} \tag{1.2}$$

The corotational ansatz which we adopt requires that

$$u(t,x) = \begin{pmatrix} \cos(h(t,|x|)) \\ \sin(h(t,|x|))\frac{x}{|x|} \end{pmatrix} \tag{1.3}$$

for a radial function $h = h(t,r)$. Under this ansatz, the problem reduces to a scalar, radially symmetric PDE:

$$\begin{cases} h_t - h_{rr} - \dfrac{d-1}{r} h_r + \dfrac{d-1}{2r^2} \sin(2h) = 0 \\ h(t=0) = h_0. \end{cases} \tag{1.4}$$

Notice right away that $h \equiv 0$, $h \equiv \frac{\pi}{2}$, and $h \equiv \pi$ are stationary solutions of this equation; these particular solutions will be important in the following, and correspond geometrically to the north pole, the equator, and the south pole of the sphere.

For $d \geq 3$ (the supercritical case, which will occupy us), self-similar solutions play a key role in the dynamics. In particular, for $3 \leq d \leq 6$, it was observed numerically by Biernat and Bizon [2] that typical solutions develop singularities at the origin in space, which are then smoothly "resolved" after the singular time. Both of these phenomena are asymptotically self-similar: when a singularity is formed at time t_0, it can be described asymptotically by a

"shrinker solution" $\Psi\left(\frac{r}{\sqrt{t_0-t}}\right)$, while the resolution of the singularity is given by an "expander solution" $\psi\left(\frac{r}{\sqrt{t-t_0}}\right)$.

Shrinkers and expanders were studied analytically by Fan [14] and Germain-Rupflin [17] respectively. An expander solution $h(t,r)=\psi\left(\frac{r}{\sqrt{t}}\right)$ corresponds to very particular Cauchy data, namely $h_0\equiv\psi(\infty)$, the limit of ψ at ∞. Germain-Rupflin observed that different expanders share the same limit $\psi(\infty)$, yielding an example of non-uniqueness for the harmonic map heat flow even if the local energy inequality is required.

One of the goals of this article is showing the nonlinear stability of these self-similar expanders, and thus the genericity of the non-uniqueness result.

1.4. **Obtained results.**

1.4.1. *Expanders.* The starting point of our investigations is expander solutions of (1.4), hence of the form $h(t,r)=\psi\left(\frac{r}{\sqrt{t}}\right)$ with constant data $h(t=0)=\psi(\infty)\in[0,\pi]$. A direct computation shows that they solve the ODE

$$\psi''+\left(\frac{d-1}{\rho}+\frac{\rho}{2}\right)\psi'-\frac{d-1}{2\rho^2}\sin(2\psi)=0. \tag{1.5}$$

For expanders to be smooth and non-constant, we need to impose boundary data which are either of the form

$$\begin{cases}\psi(0)=0\\ \psi'(0)=\alpha,\end{cases} \tag{1.6}$$

(for $\alpha\in\mathbb{R}$) or of the form

$$\begin{cases}\psi(0)=\pi\\ \psi'(0)=\alpha\end{cases} \tag{1.7}$$

(for $\alpha\in\mathbb{R}$). The former case, $\psi(0)=0$, corresponds to $\psi(0)$ on the north pole of the sphere, while the latter case $\psi(0)=\pi$, corresponds to $\psi(0)$ on the south pole of the sphere. We therefore refer to them as north pole, or south pole, boundary conditions, respectively. Note that the equation (1.5) is obviously invariant under the transform: $\psi\mapsto\pi-\psi$.

The following theorem combines the existence result from Germain-Rupflin [17] with new contribution on uniqueness part in (i) and (iii).

Theorem 1.1. *Let $\ell\in\left[0,\frac{\pi}{2}\right]$ (the situation being obviously symmetrical if $\ell\in\left[\frac{\pi}{2},\pi\right]$).*

(i) *If $3\le d\le 6$, there exists a unique profile $\psi=\psi_N[\ell]$ satisfying (1.5), with north pole boundary conditions (1.6), $\psi(\infty)=\ell$, and $\psi([0,\infty))\subset\left[0,\frac{\pi}{2}\right]$.*

(ii) *If $3\le d\le 6$, there exists a constant $\delta^*>0$ such that: for $\ell\in[\frac{\pi}{2}-\delta^*,\frac{\pi}{2}]$, there exists a profile $\psi=\psi_S[\ell]$ satisfying (1.5), with south pole boundary conditions (1.7), $\psi(\infty)=\ell$, and which crosses exactly once $\frac{\pi}{2}$.*

(iii) *If $d\ge 7$ and for $\ell\in[0,\pi/2)$, there exists exactly one profile $\psi=\psi[\ell]$ satisfying (1.5), and $\psi(\infty)=\ell$. This profile has north pole boundary conditions (1.6).*

Remark 1.2. *Regarding the existence of expanders as well as their stability properties, the different behavior in the cases $d\le 6$ and $d\ge 7$ is linked to the minimizing properties of the equator map $x\mapsto\frac{x}{|x|}$: it is a local minimum of the Dirichlet energy if and only if $d\ge 7$.*

More general equivariant setups can be considered, and we believe that results similar to the case considered here can be obtained. In particular, the minimizing properties of the equator map should be decisive for the question of uniqueness.

1.4.2. *Stability of expanders.* Our next result shows that all expanders constructed above are stable at least locally. In particular, even if the uniqueness of the expanders is lost for $3 \le d \le 6$, the expanders are shown to be stable.

Theorem 1.3. *Let $\psi_N[\ell]$ and $\psi_S[\ell]$ be the expanders given in Theorem 1.1. Consider data $h_0 \in L^\infty$ and such that*

$$h_0(0) = \ell, \quad |h_0(r) - \ell| \lesssim r, \quad |\partial_r h_0(r)| \lesssim \frac{1}{r}, \quad |\partial_r^2 h_0(r)| \lesssim \frac{1}{r^2}.$$

(i) *If $3 \le d \le 6$ and $\ell \in [0, \pi/2]$, there exist $T > 0$ and a solution $h_N \in L^\infty(0,T;L^\infty)$ of (1.4) close to $\psi_N[\ell]$ in the following sense: for all $\epsilon > 0$, there exists $\zeta > 0$ such that*

$$t + r < \zeta \quad \Longrightarrow \quad \left| h_N(t,r) - \psi_N[\ell]\left(\frac{r}{\sqrt{t}}\right)\right| < \epsilon.$$

(ii) *If $3 \le d \le 6$ and $\ell \in \left[\frac{\pi}{2} - \delta^*, \frac{\pi}{2}\right]$[1], there exist $T > 0$ and a solution $h_S \in L^\infty(0,T;L^\infty)$ of (1.4) close to $\psi_S[\ell]$ in the following sense: for all $\epsilon > 0$, there exists $\zeta > 0$ such that*

$$t + r < \zeta \quad \Longrightarrow \quad \left| h_S(t,r) - \psi_S[\ell]\left(\frac{r}{\sqrt{t}}\right)\right| < \epsilon.$$

(iii) *If $d \ge 7$ and $\ell \in [0, \pi/2)$, there exist $T > 0$ and a solution $h \in L^\infty(0,T;L^\infty)$ of (1.4) close to $\psi[\ell]$ in the following sense: for all $\epsilon > 0$, there exists $\zeta > 0$ such that*

$$t + r < \zeta \quad \Longrightarrow \quad \left| h(t,r) - \psi[\ell]\left(\frac{r}{\sqrt{t}}\right)\right| < \epsilon.$$

Furthermore, all of these solutions satisfy the local energy inequality.

Remark 1.4. *The class of singular data[2] which we consider barely miss the framework for local well-posedness* [22, 27, 34]. *As is well-known, proving solvability for large initial data in a scale-invariant space which does not contain C_0^∞ as a dense subset is a delicate question, which cannot be answered by the usual fixed-point argument.*

Remark 1.5. *An interesting point of view is to regard the data h_0 as the trace at blow up time of a solution undergoing a self-similar blow up, for instance if the shrinkers of Fan* [14] *for $3 \le d \le 6$ can be proved to be stable, which seems very likely. Therefore, the problem we consider contains the problem of continuation after blow up.*

For $d \ge 7$, Bizon and Wasserman [4] *showed that there are no self-similar shrinkers, and type II blow up solutions were constructed by Biernat and Seki* [3] *very recently.*

1.4.3. *Uniqueness.* The above theorem shows that, for singular data, non-uniqueness occurs as soon as $3 \le d \le 6$ and $h_0(0) \in \left[\frac{\pi}{2} - \delta^*, \frac{\pi}{2}\right]$. This raises the question of uniqueness: when does unconditional, or conditional uniqueness hold?

Theorem 1.6. (i) *If $3 \le d \le 6$, then, for any continuous h_0 with*

$$0 \le h_0(0) < \frac{\pi}{2}, \quad |\partial_r h_0(r)| \lesssim \frac{1}{r}, \quad |\partial_r^2 h_0(r)| \lesssim \frac{1}{r^2},$$

there exists $T > 0$ such that for any sufficiently small $\delta > 0$, there is at most one solution to (1.4) in $L^\infty(0,T;L^\infty)$ satisfying the local energy inequality and

$$0 \le h(t,r) < \frac{\pi}{2} - \delta \quad \text{for } t + r < \delta. \tag{1.8}$$

(ii) *If $d \ge 7$, for any $h_0 \in L^\infty$ and $T > 0$, there is at most one solution $h \in L^\infty(0,T;L^\infty)$.*

[1] δ^* is the constant appearing in Theorem 1.1.

[2] Notice that if $h_0(0) \notin \pi\mathbb{Z}$, the map $u(0,r)$ defined by (1.3) cannot be continuous at $r = 0$.

Remark 1.7. *Note that time continuity in L^∞ is not required for uniqueness; it is important because the solutions we are interested in are not continuous at $t = 0$ in general.*

We did not try to give optimal conditions for the initial data, in order to have a concise statement. For instance, it suffices in (i) that h_0 is continuous at the origin, and that $|\partial_r h_0(r)| \lesssim \frac{1}{r}$ and $|\partial_r^2 h_0(r)| \lesssim \frac{1}{r^2}$ hold in a neighbourhood of the origin.

1.5. **Summary of obtained results and link with Ginzburg-Landau.** As a conclusion, we find that, if $d \geq 7$, uniqueness holds in the largest class possible, $L_t^\infty L_x^\infty$.

If $3 \leq d \leq 6$, the situation is much more intricate:

- For generic data (say h smooth, with $h(0)$ close to $\frac{\pi}{2}$), two distinct solutions can be built up in $L_t^\infty L_x^\infty$ (Theorem 1.3).
- These two distinct solutions satisfy the local energy inequality, and are stable; thus both seem to be physically relevant.
- A means of selecting one of the two is given by Theorem 1.6: requiring that a "local maximum principle" holds: namely, ask that, for t and r small, $h(t, r) - \frac{\pi}{2}$ has the same sign as $h_0(0) - \frac{\pi}{2}$.

This last criterion can be reformulated in terms of the Ginzburg-Landau regularization: the penalization problem

$$\partial_t u^\epsilon - \Delta u^\epsilon + \frac{1}{\varepsilon^2}(|u^\epsilon|^2 - 1)u^\epsilon = 0$$

converges, as $\epsilon \to 0$, to a solution u of the harmonic map heat flow (1.2).

This selection principle is of course very reminiscent of the situation for scalar conservation laws: uniqueness holds for entropy solutions, which can be constructed by viscous regularization, as was showed by Kružkov [25].

1.6. **Related results and questions.**

1.6.1. *The Navier-Stokes equation.* The theory of the Navier-Stokes equation is in many respects parallel to that of the harmonic map heat flow, but it scales differently: self-similar data are invariant by the transformation $u_0 \mapsto \lambda u_0(\lambda x, \lambda^2 t)$, and L^d is a scale-invariant space for the data.

Similar to the harmonic map heat flow, the Navier-Stokes equation is well-posed for small self-similar data, see for instance Giga-Miyakawa [18], Kozono-Yamazaki [24], Cannone-Planchon [5] and Koch-Tataru [23]. For large self-similar data, existence of a smooth self-similar solution has been recently shown by Jia-Sverak [19] by applying the Leray-Schauder theorem (see also Tsai [33] for discreately self-similar data). These authors also suggest in [20] non-uniqueness of the solution for self-similar data as a possible mechanism for ill-posedness of the Navier-Stokes equation in the energy space; such a possibility would be very similar to the behavior established here for the harmonic map heat flow.

1.6.2. *The Wave Map equation.* The hyperbolic counterpart of the harmonic map heat flow is the wave map; since it is time reversible, there is no difference between shrinkers and expanders. Such self-similar solutions have been constructed by Shatah [30], Cazenave-Shatah-Tahvildar-Zadeh [6], see also Germain [15, 16] and Widmayer [35] for their uniqueness properties. The nonlinear stability of these solutions (with a perturbation at the blow up time) seems to be an open problem, but stability of blow up has been established by Donninger [12].

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Scattering by the ring magnetic field in the three dimensional space

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1 Introduction

In 1959, Y. Aharonov and D. Bohm [Ah-Bo] stated that the magnetic field enclosed in an electrically shielded material can affect the electron moving outside the material through the magnetic vector potential. Since then, numerous experiments are devoted to prove the Aharonov-Bohm effect, and the most famous one is done by A. Tonomura et. al. [To]. They prepare toroidal apparatus in which they can completely enclose the magnetic field with the aid of the super-conductive effect. They observe the interference pattern caused by a plane wave going through inside / outside the hole of the torus, and the result clearly shows that the effect of the magnetic flux inside the torus.

The paper [Ah-Bo] also gives an interesting mathematical problem in the scattering theory. They consider the magnetic field perpendicular to the xy-plane enclosed inside an infinitesimally thin cylinder along z-axis. By using the cylindrical coordinate, the problem is reduced to analyze the two-dimensional Schrödinger operator with the one-point magnetic field at the origin (sometimes called the Aharonov-Bohm magnetic field, or the AB-magnetic field). The scattering theory for the AB-magnetic field in two dimensions is further developed by the following authors. S. N. M. Ruijsenaars [Ru] gives a mathematical foundation of the scattering theory for the AB-magnetic field. P. Šťovíček [St] tries to give the Green function for the two-point AB-magnetic field by the path integral approach. His method is also developed in the papers [Ko-St1, Ko-St2]. Y. Nambu [Na] tries to calculate the scattering amplitude for the n-point AB-magnetic field, but not fully successful. H. T. Ito and H. Tamura [It-Ta1, It-Ta2] give an asymptotic formula for the scattering amplitude for n-point AB-magnetic fields, in the limit the distance between the points tends to infinity.

In the papers above, the paper by Z.-Y. Gu and S.-W. Qian [Gu-Qi] seems to be missed. They consider the case of two-point AB-magnetic field with the same magnetic fluxes in the two-dimensional plane, and calculate the scattering amplitude by using the elliptic coordinate and the corresponding Mathieu functions [Ma]. However, as the author pointed out in [Mi1], the result by Gu-Qian is valid only in the case the *magnetic quantization condition*

$$\Phi \in \Phi_0 \mathbb{Z} \tag{1}$$

holds, where Φ is the flux of each point magnetic field and $\Phi_0 = h/(2e)$ is the magnetic flux quantum. The author [Mi2] also gives an explicit formula for the scattering amplitude when the magnetic quantization condition (1) holds.

On the other hand, there are not so many results about the Aharonov-Bohm effect in the three dimensional space $\mathbb{R}^3$. Part of the reason is that we can choose compactly supported vector potential for any compactly supported magnetic field in $\mathbb{R}^3$, which makes the problem qualitatively different from the two-dimensional one (see e.g. the

review by Yafaev [Ya]). Another reason is that we did not have an explicitly solvable model for the Aharonov-Bohm effect in $\mathbb{R}^3$, except the translational invariant model like Aharonov-Bohm's one. An important contribution is done by M. Ballesteros and R. Weder [Ba-We1, Ba-We2, Ba-We3]; they consider the magnetic field in $\mathbb{R}^3$ enclosed in several tori, and study the inverse problem in the high velocity limit. Their results can be thought of as a mathematical justification of the Tonomura experiment.

In the present note, we take a different approach to the mathematical analysis of the Tonomura experiment. We consider the magnetic field enclosed inside a ring of thickness 0, satisfying the magnetic quantization condition (1). Actually, in the Tonomura experiment, the magnetic flux is quantized to be an integer multiple of the magnetic flux quantum, by the super-conductive effect. We give the explicit form of the generalized eigenfunctions and the scattering amplitude for this model, by using the oblate spheroidal coordinate and the corresponding spheroidal wave functions. This method is an extension of the method used in the preceding paper [Mi2]. At present, we can only treat the idealized ring of thickness 0, and the method does not work for the ring of positive thickness.

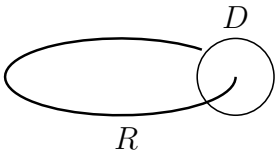

Figure 1: Infinitesimally thin ring R.

The Mathieu functions and the spheroidal wave functions are extensively studied before 1950, and the formulas are collected in N. W. McLachlan [Mc], C. Flammer [Fl], or M. Abramowitz and I. A. Stegun [Ab-St]. The elliptic coordinate and the spheroidal coordinate are also used in some recent papers; besides the one stated above, M. Seri et. al. [Se] analyze the resonances for the two-center Coulomb system by using these coordinates. Nowadays the numerical method becomes so powerful, and the treatment of these higher transcendental function also becomes feasible. We hope the analysis of these functions gives us new insight in the theory of mathematical physics.

2 Schrödinger operator with quantized ring magnetic field

We consider the quantum magnetic Hamiltonian on $\mathbb{R}^3$

$$H = \frac{1}{2m}(p - qA)^2 = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - qA\right)^2, \tag{2}$$

where m is the mass of an electron, $\hbar$ the Planck constant h divided by 2π, $p = (\hbar/i)\nabla$ the momentum operator, $q = -e$ the charge of an electron, $A = (A_1, A_2, A_3)$ the magnetic vector potential. The corresponding magnetic field B is given by

$$B = \nabla \times A = \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3},\ \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1},\ \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right).$$

Let R be an infinitesimally thin ring in $\mathbb{R}^3$ with radius a like Figure 1, given by

$$R = \{x \in \mathbb{R}^3 \mid |x| = a,\ x_3 = 0\}.$$

We assume the magnetic field B is invariant by the rotation along x_3-axis, and is enclosed inside R, that is,

$$\operatorname{supp} B = R. \tag{3}$$

We also assume the magnetic flux inside the ring is just the magnetic flux quantum, that is,

$$\int_D B \cdot \mathbf{n}\, dS = \int_{\partial D} A \cdot d\ell = \frac{h}{2e} = \frac{\pi\hbar}{e}, \tag{4}$$

where D is any small disc pierced by the ring R (see Figure 1), ∂D the boundary of D, $\mathbf{n}$ the unit normal vector on D (the orientation is fixed appropriately), dS the surface element on D, and $d\ell$ the line element on ∂D. Actually, the result is unchanged even if the magnetic flux is an odd multiple of $h/(2e)$ (see (23) below), but we assume (4) for simplicity. The conditions (3) and (4) should be interpreted in the distributional sense; in particular, the left hand side in the first equality of (4) is defined by the right hand side. We can actually construct the magnetic vector potential $A \in C^\infty(\mathbb{R}^3 \backslash R; \mathbb{R}^3) \cap L^1_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ satisfying (3) and (4), so that $\operatorname{supp} A$ is a bounded set in $\mathbb{R}^3$.

We put $\tilde{A} = (q/\hbar)A$. Then the Hamiltonian H in (2) is written as $H = (\hbar^2/(2m))\widetilde{H}$, where

$$\widetilde{H} = \left(\frac{1}{i}\nabla - \tilde{A}\right)^2, \tag{5}$$

$$\int_{\partial D} \tilde{A} \cdot d\ell = \frac{q}{\hbar} \cdot \frac{\pi\hbar}{e} = -\pi. \tag{6}$$

The motion of a quantum particle is essentially determined by the operator $\widetilde{H}$, so the equality (6) shows that the magnetic quantization condition (4) is essentially independent of the value of the physical constants $\hbar$, e, and m. For simplicity, we assume

$$\hbar = 1,\ e = 1,\ m = 1/2 \tag{7}$$

in the sequel. Then the equalities (5) and (6) hold without tilde mark $\tilde{}$.

3 Review of the scattering theory in $\mathbb{R}^3$

3.1 Basic definitions

Under the normalization (7), the wave function ψ of a quantum mechanical particle is governed by the Schrödinger equation

$$i\frac{\partial\psi}{\partial t}(t,x) = H\psi(t,x), \quad \psi(0,x) = u(x),$$

where u is the initial state at time $t = 0$. The solution is given by $\psi(t,x) = e^{-itH}u(x)$, defined by the functional calculus. Let $H_0 = -\Delta$ be the free Hamiltonian, where Δ is the

three dimensional Laplacian. Then, the wave operators $W_\pm$ and the scattering operator S are defined by

$$W_\pm = \text{s-}\lim_{t\to\pm\infty} e^{itH}e^{-itH_0},$$
$$S = W_+^* W_-,$$

where s-lim denotes the strong limit, that is, $e^{itH}e^{-itH_0}u$ converges to $W_\pm u$ as $t \to \pm\infty$ in $L^2(\mathbb{R}^3)$ for each $u \in L^2(\mathbb{R}^3)$. The physical meaning of S is as follows. For any initial state u_- in the free system, put $u = W_- u_-$ and $u_+ = W_+^* u = Su_-$. Then, we have by definition

$$e^{-itH}u - e^{-itH_0}u_\pm \to 0 \quad \text{as} \quad t \to \pm\infty,$$

that is, the asymptotic behavior of $\psi = e^{-itH}u$ in the perturbed system is approximated by $\psi_\pm = e^{-itH_0}u_\pm$ in the free system, as $t \to \pm\infty$. Thus the scattering operator S describes the scattering procedure in the system.

Next we explain the definition of the scattering amplitude. By the conservation of energy, the scattering operator S maps a state with energy E into a state with energy E. Thus the operator $\mathcal{F}S\mathcal{F}^*$ ($\mathcal{F}$ is the Fourier transform) is decomposed into the direct integral of the operators $S(E)$ $(E > 0)$ acting on $L^2(S_E)$, where S_E is the energy shell

$$S_E = \{\xi \in \mathbb{R}^3 \mid |\xi|^2 = E\} = \{\sqrt{E}\omega \mid \omega \in S^2\}.$$

Then the *scattering amplitude* $f(k^2;\omega,\omega')$ is defined by the formula

$$(S(k^2) - I)(\omega,\omega') = \frac{ki}{2\pi} f(k^2;\omega,\omega') \quad (\omega,\omega' \in S^2), \tag{8}$$

where the left hand side is the integral kernel of the operator $S(k^2) - I$. The quantity $|f(k^2;\omega,\omega')|^2$ is called the *differential scattering cross section*, which is proportional to the ratio of the particles with energy k^2, coming from the incident direction ω', and scattered into the final direction ω. The main purpose of the present note is to give an explicit formula for the scattering amplitude $f(k^2;\omega,\omega')$ for our ring magnetic field.

3.2 Scattering amplitude for radial scalar potentials

Before calculating the scattering amplitude for our system, we review how to calculate the scattering amplitude for the Schrödinger operators $H = -\Delta + V$ with radial scalar potentials $V = V(r)$ decaying sufficiently fast at ∞.

As is well-known, the spherical coordinate in $\mathbb{R}^3$ is defined as follows.

$$\begin{cases} x_1 = r\sin\theta\cos\phi, \\ x_2 = r\sin\theta\sin\phi, \\ x_3 = r\cos\theta, \end{cases}$$
$$r \geq 0,\ 0 \leq \eta \leq \pi,\ -\pi < \phi \leq \pi.$$

In the spherical coordinate, the Laplacian is written as

$$\Delta = \frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right).$$

The Helmholtz equation $-\Delta u = k^2 u$ $(k > 0)$ can be solved by the separation of variables $u = f(r)g(\theta)h(\phi)$. We require f is finite at $r = 0$, g is finite at $\theta = 0, \pi$, and h has period 2π. Then the solutions are written as

$$\begin{aligned} &u = j_\ell(kr) P_\ell^m(\cos\theta)\cos(m\phi), \\ &u = j_\ell(kr) P_\ell^m(\cos\theta)\sin(m\phi) \quad (m \neq 0), \\ &\ell = 0, 1, 2, \ldots, \quad m = 0, 1, \ldots, \ell, \end{aligned} \tag{9}$$

where $j_\ell(z) = \sqrt{\pi/(2z)} J_{\ell+1/2}(z)$ is the spherical Bessel function, and P_ℓ^m is the associated Legendre function. The number ℓ is called the azimuthal quantum number, and m the magnetic quantum number. The completeness of these solutions is guaranteed by the following formula.

Proposition 3.1 (Rayleigh's plane wave expansion formula)**.** *Let (r, θ, ϕ) and (k, τ, ψ) be the spherical coordinates for x and p, respectively, that is,*

$$\begin{aligned} x &= (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta), \\ p &= (k\sin\tau\cos\psi, k\sin\tau\sin\psi, k\cos\tau). \end{aligned}$$

Then, we have

$$\begin{aligned} e^{ix\cdot p} &= \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\Big[i^\ell c_{\ell,m}^2 j_\ell(kr) P_\ell^m(\cos\theta) P_\ell^m(\cos\tau)\cos(m(\phi-\psi))\Big], \\ c_{\ell,m} &= \sqrt{(2-\delta_{0,m})(2\ell+1)\frac{(\ell-m)!}{(\ell+m)!}}. \end{aligned} \tag{10}$$

Next, let us explain how to calculate the scattering amplitude for the Schrödinger operator $H = -\Delta + V(r)$ with a radial potential $V(r)$ decaying sufficiently fast at ∞. The radial part $j_\ell(kr)$ of the free generalized eigenfunctions (9) has the asymptotics

$$j_\ell(kr) \sim \frac{1}{kr}\cos\left(kr - \frac{(\ell+1)\pi}{2}\right) \quad (r \to \infty). \tag{11}$$

If $V = V(r)$ decays sufficiently fast as $r \to \infty$, we can prove that the perturbed operator $H = -\Delta + V$ has generalized eigenfunctions with eigenvalue k^2 $(k > 0)$

$$\begin{aligned} &u = u_{\ell,k}(r) P_\ell^m(\cos\theta)\cos(m\phi), \\ &u = u_{\ell,k}(r) P_\ell^m(\cos\theta)\sin(m\phi) \quad (m \neq 0), \\ &\ell = 0, 1, 2, \ldots, \quad m = 0, 1, \ldots, \ell, \end{aligned}$$

and $u_{\ell,k}(r)$ has the asymptotics

$$u_{\ell,k}(r) \sim \frac{1}{kr}\cos\left(kr - \frac{(\ell+1)\pi}{2} + \delta_{\ell,k}\right) \quad (r \to \infty), \tag{12}$$

where $\delta_{\ell,k}$ is a real constant called the *scattering phase shift*, in comparison with (11). Then, the scattering amplitude defined by (8) is calculated as follows.

Proposition 3.2 (Scattering amplitude for a radial potential)**.** *Let $H = -\Delta + V(r)$ and $V(r)$ decays sufficiently fast at infinity. Assume all the scattering phase shifts $\delta_{\ell,k}$ in (12) are known. Then, the scattering amplitude is given by*

$$\begin{aligned} f(k^2;\omega,\omega') &= \frac{1}{2ik}\sum_{\ell=0}^{\infty}(e^{2i\delta_{\ell,k}}-1)\sum_{m=0}^{\ell} c_{\ell,m}^2 P_\ell^m(\cos\theta)P_\ell^m(\cos\tau)\cos(m(\phi-\psi)) \\ &= \sum_{\ell=0}^{\infty}\frac{e^{2i\delta_{\ell,k}}-1}{2ik}(2\ell+1)P_\ell(\omega\cdot\omega') \end{aligned}$$

for $\omega,\omega' \in S^2$, where P_ℓ is the Legendre function and $\omega\cdot\omega'$ is the scalar product of ω and ω'.

The final result depends only on the angle between ω and ω', because of the spherical symmetry of the system.

4 The oblate spheroidal coordinate

In order to analyze the ring magnetic field, we introduce the *oblate spheroidal coordinate*, defined as follows.

$$\begin{cases} x_1 = a\cosh\xi\sin\eta\cos\phi, \\ x_2 = a\cosh\xi\sin\eta\sin\phi, \\ x_3 = a\sinh\xi\cos\eta), \end{cases}$$
$$\xi \geq 0,\ 0 \leq \eta \leq \pi,\ -\pi < \phi \leq \pi,$$

where a is a positive constant. The equation $\xi = 0$, $\eta = \pi/2$ corresponds the ring R (see Figure 1).

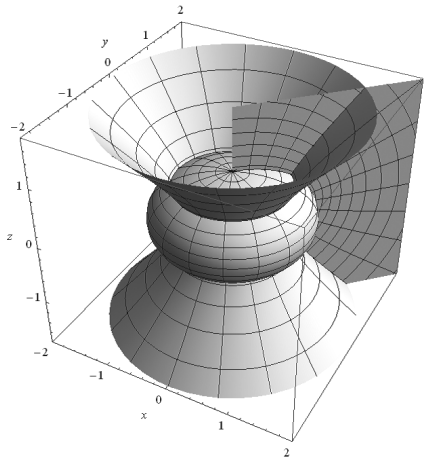

Figure 2: The coordinate surfaces for the oblate spheroidal coordinates.

In the oblate spheroidal coordinate, we have

$$\begin{aligned}\Delta u &= \frac{1}{a^2\cosh\xi\sin\eta(\cosh^2\xi-\sin^2\eta)} \\ &\quad\cdot\left(\sin\eta\frac{\partial}{\partial\xi}\left(\cosh\xi\frac{\partial u}{\partial\xi}\right)+\cosh\xi\frac{\partial}{\partial\eta}\left(\sin\eta\frac{\partial u}{\partial\eta}\right)+\frac{\cosh^2\xi-\sin^2\eta}{\cosh\xi\sin\eta}\frac{\partial^2 u}{\partial\phi^2}\right).\end{aligned}$$

By the separation of variable $u = f(\xi)g(\eta)h(\phi)$, the Helmholtz equation $-\Delta u = k^2 u$ is reduced to the following ordinary differential equations.

$$\frac{1}{\cosh\xi}\frac{d}{d\xi}\left(\cosh\xi\frac{df}{d\xi}\right)+\left(a^2k^2\cosh^2\xi-\mu+\frac{m^2}{\cosh^2\xi}\right)f=0, \tag{13}$$

$$\frac{1}{\sin\eta}\frac{d}{d\eta}\left(\sin\eta\frac{dg}{d\eta}\right)+\left(-a^2k^2\sin^2\eta+\mu-\frac{m^2}{\sin^2\eta}\right)g=0, \tag{14}$$

$$\frac{d^2h}{d\phi^2}=-m^2h, \tag{15}$$

where m, μ are the separation constants. We require

(a) $g(\eta)$ is finite at $\eta = 0, \pi$,

(b) $h(\phi)$ has period 2π, and

(c) u is single-valued with respect to the original coordinate x.

By (b) and (15), we have

$$\begin{aligned}h(\phi)&=\cos(m\phi)\quad(m=0,1,2,\ldots),\\ h(\phi)&=\sin(m\phi)\quad(m=1,2,\ldots).\end{aligned}$$

We put $\lambda = \mu - a^2k^2$, $c = -iak$. By the change of variable $z = i\sinh\xi$ in (13), and $w = \cos\eta$ in (14), we have

$$\frac{d}{dz}\left((1-z^2)\frac{df}{dz}\right)+\left(\lambda-c^2z^2-\frac{m^2}{1-z^2}\right)f=0, \tag{16}$$

$$\frac{d}{dw}\left((1-w^2)\frac{dg}{dw}\right)+\left(\lambda-c^2w^2-\frac{m^2}{1-w^2}\right)g=0. \tag{17}$$

Thus the two equations (16) and (17) are equivalent as equations for complex variables. Especially when $c = 0$, these equations become the associated Legendre differential equation.

By the requirement (a), we need the solution g to (17) finite at $w = \pm 1$. For fixed $m = 0, 1, 2, \ldots$, there are at most countable values of λ's for which the equation (17) has a non-trivial solution finite at $w = \pm 1$. These values are denoted by

$$\lambda_{m\ell}\quad(\ell=m,m+1,m+2,\ldots),$$

and called *the spheroidal eigenvalues* (the definition of the spheroidal eigenvalue depends on the reference, and the above one is adapted in Flammer [Fl] or Abramowitz–Stegun [Ab-St]). The corresponding solution g is denoted by $S_{m\ell}(c,w)$, and called the *angular spheroidal wave function (of the first kind)*. When $c=0$, $S_{m\ell}(0,w)$ coincides with the associated Legendre function $P_\ell^m(w)$. The function $S_{m\ell}(c,w)$ is normalized as

$$\int_{-1}^{1} |S_{m\ell}(c,w)|^2 dw = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!},$$

which is the same normalization as the associated Legendre function. The function $S_{m\ell}(c,z)$ is also a solution to the radial equation (16) with the same parameter $\lambda=\lambda_{m\ell}$. We introduce another solution $R^{(1)}_{m\ell}(c,z)$, which is a constant multiple of $S_{m\ell}(c,z)$, and behaves like

$$R^{(1)}_{m\ell}(c,z) \sim \frac{1}{cz}\cos\left(cz - \frac{\ell+1}{2}\pi\right) \quad \text{as} \quad z \to i\infty. \tag{18}$$

The limit $z\to i\infty$ corresponds the limit $\xi\to\infty$ ($z=i\sinh\xi$), and

$$cz = -iak\cdot i\sinh\xi = k\cdot a\sinh\xi \sim kr \quad (\xi\to\infty).$$

Comparing with (11), we see that (18) means $R^{(1)}_{m\ell}(c,z)$ behaves like usual spherical Bessel function at infinity. We call $R^{(1)}_{m\ell}(c,z)$ the *radial spheroidal wave function of the first kind*.

Notice that the left hand side of the equation (16) preserves the parity of f. Actually, $R^{(1)}_{m\ell}(c,z)$ is an even function if $\ell-m$ is even, and an odd function if $\ell-m$ is odd (the same as $P_\ell^m(z)$). Then, there exists a non-trivial solution to (16) whose parity is opposite to that of $R^{(1)}_{m\ell}(c,z)$. We denote such solution by $R^{(5)}_{m\ell}(c,z)$ ($R^{(j)}_{m\ell}(c,z)$ for $j=2,3,4$ are already defined in Abramowitz-Stegun [Ab-St]). We normalize $R^{(5)}_{m\ell}(c,z)$ so that there exists a constant $\delta^m_{\ell,k}$ such that

$$R^{(5)}_{m\ell}(c,z) \sim \frac{1}{cz}\cos\left(cz - \frac{\ell+1}{2}\pi + \delta^m_{\ell,k}\right) \quad \text{as} \quad z\to i\infty. \tag{19}$$

Later we see that the constant $\delta^m_{\ell,k}$ serves as the scattering phase shift, in the scattering problem by the ring magnetic field.

Let us consider the requirement (c). Since the two coordinates (ξ,η,ϕ) and $(-\xi,\pi-\eta,\phi)$ in the spheroidal coordinate give the same point x in $\mathbb{R}^3$, (c) implies

$$f(\xi)g(\eta) = f(-\xi)g(\pi-\eta). \tag{20}$$

Since $z=i\sinh\xi$ and $w=\cos\eta$, the condition (20) requires f and g have the same parity with respect to z and w, respectively.

Let us summarize the results obtained above.

Proposition 4.1. *In the oblate spheroidal coordinate, the generalized eigenfunctions for the free Hamiltonian $H_0=-\Delta$ with eigenvalue k^2 are given as*

$$\begin{aligned}
&u = S_{m\ell}(-iak, i\sinh\xi)R^{(1)}_{m\ell}(-iak,\cos\eta)\cos m\phi,\\
&u = S_{m\ell}(-iak, i\sinh\xi)R^{(1)}_{m\ell}(-iak,\cos\eta)\sin m\phi \quad (m\neq 0),\\
&m=0,1,2,\dots, \quad \ell = m, m+1, m+2,\dots.
\end{aligned}$$

The completeness of the above solutions is guaranteed by the following expansion formula, taken from Flammer's book [Fl].

Proposition 4.2 (Plane wave expansion formula in the oblate spheroidal coordinate)**.** *We use the oblate spheroidal coordinate in x-space and the spherical coordinate in p-space, that is,*

$$\begin{cases} x_1 = a\cosh\xi\sin\eta\cos\phi, \\ x_2 = a\cosh\xi\sin\eta\sin\phi, \\ x_3 = a\sinh\xi\cos\eta, \end{cases} \qquad \begin{cases} p_1 = k\sin\tau\cos\psi, \\ p_2 = k\sin\tau\sin\psi, \\ p_3 = k\cos\tau. \end{cases}$$

Then, we have

$$\begin{aligned} e^{ix\cdot p} &= \sum_{m=0}^{\infty}\sum_{\ell=m}^{\infty} i^\ell c_{\ell,m}^2 R_{m\ell}^{(1)}(-iak, i\sinh\xi) \\ &\quad \cdot S_{m\ell}(-iak,\cos\eta) S_{m\ell}(-iak,\cos\tau)\cdot\cos m(\phi-\psi). \end{aligned} \tag{21}$$

Here, the normalization constants $c_{\ell,m}$ is the one given in (10) (notice that $R_{m\ell}^{(1)}(-iak, i\sinh\xi)$ and $j_\ell(kr)$ has the same asymptotics as $\xi\to\infty$ or $r\to\infty$).

The above results are useful in solving the Dirichlet eigenvalue problem in a spheroidal region, or the scattering problem outside a spheroidal body. In the next section, we use the above results for the calculation of the scattering amplitude by the ring magnetic field.

5 Scattering by the quantized ring magnetic field

Let us return to our model, the Schrödinger operator H with quantized ring magnetic field B, given by (2), (3) and (4). Take $x_0\in\mathbb{R}^3$ outside the support of the vector potential A (we assume $\operatorname{supp} A$ is bounded). Define a phase function Φ by the line integral

$$\Phi(x) = \exp\left(i\int_{x_0}^{x} A\cdot d\ell\right). \tag{22}$$

The magnetic quantization condition (6) implies the function Φ is *two-valued*, since

$$\exp\left(i\int_{\partial D} A\cdot d\ell\right) = e^{-i\pi} = -1 \tag{23}$$

for any small disc D pierced by R (see Figure 1). Moreover,

$$\Phi\left(\frac{1}{i}\nabla\right)\Phi^{-1}u = \left(\frac{1}{i}\nabla - A\right)u.$$

Thus we have the intertwining relation

$$Hu = \Phi(-\Delta)\Phi^{-1}u.$$

Put $v = \Phi^{-1}u$. Then, we have

$$Hu = k^2 u \Leftrightarrow -\Delta v = k^2 v. \tag{24}$$

But v is a *two-valued* function in the sense that $v(x)$ changes the sign when x turns around along the boundary ∂D of the disc D (Figure 1).

In order to solve (24), we again use the separation of variable $v = f(\xi)g(\eta)h(\phi)$. Then f, g, h again satisfy the equations (13), (14), and (15), but the requirement (c) must be changed into the following (c)', since v is a two-valued function stated above.

(c)' $v(x)$ changes the sign when x turns around along the boundary ∂D of a small disc D pierced by the ring R.

The requirement (c)' is equivalent to the condition

$$f(\xi)g(\eta) = -f(-\xi)g(\pi - \eta),$$

which means f and g have opposite parities with respect to $z = i\sinh\xi$ and $w = \cos\eta$, respectively. Thus we need the radial solution $R^{(5)}_{m\ell}$ introduced in the previous section. To be summarized, we obtain the following.

Proposition 5.1. *H has generalized eigenfunctions with energy k^2 ($k > 0$) given by*

$$\begin{aligned}
&u = \Phi \cdot S_{m\ell}(-iak, i\sinh\xi) R^{(5)}_{m\ell}(-iak, \cos\eta)\cos m\phi,\\
&u = \Phi \cdot S_{m\ell}(-iak, i\sinh\xi) R^{(5)}_{m\ell}(-iak, \cos\eta)\sin m\phi \quad (m \neq 0),\\
&m = 0, 1, 2, \ldots, \quad \ell = m, m+1, m+2, \ldots,
\end{aligned}$$

where Φ is the phase function in (22).

Remind that the phase shifts $\delta^m_{\ell,k}$ are introduced in (19). In this case, $\delta^m_{\ell,k}$ depends on both ℓ and m, while the phase shifts depends only on ℓ in the case of radial scalar V. Then the scattering amplitude is calculated as follows.

Theorem 5.2. *We introduce the spherical coordinate (τ, ψ) in S^2 as*

$$\begin{aligned}
&\omega = (\sin\tau\cos\psi, \sin\tau\sin\psi, \cos\tau),\\
&\omega' = (\sin\tau'\cos\psi', \sin\tau'\sin\psi', \cos\tau').
\end{aligned}$$

Then, the scattering amplitude with energy k^2 for the pair H and $H_0 = -\Delta$ is

$$\begin{aligned}
f(k^2; \omega, \omega') \;=\; & \frac{1}{2ik}\sum_{m=0}^{\infty}\sum_{\ell=m}^{\infty}(e^{2i\delta^m_{\ell,k}} - 1)c^2_{\ell,m}\\
& \cdot S_{m\ell}(-iak, \cos\tau) S_{m\ell}(-iak, \cos\tau')\cos(m(\psi - \psi')).
\end{aligned} \tag{25}$$

The proof of this theorem is similar to that of Proposition 3.2. We describe the wave operators $W_\pm$ and the scattering operator S explicitly in terms of the scattering phase shifts, and calculate the integral kernel of S with the aid of the plane wave expansion formula (21).

Using the formula (25), we can calculate the scattering amplitude $f(k^2;\omega,\omega')$ at least numerically. A numerical experiment shows that the phase shift $\delta^m_{\ell,k}$ decays rapidly as ℓ increases, so first several terms may give a good approximation, at least low energy region.

The spheroidal wave function is installed in some numerical tools, but sometimes the output seems to be unreliable (possibly because such function is not frequently used). Now the author is trying to improve the numerical method, which will give us more understanding of the Tonomura experiment.

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TIME DECAY OF SCHRÖDINGER EVOLUTIONS: THE ROLE PLAYED BY THE ANGULAR HAMILTONIAN

LUCA FANELLI

1. Abstract

Consider the free Schrödinger equation

$$\partial_t \psi = i\Delta\psi, \qquad \psi(0,x) = \psi_0(x), \tag{1.1}$$

where $\psi = \psi(t,x) : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$. The solution to (1.1) with initial datum $\psi_0(x) \in L^2(\mathbb{R}^d)$ is given by

$$\psi(t,x) = (4\pi i t)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}} * \psi_0(x) = (4\pi i t)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}} \int_{\mathbb{R}^d} e^{i\frac{x\cdot y}{2t}} e^{i\frac{|y|^2}{4t}} \psi_0(y)\, dy. \tag{1.2}$$

Let us denote by $e^{it\Delta}$ the one-parameter flow on $L^2(\mathbb{R}^d)$ defined by $e^{it\Delta}\psi_0(\cdot) = \psi(t,\cdot)$, being ψ given (1.2). We see immediately that

$$\left\|e^{it\Delta}\psi_0(\cdot)\right\|_{L^2(\mathbb{R}^d)} = \|\psi_0\|_{L^2(\mathbb{R}^d)}, \qquad \forall t\in\mathbb{R}, \tag{1.3}$$

$$\left\|e^{it\Delta}\psi_0(\cdot)\right\|_{L^\infty(\mathbb{R}^d)} \leqslant C|t|^{-\frac{d}{2}}\|\psi_0\|_{L^1(\mathbb{R}^d)}, \qquad \forall t\in\mathbb{R}, \tag{1.4}$$

for some $C>0$ independent on t and ψ_0. By Riesz-Thorin we obtain the full list of *time decay estimates* for the free Schrödinger equation

$$\left\|e^{it\Delta}\psi_0(\cdot)\right\|_{L^p(\mathbb{R}^d)} \leqslant C|t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|\psi_0\|_{L^{p'}(\mathbb{R}^d)}, \qquad \forall t\in\mathbb{R}, \qquad \forall p\geqslant 2 \tag{1.5}$$

These inequalities (1.5) turn out to be a crucial tool in Scattering Theory and Nonlinear Analysis; in particular, they imply the so called *Strichartz estimates* (see the standard reference [16]), which play a fundamental role both for fixed point results and as Restriction Theorems for the Fourier transform:

$$\left\|e^{it\Delta}\psi_0\right\|_{L^q_t L^r_x} \leqslant C\|\psi_0\|_{L^2(\mathbb{R}^d)}, \tag{1.6}$$

with $2/q = d/2 - d/r$, $q\geqslant 2$ and $(q,r,d)\neq(2,\infty,2)$, and

$$\left\|e^{it\Delta}\psi_0(\cdot)\right\|_{L^p(\mathbb{R}^d)} := \left\|\left\|e^{it\Delta}\psi_0(\cdot)\right\|_{L^r(\mathbb{R}^d)}\right\|_{L^q(\mathbb{R})}.$$

We o give it a deeper insight on estimate (1.4). It is clear by (1.2) that a crucial role is played by the plane wave $K(x,y) := e^{i\frac{x\cdot y}{2t}}$ which satisfies

$$\sup_{x,y\in\mathbb{R}^d}\left|e^{i\frac{x\cdot y}{2t}}\right| = 1 < \infty, \qquad \forall t\neq 0. \tag{1.7}$$

We stress that the Heat flow shares a completely analogous behavior, for positive times:

$$\partial_t u = \Delta u, \qquad u(0,x) = u_0(x) \in L^p(\mathbb{R}^d), \tag{1.8}$$

since the solution is given by

$$u(t,x) = (4\pi t)^{-\frac{d}{2}} e^{\frac{-|x|^2}{4t}} * u_0(x), \qquad (t>0) \tag{1.9}$$

for all $p \in [1,+\infty]$. This shows that (1.8) satisfies the same a priori estimates (1.5) as equation (1.1). Notice that (1.1) and (1.8) enjoy the same scaling invariance:

$$f_\lambda(t,x) := f\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \qquad \lambda > 0.$$

In addition, the Gaussian fundamental solution in (1.9) is much smoother than the oscillating character of the fundamental solution in (1.2), and leads to much stronger phenomena than the ones led by the dispersive flow $e^{it\Delta}$. Nevertheless, from the point of view of estimate (1.4) the behavior is the same for the flows $e^{t\Delta}, e^{it\Delta}$, when $t>0$. Our first question is the following:

Q1 *is the time decay of the flows $e^{t\Delta}, e^{it\Delta}$ related to the lowest frequency behavior of the corresponding fundamental solutions?*

Let us now recall the *Jacobi-Anger* expansion of plane waves, which combined with the Addition Theorem for spherical harmonics (see for example [17, formula (4.8.3), p. 116] and [1, Corollary 1]) yields

$$e^{ix\cdot y} = (2\pi)^{d/2}\big(|x||y|\big)^{-\frac{d-2}{2}} \sum_{\ell=0}^{\infty} i^\ell J_{\ell+\frac{d-2}{2}}\big(|x||y|\big)\left(\sum_{m=1}^{m_\ell} Y_{\ell,m}\big(\tfrac{x}{|x|}\big)\overline{Y_{\ell,m}\big(\tfrac{y}{|y|}\big)}\right) \tag{1.10}$$

for all $x,y \in \mathbb{R}^d$. Here J_ν denotes the $\nu-$th Bessel function of the first kind

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)}\left(\frac{t}{2}\right)^{2k}$$

and the $Y_{\ell,m}$ are usual spherical harmonics. Recalling that $J_\nu(t) \sim t^\nu$, for $\nu \geqslant 0$, as t goes to 0, we see that an additional time-decay, for t large is hidden in formula (1.2), in the term $e^{i\frac{x\cdot y}{t}}$. We hence expect that high-frequency initial data decay polynomially faster along a Schrödinger evolution, in suitable topologies. This leads to our second question:

Q2 *how can the above described phenomenon be quantified, and how stable is it under lower-order perturbations?*

The aim of this seminar is to address some quantitative answers to the previous questions, by showing some recent results we obtained in [4, 9, 10, 11, 12].

2. Main results

From now on, for any $x \in \mathbb{R}^d$, we denote by $x = r\omega$, $r = |x|$. Let

$$\mathbf{A} = \mathbf{A}(\omega) : \mathbb{S}^{d-1} \to \mathbb{R}^d, \qquad a = a(\omega) : \mathbb{S}^{d-1} \to \mathbb{R}$$

be 0-degree homogeneous functions, and consider the quadratic form

$$q[\psi] := \int_{\mathbb{R}^d} \left|\left(-i\nabla + \frac{A(\omega)}{r}\right)\psi(x)\right|^2 dx + \int_{\mathbb{R}^d} \frac{a(\omega)}{r^2}|\psi(x)|^2\, dx. \tag{2.1}$$

and the L^2-initial value problem

$$\begin{cases} i\partial_t\psi = -iH\psi, \\ \psi(0) = \psi_0 \in L^2(\mathbb{R}^d), \end{cases} \tag{2.2}$$

where H is a self-adjoint Hamiltonian associated to q. Here $d \geqslant 2$, and we choose a *transversal gauge* for the magnetic vector potential, i.e. we assume

$$\mathbf{A}(\omega) \cdot \omega = 0 \quad \text{for all } \omega \in \mathbb{S}^{d-1}. \tag{2.3}$$

Notice that equation (2.2) is invariant under the scaling $u_\lambda(x,t) := u(x/\lambda, t/\lambda^2)$, which is the same of the free Schrödinger equation.

The aim is to understand the role of the spherical operator L associated to H, defined by

$$L = \big(- i\, \nabla_{\mathbb{S}^{d-1}} + \mathbf{A}\big)^2 + a(\omega), \tag{2.4}$$

where $\nabla_{\mathbb{S}^{d-1}}$ is the spherical gradient on the unit sphere $\mathbb{S}^{d-1}$. Usually the spectrum of the operator L is formed by a diverging sequence of real eigenvalues with finite multiplicity $\mu_0(\mathbf{A},a) \leqslant \mu_1(\mathbf{A},a) \leqslant \cdots \leqslant \mu_k(\mathbf{A},a) \leqslant \cdots$ (see e.g. [**?**, Lemma A.5]), where each eigenvalue is repeated according to its multiplicity. Moreover we have that $\lim_{k\to\infty} \mu_k(\mathbf{A},a) = +\infty$. To each $k \geqslant 1$, we can associate a $L^2\big(\mathbb{S}^{d-1},\mathbb{C}\big)$-normalized eigenfunction φ_k of the operator L on $\mathbb{S}^{d-1}$ corresponding to the k-th eigenvalue $\mu_k(\mathbf{A},a)$, i.e. satisfying

$$\begin{cases} L\varphi_k = \mu_k(\mathbf{A},a)\, \varphi_k, & \text{in } \mathbb{S}^{d-1}, \\ \int_{\mathbb{S}^{d-1}} |\varphi_k|^2 \, dS(\theta) = 1. \end{cases} \tag{2.5}$$

In particular, if $d = 2$, φ_k are one-variable 2π-periodic functions, i.e. $\varphi_k(0) = \varphi_k(2\pi)$. Since the eigenvalues $\mu_k(\mathbf{A},a)$ are repeated according to their multiplicity, exactly one eigenfunction φ_k corresponds to each index $k \geqslant 1$. We can choose the functions φ_k in such a way that they form an orthonormal basis of $L^2(\mathbb{S}^{d-1},\mathbb{C})$. We also introduce

$$\alpha_k := \frac{d-2}{2} - \sqrt{\left(\frac{d-2}{2}\right)^2 + \mu_k(\ \mathbf{A},a)}, \quad \beta_k := \sqrt{\left(\frac{d-2}{2}\right)^2 + \mu_k(\mathbf{A},a)}, \tag{2.6}$$

so that $\beta_k = \frac{d-2}{2} - \alpha_k$, for $k = 1, 2, \ldots$.

Under the condition

$$\mu_0(\mathbf{A},a) > -\frac{(d-2)^2}{4} \tag{2.7}$$

the quadratic form q in (2.1) associated to H is positive definite, and the Friedrichs' extension of H is well defined, with domain

$$\mathcal{D} := \left\{ f \in H^1_*(\mathbb{R}^d) :\ Hf \in L^2(\mathbb{R}^d) \right\}, \tag{2.8}$$

where $H^1_*(\mathbb{R}^d)$ is the completion of $C^\infty_{\mathrm{c}}(\mathbb{R}^d \setminus \{0\},\mathbb{C})$ with respect to the norm

$$\|f\|_{H^1_*(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^N} \left(|\nabla f(x)|^2 + \frac{|f(x)|^2}{|x|^2} + |f(x)|^2 \right) \right) dx \Bigg)^{1/2}.$$

By the Hardy's inequality, $H^1_*(\mathbb{R}^d) = H^1(\mathbb{R}^d)$ with equivalent norms if $d \geqslant 3$, while $H^1_*(\mathbb{R}^d)$ is strictly smaller than $H^1(\mathbb{R}^d)$ if $d = 2$. Furthermore, from condition (2.7), it follows that $H^1_*(\mathbb{R}^d)$ coincides with the space obtained by completion of $C^\infty_{\mathrm{c}}(\mathbb{R}^d \setminus \{0\},\mathbb{C})$ with respect to the norm naturally associated to H, i.e.

$$q[\psi] + \|\psi\|_2^2.$$

We remark that H could be not essentially self-adjoint. Indeed, in the case $\mathbf{A} \equiv 0$, Kalf, Schmincke, Walter, and Wüst [15] and Simon [21] proved that H

is essentially self-adjoint if and only if $\mu_0(\mathbf{0},a) \geqslant -\left(\frac{d-2}{2}\right)^2 + 1$ and, consequently, admits a unique self-adjoint extension (which coincides with the Friedrichs' extension); otherwise, i.e. if $\mu_0(\mathbf{0},a) < -\left(\frac{d-2}{2}\right)^2 + 1$, H is not essentially self-adjoint and admits infinitely many self-adjoint extensions, among which the Friedrichs' extension is the only one whose domain is included in the domain of the associated quadratic form. The Friedrichs' extension H naturally extends to a self adjoint operator on the dual $\mathcal{D}^\star$ of $\mathcal{D}$ and the unitary group e^{-itH} extends to a group of isometries on the dual of $\mathcal{D}$ which will be still denoted as e^{-itH}. Then for every $\psi_0 \in L^2(\mathbb{R}^d)$,

$$\psi(t,x) := e^{-itH}\psi_0(x) \in \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^d)) \cap \mathcal{C}^1(\mathbb{R}; \mathcal{D}^\star),$$

is the unique solution to (2.2).

Now, by means of (2.5) and (2.6) define the following kernel:

$$K(x,y) = \sum_{k=-\infty}^{\infty} i^{-\beta_k} j_{-\alpha_k}(|x||y|)\varphi_k\left(\tfrac{x}{|x|}\right)\overline{\varphi_k\left(\tfrac{y}{|y|}\right)}, \tag{2.9}$$

where

$$j_\nu(r) := r^{-\frac{d-2}{2}} J_{\nu+\frac{d-2}{2}}(r)$$

and J_ν denotes the usual Bessel function of the first kind

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{t}{2}\right)^{2k}.$$

Notice that (2.9) reduces to (1.10), in the free case $\mathbf{A} \equiv a \equiv 0$. The first result we mention in this survey is the following representation formula for e^{-itH}:

Theorem 2.1 (L. Fanelli, V. Felli, M. Fontelos, A. Primo - [9])**.** *Let $d \geqslant 3$, $a \in L^\infty(\mathbb{S}^{d-1},\mathbb{R})$ and $\mathbf{A} \in C^1(\mathbb{S}^{d-1},\mathbb{R}^N)$, and assume* (2.3) *and* (2.7)*. Then, for any $\psi_0 \in L^2(\mathbb{R}^d)$,*

$$e^{-itH}\psi_0(x) = \frac{e^{\frac{i|x|^2}{4t}}}{i(2t)^{d/2}} \int_{\mathbb{R}^d} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right) e^{i\frac{|y|^2}{4t}} \psi_0(y)\, dy. \tag{2.10}$$

As an immediate consequence, we see by (2.10) that the analog to condition (1.7) gives for H the complete list of usual time decay estimates (1.5):

Corollary 2.2. *Let $d \geqslant 3$, $a \in L^\infty(\mathbb{S}^{d-1},\mathbb{R})$ and $\mathbf{A} \in C^1(\mathbb{S}^{d-1},\mathbb{R}^N)$, and assume* (2.3) *and* (2.7)*. If*

$$\sup_{x,y\in\mathbb{R}^d} |K(x,y)| < \infty, \tag{2.11}$$

then

$$\left\|e^{-itH}\psi_0(\cdot)\right\|_{L^p(\mathbb{R}^d)} \leqslant C|t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|\psi_0\|_{L^{p'}(\mathbb{R}^d)}, \qquad \forall t \in \mathbb{R}, \qquad \forall p \geqslant 2, \tag{2.12}$$

for some $C > 0$ independent on ψ_0.

In the two last decades, estimates (2.12) were intensively studied by several authors (see e.g. [5, 6, 7, 8, 13, 14, 18, 19, 20, 22, 23, 24, 25, 26, 27]). In all these papers, the potentials are sub-critical with respect to the functional scale of the Hardy's inequality: in other words, the critical potentials in (**??**) are never considered, and it does not seem that one could handle them by perturbation techniques, which are a common factor of all the above mentioned papers. Now,

formula (2.10) and Corollary 2.2 give a usual tool to reduce matters to prove time decay, to a spectral analysis problem. This allowed us to prove some new positive results concerning with estimates (2.12). In 2D, the picture is quite well understood, thanks to the following theorem.

Theorem 2.3 (L. Fanelli, V. Felli, M. Fontelos, A. Primo - [10]). *Let $d = 2$, $a \in W^{1,\infty}(\mathbb{S}^1, \mathbb{R})$, $\mathbf{A} \in W^{1,\infty}(\mathbb{S}^1, \mathbb{R}^2)$ satisfying (2.3) and $\mu_1(\mathbf{A}, a) > 0$. Then, for any $\psi_0 \in L^2(\mathbb{R}^d) \cap L^{p'}(\mathbb{R}^d)$,*

$$\left\|e^{-itH}\psi_0(\cdot)\right\|_{L^p(\mathbb{R}^2)} \leqslant C|t|^{-2\left(\frac{1}{2}-\frac{1}{p}\right)}\|\psi_0\|_{L^{p'}(\mathbb{R}^2)}, \qquad \forall t \in \mathbb{R}, \qquad \forall p \geqslant 2, \tag{2.13}$$

for some $C > 0$ independent on ψ_0.

The higher dimensional scenario is quite more complicate, and some chaotic behavior of the eigenvalues of L can occur. This makes the generic validity of (2.12) completely unclear in dimension $d \geqslant 3$. In this direction, the only result which is available at the moment is concerned with the 3D-inverse square electric potential, and reads as follows:

Theorem 2.4 (L. Fanelli, V. Felli, M. Fontelos, A. Primo - [9]). *Let $d = 3$, $\mathbf{A} \equiv 0$ and $a(\omega) \equiv a \in \mathbb{R}$, with $a > -\frac{1}{4}$.*

i) *If $a \geqslant 0$, then, for any $\psi_0 \in L^2(\mathbb{R}^3) \cap L^{p'}(\mathbb{R}^3)$,*

$$\left\|e^{-itH}\psi_0(\cdot)\right\|_{L^p(\mathbb{R}^2)} \leqslant C|t|^{-3\left(\frac{1}{2}-\frac{1}{p}\right)}\|\psi_0\|_{L^{p'}(\mathbb{R}^2)}, \qquad \forall t \in \mathbb{R}, \qquad \forall p \geqslant 2, \tag{2.14}$$

for some $C > 0$ which does not depend on ψ_0.

ii) *If $-\frac{1}{4} < a < 0$, let α_1 as in (2.6), and define*

$$\|\psi\|_{p,\alpha_1} := \left(\int_{\mathbb{R}^3} (1 + |x|^{-\alpha_1})^{2-p}|\psi(x)|^p\, dx\right)^{1/p}, \quad p \geqslant 1.$$

Then the following estimates hold

$$\left\|e^{-itH}\psi_0(\cdot)\right\|_{p,\alpha_1} \leqslant \frac{C(1 + |t|^{\alpha_0})^{1-\frac{2}{p}}}{|t|^{3\left(\frac{1}{2}-\frac{1}{p}\right)}}\|\psi\|_{p',\alpha_0}, \quad p \geqslant 2, \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{2.15}$$

for some constant $C > 0$ which does not depend on ψ_0.

Remark 2.5. We remark that, in the range $-1/4 < a < 0$, (2.14) does not hold, while the full set of usual Strichartz estimates hold (see [2, 3]). This is now clearly understood in terms of formula (2.10): notice that, if $a = \mu_0 < 0$, then $\alpha_0 > 0$ and a negative-index Bessel function appears in the kernel K given by (2.9); since negative-index functions J_ν are singular at the origin, one cannot either expect the solution (2.10) to be in L^∞.

This can be proved as a general fact:

Theorem 2.6 (L. Fanelli, V. Felli, M. Fontelos, A. Primo - [10]). *Let $d \geqslant 3$, $a \in L^\infty(\mathbb{S}^{d-1}, \mathbb{R})$, $\mathbf{A} \in C^1(\mathbb{S}^{d-1}, \mathbb{R}^d)$, and assume (2.3), (2.7), and $\mu_0 < 0$. Then, for almost every $t \in \mathbb{R}$, $e^{-itH}(L^1) \not\subseteq L^\infty$; in particular e^{-itH} is not a bounded operator from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$.*

The above phenomenon can be quantified. To this aim, let us restrict our attention to the case

$$H = -\Delta + \frac{a}{|x|^2}, \qquad x \in \mathbb{R}^3.$$

Let us define

$$V_{n,j}(x) = |x|^{-\alpha_j} e^{-\frac{|x|^2}{4}} P_{j,n}\Big(\frac{|x|^2}{2}\Big)\psi_j\Big(\frac{x}{|x|}\Big), \quad n,j \in \mathbb{N},\ j \geqslant 1, \tag{2.16}$$

where $P_{j,n}$ is the polynomial of degree n given by

$$P_{j,n}(t) = \sum_{i=0}^{n} \frac{(-n)_i}{\big(\frac{d}{2}-\alpha_j\big)_i}\,\frac{t^i}{i!},$$

denoting as $(s)_i$, for all $s \in \mathbb{R}$, the Pochhammer's symbol

$$(s)_i = \prod_{j=0}^{i-1}(s+j), \qquad (s)_0 = 1.$$

Moreover, for all $k > 1$, define

$$\mathcal{U}_k = \operatorname{span}\{V_{n,j} : n \in \mathbb{N}, 1 \leqslant j < k\} \subset L^2(\mathbb{R}^N).$$

The functions $V_{n,j}$ spans $L^2(\mathbb{R}^3)$ (see [11] for details). Moreover, as initial data for (1.1), these functions have a quite explicit evolution: indeed, denoting by $\widetilde{V}_{n,j} := V_{n,j}/\|V_{n,j}\|_2$, the following identity holds:

(2.17)

$$\begin{aligned} e^{-itH}\widetilde{V}_{n,j}(x) &= e^{it\left(-\Delta+\frac{a}{|x|^2}\right)}V_{n,j}(x) \\ &= (1+t^2)^{-\frac{d}{4}+\frac{\alpha_j}{2}}\,|x|^{-\alpha_j}\,\frac{e^{\frac{-|x|^2}{4(1+t^2)}}}{\|V_{n,j}\|_{L^2(\mathbb{R}^d)}} e^{i\frac{|x|^2 t}{4(1+t^2)}} e^{-i\gamma_{n,j}\arctan t}\psi_j\big(\tfrac{x}{|x|}\big)P_{n,j}\big(\tfrac{|x|^2}{2(1+t^2)}\big). \end{aligned}$$

Formula (2.17) has been proved in [11]. Clearly, if $a = \mu_0 \geqslant 0$, then $\alpha_0 \leqslant 0$ and the first function $\widetilde{V}_{1,0}$ decays polynomially faster than usual, in a weighted space. This is reminiscent to question **B** in the Introduction, and gives us the following evolution version of the frequency-dependent Hardy's inequality (**??**):

Theorem 2.7 (L. Fanelli, V. Felli, M. Fontelos, A. Primo - [11])**.** *Let $d = 3$, $a = \mu_0 \geqslant 0$, α_0 as in (2.6) .*

(i) *There exists $C > 0$ such that, for all $\psi_0 \in L^2(\mathbb{R}^3)$ with $|x|^{-\alpha_0}\psi_0 \in L^1(\mathbb{R}^3)$,*

$$\big\||x|^{\alpha_0}e^{-itH}\psi_0(\cdot)\big\|_{L^\infty} \leqslant C|t|^{-\frac{3}{2}+\alpha_0}\||x|^{-\alpha_0}\psi_0\|_{L^1}.$$

(ii) *For all $k \in \mathbb{N}$, $k \geqslant 1$, there exists $C_k > 0$ such that, for all $\psi_0 \in \mathcal{U}_k^\perp$ with $|x|^{-\alpha_k}\psi_0 \in L^1(\mathbb{R}^3)$,*

$$\big\||x|^{\alpha_k}e^{-itH}\psi_0(\cdot)\big\|_{L^\infty} \leqslant C_k|t|^{-\frac{3}{2}+\alpha_k}\||x|^{-\alpha_k}\psi_0\|_{L^1}.$$

Some analogous results, only concerning with the decay of the first frequency space, had been previously proven in [12], while in [4] we study analogous problems for the Dirac Hamiltonian.

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