



HOKKAIDO UNIVERSITY

Title	Theory of solvability of generalized Hamiltonian systems and its application to sub-Riemmanian geometry
Author(s)	土田, 旭
Degree Grantor	北海道大学
Degree Name	博士(理学)
Dissertation Number	甲第12868号
Issue Date	2017-09-25
DOI	https://doi.org/10.14943/doctoral.k12868
Doc URL	https://hdl.handle.net/2115/67394
Type	doctoral thesis
File Information	Asahi_Tsuchida.pdf



博士学位論文

Theory of solvability of generalized Hamiltonian systems
and its application to sub-Riemmanian geometry
(一般化ハミルトン系の可解性の理論と
そのサブリーマン幾何学への応用)

土田 旭

北海道大学大学院理学院

数学専攻

2017 年 9 月

Theory of solvability of generalized Hamiltonian systems and its application to sub-Riemmanian geometry

Asahi TSUCHIDA

Contents

1	Introduction	2
2	Implicit differential systems and their solvability	4
2.1	Implicit differential systems	4
2.2	Implicit Hamiltonian systems	10
2.3	Smooth solvability of generalized Hamiltonian systems and their submanifolds	18
3	Control systems	24
3.1	Control systems	24
3.2	Affine control systems	25
3.3	Singular controls	26
3.4	Controllability	28
3.5	Sub-Riemmanian geometry	28
3.6	Normal extremals and abnormal extremals	31
4	Applications to the study of singular curves on sub-Riemannian manifolds	34
4.1	The case the growth vector $(2, 3, 3, \dots)$	37
4.2	The case the growth vector $(2, 3, 4, \dots)$	37

1 Introduction

The concept of generalized Hamiltonian systems is introduced to mathematics by P.A.M. Dirac in 1950 [8]. A generalized Hamiltonian system is a particular type of an implicit differential system which is defined on a symplectic manifold (see section 2.2 in detail). An implicit differential system is a subset S of the tangent bundle TM on a smooth manifold M . A solution of S is a curve $\gamma: I \rightarrow M$ on M of class C^1 which has a canonical lift by differentiation, i.e., $(\gamma(t), \frac{d\gamma}{dt}(t))$ belongs to S for every $t \in I$. For a first order ordinary differential equation defined on a smooth manifold M , we may define an implicit differential system S of the graph of the vector field X on M which corresponds to the ordinary differential equation. In this case, a solution of S is an integral curve of X . Thus the concept of implicit differential systems is one of a generalization of ordinary differential equations. There arises several natural questions those are about the existence of a (local) solution, uniqueness of the solution of the Cauchy problem, smoothness of solutions, a range of the largest domain of prolongation of solutions, and so on.

We consider the existence and smoothness of solutions of implicit differential systems, especially, of generalized Hamiltonian systems. For a symplectic manifold (M, ω) , there is a natural symplectic form $\dot{\omega}$ on the tangent bundle TM which is induced by the Liouville 1-form on the cotangent bundle T^*M by using an interior product. An implicit Hamiltonian system is defined as a Lagrangian submanifold of $(TM, \dot{\omega})$. A generalized Hamiltonian systems is a particular type of an implicit Hamiltonian system which is generated by a Morse family of an affine functions related to interior variables. If the Morse family is homogeneous function for interior variables, the generalized Hamiltonian system corresponds to a co-normal bundle in the cotangent bundle. We study their solvability and smoothness of solutions of generalized Hamiltonian systems and their isotropic submanifolds, according to Fukuda and Janeczko's papers[9], [13], [10] and [12].

In this paper, the main interest is to study smooth solvability of generalized Hamiltonian systems and of their subsystems. Our aim is to study abnormal extremals of sub-Riemannian manifolds as an application.

First, we present a variety of useful sufficient conditions for smooth solvability of generalized Hamiltonian systems and its variants. Then by using these conditions, we explore a particularly interesting topic in control theory. The topic is about singular curves on control systems, especially abnormal

curves on sub-Riemannian manifolds.

A control system is, locally, a family of sections of tangent bundle. In a control system, an associated implicit Hamiltonian system takes an important role : it characterizes singular controls, which are defined by singular points of an end-point mapping (see section 3.3 in detail). If a control system is locally expressed by linear family of sections, then an associated implicit Hamiltonian system is a generalized Hamiltonian system.

In sub-Riemannian geometry, every minimizer is an extremal. There are two types of extremals on a sub-Riemannian manifold $(M, \mathcal{D}, g)$; normal extremal and abnormal extremal. Normal extremals are defined by geodesic equations on $(M, \mathcal{D}, g)$. It is known that normal extremals are minimizers. A natural question arises to ask whether any extremal is a local minimizer or not. This question is negatively solved by R. Montgomery by giving an example of a local minimizer which is an abnormal extremal and is not a normal extremal. [20]. But not much is known about abnormal extremals and abnormal minimizers so far. Then, we give several new examples of abnormal curves on a sub-Riemannian manifold with rank 2 distribution by applying our conditions for smooth solvability of generalized Hamiltonian systems and its subsystems.

The thesis is organized as follows.

In Section 2, we give a summary of the basics in the theory of implicit differential systems and especially of generalized Hamiltonian systems. We generalize preceding works given by Fukuda and Janeczko in [13][9]. It is stated that sufficient conditions of smooth solvability of sub systems of a generalized Hamiltonian system which correspond to co-normal bundle of codimension two submanifolds in the cotangent bundle T^*M . In Section 3, we review the theory of geometric control theory and sub-Riemannian geometry[1][19]. Extremals on sub-Riemannian manifolds are explained. In Section 4, we apply the results obtained in Section 2 to the study of abnormal curves of sub-Riemannian manifolds and we give sufficient conditions for existence of smooth families of abnormal bi-extremals for a rank two distribution of a certain type.

Throughout this thesis, we assume, as smoothness, the differentiability of class C^∞ .

2 Implicit differential systems and their solvability

We refer to mainly Fukuda and Janeczko's papers [9][13] in this section. An implicit differential system is a subset S of the tangent bundle TM of a smooth manifold M . The theory of implicit differential systems is one of a generalization of ordinary differential equations. Thus we have similar type of questions for implicit differential systems to ordinary differential equations. For example, existence of solution of S , uniqueness of solution of Cauchy problem and prolongation problem for solutions, etc. We consider first two problems according to Fukuda and Janeczko's results for general implicit differential systems and for implicit Hamiltonian systems which are defined on a symplectic manifold.

2.1 Implicit differential systems

We recall the concept, definitions and theorems of implicit differential systems.

Definition 2.1. Let M be a C^∞ manifold. A subset S of tangent bundle TM is called an *implicit differential system*.

An element of S can be expressed by a pair (x, v) of a point x and a tangent vector v at x .

Definition 2.2 ([9]). A *solution* of implicit differential system S is a C^1 curve $\gamma: (a, b) \rightarrow M$ such that

$$(\gamma(t), \frac{d\gamma}{dt}(t)) \in S$$

for all t in (a, b) .

A point $(x_0, \dot{x}_0)$ of S is called a *solvable point* if there exists a positive number ε and C^1 curve $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$ such that

$$(\gamma(0), \frac{d\gamma}{dt}(0)) = (x_0, \dot{x}_0), \quad (\gamma(t), \frac{d\gamma}{dt}(t)) \in S \text{ for all } t \in (-\varepsilon, \varepsilon).$$

We assume that a subset S of TM is a smooth submanifold to define smooth solvability. Let $Proj_2: S \times \mathbb{R} \rightarrow \mathbb{R}$ be a canonical projection.

Definition 2.3 ([9]). A point $(x_0, \dot{x}_0)$ of S is called a *smoothly solvable point* if there exists an open neighborhood W of $(x_0, \dot{x}_0, 0)$ in $S \times \mathbb{R}$ and a smooth curve $\bar{\gamma}: W \rightarrow M$ such that for $\gamma_{(x_0, \dot{x}_0)}(t) := \bar{\gamma}(x_0, \dot{x}_0, t)$,

$$\begin{aligned} &(\gamma_{(x_0, \dot{x}_0)}(0), \frac{d\gamma_{(x_0, \dot{x}_0)}}{dt}(0)) = (x_0, \dot{x}_0), \\ &(\gamma_{(x_0, \dot{x}_0)}(t), \frac{d\gamma_{(x_0, \dot{x}_0)}}{dt}(t)) \in S \text{ for all } t \in Proj_2(W). \end{aligned}$$

Here we define solutions over a submanifold $N \subset M$.

Definition 2.4 ([26]). A *solution of implicit differential system S over N* is a C^1 curve $\gamma: (a, b) \rightarrow M$ such that

$$\gamma(t) \in N, \quad (\gamma(t), \frac{d\gamma}{dt}(t)) \in S$$

for all t in (a, b) .

A point $(x_0, \dot{x}_0)$ of S is called a *solvable point of S over N* if there exists a positive number ε and a solution $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$ over N such that

$$(\gamma(0), \frac{d\gamma}{dt}(0)) = (x_0, \dot{x}_0).$$

Definition 2.5 ([26]). A point $(x_0, \dot{x}_0)$ of S is called a *smoothly solvable point over N* if there exists an open neighborhood W of $(x_0, \dot{x}_0, 0)$ in $S \times \mathbb{R}$ and a smooth curve $\bar{\gamma}: W \rightarrow M$ such that for $\gamma_{(x_0, \dot{x}_0)}(t) := \bar{\gamma}(x_0, \dot{x}_0, t)$,

$$\begin{aligned} &\gamma_{(x_0, \dot{x}_0)}(t) \in N, \quad (\gamma_{(x_0, \dot{x}_0)}(0), \frac{d\gamma_{(x_0, \dot{x}_0)}}{dt}(0)) = (x_0, \dot{x}_0), \\ &(\gamma_{(x_0, \dot{x}_0)}(t), \frac{d\gamma_{(x_0, \dot{x}_0)}}{dt}(t)) \in S \text{ for all } t \in Proj_2(W). \end{aligned}$$

Definition 2.6 ([26]). An implicit differential system S over N is called a *(smoothly) solvable submanifold over N* if S consists only of (smoothly) solvable points of S over N .

When a submanifold N is M itself, definitions above are given in Fukuda and Janeczko's papers [9][13]. A solution of S over M is just called a *solution of S* . We say, simply, an implicit differential system S is *(smoothly) solvable* if S is (smoothly) solvable over M .

Here we introduce some sufficient conditions and necessary conditions for implicit differential systems to be smoothly solvable. Let $\pi: TM \rightarrow M$ denote the tangent bundle projection. The following is a necessary condition which is called tangential solvability condition.

Theorem 2.7 ([9]). If $(x_0, \dot{x}_0) \in S$ is a solvable point of S , then

$$\dot{x} \in d(\pi|_S)_{(x_0, \dot{x}_0)}(T_{(x_0, \dot{x}_0)}S).$$

Proof. From the assumption, there exists a smooth curve $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$ such that $(\gamma(0), \dot{\gamma}(0)) = (x_0, \dot{x}_0)$ and $(\gamma(t), \dot{\gamma}(t)) \in S$ for all $t \in (-\varepsilon, \varepsilon)$. Then we have

$$\begin{aligned} \dot{x}_0 = \dot{\gamma}(0) &= \frac{d(\pi(\gamma, \dot{\gamma}))}{dt}(0) \\ &= d(\pi|_S)_{(x_0, \dot{x}_0)}(\dot{\gamma}(0), \ddot{\gamma}(0)) \in d(\pi|_S)_{(x_0, \dot{x}_0)}(T_{(x_0, \dot{x}_0)}S) \end{aligned}$$

□

Theorem 2.8 ([13]). A point $(x_0, \dot{x}_0) \in S$ is a smoothly solvable point of S if and only if for a small neighborhood U of $(x_0, \dot{x}_0)$ there exists a smooth vector field along U of the form

$$X = \sum_{i=1}^n \dot{x}_i \frac{\partial}{\partial x_i} + f_i(x, \dot{x}) \frac{\partial}{\partial \dot{x}_i}, \quad \text{for } (x, \dot{x}) \in U$$

which is tangent to U .

Proof. Suppose that there exists a smooth vector field X of the form on a small neighborhood U of $(x_0, \dot{x}_0)$. We have a one parameter group $\gamma: U \times (-\varepsilon, \varepsilon) \rightarrow M$ of diffeomorphism generated by X . Then the curves

$$\gamma_{(x, \dot{x})}(t) := \gamma(x, \dot{x}, t)$$

are integral curves of X with $\gamma_{(x, \dot{x})}(0) = (x, \dot{x})$ and the curves

$$\bar{\gamma}_{(x, \dot{x})} := \pi \circ \gamma_{(x, \dot{x})}: (-\varepsilon, \varepsilon) \rightarrow M$$

are solution of S satisfying the initial condition

$$(\bar{\gamma}_{(x, \dot{x})}(0), \frac{\bar{\gamma}_{(x, \dot{x})}}{dt}(0)) = (x, \dot{x}).$$

Thus existence of family $\{\bar{\gamma}_{(x,\dot{x})}\}_{(x,\dot{x}) \in U}$ of solutions of S shows smooth solvability of $(x_0, \dot{x}_0)$.

Conversely, suppose that $(x_0, \dot{x}_0)$ is a smoothly solvable point of S . Then there exist a small neighborhood of U of $(x_0, \dot{x}_0)$ in S and a family

$$\bar{\gamma}_{(x,\dot{x})}: (-\varepsilon, \varepsilon) \rightarrow M$$

of solutions of S smoothly depending on $(x, \dot{x}) \in U$ such that

$$(\bar{\gamma}_{(x,\dot{x})}(0), \frac{d\bar{\gamma}_{(x,\dot{x})}}{dt}(0)) = (x, \dot{x}).$$

By considering differentiation of $(\bar{\gamma}_{(x,\dot{x})}(t), \frac{d\bar{\gamma}_{(x,\dot{x})}}{dt}(t))$ at zero, we have the vector field

$$X = \sum_{i=1}^n \frac{d\bar{\gamma}_{(x,\dot{x})}}{dt}(0) \frac{\partial}{\partial x_i} + \frac{d^2\bar{\gamma}_{(x,\dot{x})}}{dt^2}(0) \frac{\partial}{\partial \dot{x}_i} = \sum_{i=1}^n \dot{x}_i \frac{\partial}{\partial x_i} + \frac{d^2\bar{\gamma}_{(x,\dot{x})}}{dt^2}(0) \frac{\partial}{\partial \dot{x}_i}$$

which is tangent to U and have the form appeared in the theorem. $\square$

The idea of the Theorem plays a fundamental role in considering smooth solvability of implicit differential systems.

There is a solvability condition given by a linear equation for an implicit differential system which expressed by an embedding. Let O be a m dimensional smooth manifold. Let $f: O \rightarrow M$ and $g: O \rightarrow TM$ be embeddings which satisfy

$$f(y) = (f_1(y), \dots, f_n(y)) \in M, \quad (g_1(y), \dots, g_n(y)) \in T_{f(y)}M$$

with coordinates $y = (y_1, \dots, y_k)$ of O . We may define an embedding

$$(f, g): O \rightarrow TM, \quad (f, g)(y) = (f(y), g(y))$$

and consider the case S is given by $S = (f, g)(O)$. Tangential solvability is rewritten by the following condition

$$g(y) \in Jf(y)(T_y O), \quad \text{for all } y \in O.$$

Thus we have a necessary condition as following

Theorem 2.9 ([9]). Let $S = (f, g)(O)$ is a submanifold of TM . If S is solvable at $q \in S$, then the linear equation

$$\begin{pmatrix} g_1(y) \\ \vdots \\ g_n(y) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(y) & \cdots & \frac{\partial f_1}{\partial y_n}(y) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1}(y) & \cdots & \frac{\partial f_n}{\partial y_n}(y) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$$

has a solution $a = (a_1, \dots, a_m) \in T_y O$ at y which satisfies $(f, g)(y) = q$.

From the sufficient condition before, we have another sufficient condition for $S = (f, g)(O)$ to be smoothly solvable.

Theorem 2.10 ([9]). Let $S = (f, g)(O)$ is a submanifold of TM . The implicit differential system S is smoothly solvable if there exist smooth functions $a_1(y), \dots, a_m(y)$ and they satisfy the following linear equation

$$\begin{pmatrix} g_1(y) \\ \vdots \\ g_n(y) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(y) & \cdots & \frac{\partial f_1}{\partial y_n}(y) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1}(y) & \cdots & \frac{\partial f_n}{\partial y_n}(y) \end{pmatrix} \begin{pmatrix} a_1(y) \\ \vdots \\ a_m(y) \end{pmatrix}.$$

Proof. Let $a_1(y), \dots, a_m(y)$ be a smooth solution of the linear equation and

$$X(y) = \sum_{i=1}^m a_i(y) \frac{\partial}{\partial y_i}$$

a smooth vector field on O . Let

$$\phi = O \times I \rightarrow O$$

be the local 1-parameter group of diffeomorphisms generated by X where $I \subset \mathbb{R}$ is an open sub interval of $\mathbb{R}$. Then we see that

$$S = \{(f(y), Jf(X(y))) \mid y \in O\}$$

and the family $\{\gamma_y(t)\}_{y \in O}$ of curves on M defined by

$$\gamma_y(t) = f(\phi(y, t))$$

is a family of solution of the implicit differential equation S , which is smoothly depending on the initial condition

$$(\gamma_y(0) = f(y), \frac{d\gamma_y}{dt}(0) = g(y)).$$

Thus S is smoothly solvable. □

The concept of germ may be useful when we consider local properties of S , for example, smooth solvability of S . We here recall the concept of germs.

Definition 2.11. Let M be a topological space and $X, Y \subset M$ be subspaces of M . We say that two subspaces X and Y define a same set germ at $x_0 \in M$ if there exists an open set $U \in M$ such that

$$X \cap U = Y \cap U.$$

This is an equivalence relation among subspaces in a topological space. An equivalence class including X is called a set germ of X at x_0 . We have a similar concept for maps on a topological space.

Definition 2.12. Let N, P be topological spaces and let U, V be open neighborhoods of $x_0 \in N$. Two maps $f: U \rightarrow P$ and $g: V \rightarrow P$ are said to define a same map germ if there exists an open neighborhood W of x_0 such that

$$f|_W = g|_W.$$

This relation is also an equivalence relation among maps. An equivalence class including f is called a map germ of f at x_0 .

We repeat a definition of smooth solvability for implicit differential systems as follows.

Definition 2.2 (again). A point $q_0 \in S$ is called a smoothly solvable point if there exists a smooth map germ

$$[\bar{\gamma}]: (S \times \mathbb{R}, (q_0, 0)) \rightarrow M$$

at $(q_0, 0)$ such that for any representative $\bar{\gamma}: W \rightarrow M$ of $[\bar{\gamma}]$, by setting $\gamma_{q_0}(t) := \bar{\gamma}(q_0, t)$, it holds that

$$(\gamma_{q_0}(0), \frac{d\gamma_{q_0}}{dt}(0)) = q_0, \quad (\gamma_{q_0}(t), \frac{d\gamma_{q_0}}{dt}(t)) \in S \text{ for all } t \in \text{Proj}_2(W).$$

Since implicit differential systems are submanifolds of tangent bundle TM , for an implicit differential system S there exist a smooth manifold N and an embedding $F: N \rightarrow TM$ such that $S = F(N)$. To consider a submanifold germ of tangent bundle TM at q_0 is similar to consider an embedding germ $F: (N, F^{-1}(q_0)) \rightarrow (TM, q_0)$. The embedding germ is also regarded as an embedding germ $(\mathbb{R}^m, F^{-1}(q_0)) \mapsto (\mathbb{R}^n \times \mathbb{R}^n, q_0)$ where $\dim N = m$. In following section, we consider smooth solvability of germs of implicit differential systems at any fixed point and we often omit the point from notations.

2.2 Implicit Hamiltonian systems

Let (M, ω) be a symplectic manifold. Then there is the induced symplectic structure $\dot{\omega}$ on the tangent bundle TM : We have a bundle isomorphism induced from interior product $\flat: TM \rightarrow T^*M, \flat_x(v_q) = \iota_{v_q}\omega_q$ for each point $q \in M$. The symplectic structure $\dot{\omega}$ is given by the pullback of the Liouville form θ on T^*M , i.e., $\dot{\omega} := \flat^*d\theta$. The induced symplectic structure $\dot{\omega}$ is locally written by

$$\dot{\omega} = \sum_{i=1}^n d\dot{p}_i \wedge dx_i - d\dot{x}_i \wedge dp_i$$

with the canonical coordinates $(x, p, \dot{x}, \dot{p})$ of tangent bundle TM related to Darboux coordinates $(x, p) = (x_1, \dots, x_n, p_1, \dots, p_n)$ for the standard symplectic form $\omega = \sum_{i=1}^n dp_i \wedge dx_i$ of M .

We will define the notion of implicit Hamiltonian systems as Lagrangian submanifolds of $(TM, \dot{\omega})$. Then they are regarded as a generalization of Hamiltonian vector fields, i.e., of Hamiltonian dynamical systems. In what follows we set $M = \mathbb{R}^{2n}$ with the standard symplectic form ω as above.

Definition 2.12 ([9, 13]). A Lagrangian submanifold L of $(T\mathbb{R}^{2n}, \dot{\omega})$ (i.e., $\dim L = 2n$ and $\omega|_L = 0$) is called an *implicit Hamiltonian system*.

There is a well-known result that a Lagrangian submanifold is locally generated by a Morse family;

Theorem 2.13 ([2]). Let L be a Lagrangian submanifold of $T\mathbb{R}^{2n}$ and $(q_0, \dot{q}_0) = (x_0, p_0, \dot{x}_0, \dot{p}_0) \in L$. Suppose

$$\text{corank } d(\pi|_L)(q_0, \dot{q}_0) = k > 0.$$

Then there exist an open neighborhood O of $(q_0, \dot{q}_0)$ in $T\mathbb{R}^{2n}$, an open neighborhood W of $(q_0, 0) \in \mathbb{R}^{2n} \times \mathbb{R}^k$ and a smooth function $F: W \rightarrow \mathbb{R}$ such that

$$L \cap O = \left\{ (x_0, p_0, \dot{x}_0, \dot{p}_0) \in O \mid \exists u \in \mathbb{R}^k \text{ s.t. } (x, p, u) \in W, \frac{\partial F}{\partial u_l}(x, p, u) = 0, \right. \\ \left. \dot{x}_i = \frac{\partial F}{\partial p_i}(x, p, u), \dot{p}_i = -\frac{\partial F}{\partial x_i}(x, p, u), 1 \leq i \leq n, 1 \leq l \leq k \right\},$$

and that

$$\text{rank} \left(\frac{\partial^2 F}{\partial x_i \partial u_l}(q_0, 0), \frac{\partial^2 F}{\partial p_i \partial u_l}(q_0, 0) \right)_{1 \leq i \leq n, 1 \leq l \leq k} = k, \frac{\partial^2 F}{\partial u_r \partial u_s}(q_0, 0) = 0$$

for $1 \leq r, s \leq k$.

Recall that the family of functions $F: \mathbb{R}^{2n} \times \mathbb{R}^k \rightarrow \mathbb{R}$ with $2n$ parameters $(x_1, \dots, x_n, p_1, \dots, p_n)$ on $(\mathbb{R}^k; u_1, \dots, u_k)$ is called a *Morse family* if $0 \in \mathbb{R}^k$ is a critical point of the map $F(q_0, u)$ and the map

$$\left(\frac{\partial F}{\partial u_1}, \dots, \frac{\partial F}{\partial u_k} \right) : \mathbb{R}^{2n} \times \mathbb{R}^k \rightarrow \mathbb{R}^k$$

is submersive at $(q_0, 0)$. We denote L_F a Lagrangian submanifold generated by Morse family $F: \mathbb{R}^{2n} \times \mathbb{R}^k \rightarrow \mathbb{R}$. That is, for the catastrophe set

$$C(F) = \left\{ (x, p, u) \in \mathbb{R}^{2n} \times \mathbb{R}^k \mid \frac{\partial F}{\partial u_i}(x, p, u) = 0, i = 1, \dots, k \right\}$$

of F and C^∞ map $\phi_F: \mathbb{R}^{2n} \times \mathbb{R}^k \rightarrow T\mathbb{R}^{2n}$ defined by

$$\phi(x, p, u) = (x, p, \frac{\partial F}{\partial p_i}(x, p, u), -\frac{\partial F}{\partial x_i}(x, p, u)),$$

we set $L_F = \phi_F(C(F))$.

The following propositions are given in [9] which are a necessary condition and a sufficient condition for L_F to be (smoothly) solvable in the sense of Definition 2.6.

Proposition 2.14 ([9]). Let $(x, p, \dot{x}, \dot{p})$ be a solvable point of L_F . We set $\phi_F(x, p, u) = (x, p, \dot{x}, \dot{p})$. Then there exists a real vector $\mu = (\mu_1, \dots, \mu_k)$ in $\mathbb{R}^k$ such that

$$\begin{pmatrix} \frac{\partial^2 F}{\partial u_1 \partial u_1}(x, p, u) & \cdots & \frac{\partial^2 F}{\partial u_1 \partial u_k}(x, p, u) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial u_k \partial u_1}(x, p, u) & \cdots & \frac{\partial^2 F}{\partial u_k \partial u_k}(x, p, u) \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix} = \begin{pmatrix} \left\{ \frac{\partial F}{\partial u_1}, F \right\}(x, p, u) \\ \vdots \\ \left\{ \frac{\partial F}{\partial u_k}, F \right\}(x, p, u) \end{pmatrix}.$$

Here the bracket $\{, \}$ is the Poisson bracket associated to the symplectic form ω .

Proposition 2.14 is proved to show an equivalence condition to tangential solvability condition for L_F .

Proof of Proposition 2.14. Let $\Pi: \mathbb{R}^{2n} \times \mathbb{R}^k \rightarrow \mathbb{R}^{2n}$ be a canonical projection, a point $(x, p, u) \in \mathbb{R}^{2n} \times \mathbb{R}^k$ be a point which satisfies $\phi(x, p, u) = (x, p, \dot{x}, \dot{p})$. A point $(x, p, \dot{x}, \dot{p})$ is a solvable point of L_F , thus L_F satisfies tangential solvability condition at $(x, p, \dot{x}, \dot{p})$. By definition, we have

$$(\dot{x}, \dot{p}) = \left(\frac{\partial F}{\partial p_i}(x, p, u), -\frac{\partial F}{\partial x_i}(x, p, u) \right) \in d\pi(T_{(x, p, \dot{x}, \dot{p})}L_F) = d\Pi(T_{(x, p, u)}C(F)).$$

Then there exists a tangent vector $v \in T_{(x, p, u)}C(F)$ such that $(\dot{x}, \dot{p}) = d\Pi(v)$. Since the tangent space $T_{(x, p, u)}C(F)$ is a subspace of $T_{(x, p)}\mathbb{R}^{2n} \times T_u\mathbb{R}^k$, the condition of existence of tangent vector v is reduced to existence of a tangent vector $\mu \in T_u\mathbb{R}^k$ such that

$$v = \sum_{i=1}^n \left(\dot{x}_i \frac{\partial}{\partial x_i} + \dot{p}_i \frac{\partial}{\partial p_i} \right) + \sum_{i=1}^k \left(\mu_i \frac{\partial}{\partial u_i} \right) \quad (*)$$

The condition which vector v is tangent to $C(F)$ implies

$$v \left(\frac{\partial F}{\partial u_j} \right) (x, p, u) = 0, \quad j = 1, \dots, k$$

By submitting $(*)$ to above equation, we obtain

$$\left(\frac{\partial^2 F}{\partial u_l \partial u_m} \right)_{1 \leq l \leq k, 1 \leq m \leq k} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix} + \begin{pmatrix} \sum_{i=1}^n \left(\dot{x}_i \frac{\partial^2 F}{\partial x_i \partial u_1} + \dot{p}_i \frac{\partial^2 F}{\partial p_i \partial u_1} \right) \\ \vdots \\ \sum_{i=1}^n \left(\dot{x}_i \frac{\partial^2 F}{\partial x_i \partial u_k} + \dot{p}_i \frac{\partial^2 F}{\partial p_i \partial u_k} \right) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $(\dot{x}, \dot{p}) = \left(\frac{\partial F}{\partial p_i}(x, p, u), -\frac{\partial F}{\partial x_i}(x, p, u) \right)$, we may simplify the equation using Poisson bracket as

$$\left(\frac{\partial^2 F}{\partial u_l \partial u_m} \right)_{1 \leq l \leq k, 1 \leq m \leq k} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix} = \begin{pmatrix} \left\{ \frac{\partial F}{\partial u_1}, F \right\}(x, p, u) \\ \vdots \\ \left\{ \frac{\partial F}{\partial u_k}, F \right\}(x, p, u) \end{pmatrix}.$$

□

A sufficient condition for L_F to be solvable is the following

Proposition 2.15 ([9]). A point $(x, p, \dot{x}, \dot{p})$ in L_F is (smoothly) solvable if a linear equation

$$\begin{pmatrix} \frac{\partial^2 F}{\partial u_1 \partial u_1}(x, p, u) & \cdots & \frac{\partial^2 F}{\partial u_1 \partial u_k}(x, p, u) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial u_k \partial u_1}(x, p, u) & \cdots & \frac{\partial^2 F}{\partial u_k \partial u_k}(x, p, u) \end{pmatrix} \begin{pmatrix} \mu_1(x, p, u) \\ \vdots \\ \mu_k(x, p, u) \end{pmatrix} = \begin{pmatrix} \{\frac{\partial F}{\partial u_1}, F\}(x, p, u) \\ \vdots \\ \{\frac{\partial F}{\partial u_k}, F\}(x, p, u) \end{pmatrix}$$

has a (smooth) solution on a neighborhood of $(x, p, u) = \phi_F(x, p, \dot{x}, \dot{p})$ in $C(F)$.

Proof. Suppose that the linear equation has a smooth solution $\mu(x, p, u) = (\mu_1(x, p, u), \dots, \mu_k(x, p, u))$ defined on $C(F)$. Then as seen in the proof of Proposition 2.14, the vector field

$$Y_\mu = \sum_{i=1}^n \frac{\partial F}{\partial p_i}(x, p, u) \frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i}(x, p, u) \frac{\partial}{\partial p_i} + \sum_{l=1}^k \mu_l(x, p, u) \frac{\partial}{\partial u_l}$$

is a tangent vector field on C_F . Consider the following tangent vector field X_μ on L_F defined by

$$X_\mu(\phi_F(x, p, u)) := d\phi_F(x, p, u)(Y_\mu(x, p, u)).$$

Since

$$\begin{aligned} \phi_F(x, p, u) &= (x, p, \partial F / \partial p(x, p, u), -\partial F / \partial x(x, p, u)), \\ d\phi_F \left(\frac{\partial}{\partial x_i} \right) &= \frac{\partial}{\partial x_i} + \sum_{j=1}^n \frac{\partial^2 F}{\partial p_j \partial x_i}(x, p, u) \frac{\partial}{\partial \dot{x}_j} - \frac{\partial^2 F}{\partial x_j \partial x_i}(x, p, u) \frac{\partial}{\partial \dot{p}_j}, \\ d\phi_F \left(\frac{\partial}{\partial p_i} \right) &= \frac{\partial}{\partial p_i} + \sum_{j=1}^n \frac{\partial^2 F}{\partial p_j \partial p_i}(x, p, u) \frac{\partial}{\partial \dot{x}_j} - \frac{\partial^2 F}{\partial x_j \partial p_i}(x, p, u) \frac{\partial}{\partial \dot{p}_j}, \\ d\phi_F \left(\frac{\partial}{\partial u_l} \right) &= \sum_{j=1}^n \frac{\partial^2 F}{\partial p_j \partial u_l}(x, p, u) \frac{\partial}{\partial \dot{x}_j} - \frac{\partial^2 F}{\partial x_j \partial u_l}(x, p, u) \frac{\partial}{\partial \dot{p}_j} \end{aligned}$$

Set $(x, p, \dot{x}, \dot{p}) = \phi_F(x, p, u) = (x, p, \partial F / \partial p(x, p, u) - \partial F / \partial x(x, p, u))$ and we have

$$\begin{aligned}
X_\mu &= \sum_{i=1}^n \dot{x}_i \frac{\partial}{\partial x_i} + \dot{p}_i \frac{\partial}{\partial p_i} \\
&+ \sum_{j=1}^n \left(\sum_{i=1}^n \left(\frac{\partial^2 F}{\partial p_j \partial x_i}(x, p, u) + \frac{\partial^2 F}{\partial p_j \partial p_i}(x, p, u) \right) + \sum_{l=1}^k \frac{\partial^2 F}{\partial p_j \partial u_l}(x, p, u) \right) \frac{\partial}{\partial \dot{x}_j} \\
&- \sum_{j=1}^n \left(\sum_{i=1}^n \left(\frac{\partial^2 F}{\partial x_j \partial x_i}(x, p, u) + \frac{\partial^2 F}{\partial x_j \partial p_i}(x, p, u) \right) + \sum_{l=1}^k \frac{\partial^2 F}{\partial x_j \partial u_l}(x, p, u) \right) \frac{\partial}{\partial \dot{y}_j}.
\end{aligned}$$

Thus X_μ is a smooth tangent vector field on L_F of the above form. Therefore, from Proposition 2.8, L_F is smoothly solvable.

Conversely suppose that L_F is smoothly solvable. Then by Proposition 2.8, there exists a smooth tangent vector field X on L_F of the form

$$X = \sum_{i=1}^n \left(\dot{x}_i \frac{\partial}{\partial x_i} + \dot{p}_i \frac{\partial}{\partial p_i} + a_i(x, p, \dot{x}, \dot{p}) \frac{\partial}{\partial x_i} + b_i(x, p, \dot{x}, \dot{p}) \frac{\partial}{\partial p_i} \right).$$

Let Y be a smooth tangent vector field on C_F defined by

$$Y(x, p, u) = d\phi_F(x, p, u)^{-1}(X(\phi_F(x, p, u))).$$

Then from the form of X_μ which is calculated above, Y has the form

$$\begin{aligned}
Y &= \sum_{i=1}^n \dot{x}_i \frac{\partial}{\partial x_i} + \dot{p}_i \frac{\partial}{\partial p_i} + \sum_{l=1}^k \mu_l(x, p, u) \frac{\partial}{\partial u_l} \\
&= \sum_{i=1}^n \frac{\partial F}{\partial p_i}(x, p, u) \frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i}(x, p, u) \frac{\partial}{\partial p_i} + \sum_{l=1}^k \mu_l(x, p, u) \frac{\partial}{\partial u_l}
\end{aligned}$$

The vector field Y and functions $\mu_1, \dots, \mu_k$ are smooth since X is smooth. From the latter part of proof of Proposition 2.14, we see that $\mu(x, p, u) = (\mu_1(x, p, u), \dots, \mu_k(x, p, u))$ is a smooth solution of the linear equation. $\square$

Now we consider a Morse family of particular type:

$$F: \mathbb{R}^{2n} \times \mathbb{R}^k \rightarrow \mathbb{R}, \quad F(x, p, u) = \sum_{j=1}^k a_j(x, p) u_j + b(x, p).$$

Note that functions $a_1, \dots, a_k$ are independent, i.e., differential one forms $da_1(x, p), \dots, da_k(x, p)$ are linearly independent in $T_{(x,p)}^* \mathbb{R}^{2n}$ at each point

$(x, p) \in \mathbb{R}^{2n}$ because F is a Morse family. The catastrophe set of F is given by $C(F) = K \times \mathbb{R}^k$ with

$$K := \{(x, p) \in \mathbb{R}^{2n} \mid a_i(x, p) = 0, i = 1, \dots, k\}.$$

We define a generalized Hamiltonian system as follows.

Definition 2.16 ([13]). Let $F: \mathbb{R}^{2n} \times \mathbb{R}^k \rightarrow \mathbb{R}$ be a Morse family of functions defined by

$$F(x, p, u) = \sum_{j=1}^k a_j(x, p)u_j + b(x, p).$$

A Lagrangian submanifold L_F which is generated by Morse family F is called a generalized Hamiltonian system. Namely, we call

$$L_F := \phi_F(C(F))$$

a generalized Hamiltonian system.

We consider solvability of L_F and that of submanifolds of L_F . Applying Proposition 2.14 and Proposition 2.15, we know that L_F is smoothly solvable if and only if $\{a_i, a_j\}(x, p) = 0$, $\{b, a_i\}(x, p) = 0$ on K , $(1 \leq i, j \leq k)$ [13].

Considering conditions in the propositions, we set

$$\widetilde{S}_F := \{(x, p, u) \in C(F) \mid \sum_{j=1}^k \{a_i, a_j\}(x, p)u_j = \{b, a_i\}(x, p), 1 \leq i \leq k\},$$

$$S_F := \phi_F(\widetilde{S}_F).$$

Then we see that every smoothly solvable submanifold of L_F is contained in S_F ([13]). The following two lemmas are sufficient conditions of (smooth) solvability of submanifolds of L_F .

Lemma 2.17 ([26]). Let A be a submanifold of K . For a point $q_0 \in A$, if there exist a smooth map $u: A \rightarrow \mathbb{R}^k$ and an open neighborhood V_0 of q_0 such that X_u is tangent to V_0 , then $\phi_F(q_0, u(q_0))$ is a solvable point of L_F over A .

Proof. There are a smooth map $u, u: A \rightarrow \mathbb{R}^k$, an open neighborhood V_0 of q_0 and a tangent vector field X_u on V_0 . We have a integral curve $\gamma: (-\varepsilon, \varepsilon) \rightarrow V_0$ of X_u , that is, γ satisfies

$$(\gamma(t), \dot{\gamma}(t)) = (\gamma(t), X_u(\gamma(t))) \in L_F$$

for every $t \in (-\varepsilon, \varepsilon)$. Thus γ is a solution of L_F passing through $\phi_F(q_0, u(q_0))$. $\square$

Lemma 2.17 asserts solvability not only of L_F but also $\phi_F(V_0 \times u(A))$.

We have a similar type of lemma for smooth solvability. Let A be a submanifold of K and $C \subset \mathbb{R}^k$ a submanifold. Let $\tilde{A}$ a submanifold of $A \times \mathbb{R}^k$ and $\alpha: \tilde{A} \rightarrow A$ a locally trivial fibration which fibre is C . Here $\alpha(q, v) := q$ for $(q, v) \in \tilde{A} \subset A \times \mathbb{R}^k$.

Lemma 2.18 ([26]). A point $(q_0, \dot{q}_0) \in \phi_F(\tilde{A})$ is a smoothly solvable point of $\phi_F(\tilde{A})$ over A if there exist an open neighborhood V_0 of q_0 in A and a smooth map $s: V_0 \times C \rightarrow \tilde{A}$ of smooth family of sections

$$s_c := s(\cdot, c): V_0 \rightarrow \tilde{A}$$

for each $c \in C$ such that for any $(q, \dot{q}) \in \phi_F(\alpha^{-1}(V_0))$ there exists $c \in C$ which satisfies

$$\phi_F(s_c(q)) = \dot{q}$$

and X_{s_c} is tangent vector field on V_0 .

Proof. For a point $(q_0, \dot{q}_0) \in \phi_F(\tilde{A})$, there are V_0 of q_0 in A and a smooth map $s: V_0 \times C \rightarrow \tilde{A}$ such that $s_c: V_0 \rightarrow \tilde{A}$ is a smooth section of the fibration $\alpha: \tilde{A} \rightarrow A$ for each $c \in C$. Since the vector field X_{s_c} is tangent to V_0 , there is an integral curve γ_c which satisfies

$$(\gamma_c(t), \dot{\gamma}_c(t)) = (\gamma_c(t), X_{s_c}(\gamma_c(t))) \in \phi_F(\tilde{A})$$

for each $c \in C$. The vector field X_{s_c} is smoothly depending on $c \in C$ as a parameter and $(q, \dot{q}) \in \pi^{-1}(V_0)$ is also smoothly depending on (q, c) via a smooth map $\phi_F \circ s$ where π is a tangent bundle projection. Hence from the well-known theorem about uniqueness and smoothness of the solution of an ordinary differential equation, the point $(q_0, \dot{q}_0)$ is a smoothly solvable point over A . $\square$

We show a sufficient condition of S_F itself to be smoothly solvable by using the Lemma 2.18.

Theorem 2.19 ([13]). S_F is a smoothly solvable submanifold of L_F if

$$\begin{aligned} \text{rank}(\{a_i, a_j\}(x, p))_{1 \leq i, j \leq k} &= r \text{ (constant)} \quad \text{and} \\ \begin{pmatrix} \{b, a_1\}(x, p) \\ \vdots \\ \{b, a_k\}(x, p) \end{pmatrix} &\in \text{Im}(\{a_i, a_j\}(x, p))_{1 \leq i, j \leq k}, \end{aligned}$$

holds for every $(x, p) \in K = \{(x, p) \in \mathbb{R}^{2n} \mid \frac{\partial F}{\partial u_i}(x, p, u) = 0, 1 \leq i \leq k\}$.

Proof. Let $q_0 = (x_0, p_0) \in K$ be a point of which $p_0 \neq 0$. Let U_{q_0} is a small ball neighborhood of q_0 in K . We prove that $S_F \cap TU_{q_0}$ is a smoothly solvable submanifold of L_F . First we prove the following linear equation

$$(\{a_l, a_m\}(x, p)) \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x, p) \\ \vdots \\ \{b, a_k\}(x, p) \end{pmatrix}, \quad (x, p) \in \mathbb{R}^{2n} \quad (\star)$$

has a smooth solution on K . Since the rank of the matrix $(\{a_i, a_j\}(q_0))$ is r , there exist linearly independent r vectors $e_1, \dots, e_r \in \mathbb{R}^k$ such that

$$(\{a_l, a_m\}(q_0))(e_1), \dots, (\{a_l, a_m\}(q_0))(e_r) \text{ spans } (\{a_l, a_m\}(q_0))(\mathbb{R}^k).$$

Since the rank of the matrix $(\{a_l, a_m\}(q))$ is constantly r , vectors

$$(\{a_l, a_m\}(q))(e_1), \dots, (\{a_l, a_m\}(q))(e_r)$$

span an image of $\{a_l, a_m\}(q)$ at every point $q \in U_{q_0}$. Then the vector ${}^t(\{b, a_1\}(q), \dots, \{b, a_k\}(q))$ is uniquely expressed as a linear combination of $(\{a_l, a_m\}(q)(e_1), \dots, \{a_l, a_m\}(q)(e_r))$, that is, there exist numbers $\gamma_1(q), \dots, \gamma_r(q)$ for each q such that

$${}^t(\{b, a_1\}(q), \dots, \{b, a_k\}(q)) = \sum_{i=1}^r \gamma_i(q) (\{a_l, a_m\}(q))(e_i).$$

Since $\{b, a_1\}(x, p), \dots, \{b, a_k\}(x, p)$ are smooth functions, $\gamma_1(q), \dots, \gamma_r(q)$ are also smooth. Then a vector

$${}^t(u_1(q), \dots, u_k(q)) = \sum_{i=1}^r \gamma_i(q) e_i$$

is a smooth solution of the linear equation $(\star)$. Because the matrix $\{a_i, a_j\}_{i,j}$ has rank r , thus there exist a basis

$$\nu_1(q), \dots, \nu_{k-r}(q)$$

of the kernel of $\{a_i, a_j\}_{i,j}$. For a smooth solution $u(x, p) = (u_1(q), \dots, u_k(q))$ of $(\star)$ defined on $K \cap U_{q_0}$, we have a solution of $(\star)$ as

$$u_c(q) = u(q) + \sum_{i=1}^{k-r} c_i \nu_i(q)$$

for each $c = (c_1, \dots, c_{k-r}) \in \mathbb{R}^{k-r}$. We set a smooth family of sections by using $u_c(q)$:

$$s: U_{q_0} \times \mathbb{R}^{k-r} \rightarrow K \times \mathbb{R}^k, s(q, c) = (q, u_c(q)).$$

For each $(q, \dot{q}) \in \phi_F(U_{q_0} \times \mathbb{R}^k)$, there exists a vector $c \in \mathbb{R}^{k-r}$ such that

$$\dot{q} = (\dot{x}, \dot{p}) = \left(\frac{\partial F}{\partial p}(x, p, u_c(x, p)), -\frac{\partial F}{\partial x}(x, p, u_c(x, p)) \right).$$

Since $u_c(q) = (u_c^1(q), \dots, u_c^k(q))$ is a solution of $(\star)$, a vector field X_s satisfies

$$\begin{aligned} \begin{pmatrix} X_s(a_1)(q) \\ \vdots \\ X_s(a_k)(q) \end{pmatrix} &= - \begin{pmatrix} \{a_1, a_1\}(q) & \cdots & \{a_1, a_k\}(q) \\ \vdots & \ddots & \vdots \\ \{a_k, a_1\}(q) & \cdots & \{a_k, a_k\}(q) \end{pmatrix} \begin{pmatrix} u_c^1(q) \\ \vdots \\ u_c^k(q) \end{pmatrix} + \begin{pmatrix} \{a_1, b\}(q) \\ \vdots \\ \{a_k, b\}(q) \end{pmatrix} \\ &= 0. \end{aligned}$$

Thus X_s is a tangent vector field on U_{q_0} . From the Lemma 2.18, existence of such a smooth family s of sections implies smooth solvability of S_F . $\square$

Remark 1. It is partially mentioned about Theorem 2.19 in [17] for the case $b \equiv 0$. The sufficient condition of solvability is given by $\{a_i, a_j\} = 0$ for all $1 \leq i, j \leq k$. It is tried to describe a necessary and sufficient condition of solvability by stratifying K by the rank of matrix $\{a_i, a_j\}$ for the case $b \equiv 0$.

2.3 Smooth solvability of generalized Hamiltonian systems and their submanifolds

Now we pose a question; for which submanifold S of L_F does there exist a submanifold A of K such that S is smoothly solvable over A ? We consider

the question for the case $k = 2$, namely, we consider the case the generalized Hamiltonian system is induced from conormal bundle of codimension two submanifold in T^*M .

Let $F: \mathbb{R}^{2n} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a Morse family which is defined by

$$F(x, p, u) = a_1(x, p)u_1 + a_2(x, p)u_2.$$

We consider solvability of the Lagrangian submanifold L_F which is generated by F . Moreover we consider solvability of submanifolds of L_F . In detail, we consider a map $\phi_F: \mathbb{R}^{2n} \times \mathbb{R}^2 \rightarrow T(\mathbb{R}^{2n})$ which is defined by

$$\phi_F(x, p, u) = (x, p, \frac{\partial F}{\partial p}(x, p, u), -\frac{\partial F}{\partial x}(x, p, u))$$

and the catastrophe set

$$C(F) = \{(x, p, u) \mid a_1(x, p) = a_2(x, p) = 0\} = K \times \mathbb{R}^2$$

of F for $K = \{(x, p) \mid a_1(x, p) = a_2(x, p) = 0\} \subset \mathbb{R}^{2n}$, then we define L_F by the image $\phi_F(C(F))$. According to Fukuda–Janeczko’s Theorem 2.19, L_F is smoothly solvable if and only if $\{a_1, a_2\} = 0$ locally on K .

Now we consider the cases where the assumptions of Theorem 2.19 are not fulfilled. We consider the family of vector fields $X_u: K \rightarrow T(\mathbb{R}^{2n})$ along K with parameter u

$$X_u(x, p) = (x, p, \frac{\partial F}{\partial p}(x, p, u), -\frac{\partial F}{\partial x}(x, p, u)),$$

and detect submanifolds of K on which X_u are tangent to K . We are going to give smoothly solvable submanifolds over submanifolds of K in the following series of Propositions 2.20 – 2.25.

Let $A_0 = K$. The vector field X_u is tangent to A_0 if and only if

$$X_u(a_1) = X_u(a_2) = 0 \quad \text{i.e.} \quad u_1\{a_1, a_2\} = u_2\{a_1, a_2\} = 0.$$

Hence $\phi_F(A_0 \times \mathbb{R}^2)$ is smoothly solvable if and only if $\{a_1, a_2\} = 0$ on $A_0 = K$. This fact is also obtained as a corollary of Theorem 2.19.

Then we consider a submanifold A_1 of A_0 consisting of points at which X_u is tangent to A_0 ;

$$A_1 := \{(x, p) \mid a_1(x, p) = a_2(x, p) = \{a_1, a_2\}(x, p) = 0\}.$$

We assume that the functions $a_1, a_2, \{a_1, a_2\}$ are independent. The vector field X_u is tangent to A_1 if and only if

$$X_u(\{a_1, a_2\}) = 0 \quad i.e. \quad u_1\{a_1, \{a_1, a_2\}\}(x, p) + u_2\{a_2, \{a_1, a_2\}\}(x, p) = 0$$

on A_1 . Note that we have $X_u(a_1) = X_u(a_2) = 0$ from the definition of A_1 . Let $\mathcal{E}_{\mathbb{R}^{2n}, q_0}$ be the $\mathbb{R}$ -algebra of C^∞ function germs at q_0 on $\mathbb{R}^{2n}$. We denote by $\langle a_1, a_2, \{a_1, a_2\} \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$ the $\mathcal{E}_{\mathbb{R}^{2n}, q_0}$ -module generated by a_1, a_2 and $\{a_1, a_2\}$. We set

$$\xi_1 := \{a_1, \{a_1, a_2\}\}, \quad \xi_2 := \{a_2, \{a_1, a_2\}\}.$$

Then the vector field X_u is tangent to A_1 if the functions ξ_1 and ξ_2 belong to the $\mathcal{E}_{\mathbb{R}^{2n}, q_0}$ -module $\langle a_1, a_2, \{a_1, a_2\} \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$ for any point $q_0 \in A_1$.

Proposition 2.20 ([26]). Assume that a_1, a_2 , and $\{a_1, a_2\}$ are independent. Then $\phi_F(A_1 \times \mathbb{R}^2)$ is a smoothly solvable submanifold of L_F over A_1 if and only if $\xi_1, \xi_2 \in \langle a_1, a_2, \{a_1, a_2\} \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$ for any point q_0 in A_1 .

To find smoothly solvable submanifolds of L_F over submanifolds of A_1 , we construct fiber bundles as follows. Let

$$C_{(x,p)} := \{(u_1, u_2) \mid u_1\{a_1, \{a_1, a_2\}\}(x, p) + u_2\{a_2, \{a_1, a_2\}\}(x, p) = 0\}$$

for $(x, p) \in K$ and define line bundles

$$\begin{aligned} \overline{A_2^1} &:= \{(x, p, u) \mid u \in C_{(x,p)}^1, (x, p) \in A_2^1\}, \\ \overline{A_2^2} &:= \{(x, p, u) \mid u \in C_{(x,p)}^2, (x, p) \in A_2^1\}, \\ \overline{A_2^1} &:= \{(x, p, u) \mid u \in C_{(x,p)}^1, (x, p) \in A_2^2\}, \\ \overline{A_2^2} &:= \{(x, p, u) \mid u \in C_{(x,p)}^2, (x, p) \in A_2^2\}, \\ \overline{A_{1,1}^2} &:= \{(x, p, u) \mid u \in C_{(x,p)}^2, (x, p) \in A_{1,1}\}, \\ \overline{A_{1,2}^1} &:= \{(x, p, u) \mid u \in C_{(x,p)}^1, (x, p) \in A_{1,2}\}, \\ \overline{A_{1,(1,2)}^{1,2}} &:= \{(x, p, u) \mid u \in C_{(x,p)}^{1,2}, (x, p) \in A_{1,(1,2)}\}, \end{aligned}$$

with

$$\begin{aligned}
A_2^1 &:= A_1 \cap \{(x, p) \mid \xi_1 = 0\}, & C_{(x,p)}^1 &= \{(u_1, 0) \in C_{(x,p)}\}, \\
A_2^2 &:= A_1 \cap \{(x, p) \mid \xi_2 = 0\}, & C_{(x,p)}^2 &= \{(0, u_2) \in C_{(x,p)}\}, \\
A_{1,1} &:= A_1 \cap \{(x, p) \mid \xi_1 \neq 0\}, & C_{(x,p)}^{1,2} &= C_{(x,p)} \setminus \{0\}, \\
A_{1,2} &:= A_1 \cap \{(x, p) \mid \xi_2 \neq 0\}, \\
A_{1,(1,2)} &:= A_1 \cap \{(x, p) \mid \xi_1 \neq 0, \xi_2 \neq 0\}.
\end{aligned}$$

Let us consider the case that one of ξ_1 and ξ_2 belongs to the $\mathcal{E}_{\mathbb{R}^{2n}, q_0}$ -module $\langle a_1, a_2, \{a_1, a_2\} \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$ and the other does not.

Proposition 2.21 ([26]). Assume that a_1, a_2 and $\{a_1, a_2\}$ are independent. Assume also that

$$\xi_2 \in \langle a_1, a_2, \{a_1, a_2\} \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}} \text{ and } \xi_1 \notin \langle a_1, a_2, \{a_1, a_2\} \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$$

at every point q_0 of A_1 . Then the followings hold.

1. $\phi_F(\overline{A_{1,1}}^2)$ is a smoothly solvable submanifold of L_F over $A_{1,1}$.
2. Assume, furthermore, that $\xi_1, a_1, a_2, \{a_1, a_2\}$ are independent.
 - (a) $\phi_F(\overline{A_2^1}^2)$ is a smoothly solvable submanifold of L_F over A_2^1 .
 - (b) $\phi_F(A_2^1 \times \mathbb{R}^2)$ is a smoothly solvable submanifold of L_F over A_2^1 if $\{a_1, \xi_1\} \in \langle a_1, a_2, \{a_1, a_2\}, \xi_1 \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$ for any point q_0 in A_2^1 .

Proof. (1): $A_{1,1}$ is an open submanifold of A_1 from the definition. Since there exist β_1, β_2 and $\beta_3 \in \mathcal{E}_{\mathbb{R}^{2n}, q_0}$ such that $\xi_2 = \beta_1 a_1 + \beta_2 a_2 + \beta_3 \{a_1, a_2\}$, the vector $X_u(x, p)$ with $u \in C_{(x,p)}^2$ is tangent to $A_{1,1}$ at each point $(x, p) \in A_{1,1}$ because

$$\begin{aligned}
X_u(a_1) &= u_2 \{a_2, a_1\} = 0, \\
X_u(a_2) &= 0 \cdot \{a_1, a_2\} = 0, \\
X_u(\{a_1, a_2\}) &= 0 \cdot \xi_1 + u_2 \xi_2 = u_2 \xi_2 = \beta_1 a_1 + \beta_2 a_2 + \beta_3 \{a_1, a_2\} = 0
\end{aligned}$$

on $A_{1,1}$.

(2)-(a): We check the condition that $X_u(x, p)$ is tangent to A_2^1 with $u \in C_{(x,p)}^2$ at each point $(x, p) \in A_2^1$. Note that $\{a_1, \xi_2\} = \{a_2, \xi_1\}$ from Jacobian identity:

$$\begin{aligned}\{a_1, \{a_2, \{a_1, a_2\}\}\} &= \{a_2, \{a_1, \{a_1, a_2\}\}\} + \{\{a_1, a_2\}, \{a_1, a_2\}\} \\ &= \{a_2, \{a_1, \{a_1, a_2\}\}\}.\end{aligned}$$

Then

$$X_u(\{a_1, a_2\}) = 0 \cdot \xi_1 + u_2 \xi_2 = 0,$$

$$\begin{aligned}X_u(\xi_1) &= u_1 \{a_1, \xi_1\} + u_2 \{a_2, \xi_1\} = u_1 \{a_1, \xi_1\} + u_2 \{a_1, \xi_2\} \quad \cdots (\star) \\ &= u_2 (\{a_1, \beta_1 a_1\} + \{a_1, \beta_2 a_2\} + \{a_1, \beta_3 \{a_1, a_2\}\}) \\ &= u_2 (a_1 \{a_1, \beta_1\} + \beta_2 \{a_1, a_2\} + a_2 \{a_1, \beta_2\} \\ &\quad + \beta_3 \xi_1 + \{a_1, a_2\} \{a_1, \beta_3\}) \\ &= 0\end{aligned}$$

on A_2^1 .

(2)-(b): From an equality $(\star)$ we have

$$X_u(\xi_1) = u_1 \{a_1, \xi_1\} + u_2 \{a_2, \xi_1\} = u_1 \{a_1, \xi_1\} + u_2 \{a_1, \xi_2\}.$$

Since $\xi_2 \in \langle a_1, a_2, \{a_1, a_2\} \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$, it holds that

$$\{a_1, \xi_2\} \in \langle a_1, a_2, \{a_1, a_2\}, \xi_1 \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}.$$

Consequently, by using $\{a_1, \xi_1\} \in \langle a_1, a_2, \{a_1, a_2\}, \xi_1 \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$, we obtain $X_u(\xi_1) = 0$ on A_2^1 . $\square$

In the same way we have the counterpart of Proposition 2.21.

Proposition 2.22 ([26]). Assume that a_1, a_2 and $\{a_1, a_2\}$ are independent. Assume also that

$$\xi_1 \in \langle a_1, a_2, \{a_1, a_2\} \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}} \text{ and } \xi_2 \notin \langle a_1, a_2, \{a_1, a_2\} \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$$

at every point q_0 of A_1 . Then the followings hold.

1. $\phi_F(\overline{A_{1,2}}^1)$ is a smoothly solvable submanifold of L_F over $A_{1,2}$.

2. Assume, furthermore, that $\xi_2, a_1, a_2, \{a_1, a_2\}$ are independent.

- (a) $\phi_F(\overline{A_2^2}^1)$ is a smoothly solvable submanifold of L_F over A_2^2 .
- (b) $\phi_F(A_2^2 \times \mathbb{R}^2)$ is a smoothly solvable submanifold of L_F over A_2^2 if $\{a_2, \xi_2\} \in \langle a_1, a_2, \{a_1, a_2\}, \xi_2 \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$ for any point q_0 in A_2^2 .

In the case $\xi_1, \xi_2 \notin \langle a_1, a_2, \{a_1, a_2\} \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$ we have

Proposition 2.23 ([26]). Assume that a_1, a_2 and $\{a_1, a_2\}$ are independent. Then $\phi_F(\overline{A_{1,(1,2)}}^{1,2})$ is a smoothly solvable submanifold of L_F over $A_{1,(1,2)}$ if $\xi_1, \xi_2 \notin \langle a_1, a_2, \{a_1, a_2\} \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$ for every point q_0 in $A_{1,(1,2)}$.

Proof. $A_{1,(1,2)}$ is an open submanifold of A_1 from the definition. The vector $X_u(x, p)$ with $u \in C_{(x,p)}^{1,2}$ is tangent to $A_{1,(1,2)}$ at each point $(x, p) \in A_{1,(1,2)}$ since

$$\begin{aligned} X_u(a_1) &= u_2 \{a_2, a_1\} = 0, \\ X_u(a_2) &= u_1 \{a_1, a_2\} = 0, \\ X_u(\{a_1, a_2\}) &= u_1 \xi_1 + u_2 \xi_2 = 0 \end{aligned}$$

on $A_{1,(1,2)}$. □

In Proposition 2.20-2.23, we gave sufficient conditions for existence of smoothly solvable submanifolds of L_F over $A_{1,1}$, A_2^1 , $A_{1,2}$, A_2^2 and $A_{1,(1,2)}$ and examples of smoothly solvable submanifolds of L_F over them. The following two propositions give different sufficient conditions for existence of smoothly solvable submanifolds of L_F over A_2^1 and A_2^2 and examples of smoothly solvable submanifolds over them respectively.

Proposition 2.24 ([26]). Assume that $a_1, a_2, \{a_1, a_2\}$ and ξ_1 are independent. Then $\phi_F(\overline{A_2^1}^1)$ is a smoothly solvable submanifold of L_F over A_2^1 if $\{a_1, \xi_1\} \in \langle a_1, a_2, \{a_1, a_2\}, \xi_1 \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$ for any point q_0 in A_2^1 .

Proof. Since $a_1, a_2, \{a_1, a_2\}$ and ξ_1 are independent, A_2^1 is a submanifold of K . For the vector field X_u along A_2^1 with $(u_1, 0)$,

$$\begin{aligned} X_u(a_1) &= 0 \cdot \{a_1, a_2\} = 0, \\ X_u(a_2) &= u_1 \{a_2, a_1\} = 0, \\ X_u(\{a_1, a_2\}) &= u_1 \xi_1 = 0 \end{aligned}$$

hold on A_2^1 . Since $\{a_1, \xi_1\} \in \langle a_1, a_2, \{a_1, a_2\}, \xi_1 \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$, there exist $\beta_1, \beta_2, \beta_3$ and $\beta_4 \in \mathcal{E}_{\mathbb{R}^{2n}, q_0}$ such that $\{a_1, \xi_1\} = \beta_1 a_1 + \beta_2 a_2 + \beta_3 \{a_1, a_2\} + \beta_4 \xi_1$. Hence we have

$$X_u(\xi_1) = u_1 \{a_1, \xi_1\} = u_1 (\beta_1 a_1 + \beta_2 a_2 + \beta_3 \{a_1, a_2\} + \beta_4 \xi_1) = 0$$

on A_2^1 . Thus the vector field X_u with $(u_1, 0)$ is tangent to A_2^1 . $\square$

In the same way we have

Proposition 2.25 ([26]). Assume that $a_1, a_2, \{a_1, a_2\}$ and ξ_2 are independent. Then $\phi_F(\overline{A_2^2})$ is a smoothly solvable submanifold of L_F over A_2^2 if $\{a_2, \xi_2\} \in \langle a_1, a_2, \{a_1, a_2\}, \xi_2 \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$ for any point q_0 in A_2^2 .

3 Control systems

Toward applications of the results in previous section, we give a review of the theory of control systems [1][16][24].

One of the purpose of the control theory is to study how a state reach translate to another. Then there are natural questions; How does a point reach to another, what is the optimal route of them, etc. We review mathematical formulation of control theory and sub-Riemannian geometry as a kind of geometric control theory.

3.1 Control systems

We introduce a concept of control systems on a manifold. Let M be a smooth manifold and $\{B, M, \alpha\}$ be a locally trivial fibration of which fiber is an open subset $U \subset \mathbb{R}^k$, here we assume $k \leq \dim M$.

Definition 3.1 ([16, 24]). A control system on a smooth manifold M is a tuple of a locally trivial fibration $\{B, M, \alpha\}$ and a smooth map $F: B \rightarrow TM$ from the total space B to the tangent bundle TM of M which commutes the diagram

$$\begin{array}{ccc}
B & \xrightarrow{F} & TM \\
& \searrow \alpha & \swarrow \pi_{TM} \\
& M &
\end{array}$$

where $\pi_{TM}: TM \rightarrow M$ is a canonical projection of the tangent bundle TM .

For an open set $V \subset M$ on which local triviality holds, we have a commuting diagram;

$$\begin{array}{ccc}
B \supset \alpha^{-1}(V) & \xrightarrow{\cong} & V \times U \\
\alpha|_{\alpha^{-1}(V)} \downarrow & \swarrow \text{proj} & \\
M \supset V & &
\end{array}$$

Unless causing no misunderstanding, we write $F: V \times U \rightarrow TV$ instead of $F|_{\alpha^{-1}(V)} \circ h: V \times U \rightarrow TV$ with diffeomorphism $h: V \times U \rightarrow \alpha^{-1}(V)$.

With the local expression of the control system, the map F is regarded as a family of vector fields on V ; for $x \in M$ and $u \in U$, $F(x, u)$ can be written as $(x, f(x, u))$ by definition of the control system. Then we have a section $(x, f_u(x))$ of tangent bundle TV on V for fixed $u \in U$.

3.2 Affine control systems

A fibration $\{B, M, \alpha\}$ is called an affine bundle if all the fibres are affine space, transformation group is affine group and homeomorphism which realize local trivialization is affine.

Definition 3.2 ([16, 24]). Let $\{B, M, \alpha\}$ be an affine bundle. An affine control system is a pair $(\{B, M, \alpha\}, F)$ of the affine bundle and an affine bundle homomorphism $F: B \rightarrow TM$.

For an affine control system, there exist smooth vector fields $X_0, \dots, X_k$ on a open neighborhood V_x of any $x \in M$ such that for $F(x, u) = (x, f(x, u))$,

$$f(x, u) = X_0(x) + \sum_{i=1}^k u_i X_i(x).$$

holds. Here u is a fiber coordinate.

The vector field X_0 is called a *drift*, and $X_1, \dots, X_k$ *input vector fields*. A parameter $u = (u_1, \dots, u_r)$ is called a *control parameter*.

For a local trivialization, affine control systems are represented by the following differential equations:

$$\Sigma : \dot{x}(t) = F(x(t), u(t)) = X_0(x(t)) + \sum_{i=1}^r u_i(t) X_i(x(t)).$$

We also regard the differential equations Σ as an affine control system.

Curves on the total space B are called controls. Let us define a class of the controls.

For a local trivialization $B|_V \cong V \times U$, a curve $c: (a, b) \rightarrow B$ is expressed by $c(t) = (x(t), u(t))$ on whenever the image of the curve lies on $\alpha^{-1}(V)$.

Definition 3.3 ([1]). We define the set of controls as

$$\mathcal{U} := \{c: [0, T] \rightarrow B \mid u : \text{measurable, bounded}\}.$$

For a given point $x_0 \in M$ and a control $c(t)$, there is a curve on M which starts from x_0 . If the trajectory can be extend to whole closed interval $[0, T]$, then the control $c(t)$ is called admissible [24];

$$\mathcal{U} \supset \mathcal{U}_{x_0} := \{c: [0, T] \rightarrow B \mid x(t) = \alpha \circ c(t): [0, T] \rightarrow M, x(0) = x_0\}$$

We may consider a map from admissible controls to a manifold M which assigns an admissible control a point of trajectory $x(t)$ at some time $t \in [0, T]$, especially at the end of the time T .

Definition 3.4 ([6]). An *Endpoint mapping* E is a map from admissible controls to the manifold M which assigns a control c an endpoint $x(T)$;

$$\begin{aligned} E : \mathcal{U}_{x_0} &\longrightarrow M \\ c &\longmapsto x(T). \end{aligned}$$

The end-point mapping is differentiable in terms of Fréchet derivative.

3.3 Singular controls

Since the set of admissible control $\mathcal{U}_{x_0}$ is a Banach manifold and the endpoint mapping is differentiable, we may define a singular control as a singular point of an endpoint mapping.

Definition 3.5 ([1, 6]). A *singular control* is a singular point of an endpoint mapping, that is, a singular control is a point where the differential map

$$E_{*c} : T_c \mathcal{U}_{x_0} \rightarrow T_{x(T)} M$$

is not surjective.

If a control c is a singular point of the end-point mapping, then controls which are restricted to sub interval $[0, T']$ for $T' \leq T$ are also singular controls for related end-point mapping. Thus the singularity of the end-point mapping is micro local property of time.

We give an equivalence condition to the above definition as a constrained Hamiltonian system. For writing the condition, we define a Hamiltonian function

$$\hat{H} : T^*V \times U \rightarrow \mathbb{R}$$

for local triviality $V \times U$ of the fibration $\alpha : B \rightarrow M$ as

$$\hat{H}(x, p, u) := \langle p, f_u(x) \rangle.$$

For an affine control system, $\hat{H}(x, p, u)$ can be written by

$$\hat{H}(x, p, u) = \langle p, X_0(x) + \sum_{i=1}^r u_i X_i(x) \rangle.$$

Proposition 3.6 ([15]). A control $c(t) = (x(t), u(t))$ is a singular control if and only if there exist a positive number $\varepsilon > 0$ and a curve $p(t)$ on $T_{x(t)}^* V \setminus \{0\}$ such that the curve $(x(t), p(t), u(t))$ satisfies the following equation (which is called the *constrained Hamiltonian system*) for all $t \in [0, \varepsilon]$;

$$\begin{cases} \dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), u(t)), \\ \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), u(t)), \\ \frac{\partial H}{\partial u_i}(x(t), p(t), u(t)) = 0 \quad (1 \leq i \leq r). \end{cases}$$

The third equation is called constrained condition. For an affine control system, the constrained condition becomes

$$\frac{\partial H}{\partial u_i}(x(t), p(t), u(t)) = \langle p(t), X_i(x(t)) \rangle = 0 \quad (1 \leq i \leq r).$$

3.4 Controllability

One of problems on the control theory is to study how a state reach to another state under some conditions. The controllability problem is to study what extent states of the system may reach to. We recall controllability here to discuss the problem quantitatively. As we may not observe the trajectory of the system per se, we notice that the controllability is the “intrinsic” property of control systems.

Definition 3.7 ([16, 24]). The *accessibility algebra* $\mathcal{C}$ of an affine control system Σ is the collection of the vector fields which generated by Lie bracket among vector fields X_i and the drift X_0 .

$$\mathcal{C}_\Sigma := \{X_i, [X_j, X_i], [X_k, [X_j, X_i]], \dots \mid i = 0, \dots, r\}.$$

We denote by Z_l the elements of accessibility algebra $\mathcal{C}$.

Definition 3.8 ([24]). The *accessibility distribution* $Acc(\mathcal{C})$ is the distribution which generated by the accessibility algebra $\mathcal{C}_\Sigma$

$$Acc(\mathcal{C}_\Sigma) = \text{span}_{\mathbb{R}}\{Z \mid Z \in \mathcal{C}_\Sigma\}$$

We can say the system is controllable if one may transfer arbitrary state to every state by the system.

Definition 3.9 ([16, 24]). Let x_1, x_2 be any arbitrary points of the manifold M . A control system Σ is *controllable* if there exists a control function $u(t) \in \mathcal{U}_{x_1, T}$ and a trajectory $x(t)$ for the control $u(t)$ such that

$$x(0) = x_1, E(u) = x(T) = x_2.$$

A sufficient condition for controllability, which is given by Chow and Rashevskii, is stated in the next section for driftless control systems.

3.5 Sub-Riemmanian geometry

In this section [21] and [19] are mainly referred. For a distribution $\mathcal{D}$ on a smooth manifold M , which is a subbundle of the tangent bundle TM , there arises a natural control system by an inclusion $\mathcal{D} \rightarrow TM$. In this section we review the geometry of distribution $\mathcal{D}$ endowed with metric which is sub-Riemannian geometry.

The subject of the sub-Riemannian geometry is to study a triple $(M, \mathcal{D}, g)$ of a manifold M , a distribution $\mathcal{D}$ on M and a bi-linear positive definite form g on $\mathcal{D}$, which is called a *sub-Riemannian manifold*. The object appears naturally as a collapsing Riemannian manifold and is a generalization of a Riemannian manifold. However the properties of the sub-Riemannian manifolds are much different from those of Riemannian manifolds. In fact, it is not much known about the properties of exponential maps and even about the smoothness of minimizers.

We start this section with recalling notations and results in sub-Riemannian geometry. A distribution $\mathcal{D}$ is also called a horizontal distribution. A vector field or a curve which is tangent to $\mathcal{D}$ is called horizontal. A *horizontal curve* is an absolutely continuous curve $\gamma: I \rightarrow M$ such that $\dot{\gamma}(t)$ is a measurable and bounded map which satisfies $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for almost every $t \in I$. We define the length

$$L(\gamma) := \int_{[a,b]} \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} dt$$

for a horizontal curve $\gamma: [a, b] \rightarrow M$. A distribution $\mathcal{D}$ of rank k is said to satisfy Hörmander's condition if for every $p \in M$ there exists a number $d \in \mathbb{N}$ with local framing $\{X_1, \dots, X_k\}$ around p such that

$$\text{span}_p\{X_1, \dots, X_d, [X_i, X_j], \dots, [X_{i_1}, [X_{i_2}, [\dots, [X_{i_{d-1}}, X_d], \dots,]]]\} = T_p M.$$

By means of the notation of accessibility distribution, Hörmander's condition is written by

$$\text{Acc}(\mathcal{C}_{\mathcal{D}})_p = T_p M,$$

here a distribution $\mathcal{D}$ is regarded as a control system on M and we denote $\text{Acc}(\mathcal{C}_{\mathcal{D}})_p = \text{span}_{\mathbb{R}}\{Z_p \mid Z \in \mathcal{C}_{\mathcal{D}}\}$.

According to Chow–Rashevskii's theorem, if a distribution $\mathcal{D}$ on a connected manifold M satisfies Hörmander's condition, every two points are connected by a horizontal curve.

Theorem 3.10 ([7]). If a distribution $\mathcal{D}$ on a connected manifold M satisfies Hörmander's condition, then for any two points p, q in M there exists a horizontal curve $\gamma: [a, b] \rightarrow M$ such that $\gamma(a) = p$ and $\gamma(b) = q$.

We remark that Theorem 3.10 states controllability for a control system given by a distribution $\mathcal{D}$.

For a sub-Riemannian manifold which satisfies Hörmander's condition, we may define a distance

$$d_{CC}(p, q) := \inf_{\gamma} \left\{ L(\gamma) \mid \gamma: [a, b] \rightarrow M : \text{horizontal}, \gamma(a) = p, \gamma(b) = q \right\}$$

which is called a *Carnot–Carathéodory* (or *sub-Riemannian*) *distance*. It is known that the topology from Carnot–Carathéodory distance agrees with the original one (ball-box theorem[23]).

To state the ball-box theorem, we give some notations. The *Lie flag* of a distribution $\mathcal{D}$ is the sequence $\mathcal{D}_0 \subset \mathcal{D}_1 \subset \cdots$ defined inductively by

$$\mathcal{D}_0 := \mathcal{D}, \quad \mathcal{D}_{i+1} := \mathcal{D}_i + [\mathcal{D}_0, \mathcal{D}_i], \quad i \geq 0.$$

The *small growth vector* of a distribution $\mathcal{D}$ at $q \in M$ is the sequence of the dimension of Lie flags;

$$(\dim \mathcal{D}_0(q), \dim \mathcal{D}_1(q), \dim \mathcal{D}_2(q), \dots).$$

For a distribution which satisfies Hörmander's condition, there exists a number l for each point q in M such that

$$(\dim \mathcal{D}_0(q), \dim \mathcal{D}_1(q), \dim \mathcal{D}_2(q), \dots, \dim \mathcal{D}_l(q) = n).$$

We denote $d_i(q)$ by $\dim \mathcal{D}_i(q)$. Let $\{Y_1, \dots, Y_{d_i}\}$ be a local frame of the flag $\mathcal{D}^i$. Local coordinates $(y_1, \dots, y_n)$ of M is said to be linearly adopted to $\mathcal{D}$ if

$$\text{span}\{dy_{d_i+1}, \dots, dy_n\}(q) = \mathcal{D}_i^\perp(q)$$

for each point q in M .

Theorem 3.11 ([23]). There exist linearly adapted coordinates $(y_1, \dots, y_n)$ and positive constants $c < C$ and $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$

$$\text{Box}(c\varepsilon) \subset B(\varepsilon, q_0) \subset \text{Box}(C\varepsilon)$$

where $\text{Box}(r) = \{y \in \mathbb{R}^n \mid |y_i| < r, i = 1, \dots, n\}$ and $B(\varepsilon, q_0) = \{q \in M \mid d_{CC}(q_0, q) < \varepsilon\}$.

A horizontal curve γ connecting p and q is called a *minimizer* if $d_{CC}(p, q) = L(\gamma)$. A horizontal curve $\gamma: I \rightarrow M$ is a *local minimizer* if for any $t_0 \in I$, there exists $\varepsilon > 0$ such that for all closed sub-interval J of $[t_0 - \varepsilon, t_0 + \varepsilon]$,

$\gamma|_{J \cap I}$ is a minimizer between the end points. Note that any minimizer is necessarily a local minimizer.

To consider local minimizers, we classify horizontal curves on $(M, \mathcal{D}, g)$ by using the end-point mapping.

For a bounded measurable curve $c: [0, T] \rightarrow \mathcal{D}$, if $\gamma := \pi_{\mathcal{D}} \circ c: [0, T] \rightarrow M$ satisfies $\dot{\gamma}(t) = c(t)$ for almost everywhere on $[0, T]$, then γ is a horizontal curve and c is called an *admissible velocity*. Here $\pi_{\mathcal{D}}: \mathcal{D} \rightarrow M$ is the canonical projection. For a point q_0 on M , we set

$$\mathcal{V}_{q_0} := \{c \mid c: [0, T] \rightarrow \mathcal{D} : \text{admissible}, \gamma(0) = q_0\}$$

a set of admissible velocities and we regard it as a set of admissible controls for control system $\mathcal{D}$.

A singular (resp. regular) point of the end-point mapping is called a *singular (resp. ordinary) velocity*. The trajectory corresponds to singular (resp. ordinary) velocity is called a *singular (resp. ordinary) curve*. Since every curve is ordinary or singular, every minimizer is also a ordinary curve or a singular curve.

3.6 Normal extremals and abnormal extremals

For a sub-Riemannian manifold, there is a geodesic equation given as Hamiltonian formulation, not as Lagrangian, with a function

$$H(x, p) = \sum_{ij} g^{ij} \langle p, X_i \rangle \langle p, X_j \rangle \text{ on } T^*M,$$

where $g_{ij} = g(X_i, X_j)$ and $(g^{ij})_{i,j}$ is an inverse matrix of $(g_{ij})_{i,j}$. A solution of the geodesic equation is called a *normal geodesics* of a *normal bi-extremal*.

It is known that regular local minimizers are normal geodesics, and so is smooth since they are solutions of the geodesic equation. Singular minimizers are sometimes also normal geodesic depending on the metric g . A singular (local) minimizer which is not normal geodesics is called *strictly singular (local) minimizer*. The examples of singular minimizers which are not normal on Martinet distribution are given by R. Montgomery in 1994 [20]. This guarantees the existence of strictly singular minimizer, so the problem is to study strictly singular minimizers.

As a strategy to approach the problem which we mentioned, we consider two problems; to detect singular curves, and to find local minimizers among them.

Now, we introduce the result of foothold in the study of singular curves. Take a function $H: T^*M \times_M \mathcal{D} \rightarrow \mathbb{R}$, $H(x, p, u) := \langle p, u \rangle$ for $x \in M, p \in T^*M$ and $u \in \mathcal{D}_x$, here $\times_M$ is a fibre product. Let $\{X_1, \dots, X_k\}$ be a local framing of $\mathcal{D}$ on an open neighborhood U_{x_0} of x_0 in M and $(u_1, \dots, u_k)$ the fibre coordinate related to the local framing. Then we have locally

$$H(x, p, u) = \sum_{i=1}^k u_i \langle p, X_i(x) \rangle.$$

Then singular curves are characterized by the constrained Hamiltonian system from Proposition 3.6. We state Proposition 3.6 again by means of a notation of distributions. A curve $x(t)$ on M is a singular curve if and only if there exists a curve $p(t)$ on $T^*_{x(t)}M \setminus \{0\}$ and $u(t) \in \mathcal{D}_{x(t)}$ such that the curve $(x(t), p(t), u(t))$ satisfies the following equation

$$\begin{cases} \dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), u(t)), \\ \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), u(t)), \\ \frac{\partial H}{\partial u_i}(x(t), p(t), u(t)) = 0 \quad (1 \leq i \leq k). \end{cases}$$

In this sense, a singular curve $x(t)$ is also called an abnormal extremal and a solution of the constrained Hamiltonian system $(x(t), p(t))$ is called an abnormal bi-extremal.

We introduce results on abnormal extremals of rank two distribution which are given by Liu and Sussmann [19].

We start with giving some notations. Let $\mathcal{D}$ be a distribution on a smooth manifold M . We set

$$T_x^\sharp M := T_x M \setminus \{0\}, \quad T^\sharp M := \bigcup_{x \in M} T_x^\sharp M,$$

a cotangent bundle without zero section. For a distribution $\mathcal{D}$, its annihilator $\mathcal{D}^\perp$ is defined by

$$\begin{aligned} \mathcal{D}_x^\perp &:= \{p \in T_x^*M \mid \langle p, v \rangle = 0, \text{ for any } v \in \mathcal{D}_x\}, \\ \mathcal{D}^\perp &:= \bigcup_{x \in M} \mathcal{D}_x^\perp. \end{aligned}$$

The annihilator $\mathcal{D}^\perp$ of a distribution $\mathcal{D}$ is a submanifold of T^*M with codimension $\text{rank } \mathcal{D}$. We define two important subsets of cotangent bundle T^*M .

Definition 3.12 ([19]). An abnormal extremal carrier $AEC(\mathcal{D})$ of a distribution D on M is defined by

$$AEC(\mathcal{D}) := (\mathcal{D}_1)^\perp \cap T^\sharp M.$$

A regular abnormal set of $\mathcal{D}$ is defined by

$$RA(\mathcal{D}) := (\mathcal{D}_1)^\perp \setminus (\mathcal{D}_2)^\perp.$$

We say $(x, p) \in T^*M$ is a regular abnormal point if

$$p \in (\mathcal{D}_1)_x^\perp \setminus (\mathcal{D}_2)_x^\perp.$$

A curve in M is called regular abnormal extremal if it is the projection of an abnormal bi-extremal of which image consists only of regular abnormal points.

For a rank two distribution, a condition $(x, p) \in (\mathcal{D}_2)^\perp$ is equivalent to

$$\langle p, [X_1, [X_1, X_2]](x) \rangle \neq 0 \text{ or } \langle p, [X_2, [X_1, X_2]](x) \rangle \neq 0$$

for basis of sections X_1, X_2 of $\mathcal{D}$.

The following Proposition is about where non constant abnormal bi-extremals lie on.

Proposition 3.13 ([19]). Let $\mathcal{D}$ be a rank two distribution on M . Any non constant abnormal bi-extremal $(x(t), p(t))$ satisfies

$$p(t) \in (\mathcal{D}_1)_{x(t)}^\perp \cap T_{x(t)}^\sharp M$$

on the domain of the abnormal bi-extremal.

The following is a sufficient condition of existence of abnormal extremals.

Proposition 3.14 ([19]). Let $\mathcal{D}$ be a rank two distribution on M . If $(\mathcal{D}_1)_x^\perp \neq (\mathcal{D}_2)_x^\perp$ for $x \in M$, then there exists an abnormal extremal passing through x .

For a rank two distribution, with small growth vector $(2, 3, 4, \dots)$, there is a sufficient condition of existence of abnormal extremals of particular types.

Proposition 3.15 ([19]). Let $\mathcal{D}$ be a distribution of rank two which satisfies

$$\dim \mathcal{D}_x = 2, \quad \dim \mathcal{D}_x^1 = 3, \quad \dim \mathcal{D}_x^2 = 4$$

at any point $x \in M$. Then there exist a line subbundle $l_{\mathcal{D}}$ of $\mathcal{D}$ such that, if U is any open subset of M on which $\mathcal{D}$ admits a basis X_1, X_2 of sections, and u_1, u_2 a smooth functions of U such that

$$u_1[X_1, [X_1, X_2]] + u_2[X_2, [X_1, X_2]] \equiv 0 \pmod{D^1}$$

and $u_1(x)^2 + u_2(x)^2 \neq 0$ for all $x \in U$, then the vector field $Z = u_1X_1 + u_2X_2$ is a section of $l_{\mathcal{D}}$ on U . Every locally simple abnormal extremal parametrized by arc length is either regular or totally singular.

A curve in M is called a totally singular abnormal extremal if it is the projection of an abnormal bi-extremal of which is entirely contained in $(\mathcal{D}^2)^\perp$. In the next section we give propositions which are strongly related to Proposition 3.15.

4 Applications to the study of singular curves on sub-Riemannian manifolds

We apply the results which are obtained in Section 2 to the study of abnormal curves of distributions.

In this section we consider distributions of rank two. The following lemma plays an essential role throughout the section.

Lemma 4.1 ([26]). Let $\mathcal{D}$ be a rank two distribution with small growth vector $(2, 3, 4, \dots)$ at any point in an open neighborhood of $x_0 \in M$ and g a bi-linear positive definite form on $\mathcal{D}$. Then there exist an open neighborhood U_{x_0} and local orthonormal frame X_1, X_2 of $\mathcal{D}$ on U_{x_0} such that

$$X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]]$$

are linearly independent at q and $[X_2, [X_1, X_2]]$ is a functional linear combination of X_1, X_2 and $[X_1, X_2]$ on U_{x_0} .

Proof. $\mathcal{E}_{M, x_0}$ denotes the $\mathbb{R}$ -algebra of C^∞ function germs at x_0 on M . Let X_1, X_2 be any local frame of $\mathcal{D}$ around x_0 . We may suppose that

$$X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]]$$

are linearly independent at x_0 . From the assumption we have

$$[X_2, [X_1, X_2]] \in \langle X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]] \rangle_{\mathcal{E}_{M, x_0}}.$$

Then there exists a $\lambda \in \mathcal{E}_{M,x_0}$ such that

$$[X_2, [X_1, X_2]] \equiv \lambda[X_1, [X_1, X_2]] \mod \langle X_1, X_2, [X_1, X_2] \rangle_{\mathcal{E}_{M,x_0}}$$

Set $\tilde{X}_2 = X_2 - \lambda X_1$. Then $(X_1, \tilde{X}_2)$ is a local frame of $\mathcal{D}$ around x_0 . Then

$$\begin{aligned} [\tilde{X}_2, [X_1, \tilde{X}_2]] &= [\tilde{X}_2, [X_1, X_2 - \lambda X_1]] \\ &= [\tilde{X}_2, [X_1, X_2]] - [\tilde{X}_2, X_1(\lambda)X_1] \\ &= [\tilde{X}_2, [X_1, X_2]] - X_1(\lambda)[\tilde{X}_2, X_1] - \tilde{X}_2(X_1(\lambda))X_1 \\ &\equiv [\tilde{X}_2, [X_1, X_2]] = [X_2 - \lambda X_1, [X_1, X_2]] \\ &\equiv 0 \mod \langle X_1, \tilde{X}_2, [X_1, \tilde{X}_2] \rangle_{\mathcal{E}_{M,x_0}}. \end{aligned}$$

For functions

$$g_{11} := g(X_1, X_1), g_{12} := g(X_1, \tilde{X}_2) \text{ and } g_{22} := g(\tilde{X}_2, \tilde{X}_2),$$

we set

$$X'_1 = \frac{\sqrt{g_{22}}}{\sqrt{g_{11}g_{22} - g_{12}^2}} \left(X_1 - \frac{g_{12}}{g_{22}} \tilde{X}_2 \right), X'_2 = \frac{1}{\sqrt{g_{22}}} \tilde{X}_2.$$

Then (X'_1, X'_2) is a local orthonormal basis of $\mathcal{D}$ around x_0 i.e.,

$$g(X'_1, X'_1) = 1, g(X'_1, X'_2) = 0, g(X'_2, X'_2) = 1,$$

and

$$X'_1, X'_2, [X'_1, X'_2], [X'_1, [X'_1, X'_2]]$$

are linearly independent since X'_1 and X'_2 is a functional linear combination of X_1 and $\tilde{X}_2$. Moreover (X'_1, X'_2) satisfies

$$[X'_2, [X'_1, X'_2]] \equiv 0 \mod \langle X'_1, X'_2, [X'_1, X'_2] \rangle_{\mathcal{E}_{M,x_0}}$$

because for functions

$$\alpha_1 = \frac{\sqrt{g_{22}}}{\sqrt{g_{11}g_{22} - g_{12}^2}}, \alpha_2 = -\frac{\sqrt{g_{22}}}{\sqrt{g_{11}g_{22} - g_{12}^2}} \frac{g_{12}}{g_{22}} \text{ and } \alpha_3 = \frac{1}{\sqrt{g_{22}}},$$

we have the following by straight calculation:

$$\begin{aligned}
[X'_2, [X'_1, X'_2]] &= [\alpha_3 \tilde{X}_2, [\alpha_1 X_1, \alpha_3 \tilde{X}_2]] - [\alpha_3 \tilde{X}_2, [\alpha_2 \tilde{X}_2, \alpha_3 \tilde{X}_2]] \\
&= [\alpha_3 \tilde{X}_2, \alpha_1 \alpha_3 [X_1, \tilde{X}_2]] + [\alpha_3 \tilde{X}_2, \alpha_1 X_1 (\alpha_3) \tilde{X}_2] \\
&\quad - [\alpha_3 \tilde{X}_2, \alpha_3 \tilde{X}_2 (\alpha_1) X_1] \\
&\equiv [\alpha_3 \tilde{X}_2, \alpha_1 \alpha_3 [X_1, \tilde{X}_2]] \mod \langle X'_1, X'_2, [X'_1, X'_2] \rangle_{\mathcal{E}_{M, x_0}} \\
&\equiv \alpha_1 \alpha_3^2 [\tilde{X}_2, [X_1, \tilde{X}_2]] \mod \langle X'_1, X'_2, [X'_1, X'_2] \rangle_{\mathcal{E}_{M, x_0}} \\
&\equiv 0 \mod \langle X'_1, X'_2, [X'_1, X'_2] \rangle_{\mathcal{E}_{M, x_0}}
\end{aligned}$$

□

Let $\{X_1, X_2\}$ be a local frame of $\mathcal{D}$ on U_{x_0} for any $x_0 \in M$ with the property of Lemma 4.1 and define a function $H: T^*U_{x_0} \times_{U_{x_0}} \mathcal{D} \rightarrow \mathbb{R}$ for the distribution $\mathcal{D}$ locally by

$$H(x, p, u) = u_1 \langle p, X_1(x) \rangle + u_2 \langle p, X_2(x) \rangle.$$

We set functions a_1, a_2 on $T^*U_{x_0}$ as

$$a_1(x, p) := \langle p, X_1(x) \rangle, \quad a_2(x, p) := \langle p, X_2(x) \rangle.$$

The set of sections $\Gamma(TM)$ of the tangent bundle is equipped with Lie bracket. A map

$$X \mapsto \langle p, X(x) \rangle \in C^\infty(T^*M, \mathbb{R})$$

is a Lie algebra homomorphism between $\Gamma(TM)$ and the set of smooth functions on T^*M endowed with Poisson bracket induced from standard symplectic structure on T^*M . Namely we have

$$\{a_1, a_2\}(x, p) = \langle p, [X_1, X_2](x) \rangle.$$

From this correspondence we have the following relations:

$$\begin{aligned}
a_1 &= \langle p, X_1(x) \rangle, \quad a_2 = \langle p, X_2(x) \rangle \\
\{a_1, a_2\}(x, p) &= \langle p, [X_1, X_2](x) \rangle, \\
\xi_1 &= \{a_1, \{a_1, a_2\}\}(x, p) = \langle p, [X_1, [X_1, X_2]](x) \rangle, \\
\xi_2 &= \{a_2, \{a_1, a_2\}\}(x, p) = \langle p, [X_2, [X_1, X_2]](x) \rangle.
\end{aligned}$$

Then we apply the series of Propositions 2.20 - 2.25 for these functions and we obtain the series of theorems. We do not concentrate on regular abnormal bi-extremals. The difference from Liu and Sussmann's results is existence of smooth family of smooth abnormal bi-extremals.

4.1 The case the growth vector $(2, 3, 3, \dots)$

The first one is existence of a smooth $(2n - 3)$ -parameter family of totally singular abnormal bi-extremals for distribution which has a small growth vector $(2, 3, 3, \dots)$.

Theorem 4.2. Let $(M, \mathcal{D})$ be a pair of a smooth manifold M and a rank two distribution $\mathcal{D}$ on M . For a point $x_0 \in M$, we assume there exist an open neighborhood U_{x_0} and smooth sections of the basis X_1, X_2 to $\mathcal{D}$ such that X_1, X_2 and $[X_1, X_2]$ are linearly independent at each point of U_{x_0} . If the sections $[X_1, [X_1, X_2]], [X_2, [X_1, X_2]]$ are functional linear combinations of X_1, X_2 and $[X_1, X_2]$ for any point $x \in U_{x_0}$, then there exist an open neighborhood V_0 of q_0 which is an element of a set

$$\{q = (x, p) \in \pi_{T^*M}^{-1}(U_{x_0}) \mid \langle p, X_1(x) \rangle = \langle p, X_2(x) \rangle = \langle p, [X_1, X_2](x) \rangle = 0\}$$

and a smooth family of abnormal bi-extremals $\{\Gamma_q: I \rightarrow T^*M\}_{q \in V_0}$ such that $\Gamma_q(0) = q$ and each $\Gamma_q(t)$ satisfies the following equations

$$\begin{aligned} \dot{x}_q(t) &= u_1(t)X_1(x_q(t)) + u_2(t)X_2(x_q(t)), \\ \dot{p}_q(t) &= -u_1(t)\frac{\partial \langle p, X_1(x) \rangle}{\partial x}(x_q(t), p_q(t)) - u_2(t)\frac{\partial \langle p, X_2(x) \rangle}{\partial x}(x_q(t), p_q(t)) \end{aligned}$$

for any $u_1(t), u_2(t)$ and

$$\begin{aligned} \langle p_q(t), X_1(x_q(t)) \rangle &= 0, & \langle p_q(t), X_2(x_q(t)) \rangle &= 0, \\ \langle p_q(t), [X_1, X_2](x_q(t)) \rangle &= 0, & \langle p_q(t), [X_1, [X_1, X_2]](x_q(t)) \rangle &= 0, \\ \langle p_q(t), [X_2, [X_1, X_2]](x_q(t)) \rangle &= 0. \end{aligned}$$

Thus we have the following

Proposition 4.3. Let $\mathcal{D}$ be a rank 2 distribution on M^n . If a small growth vector is $(2, 3, 3, \dots)$ at each point x_0 in M , then there exist a closed neighborhood V_{q_0} of $q_0 \in \pi_{T^*M}^{-1}(x_0)$ in T^*M and a smooth $(2n - 3)$ -parameter family of totally singular abnormal bi-extremals $\{(x_q(t), p_q(t))\}_{q \in V_{q_0}}$ in $T^*M \setminus \{o\}$.

4.2 The case the growth vector $(2, 3, 4, \dots)$

The following four theorems are on a distribution $\mathcal{D}$ with small growth vector $(2, 3, 4, \dots)$. The next one is of existence of a smooth $(2n - 3)$ -parameter family of regular abnormal bi-extremals.

Theorem 4.4 ([26]). Let $(M, \mathcal{D})$ be a pair of a smooth manifold M and a rank two distribution $\mathcal{D}$ on M . For a point $x_0 \in M$, we assume that there exist an open neighborhood U_{x_0} and smooth sections of basis X_1, X_2 to $\mathcal{D}$ such that X_1, X_2 and $[X_1, X_2]$ are linearly independent at each point of U_{x_0} . If $X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]]$ are linearly independent for any point $x \in U_{x_0}$ and the section $[X_2, [X_1, X_2]]$ is a functional linear combination of X_1, X_2 and $[X_1, X_2]$, then there exist an open neighborhood V_0 of q_0 which is an element of a set

$$\{q = (x, p) \in \pi_{T^*M}^{-1}(U_{x_0}) \mid \langle p, X_1(x) \rangle = \langle p, X_2(x) \rangle = \langle p, [X_1, X_2](x) \rangle = 0, \\ \langle p, [X_1, [X_1, X_2]](x) \rangle \neq 0 \}$$

and a smooth family of abnormal bi-extremals $\{\Gamma_q: I \rightarrow T^*M\}_{q \in V_0}$ such that $\Gamma_q(0) = q$ and each $\Gamma_q(t)$ satisfies the following equations

$$\dot{x}_q(t) = u_2(t)X_2(x_q(t)), \\ \dot{p}_q(t) = -u_2(t)\frac{\partial \langle p, X_2(x) \rangle}{\partial x}(x_q(t), p_q(t))$$

and

$$\langle p_q(t), X_1(x_q(t)) \rangle = 0, \quad \langle p_q(t), X_2(x_q(t)) \rangle = 0, \\ \langle p_q(t), [X_1, X_2](x_q(t)) \rangle = 0, \quad \langle p_q(t), [X_2, [X_1, X_2]](x_q(t)) \rangle = 0,$$

for a function $u_2(t)$ which satisfies

$$u_2(t)\langle p_q(t), [X_2, [X_1, X_2]](x_q(t)) \rangle = 0.$$

The next one is of existence of a smooth $(2n - 4)$ -parameter family of totally singular abnormal bi-extremals.

Theorem 4.5. Let $(M, \mathcal{D})$ be a pair of a smooth manifold M and a rank two distribution $\mathcal{D}$ on M . For a point $x_0 \in M$, we assume that there exist an open neighborhood U_{x_0} and smooth sections of the basis X_1, X_2 to $\mathcal{D}$ such that X_1, X_2 and $[X_1, X_2]$ are linearly independent at each point of U_{x_0} . If $X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]]$ are linearly independent for any point $x \in U_{x_0}$ and the section $[X_2, [X_1, X_2]]$ is a functional linear combination of X_1, X_2 and $[X_1, X_2]$, then there exist an open neighborhood V_0 of q_0 which is an element of a set

$$\{(x, p) \in \pi_{T^*M}^{-1}(U_{x_0}) \mid \langle p, X_1(x) \rangle = \langle p, X_2(x) \rangle = \langle p, [X_1, X_2](x) \rangle = 0, \\ \langle p, [X_1, [X_1, X_2]](x) \rangle = 0 \}$$

and a smooth family of abnormal bi-extremals $\{\Gamma_q: I \rightarrow T^*M\}_{q \in V_0}$ such that $\Gamma_q(0) = q$ and each $\Gamma_q(t)$ satisfies the following equations

$$\begin{aligned}\dot{x}_q(t) &= u_2(t)X_2(x_q(t)), \\ \dot{p}_q(t) &= -u_2(t)\frac{\partial \langle p, X_2(x) \rangle}{\partial x}(x_q(t), p_q(t))\end{aligned}$$

and

$$\begin{aligned}\langle p_q(t), X_1(x_q(t)) \rangle &= 0, & \langle p_q(t), X_2(x_q(t)) \rangle &= 0, \\ \langle p_q(t), [X_1, X_2](x_q(t)) \rangle &= 0, & \langle p_q(t), [X_1, [X_1, X_2]](x_q(t)) \rangle &= 0, \\ \langle p_q(t), [X_2, [X_1, X_2]](x_q(t)) \rangle &= 0,\end{aligned}$$

for a function $u_2(t)$ which satisfies

$$u_2(t)\langle p_q(t), [X_2, [X_1, X_2]](x_q(t)) \rangle = 0.$$

The next one is of existence of a smooth $(2n - 4)$ -parameter family of totally singular abnormal bi-extremals.

Theorem 4.6. Let $(M, \mathcal{D})$ be a pair of a smooth manifold M and a rank two distribution $\mathcal{D}$ on M . For a point $x_0 \in M$, we assume that there exist an open neighborhood U_{x_0} and smooth sections X_1, X_2 to $\mathcal{D}$ of the basis such that X_1, X_2 and $[X_1, X_2]$ are linearly independent at each point of U_{x_0} . If $X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]]$ are linearly independent for any point $x \in U_{x_0}$ and the section $[X_2, [X_1, X_2]]$ is a functional linear combination of X_1, X_2 and $[X_1, X_2]$, furthermore, a section $[X_1, [X_1, [X_1, X_2]]]$ is a functionally linear combination of $X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]]$ on U_{x_0} , then there exist an open neighborhood V_0 of q_0 which is an element of a set

$$\begin{aligned}\{q = (x, p) \in \pi_{T^*M}^{-1}(U_{x_0}) \mid \langle p, X_1(x) \rangle = \langle p, X_2(x) \rangle = \langle p, [X_1, X_2](x) \rangle = 0, \\ \langle p, [X_1, [X_1, X_2]](x) \rangle = 0 \}\end{aligned}$$

and a smooth family of abnormal bi-extremals $\{\Gamma_q: I \rightarrow T^*M\}_{q \in V_0}$ such that $\Gamma_q(0) = q$ and each $\Gamma_q(t)$ satisfies the following equations

$$\begin{aligned}\dot{x}_q(t) &= u_1(t)X_1(x_q(t)) + u_2(t)X_2(x_q(t)), \\ \dot{p}_q(t) &= -u_1(t)\frac{\partial \langle p, X_1(x) \rangle}{\partial x}(x_q(t), p_q(t)) - u_2(t)\frac{\partial \langle p, X_2(x) \rangle}{\partial x}(x_q(t), p_q(t))\end{aligned}$$

for any $u_1(t), u_2(t)$ and

$$\begin{aligned}\langle p_q(t), X_1(x_q(t)) \rangle &= 0, & \langle p_q(t), X_2(x_q(t)) \rangle &= 0, \\ \langle p_q(t), [X_1, X_2](x_q(t)) \rangle &= 0, & \langle p_q(t), [X_1, [X_1, X_2]](x_q(t)) \rangle &= 0, \\ \langle p_q(t), [X_2, [X_1, X_2]](x_q(t)) \rangle &= 0, & \langle p_q(t), [X_1, [X_1, [X_1, X_2]](x_q(t)) \rangle &= 0.\end{aligned}$$

The following is of existence of a smooth $(2n - 3)$ -parameter family of regular abnormal bi-extremals.

Theorem 4.7. Let $(M, \mathcal{D})$ be a pair of a smooth manifold M and a rank two distribution $\mathcal{D}$ on M . For a point $x_0 \in M$, we assume that there exist an open neighborhood U_{x_0} and smooth sections of the basis X_1, X_2 to $\mathcal{D}$ such that X_1, X_2 and $[X_1, X_2]$ are linearly independent at each point of U_{x_0} . If the sections $[X_1, [X_1, X_2]]$ and $[X_2, [X_1, X_2]]$ are unable to be expressed by a functional linear combination of X_1, X_2 and $[X_1, X_2]$ and two sections $[X_1, [X_1, X_2]], [X_2, [X_1, X_2]]$ are linearly dependent for any point $x \in U_{x_0}$, then there exist an open neighborhood V_0 of q_0 which is an element of a set

$$\begin{aligned}\{q = (x, p) \in \pi_{T^*M}^{-1}(U_{x_0}) \mid \langle p, X_1(x) \rangle = \langle p, X_2(x) \rangle = \langle p, [X_1, X_2](x) \rangle = 0, \\ \langle p, [X_2, [X_1, X_2]](x) \rangle \neq 0, \langle p, [X_1, [X_1, X_2]](x) \rangle \neq 0\}\end{aligned}$$

and a smooth family of abnormal bi-extremals $\{\Gamma_q: I \rightarrow T^*M\}_{q \in V_0}$ such that $\Gamma_q(0) = q$ and each $\Gamma_q(t)$ satisfies the following equations

$$\begin{aligned}\dot{x}_q(t) &= u_1(t)X_1(x_q(t)) + u_2(t)X_2(x_q(t)), \\ \dot{p}_q(t) &= -u_1(t)\frac{\partial \langle p, X_1(x) \rangle}{\partial x}(x_q(t), p_q(t)) - u_2(t)\frac{\partial \langle p, X_2(x) \rangle}{\partial x}(x_q(t), p_q(t))\end{aligned}$$

and

$$\begin{aligned}\langle p_q(t), X_1(x_q(t)) \rangle &= 0, & \langle p_q(t), X_2(x_q(t)) \rangle &= 0, \\ \langle p_q(t), [X_1, X_2](x_q(t)) \rangle &= 0, & \langle p_q(t), [X_1, [X_1, X_2]](x_q(t)) \rangle &\neq 0, \\ \langle p_q(t), [X_2, [X_1, X_2]](x_q(t)) \rangle &\neq 0,\end{aligned}$$

for functions $u_1(t)$ and $u_2(t)$ which satisfy

$$u_1(t)\langle p_q(t), [X_1, [X_1, X_2]](x_q(t)) \rangle + u_2(t)\langle p_q(t), [X_2, [X_1, X_2]](x_q(t)) \rangle = 0.$$

Because they have similar type of proof, we give a proof of Theorem 4.5.

Proof of Theorem 4.5. From the property in Lemma 4.1 we take a local frame $\{X_1, X_2\}$ of $\mathcal{D}$ on an open neighborhood U_{x_0} of x_0 in M such that

$$X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]]$$

are linearly independent and $[X_2, [X_1, X_2]]$ is a functional linear combination of X_1, X_2 and $[X_1, X_2]$. We take the function $H: T^*U_{x_0} \times_{U_{x_0}} \mathcal{D} \rightarrow \mathbb{R}$ as

$$H(x, p, u) = \langle p, X_1(x) \rangle u_1 + \langle p, X_2(x) \rangle u_2$$

and set $a_1(x, p) := \langle p, X_1(x) \rangle$ and $a_2(x, p) := \langle p, X_2(x) \rangle$. Then H is a Morse family because X_1 and X_2 are linearly independent at each point. In conformity with the property of vector fields X_1 and X_2 , functions $a_1, a_2, \{a_1, a_2\}$ and $\{a_1, \{a_1, a_2\}\}$ are independent and we have

$$a_1, a_2, \{a_1, a_2\}, \{a_2, \{a_1, a_2\}\} \in \langle a_1, a_2, \{a_1, \{a_1, a_2\}\} \rangle_{\mathcal{E}_{T^*U_{x_0}, q}}$$

for any $q \in T^*U_{x_0}$.

According to the proof of Proposition 2.21, X_u is a tangent vector field to a submanifold

$$\begin{aligned} A_2^1 &= \{(x, p) \in T^*U_{x_0} \mid a_1(x, p) = a_2(x, p) = \{a_1, a_2\}(x, p) \\ &= \{a_1, \{a_1, a_2\}\}(x, p) = 0\} \end{aligned}$$

of $T^*U_{x_0}$ for $u = (0, u_2)$. Thus there exist an open neighborhood V_{q_0} of $q_0 \in (\pi_{T^*M}|_{A_2^1})^{-1}(x_0)$ and a family of local integral curves $\{(x_q(t), p_q(t))\}_{q \in V_{q_0}}$ of X_u starting from a point q in A_2^1 , that is, for each q the curve $(x_q(t), p_q(t))$ satisfies ordinary differential equations

$$\begin{aligned} \dot{x}_q(t) &= \frac{\partial H}{\partial p}(x_q(t), p_q(t), (0, u_2)) = \frac{\partial \langle p, X_2(x) \rangle}{\partial p}(x_q(t), p_q(t)) = u_2(t) X_2(x_q(t)) \\ \dot{p}_q(t) &= -\frac{\partial H}{\partial x}(x_q(t), p_q(t), (0, u_2)) = -\frac{\partial \langle p, X_2(x) \rangle}{\partial x}(x_q(t), p_q(t)) \end{aligned}$$

and following conditions;

$$\begin{aligned} a_1(x_q(t), p_q(t)) &= \langle p_q(t), X_1(x_q(t)) \rangle = 0, \quad \langle p_q(t), X_2(x_q(t)) \rangle = 0, \\ \langle p_q(t), [X_1, X_2](x_q(t)) \rangle &= 0, \quad \langle p_q(t), [X_1, [X_1, X_2]](x_q(t)) \rangle = 0. \end{aligned}$$

□

The following is a summary of the above theorems.

Proposition 4.8. Let $\mathcal{D}$ be a rank 2 distribution on M^n . If a small growth vector is $(2, 3, 4, \dots)$ at each point x_0 in M , then there exist a closed neighborhood V_{q_0} of $q_0 \in \pi_{T^*M}^{-1}(x_0)$ in T^*M and a smooth family of totally singular abnormal bi-extremals $\{(x_q(t), p_q(t))\}_{q \in V_{q_0}}$ in $T^*M \setminus \{o\}$.

Each element of the family of abnormal extremals in Theorem 4.5 are not normal geodesics on a sub-Riemannian manifold. This is also a different point of Liu and Sussmann's results.

Theorem 4.9 ([26]). Let $(M, \mathcal{D}, g)$ be a sub-Riemannian smooth manifold with a distribution $\mathcal{D}$ of rank two. Suppose that $\mathcal{D}_1 := \mathcal{D} + [\mathcal{D}, \mathcal{D}]$ is a sub-bundle of rank three and $\mathcal{D}_2 := \mathcal{D}_1 + [\mathcal{D}, \mathcal{D}_1]$ is a sub-bundle of rank four. Then for any point x_0 in M , there exist an open neighborhood U_{x_0} of x_0 in M , a closed neighborhood V_{q_0} of $q_0 \in \pi_{T^*M}^{-1}(x_0)$ in $T^*U_{x_0}$ and a smooth $(2n - 4)$ -parameter family of immersive abnormal bi-extremals $\{x_q(t), p_q(t)\}_{q \in V_{q_0}}$ of which projections are not normal geodesics in U_{x_0} .

Proof. Let x_0 be a point of M and let $x_q(t)$ be the smooth immersive abnormal extremal in a neighborhood U_{x_0} of x_0 obtained in Theorem 4.5. Let $(x_q(t), p_q(t))$ be the abnormal bi-extremal considered in the proof of Theorem 4.5. We are going to prove that the curve $x_q(t)$ is not a normal extremal. From Lemma 4.1 we may take a local orthonormal frame $\{X_1, X_2\}$ of $\mathcal{D}$ on U_{x_0} . We consider the Hamiltonian function in terms of the orthonormal frame $\{X_1, X_2\}$;

$$H_E(x, p) = -\frac{1}{2} \sum_{i=1}^2 \langle p, X_i(x) \rangle^2.$$

Suppose that $x_q(t)$ is a normal extremal. Then there must exist a normal bi-extremal of the form $(x_q(t), \tilde{p}_q(t))$ which satisfies the following differential equation;

$$\begin{aligned} \dot{x}_q(t) &= \frac{\partial H_E}{\partial p}(x_q(t), \tilde{p}_q(t)) = -\sum_{i=1}^2 X_i(x_q(t)), \\ \dot{\tilde{p}}_q(t) &= -\frac{\partial H_E}{\partial x}(x_q(t), \tilde{p}_q(t)) = \sum_{i=1}^2 \frac{\partial \langle p, X_i(x) \rangle}{\partial x}(x_q(t), \tilde{p}_q(t)). \end{aligned}$$

Since the abnormal extremal $x_q(t)$ satisfies $\dot{x}_q(t) = u_2(t)X_2(x_q(t))$ by Theorem 4.5

$$u_2(t)X_2(x_q(t)) = \dot{x}_q(t) = -X_1(x_q(t)) - X_2(x_q(t))$$

holds. Thus $X_1(x_q(t)) + (1 + u_2(t))X_2(x_q(t)) = 0$ holds. This is a contradiction to $\{X_1, X_2\}$ being a local frame of $\mathcal{D}$.

□

Remark 2 ([6], Theorem 2.8). It is known that there are no singular minimizer for a generic sub-Riemannian manifold (the genericity is used for the distribution as map-germs) with rank greater than 2.

Acknowledgment.

I would like to thank my supervisor Goo ISHIKAWA for his helps and encouragements during the PhD studies.

I also thanks to Takuo FUKUDA for fruitful discussion. I am grateful to Shyuichi IZUMIYA and Toru OHMOTO for their every support.

References

- [1] A. A. Agrachev, Y. L. Sachkov. Y, *Control theory from the geometric viewpoint*, Springer, (2004).
- [2] V. I. Arnold, V, A. Varchenko, S. M. Gusein-Zade, *Singularities of Differentiable Maps, Vol. I*, Springer, (1985).
- [3] B. Bonnard, M. Chyba, *Singular trajectories and their role in control theory*, Springer, (2003).
- [4] R. Bryant, L. Hsu, Rigidity of integral curves of rank 2 distributions, *invent. math.* **114** (1993) 435-461.
- [5] Y. Chitour, F. Jean, E. Trélat, Singular trajectories of control-affine systems, *Siam J. Control Optim.*, Vol.47, No.2, (2008), pp.1078–1095.
- [6] Y. Chitour, F. Jean, E. Trélat, Genericity results for singular curves, *J. Diff. Geom.*, **73** (2006) 45-73.
- [7] W. L. Chow, Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung. (German) *Math. Ann.* **117** (1939) 98-105.
- [8] P. A. M. Dirac, Generalized Hamiltonian dynamics, *Canadian J. Math.*, **2** (1950) 129 -148.
- [9] T. Fukuda, S. Janeczko, Singularities of implicit differential systems and their integrability, *Banach center publications*, **65** (2004), pp. 23–47.
- [10] T. Fukuda, S. Janeczko, Global properties of integrable implicit Hamiltonian systems, *Proc. of the 2005 Marseille Singularity School and conference*, *World Scientific*, (2007), pp. 111–124.
- [11] T. Fukuda, S. Janeczko, On singularities of Hamiltonian mappings, Geometry and Topology of Caustics- CAUTICS'06, *Banach center publications*, **82** (2008), pp. 55–69.
- [12] T. Fukuda, S. Janeczko, On the Poisson algebra of a singular map, *J. Geom. Phys.*, **86** (2014), pp. 194–202.

- [13] T. Fukuda, S. Janeczko, *A résumé on Workshop on singularities, geometry, topology and related topics (2014, September 1st – 3rd)*, personal communication.
- [14] T. Fukuda, S. Janeczko, Symplectic singularities and solvable Hamiltonian mappings, *Demonstr. Math.* **48** (2015), no. 2, pp. 118–146.
- [15] L. Hsu, Calculus of variations via the Griffiths formalism, *J. Diff. Geom.* **36** (1992), pp. 551–589.
- [16] A. Isidori, Nonlinear control systems (third edition), *Springer-Verlag*, (1995).
- [17] S. Janeczko, On implicit Lagrangian differential systems, *Ann. Polon. Math.* **74** (2000), 133–141.
- [18] S. Janeczko, A note on singular lagrangian submanifolds, *J. Geom. Phys.* **2** (1985), no. 3, pp. 33–59.
- [19] W. Liu, H. J. Sussman, Shortest Paths for sub-Riemannian metrics on rank-two distributions, *American Mathematical Society*, Memoirs of the AMS, vol. **118**, no. 564 (1995).
- [20] R. Montgomery, Abnormal minimizers, SIAM, *J. Control Optim.* **32** 6, (1994) 1605–1620.
- [21] R. Montgomery, A tour of subriemannian geometries, their geodesics and applications, *American Mathematical Society* Mathematical surveys and Monographs vol. **91** (2002).
- [22] P. Mormul, Real moduli in local classification of Goursat flags, *Hokkaido Mathematical Journal* **34** (2005), 1–35.
- [23] A. Nagel, E. M. Stein, S. Wainger, Balls and metrics defined by vector fields I: Basic properties, *Acta Mathematica*, **155** (1985), Issue 1, pp.103–147.
- [24] H. Nijmeijer, A. J. van der schaft, *Nonlinear Dynamical control systems*, Springer-Verlag, (1990).

- [25] A. Tsuchida, Smooth solvability of implicit Hamiltonian systems and existence of singular control for affine control systems (in Japanese), *RIMS Kôkyûroku* **1948**, 153–159.
- [26] A. Tsuchida, Implicit Hamiltonian systems and singular curves of distributions, *accepted*.
- [27] A. Weinstein, The local structure of poisson manifolds, *J. Diff. Geom.*, **18** (1983), pp. 523–557.
- [28] A. Weinstein, Poisson geometry, *Differential Geometry and its Applications*, **9** (1998), pp. 213–238.