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Timelike Sabban curves in de Sitter space
(ド・ジッター空間内の時間的サバン曲線)

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1 Introduction

It is well known that Lorentzian space forms are classified into three types depending on the value of the scalar curvature. One of them is Lorentz-Minkowski space which has zero curvature. The Lorentz space form with negative curvature is Anti-de Sitter space. De Sitter space is a Lorentzian space form with positive curvature which has rich geometric properties. De Sitter space is named after Willem de Sitter (1872-1934). In general relativity, a vacuum solution according to the Einstein field equation is a Lorentzian manifold whose Einstein tensor vanishes identically, and de Sitter space is one of the vacuum solutions of the Einstein field equation. In cosmology, the very early models of inflation are converging on a consistent model where our universe was best described as a de Sitter space, which is called de Sitter universe. In Mathematics, de Sitter space is defined to be the Lorentzian sphere in Minkowski space-time with a positive curvature, which is a good model for studying Lorentzian spherical geometry. This is the motivation for the investigation of submanifolds in de Sitter space from a mathematical viewpoint.

On the other hand, in physics, the locus of a moving particle is called a world line. In special relativity, world lines are timelike curves. World lines of particles/objects with constant speed are called geodesics which are straight lines in Minkowski space. The use of world lines in general relativity is basically the same as in special relativity, with the difference that space-time can be curved. Since the speed of a moving particle never exceed the speed of rays, in general relativity, world lines are timelike curves in space-time, where the tangent vectors of timelike curves fall within the lightcone.

In this thesis we consider timelike Sabban curves in de Sitter space. The original notion of Sabban curves belongs to spherical geometry in the Euclidean space. Let $\gamma : I \rightarrow S^2 \subset \mathbb{R}^3$ be a unit speed regular curve, then we have an orthonormal frame $\{\gamma, \mathbf{t}, \mathbf{n}\}$ along γ , where $\mathbf{t}(s)$ is the unit tangent vector of γ at s and $\mathbf{n}(s) = \gamma(s) \times \mathbf{t}(s)$. Then we have the Frenet-Serret type

formulae:

$$\begin{cases} \gamma'(s) = \mathbf{t}(s), \\ \mathbf{t}'(s) = -\gamma(s) + \kappa_g(s)\mathbf{n}(s), \\ \mathbf{n}'(s) = -\kappa_g(s)\mathbf{t}(s), \end{cases}$$

where $\kappa_g(s) = \gamma''(s) \cdot \mathbf{n}(s)$ is the *geodesic curvature* of γ at s . This frame was introduced by Sabban (cf. [10]) and it is called a *Sabban frame* of γ . For a unit speed curve γ in S^2 , we always have the Sabban frame. However, we need some assumptions to define the Sabban type frame for regular curves in S^n , where $n \geq 3$. In [9], the notion of the generalized Sabban curve was introduced in S^n . In this thesis, we consider de Sitter space S_1^n in the Minkowski space $\mathbb{R}_1^{n+1}$. For detailed notions about Minkowski sapce, see §2. We say that a non-lightlike curve $\gamma : I \rightarrow S_1^3 \subset \mathbb{R}_1^4$ is a *Sabban curve* if $\|\mathbf{t}'(s) + \delta(\mathbf{t}(s))\gamma(s)\| \neq 0$ at any point $s \in I$, where $\delta(\mathbf{x})$ is the *sign function*. Then we have an orthonormal frame $\{\gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \mathbf{n}_2(s)\}$ of $\mathbb{R}_1^4$ along γ , where $\mathbf{n}_1(s) = (\mathbf{t}'(s) + \delta(\mathbf{t}(s))\gamma(s))/\|\mathbf{t}'(s) + \delta(\mathbf{t}(s))\gamma(s)\|$ and $\mathbf{n}_2(s) = \gamma(s) \wedge \mathbf{t}(s) \wedge \mathbf{n}_1(s)$. Here, $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3$ is the *pseudo vector product* in $\mathbb{R}_1^4$. We have the following formulae (cf. §3):

$$\begin{cases} \gamma'(s) = \mathbf{t}(s), \\ \mathbf{t}'(s) = -\delta(\mathbf{t}(s))\gamma(s) + \kappa_1(s)\mathbf{n}_1(s), \\ \mathbf{n}_1'(s) = -\delta(\mathbf{t}(s))\delta(\mathbf{n}_1(s))\kappa_1(s)\mathbf{t}(s) + \kappa_2(s)\mathbf{n}_2(s), \\ \mathbf{n}_2'(s) = -\delta(\mathbf{n}_1(s))\delta(\mathbf{n}_2(s))\kappa_2(s)\mathbf{n}_1(s), \end{cases}$$

where $\kappa_1(s) = \|\mathbf{t}'(s) + \delta(\mathbf{t}(s))\gamma(s)\| \neq 0$ and $\kappa_2(s) = \delta(\mathbf{n}_2(s))\langle \mathbf{n}_1'(s), \mathbf{n}_2(s) \rangle$. We call $\{\gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \mathbf{n}_2(s)\}$ a *Sabban frame* along the Sabban curve γ . In §3, we construct a frame $\{\gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \dots, \mathbf{n}_{n-1}(s)\}$ along the non-lightlike curve $\gamma : I \rightarrow S_1^n \subset \mathbb{R}_1^{n+1}$ under some conditions on γ . We call the above frame a *generalized Lorentzian Sabban frame*. In this case, γ is called a *generalized Lorentzian Sabban curve*. We prove that the generalized Lorentzian Sabban frame is a pseudo-orthonormal frame (cf. Lemma 3.1.1). If the non-lightlike curve γ is timelike curve, then we call it a *timelike Sabban curve*. For a timelike Sabban curve in S_1^n , we define the curvature functions $\kappa_i(s)$ ($i = 1, \dots, n-1$), where the curvature function $\kappa_i(s) \neq 0$ ($i = 1, \dots, n-2$), while $\kappa_{n-1}(s)$ may be equal to zero. If $\kappa_{n-1}(s) \equiv 0$, the timelike Sabban curve γ is always lying on a great de Sitter subspace $S_1^{n-1}(\mathbf{n}, 0)$ in S_1^n (cf. Proposition 3.1.2).

On other hand, Izumiya and Liang [4, 7] introduced the mandala of Legendrian dualities between psudo-spheres in Minkowski space-time. There are three kinds of pseudo-spheres in Minkowski space-time (i.e., the hyperbolic space, the de Sitter space and the lightcone). In particular, if we investigate spacelike submanifolds or timelike submanifolds in the de Sitter space, the theorem of Legendrian dualities is a fundamental tool for the study of extrinsic differential geometry on submanifolds from the view point of singularity theory. Therefore, we discuss the dual hypersurfaces in the de Sitter space as an application of the duality theorem. We consider de Sitter dual hypersurfaces of timelike Sabban curves in S_1^n which are defined by

$$\gamma^* = \{ \xi_1 \mathbf{n}_1 + \cdots + \xi_{n-1} \mathbf{n}_{n-1} | s \in I, \xi_1^2 + \cdots + \xi_{n-1}^2 = 1 \}.$$

We give classifications and characterizations of the singularities of γ^* by using singularity theory and the geometric properties of the generalized Lorentzian Sabban frame. The main results in §3 are Theorems 3.4.2 and 3.5.2 which are analogous to the results in [9].

In §§4.1, we consider timelike Sabban curves in S_1^2 . As we know, the spherical evolute of a Euclidean spherical curve γ is naturally obtained as the envelope of the family of normal geodesics to γ . The evolute in Euclidean plane is naturally interpreted as a caustic [6]. The spherical evolute is also a caustic [11]. We define the notion of de Sitter evolute of a timelike Sabban curve in S_1^2 and study the geometric properties of those singularities from a contact viewpoint. These de Sitter evolutes are associated to timelike Sabban curves in de Sitter 2-space. The geometry of these timelike Sabban curves determine the behavior of the corresponding de Sitter evolutes. We define the *de Sitter height function* of a timelike Sabban curve, and show the bifurcation set of de Sitter height function is the de Sitter evolute. By using the techniques of the unfolding theory on such a function, we show that the singularity of the de Sitter evolute depends on the derivative of the geodesic curvature function (cf. Proposition 4.1.2). We show that the de Sitter evolute is constant if and only if the derivative of the geodesic curvature function is always equal to zero. In this case, the curve γ is a special curve on S_1^2 , which is called a D-slice. Moreover, we give the equivalence condition that γ has third order contact with D-slice (cf. Proposition 4.1.3). The main result in §§4.1 is Theorem 4.1.5, which gives a

generic classification of singularities of de Sitter evolutes, and characterizes the contact between timelike Sabban curves and model curves (D-slices).

In the Euclidean space, focal sets are useful for the study of certain optical phenomena (namely, scattering, in fact a rainbow is caused by caustics), expressing some geometrical results within fluid mechanics as well as describing many medical anomalies. Focal surfaces are formed by taking the centers of the curvature spheres, which are the tangential spheres whose radii are the reciprocals of one of the principal curvatures at the point of tangency. Equivalently, they are the surfaces formed by the centers of the circles which osculate the curvature lines. In §§4.2, we consider timelike Sabban curves in S_1^3 . We define the de Sitter evolute and the de Sitter focal surface of a timelike Sabban curve in S_1^3 and study geometric properties of these objects. We show that the de Sitter evolutes are the singularities of the de Sitter focal surfaces. We define the de Sitter height function H on the timelike Sabban curve $\gamma(s)$ in S_1^3 . By differentiating the de Sitter height function, we obtain a new invariant $\sigma_2(s)$ which characterizes the conditions on the derivatives of de Sitter height functions. For instance, $\frac{\partial H}{\partial s} = \frac{\partial^2 H}{\partial s^2} = \frac{\partial^3 H}{\partial s^3} = \frac{\partial^4 H}{\partial s^4} = 0$ if and only if $\sigma_2(s) = 0$ with some other conditions. We also show that the de Sitter evolute $\mathbf{E}_\gamma$ is singular at s_0 if and only if $\sigma_2(s_0) = 0$ (cf. Proposition 4.2.2). If $\sigma_2(s) \equiv 0$, then γ is a curve not only in S_1^3 , but also in a special kind of hyperplanes in $\mathbb{R}_1^4$. As an application of the theory of unfoldings of functions, we give a classification of singularities of both the de Sitter evolutes and de Sitter focal surfaces (cf. Theorem 4.2.7), which are also interpreted by the contact of timelike Sabban curves with tangent de Sitter subspaces.

Since the de Sitter 4-space is a cosmological model for the physical universe, the timelike curve in S_1^4 is a quite interesting subject from the view point of relativity theory. In §§4.3, we consider timelike Sabban curves in S_1^4 . We study the contact between timelike Sabban curves and de Sitter subspaces as an application of singularity theory of smooth functions. One of the basic tools that we use is the notion of the de Sitter height function in S_1^4 . By using the techniques of singularity theory on such a function, we define *tangent de Sitter subspaces* and *osculating de Sitter subspaces* along timelike Sabban curves. We remark that there are infinitely many tangent de Sitter subspaces of $\gamma(s)$ at s_0 , while there is only one osculating

de Sitter subspace of $\gamma(s)$ at s_0 . We define de Sitter focal hypersurfaces of timelike Sabban curves as the bifurcation sets of the de Sitter height functions on the curves and define the de Sitter evolute of the timelike Sabban curve under a certain generic assumption, which are different from the case in S_1^3 . Under the above assumption, de Sitter evolutes are the second order singularities of de Sitter focal hypersurfaces (for the definition, see §§2.3). We study the geometric meanings of singularities of de Sitter focal hypersurfaces and introduce a new invariant $\sigma_3(s)$. We show that $\sigma_3(s) = 0$ for all s if and only if the timelike Sabban curve is located on a special kind of hyperplanes. The main result in this subsection is Theorem 4.3.8, which gives a generic classification of singularities of de Sitter focal hypersurfaces and characterizes the contact between timelike Sabban curves and osculating de Sitter subspaces (or, tangent de Sitter subspaces). We can also consider the de Sitter evolute and the de Sitter focal surface of a timelike Sabban curve in S_1^n ($n \geq 5$) similar to the case for S_1^n ($n \leq 4$). However the calculations in this case is too hard to obtain meaningful results. In §§4.4, we study focal sets of timelike Sabban curves from the view point of Legendrian dualities. In [7], Izumiya investigated Legendrian dualities between pseudo-spheres in Minkowski space-time. It is a fundamental tool for the study of spacelike submanifolds from the view point of singularity theory. The unit tangent vector field of a timelike curve is a curve in the hyperbolic space and the regular part of the focal set of a timelike Sabban curve is a spacelike hypersurface. We show that the dual relationship between the tangent vector field of a timelike Sabban curve and the focal set.

We shall assume throughout the whole thesis that all the maps and manifolds are $\mathcal{C}^\infty$ unless the contrary is explicitly stated.

2 Preliminary knowledge

2.1 The basic concepts

In this section we introduce the basic concepts in this thesis. Let $\mathbb{R}^{n+1}$ be an $(n+1)$ -dimensional vector space. For any two vectors $\mathbf{x} = (x_1, \dots, x_{n+1})$, $\mathbf{y} = (y_1, \dots, y_{n+1})$ in $\mathbb{R}^{n+1}$, their *pseudo scalar product* is defined to be $\langle \mathbf{x}, \mathbf{y} \rangle = -x_1 y_1 + \dots + x_{n+1} y_{n+1}$. Here, $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$ is called *Lorentz-Minkowski $(n+1)$ -space* (simply, *Minkowski $(n+1)$ -space*), which is denoted by $\mathbb{R}_1^{n+1}$. For any n vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \in \mathbb{R}_1^{n+1}$, their *pseudo vector product* is defined to be

$$\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \dots \wedge \mathbf{x}_n = \begin{vmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_{n+1} \\ x_1^1 & x_1^2 & \cdots & x_1^{n+1} \\ x_2^1 & x_2^2 & \cdots & x_2^{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ x_n^1 & x_n^2 & \cdots & x_n^{n+1} \end{vmatrix},$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n+1}\}$ is the canonical basis of $\mathbb{R}_1^{n+1}$ and $\mathbf{x}_i = (x_i^1, x_i^2, \dots, x_i^{n+1})$. A non-zero vector $\mathbf{x} \in \mathbb{R}_1^{n+1}$ is called *spacelike*, *lightlike* or *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ respectively. We define the signature of $\mathbf{x}$ by

$$\text{sign}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \text{ is spacelike,} \\ 0 & \mathbf{x} \text{ is lightlike,} \\ -1 & \mathbf{x} \text{ is timelike.} \end{cases}$$

The norm of $\mathbf{x} \in \mathbb{R}_1^{n+1}$ is defined to be $\|\mathbf{x}\| = \sqrt{\text{sign}(\mathbf{x})\langle \mathbf{x}, \mathbf{x} \rangle}$.

Let $\gamma : I \rightarrow \mathbb{R}_1^{n+1}$ be a regular curve in $\mathbb{R}_1^{n+1}$ (i.e., $\dot{\gamma}(t) \neq \mathbf{0}$ for any $t \in I$), where I is an open interval. For any $t \in I$, the curve γ is called *spacelike*, *lightlike* or *timelike* if $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle > 0$, $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 0$ or $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle < 0$ respectively. We call γ a *non-lightlike curve* if γ is a spacelike or timelike curve. The arc-length of a non-lightlike curve γ measured from $\gamma(t_0)$ ($t_0 \in I$) is $s(t) = \int_{t_0}^t \|\dot{\gamma}(t)\| dt$.

The parameter s is determined such that $\|\gamma'(s)\| = 1$ for the non-lightlike curve, where $\gamma'(s) = d\gamma/ds(s)$ is the derivative of γ by s .

We define the *de Sitter n-space* by

$$S_1^n = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}.$$

We define the *hyperbolic n-space* by

$$H^n(-1) = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1\}.$$

We define the *closed lightcone with the vertex $\mathbf{a}$* by

$$LC_a = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = 0\}.$$

Moreover the *open lightcone at the origin* is defined to be

$$LC^* = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \setminus \{\mathbf{0}\} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0\}.$$

2.2 The Legendrian duality theorem

We now review some properties of contact manifolds and Legendrian submanifolds. Let N be a $(2n+1)$ -dimensional smooth manifold and K be a tangent hyperplane field on N . Locally, such a field is defined as the field of zeros of a 1-form α . The tangent hyperplane field on K is non-degenerate if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of N . We say that (N, K) is a contact manifold if K is a non-degenerate hyperplane field. In this case, K is called a contact structure and α is a contact form. Let $\phi : N \rightarrow N'$ be a diffeomorphism between contact manifolds (N, K) and (N', K') . We say that ϕ is a contact diffeomorphism if $d\phi(K) = K'$. Two contact manifolds (N, K) and (N', K') are contact diffeomorphic if there exists a contact diffeomorphism $\phi : N \rightarrow N'$. A submanifold $i : L \subset N$ of a contact manifold (N, K) is said to be Legendrian if $\dim L = n$ and $di_x(T_x L) \subset K_{i(x)}$ at any $x \in L$. We say that a smooth fiber bundle $\pi : E \rightarrow M$ is called a Legendrian fibration if its total space E is furnished with a contact structure and its fibers are Legendrian submanifolds. Let $\pi : E \rightarrow M$ be a Legendrian fibration. For a Legendrian submanifold $i : L \subset E$, $\pi \circ i : L \rightarrow M$ is called a Legendrian map. The image of the Legendrian map $\pi \circ i$ is called a wavefront set of i which is denoted by $W(i)$. For any

$p \in E$, it is known that there is a local coordinate system $(x_1, \dots, x_m, p_1, \dots, p_m, z)$ around p such that $\pi(x_1, \dots, x_m, p_1, \dots, p_m, z) = (x_1, \dots, x_m, z)$ and the contact structure is given by the 1-form

$$\alpha = dz - \sum_{i=1}^m p_i dx_i$$

(cf. [1], 20.3). One of the examples of Legendrian fibrations is given by the unit spherical tangent bundle of a Riemannian manifold. Let M be a Riemannian manifold and TM is its tangent bundle. Let $(x_1, \dots, x_n)$ be local coordinates on a neighbourhood U of M and $(v_1, \dots, v_n)$ be coordinates on the fiber over U . Let g_{ij} be the components of the metric $\langle, \rangle$ with respect to the above coordinates. Then the canonical 1-form can be locally given by $\theta = \sum_{i,j} g_{ij} v_j dq_i$ where $q_i = x_i \circ \pi$ for the projection $\pi : TM \longrightarrow M$. Let $\tilde{\pi} : S(TM) \longrightarrow M$ be the unit spherical tangent bundle with respect to the metric $\langle, \rangle$. Then the restriction of θ onto $S(TM)$ gives a contact structure and $\tilde{\pi} : S(TM) \longrightarrow M$ is a Legendrian fibration (cf., [2]).

We now formulate the basic theorem in this thesis which is the fundamental tool for the study of spacelike submanifolds in pseudo-spheres in Minkowski space. We define 1-forms $\langle d\mathbf{v}, \mathbf{w} \rangle = -w_0 dv_0 + \sum_{i=1}^n w_i dv_i$, $\langle \mathbf{v}, d\mathbf{w} \rangle = -v_0 dw_0 + \sum_{i=1}^n v_i dw_i$ on $\mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1}$ and consider the following four double fibrations with 1-forms:

- (1)(a) $H^n(-1) \times S_1^n \supset \Delta_1 = \{(\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0\}$,
 - (b) $\pi_{11} : \Delta_1 \longrightarrow H^n(-1), \pi_{12} : \Delta_1 \longrightarrow S_1^n$,
 - (c) $\theta_{11} = \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_1}, \theta_{12} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_1}$.
- (2)(a) $H^n(-1) \times LC^* \supset \Delta_2 = \{(\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w} \rangle = -1\}$,
 - (b) $\pi_{21} : \Delta_2 \longrightarrow H^n(-1), \pi_{22} : \Delta_2 \longrightarrow LC^*$,
 - (c) $\theta_{21} = \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_2}, \theta_{22} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_2}$.
- (3)(a) $LC^* \times S_1^n \supset \Delta_3 = \{(\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w} \rangle = 1\}$,
 - (b) $\pi_{31} : \Delta_3 \longrightarrow LC^*, \pi_{32} : \Delta_3 \longrightarrow S_1^n$,
 - (c) $\theta_{31} = \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_3}, \theta_{32} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_3}$.
- (4)(a) $LC^* \times LC^* \supset \Delta_4 = \{(\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w} \rangle = -2\}$,
 - (b) $\pi_{41} : \Delta_4 \longrightarrow LC^*, \pi_{42} : \Delta_4 \longrightarrow LC^*$,
 - (c) $\theta_{41} = \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_4}, \theta_{42} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_4}$.

Here, $\pi_{i1}(v, w) = v$, $\pi_{i2}(v, w) = w$ are the canonical projections. Moreover, $\theta_{i1} = \langle dv, w \rangle|_{\Delta_i}$ and $\theta_{i2} = \langle v, dw \rangle|_{\Delta_i}$ are the restrictions of the 1-forms $\langle dv, w \rangle$ and $\langle v, dw \rangle$ on Δ_i . We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over Δ_i which is denoted by K_i . In [7], the following theorem was shown:

Theorem 2.2.1. Under the same notations as the previous paragraph, each $(\Delta_i; K_i)$ ($i = 1, 2, 3, 4$) is a contact manifold and both of π_{ij} ($j = 1; 2$) are Legendrian fibrations. Moreover those contact manifolds are contact diffeomorphic each other. In this thesis, we only consider (Δ_1, K_1) and (Δ_5, K_5) .

In §3 we consider the following double fibrations:

$$\begin{aligned} S_1^n \times S_1^n \supset \Delta_5 &= \{(\mathbf{v}, \mathbf{w}) | \langle \mathbf{v}, \mathbf{w} \rangle = 0\}, \\ \pi_{51} : \Delta_5 &\rightarrow S_1^n, \pi_{52} : \Delta_5 \rightarrow S_1^n, \\ \theta_{51} &= \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_5}, \theta_{52} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_5}. \end{aligned}$$

Here, $\pi_{51}(\mathbf{v}, \mathbf{w}) = \mathbf{v}$, $\pi_{52}(\mathbf{v}, \mathbf{w}) = \mathbf{w}$ (see, [4]). In §4 we also consider the following double fibrations in S_1^n for $n = 2, 3, 4$:

$$\begin{aligned} H^n(-1) \times S_1^n \supset \Delta_1 &= \{(\mathbf{v}, \mathbf{w}) | \langle \mathbf{v}, \mathbf{w} \rangle = 0\}, \\ \pi_{11} : \Delta_1 &\rightarrow H^n(-1), \pi_{12} : \Delta_1 \rightarrow S_1^n, \\ \theta_{11} &= \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_1}, \theta_{12} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_1}. \end{aligned}$$

If we have an isotropic mapping $i : L \rightarrow \Delta_i$ (i.e., $i^*\theta_{i1} = 0$), we say that $\pi_{i1}(i(L))$ and $\pi_{i2}(i(L))$ are Δ_i -dual to each other ($i = 1, 5$). For detailed properties of Legendrian fibrations, see [1].

2.3 Unfolding of function germs

We review some results of function germs as an application of the unfolding theory of functions. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \rightarrow \mathbb{R}$ be a function germ, $f(s) = F(s, \mathbf{x}_0)$. We call F an r -parameter unfolding of f . If $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$ and $f^{(k+1)}(s_0) \neq 0$, we say f has A_k -singularity at s_0 . We also say f has $A_{\geq k}$ -singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$.

Let F be a r -parameter unfolding of f and f has A_k -singularity ($k \geq 1$) at s_0 , we define the $(k-1)$ -jet of the partial derivative $\partial F/\partial x_i$ at s_0 as

$$j^{(k-1)}\left(\frac{\partial F}{\partial x_i}(s, \mathbf{x}_0)\right)(s_0) = \sum_{j=1}^{k+1} \alpha_{ji}(s - s_0)^j, \quad (i = 1, \dots, r).$$

If the rank of $k \times r$ matrix $(\alpha_{0i}, \alpha_{ji})$ is k ($k \leq r$), then F is called a *versal unfolding* of f , where $\alpha_{0i} = \partial F/\partial x_i(s_0, \mathbf{x}_0)$. Under the same condition as the above, F is called a *p-versal unfolding* if the $(k-1) \times r$ matrix of coefficients (α_{ji}) has rank $k-1$ ($k-1 \leq r$). The *discriminant set* of F is defined by

$$D_F = \{\mathbf{x} \in \mathbb{R}^r \mid \exists s \in \mathbb{R}, F(s, \mathbf{x}) = \frac{\partial F}{\partial s}(s, \mathbf{x}) = 0\}.$$

The *bifurcation set* of F is

$$B_F = \{\mathbf{x} \in \mathbb{R}^r \mid \exists s; \frac{\partial F}{\partial s}(s, \mathbf{x}) = \frac{\partial^2 F}{\partial s^2}(s, \mathbf{x}) = 0\}.$$

We also define

$$D_F^i = \{\mathbf{x} \in \mathbb{R}^r \mid \exists s \in \mathbb{R}, F(s, \mathbf{x}) = \frac{\partial F}{\partial s}(s, \mathbf{x}) = \dots = \frac{\partial^i F}{\partial s^i}(s, \mathbf{x}) = 0\},$$

which is called an *ith-order discriminant set* of F . We are interested in classification of D_F^i by diffeomorphisms. We say that two function germs $f, g : (\mathbb{R}, 0) \rightarrow \mathbb{R}$ are $\mathcal{R}$ -equivalent if there exist a diffeomorphism germ $\phi : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ such that $f \circ \phi(s) = g(s) + (f(0) - g(0))$ for any $s \in (\mathbb{R}, 0)$. If $f : (\mathbb{R}, 0) \rightarrow \mathbb{R}$ has an A_k -singularity at 0, then f is $\mathcal{R}$ -equivalent to $g(s) = \pm s^{k+1}$. We can also show that the following proposition [3].

Proposition 2.3.1. The unfolding $G : (\mathbb{R} \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}, 0)$ given by

$$G(s, x) = \pm s^{k+1} + x_1 + x_2 s + \dots + x_k s^{k-1}$$

is a versal unfolding of $g(s) = \pm s^{k+1}$ at 0.

Let $F, G : (\mathbb{R} \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}, 0)$ be unfolding of $f, g : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$, respectively. We say that F, G are P - $\mathcal{R}$ -equivalent if there exists a diffeomorphism germ $\Phi : (\mathbb{R} \times \mathbb{R}^r, (0, 0)) \rightarrow$

$(\mathbb{R} \times \mathbb{R}^r, (0, 0))$ of the form $\Phi(s, x) = (\phi_1(s, x), \phi_2(x))$ such that $F \circ \Phi = G$. So, if F, G are $P\text{-}\mathcal{R}$ -equivalent, then f, g are $\mathcal{R}$ -equivalent. We also say F, G are $P\text{-}\mathcal{R}^+$ -equivalent if there exist a germ of a diffeomorphism $\Phi : (\mathbb{R} \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R} \times \mathbb{R}^r, (0, 0))$ and a germ of a function $c : (\mathbb{R}^r, 0) \rightarrow \mathbb{R}$ such that $G(s, x) = F(\Phi(s, x)) + c(x)$. By the uniqueness of versal unfolding [5], we have the following theorem.

Theorem 2.3.2. Let $F, G : (\mathbb{R} \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}, 0)$ be unfolding of $f, g : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$, respectively. Suppose that F, G are versal unfolding of f, g respectively. If f, g are $\mathcal{R}$ -equivalent, then F, G are $P\text{-}\mathcal{R}$ -equivalent.

By an induction, we show the following proposition.

Proposition 2.3.3. Let $F, G : (\mathbb{R} \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}, 0)$ be unfolding of $f, g : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$, respectively. If F, G are $P\text{-}\mathcal{R}$ -equivalent, then there exists a diffeomorphism germ $\phi : (\mathbb{R}^r, 0) \rightarrow (\mathbb{R}^r, 0)$ such that $\phi(D_G^i) = D_F^i$ as set germs for any i .

For the unfolding $G : (\mathbb{R} \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}, 0)$ given by

$$G(s, x) = \pm s^{k+1} + x_1 + x_2 s + \cdots + x_k s^{k-1},$$

we show that $\mathbf{x} = (x_1, x_2, \dots, x_k) \in D_G^i$ ($1 \leq i \leq k-1$) if and only if

$$\begin{cases} x_1 = \mp s^{k+1} - x_2 s - \cdots - x_k s^{k-1}, \\ x_2 = \mp(k+1)s^k - 2x_3 s - \cdots - (k-1)x_k s^{k-2}, \\ x_3 = \mp \frac{k(k+1)}{2} s^k - 3x_4 s - \cdots - \frac{(k-1)(k-2)}{2} s^{k-3}, \\ \dots = \dots \\ x_{i+1} = \mp \frac{(k+1)k \cdots (k-i+3)}{i!} s^{k+1-i} - (i+1)x_{i+2} s - \cdots - \frac{(k-1)(k-2) \cdots (k-i+1)}{(i-1)!} x_k s^{k-i}. \end{cases} \quad (1)$$

We now consider a map-germ $DA_k^\pm : (\mathbb{R}^{k-1}, 0) \rightarrow (\mathbb{R}^k, 0)$ defined by

$$DA_k^\pm(u_1, \dots, u_{k-1}) = (\pm u_1^{k+1} + \sum_{i=2}^{k-1} (i-1)u_1^i u_i, \mp(k+1)u_1^k - \sum_{i=2}^{k-1} i u_1^{i-1} u_i, u_2, \dots, u_{k-1}).$$

We remark that $(\text{Im} DA_2^\pm, 0)$ is diffeomorphic to the cusp $C = \{(t^2, t^3) | t \in (\mathbb{R}, 0)\}$ and $(\text{Im} DA_3^\pm, 0)$ is diffeomorphic to the swallowtail $SW = \{(3u^4 + u^2 v, 4u^3 + 2uv, v) | (u, v) \in (\mathbb{R}^2, 0)\}$ as set-germs. we show the following proposition [8].

Proposition 2.3.4. Let $F, G : (\mathbb{R} \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}, 0)$ be unfolding of $f, g : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$, respectively. If F, G are $P\text{-}\mathcal{R}^+$ -equivalent, then there exists a diffeomorphism germ $\phi : (\mathbb{R}^r, 0) \rightarrow (\mathbb{R}^r, 0)$ such that $\phi(B_F) = B_G$.

By Theorem 2.3.2 and Proposition 2.3.3 and 2.3.4, We have the following theorems.

Theorem 2.3.5. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \rightarrow \mathbb{R}$ be an r -parameter unfolding of $f(s)$ which has an A_k -singularity ($k \leq r$) at s_0 . Suppose that F is a versal unfolding, then $(D_F, \mathbf{x}_0)$ is diffeomorphic to $(\text{Im}DA_k^\pm \times \mathbb{R}^{r-k}, 0)$ as set-germs. Moreover, $(D_F^{k-1}, \mathbf{x}_0)$ is diffeomorphic to $(\text{Im}\sigma_1[2, 3, \dots, k, k+1] \times \mathbb{R}^{r-k}, 0)$ as set-germs, where $\sigma_1[2, 3, \dots, k, k+1] : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^k, 0)$ is a curve defined by

$$\sigma_1[2, 3, \dots, k, k+1](t) = (t^2, t^3, \dots, t^k, t^{k+1}).$$

Proof. . By the equation 1 for $i = 1$, we have

$$\begin{cases} x_1 = \mp s^{k+1} - x_2 s - \dots - x_k s^{k-1}, \\ x_2 = \mp(k+1)s^k - 2x_3 s - \dots - (k-1)x_k s^{k-2}. \end{cases}$$

Then

$$\begin{cases} x_1 = \mp s^{k+1} - x_3 s^2 + \dots + (k-2)x_k s^{k-1}, \\ x_2 = \mp(k+1)s^k - 2x_3 s - \dots - (k-1)x_k s^{k-2}. \end{cases}$$

If we put $s = u_1, x_3 = u_2, \dots, x_k = u_{k-1}$, the above equations means that $(D_G, 0) = (\text{Im}DA_k^\pm \times \mathbb{R}^{r-k}, 0)$. Since $f(s)$ has an A_k singularity at $s = s_0$, f is $\mathcal{R}$ -equivalent to $\pm t^{k+1}$. By Theorem 2.3.2, F and G are $P\text{-}\mathcal{R}$ -equivalent, so $(D_F, \mathbf{x}_0)$ and $(D_G, 0)$ are diffeomorphic as set-germs.

On the other hand, if we continue the above calculation until $i = k - 1$, we can show that

$$x_1 = \lambda_1(k)s^{k+1}, \quad x_2 = \lambda_2(k)s^k, \quad x_3 = \lambda_3(k)s^{k-1}, \dots, \quad x_k = \lambda_k(k)s^2,$$

for some $\lambda_i(k) \in \mathbb{Q}$. By an coordinate change on $\mathbb{R}^r$, we have

$$x_1 = s^2, \quad x_2 = s^3, \quad x_3 = s^4, \dots, \quad x_k = s^{k+1}.$$

This means that $(D_G^{k-1}, 0)$ is diffeomorphic to $(\text{Im}\sigma_1[2, 3, \dots, k, k+1] \times \mathbb{R}^{r-k}, 0)$ as set germs. By Theorem 2.3.2 and Proposition 2.3.3, we have the assertion. This completes the proof. $\square$

Theorem 2.3.6. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \rightarrow \mathbb{R}$ be an r -parameter unfolding of f which has the type A_k at s_0 . If F is a p -versal unfolding of f , then we have the following:

- (1) If $k = 3$, then B_F is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.
- (2) If $k = 4$, then B_F is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.
- (3) If $k = 5$, then B_F is locally diffeomorphic to $BF \times \mathbb{R}^{r-4}$.

Here $C = \{(x_1, x_2) | x_1^2 = x_2^3\}$ is the cusp, $SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the swallowtail and $BF = \{(x_1, x_2, x_3, x_4) | x_1 = 4u^5 + 2u^3v + u^2w, x_2 = -5u^4 - 3u^2v - 2uw, x_3 = v, x_4 = w\}$ is the butterfly.

3 Lorentzian Sabban curves in de Sitter n-space

3.1 Differential geometry of curves in de Sitter space

Let $\gamma : I \longrightarrow \mathbb{R}_1^{n+1}$ be a regular curve (i.e., $\dot{\gamma}(t) = \frac{d\gamma}{dt} \neq 0$), where I is an open interval. Let $S_1^n = \{\mathbf{x} \in \mathbb{R}_1^{n+1} | \langle \mathbf{x}, \mathbf{x} \rangle = 1\}$ be the n -dimensional de Sitter space. Given a vector $\mathbf{n} \in \mathbb{R}_1^{n+1} \setminus \{0\}$ and a real number c , the hyperplane with a normal vector $\mathbf{n}$ is defined to be $HP(\mathbf{n}, c) = \{\mathbf{x} \in \mathbb{R}_1^{n+1} | \langle \mathbf{x}, \mathbf{n} \rangle = c\}$. A subspace in S_1^n is given by

$$S_q^{n-1}(\mathbf{n}, c) = S_1^n \cap HP(\mathbf{n}, c) = \{\mathbf{x} \in S_1^n | \langle \mathbf{x}, \mathbf{n} \rangle = c\},$$

where $q = 0$ if $\mathbf{n}$ is a timelike vector or a lightlike vector, $q = 1$ if $\mathbf{n}$ is a spacelike vector. We say that $S_q^{n-1}(\mathbf{n}, c)$ is a *great de Sitter subspace* if $q = 1$ and a *small de Sitter subspace* if $q = 0$, respectively.

We now define generalized Lorentzian Sabban frame of a non-lightlike curve in S_1^n . Let $\gamma : I \rightarrow S_1^n$ be a regular non-lightlike curve. We can reparametrize γ by the arc-length s , so that we have the unit tangent vector $\mathbf{t}(s) = \gamma'(s)$. When $\|\mathbf{t}'(s) + \delta(\mathbf{t}(s))\gamma(s)\| \neq 0$, where $\delta(\mathbf{x}) = \text{sign}(\mathbf{x})$, we define a unit vector $\mathbf{n}_1(s) = (\mathbf{t}'(s) + \delta(\mathbf{t}(s))\gamma(s)) / \|\mathbf{t}'(s) + \delta(\mathbf{t}(s))\gamma(s)\|$. We write $\kappa_1(s) = \|\mathbf{t}'(s) + \delta(\mathbf{t}(s))\gamma(s)\|$, we consider $\kappa_2(s) = \|\mathbf{n}_1'(s) + \delta(\mathbf{t}(s))\delta(\mathbf{n}_1(s))\kappa_1(s)\mathbf{t}(s)\|$. In the case when $\kappa_2(s) \neq 0$, we have another unit vector $\mathbf{n}_2(s) = (\mathbf{n}_1'(s) + \delta(\mathbf{n}_1(s))\kappa_1(s)\mathbf{t}(s)) / \|\mathbf{n}_1'(s) + \delta(\mathbf{t}(s))\delta(\mathbf{n}_1(s))\kappa_1(s)\mathbf{t}(s)\|$. By the same way, we have the following functions and unit vectors.

$$\begin{aligned} \kappa_i(s) &= \|\mathbf{n}_{i-1}'(s) + \delta(\mathbf{n}_{i-1}(s))\delta(\mathbf{n}_{i-2}(s))\kappa_{i-1}(s)\mathbf{n}_{i-2}(s)\|, \\ \mathbf{n}_i(s) &= \frac{\mathbf{n}_{i-1}'(s) + \delta(\mathbf{n}_{i-1}(s))\delta(\mathbf{n}_{i-2}(s))\kappa_{i-1}(s)\mathbf{n}_{i-2}(s)}{\kappa_i(s)}, \end{aligned}$$

for $i = 1, \dots, n-2$. We assume that $\kappa_i(s) \neq 0$ for all i . At last, we define

$$\begin{aligned} \mathbf{n}_{n-1}(s) &= \frac{\gamma(s) \wedge \mathbf{t}(s) \wedge \mathbf{n}_1(s) \wedge \dots \wedge \mathbf{n}_{n-2}(s)}{\|\gamma(s) \wedge \mathbf{t}(s) \wedge \mathbf{n}_1(s) \wedge \dots \wedge \mathbf{n}_{n-2}(s)\|}, \\ \kappa_{n-1}(s) &= \langle \mathbf{n}_{n-1}(s), \mathbf{n}_{n-2}'(s) \rangle. \end{aligned}$$

We call γ a *generalized Lorentzian Sabban curve* in S_1^n , $\kappa_i(s)$ a *ith-curvature* of $\gamma(s)$.

With the above notation, by a straightforward calculation, we can prove the set of vectors

$$\{\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}_1(s), \dots, \boldsymbol{n}_{n-1}(s)\}$$

is a frame of $\mathbb{R}_1^{n+1}$ along $\boldsymbol{\gamma}$. We call this frame a *generalized Lorentzian Sabban frame* along $\boldsymbol{\gamma}$.

We remark that $\kappa_i(s) \neq 0$ for $i = 1, \dots, n-2$, while $\kappa_{n-1}(s)$ might be equal to 0.

Lemma 3.1.1. With the above notation, vectors $\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}_1(s), \dots, \boldsymbol{n}_{n-1}(s)$ are orthogonal to each other.

Proof. By definition, $\boldsymbol{n}_{n-1}(s)$ is orthogonal to $\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}_1(s), \dots, \boldsymbol{n}_{n-2}(s)$. For other vectors, we can prove by calculation. For example,

$$\langle \boldsymbol{\gamma}(s), \boldsymbol{n}_1(s) \rangle = \frac{1}{\kappa_1} (\langle \boldsymbol{\gamma}(s), \boldsymbol{t}'(s) \rangle + \delta(\boldsymbol{t}(s)) \langle \boldsymbol{\gamma}(s), \boldsymbol{\gamma}(s) \rangle).$$

Since $\langle \boldsymbol{\gamma}(s), \boldsymbol{\gamma}(s) \rangle = 1$, so we have $\langle \boldsymbol{\gamma}(s), \boldsymbol{t}(s) \rangle = 0$. Then $\langle \boldsymbol{\gamma}(s), \boldsymbol{t}(s) \rangle' = \langle \boldsymbol{\gamma}'(s), \boldsymbol{t}(s) \rangle + \langle \boldsymbol{\gamma}(s), \boldsymbol{t}'(s) \rangle = 0$, so

$$\langle \boldsymbol{\gamma}(s), \boldsymbol{t}'(s) \rangle = -\langle \boldsymbol{\gamma}'(s), \boldsymbol{t}(s) \rangle = -\delta(\boldsymbol{t}(s)).$$

Then

$$\begin{aligned} \langle \boldsymbol{\gamma}(s), \boldsymbol{n}_1(s) \rangle &= \frac{1}{\kappa_1} (-\langle \boldsymbol{\gamma}'(s), \boldsymbol{t}(s) \rangle + \delta(\boldsymbol{t}(s))) \\ &= \frac{1}{\kappa_1} (-\delta(\boldsymbol{t}(s)) + \delta(\boldsymbol{t}(s))) = 0. \end{aligned}$$

Therefore $\boldsymbol{\gamma}(s)$ and $\boldsymbol{n}_1(s)$ are orthogonal. We can prove all vectors are orthogonal to each other by the same way. $\square$

We have the following *Frenet-Serret type formulae* of non-lightlike curves in de Sitter n-

space.

$$\left\{ \begin{array}{l} \gamma'(s) = \mathbf{t}(s), \\ \mathbf{t}'(s) = -\delta(\mathbf{t}(s))\gamma(s) + \kappa_1(s)\mathbf{n}_1(s), \\ \mathbf{n}'_1(s) = -\delta(\mathbf{t}(s))\delta(\mathbf{n}_1(s))\kappa_1(s)\mathbf{t}(s) + \kappa_2(s)\mathbf{n}_2(s), \\ \dots = \dots \\ \mathbf{n}'_i(s) = -\delta(\mathbf{n}_{i-1}(s))\delta(\mathbf{n}_i(s))\kappa_i(s)\mathbf{n}_{i-1}(s) + \kappa_{i+1}(s)\mathbf{n}_{i+1}(s), \\ \dots = \dots \\ \mathbf{n}'_{n-2}(s) = -\delta(\mathbf{n}_{n-3}(s))\delta(\mathbf{n}_{n-2}(s))\kappa_{n-2}(s)\mathbf{n}_{n-3}(s) + \kappa_{n-1}(s)\mathbf{n}_{n-1}(s), \\ \mathbf{n}'_{n-1}(s) = -\delta(\mathbf{n}_{n-2}(s))\delta(\mathbf{n}_{n-1}(s))\kappa_{n-1}(s)\mathbf{n}_{n-2}(s). \end{array} \right.$$

If γ is timelike, then a generalized Lorentzian Sabban curve is called a *timelike Sabban curve* which have the following Frenet-Serret type formulae in de Sitter n-space.

$$\left\{ \begin{array}{l} \gamma'(s) = \mathbf{t}(s), \\ \mathbf{t}'(s) = \gamma(s) + \kappa_1(s)\mathbf{n}_1(s), \\ \mathbf{n}'_1(s) = \kappa_1(s)\mathbf{t}(s) + \kappa_2(s)\mathbf{n}_2(s), \\ \mathbf{n}'_2(s) = -\kappa_2(s)\mathbf{n}_1(s) + \kappa_3(s)\mathbf{n}_3(s), \\ \dots = \dots \\ \mathbf{n}'_i(s) = -\kappa_i(s)\mathbf{n}_{i-1}(s) + \kappa_{i+1}(s)\mathbf{n}_{i+1}(s), \\ \dots = \dots \\ \mathbf{n}'_{n-2}(s) = -\kappa_{n-2}(s)\mathbf{n}_{n-3}(s) + \kappa_{n-1}(s)\mathbf{n}_{n-1}(s), \\ \mathbf{n}'_{n-1}(s) = -\kappa_{n-1}(s)\mathbf{n}_{n-2}(s). \end{array} \right. \quad (2)$$

We have the geometric meaning of the curvature $\kappa_{n-1}(s)$ of timelike Sabban curve $\gamma(s)$ as follows.

Proposition 3.1.2. Let $\gamma : I \rightarrow S_1^n$ be a timelike Sabban curve. Then there exists a great de Sitter subspace $S_1^{n-1}(\mathbf{n}, 0)$ such that $\gamma(I) \subset S_1^{n-1}(\mathbf{n}, 0)$ if and only if $\kappa_{n-1}(s) \equiv 0$.

Proof. Suppose that $\kappa_{n-1}(s) \equiv 0$. By the Frenet-Serret type formula (2), $\mathbf{n}_{n-1}(s)$ is a constant vector. We denote that $\mathbf{n}_{n-1}(s) = \mathbf{n}$. We consider a function $f : I \rightarrow \mathbb{R}$ defined by $f(s) = \langle \gamma(s), \mathbf{n} \rangle$, then we have $f(s) = \langle \gamma(s), \mathbf{n} \rangle = \langle \gamma(s), \mathbf{n}_{n-1}(s) \rangle = 0$ and $f'(s) = \langle \mathbf{t}(s), \mathbf{n} \rangle = \langle \mathbf{t}(s), \mathbf{n}_{n-1}(s) \rangle = 0$. Therefore $f(s)$ is constantly equal to 0, so $\gamma(s) \in S_1^{n-1}(\mathbf{n}, 0)$.

For the converse, suppose that there exists $S_1^{n-1}(\mathbf{n}, 0)$ such that $\gamma(I) \subset S_1^{n-1}(\mathbf{n}, 0)$. Then the function f as above is constantly equal to 0. It follows that $f'(s) = \langle \mathbf{t}(s), \mathbf{n} \rangle = 0$. Thus $f''(s) = \langle \mathbf{t}'(s), \mathbf{n} \rangle = \langle \gamma(s) + \kappa_1(s)\mathbf{n}_1(s), \mathbf{n} \rangle = \kappa_1 \langle \mathbf{n}_1(s), \mathbf{n} \rangle = 0$. Since $\kappa_1(s) \neq 0$, we have $\langle \mathbf{n}_1(s), \mathbf{n} \rangle = 0$. It follows that

$$\langle \mathbf{n}'_1(s), \mathbf{n} \rangle = \langle \kappa_1(s)\mathbf{t}(s) + \kappa_2(s)\mathbf{n}_2(s), \mathbf{n} \rangle = \kappa_2 \langle \mathbf{n}_2(s), \mathbf{n} \rangle = 0.$$

Since $\kappa_2(s) \neq 0$, we have $\langle \mathbf{n}_2(s), \mathbf{n} \rangle = 0$. It follows that

$$\langle \mathbf{n}'_2(s), \mathbf{n} \rangle = \langle -\kappa_2(s)\mathbf{n}_1(s) + \kappa_3(s)\mathbf{n}_3(s), \mathbf{n} \rangle = \kappa_3 \langle \mathbf{n}_3(s), \mathbf{n} \rangle = 0.$$

Since $\kappa_3(s) \neq 0$, we have $\langle \mathbf{n}_3(s), \mathbf{n} \rangle = 0$. We continue this procedure. Finally, we have $\kappa_{n-1}(s) \langle \mathbf{n}_{n-1}(s), \mathbf{n} \rangle = 0$. If $\langle \mathbf{n}_{n-1}(s), \mathbf{n} \rangle = 0$, then $\mathbf{n}$ is orthogonal to all vectors of the generalized Sabban frame $\{\gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \dots, \mathbf{n}_{n-1}(s)\}$ which contradicts to the fact that the generalized Sabban frame is a basis of $\mathbb{R}_1^{n+1}$ and $\mathbf{n} \neq 0$. Thus $\kappa_{n-1}(s) = 0$ for any $s \in I$. $\square$

3.2 De Sitter height function

In this section we introduce a family of functions which is useful for the study of invariants of timelike curves in the S_1^n . We consider $\gamma^{(k)}(s)$, by the Frenet-Serret type formulae, we have the following lemma.

Lemma 3.2.1. Let $\gamma : I \rightarrow S_1^n$ be a timelike Sabban curve. Then $\gamma^{(k)}(s)$ has the following form:

- (1) $\gamma'(s) = \mathbf{t}(s)$.
- (2) $\gamma''(s) = \gamma(s) + \kappa_1 \mathbf{n}_1(s)$.
- (3) For $3 \leq k \leq n$, there exist functions $x(s), y(s), a_1(s), \dots, a_{k-2}(s)$ such that

$$\gamma^{(k)}(s) = x(s)\gamma(s) + y(s)\mathbf{t}(s) + a_1(s)\mathbf{n}_1(s) + \dots + a_{k-2}(s) + \kappa_1(s)\kappa_2(s) \dots \kappa_{k-1}(s)\mathbf{n}_{k-1}(s). \quad (3)$$

Proof. By the Frenet-Serret type formulae, (1),(2) hold.

Now we prove (3). Let $k = 3$, we have

$$\begin{aligned}
\gamma^{(3)}(s) &= \{\gamma(s) + \kappa_1(s)\mathbf{n}_1(s)\}' \\
&= \mathbf{t}(s) + \kappa_1'(s)\mathbf{n}_1(s) + \kappa_1(s)\mathbf{n}_1'(s) \\
&= \mathbf{t}(s) + \kappa_1'(s)\mathbf{n}_1(s) + \kappa_1(s)(\kappa_1(s)\mathbf{t}(s) + \kappa_2(s)\mathbf{n}_2(s)) \\
&= (1 + \kappa_1^2(s))\mathbf{t}(s) + \kappa_1'(s)\mathbf{n}_1(s) + \kappa_1(s)\kappa_2(s)\mathbf{n}_2(s).
\end{aligned}$$

So (3) holds for $k = 3$.

We assume that (3) holds for $k - 1$, there exist $x(s), y(s), a_1(s), \dots, a_{k-3}(s)$ such that

$$\begin{aligned}
\gamma^{(k-1)}(s) &= x(s)\gamma(s) + y(s)\mathbf{t}(s) + a_1(s)\mathbf{n}_1(s) + \dots + a_{k-3}(s)\mathbf{n}_{k-3}(s) \\
&\quad + \kappa_1(s)\kappa_2(s) \dots \kappa_{k-2}(s)\mathbf{n}_{k-2}(s).
\end{aligned}$$

We can calculate $\gamma^{(k)}(s)$ by using the Frenet-Serret type formulae,

$$\begin{aligned}
\gamma^{(k)}(s) &= \{\gamma^{(k-1)}\}'(s) \\
&= \{x(s)\gamma(s) + y(s)\mathbf{t}(s) + a_1(s)\mathbf{n}_1(s) + \dots + a_{k-3}(s)\mathbf{n}_{k-3}(s) \\
&\quad + \kappa_1(s)\kappa_2(s) \dots \kappa_{k-2}(s)\mathbf{n}_{k-2}(s)\}' \\
&= (x'(s) + y(s))\gamma(s) + (x(s) + y'(s) + a_1(s)\kappa_1(s))\mathbf{t}(s) + \dots \\
&\quad + (a_{k-3}(s)\kappa_{k-2}(s) + (\kappa_1(s)\kappa_2(s) \dots \kappa_{k-2}(s))')\mathbf{n}_{k-2}(s) \\
&\quad + \kappa_1(s)\kappa_2(s) \dots \kappa_{k-2}(s)\kappa_{k-1}(s)\mathbf{n}_{k-1}(s).
\end{aligned}$$

The coefficient of $\mathbf{n}_{k-1}(s)$ is $\kappa_1(s)\kappa_2(s) \dots \kappa_{k-2}(s)\kappa_{k-1}(s)$, then (3) holds for all $k = 3, 4, \dots, n$. □

For a timelike curve $\gamma : I \rightarrow S_1^n$, we define a family of *de Sitter height function* $H^s : I \times S_1^n \rightarrow \mathbb{R}$ by $H^s(s, \mathbf{v}) = \langle \gamma(s), \mathbf{v} \rangle$. We write $h_v^s(s) = H^s(s, \mathbf{v})$. Then we have the following proposition.

Proposition 3.2.2. Let $\gamma : I \rightarrow S_1^n$ be a timelike Sabban curve. Then we have the following:

(1) $h_v^s(s) = 0$ if and only if there exist $y, a_1, \dots, a_{n-2}, a_{n-1} \in \mathbb{R}$ such that $\mathbf{v} = y\mathbf{t}(s) + a_1\mathbf{n}_1(s) +$

$\cdots + a_{n-1}\mathbf{n}_{n-1}(s)$ and $-y^2 + a_1^2 + \cdots + a_{n-1}^2 = 1$.

(2) For $k < n$, $h_v^s(s) = (h_v^s)'(s) = \cdots = (h_v^s)^{(k)}(s) = 0$ if and only if there exist $a_k, \cdots, a_{n-2}, a_{n-1} \in \mathbb{R}$ such that $\mathbf{v} = a_k\mathbf{n}_k(s) + \cdots + a_{n-1}\mathbf{n}_{n-1}(s)$ and $a_k^2 + \cdots + a_{n-1}^2 = 1$.

(3) $h_v^s(s) = (h_v^s)'(s) = \cdots = (h_v^s)^{(n)}(s) = 0$ if and only if $\mathbf{v} = \pm\mathbf{n}_{n-1}(s)$ and $\kappa_{n-1}(s) = 0$.

(4) For $k > n$, $h_v^s(s) = (h_v^s)'(s) = \cdots = (h_v^s)^{(k)}(s) = 0$ if and only if $\mathbf{v} = \pm\mathbf{n}_{n-1}(s)$ and $\kappa_{n-1}(s) = \kappa'_{n-1}(s) = \cdots = \kappa_{n-1}^{(k-n)}(s) = 0$.

Proof. (1) By using the Frenet-Serret type formulae of timelike curve, there exist $x, y, a_1, \cdots, a_{n-1} \in \mathbb{R}$ such that

$$\mathbf{v} = x\boldsymbol{\gamma}(s) + y\mathbf{t}(s) + a_1\mathbf{n}_1(s) + \cdots + a_{n-1}\mathbf{n}_{n-1}(s).$$

Then

$$h_v^s(s) = \langle \boldsymbol{\gamma}(s), \mathbf{v} \rangle = \langle \boldsymbol{\gamma}(s), x\boldsymbol{\gamma}(s) + y\mathbf{t}(s) + a_1\mathbf{n}_1(s) + \cdots + a_{n-1}\mathbf{n}_{n-1}(s) \rangle = x,$$

we have $x = 0$. Since $\mathbf{v} \in S_1^n$, then $-y^2 + a_1^2 + \cdots + a_{n-1}^2 = 1$.

(2) Let $k = 1$, we have

$$(h_v^s)'(s) = \langle \mathbf{t}(s), \mathbf{v} \rangle = -y.$$

Thus $h_v^s(s) = (h_v^s)'(s) = 0$ means $x = y = 0$ and $a_1^2 + \cdots + a_{n-1}^2 = 1$. Then (2) holds for $k = 1$.

We now assume that (2) for $k - 1$, it means that $h_v^s(s) = (h_v^s)'(s) = \cdots = (h_v^s)^{(k-1)}(s) = 0$ if and only if there exist $a_{k-1}, \cdots, a_{n-1} \in \mathbb{R}$ such that $\mathbf{v} = a_{k-1}\mathbf{n}_{k-1}(s) + \cdots + a_{n-1}\mathbf{n}_{n-1}(s)$ and $a_{k-1}^2 + \cdots + a_{n-1}^2 = 1$. By assertion 3.1, we have

$$\boldsymbol{\gamma}^{(k)}(s) = (x'(s) + y(s))\boldsymbol{\gamma}(s) + \cdots + \kappa_1(s)\kappa_2(s) \cdots \kappa_{k-2}(s)\kappa_{k-1}(s)\mathbf{n}_{k-1}(s).$$

Therefore, $h_v^s(s) = (h_v^s)'(s) = \cdots = (h_v^s)^{(k)}(s) = 0$ if and only if

$$\langle \boldsymbol{\gamma}^{(k)}(s), a_{k-1}\mathbf{n}_{k-1}(s) + \cdots + a_{n-1}\mathbf{n}_{n-1}(s) \rangle = a_{k-1}\kappa_1(s)\kappa_2(s) \cdots \kappa_{k-2}(s)\kappa_{k-1}(s) = 0.$$

Since $k-1 < n-1$, we have $\kappa_1(s)\kappa_2(s) \cdots \kappa_{k-2}(s)\kappa_{k-1}(s) \neq 0$, so $a_{k-1} = 0$ and $a_k^2 + \cdots + a_{n-1}^2 = 1$.

This completes the induction step.

When $k = n$, by the calculation as above, $h_v^s(s) = (h_v^s)'(s) = \dots = (h_v^s)^{(n)}(s) = 0$ if and only if

$$(h_v^s)^{(n)}(s) = \langle \gamma^{(n)}(s), \pm \mathbf{n}_{n-1}(s) \rangle = \pm \kappa_1(s) \kappa_2(s) \cdots \kappa_{k-2}(s) \kappa_{n-1}(s),$$

it means that $\kappa_{n-1}(s) = 0$. This completes the proof of (3).

By assertion (3), $h_v^s(s) = (h_v^s)'(s) = \dots = (h_v^s)^{(n+1)}(s) = 0$ if and only if $\mathbf{v} = \pm \mathbf{n}_{n-1}(s)$, $\kappa_{n-1}(s) = 0$ and $(h_v^s)^{(n+1)}(s) = 0$. By Lemma 3.1, we have

$$\begin{aligned} \gamma^{(n)}(s) &= \bar{x}(s)\gamma(s) + \bar{y}(s)\mathbf{t}(s) + \bar{a}_1(s)\mathbf{n}_1(s) + \cdots + \bar{a}_{n-2}(s)\mathbf{n}_{n-2}(s) \\ &\quad + \kappa_1(s)\kappa_2(s) \cdots \kappa_{k-2}(s)\kappa_{n-1}(s)\mathbf{n}_{n-1}(s). \end{aligned}$$

We put $K(s) = \kappa_1(s)\kappa_2(s) \cdots \kappa_{k-2}(s)\kappa_{n-1}(s)$, then we have

$$\begin{aligned} \gamma^{(n+1)}(s) &= \bar{x}'(s)\gamma(s) + \bar{x}(s)\mathbf{t}(s) + \bar{y}'(s)\mathbf{t}(s) + \bar{y}(s)(\gamma(s) + \kappa_1(s)\mathbf{n}_1(s)) \\ &\quad \bar{a}_1'(s)\mathbf{n}_1(s) + \bar{a}_1(s)(\kappa_1(s)\mathbf{t}(s) + \kappa_2(s)\mathbf{n}_2(s)) + \cdots \\ &\quad \bar{a}_{n-2}'(s)\mathbf{n}_{n-2}(s) + \bar{a}_{n-2}(s)(-\kappa_{n-2}(s)\mathbf{n}_{n-3}(s) + \kappa_{n-1}(s)\mathbf{n}_{n-1}(s)) \\ &\quad (K'(s)\kappa_{n-1}(s) + K(s)\kappa'_{n-1}(s))\mathbf{n}_{n-1}(s) - K(s)\kappa_{n-1}^2(s)\mathbf{n}_{n-2}(s) \\ &= \mathbf{V}(s) + \mathbf{W}(s), \end{aligned}$$

for some $\mathbf{V}(s) \in \{\gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \dots, \mathbf{n}_{n-2}(s)\}_{\mathbb{R}}$ and

$$\mathbf{W}(s) = (\bar{a}_{n-2}(s)\kappa_{n-1}(s) + K'(s)\kappa_{n-1}(s) + K(s)\kappa'_{n-1}(s))\mathbf{n}_{n-1}(s).$$

Since $\langle \mathbf{V}(s), \mathbf{n}_{n-1}(s) \rangle = 0$ and $\kappa_1(s)\kappa_2(s) \cdots \kappa_{k-2}(s)\kappa_{n-2}(s) \neq 0$, $\mathbf{v}(s) = \pm \mathbf{n}_{n-1}(s)$, $\kappa_{n-1}(s) = 0$ and $(h_v^s)^{(n+1)}(s) = 0$ if and only if $\mathbf{v}(s) = \pm \mathbf{n}_{n-1}(s)$, $\kappa_{n-1}(s) = 0$ and $\kappa'_{n-1}(s) = 0$. Thus assertion (4) for $k = n + 1$ holds. We now assume that the following conditions hold for $k = n + (r - 1)$: $h_v^s(s) = (h_v^s)'(s) = \dots = (h_v^s)^{(k)}(s) = 0$ if and only if $\mathbf{v}(s) = \pm \mathbf{n}_{n-1}(s)$ and $\kappa_{n-1}(s) = \kappa'_{n-1}(s) = \dots = \kappa_{n-1}^{(r-1)}(s) = 0$ and $\gamma^{(k)}(s) = \mathbf{V}(s) + \mathbf{W}(s)$, where $\mathbf{V}(s) \in \{\gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \dots, \mathbf{n}_{n-2}(s)\}_{\mathbb{R}}$ and there exist functions $\eta_0(s), \eta_1(s), \dots, \eta_{r-2}(s)$ such that

$$\mathbf{W}(s) = \left(\sum_{i=0}^{r-2} \eta_i(s) \kappa_{n-1}^{(i)}(s) + K(s) \kappa_{n-1}^{(r-1)}(s) \right) \mathbf{n}_{n-1}(s).$$

Therefore, we have $\mathbf{V}(s) = \overline{\mathbf{V}}(s) + \zeta(s)\mathbf{n}_{n-2}(s)$ for some $\overline{\mathbf{V}}(s) \in \{\boldsymbol{\gamma}(s), \mathbf{t}(s), \mathbf{n}_1(s), \dots, \mathbf{n}_{n-3}(s)\}_{\mathbb{R}}$.

It means that

$$\mathbf{V}'(s) = \overline{\mathbf{V}}'(s) + \zeta'(s)\mathbf{n}_{n-2}(s) + \zeta(s)(-\kappa_{n-2}(s)\mathbf{n}_{n-3}(s) + \kappa_{n-1}(s)\mathbf{n}_{n-1}(s))$$

and

$$\begin{aligned} \mathbf{W}'(s) = & \left(\sum_{i=0}^{r-2} (\eta_i'(s)\kappa_{n-1}^{(i)}(s) + \eta_i(s)\kappa_{n-1}^{(i+1)}(s)) + K'(s)\kappa_{n-1}^{(r-1)}(s) + K(s)\kappa_{n-1}^{(r)}(s) \right) \mathbf{n}_{n-1}(s) \\ & - \left(\sum_{i=0}^{r-2} (\eta_i(s)\kappa_{n-1}^{(i)}(s) + K(s)\kappa_{n-1}^{(r-1)}(s)) \right) \kappa_{n-2}(s)\mathbf{n}_{n-2}(s). \end{aligned}$$

It follows that there exist $\tilde{\mathbf{V}}(s) \in \{\boldsymbol{\gamma}(s), \mathbf{t}(s), \mathbf{n}_1(s), \dots, \mathbf{n}_{n-2}(s)\}_{\mathbb{R}}$ and functions $\bar{\eta}_0(s), \bar{\eta}_1(s), \dots, \bar{\eta}_{r-1}(s)$ such that

$$\boldsymbol{\gamma}^{(k+1)}(s) = \tilde{\mathbf{V}}(s) + \left(\sum_{i=0}^{r-1} \bar{\eta}_i(s)\kappa_{n-1}^{(i)}(s) + K(s)\kappa_{n-1}^{(r)}(s) \right) \mathbf{n}_{n-1}(s).$$

Since $K(s) \neq 0$ and $\langle \mathbf{n}_{n-1}(s), \tilde{\mathbf{V}}(s) \rangle = 0$, $h_v^s(s) = (h_v^s)'(s) = \dots = (h_v^s)^{(k+1)}(s) = 0$ if and only if $\mathbf{v}(s) = \pm \mathbf{n}_{n-1}(s)$ and $\kappa_{n-1}(s) = \kappa'_{n-1}(s) = \dots = \kappa_{n-1}^{(r)}(s) = 0$. This completes the proof of (4). $\square$

3.3 De Sitter Legendrian duality

In [7], Izumiya introduced the Legendrian dualities between pseudo-spheres in Minkowski space which is a basic tool for the study of spacelike submanifolds in pseudo-spheres in Minkowski space. In this section we study the de Sitter Legendrian duality of timelike Sabban curve in the S_1^n . We define 1-forms $\langle d\mathbf{v}, \mathbf{w} \rangle = -w_1 dv_1 + \sum_{i=2}^{n+1} w_i dv_i$, $\langle \mathbf{v}, d\mathbf{w} \rangle = -v_1 dw_1 + \sum_{i=2}^{n+1} v_i dw_i$ in $\mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1}$ and consider the following double fibrations:

$$S_1^n \times S_1^n \supset \Delta_5 = \{(\mathbf{v}, \mathbf{w}) | \langle \mathbf{v}, \mathbf{w} \rangle = 0\},$$

$$\pi_{51} : \Delta_5 \rightarrow S_1^n, \pi_{52} : \Delta_5 \rightarrow S_1^n,$$

$$\theta_{51} = \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_5}, \theta_{52} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_5}.$$

Here, $\pi_{51}(\mathbf{v}, \mathbf{w}) = \mathbf{v}$, $\pi_{52}(\mathbf{v}, \mathbf{w}) = \mathbf{w}$. We remark that $\theta_{51}^{-1}(0)$ and $\theta_{52}^{-1}(0)$ define the same tangent hyperplane field over Δ_5 which is denoted by K . In [4], the following theorem was shown:

Theorem 3.3.1. Under the above notation, both of π_{5i} ($i = 1, 2$) are Legendrian fibrations and (Δ_5, K) is a contact manifold.

If we have a Legendrian submanifold $L \subset \Delta_5$, then we say that $\pi_{51}(L) \subset S_1^n$ and $\pi_{52}(L) \subset S_1^n$ are *de Sitter Legendrian dual* to each other. Since both of $\pi_{51}^{-1}(\mathbf{v}) = \mathbf{v} \times S_1^{n-1}(\mathbf{v}, 0)$ and $\pi_{52}^{-1}(\mathbf{v}) = S_1^{n-1}(\mathbf{v}, 0) \times \mathbf{v}$ are Legendrian submanifolds of Δ_5 , $\mathbf{v} \in S_1^n$ and $S_1^{n-1}(\mathbf{v}, 0) \in S_1^n$ are de Sitter Legendrian dual to each other.

We now define a mapping $D_\gamma : I \times S^{n-2} \rightarrow S_1^{n-1}$ by

$$D_\gamma(s, \xi) = \xi_1 \mathbf{n}_1 + \cdots + \xi_{n-1} \mathbf{n}_{n-1},$$

where $\xi = (\xi_1, \dots, \xi_{n-1}) \in S^{n-2}$. We denote that $\gamma^* = D_\gamma(s, \xi)$, which is called a *de Sitter dual hypersurface* of γ . We have the following theorem:

Theorem 3.3.2. Let $\gamma : I \rightarrow S_1^n$ be a timelike Sabban curve. Then γ and γ^* are de Sitter Legendrian dual to each other.

Proof. We define a mapping $\mathcal{L} : I \times S_1^n \rightarrow \Delta_5$ by $\mathcal{L}(s, \xi) = (\gamma(s), D_\gamma(s, \xi))$. By definition, we have

$$\frac{\partial \mathcal{L}}{\partial s} = (\mathbf{t}(s), \frac{\partial D_\gamma}{\partial s}), \quad \frac{\partial \mathcal{L}}{\partial \xi_i} = (\mathbf{0}, \mathbf{n}_i(s)).$$

Therefore, $\{\partial \mathcal{L} / \partial s, \partial \mathcal{L} / \partial \xi_1, \dots, \partial \mathcal{L} / \partial \xi_{n-1}\}$ are linearly independent. This means that $\mathcal{L} : I \times \mathbb{R}_1^{n+1} \rightarrow \Delta_5$ is an immersion, so that $\mathcal{L}|_{I \times S_1^n}$ is an immersion. Moreover, we have $\mathcal{L}^* \theta_{51} = \langle d\gamma(s), D_\gamma(s, \xi) \rangle = 0$, so that $\mathcal{L}(s, \xi)$ is a Legendrian submanifold of Δ_5 . This completes the proof. $\square$

Corollary 3.3.3. Let $\gamma : I \rightarrow S_1^n$ be a regular timelike curve. Then γ^* is a wave front of $\mathcal{L}(s, \xi)$ respect to the Legendrian fibration π_{52} .

3.4 An application of the theory of unfoldings of function germs

In this section we classify the families of function germs as an application of the unfolding theory of functions. We consider the family of de Sitter height functions on timelike Sabban curve. Then we have the following proposition.

Proposition 3.4.1. Let $\gamma : I \rightarrow S_1^n$ be a timelike Sabban curve and $h_{v_0}^s$ has A_k -singularity at s_0 for any $k \leq n$. Then H^s is an R-versal unfolding of $h_{v_0}^s$.

Proof. Let

$$\gamma(s) = (x_1, x_2, x_3, \dots, x_{n+1})$$

and

$$\mathbf{v}_0 = (\lambda, \mu, \xi_1, \dots, \xi_{n-1})$$

with $\lambda^2 - \mu^2 + \sum_{i=1}^{n-1} \xi_i^2 = 1$. Then we have

$$H^s(s, \mathbf{v}_0) = x_1(s)\lambda - x_2(s)\mu + x_3(s)\xi_1 + \dots + x_{n+1}(s)\xi_{n-1}.$$

Suppose that $\xi_{n-1} > 0$ and $\xi_{n-1} = \sqrt{1 - \lambda^2 + \mu^2 - \sum_{i=1}^{n-2} \xi_i^2}$. Then we adopt the local coordinates $(\lambda, \mu, \xi_1, \dots, \xi_{n-2})$ for S_1^n , it follows that

$$\begin{aligned} \frac{\partial H^s}{\partial \lambda}(s, \mathbf{v}_0) &= x_1(s) - \frac{\lambda}{\xi_{n-1}} x_{n+1}(s), \\ \frac{\partial H^s}{\partial \mu}(s, \mathbf{v}_0) &= -x_2(s) + \frac{\mu}{\xi_{n-1}} x_{n+1}(s), \\ \frac{\partial H^s}{\partial \xi_1}(s, \mathbf{v}_0) &= x_3(s) - \frac{\xi_1}{\xi_{n-1}} x_{n+1}(s), \\ &\vdots \\ \frac{\partial H^s}{\partial \xi_{n-2}}(s, \mathbf{v}_0) &= x_n(s) - \frac{\xi_{n-2}}{\xi_{n-1}} x_{n+1}(s). \end{aligned}$$

We consider the following matrix:

$$A = \begin{pmatrix} x_1(s) - \frac{\lambda}{\xi_{n-1}} x_{n+1}(s) & -x_2(s) + \frac{\mu}{\xi_{n-1}} x_{n+1}(s) & \dots & x_n(s) - \frac{\xi_{n-2}}{\xi_{n-1}} x_{n+1}(s) \\ x'_1(s) - \frac{\lambda}{\xi_{n-1}} x'_{n+1}(s) & -x'_2(s) + \frac{\mu}{\xi_{n-1}} x'_{n+1}(s) & \dots & x'_n(s) - \frac{\xi_{n-2}}{\xi_{n-1}} x'_{n+1}(s) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{(n-1)}(s) - \frac{\lambda}{\xi_{n-1}} x_{n+1}^{(n-1)}(s) & -x_2^{(n-1)}(s) + \frac{\mu}{\xi_{n-1}} x_{n+1}^{(n-1)}(s) & \dots & x_n^{(n-1)}(s) - \frac{\xi_{n-2}}{\xi_{n-1}} x_{n+1}^{(n-1)}(s) \end{pmatrix}.$$

If we put $\mathbf{a}_i = (x_i(s), x'_i(s), \dots, x_i^{(n-1)}(s))$, then we have

$$A = (\mathbf{a}_1 - \frac{\lambda}{\xi_{n-1}} \mathbf{a}_{n+1}, -\mathbf{a}_2 + \frac{\mu}{\xi_{n-1}} \mathbf{a}_{n+1}, \mathbf{a}_3 - \frac{\xi_1}{\xi_{n-1}} \mathbf{a}_{n+1}, \dots, \mathbf{a}_n - \frac{\xi_{n-2}}{\xi_{n-1}} \mathbf{a}_{n+1}).$$

By a straightforward calculation, we have

$$\begin{aligned}
-\det A &= \det(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n) + \frac{\lambda}{\xi_{n-1}} \det(\mathbf{a}_{n+1}, \mathbf{a}_2, \dots, \mathbf{a}_n) \\
&\quad + \frac{\mu}{\xi_{n-1}} \det(\mathbf{a}_1, \mathbf{a}_{n+1}, \mathbf{a}_3, \dots, \mathbf{a}_n) + \dots + \frac{\xi_{n-2}}{\xi_{n-1}} \det(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n-1}, \mathbf{a}_{n+1}) \\
&= \frac{(-1)^n}{\xi_{n-1}} \langle \mathbf{v}_0, \gamma(s) \wedge \gamma'(s) \wedge \dots \wedge \gamma^{(n-1)}(s) \rangle.
\end{aligned}$$

Since

$$\gamma(s) \wedge \gamma'(s) \wedge \dots \wedge \gamma^{(n-1)}(s) = \kappa_1^{n-3}(s) \kappa_2^{n-4}(s) \dots \kappa_{n-2}(s).$$

As $h_{v_0}^s$ has A_k -singularity, so that

$$\det A = (-1)^{n+1} \kappa_1^{n-3}(s) \kappa_2^{n-4}(s) \dots \kappa_{n-2}(s) \neq 0.$$

It means that $\text{rank} A = n$. Thus the assertion for $k = n$ holds. It follows that the assertion for $k < n$ also holds. For other coordinates, we have the arguments similar to above. This completes the proof. $\square$

By Proposition 3.2.2, we have $\gamma^* = D_{H^s}$ and $D_{H^s}^{n-1} = \{\pm \mathbf{n}_{n-1}(s) | s \in I\}$. Proposition 3.2.2 also asserts that $h_{v_0}^s$ has A_k -singularity at s_0 for $k < n$ if and only if exist

$$(0, \dots, 0, \xi_k, \xi_{k+1}, \dots, \xi_{n-1}) \in S^{n-2} \text{ and } (\xi_k \neq 0)$$

such that

$$\mathbf{v}_0 = \xi_k \mathbf{n}_k(s_0) + \dots + \xi_{n-1} \mathbf{n}_{n-1}(s_0).$$

We define a set $\gamma_k^* \subset S^n$ by

$$\gamma_k^* = \{\xi_k \mathbf{n}_k(s) + \dots + \xi_{n-1} \mathbf{n}_{n-1}(s) | \xi_k^2 + \dots + \xi_{n-1}^2 = 1, s \in I\},$$

so that $\gamma_k^* = D_{H^s}^k$. By Theorem 2.3.5 and Proposition 3.4.1, we have the following theorem.

Theorem 3.4.2. Let $\gamma : I \rightarrow S_1^n$ be a timelike Sabban curve. Then we have the following:

(1) For $k < n-1$, the germ of de Sitter dual γ^* of γ at $(s_0, \xi) = (s_0, (\xi_1, \dots, \xi_{n-1})) \in I \times S^{n-1}$ is

diffeomorphic to $(\text{Im}DA_k^\pm \times \mathbb{R}^{n-k}, 0)$ as set-germs if and only if $\xi_1 = \dots = \xi_{k-1} = 0$ and $\xi_k \neq 0$. In this case the germ of γ_k^* at (s_0, ξ) is diffeomorphic to $(\text{Im}\sigma_1[2, 3, \dots, k, k+1] \times \mathbb{R}^{n-k}, 0)$ as set-germs.

(2) The germ of de Sitter dual γ^* of γ at $(s_0, \xi) \in I \times S^{n-1}$ is diffeomorphic to $(\text{Im}DA_k^\pm \times \mathbb{R}, 0)$ as set-germs if and only if $\xi = (0, \dots, 0, \pm 1)$ and $\kappa_{n-1}(s_0) \neq 0$. In this case the germ of the image of $\mathbf{n}_{n-1}(s)$ at s_0 is diffeomorphic to $(\mathbb{R}, 0)$ as set-germs.

(3) The germ of de Sitter dual γ^* of γ at $(s_0, \xi) \in I \times S^{n-1}$ is diffeomorphic to $(\text{Im}DA_k^\pm, 0)$ as set-germs if and only if $\xi = (0, \dots, 0, \pm 1)$, $\kappa_{n-1}(s_0) = 0$ and $\kappa'_{n-1}(s_0) \neq 0$. In this case the germ of the image of $\mathbf{n}_{n-1}(s)$ at s_0 is diffeomorphic to $(\text{Im}\sigma_1[2, 3, \dots, n-1, n], 0)$ as set-germs.

Proof. By Proposition 3.4.1, if $h_{v_0}^s$ has A_k -singularity at s_0 for any $k \leq n$. Then H^s is an R-versal unfolding of $h_{v_0}^s$. Since $\gamma^* = D_{H^s}^k$, the assertions holds by Proposition 3.2.2, Theorem 2.3.5 and Proposition 3.4.1. $\square$

3.5 Contact with great de Sitter subspaces

In this section, we discuss the contact of γ with a great de Sitter subspace. By Proposition 3.1.2, there exist a spacelike vector $\mathbf{v} \in S_1^n$ such that $\gamma(I) \subset S_1^{n-1}(\mathbf{v}, 0)$ if and only if $\kappa_{n-1}(s) \equiv 0$. In this case, $\mathbf{n}_{n-1}(s) \equiv \mathbf{v}$ is a constant vector. By definition, $\mathbf{v} \in D_{H^s}$ if and only if there exists $s_0 \in I$ such that $h_v^s(s_0) = (h_v^s)'(s_0) = 0$. If we consider a function $\mathbf{g}_v : S_1^n \rightarrow \mathbb{R}$ defined by $\mathbf{g}_v(\mathbf{x}) = \langle \mathbf{v}, \mathbf{x} \rangle$, then $\mathbf{g}_v^{-1}(0) = S_1^{n-1}(\mathbf{v}, 0)$ and $\mathbf{g}_v \circ \gamma(s) = h_v^s(s)$, so that $h_v^s(s_0) = (h_v^s)'(s_0) = 0$ if and only if $\gamma(s)$ is tangent to $S_1^{n-1}(\mathbf{v}, 0)$ at s_0 . We call $S_1^{n-1}(\mathbf{v}, 0)$ a *tangent great de Sitter subspace* of γ at s_0 . There are infinite tangent great de Sitter subspaces of γ at s_0 . We say that $S_1^{n-1}(\mathbf{v}, 0)$ has *at least $k+1$ -point contact* with γ at s_0 if $h_v^s(s_0) = (h_v^s)'(s_0) = \dots = (h_v^s)^k(s_0) = 0$, we also say that $S_1^{n-1}(\mathbf{v}, 0)$ has *$k+1$ -point contact* with γ at s_0 if $h_v^s(s_0) = (h_v^s)'(s_0) = \dots = (h_v^s)^k(s_0) = 0$ and $(h_v^s)^{k+1}(s_0) \neq 0$. By Proposition 3.2.2, we have the following proposition.

Proposition 3.5.1. Let $\gamma : I \rightarrow S_1^n$ be a timelike Sabban curve and $\mathbf{v}_0 \in S_1^n$. Then we have the following:

(1) $S_1^{n-1}(\mathbf{v}, 0)$ is a tangent great de Sitter subspace of γ at s_0 if and only if there exists

$\xi = (\xi_1, \dots, \xi_{n-1}) \in S^{n-2}$ such that $\mathbf{v}_0 = \xi_1 \mathbf{n}_1(s_0) + \dots + \xi_{n-1} \mathbf{n}_{n-1}(s_0)$.

(2) For $k < n$, $S_1^{n-1}(\mathbf{v}, 0)$ has at least $k+1$ -point contact with γ at s_0 if and only if there exists

$\xi = (0, \dots, 0, \xi_k, \xi_{k+1}, \dots, \xi_{n-1}) \in S^{n-2}$ such that $\mathbf{v}_0 = \xi_k \mathbf{n}_k(s_0) + \dots + \xi_{n-1} \mathbf{n}_{n-1}(s_0)$.

(3) $S_1^{n-1}(\mathbf{v}, 0)$ has at least n -point contact with γ at s_0 if and only if $\mathbf{v}_0 = \pm \mathbf{n}_{n-1}(s_0)$.

(4) For $k > n$, $S_1^{n-1}(\mathbf{v}, 0)$ has at least $k+1$ -point contact with γ at s_0 if and only if $\mathbf{v}_0 = \pm \mathbf{n}_{n-1}(s_0)$ and $\kappa_{n-1}(s_0) = \kappa'_{n-1}(s_0) = \dots = \kappa_{n-1}^{(k-n)}(s_0) = 0$.

For $k < n$, γ_k^* is the locus of the vectors $\mathbf{v} \in S_1^n$ such that $S_1^{n-1}(\mathbf{v}, 0)$ has at least $k+1$ -point contact with γ . In particular, $\gamma_n^* = \{\pm \mathbf{n}_{n-1}(s) | s \in I\}$ is the locus of the vectors $\mathbf{v} \in S_1^n$ such that $S_1^{n-1}(\mathbf{v}, 0)$ has at least $n+1$ -point contact with γ . So if $\mathbf{v} = \pm \mathbf{n}_{n-1}(s_0)$, we call $S_1^{n-1}(\mathbf{v}, 0)$ an *osculating tangent great de Sitter subspace* of γ at s_0 . Then, as a corollary of Theorem 3.4.2 and Proposition 3.5.1, we have the following theorem.

Theorem 3.5.2. Let $\gamma : I \rightarrow S_1^n$ be a timelike Sabban curve and $S_1^{n-1}(\mathbf{v}_0, 0)$ a tangent great de Sitter subspace of γ at s_0 . Then we have the following:

(1) For $k < n-1$, the germ of de Sitter dual γ^* of γ at $(s_0, \xi) = (s_0, (\xi_1, \dots, \xi_{n-1})) \in I \times S^{n-1}$ is diffeomorphic to $(\text{Im}DA_k^\pm \times \mathbb{R}^{n-k}, 0)$ as set-germs if and only if $S_1^{n-1}(\mathbf{v}_0, 0)$ has at least $k+1$ -point contact with γ at s_0 . In this case the germ of γ_k^* at (s_0, ξ) is diffeomorphic to $(\text{Im}\sigma_1[2, 3, \dots, k, k+1] \times \mathbb{R}^{n-k}, 0)$ as set-germs.

(2) The germ of de Sitter dual γ^* of γ at $(s_0, \xi) \in I \times S^{n-1}$ is diffeomorphic to $(\text{Im}DA_k^\pm \times \mathbb{R}, 0)$ as set-germs if and only if $S_1^{n-1}(\mathbf{v}_0, 0)$ is an osculating tangent great de Sitter subspace of γ at s_0 and $\kappa_{n-1}(s_0) \neq 0$. In this case the germ of the image of $\mathbf{n}_{n-1}(s)$ at s_0 is diffeomorphic to $(\mathbb{R}, 0)$ as set-germs.

(3) The germ of de Sitter dual γ^* of γ at $(s_0, \xi) \in I \times S^{n-1}$ is diffeomorphic to $(\text{Im}DA_k^\pm, 0)$ as set-germs if and only if $S_1^{n-1}(\mathbf{v}_0, 0)$ is an osculating tangent great de Sitter subspace of γ at s_0 , $\kappa_{n-1}(s_0) = 0$ and $\kappa'_{n-1}(s_0) \neq 0$. In this case the germ of the image of $\mathbf{n}_{n-1}(s)$ at s_0 is diffeomorphic to $(\text{Im}\sigma_1[2, 3, \dots, n-1, n], 0)$ as set-germs.

4 De Sitter evolutes and de Sitter focal surfaces of time-like Sabban curves

4.1 De Sitter evolutes of timelike Sabban curves in S_1^2

In this section we consider a timelike Sabban curve in S_1^2 . Let $\gamma : I \rightarrow S_1^2$ be a timelike Sabban curve, we can reparameterize it by the arc-length s . Then we have a orthogonal frame $\{\gamma, \mathbf{t}, \mathbf{n}\}$ along γ , where $\mathbf{t}(s)$ is the unit tangent vector of γ at s and $\mathbf{n}(s) = \gamma(s) \wedge \mathbf{t}(s)$. We have the Frenet-type formulae:

$$\begin{cases} \gamma'(s) = \mathbf{t}(s), \\ \mathbf{t}'(s) = \gamma(s) + \kappa \mathbf{n}(s), \\ \mathbf{n}'(s) = \kappa \mathbf{t}(s). \end{cases}$$

Here, $\kappa = \|\gamma''(s) - \gamma(s)\|$ is the *geodesic curvature* of γ .

We define a function $H : I \times S_1^2 \rightarrow \mathbb{R}$ by $H(s, \mathbf{v}) = \langle \gamma(s), \mathbf{v} \rangle$. We call H a *de Sitter height function* on γ and write $h_{v_0}(s) = H(s, v_0)$ for any fixed vector $\mathbf{v}_0 \in S_1^2$.

Here we calculate the derivative of H , by Frenet-type formulae, we have

$$\frac{\partial H}{\partial s} = \langle \gamma'(s), \mathbf{v} \rangle = \langle \mathbf{t}(s), \mathbf{v} \rangle.$$

Since $\mathbf{v} \in S_1^2$, there exists $\theta \in [0, 2\pi)$ such that $\mathbf{v} = \cos \theta \gamma(s) + \sin \theta \mathbf{n}(s)$ if and only if $\frac{\partial H}{\partial s} = 0$.

We also have

$$\frac{\partial^2 H}{\partial s^2} = \langle \mathbf{t}'(s), \mathbf{v} \rangle = \langle \gamma(s) + \kappa \mathbf{n}(s), \mathbf{v} \rangle.$$

Therefore, $\frac{\partial H}{\partial s} = \frac{\partial^2 H}{\partial s^2} = 0$ if and only if $\mathbf{v} = \cos \theta \gamma(s) + \sin \theta \mathbf{n}(s)$ and $\cos \theta + \kappa \sin \theta = 0$.

Moreover, we have

$$\frac{\partial^3 H}{\partial s^3} = \langle (1 + \kappa^2) \mathbf{t}(s) + \kappa' \mathbf{n}(s), \mathbf{v} \rangle.$$

Under the condition $\frac{\partial H}{\partial s} = \frac{\partial^2 H}{\partial s^2} = 0$, $\frac{\partial^3 H}{\partial s^3} = 0$ if and only if $\mathbf{v} = \cos \theta \gamma(s) + \sin \theta \mathbf{n}(s)$, $\cos \theta + \kappa \sin \theta = 0$ and $\kappa'(s) = 0$. If we calculate the 4th derivative of H , we can show that the above

conditions with the extra condition

$$\frac{\partial^4 H}{\partial s^4} = \langle (1 + \kappa^2)\boldsymbol{\gamma}(s) + 3\kappa\kappa'\mathbf{t}(s) + (\kappa + \kappa^3 + \kappa'')\mathbf{n}(s), \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\mathbf{n}(s) \rangle = 0$$

which is equivalent to the $\kappa''(s) = 0$. As a consequence, we have the following proposition.

Proposition 4.1.1. For any $(s, \mathbf{v}) \in I \times S_1^2$, we have the followings:

- (1) $h'_v(s) = 0$ if and only if there exists $\theta \in [0, 2\pi)$ such that $\mathbf{v} = \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\mathbf{n}(s)$.
- (2) $h'_v(s) = h''_v(s) = 0$ if and only if $\mathbf{v} = \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\mathbf{n}(s)$ and $\cos\theta + \kappa\sin\theta = 0$.
- (3) $h'_v(s) = h''_v(s) = h'''_v(s) = 0$ if and only if $\mathbf{v} = \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\mathbf{n}(s)$, $\cos\theta + \kappa\sin\theta = 0$ and $\kappa'(s) = 0$.
- (4) $h'_v(s) = h''_v(s) = h'''_v(s) = h_v^{(4)}(s) = 0$ if and only if $\mathbf{v} = \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\mathbf{n}(s)$, $\cos\theta + \kappa\sin\theta = 0$, $\kappa'(s) = 0$ and $\kappa''(s) = 0$.

By the above proposition, we have an invariant κ' . $h'_v(s) = h''_v(s) = 0$ if and only if $\mathbf{v} = \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\mathbf{n}(s)$ and $\cos\theta + \kappa\sin\theta = 0$. Since $\cos\theta + \kappa\sin\theta = 0$ is equal to $\sin\theta = \pm \frac{1}{\sqrt{1+\kappa^2}}$ and $\cos\theta = \mp \frac{\kappa}{\sqrt{1+\kappa^2}}$, then $h'_v(s) = h''_v(s) = 0$ if and only if $\mathbf{v} = \mp \frac{\kappa}{\sqrt{1+\kappa^2}}\boldsymbol{\gamma}(s) \pm \frac{1}{\sqrt{1+\kappa^2}}\mathbf{n}(s)$. We now define curves $\mathbf{E}_\gamma^\pm : I \rightarrow S_1^2$ by

$$\mathbf{E}_\gamma^\pm(s) = \mp \frac{\kappa}{\sqrt{1+\kappa^2}}\boldsymbol{\gamma}(s) \pm \frac{1}{\sqrt{1+\kappa^2}}\mathbf{n}(s).$$

We call $\mathbf{E}_\gamma^\pm(s)$ the *de Sitter evolute* of $\boldsymbol{\gamma}(s)$. By straightforward calculations, we have the following proposition.

Proposition 4.1.2. The evolute $\mathbf{E}_\gamma^\pm(s)$ of $\boldsymbol{\gamma}(s)$ is singular at s_0 if and only if $\kappa'(s_0) = 0$.

Proof. The evolute $\mathbf{E}_\gamma^\pm(s)$ of $\boldsymbol{\gamma}(s)$ is given by

$$\mathbf{E}_\gamma^\pm(s) = \mp \frac{\kappa}{\sqrt{1+\kappa^2}}\boldsymbol{\gamma}(s) \pm \frac{1}{\sqrt{1+\kappa^2}}\mathbf{n}(s).$$

We have

$$\begin{aligned} (\mathbf{E}_\gamma^\pm)'(s) &= \mp \left(\frac{\kappa'(s)}{\sqrt{1+\kappa^2(s)}} + \frac{\kappa'(s)\kappa^2(s)}{(1+\kappa^2(s))\sqrt{1+\kappa^2(s)}} \right) \boldsymbol{\gamma}(s) \mp \frac{\kappa(s)}{\sqrt{1+\kappa^2(s)}} \mathbf{t}(s) \\ &\quad \mp \frac{\kappa'(s)\kappa(s)}{(1+\kappa^2(s))\sqrt{1+\kappa^2(s)}} \mathbf{n}(s) \pm \frac{\kappa(s)}{\sqrt{1+\kappa^2(s)}} \mathbf{t}(s) \\ &= \mp \frac{\kappa'(s)}{(1+\kappa^2(s))\sqrt{1+\kappa^2(s)}} [(1+2\kappa^2(s))\boldsymbol{\gamma}(s) + \kappa(s)\mathbf{n}(s)]. \end{aligned}$$

It follows that $(\mathbf{E}_\gamma^\pm)'(s) = 0$ if and only if $\kappa'(s) = 0$. This completes the proof. $\square$

By the above proposition, $\mathbf{E}_\gamma^\pm(s) = \mathbf{v}_0$ is constant if and only if $\kappa'(s) \equiv 0$. In this case, by Proposition 4.1.1, h_{v_0} is constant, that is, there is a real number $c \in \mathbb{R}$ such that $\langle \gamma(s), \mathbf{v}_0 \rangle = c$. It means that $\text{Im}\gamma(s) \in HP(\mathbf{v}_0, c) \cap S_1^2$. It suggests that curves of the form $HP(\mathbf{v}_0, c) \cap S_1^2$ for $\mathbf{v}_0 \in S_1^2$ are the candidates of model curves on S_1^2 . They might play a similar role to lines in Euclidean plane. We call it *de Sitter slice* (or, a *D-slice*) of S_1^2 .

We introduce another family of functions:

$$\mathcal{H} : \mathbb{R}_1^3 \times S_1^2 \rightarrow \mathbb{R}, \quad (\mathbf{x}, \mathbf{v}) \mapsto \langle \mathbf{x}, \mathbf{v} \rangle.$$

We denote $\mathfrak{h}_{v_0} = \mathcal{H}(\mathbf{x}, \mathbf{v}_0)$ for any fixed $\mathbf{v}_0 \in S_1^2$, then we have

$$h_{v_0}(s) = \langle \gamma(s), \mathbf{v}_0 \rangle = \mathcal{H}(\gamma(s), \mathbf{v}_0) = \mathfrak{h}_{v_0}(\gamma(s)).$$

Moreover, for any $s_0 \in I$ and $\mathbf{E}_\gamma^\pm(s_0) = \mathbf{v}_0$, $(\mathfrak{h}_{v_0}|_{S_1^2})^{-1}(c)$ is a D-slice of S_1^2 . On the other hand, let $F : \mathbb{R}_1^3 \rightarrow \mathbb{R}$ be a submersion. We say that γ and $F^{-1}(0)$ have *contact of order k* at s_0 if the function $f(s) = F \circ \gamma(s)$ satisfies $f(s_0) = f'(s_0) = \dots = f^{(k)}(s_0) = 0$ and $f^{(k+1)}(s_0) \neq 0$. Under the above notations and Proposition 4.1.1, we have the following proposition.

Proposition 4.1.3. Let $\gamma : I \rightarrow S_1^2$ be a timelike Sabban curve and $\mathbf{E}_\gamma^\pm(s_0) = \mathbf{v}_0$. Then the following conditions are equivalent:

- (1) γ and $(\mathfrak{h}_{v_0}|_{S_1^2})^{-1}(c_0)$ have contact of order three,
- (2) $\kappa'(s_0) = 0$ and $\kappa''(s_0) \neq 0$.

In order to investigate the singularities of de Sitter evolutes of timelike Sabban curves, we apply the theory of unfolding of functions. In §2. we defined the bifurcation set B_F of F by

$$B_F = \left\{ \mathbf{x} \in \mathbb{R}^r \mid \exists s, \text{ with } \frac{\partial F}{\partial s}(s, \mathbf{x}) = \frac{\partial^2 F}{\partial s^2}(s, \mathbf{x}) = 0 \right\}.$$

Here we have the following proposition.

Proposition 4.1.4. Let $\gamma : I \rightarrow S_1^2$ be a timelike Sabban curve. If $h_v(s)$ has type A_3 at s_0 , then H is a p -versal unfolding of h_v .

Proof. We denote $\boldsymbol{\gamma}(s) = (x_0(s), x_1(s), x_2(s))$, $\mathbf{v} = (v_0, v_1, \sqrt{v_0^2 - v_1^2 + 1})$. Then we adopt the local coordinates (v_0, v_1) for S_1^2 . We have

$$H(s, \mathbf{v}) = -x_0(s)v_0 + x_1(s)v_1 + x_2(s)\sqrt{v_0^2 - v_1^2 + 1}$$

and

$$\begin{aligned} \frac{\partial H}{\partial v_0} &= -x_0(s) + \frac{v_0}{\sqrt{v_0^2 - v_1^2 + 1}}x_2(s), & \frac{\partial H}{\partial v_1} &= x_1(s) - \frac{v_1}{\sqrt{v_0^2 - v_1^2 + 1}}x_2(s), \\ \frac{\partial^2 H}{\partial s \partial v_0} &= -x'_0(s) + \frac{v_0}{\sqrt{v_0^2 - v_1^2 + 1}}x'_2(s), & \frac{\partial^2 H}{\partial s \partial v_1} &= x'_1(s) - \frac{v_1}{\sqrt{v_0^2 - v_1^2 + 1}}x'_2(s), \\ \frac{\partial^3 H}{\partial^2 s \partial v_0} &= -x''_0(s) + \frac{v_0}{\sqrt{v_0^2 - v_1^2 + 1}}x''_2(s), & \frac{\partial^3 H}{\partial^2 s \partial v_1} &= x''_1(s) - \frac{v_1}{\sqrt{v_0^2 - v_1^2 + 1}}x''_2(s). \end{aligned}$$

By Proposition 4.1.1, $h_v(s)$ has type A_3 at s_0 if and only if $\mathbf{v} = \cos \theta \boldsymbol{\gamma}(s_0) + \sin \theta \mathbf{n}(s_0)$, $\cos \theta + \kappa \sin \theta = 0$ and $\kappa'(s_0) = 0$. Since $\cos \theta + \kappa \sin \theta = 0$ is equal to $\sin \theta = \pm \frac{1}{\sqrt{1+\kappa^2}}$ and $\cos \theta = \mp \frac{\kappa}{\sqrt{1+\kappa^2}}$, then $h_v(s)$ has type A_3 at s_0 if and only if

$$\mathbf{v} = \mp \frac{\kappa}{\sqrt{1+\kappa^2}} \boldsymbol{\gamma}(s_0) \pm \frac{1}{\sqrt{1+\kappa^2}} \mathbf{n}(s_0) \text{ and } \kappa'(s_0) = 0.$$

For the purpose, we require the matix

$$A = \begin{pmatrix} -x'_0(s) + \frac{v_0}{\sqrt{v_0^2 - v_1^2 + 1}}x'_2(s) & x'_1(s) - \frac{v_1}{\sqrt{v_0^2 - v_1^2 + 1}}x'_2(s) \\ -x''_0(s) + \frac{v_0}{\sqrt{v_0^2 - v_1^2 + 1}}x''_2(s) & x''_1(s) - \frac{v_1}{\sqrt{v_0^2 - v_1^2 + 1}}x''_2(s) \end{pmatrix}$$

to have rank 2. Therefore we calculate the determinant of this matrix as follows:

$$\begin{aligned}
\det A &= (-(x'_1 x''_2 - x'_2 x''_1), x'_2 x''_0 - x'_0 x''_2, x'_0 x''_1 - x'_1 x''_0) \begin{pmatrix} \frac{v_0}{\sqrt{v_0^2 - v_1^2 + 1}} \\ -\frac{v_1}{\sqrt{v_0^2 - v_1^2 + 1}} \\ 1 \end{pmatrix} \\
&= ((x'_0, x'_1, x'_2) \wedge (x''_0, x''_1, x''_2)) \begin{pmatrix} \frac{v_0}{\sqrt{v_0^2 - v_1^2 + 1}} \\ -\frac{v_1}{\sqrt{v_0^2 - v_1^2 + 1}} \\ 1 \end{pmatrix} \\
&= -\frac{1}{\sqrt{v_0^2 - v_1^2 + 1}} (\mathbf{t} \wedge (\boldsymbol{\gamma} + \kappa \mathbf{n})) \begin{pmatrix} -v_0 \\ v_1 \\ \sqrt{v_0^2 - v_1^2 + 1} \end{pmatrix} \\
&= -\frac{1}{\sqrt{v_0^2 - v_1^2 + 1}} \langle -\mathbf{n} + \kappa \boldsymbol{\gamma}, \mp \frac{\kappa}{\sqrt{1 + \kappa^2}} \boldsymbol{\gamma}(s) \pm \frac{1}{\sqrt{1 + \kappa^2}} \mathbf{n}(s) \rangle \\
&= \pm \frac{\sqrt{1 + \kappa^2}}{\sqrt{v_0^2 - v_1^2 + 1}} \neq 0.
\end{aligned}$$

For other local coordinates, we can apply the arguments similar to above. This completes the proof. $\square$

As a consequence, we have the following theorem.

Theorem 4.1.5. Let $\boldsymbol{\gamma} : I \rightarrow S_1^2$ be a timelike Sabban curve. Then we have the following assertions:

(A1) The evolute $\mathbf{E}_\gamma^\pm(s)$ is regular at s_0 if and only if $\kappa'(s_0) \neq 0$.

(A2) The following conditions are equivalent:

(i) $\kappa'(s_0) = 0$ and $\kappa''(s_0) \neq 0$,

(ii) $\boldsymbol{\gamma}$ and D-slice $(\mathfrak{h}_{v_0}|_{S_1^2})^{-1}(c_0)$ have contact of order three. In this case the germ of $\mathbf{E}_\gamma^\pm(s)$ at s_0 is diffeomorphic to the ordinary cusp.

Proof. (A1) By Proposition 4.1.2, $(\mathbf{E}_\gamma^\pm)'(s) = 0$ if and only if $\kappa'(s) = 0$. It means that the evolute $\mathbf{E}_\gamma^\pm(s)$ is regular at s_0 if and only if $\kappa'(s_0) \neq 0$.

(A2) By Proposition 4.1.1, the bifurcation set of H is $\mathbf{E}_\gamma^\pm(s)$. By Theorem 2.3.6 and Proposition 4.2.2, the germ of the bifurcation set is diffeomorphic to the cusp if $\kappa'(s_0) = 0$ and $\kappa''(s_0) \neq 0$. By Proposition 4.1.3, (A2),(i) is equivalent to (A2),(ii). This completes the proof. $\square$

4.2 De Sitter evolutes and de Sitter focal surfaces of timelike Sabban curves in S_1^3

In this section we consider timelike Sabban curves in S_1^3 . Let $\gamma : I \rightarrow S_1^3$ be a timelike Sabban curve, we can reparameterize it by the arc-length s . Then we have the tangent vector $\mathbf{t}(s) = \gamma'(s)$, obviously $\|\mathbf{t}(s)\| = 1$. When $\langle \mathbf{t}'(s), \mathbf{t}'(s) \rangle \neq 1$, we define a unit vector

$$\mathbf{n}_1(s) = \frac{\mathbf{t}'(s) - \gamma(s)}{\|\mathbf{t}'(s) - \gamma(s)\|},$$

let $\mathbf{n}_2(s) = \gamma(s) \wedge \mathbf{t}(s) \wedge \mathbf{n}_1(s)$. Then we have the Sabban frame $\{\gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \mathbf{n}_2(s)\}$ along γ . As a special case of the Frenet-Serret type formulae in §3, we have the following:

$$\begin{cases} \gamma'(s) = \mathbf{t}(s), \\ \mathbf{t}'(s) = \gamma(s) + \kappa_1(s)\mathbf{n}_1(s), \\ \mathbf{n}_1'(s) = \kappa_1(s)\mathbf{t}(s) + \kappa_2(s)\mathbf{n}_2(s), \\ \mathbf{n}_2'(s) = -\kappa_2(s)\mathbf{n}_1(s). \end{cases}$$

Here, $\kappa_1(s) = \|\mathbf{t}'(s) - \gamma(s)\|$ and $\kappa_2(s) = -\kappa_1^{-2}(s)\det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))$. We remark the condition $\langle \mathbf{t}'(s), \mathbf{t}'(s) \rangle \neq 1$ is equivalent to $\kappa_1(s) \neq 0$.

Now we define a family of functions on a timelike Sabban curve. Let $\gamma : I \rightarrow S_1^3$ be a timelike Sabban curve. We define a function $H : I \times S_1^3 \rightarrow \mathbb{R}$ by $H(s, \mathbf{v}) = \langle \gamma(s), \mathbf{v} \rangle$, we call H a *de Sitter height function* on the timelike Sabban curve γ . We denote $h_v(s) = H(s, \mathbf{v})$ for any fixed $\mathbf{v} \in S_1^3$. Then we have the following proposition.

Proposition 4.2.1. Let $\gamma : I \rightarrow S_1^3$ be a timelike Sabban curve. Then we have the following assertions:

(1) $h'_v(s) = 0$ if and only if there exist $a, c, d \in \mathbb{R}$ with $a^2 + c^2 + d^2 = 1$ such that $\mathbf{v} =$

$$a\boldsymbol{\gamma}(s) + c\mathbf{n}_1(s) + d\mathbf{n}_2(s).$$

(2) $h'_v(s) = h''_v(s) = 0$ if and only if there exists $\theta \in [0, 2\pi)$ such that

$$\mathbf{v} = \frac{\kappa_1 \cos \theta}{\sqrt{1 + \kappa_1^2}}(\boldsymbol{\gamma}(s) - \frac{1}{\kappa_1}\mathbf{n}_1(s)) + \sin \theta \mathbf{n}_2(s).$$

(3) $h'_v(s) = h''_v(s) = h'''_v(s) = 0$ if and only if there exists $\theta \in [0, 2\pi)$ such that

$$\mathbf{v} = \frac{\kappa_1 \cos \theta}{\sqrt{1 + \kappa_1^2}}(\boldsymbol{\gamma}(s) - \frac{1}{\kappa_1}\mathbf{n}_1(s)) + \sin \theta \mathbf{n}_2(s)$$

and $\frac{\kappa'_1}{\sqrt{1 + \kappa_1^2}} \cos \theta = \kappa_1 \kappa_2 \sin \theta$.

(4) $h'_v(s) = h''_v(s) = h'''_v(s) = h_v^{(4)}(s) = 0$ if and only if there exists $\theta \in [0, 2\pi)$ such that

$$\mathbf{v} = \frac{\kappa_1 \cos \theta}{\sqrt{1 + \kappa_1^2}}(\boldsymbol{\gamma}(s) - \frac{1}{\kappa_1}\mathbf{n}_1(s)) + \sin \theta \mathbf{n}_2(s), \quad \frac{\kappa'_1}{\sqrt{1 + \kappa_1^2}} \cos \theta = \kappa_1 \kappa_2 \sin \theta$$

and

$$2(\kappa'_1(s))^2 \kappa_2(s) + \kappa_1(s) \kappa'_1(s) \kappa'_2(s) + \kappa_1^2(s) \kappa_2^3(s) - \kappa_1(s) \kappa_2(s) \kappa''_1(s) = 0.$$

(5) $h'_v(s) = h''_v(s) = h'''_v(s) = h_v^{(4)}(s) = h_v^{(5)}(s) = 0$ if and only if there exists $\theta \in [0, 2\pi)$ such that

$$\begin{aligned} \mathbf{v} &= \frac{\kappa_1 \cos \theta}{\sqrt{1 + \kappa_1^2}}(\boldsymbol{\gamma}(s) - \frac{1}{\kappa_1}\mathbf{n}_1(s)) + \sin \theta \mathbf{n}_2(s), \quad \frac{\kappa'_1}{\sqrt{1 + \kappa_1^2}} \cos \theta = \kappa_1 \kappa_2 \sin \theta, \\ 2(\kappa'_1(s))^2 \kappa_2(s) &+ \kappa_1(s) \kappa'_1(s) \kappa'_2(s) + \kappa_1^2(s) \kappa_2^3(s) - \kappa_1(s) \kappa_2(s) \kappa''_1(s) = 0, \end{aligned}$$

and

$$\begin{aligned} & \left(2(\kappa'_1(s))^2 \kappa_2(s) + \kappa_1(s) \kappa'_1(s) \kappa'_2(s) + \kappa_1^2(s) \kappa_2^3(s) - \kappa_1(s) \kappa_2(s) \kappa''_1(s) \right)' \\ &= 3\kappa'_1(s) \kappa''_1(s) \kappa_2(s) + 3(\kappa'_1(s))^2 \kappa'_2(s) + \kappa_1(s) \kappa'_1(s) \kappa''_2(s) - \kappa_1(s) \kappa_1'''(s) \kappa_2(s) \\ &+ 2\kappa_1(s) \kappa'_1(s) \kappa_2^3(s) + 3\kappa_2^2(s) \kappa'_2(s) \kappa_1^2(s) = 0. \end{aligned}$$

Proof. (1) Since $\mathbf{v} \in S_1^3$, there exist $a, b, c, d \in \mathbb{R}$ such that $\mathbf{v} = a\boldsymbol{\gamma}(s) + b\mathbf{t}(s) + c\mathbf{n}_1(s) + d\mathbf{n}_2(s)$ with $a^2 + b^2 + c^2 + d^2 = 1$. Because $h'_v(s) = 0$, we have $b = 0$, so that $a^2 + c^2 + d^2 = 1$. The converse direction also holds.

(2) By (1), we show that $\langle \gamma(s) + \kappa_1 \mathbf{n}_1(s), \mathbf{v} \rangle = 0$. Then $c = -\frac{a}{\kappa_1}$. Since $a^2 + c^2 + d^2 = 1$, there exists $\theta \in [0, 2\pi)$ such that

$$a = \frac{\kappa_1 \cos \theta}{\sqrt{1 + \kappa_1^2}}, \quad c = -\frac{\cos \theta}{\sqrt{1 + \kappa_1^2}}, \quad d = \sin \theta.$$

Therefore

$$\mathbf{v} = \frac{\kappa_1 \cos \theta}{\sqrt{1 + \kappa_1^2}}(\gamma(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) + \sin \theta \mathbf{n}_2(s).$$

(3) Under the assumption that $h'_v(s) = h''_v(s) = 0$, we have

$$h'''_v(s) = \langle (1 + \kappa_1^2) \mathbf{t}(s) + \kappa'_1 \mathbf{n}_1(s) + \kappa_1 \kappa_2 \mathbf{n}_2(s), \mathbf{v} \rangle = 0.$$

It follows that $\frac{\kappa'_1}{\sqrt{1 + \kappa_1^2}} \cos \theta = \kappa_1 \kappa_2 \sin \theta$.

(4) With the assumption $h'_v(s) = h''_v(s) = h'''_v(s) = 0$ and a formula

$$h_v^{(4)}(s) = \langle (1 + \kappa_1^2) \gamma(s) + 3\kappa_1 \kappa'_1 \mathbf{t}(s) + (\kappa_1 + \kappa_1^3 + \kappa'_1 - \kappa_1 \kappa_2^2) \mathbf{n}_1(s) + (2\kappa'_1 \kappa_2 + \kappa_1 \kappa'_2) \mathbf{n}_2(s), \mathbf{v} \rangle,$$

$h_v^{(4)}(s) = 0$ is equivalent to

$$(\kappa_1 \kappa_2^2 - \kappa''_1) \cos \theta = -\sqrt{1 + \kappa_1^2} (2\kappa'_1 \kappa_2 + \kappa_1 \kappa'_2) \sin \theta.$$

Since we have $\frac{\kappa'_1}{\sqrt{1 + \kappa_1^2}} \cos \theta = \kappa_1 \kappa_2 \sin \theta$,

$$2(\kappa'_1(s))^2 \kappa_2(s) + \kappa_1(s) \kappa'_1(s) \kappa'_2(s) + \kappa_1^2(s) \kappa_2^3(s) - \kappa_1(s) \kappa_2(s) \kappa''_1(s) = 0.$$

This proves assertion (4).

(5) When $h'_v(s) = h''_v(s) = h'''_v(s) = h_v^{(4)}(s) = 0$, the fifth derivative is

$$\begin{aligned} h_v^{(5)}(s) = & \langle 5\kappa_1 \kappa'_1 \gamma(s) + (1 + 2\kappa_1^2 + 3(\kappa'_1)^2 + 4\kappa_1 \kappa''_1 + \kappa_1^4 - \kappa_1^2 \kappa_2^2) \mathbf{t}(s) \\ & + (6\kappa_1^2 \kappa'_1 + \kappa'_1 + \kappa'''_1 - 3\kappa'_1 \kappa_2^2 - 3\kappa_1 \kappa_2 \kappa'_2) \mathbf{n}_1(s) \\ & + (\kappa_1 \kappa_2 + \kappa_1^3 \kappa_2 + 3\kappa''_1 \kappa_2 - \kappa_1 \kappa_2^3 + 3\kappa'_1 \kappa'_2 + \kappa_1 \kappa''_2) \mathbf{n}_2(s), \mathbf{v} \rangle. \end{aligned}$$

It follows that $h_v^{(5)}(s) = 0$ is equivalent to

$$\begin{aligned} & \sqrt{1 + \kappa_1^2} (\kappa_1 \kappa_2 + \kappa_1^3 \kappa_2 + 3\kappa''_1 \kappa_2 - \kappa_1 \kappa_2^3 + 3\kappa'_1 \kappa'_2 + \kappa_1 \kappa''_2) \sin \theta \\ & + (-\kappa_1^2 \kappa'_1 - \kappa'_1 - \kappa'''_1 + 3\kappa'_1 \kappa_2^2 + 3\kappa_1 \kappa_2 \kappa'_2) \cos \theta = 0. \end{aligned}$$

Since $\frac{\kappa_1'}{\sqrt{1+\kappa_1^2}} \cos \theta = \kappa_1 \kappa_2 \sin \theta$ and

$$2(\kappa_1'(s))^2 \kappa_2(s) + \kappa_1(s) \kappa_1'(s) \kappa_2'(s) + \kappa_1^2(s) \kappa_2^3(s) - \kappa_1(s) \kappa_2(s) \kappa_1''(s) = 0,$$

we have

$$\begin{aligned} & 3\kappa_1' \kappa_1'' \kappa_2 + 3(\kappa_1')^2 \kappa_2' + \kappa_1 \kappa_1' \kappa_2'' - \kappa_1 \kappa_1''' \kappa_2 + 2\kappa_1 \kappa_1' \kappa_2^3 + 3\kappa_2^2 \kappa_2' \kappa_1^2 \\ &= (2(\kappa_1'(s))^2 \kappa_2(s) + \kappa_1(s) \kappa_1'(s) \kappa_2'(s) + \kappa_1^2(s) \kappa_2^3(s) - \kappa_1(s) \kappa_2(s) \kappa_1''(s))' = 0. \end{aligned}$$

This completes the proof. $\square$

By the above proposition, we define a new invariant $\sigma_2(s)$ as follows:

$$\sigma_2(s) = 2(\kappa_1'(s))^2 \kappa_2(s) + \kappa_1(s) \kappa_1'(s) \kappa_2'(s) + \kappa_1^2(s) \kappa_2^3(s) - \kappa_1(s) \kappa_2(s) \kappa_1''(s).$$

We now respectively define a curve $\mathbf{E}_\gamma : I \rightarrow S_1^3$ and a surface $\mathbf{FS} : I \times S^1 \rightarrow S_1^3$ by

$$\begin{aligned} \mathbf{E}_\gamma(s) &= \frac{\kappa_1^2 \kappa_2}{\sqrt{(\kappa_1')^2 + \kappa_1^2 \kappa_2^2 (1 + \kappa_1^2)}} (\gamma(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) + \frac{\kappa_1'}{\sqrt{(\kappa_1')^2 + \kappa_1^2 \kappa_2^2 (1 + \kappa_1^2)}} \mathbf{n}_2(s), \\ \mathbf{FS}(s, \mathbf{u}) &= u_1 \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\gamma(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) + u_2 \mathbf{n}_2(s). \end{aligned}$$

Here $\mathbf{u} = (u_1, u_2) \in S^1$, we call $\mathbf{E}_\gamma(s)$ the *de Sitter evolute* of γ and $\mathbf{FS}(s, \mathbf{u})$ the *de Sitter focal surface* of γ .

We consider some properties of the de Sitter evolutes and de Sitter focal surfaces of timelike Sabban curves in S_1^3 . Moreover we investigate the geometric meanings of the invariant $\sigma_2(s)$. We have the following proposition.

Proposition 4.2.2. Let $\gamma : I \rightarrow S_1^3$ be a timelike Sabban curve. Then we have the following assertions:

- (1) The set $\left\{ (s, \theta) \mid \frac{\kappa_1'}{\sqrt{1+\kappa_1^2}} \cos \theta = \kappa_1 \kappa_2 \sin \theta \right\}$ is the singularities of $\mathbf{FS}(s, \theta)$.
- (2) The de Sitter evolute $\mathbf{E}_\gamma(s)$ of γ is singular at s_0 if and only if $\sigma_2(s_0) = 0$.

Proof. (1) Since $\mathbf{FS}(s, \mathbf{u}) = u_1 \frac{\kappa_1}{\sqrt{1+\kappa_1^2}} (\gamma(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) + u_2 \mathbf{n}_2(s)$ and $\mathbf{u} = (u_1, u_2) \in S^1$, we denote $u_1 = \cos \theta$ and $u_2 = \sin \theta$, where $\theta \in [0, 2\pi)$. Then the focal surface is parameterized by

$$\mathbf{FS}(s, \theta) = \cos \theta \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\gamma(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) + \sin \theta \mathbf{n}_2(s).$$

By straightforward calculations, we have

$$\begin{aligned}\frac{\partial \mathbf{FS}}{\partial \theta} &= -\frac{\kappa_1 \sin \theta}{\sqrt{1 + \kappa_1^2}} \boldsymbol{\gamma}(s) + \frac{\sin \theta}{\sqrt{1 + \kappa_1^2}} \mathbf{n}_1(s) + \cos \theta \mathbf{n}_2(s), \\ \frac{\partial \mathbf{FS}}{\partial s} &= \left(\frac{\kappa_1' \cos \theta}{\kappa_1 \sqrt{1 + \kappa_1^2}} - \frac{\kappa_1' \cos \theta}{\kappa_1 (1 + \kappa_1^2) \sqrt{1 + \kappa_1^2}} - \kappa_2 \sin \theta \right) \mathbf{n}_1(s) \\ &\quad + \frac{\kappa_1' \cos \theta}{(1 + \kappa_1^2) \sqrt{1 + \kappa_1^2}} \boldsymbol{\gamma}(s) - \frac{\kappa_2 \cos \theta}{\sqrt{1 + \kappa_1^2}} \mathbf{n}_2(s).\end{aligned}$$

Then the above two vectors are linearly dependent if and only if $\frac{\kappa_1'}{\sqrt{1 + \kappa_1^2}} \cos \theta = \kappa_1 \kappa_2 \sin \theta$. This is the condition for the singularities of $\mathbf{FS}$.

(2) Since

$$\mathbf{E}_\gamma(s) = \frac{\kappa_1^2 \kappa_2}{\sqrt{(\kappa_1')^2 + \kappa_1^2 \kappa_2^2 (1 + \kappa_1^2)}} \left(\boldsymbol{\gamma}(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s) \right) + \frac{\kappa_1'}{\sqrt{(\kappa_1')^2 + \kappa_1^2 \kappa_2^2 (1 + \kappa_1^2)}} \mathbf{n}_2(s),$$

we have

$$\begin{aligned}\frac{d\mathbf{E}_\gamma}{ds} &= -\frac{\kappa_1' \kappa_1'' + \kappa_1 \kappa_1' \kappa_2^2 + 2\kappa_1^3 \kappa_1' \kappa_2^2 + \kappa_1^2 \kappa_2 \kappa_2' + \kappa_1^4 \kappa_2 \kappa_2'}{((\kappa_1')^2 + \kappa_1^4 \kappa_2^2 + \kappa_1^2 \kappa_2^2)^{\frac{3}{2}}} (\kappa_1^2 \kappa_2 \boldsymbol{\gamma}(s) - \kappa_1 \kappa_2 \mathbf{n}_1(s) + \kappa_1' \mathbf{n}_2(s)) \\ &\quad + \frac{1}{((\kappa_1')^2 + \kappa_1^4 \kappa_2^2 + \kappa_1^2 \kappa_2^2)^{\frac{1}{2}}} [(2\kappa_1 \kappa_1' \kappa_2 + \kappa_1^2 \kappa_2') \boldsymbol{\gamma}(s) + (-2\kappa_1' \kappa_2 - \kappa_1 \kappa_2') \mathbf{n}_1(s) \\ &\quad + (-\kappa_1 \kappa_2^2 + \kappa_1'') \mathbf{n}_2(s)] \\ &= ((\kappa_1')^2 + \kappa_1^4 \kappa_2^2 + \kappa_1^2 \kappa_2^2)^{-\frac{3}{2}} [(-\kappa_1^2 \kappa_1' \kappa_1'' \kappa_2 + \kappa_1^3 \kappa_1' \kappa_2^3 + 2\kappa_1 (\kappa_1')^3 \kappa_2 + \kappa_1^2 (\kappa_1')^2 \kappa_2') \boldsymbol{\gamma}(s) \\ &\quad + (\kappa_1 \kappa_1' \kappa_1'' \kappa_2 - \kappa_1^2 \kappa_1' \kappa_2^3) \mathbf{n}_1(s) + (-2\kappa_1 (\kappa_1')^2 \kappa_2^2 - \kappa_1^2 \kappa_1' \kappa_2 \kappa_2' - \kappa_1^3 \kappa_2^4 + \kappa_1^2 \kappa_1'' \kappa_2^2 \\ &\quad - 2\kappa_1^3 (\kappa_1')^2 \kappa_2^2 - \kappa_1^4 \kappa_1' \kappa_2 \kappa_2' - \kappa_1^5 \kappa_2^4 + \kappa_1^4 \kappa_1'' \kappa_2^2) \mathbf{n}_2(s)] \\ &= ((\kappa_1')^2 + \kappa_1^4 \kappa_2^2 + \kappa_1^2 \kappa_2^2)^{-\frac{3}{2}} [\kappa_1 \kappa_1' \sigma_2(s) \boldsymbol{\gamma}(s) - \kappa_1' \sigma_2(s) \mathbf{n}_1(s) - (\kappa_1 \kappa_2 + \kappa_1^3 \kappa_2) \sigma_2(s) \mathbf{n}_2(s)].\end{aligned}$$

Therefore, $\frac{d\mathbf{E}_\gamma}{ds} = 0$ if and only if $\sigma_2(s) = 0$. This completes the proof. $\square$

In order to investigate the singularities of de Sitter evolutes and de Sitter focal surfaces of timelike Sabban curves, we apply the theory of unfolding of functions. Here we have the following proposition.

Proposition 4.2.3. Let $\gamma : I \rightarrow S_1^3$ be a timelike Sabban curve with $\kappa_1'(s) \neq 0$. If $h_v(s)$ has type A_k at s_0 ($k = 3, 4$), then H is a p -versal unfolding of h_v .

Proof. Let $\gamma(s) = (x_0(s), x_1(s), x_2(s), x_3(s))$. Since $\mathbf{v} \in S_1^3$, let

$$\mathbf{v} = (v_0, v_1, v_2, v_3) = (v_0, v_1, v_2, \sqrt{1 + v_0^2 - v_1^2 - v_2^2}).$$

Then we adopt the local coordinates (v_0, v_1, v_2) for S_1^3 . We have

$$H(s, \mathbf{v}) = -x_0(s)v_0 + x_1(s)v_1 + x_2(s)v_2 + x_3(s)\sqrt{1 + v_0^2 - v_1^2 - v_2^2},$$

so that

$$\begin{aligned} \frac{\partial H}{\partial v_0} &= -x_0(s) + \frac{v_0}{v_3}x_3(s), \quad \frac{\partial H}{\partial v_1} = x_1(s) - \frac{v_1}{v_3}x_3(s), \quad \frac{\partial H}{\partial v_2} = x_2(s) - \frac{v_2}{v_3}x_3(s), \\ \frac{\partial^2 H}{\partial s \partial v_0} &= -x_0'(s) + \frac{v_0}{v_3}x_3'(s), \quad \frac{\partial^2 H}{\partial s \partial v_1} = x_1'(s) - \frac{v_1}{v_3}x_3'(s), \quad \frac{\partial^2 H}{\partial s \partial v_2} = x_2'(s) - \frac{v_2}{v_3}x_3'(s), \\ \frac{\partial^3 H}{\partial s^2 \partial v_0} &= -x_0''(s) + \frac{v_0}{v_3}x_3''(s), \quad \frac{\partial^3 H}{\partial s^2 \partial v_1} = x_1''(s) - \frac{v_1}{v_3}x_3''(s), \quad \frac{\partial^3 H}{\partial s^2 \partial v_2} = x_2''(s) - \frac{v_2}{v_3}x_3''(s), \\ \frac{\partial^4 H}{\partial s^3 \partial v_0} &= -x_0'''(s) + \frac{v_0}{v_3}x_3'''(s), \quad \frac{\partial^4 H}{\partial s^3 \partial v_1} = x_1'''(s) - \frac{v_1}{v_3}x_3'''(s), \quad \frac{\partial^4 H}{\partial s^3 \partial v_2} = x_2'''(s) - \frac{v_2}{v_3}x_3'''(s). \end{aligned}$$

By Proposition 4.2.1, h_v has the A_3 singularity if and only if

$$\mathbf{v} = \frac{\kappa_1 \cos \theta}{\sqrt{1 + \kappa_1^2}}(\gamma(s) - \frac{1}{\kappa_1}\mathbf{n}_1(s)) + \sin \theta \mathbf{n}_2(s)$$

and $\frac{\kappa_1'}{\sqrt{1 + \kappa_1^2}} \cos \theta = \kappa_1 \kappa_2 \sin \theta$. When h_v has the A_3 singularity, we require the 2×3 matrix

$$\begin{pmatrix} x_0'(s) - \frac{v_0}{v_3}x_3'(s) & x_1'(s) - \frac{v_1}{v_3}x_3'(s) & x_2'(s) - \frac{v_2}{v_3}x_3'(s) \\ x_0''(s) - \frac{v_0}{v_3}x_3''(s) & x_1''(s) - \frac{v_1}{v_3}x_3''(s) & x_2''(s) - \frac{v_2}{v_3}x_3''(s) \end{pmatrix}$$

to have rank 2, which follows from the proof of the next case.

By Proposition 4.2.1, h_v has the A_4 singularity if and only if

$$\mathbf{v} = \frac{\kappa_1 \cos \theta}{\sqrt{1 + \kappa_1^2}}(\gamma(s) - \frac{1}{\kappa_1}\mathbf{n}_1(s)) + \sin \theta \mathbf{n}_2(s), \quad \frac{\kappa_1'}{\sqrt{1 + \kappa_1^2}} \cos \theta = \kappa_1 \kappa_2 \sin \theta$$

and $\sigma_2(s) = 0$. In this case, we need the 3×3 matrix

$$A = \begin{pmatrix} x'_0(s) - \frac{v_0}{v_3}x'_3(s) & x'_1(s) - \frac{v_1}{v_3}x'_3(s) & x'_2(s) - \frac{v_2}{v_3}x'_3(s) \\ x''_0(s) - \frac{v_0}{v_3}x''_3(s) & x''_1(s) - \frac{v_1}{v_3}x''_3(s) & x''_2(s) - \frac{v_2}{v_3}x''_3(s) \\ x'''_0(s) - \frac{v_0}{v_3}x'''_3(s) & x'''_1(s) - \frac{v_1}{v_3}x'''_3(s) & x'''_2(s) - \frac{v_2}{v_3}x'''_3(s) \end{pmatrix}$$

to have rank 3. Put $a_i = (x'_i, x''_i, x'''_i)^t$, then we have

$$A = (a_0 - \frac{v_0}{v_3}a_3, a_1 - \frac{v_1}{v_3}a_3, a_2 - \frac{v_2}{v_3}a_3),$$

$$\begin{aligned} -\det A &= \det(a_0, a_1, a_2) + \frac{v_0}{v_3}\det(a_3, a_1, a_2) + \frac{v_1}{v_3}\det(a_0, a_3, a_2) + \frac{v_2}{v_3}\det(a_0, a_1, a_3) \\ &= -\left\langle \frac{\mathbf{v}}{v_3}, (\gamma'(s) \wedge \gamma''(s) \wedge \gamma'''(s)) \right\rangle \\ &= \kappa_1 \sin^2 \theta = \frac{\kappa_1^2(\kappa'_1)^2}{(\kappa'_1)^2 + \kappa_1^2\kappa_2^2(1 + \kappa_1^2)} \neq 0. \end{aligned}$$

For other local coordinates, we can apply the arguments similar to above. This completes the proof. $\square$

By Proposition 4.2.1, Proposition 4.2.3 and Theorem 2.3.6, we have the following theorem.

Theorem 4.2.4. Let $\gamma : I \rightarrow S_1^3$ be a timelike Sabban curve with $\kappa'_1(s) \neq 0$. Then we have the following:

- (1) If h_v has A_3 singularity at s_0 , the germ of de Sitter focal surface $\mathbf{FS}$ at (s_0, θ_0) is diffeomorphic to $C \times \mathbb{R}$.
- (2) If h_v has A_4 singularity at s_0 , the germ of de Sitter focal surface $\mathbf{FS}$ at (s_0, θ_0) is diffeomorphic to SW and the germ of de Sitter evolute $\mathbf{E}_\gamma$ at s_0 is diffeomorphic to C .

We consider the *hyperplane* in $\mathbb{R}_1^4$, which is defined by $HP(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}_1^4 | \langle \mathbf{x}, \mathbf{v} \rangle = c\}$, where $\mathbf{v} \in S_1^3$ is a fixed vector and $c \in \mathbb{R}$ is a constant. We define

$$S_1^2(\mathbf{v}, c) = S_1^3 \cap HP(\mathbf{v}, c)$$

and call $S_1^2(\mathbf{v}, c)$ a *2-dimensional de Sitter subspace* in S_1^3 . For any fixed $s_0 \in I$, we set $\langle \gamma(s_0), \mathbf{v} \rangle = \varepsilon$. If we consider a function $\mathfrak{h}_v : S_1^3 \rightarrow \mathbb{R}$ defined by $\mathfrak{h}_v(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v} \rangle$, then

$\mathfrak{h}_v^{-1}(\varepsilon) = S_1^2(\mathbf{v}, \varepsilon)$ and $\mathfrak{h}_v \circ \gamma(s) = h_v(s)$, so that $h_v(s_0) = \varepsilon$ and $h'_v(s_0) = 0$ if and only if $S_1^2(\mathbf{v}, \varepsilon)$ is tangent to $\gamma(s)$ at s_0 . In this case, we call $S_1^2(\mathbf{v}, \varepsilon)$ a *tangent de Sitter subspace* at s_0 , we remark that there are infinitely many tangent de Sitter subspaces of $\gamma(s)$ at s_0 . We say that a tangent de Sitter subspace $S_1^2(\mathbf{v}, \varepsilon)$ of $\gamma(s)$ at s_0 has *at least $k+1$ -point contact* with γ if $h_v(s_0) = \varepsilon$ and $h'_v(s_0) = \cdots = h_v^{(k)}(s_0) = 0$. We also say that tangent de Sitter subspace $S_1^2(\mathbf{v}, \varepsilon)$ of $\gamma(s)$ at s_0 has *$k+1$ -point contact* with γ if $h_v(s_0) = \varepsilon$, $h'_v(s_0) = \cdots = h_v^{(k)}(s_0) = 0$ and $h_v^{(k+1)}(s_0) \neq 0$. By Propositions 4.2.1, 4.2.2 and arguments similar to the proof of Proposition 3.1.2, we have the following propositions.

Proposition 4.2.5. Let $\gamma : I \rightarrow S_1^3$ be a timelike Sabban curve. Then the following conditions are equivalent:

- (1) The de Sitter evolute $\mathbf{E}_\gamma(s)$ is a constant vector.
- (2) There exist $\mathbf{v} \in S_1^3$ and $c \in \mathbb{R}$ such that $\gamma(I) \subset S_1^2(\mathbf{v}, c)$.
- (3) $\sigma_2(s) \equiv 0$.

Proposition 4.2.6. Let $\gamma : I \rightarrow S_1^3$ be a timelike Sabban curve. Then we have the following:

- (1) $S_1^2(\mathbf{v}, \varepsilon)$ is a tangent de Sitter subspace of $\gamma(s)$ at s_0 if and only if there exist $a, b, c \in \mathbb{R}$ with $a^2 + b^2 + c^2 = 1$ such that $\mathbf{v} = a\gamma(s_0) + b\mathbf{n}_1(s_0) + c\mathbf{n}_2(s_0)$.
- (2) The tangent de Sitter subspace $S_1^2(\mathbf{v}, \varepsilon)$ has at least 3-point contact with $\gamma(s)$ at s_0 if and only if there exists $\theta_0 \in [0, 2\pi)$ such that

$$\mathbf{v} = \frac{\kappa_1 \cos \theta_0}{\sqrt{1 + \kappa_1^2}}(\gamma(s_0) - \frac{1}{\kappa_1}\mathbf{n}_1(s_0)) + \sin \theta_0 \mathbf{n}_2(s_0).$$

- (3) The tangent de Sitter subspace $S_1^2(\mathbf{v}, \varepsilon)$ has at least 4-point contact with $\gamma(s)$ at s_0 if and only if there exists $\theta_0 \in [0, 2\pi)$ such that

$$\mathbf{v} = \frac{\kappa_1 \cos \theta_0}{\sqrt{1 + \kappa_1^2}}(\gamma(s_0) - \frac{1}{\kappa_1}\mathbf{n}_1(s_0)) + \sin \theta_0 \mathbf{n}_2(s_0)$$

and $\frac{\kappa'_1}{\sqrt{1 + \kappa_1^2}} \cos \theta_0 = \kappa_1 \kappa_2 \sin \theta_0$.

- (4) The tangent de Sitter subspace $S_1^2(\mathbf{v}, \varepsilon)$ has at least 5-point contact with $\gamma(s)$ at s_0 if and only if there exists $\theta_0 \in [0, 2\pi)$ such that

$$\mathbf{v} = \frac{\kappa_1 \cos \theta_0}{\sqrt{1 + \kappa_1^2}}(\gamma(s_0) - \frac{1}{\kappa_1}\mathbf{n}_1(s_0)) + \sin \theta_0 \mathbf{n}_2(s_0), \quad \frac{\kappa'_1}{\sqrt{1 + \kappa_1^2}} \cos \theta_0 = \kappa_1 \kappa_2 \sin \theta_0$$

and $\sigma_2(s_0) = 0$.

(5) The tangent de Sitter subspace $S_1^2(\mathbf{v}, \varepsilon)$ has at least 6-point contact with $\gamma(s)$ at s_0 if and only if there exists $\theta_0 \in [0, 2\pi)$ such that

$$\mathbf{v} = \frac{\kappa_1 \cos \theta_0}{\sqrt{1 + \kappa_1^2}}(\gamma(s_0) - \frac{1}{\kappa_1}\mathbf{n}_1(s_0)) + \sin \theta_0 \mathbf{n}_2(s_0), \quad \frac{\kappa_1'}{\sqrt{1 + \kappa_1^2}} \cos \theta_0 = \kappa_1 \kappa_2 \sin \theta_0$$

and $\sigma_2(s_0) = \sigma_2'(s_0) = 0$.

As a consequence, $\mathbf{E}_\gamma(s)$ is the locus of the polar vector $\mathbf{v} \in S_1^3$ such that $S_1^2(\mathbf{v}, \varepsilon)$ has at least 4-point contact with $\gamma(s)$ at s_0 . We call $S_1^2(\mathbf{v}, \varepsilon)$ an *osculating de Sitter subspace* of $\gamma(s)$ at s_0 if $\mathbf{v} = \mathbf{E}_\gamma(s_0)$. As a corollary of Theorem 4.2.4 and Proposition 4.2.6, we have the following theorem.

Theorem 4.2.7. Let $\gamma : I \rightarrow S_1^3$ be a timelike Sabban curve with $\kappa_1'(s) \neq 0$. Then we have the following:

- (1) If $\mathbf{v}_0 = \mathbf{FS}(s_0, \theta_0)$, then the tangent de Sitter subspace $S_1^2(\mathbf{v}_0, \varepsilon)$ and $\gamma(s)$ have at least 3-point contact at s_0 .
- (2) The osculating de Sitter subspace $S_1^2(\mathbf{v}_0, \varepsilon)$ and $\gamma(s)$ have 4-point contact at s_0 if and only if there exists $\theta_0 \in [0, 2\pi)$ such that

$$\mathbf{v} = \frac{\kappa_1 \cos \theta_0}{\sqrt{1 + \kappa_1^2}}(\gamma(s_0) - \frac{1}{\kappa_1}\mathbf{n}_1(s_0)) + \sin \theta_0 \mathbf{n}_2(s_0),$$

$\frac{\kappa_1'}{\sqrt{1 + \kappa_1^2}} \cos \theta_0 = \kappa_1 \kappa_2 \sin \theta_0$ and $\sigma_2(s_0) \neq 0$. In this case, the germ of $\mathbf{FS}$ at $\mathbf{FS}(s_0, \theta_0)$ is diffeomorphic to $C \times \mathbb{R}$.

- (3) The osculating de Sitter subspace $S_1^2(\mathbf{v}_0, \varepsilon)$ and $\gamma(s)$ have 5-point contact at s_0 if and only if there exists $\theta_0 \in [0, 2\pi)$ such that

$$\mathbf{v} = \frac{\kappa_1 \cos \theta_0}{\sqrt{1 + \kappa_1^2}}(\gamma(s_0) - \frac{1}{\kappa_1}\mathbf{n}_1(s_0)) + \sin \theta_0 \mathbf{n}_2(s_0),$$

$\frac{\kappa_1'}{\sqrt{1 + \kappa_1^2}} \cos \theta_0 = \kappa_1 \kappa_2 \sin \theta_0$, $\sigma_2(s_0) = 0$ and $\sigma_2'(s_0) \neq 0$. In this case, the germ of $\mathbf{FS}$ at $\mathbf{FS}(s_0, \theta_0)$ is diffeomorphic to SW , the germ of $\mathbf{E}_\gamma$ at $\mathbf{E}_\gamma(s_0)$ is diffeomorphic to C .

4.3 De Sitter evolutes and de Sitter focal hypersurfaces of timelike Sabban curves in S_1^4

In this section we consider the timelike Sabban curve in S_1^4 . Let $\gamma : I \rightarrow S_1^4$ be a timelike Sabban curve, we can reparameterize it by the arc-length s . Then we have the tangent vector $\mathbf{t}(s) = \gamma'(s)$, obviously $\|\mathbf{t}(s)\| = 1$. When $\langle \mathbf{t}'(s), \mathbf{t}'(s) \rangle \neq 1$, we define a unit vector

$$\mathbf{n}_1(s) = \frac{\mathbf{t}'(s) - \gamma(s)}{\|\mathbf{t}'(s) - \gamma(s)\|}.$$

Let $\kappa_1(s) = \|\mathbf{t}'(s) - \gamma(s)\|$, $\langle \mathbf{t}'(s), \mathbf{t}'(s) \rangle \neq 1$ is equal to $\kappa_1(s) \neq 0$. If $\|\mathbf{n}_1'(s) - \kappa_1(s)\mathbf{t}(s)\| \neq 0$, we define a unit vector

$$\mathbf{n}_2(s) = \frac{\mathbf{n}_1'(s) - \kappa_1(s)\mathbf{t}(s)}{\|\mathbf{n}_1'(s) - \kappa_1(s)\mathbf{t}(s)\|}$$

and denote $\kappa_2(s) = \|\mathbf{n}_1'(s) - \kappa_1(s)\mathbf{t}(s)\|$. Let $\mathbf{n}_3(s) = \gamma(s) \wedge \mathbf{t}(s) \wedge \mathbf{n}_1(s) \wedge \mathbf{n}_2(s)$, then we have the Sabban frame $\{\gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \mathbf{n}_2(s), \mathbf{n}_3(s)\}$ along γ . In §3, we have shown the Frenet-Serret type of timelike Sabban curves, under the assumption that $\kappa_1(s) \neq 0$ and $\kappa_2(s) \neq 0$, the Frenet-Serret type is as follows:

$$\begin{cases} \gamma'(s) = \mathbf{t}(s), \\ \mathbf{t}'(s) = \gamma(s) + \kappa_1(s)\mathbf{n}_1(s), \\ \mathbf{n}_1'(s) = \kappa_1(s)\mathbf{t}(s) + \kappa_2(s)\mathbf{n}_2(s), \\ \mathbf{n}_2'(s) = -\kappa_2(s)\mathbf{n}_1(s) + \kappa_3(s)\mathbf{n}_3(s), \\ \mathbf{n}_3'(s) = -\kappa_3(s)\mathbf{n}_2(s). \end{cases}$$

Here $\kappa_1(s)$ and $\kappa_2(s)$ can't be zero, $\kappa_3(s)$ may equal to zero.

We define a family of functions on timelike Sabban curve. Let $\gamma : I \rightarrow S_1^4$ be a timelike Sabban curve. We define a function $H : I \times S_1^4 \rightarrow \mathbb{R}$ by $H(s, \mathbf{v}) = \langle \gamma(s), \mathbf{v} \rangle$, which is called a *de Sitter height function* on the timelike Sabban curve γ . We denote that $h_v(s) = H(s, \mathbf{v})$ for any fixed $\mathbf{v} \in S_1^4$. Then we have the following proposition.

Proposition 4.3.1. Let $\gamma : I \rightarrow S_1^4$ be a timelike Sabban curve. Then we have the following assertions:

(1) $h'_v(s) = 0$ if and only if there exist $a, b, c, d \in \mathbb{R}$ with $a^2 + b^2 + c^2 + d^2 = 1$ such that $\mathbf{v} = a\boldsymbol{\gamma}(s) + b\mathbf{n}_1(s) + c\mathbf{n}_2(s) + d\mathbf{n}_3(s)$.

(2) $h'_v(s) = h''_v(s) = 0$ if and only if there exist $\theta \in [0, 2\pi)$ and $\varphi \in [-\pi, \pi)$ such that

$$\mathbf{v} = \sin \theta \cos \varphi \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\boldsymbol{\gamma}(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) + \cos \theta \cos \varphi \mathbf{n}_2(s) + \sin \varphi \mathbf{n}_3(s).$$

(3) $h'_v(s) = h''_v(s) = h'''_v(s) = 0$ if and only if there exist $\theta \in [0, 2\pi)$ and $\varphi \in [-\pi, \pi)$ such that

$$\mathbf{v} = \sin \theta \cos \varphi \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\boldsymbol{\gamma}(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) + \cos \theta \cos \varphi \mathbf{n}_2(s) + \sin \varphi \mathbf{n}_3(s),$$

$$\sin \theta = \frac{\kappa_1 \kappa_2 \sqrt{1 + \kappa_1^2}}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa'_1)^2}} \text{ and } \cos \theta = \frac{\kappa'_1}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa'_1)^2}}.$$

(4) $h'_v(s) = h''_v(s) = h'''_v(s) = h_v^{(4)}(s) = 0$ if and only if there exist $\theta \in [0, 2\pi)$ and $\varphi \in [-\pi, \pi)$ such that

$$\mathbf{v} = \sin \theta \cos \varphi \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\boldsymbol{\gamma}(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) + \cos \theta \cos \varphi \mathbf{n}_2(s) + \sin \varphi \mathbf{n}_3(s),$$

$$(\kappa_1^2 \kappa_2^3 - \kappa_1'' \kappa_1 \kappa_2 + 2(\kappa'_1)^2 \kappa_2 + \kappa_1 \kappa'_1 \kappa'_2) \cos \varphi + \kappa_1 \kappa_2 \kappa_3 \sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa'_1)^2} \sin \varphi = 0,$$

$$\sin \theta = \frac{\kappa_1 \kappa_2 \sqrt{1 + \kappa_1^2}}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa'_1)^2}} \text{ and } \cos \theta = \frac{\kappa'_1}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa'_1)^2}}.$$

(5) $h'_v(s) = h''_v(s) = h'''_v(s) = h_v^{(4)}(s) = h_v^{(5)}(s) = 0$ if and only if there exist $\theta \in [0, 2\pi)$ and $\varphi \in [-\pi, \pi)$ such that

$$\mathbf{v} = \sin \theta \cos \varphi \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\boldsymbol{\gamma}(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) + \cos \theta \cos \varphi \mathbf{n}_2(s) + \sin \varphi \mathbf{n}_3(s),$$

$$(\kappa_1^2 \kappa_2^3 - \kappa_1'' \kappa_1 \kappa_2 + 2(\kappa'_1)^2 \kappa_2 + \kappa_1 \kappa'_1 \kappa'_2) \cos \varphi + \kappa_1 \kappa_2 \kappa_3 \sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa'_1)^2} \sin \varphi = 0,$$

$$\begin{aligned} & - \kappa_1^2 \kappa_1''' \kappa_2^2 \kappa_3 - \kappa_1^2 \kappa_1' \kappa_2^4 \kappa_3 + \kappa_1^3 \kappa_2^3 \kappa'_2 \kappa_3 + 6\kappa_1 \kappa_1' \kappa_1'' \kappa_2^2 \kappa_3 - 4\kappa_1 (\kappa'_1)^2 \kappa_2 \kappa'_2 \kappa_3 + \kappa_1^2 \kappa_1' \kappa_2 \kappa_2'' \kappa_3 \\ & - \kappa_1^2 \kappa_1' \kappa_2^2 \kappa_3^3 - 6(\kappa'_1)^3 \kappa_2^2 \kappa_3 + 2\kappa_1^2 \kappa_1'' \kappa_2 \kappa'_2 \kappa_3 - 2\kappa_1^2 \kappa_1' (\kappa'_1)^2 \kappa_3 - \kappa_1^3 \kappa_2^4 \kappa'_3 + \kappa_1^2 \kappa_1'' \kappa_2^2 \kappa'_3 \\ & - 2\kappa_1 (\kappa'_1)^2 \kappa_2^2 \kappa'_3 - \kappa_1^2 \kappa_1' \kappa_2 \kappa_2' \kappa'_3 = 0, \end{aligned}$$

$$\sin \theta = \frac{\kappa_1 \kappa_2 \sqrt{1 + \kappa_1^2}}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa'_1)^2}} \text{ and } \cos \theta = \frac{\kappa'_1}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa'_1)^2}}.$$

Proof. (1) Since $\mathbf{v} \in S_1^4$, there exist $a, x, b, c, d \in \mathbb{R}$ such that $\mathbf{v} = a\boldsymbol{\gamma}(s) + x\mathbf{t}(s) + b\mathbf{n}_1(s) + c\mathbf{n}_2(s) + d\mathbf{n}_3(s)$ with $a^2 + x^2 + b^2 + c^2 + d^2 = 1$. Since $h'_v(s) = 0$, $x = 0$ so that $a^2 + b^2 + c^2 + d^2 = 1$.

The converse assertion also holds.

(2) By (1), we can show that $\langle \boldsymbol{\gamma}(s) + \kappa_1 \mathbf{n}_1(s), \mathbf{v} \rangle = 0$. Thus $b = -\frac{a}{\kappa_1}$. Since $a^2 + b^2 + c^2 + d^2 = 1$, there exist $\theta \in [0, 2\pi)$ and $\varphi \in [-\pi, \pi)$ such that

$$a = \sin \theta \cos \varphi \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}}, \quad b = -\sin \theta \cos \varphi \frac{1}{\sqrt{1 + \kappa_1^2}}, \quad c = \cos \theta \cos \varphi, \quad d = \sin \varphi.$$

Therefore

$$\mathbf{v} = \sin \theta \cos \varphi \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\boldsymbol{\gamma}(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) + \cos \theta \cos \varphi \mathbf{n}_2(s) + \sin \varphi \mathbf{n}_3(s).$$

(3) We assume that $h'_v(s) = h''_v(s) = 0$. Since

$$h'''_v(s) = \langle (1 + \kappa_1^2) \mathbf{t}(s) + \kappa'_1 \mathbf{n}_1(s) + \kappa_1 \kappa_2 \mathbf{n}_2(s), \mathbf{v} \rangle = 0,$$

$$\frac{\kappa'_1}{\sqrt{1 + \kappa_1^2}} \sin \theta = \kappa_1 \kappa_2 \cos \theta. \quad \text{The last condition is equivalent to } \sin \theta = \frac{\kappa_1 \kappa_2 \sqrt{1 + \kappa_1^2}}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa'_1)^2}} \text{ and}$$

$$\cos \theta = \frac{\kappa'_1}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa'_1)^2}}.$$

(4) We assume that $h'_v(s) = h''_v(s) = h'''_v(s) = 0$. Then we have

$$h_v^{(4)}(s) = \langle (1 + \kappa_1^2) \boldsymbol{\gamma}(s) + 3\kappa_1 \kappa'_1 \mathbf{t}(s) + (\kappa_1 + \kappa_1^3 + \kappa_1'' - \kappa_1 \kappa_2^2) \mathbf{n}_1(s) + (2\kappa'_1 \kappa_2 + \kappa_1 \kappa'_2) \mathbf{n}_2(s) \\ + \kappa_1 \kappa_2 \kappa_3 \mathbf{n}_3(s), \mathbf{v} \rangle.$$

Therefore $h_v^{(4)}(s) = 0$ if and only if

$$\sin \theta \cos \varphi \frac{\kappa_1 \kappa_2^2 - \kappa_1''}{\sqrt{1 + \kappa_1^2}} + \cos \theta \cos \varphi (2\kappa'_1 \kappa_2 + \kappa_1 \kappa'_2) + \kappa_1 \kappa_2 \kappa_3 \sin \varphi = 0.$$

Since we have $\sin \theta = \frac{\kappa_1 \kappa_2 \sqrt{1 + \kappa_1^2}}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa'_1)^2}}$ and $\cos \theta = \frac{\kappa'_1}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa'_1)^2}}$, then

$$(\kappa_1^2 \kappa_2^3 - \kappa_1'' \kappa_1 \kappa_2 + 2(\kappa'_1)^2 \kappa_2 + \kappa_1 \kappa'_1 \kappa'_2) \cos \varphi + \kappa_1 \kappa_2 \kappa_3 \sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa'_1)^2} \sin \varphi = 0.$$

This proves assertion (4).

(5) When $h'_v(s) = h''_v(s) = h'''_v(s) = h^{(4)}_v(s) = 0$, we have the fifth derivative:

$$\begin{aligned} h^{(5)}_v(s) = & \langle 5\kappa_1\kappa'_1\gamma(s) + (1 + 2\kappa_1^2 + 3(\kappa'_1)^2 + 4\kappa_1\kappa''_1 + \kappa_1^4 - \kappa_1^2\kappa_2^2)\mathbf{t}(s) \\ & + (6\kappa_1^2\kappa'_1 + \kappa'_1 + \kappa'''_1 - 3\kappa'_1\kappa_2^2 - 3\kappa_1\kappa_2\kappa'_2)\mathbf{n}_1(s) \\ & + (\kappa_1\kappa_2 + \kappa_1^3\kappa_2 + 3\kappa''_1\kappa_2 - \kappa_1\kappa_2^3 + 3\kappa'_1\kappa'_2 + \kappa_1\kappa''_2 - \kappa_1\kappa_2\kappa_3^2)\mathbf{n}_2(s) \\ & + (3\kappa'_1\kappa_2\kappa_3 + 2\kappa_1\kappa'_2\kappa_3 + \kappa_1\kappa_2\kappa'_3)\mathbf{n}_3(s), \mathbf{v} \rangle. \end{aligned}$$

It follows that $h^{(5)}_v(s) = 0$ if and only if

$$\begin{aligned} \sin \varphi \sqrt{\kappa_1^2\kappa_2^2(1 + \kappa_1^2) + (\kappa'_1)^2(3\kappa'_1\kappa_2\kappa_3 + 2\kappa_1\kappa'_2\kappa_3 + \kappa_1\kappa_2\kappa'_3)} = \cos \varphi (-\kappa_1\kappa'''_1\kappa_2 + 2\kappa_1\kappa'_1\kappa_2^3 \\ + 3\kappa_1^2\kappa_2^2\kappa'_2 + 3\kappa'_1\kappa''_1\kappa_2 + 3(\kappa'_1)^2\kappa'_2 + \kappa_1\kappa'_1\kappa''_2 - \kappa_1\kappa'_1\kappa_2\kappa_3^2). \end{aligned}$$

Since

$$(\kappa_1^2\kappa_2^3 - \kappa''_1\kappa_1\kappa_2 + 2(\kappa'_1)^2\kappa_2 + \kappa_1\kappa'_1\kappa'_2) \cos \varphi + \kappa_1\kappa_2\kappa_3 \sqrt{\kappa_1^2\kappa_2^2(1 + \kappa_1^2) + (\kappa'_1)^2} \sin \varphi = 0,$$

we have

$$\begin{aligned} & -\kappa_1^2\kappa'''_1\kappa_2^2\kappa_3 - \kappa_1^2\kappa'_1\kappa_2^4\kappa_3 + \kappa_1^3\kappa_2^3\kappa'_2\kappa_3 + 6\kappa_1\kappa'_1\kappa''_1\kappa_2^2\kappa_3 - 4\kappa_1(\kappa'_1)^2\kappa_2\kappa'_2\kappa_3 + \kappa_1^2\kappa'_1\kappa_2\kappa''_2\kappa_3 \\ & - \kappa_1^2\kappa'_1\kappa_2^2\kappa_3^3 - 6(\kappa'_1)^3\kappa_2^2\kappa_3 + 2\kappa_1^2\kappa''_1\kappa_2\kappa'_2\kappa_3 - 2\kappa_1^2\kappa'_1(\kappa'_1)^2\kappa_3 - \kappa_1^3\kappa_2^4\kappa'_3 + \kappa_1^2\kappa''_1\kappa_2^2\kappa'_3 \\ & - 2\kappa_1(\kappa'_1)^2\kappa_2^2\kappa'_3 - \kappa_1^2\kappa'_1\kappa_2\kappa'_2\kappa'_3 = 0. \end{aligned}$$

This completes the proof. □

We now define a new invariant as follows:

$$\begin{aligned} \sigma_3(s) = & -\kappa_1^2\kappa'''_1\kappa_2^2\kappa_3 - \kappa_1^2\kappa'_1\kappa_2^4\kappa_3 + \kappa_1^3\kappa_2^3\kappa'_2\kappa_3 + 6\kappa_1\kappa'_1\kappa''_1\kappa_2^2\kappa_3 \\ & - 4\kappa_1(\kappa'_1)^2\kappa_2\kappa'_2\kappa_3 + \kappa_1^2\kappa'_1\kappa_2\kappa''_2\kappa_3 - \kappa_1^2\kappa'_1\kappa_2^2\kappa_3^3 - 6(\kappa'_1)^3\kappa_2^2\kappa_3 \\ & + 2\kappa_1^2\kappa''_1\kappa_2\kappa'_2\kappa_3 - 2\kappa_1^2\kappa'_1(\kappa'_1)^2\kappa_3 - \kappa_1^3\kappa_2^4\kappa'_3 + \kappa_1^2\kappa''_1\kappa_2^2\kappa'_3 \\ & - 2\kappa_1(\kappa'_1)^2\kappa_2^2\kappa'_3 - \kappa_1^2\kappa'_1\kappa_2\kappa'_2\kappa'_3. \end{aligned}$$

Conjecture 4.3.2. Let $\gamma : I \rightarrow S_1^4$ be a timelike Sabban curve. then $h'_v(s) = h''_v(s) = h'''_v(s) = h_v^{(4)}(s) = h_v^{(5)}(s) = h_v^{(6)}(s) = 0$ if and only if there exist $\theta \in [0, 2\pi)$ and $\varphi \in [-\pi, \pi)$ such that

$$\mathbf{v} = \sin \theta \cos \varphi \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\gamma(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) + \cos \theta \cos \varphi \mathbf{n}_2(s) + \sin \varphi \mathbf{n}_3(s),$$

$$(\kappa_1^2 \kappa_2^3 - \kappa_1'' \kappa_1 \kappa_2 + 2(\kappa_1')^2 \kappa_2 + \kappa_1 \kappa_1' \kappa_2') \cos \varphi + \kappa_1 \kappa_2 \kappa_3 \sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2} \sin \varphi = 0,$$

$$\sin \theta = \frac{\kappa_1 \kappa_2 \sqrt{1 + \kappa_1^2}}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2}}, \cos \theta = \frac{\kappa_1'}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2}} \text{ and } \sigma_3(s) = \sigma'_3(s) = 0.$$

By the above proposition, we get a function $\sigma_3(s)$ which is useful in following investigation.

We define another function $\rho(s)$ by

$$\rho(s) = (\kappa_1^2 \kappa_2^3 - \kappa_1'' \kappa_1 \kappa_2 + 2(\kappa_1')^2 \kappa_2 + \kappa_1 \kappa_1' \kappa_2')^2 + \kappa_1^4 \kappa_2^4 \kappa_3^2 (1 + \kappa_1^2) + (\kappa_1')^2 \kappa_1^2 \kappa_2^2 \kappa_3^2.$$

Now we define a hypersurface $\mathbf{FHS} : I \times S^2 \rightarrow S_1^4$ by

$$\mathbf{FHS}(s, \mathbf{u}) = u_1 \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\gamma(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) + u_2 \mathbf{n}_2(s) + u_3 \mathbf{n}_3(s).$$

Here $\mathbf{u} = (u_1, u_2, u_3) \in S^2$, we call $\mathbf{FHS}(s, \mathbf{u})$ the *de Sitter focal hypersurface* of γ . Since

$\mathbf{u} = (u_1, u_2, u_3) \in S^2$, there exist $\theta \in [0, 2\pi)$ and $\varphi \in [-\pi, \pi)$ such that

$$\mathbf{FHS}(s, \theta, \varphi) = \sin \theta \cos \varphi \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\gamma(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) + \cos \theta \cos \varphi \mathbf{n}_2(s) + \sin \varphi \mathbf{n}_3(s).$$

If $\rho(s) \neq 0$, we define a curve $\mathbf{E}_\gamma : I \rightarrow S_1^4$ by

$$\begin{aligned} \mathbf{E}_\gamma(s) = & \frac{1}{\sqrt{\rho(s)}} [\kappa_1^3 \kappa_2^2 \kappa_3 (\gamma(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) + \kappa_1 \kappa_1' \kappa_2 \kappa_3 \mathbf{n}_2(s) \\ & - (\kappa_1^2 \kappa_2^3 - \kappa_1'' \kappa_1 \kappa_2 + 2(\kappa_1')^2 \kappa_2 + \kappa_1 \kappa_1' \kappa_2') \mathbf{n}_3(s)]. \end{aligned}$$

We call $\mathbf{E}_\gamma(s)$ the *de Sitter evolute* of γ .

We consider some properties of the de Sitter evolute and the de Sitter focal hypersurface of a timelike Sabban curve in S_1^4 . Moreover we investigate the geometric meaning of the function $\sigma_3(s)$. We have the following proposition.

Proposition 4.3.3. Let $\gamma : I \rightarrow S_1^4$ be a timelike Sabban curve. Then we have the following assertions:

- (1) The set $\{(s, \theta, \varphi) | \frac{\kappa_1'}{\sqrt{1+\kappa_1^2}} \sin \theta = \kappa_1 \kappa_2 \cos \theta\}$ is the singularities of $\mathbf{FHS}(s, \theta, \varphi)$.
- (2) If $\rho(s) \neq 0$, the de Sitter evolute $\mathbf{E}_\gamma(s)$ of γ is singular at s_0 if and only if $\sigma_3(s_0) = 0$.

Proof. (1) Since

$$\mathbf{FHS}(s, \theta, \varphi) = \sin \theta \cos \varphi \frac{\kappa_1}{\sqrt{1+\kappa_1^2}} (\gamma(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) + \cos \theta \cos \varphi \mathbf{n}_2(s) + \sin \varphi \mathbf{n}_3(s).$$

By straightforward calculations, we have

$$\begin{aligned} \frac{\partial \mathbf{FHS}}{\partial \theta} &= \frac{\kappa_1 \cos \theta \cos \varphi}{\sqrt{1+\kappa_1^2}} \gamma(s) - \frac{\cos \theta \cos \varphi}{\sqrt{1+\kappa_1^2}} \mathbf{n}_1(s) - \sin \theta \cos \varphi \mathbf{n}_2(s), \\ \frac{\partial \mathbf{FHS}}{\partial \varphi} &= -\frac{\kappa_1 \sin \theta \sin \varphi}{\sqrt{1+\kappa_1^2}} \gamma(s) + \frac{\sin \theta \sin \varphi}{\sqrt{1+\kappa_1^2}} \mathbf{n}_1(s) - \cos \theta \sin \varphi \mathbf{n}_2(s) + \cos \varphi \mathbf{n}_3(s), \\ \frac{\partial \mathbf{FHS}}{\partial s} &= \left(\frac{\kappa_1 \kappa_1' \sin \theta \cos \varphi}{(1+\kappa_1^2)\sqrt{1+\kappa_1^2}} - \kappa_2 \cos \theta \cos \varphi \right) \mathbf{n}_1(s) + \frac{\kappa_1' \sin \theta \cos \varphi}{(1+\kappa_1^2)\sqrt{1+\kappa_1^2}} \gamma(s) \\ &\quad - \left(\frac{\kappa_2 \sin \theta \cos \varphi}{\sqrt{1+\kappa_1^2}} + \sin \theta \kappa_3 \right) \mathbf{n}_2(s) + \frac{\kappa_1' \kappa_3 \sin \theta \cos \varphi}{\kappa_1 \kappa_2 \sqrt{1+\kappa_1^2}} \mathbf{n}_3(s). \end{aligned}$$

The above three vectors are linearly dependent if and only if $\frac{\kappa_1'}{\sqrt{1+\kappa_1^2}} \sin \theta = \kappa_1 \kappa_2 \cos \theta$, which gives the condition for the singularities of $\mathbf{FHS}$.

(2) We define a function $f(s) = \langle \mathbf{t}(s), \mathbf{v}(s) \rangle$. Under the condition $f(s) = 0$, if $\langle \mathbf{t}'(s), \mathbf{v}(s) \rangle = 0$, we have $\langle \mathbf{t}(s), \mathbf{v}'(s) \rangle = 0$. Under the condition $\langle \mathbf{t}'(s), \mathbf{v}(s) \rangle = 0$, if $\langle \mathbf{t}''(s), \mathbf{v}(s) \rangle = 0$, we have $\langle \mathbf{t}'(s), \mathbf{v}'(s) \rangle = 0$. We continue this procedure. Finally, if

$$\langle \mathbf{t}(s), \mathbf{v}(s) \rangle = \langle \mathbf{t}'(s), \mathbf{v}(s) \rangle = \langle \mathbf{t}''(s), \mathbf{v}(s) \rangle = \langle \mathbf{t}'''(s), \mathbf{v}(s) \rangle = \langle \mathbf{t}^{(4)}(s), \mathbf{v}(s) \rangle = 0,$$

then

$$\langle \mathbf{t}(s), \mathbf{v}'(s) \rangle = \langle \mathbf{t}'(s), \mathbf{v}'(s) \rangle = \langle \mathbf{t}''(s), \mathbf{v}'(s) \rangle = \langle \mathbf{t}'''(s), \mathbf{v}'(s) \rangle = 0.$$

If $\rho(s) \neq 0$, by the Proposition 4.3.1, $\mathbf{v}(s) = \mathbf{E}_\gamma(s)$ is orthogonal to the following vectors: $\mathbf{t}(s), \mathbf{t}'(s), \mathbf{t}''(s)$ and $\mathbf{t}'''(s)$. Since $\mathbf{v}(s) \in S_1^4$, we have $\langle \mathbf{v}(s), \mathbf{v}'(s) \rangle = 0$. Then $\mathbf{v}'(s)$ is orthogonal to $\mathbf{v}(s), \mathbf{t}(s), \mathbf{t}'(s), \mathbf{t}''(s)$ and $\mathbf{t}'''(s)$, this means $\mathbf{v}'(s) = 0$. By the Proposition 4.3.1, we have that $\frac{d\mathbf{E}_\gamma}{ds} = 0$ if and only if $\sigma_3(s) = 0$. This completes the proof. $\square$

In order to investigate the singularities of de Sitter evolutes and de Sitter focal hypersurfaces of timelike Sabban curves, we apply the theory of unfolding of functions. In §2, we defined the bifurcation set B_H of H by

$$B_H = \left\{ \mathbf{x} \in \mathbb{R}^r \mid \exists s, \text{ with } \frac{\partial H}{\partial s}(s, \mathbf{x}) = \frac{\partial^2 H}{\partial s^2}(s, \mathbf{x}) = 0 \right\}.$$

Therefore, the focal hypersurface **FHS** is the bifurcation set of the de Sitter height function H . We have the following proposition.

Proposition 4.3.4. Let $\gamma : I \rightarrow S_1^4$ be a timelike Sabban curve with $\rho(s) \neq 0$. If $h_v(s)$ has type A_k at s_0 ($k = 3, 4, 5$), then H is a p -versal unfolding of h_v .

Proof. We denote $\gamma(s) = (x_0(s), x_1(s), x_2(s), x_3(s), x_4(s))$, $\mathbf{v} = (v_0, v_1, v_2, v_3, \sqrt{v_0^2 - v_1^2 - v_2^2 - v_3^2 + 1})$.

Then we adopt the local coordinates (v_0, v_1, v_2, v_3) for S_1^4 . We have

$$H(s, \mathbf{v}) = -x_0(s)v_0 + x_1(s)v_1 + x_2(s)v_2 + x_3(s)v_3 + x_4(s)\sqrt{v_0^2 - v_1^2 - v_2^2 - v_3^2 + 1},$$

so that

$$\begin{aligned} \frac{\partial H}{\partial v_0} &= -x_0(s) + \frac{v_0}{v_4}x_4(s), & \frac{\partial H}{\partial v_1} &= x_1(s) - \frac{v_1}{v_4}x_4(s), \\ \frac{\partial H}{\partial v_2} &= x_2(s) - \frac{v_2}{v_4}x_4(s), & \frac{\partial H}{\partial v_3} &= x_3(s) - \frac{v_3}{v_4}x_4(s), \\ \frac{\partial^2 H}{\partial s \partial v_0} &= -x'_0(s) + \frac{v_0}{v_4}x'_4(s), & \frac{\partial^2 H}{\partial s \partial v_1} &= x'_1(s) - \frac{v_1}{v_4}x'_4(s), \\ \frac{\partial^2 H}{\partial s \partial v_2} &= x'_2(s) - \frac{v_2}{v_4}x'_4(s), & \frac{\partial^2 H}{\partial s \partial v_3} &= x'_3(s) - \frac{v_3}{v_4}x'_4(s), \\ \frac{\partial^3 H}{\partial s^2 \partial v_0} &= -x''_0(s) + \frac{v_0}{v_4}x''_4(s), & \frac{\partial^3 H}{\partial s^2 \partial v_1} &= x''_1(s) - \frac{v_1}{v_4}x''_4(s), \\ \frac{\partial^3 H}{\partial s^2 \partial v_2} &= x''_2(s) - \frac{v_2}{v_4}x''_4(s), & \frac{\partial^3 H}{\partial s^2 \partial v_3} &= x''_3(s) - \frac{v_3}{v_4}x''_4(s), \\ \frac{\partial^4 H}{\partial s^3 \partial v_0} &= -x'''_0(s) + \frac{v_0}{v_4}x'''_4(s), & \frac{\partial^4 H}{\partial s^3 \partial v_1} &= x'''_1(s) - \frac{v_1}{v_4}x'''_4(s), \\ \frac{\partial^4 H}{\partial s^3 \partial v_2} &= x'''_2(s) - \frac{v_2}{v_4}x'''_4(s), & \frac{\partial^4 H}{\partial s^3 \partial v_3} &= x'''_3(s) - \frac{v_3}{v_4}x'''_4(s), \\ \frac{\partial^5 H}{\partial s^4 \partial v_0} &= -x^{(4)}_0(s) + \frac{v_0}{v_4}x^{(4)}_4(s), & \frac{\partial^5 H}{\partial s^4 \partial v_1} &= x^{(4)}_1(s) - \frac{v_1}{v_4}x^{(4)}_4(s), \\ \frac{\partial^5 H}{\partial s^4 \partial v_2} &= x^{(4)}_2(s) - \frac{v_2}{v_4}x^{(4)}_4(s), & \frac{\partial^5 H}{\partial s^4 \partial v_3} &= x^{(4)}_3(s) - \frac{v_3}{v_4}x^{(4)}_4(s). \end{aligned}$$

Here, $v_4 = \sqrt{v_0^2 - v_1^2 - v_2^2 - v_3^2 + 1} \neq 0$. We now consider the following matrix:

$$A = \begin{pmatrix} x'_0(s) - \frac{v_0}{v_4}x'_4(s) & x'_1(s) - \frac{v_1}{v_4}x'_4(s) & x'_2(s) - \frac{v_2}{v_4}x'_4(s) & x'_3(s) - \frac{v_3}{v_4}x'_4(s) \\ x''_0(s) - \frac{v_0}{v_4}x''_4(s) & x''_1(s) - \frac{v_1}{v_4}x''_4(s) & x''_2(s) - \frac{v_2}{v_4}x''_4(s) & x''_3(s) - \frac{v_3}{v_4}x''_4(s) \\ x'''_0(s) - \frac{v_0}{v_4}x'''_4(s) & x'''_1(s) - \frac{v_1}{v_4}x'''_4(s) & x'''_2(s) - \frac{v_2}{v_4}x'''_4(s) & x'''_3(s) - \frac{v_3}{v_4}x'''_4(s) \\ x^{(4)}_0(s) - \frac{v_0}{v_4}x^{(4)}_4(s) & x^{(4)}_1(s) - \frac{v_1}{v_4}x^{(4)}_4(s) & x^{(4)}_2(s) - \frac{v_2}{v_4}x^{(4)}_4(s) & x^{(4)}_3(s) - \frac{v_3}{v_4}x^{(4)}_4(s) \end{pmatrix}.$$

If we put $a_i = (x'_i, x''_i, x'''_i, x^{(4)}_i)^t$, we have

$$A = (a_0 - \frac{v_0}{v_4}a_4, a_1 - \frac{v_1}{v_4}a_4, a_2 - \frac{v_2}{v_4}a_4, a_3 - \frac{v_3}{v_4}a_4).$$

Then

$$\begin{aligned} \det A &= \det(a_0, a_1, a_2, a_3) + \frac{v_0}{v_4} \det(a_4, a_1, a_2, a_3) + \frac{v_1}{v_4} \det(a_0, a_4, a_2, a_3) \\ &\quad + \frac{v_2}{v_4} \det(a_0, a_1, a_4, a_3) + \frac{v_3}{v_4} \det(a_0, a_1, a_2, a_4) \\ &= \langle \frac{\mathbf{v}}{v_4}, (\gamma'(s) \wedge \gamma''(s) \wedge \gamma'''(s) \wedge \gamma^{(4)}(s)) \rangle. \end{aligned}$$

Since

$$\mathbf{v} = \sin \theta \cos \varphi \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\gamma(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) + \cos \theta \cos \varphi \mathbf{n}_2(s) + \sin \varphi \mathbf{n}_3(s),$$

by a straightforward calculation, we have $\gamma'(s) \wedge \gamma''(s) \wedge \gamma'''(s) \wedge \gamma^{(4)}(s) = \sqrt{\rho(s)} \mathbf{v}$. Therefore $\det A = \frac{\sqrt{\rho(s)}}{v_4} \neq 0$. For other local coordinates, we can apply the arguments similar to above.

This completes the proof. $\square$

By Proposition 4.3.1, Proposition 4.3.4 and Theorem 2.3.6, we have the following theorem.

Theorem 4.3.5. Let $\gamma : I \rightarrow S_1^4$ be a timelike Sabban curve with $\rho(s) \neq 0$. Then we have the following:

- (1) If h_v has A_3 singularity at s_0 , then the germ of focal hypersurface **FHS** at $(s_0, \theta_0, \varphi_0)$ is diffeomorphic to $C \times \mathbb{R}^2$.
- (2) If h_v has A_4 singularity at s_0 , then the germ of focal surface **FHS** at $(s_0, \theta_0, \varphi_0)$ is diffeomorphic to $SW \times \mathbb{R}$.
- (3) If h_v has A_5 singularity at s_0 , then the germ of focal surface **FHS** at $(s_0, \theta_0, \varphi_0)$ is diffeomorphic to BF and the germ of de Sitter evolute **E** $_\gamma$ at s_0 is diffeomorphic to C .

We consider a hyperplane $HP(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}_1^5 | \langle \mathbf{x}, \mathbf{v} \rangle = c\}$, for a fixed vector $\mathbf{v} \in S_1^4$ and a constant $c \in \mathbb{R}$. Then

$$S_1^3(\mathbf{v}, c) = S_1^4 \cap HP(\mathbf{v}, c)$$

is called a *3-dimensional de Sitter subspace* in S_1^4 . For any fixed $s_0 \in I$, put $\langle \gamma(s_0), \mathbf{v} \rangle = \varepsilon$. If we consider a function $\mathfrak{h}_v : S_1^4 \rightarrow \mathbb{R}$ defined by $\mathfrak{h}_v(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v} \rangle$, then $\mathfrak{h}_v^{-1}(\varepsilon) = S_1^3(\mathbf{v}, \varepsilon)$ and $\mathfrak{h}_v \circ \gamma(s) = h_v(s)$. Therefore, $h_v(s_0) = \varepsilon$ and $h'_v(s_0) = 0$ if and only if $S_1^3(\mathbf{v}, \varepsilon)$ is tangent to $\gamma(s)$ at s_0 . We call $S_1^3(\mathbf{v}, \varepsilon)$ a *tangent de Sitter subspace* at s_0 . We remark that there are infinitely many tangent de Sitter subspaces of $\gamma(s)$ at s_0 . We say that a tangent de Sitter subspace $S_1^3(\mathbf{v}, \varepsilon)$ of $\gamma(s)$ at s_0 has *at least $k+1$ -point contact* with γ if $h_v(s_0) = \varepsilon$ and $h'_v(s_0) = \dots = h_v^{(k)}(s_0) = 0$. We also say that tangent de Sitter subspace $S_1^3(\mathbf{v}, \varepsilon)$ of $\gamma(s)$ at s_0 has *$k+1$ -point contact* with γ if $h_v(s_0) = \varepsilon$, $h'_v(s_0) = \dots = h_v^{(k)}(s_0) = 0$ and $h_v^{(k+1)}(s_0) \neq 0$. By Propositions 4.3.1, 4.3.3 and arguments similar to the proof of Proposition 3.1.2, we have the following propositions.

Proposition 4.3.6. Let $\gamma : I \rightarrow S_1^4$ be a timelike Sabban curve with $\rho(s) \neq 0$. Then the following conditions are equivalent:

- (1) The de Sitter evolute $E_\gamma(s)$ is a constant vector.
- (2) There exist $\mathbf{v} \in S_1^4$ and $\varepsilon \in \mathbb{R}$ such that $\gamma(I) \subset S_1^3(\mathbf{v}, \varepsilon)$.
- (3) $\sigma_3(s) \equiv 0$.

Proposition 4.3.7. Let $\gamma : I \rightarrow S_1^4$ be a timelike Sabban curve with $\rho(s) \neq 0$. Then we have the following:

- (1) $S_1^3(\mathbf{v}, \varepsilon)$ is a tangent de Sitter subspace of $\gamma(s)$ at s_0 if and only if there exist $a, b, c, d \in \mathbb{R}$ with $a^2 + b^2 + c^2 + d^2 = 1$ such that $\mathbf{v} = a\gamma(s_0) + b\mathbf{n}_1(s_0) + c\mathbf{n}_2(s_0) + d\mathbf{n}_3(s_0)$.
- (2) The tangent de Sitter subspace $S_1^3(\mathbf{v}, \varepsilon)$ has at least 3-point contact with $\gamma(s)$ at s_0 if and only if there exist $\theta_0 \in [0, 2\pi)$ and $\varphi_0 \in [-\pi, \pi)$ such that

$$\mathbf{v} = \sin \theta_0 \cos \varphi_0 \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\gamma(s_0) - \frac{1}{\kappa_1} \mathbf{n}_1(s_0)) + \cos \theta_0 \cos \varphi_0 \mathbf{n}_2(s_0) + \sin \varphi_0 \mathbf{n}_3(s_0).$$

- (3) The tangent de Sitter subspace $S_1^3(\mathbf{v}, \varepsilon)$ has at least 4-point contact with $\gamma(s)$ at s_0 if and

only if there exist $\theta_0 \in [0, 2\pi)$ and $\varphi_0 \in [-\pi, \pi)$ such that

$$\mathbf{v} = \sin \theta_0 \cos \varphi_0 \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\boldsymbol{\gamma}(s_0) - \frac{1}{\kappa_1} \mathbf{n}_1(s_0)) + \cos \theta_0 \cos \varphi_0 \mathbf{n}_2(s_0) + \sin \varphi_0 \mathbf{n}_3(s_0),$$

$$\sin \theta_0 = \frac{\kappa_1 \kappa_2 \sqrt{1 + \kappa_1^2}}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2}} \text{ and } \cos \theta_0 = \frac{\kappa_1'}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2}}.$$

(4) The tangent de Sitter subspace $S_1^3(\mathbf{v}, \varepsilon)$ has at least 5-point contact with $\boldsymbol{\gamma}(s)$ at s_0 if and only if there exist $\theta_0 \in [0, 2\pi)$ and $\varphi_0 \in [-\pi, \pi)$ such that

$$\mathbf{v} = \sin \theta_0 \cos \varphi_0 \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\boldsymbol{\gamma}(s_0) - \frac{1}{\kappa_1} \mathbf{n}_1(s_0)) + \cos \theta_0 \cos \varphi_0 \mathbf{n}_2(s_0) + \sin \varphi_0 \mathbf{n}_3(s_0),$$

$$(\kappa_1^2 \kappa_2^3 - \kappa_1'' \kappa_1 \kappa_2 + 2(\kappa_1')^2 \kappa_2 + \kappa_1 \kappa_1' \kappa_2') \cos \varphi_0 + \kappa_1 \kappa_2 \kappa_3 \sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2} \sin \varphi_0 = 0,$$

$$\sin \theta_0 = \frac{\kappa_1 \kappa_2 \sqrt{1 + \kappa_1^2}}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2}} \text{ and } \cos \theta_0 = \frac{\kappa_1'}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2}}.$$

(5) The tangent de Sitter subspace $S_1^3(\mathbf{v}, \varepsilon)$ has at least 6-point contact with $\boldsymbol{\gamma}(s)$ at s_0 if and only if there exist $\theta_0 \in [0, 2\pi)$ and $\varphi_0 \in [-\pi, \pi)$ such that

$$\mathbf{v} = \sin \theta_0 \cos \varphi_0 \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\boldsymbol{\gamma}(s_0) - \frac{1}{\kappa_1} \mathbf{n}_1(s_0)) + \cos \theta_0 \cos \varphi_0 \mathbf{n}_2(s_0) + \sin \varphi_0 \mathbf{n}_3(s_0),$$

$$(\kappa_1^2 \kappa_2^3 - \kappa_1'' \kappa_1 \kappa_2 + 2(\kappa_1')^2 \kappa_2 + \kappa_1 \kappa_1' \kappa_2') \cos \varphi_0 + \kappa_1 \kappa_2 \kappa_3 \sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2} \sin \varphi_0 = 0,$$

$$\sin \theta_0 = \frac{\kappa_1 \kappa_2 \sqrt{1 + \kappa_1^2}}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2}}, \cos \theta_0 = \frac{\kappa_1'}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2}} \text{ and } \sigma_3(s_0) = 0.$$

As a consequence, $\mathbf{E}_\gamma(s)$ is the locus of the polar vector $\mathbf{v} \in S_1^4$ such that $S_1^3(\mathbf{v}, \varepsilon)$ has at least 5-point contact with $\boldsymbol{\gamma}(s)$ at s_0 . We call $S_1^3(\mathbf{v}, \varepsilon)$ an *osculating de Sitter subspace* of $\boldsymbol{\gamma}(s)$ at s_0 if $\mathbf{v} = \mathbf{E}_\gamma(s_0)$. As a corollary of Theorem 4.3.5, Proposition 4.3.3 and 4.3.7, we have the following theorem.

Theorem 4.3.8. Let $\boldsymbol{\gamma} : I \rightarrow S_1^4$ be a timelike Sabban curve with $\rho(s) \neq 0$. Then we have the following:

- (1) If $\mathbf{v}_0 = \mathbf{FHS}(s_0, \theta_0, \varphi_0)$, then the tangent de Sitter subspace $S_1^3(\mathbf{v}_0, \varepsilon)$ and $\boldsymbol{\gamma}(s)$ have at least 3-point contact at s_0 .
- (2) The tangent de Sitter subspace $S_1^3(\mathbf{v}_0, \varepsilon)$ and $\boldsymbol{\gamma}(s)$ have 4-point contact at s_0 if and only if

there exist $\theta_0 \in [0, 2\pi)$ and $\varphi_0 \in [-\pi, \pi)$ such that

$$\mathbf{v} = \sin \theta_0 \cos \varphi_0 \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\boldsymbol{\gamma}(s_0) - \frac{1}{\kappa_1} \mathbf{n}_1(s_0)) + \cos \theta_0 \cos \varphi_0 \mathbf{n}_2(s_0) + \sin \varphi_0 \mathbf{n}_3(s_0),$$

$$(\kappa_1^2 \kappa_2^3 - \kappa_1'' \kappa_1 \kappa_2 + 2(\kappa_1')^2 \kappa_2 + \kappa_1 \kappa_1' \kappa_2') \cos \varphi_0 + \kappa_1 \kappa_2 \kappa_3 \sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2} \sin \varphi_0 \neq 0,$$

$\sin \theta_0 = \frac{\kappa_1 \kappa_2 \sqrt{1 + \kappa_1^2}}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2}}$ and $\cos \theta_0 = \frac{\kappa_1'}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2}}$. In this case, the germ of **FHS** at **FHS**(s_0, θ_0, φ_0) is diffeomorphic to $C \times \mathbb{R}^2$.

(3) The osculating de Sitter subspace $S_1^3(\mathbf{v}_0, \varepsilon)$ and $\boldsymbol{\gamma}(s)$ have 5-point contact at s_0 if and only if there exist $\theta_0 \in [0, 2\pi)$ and $\varphi_0 \in [-\pi, \pi)$ such that

$$\mathbf{v} = \sin \theta_0 \cos \varphi_0 \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\boldsymbol{\gamma}(s_0) - \frac{1}{\kappa_1} \mathbf{n}_1(s_0)) + \cos \theta_0 \cos \varphi_0 \mathbf{n}_2(s_0) + \sin \varphi_0 \mathbf{n}_3(s_0),$$

$$(\kappa_1^2 \kappa_2^3 - \kappa_1'' \kappa_1 \kappa_2 + 2(\kappa_1')^2 \kappa_2 + \kappa_1 \kappa_1' \kappa_2') \cos \varphi_0 + \kappa_1 \kappa_2 \kappa_3 \sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2} \sin \varphi_0 = 0,$$

$\sin \theta_0 = \frac{\kappa_1 \kappa_2 \sqrt{1 + \kappa_1^2}}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2}}$, $\cos \theta_0 = \frac{\kappa_1'}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2}}$ and $\sigma(s_0) \neq 0$. In this case, the germ of **FHS** at **FHS**(s_0, θ_0, φ_0) is diffeomorphic to $SW \times \mathbb{R}$.

(4) If the osculating de Sitter subspace $S_1^3(\mathbf{v}_0, \varepsilon)$ and $\boldsymbol{\gamma}(s)$ have 6-point contact at s_0 , then the germ of **FHS** at **FHS**(s_0, θ_0, φ_0) is diffeomorphic to BF .

Remark 4.3.9. We have a conjecture as follows:

The osculating de Sitter subspace $S_1^3(\mathbf{v}_0, \varepsilon)$ and $\boldsymbol{\gamma}(s)$ have 6-point contact at s_0 if and only if there exist $\theta_0 \in [0, 2\pi)$ and $\varphi_0 \in [-\pi, \pi)$ such that

$$\mathbf{v} = \sin \theta_0 \cos \varphi_0 \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\boldsymbol{\gamma}(s_0) - \frac{1}{\kappa_1} \mathbf{n}_1(s_0)) + \cos \theta_0 \cos \varphi_0 \mathbf{n}_2(s_0) + \sin \varphi_0 \mathbf{n}_3(s_0),$$

$$(\kappa_1^2 \kappa_2^3 - \kappa_1'' \kappa_1 \kappa_2 + 2(\kappa_1')^2 \kappa_2 + \kappa_1 \kappa_1' \kappa_2') \cos \varphi_0 + \kappa_1 \kappa_2 \kappa_3 \sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2} \sin \varphi_0 = 0,$$

$$\sin \theta_0 = \frac{\kappa_1 \kappa_2 \sqrt{1 + \kappa_1^2}}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2}}, \cos \theta_0 = \frac{\kappa_1'}{\sqrt{\kappa_1^2 \kappa_2^2 (1 + \kappa_1^2) + (\kappa_1')^2}}, \sigma_3(s_0) = 0 \text{ and } \sigma_3'(s_0) \neq 0.$$

4.4 Dualities for focal sets of timelike Sabban curves

Izumiya introduced the Legendrian dualities between pseudo-spheres in Minkowski space in [7].

In this section we study the Legendrian dualities of focal sets of timelike Sabban curves. We

define 1-forms $\langle d\mathbf{v}, \mathbf{w} \rangle = -w_1 dv_1 + \sum_{i=2}^{n+1} w_i dv_i$, $\langle \mathbf{v}, d\mathbf{w} \rangle = -v_1 dw_1 + \sum_{i=2}^{n+1} v_i dw_i$ in $\mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1}$ and consider the following double fibrations:

$$H^n(-1) \times S_1^n \supset \Delta_1 = \{(\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0\},$$

$$\pi_{11} : \Delta_1 \longrightarrow H^n(-1), \pi_{12} : \Delta_1 \longrightarrow S_1^n,$$

$$\theta_{11} = \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_1}, \theta_{12} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_1}.$$

Here, $\pi_{11}(\mathbf{v}, \mathbf{w}) = \mathbf{v}$, $\pi_{12}(\mathbf{v}, \mathbf{w}) = \mathbf{w}$ are the canonical projections. Moreover, $\theta_{11} = \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_1}$ and $\theta_{12} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_1}$ are the restrictions of the 1-form $\langle d\mathbf{v}, \mathbf{w} \rangle$ and $\langle \mathbf{v}, d\mathbf{w} \rangle$ on Δ_1 . We remark that $\theta_{11}^{-1}(0)$ and $\theta_{12}^{-1}(0)$ define the same tangent hyperplane field over Δ_1 . In [7], the following theorem was shown:

Theorem 4.4.1. Under the above notation, both of π_{1i} ($i = 1, 2$) are Legendrian fibrations and (Δ_1, K) is a contact manifold.

If we have a Legendrian submanifold $L \subset \Delta_1$, then we say that $\pi_{11}(L) \subset H^n(-1)$ and $\pi_{12}(L) \subset S_1^n$ are Δ_1 -dual to each other. Since both of $\pi_{11}^{-1}(\mathbf{v}) = \mathbf{v} \times S_1^{n-1}(\mathbf{v}, 0)$ and $\pi_{12}^{-1}(\mathbf{w}) = H^n(-1)(0, \mathbf{w}) \times \mathbf{w}$ are Legendrian submanifolds of Δ_1 . We consider timelike Sabban curve $\gamma : I \rightarrow S_1^n$ ($n = 2, 3, 4$). The de Sitter evolute $\mathbf{E}_\gamma^\pm(s)$ of a timelike Sabban curve $\gamma : I \rightarrow S_1^2$, which is the *focal set* of $\gamma(s)$. Since a timelike Sabban curve γ is a unit speed timelike curve, we have the hyperbolic tangent indicatrix $\mathbf{t} : I \rightarrow H^n(-1)$ of γ . Then we have the following theorem:

Theorem 4.4.2. Let $\gamma : I \rightarrow S_1^2$ be a timelike Sabban curve. Then the hyperbolic tangent indicatrix $\mathbf{t}$ and de Sitter evolute $\mathbf{E}_\gamma^\pm$ are Δ_1 -dual to each other.

Proof. We define a mapping $\mathcal{L} : I \rightarrow \Delta_1$ by $\mathcal{L}(s) = (\mathbf{t}(s), \mathbf{E}_\gamma^\pm(s))$. By definition, we have

$$\frac{\partial \mathcal{L}}{\partial s} = \left(\gamma(s) + \kappa \mathbf{n}(s), \frac{\partial \mathbf{E}_\gamma^\pm}{\partial s} \right).$$

Since $\{\gamma, \mathbf{n}\}$ are linearly indepent, $\partial \mathcal{L} / \partial s$ can't be zero. This means that $\mathcal{L} : I \rightarrow \Delta_1$ is an immersion, so that $\mathcal{L}|_I$ is an immersion. Moreover, we have $\mathcal{L}^* \theta_{11} = \langle d\mathbf{t}(s), \mathbf{E}_\gamma^\pm(s) \rangle = 0$, so that $\mathcal{L}(s)$ is a Legendrian submanifold of Δ_1 . This completes the proof. $\square$

If we consider timelike Sabban curves in S_1^3 and S_1^4 , then the focal sets of the curve γ are de Sitter focal surfaces $\mathbf{FS}(s, \theta)$ and de Sitter focal hypersurfaces $\mathbf{FHS}(s, \theta, \varphi)$ respectively. Then we have the following theorems:

Theorem 4.4.3. Let $\gamma : I \rightarrow S_1^3$ be a timelike Sabban curve. Then the hyperbolic tangent indicatrix $\mathbf{t}$ and de Sitter focal surface $\mathbf{FS}(s, \theta)$ are Δ_1 -dual to each other.

Proof. We define a mapping $\mathcal{L} : I \times [0, 2\pi) \rightarrow \Delta_1$ by $\mathcal{L}(s) = (\mathbf{t}(s), \mathbf{FS}(s, \theta))$. By definition, we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial s} &= \left(\gamma(s) + \kappa_1 \mathbf{n}_1(s), \left(\frac{\kappa_1' \cos \theta}{\kappa_1 \sqrt{1 + \kappa_1^2}} - \frac{\kappa_1' \cos \theta}{\kappa_1 (1 + \kappa_1^2) \sqrt{1 + \kappa_1^2}} - \kappa_2 \sin \theta \right) \mathbf{n}_1(s) \right. \\ &\quad \left. + \frac{\kappa_1' \cos \theta}{(1 + \kappa_1^2) \sqrt{1 + \kappa_1^2}} \gamma(s) - \frac{\kappa_2 \cos \theta}{\sqrt{1 + \kappa_1^2}} \mathbf{n}_2(s) \right), \\ \frac{\partial \mathcal{L}}{\partial \theta} &= \left(0, -\frac{\kappa_1 \sin \theta}{\sqrt{1 + \kappa_1^2}} \gamma(s) + \frac{\sin \theta}{\sqrt{1 + \kappa_1^2}} \mathbf{n}_1(s) + \cos \theta \mathbf{n}_2(s) \right). \end{aligned}$$

Since $\{\gamma, \mathbf{n}_1\}$ are linearly indepent and $\partial \mathcal{L} / \partial \theta \neq 0$, so $\{\partial \mathcal{L} / \partial s, \partial \mathcal{L} / \partial \theta\}$ are linearly indepent. This means that $\mathcal{L} : I \times [0, 2\pi) \rightarrow \Delta_1$ is an immersion, so that $\mathcal{L}|_{I \times [0, 2\pi)}$ is an immersion. Moreover, we have

$$\mathcal{L}^* \theta_{11} = \langle d\mathbf{t}(s), \mathbf{FS}(s, \theta) \rangle = \langle \gamma(s) + \kappa_1 \mathbf{n}_1(s), \cos \theta \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\gamma(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) + \sin \theta \mathbf{n}_2(s) \rangle = 0,$$

so that $\mathcal{L}(s, \theta)$ is a Legendrian submanifold of Δ_1 . This completes the proof. $\square$

Theorem 4.4.4. Let $\gamma : I \rightarrow S_1^4$ be a timelike Sabban curve. Then the hyperbolic tangent indicatrix $\mathbf{t}$ and de Sitter focal hypersurface $\mathbf{FHS}(s, \theta, \varphi)$ are Δ_1 -dual to each other.

Proof. We define a mapping $\mathcal{L} : I \times [0, 2\pi) \times [-\pi, \pi) \rightarrow \Delta_1$ by $\mathcal{L}(s) = (\mathbf{t}(s), \mathbf{FHS}(s, \theta, \varphi))$. By definition, we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial s} &= \left(\gamma(s) + \kappa_1 \mathbf{n}_1(s), \left(\frac{\kappa_1 \kappa_1' \sin \theta \cos \varphi}{(1 + \kappa_1^2) \sqrt{1 + \kappa_1^2}} - \kappa_2 \cos \theta \cos \varphi \right) \mathbf{n}_1(s) + \frac{\kappa_1' \sin \theta \cos \varphi}{(1 + \kappa_1^2) \sqrt{1 + \kappa_1^2}} \gamma(s) \right. \\ &\quad \left. - \left(\frac{\kappa_2 \sin \theta \cos \varphi}{\sqrt{1 + \kappa_1^2}} + \sin \theta \kappa_3 \right) \mathbf{n}_2(s) + \frac{\kappa_1' \kappa_3 \sin \theta \cos \varphi}{\kappa_1 \kappa_2 \sqrt{1 + \kappa_1^2}} \mathbf{n}_3(s) \right), \\ \frac{\partial \mathcal{L}}{\partial \theta} &= \left(0, \frac{\kappa_1 \cos \theta \cos \varphi}{\sqrt{1 + \kappa_1^2}} \gamma(s) - \frac{\cos \theta \cos \varphi}{\sqrt{1 + \kappa_1^2}} \mathbf{n}_1(s) - \sin \theta \cos \varphi \mathbf{n}_2(s) \right), \\ \frac{\partial \mathcal{L}}{\partial \varphi} &= \left(0, -\frac{\kappa_1 \sin \theta \sin \varphi}{\sqrt{1 + \kappa_1^2}} \gamma(s) + \frac{\sin \theta \sin \varphi}{\sqrt{1 + \kappa_1^2}} \mathbf{n}_1(s) - \cos \theta \sin \varphi \mathbf{n}_2(s) + \cos \varphi \mathbf{n}_3(s) \right). \end{aligned}$$

Since $\{\boldsymbol{\gamma}, \mathbf{n}_1\}$ are linearly indepent and $\{\partial\mathcal{L}/\partial\theta, \partial\mathcal{L}/\partial\varphi\}$ are linearly indepent, so $\{\partial\mathcal{L}/\partial s, \partial\mathcal{L}/\partial\theta, \partial\mathcal{L}/\partial\varphi\}$ are linearly indepent. This means that $\mathcal{L} : I \times [0, 2\pi) \times [-\pi, \pi) \rightarrow \Delta_1$ is an immersion, so that $\mathcal{L}|_{I \times [0, 2\pi) \times [-\pi, \pi)}$ is an immersion. Moreover, we have

$$\begin{aligned}\mathcal{L}^*\theta_{11} &= \langle d\mathbf{t}(s), \mathbf{FHS}(s, \theta, \varphi) \rangle \\ &= \langle \boldsymbol{\gamma}(s) + \kappa_1 \mathbf{n}_1(s), \sin \theta \cos \varphi \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} (\boldsymbol{\gamma}(s) - \frac{1}{\kappa_1} \mathbf{n}_1(s)) \\ &\quad + \cos \theta \cos \varphi \mathbf{n}_2(s) + \sin \varphi \mathbf{n}_3(s) \rangle = 0,\end{aligned}$$

so that $\mathcal{L}(s, \theta, \varphi)$ is a Legendrian submanifold of Δ_1 . This completes the proof. $\square$

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