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Study on quantum mechanical drift motion and expansion of variance of a charged particle in non-uniform electromagnetic fields

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Abstract

In the field of plasma fusion, grad- B drift and $\mathbf{E} \times \mathbf{B}$ drift velocities are the well-known topics. In the recent years, motion of a charged particles is getting attention not only in the classical approach but also using the quantum approach. In this study, the quantum mechanical effects of a non-relativistic spinless charged particle in the presence of variation inhomogeneity of electromagnetic field are shown.

In Chap. 2, the two-dimensional time-dependent Schrödinger equation, for a magnetized proton in the presence of a fixed field particle and of a homogeneous magnetic field is numerically solved. In the relatively high-speed case, the fast-speed proton has the similar behaviors to those of classical ones. However, in the extension of time, the relatively high-speed case shows similar behavior to the low-speed case: the cyclotron radii in the both mechanical momentum and position are appreciably decreasing with time. However, the kinetic energy and the potential energy do not show appreciable changes. This is because of the increasing variances, i.e. uncertainty, in the both momentum and position. The increment in variance of momentum corresponds to the decrement in the magnitude of mechanical momentum in a classical sense: Part of energy is transferred from the directional (classical kinetic) energy to the uncertainty (quantum mechanical zero-point) energy.

In Chap. 3, by solving the Heisenberg equation of motion operators for a charged particle in the presence of an inhomogeneous magnetic field, the analytical solution for quantum mechanical grad- B drift velocity operator is shown. Using the time dependent operators, it is shown the analytical solution of the position. It is also numerically shown that the grad- B drift velocity operator agrees with the classical counterpart. Using the time dependent operators, it is shown the variance in position and momenta grow with time. The expressions of quantum mechanical expansion rates for position and momenta are also obtain analytically.

In Chap. 4, the Heisenberg equation of motion for the time evolution of the position and momentum operators for a charged particle in the presence of an inhomogeneous electric and magnetic field is solved. It is shown that the analytical $\mathbf{E} \times \mathbf{B}$ drift velocity obtained in this study agrees with the classical counterpart, and that, using the time dependent operators, the variances in position and momentum grow with time. It is also shown that the theoretical expansion rates of variance expansion are in good agreement with the numerical analysis. The expansion rates of variance in position and momentum are dependent on the magnetic gradient scale length, however,

independent of the electric gradient scale length. Therefore, a higher order of non-uniform electric field is introduced in the next chapter.

In Chap. 5, a charged particle in a higher order of electric field inhomogeneity is introduced and the quantum mechanical drift velocity is solved analytically. The analytical solution of the time dependent momenta operators and position operators are shown. With further combination of the operators, the quantum mechanical expansion rates of variance are shown and the results agree with the numerical results. Finally, it is analytically shown the analytical result of quantum mechanical drift velocity, which coincides with the classical drift velocity. The result implies that light particles with low energy would drift faster than classical drift theory predicts. The drift velocity and the expansion rates of variance are dependent on the both electric and magnetic gradient scale length.

In Chap. 6, this study is concluded. It is analytically shown that the variance in position reaches the square of the interparticle separation, which is the characteristic time much shorter than the proton collision time of plasma fusion. After this time the wavefunctions of the neighboring particles would overlap, thus the conventional classical analysis may lose its validity. The expansion time in position implies that the probability density function of such energetic charged particles expands fast in the plane perpendicular to the magnetic field and their Coulomb interaction with other particles becomes weaker than that expected in the classical mechanics.

Contents

Chapter 1	Introduction	1
1.1	Background	
1.2	Probability density function (PDF)	
1.3	Numerical calculation and numerical error	
1.4	Exact wavefunction in a uniform magnetic field	
1.5	Numerical calculation normalization	
1.6	Numerical error subtraction	
	References	
Chapter 2	Quantum mechanical uncertainty energy	11
2.1	Initial condition: Electrostatic potential due to a field particle	
2.2	Errors in numerical calculations	
2.2	Case 1: Repulsive force	
2.3	Expectation values and variances	
2.4	Case 2: Attractive force	
2.5	Summary	
	References	
Chapter 3	Quantum mechanical grad-B drift	22
3.1	Initial condition	
3.2	Time dependent operator for weakly non-uniform magnetic field	
3.3.1	Time average of variance	
3.3.2	Numerical time average of variance	
3.4.1	Expansion rates of variance	
3.4.2	Expansion time	
3.4.3	Numerical expansion rates of variance	
3.5	Analytical grad- B drift velocity	
3.6	Numerical grad- B drift velocity	
3.7	Summary	
	References	

Chapter 4	Quantum mechanical $E \times B$ drift	36
4.1	Initial condition	
4.2	Time dependent operators for the non-uniform electric and magnetic field	
4.2.1	Time dependent operator of expansion of variance	
4.2.2	Expansion times	
4.3	Numerical confirmation	
4.4	Summary	
	References	
Chapter 5	Quantum mechanical $E \times B$ drift at higher order of electromagnetic field inhomogeneity	46
5.1	Initial condition	
5.2	Time dependent operators for the inhomogeneous electric and magnetic field	
5.3	Expansion times	
5.4	Quantum mechanical $E \times B$ drift velocity	
5.5	Summary	
	References	
Chapter 6	Conclusion	61
	Acknowledgement	62

Chapter 1

Introduction

1.1 Background

In the field of plasma physics, grad- B and $E \times B$ drift velocity are the well-known topics. The gyration of the particle or drift motion of a charged particle gain many interest by researchers especially after Alfvén [1] and Spitzer [2] had obtained the expression for the drift velocity of a charged particle in the presence of a non-uniform magnetic field. Following that, the analytical analysis is getting attention by researchers. Seymour [3,4], Hurley [5], and Karlson [6], by using different analytical approach, the exact solutions for a charged particle drift in a static magnetic field are obtained. Different approaches are being studied by researcher, i.e. Birmingham [7] use a bounce-average guiding-center trajectory to solve the drift motion of a charged particle. On the later studies, the drift velocity of a charged particle being studied in a wide field of plasma, however by using the classical approach [8-14].

In recent years, motion of charged particles is getting attention not only in the classical approach but also in the quantum approach [15-20]. In considering the diffusion of plasmas, it was pointed out more than half a century ago [21, 22] that the wave character of a charged particle should be taken into consideration when the temperature is high, i.e. the relative speed of interacting particle are fast. The criterion on the classical theory to be valid in terms of relative speed g in hydrogen plasma is given as, $g \ll 2e^2 / (4\pi\epsilon_0\hbar) = 4.4 \times 10^6$ m/s [22], where e and $\hbar \equiv h/2\pi$ stand for the elementary electric charge and the reduced Planck constant.

As pointed out in Refs. [23-25] that; (i) for distant encounters in the plasma of a temperature $T \sim 10$ keV and $n = 10^{20} \text{ m}^{-3}$, the average potential energy $U \sim 30$ meV is as small as the uncertainty in energy $\Delta E \sim 40$ meV, and (ii) for a magnetic field $B \sim 3$ T, the spatial size of the wavefunction in the plane perpendicular to the magnetic field is as large as the magnetic length $\sigma_B = \sqrt{\hbar/eB} \sim 2 \times 10^{-8}$ m [26], which is larger than the typical electron wavelength $\lambda_e \sim 10^{-11}$ m, and is around one-tenth of the average interparticle separation $n^{-1/3} \sim 2 \times 10^{-7}$ m. Thus, for plasma with a temperature $T \sim 10$ keV or higher, ions as well as electrons should be treated quantum mechanically. In current plasma physics, the quantum mechanical effect enters as a minor correction to the Coulomb logarithm in the case of close encounter [27]. Nonetheless, the neoclassical theory [28], is capable of predicting a lot of phenomena such as related to the current conduction.

Such phenomena linearly depend on the change in momentum $\Delta \mathbf{p} = m\Delta \mathbf{v}$ or in position $\Delta \mathbf{r}$ due to the Coulomb interaction. The diffusion, however, is quadratic function changes, such as $\Delta \mathbf{p}^2$ and $\Delta \mathbf{r}^2$, and are not properly accounted for the existing classical and the neoclassical theories. Corresponding to these facts, authors conducted the quantum mechanical analyses on the both numerically [29-35] and also analytically [36-38] on a single charged particle in the presence of

external electromagnetic fields, focusing especially on the time development of variance in position and momentum. For the plasma mentioned above, the deviation $\sigma_r(t)$ of the ions would reach the interparticle separation $n^{-1/3}$ in a time interval of the order of 10^{-4} sec. After this time the wavefunctions of the neighboring particles would overlap, as a result the conventional classical analysis may lose its validity. Plasmas may behave like extremely-low-density liquids, not gases, since the size σ_r of each particle is the same order of the interparticle separation $n^{-1/3}$.

1.2 Probability density function (PDF)

It is well known that a charged particle in the presence of a non-uniform magnetic field $\mathbf{B}$ tends to move to regions of weak magnetic field via collision/interaction with other particles. In quantum mechanics, the probability density function (PDF) for a charged particle in the presence of a uniform magnetic field $\mathbf{B} = B_0 \mathbf{e}_z$ along the z -axis in the $x-y$ plane perpendicular to the field is given by

$$\rho(\mathbf{r}, t) = \frac{qB_0}{\pi\hbar} \exp\left[-\frac{qB_0}{\hbar} (\mathbf{r} - \langle \mathbf{r}(t) \rangle)^2\right],$$

where $\langle \mathbf{r}(t) \rangle$ stands for the expectation value of the position that is the same as the time-dependent position of the corresponding classical particle. The tendency towards weak $B = |\mathbf{B}|$ field region stated above makes the PDF of the particle broader than that for a uniform field case.

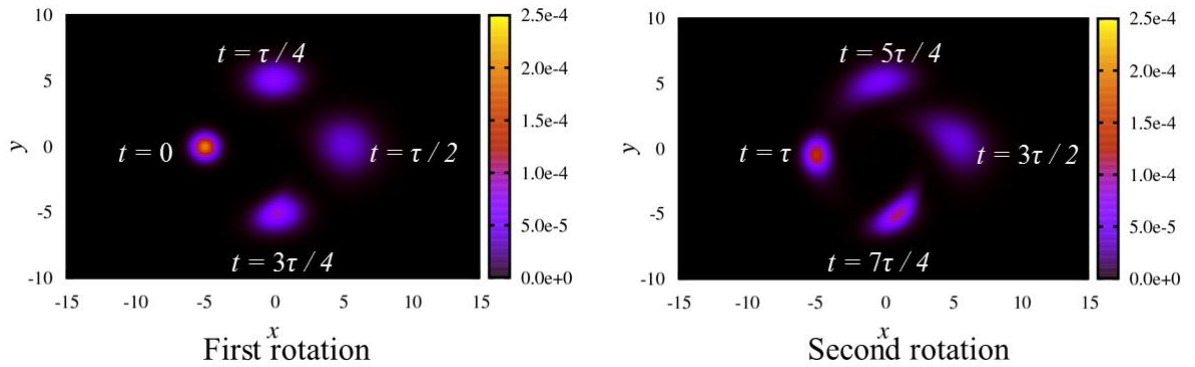


Fig. 1.1 Quantum mechanical expansion of a charged particle. Figure on the left shows the first rotation of the probability density function (PDF) and figure on the right is the second rotation of the PDF.

The circular-shaped initial PDF change its shape as time goes by. Shape changes due to the influence of the field particle at the origin. As shown in figure above, the particle gyrates in clockwise direction. The particle expands after the 1st gyration and diffused slightly on the 2nd rotation and so on for the 3rd rotation.

1.3 Numerical calculation and numerical error

The two-dimensional Schrödinger equation is solved numerically [29-35] and theoretically [36-38] for a wavefunction ψ at position $\mathbf{r}$ and time t ,

$$i\hbar \frac{\partial \psi}{\partial t} = \left\{ \frac{1}{2m} (-i\hbar \nabla - q\mathbf{A})^2 + qV \right\} \psi, \quad (1.1)$$

where V and $\mathbf{A}$ stand for the scalar and vector potentials, m and q the mass and electric charge of the particle under consideration and $i \equiv \sqrt{-1}$ is the imaginary unit. The initial condition for wavefunction at $\mathbf{r} = \mathbf{r}_0$ with $\mathbf{r}_0$ being the initial center of wavefunction ψ is given by

$$\psi(\mathbf{r}, 0) = \frac{1}{\sqrt{\pi}\sigma_B} \exp \left\{ -\frac{(\mathbf{r} - \mathbf{r}_0)^2}{2\sigma_B^2} + i\mathbf{k}_0 \cdot \mathbf{r} \right\}, \quad (1.2)$$

where the magnetic length σ_B is the initial standard deviation, and $\hbar\mathbf{k}_0$ is the initial canonical momentum.

For the numerical calculations, a two-dimensional Schrödinger equation code is developed and the calculations are done on a GPU (Nvidia GTX-980: 2048cores/4GB @1.126GHz), using CUDA [39]. Successive over relaxation (SOR) scheme for time integration in is use in the numerical calculation. Furthermore, the numerical errors had removed from the numerical calculation by subtracting the variances in the uniform magnetic field to the non-uniform magnetic field [29, 30]. By using the finite difference method in space with the Crank-Nicolson scheme for the time integration, the Schrödinger equation above become as

$$\left(\mathbf{I} - \frac{\Delta t}{2i\hbar} \mathbf{H} \right) \{\psi^{n+1}\} = \left(\mathbf{I} + \frac{\Delta t}{2i\hbar} \mathbf{H} \right) \{\psi^n\}, \quad (1.6)$$

where $\mathbf{I}$ is a unit matrix, $\mathbf{H}$ the numerical Hamiltonian matrix, and $\{\psi^n\}$ stands for the discretized set of the two dimensional time-dependent wavefunction $\psi(x, y, t)$ at a discrete time $t_n = n\Delta t$ to be solved numerically.

The canonical momentum in x -direction, $P_x = mv_x + qA_x$, is conserved as much as 10–11 digits. In this case, we used a normalized initial momentum of $mv = (0, 1)$. Results shown in Fig. 1.2, errors of momentum in x -direction, are sufficiently small enough for validity. We obtain energy error in the particle at a certain time by comparing the numerical results with the initial value. The results are proven small enough for validity as shown in Fig. 1.3. Energy in our calculation is conserved as much as 10–11 digits. In this research, our normalized initial energy, $E \approx 95.3 \sim 10^2$, is used to compare with our numerical calculation. Note that the initial energy is the order of $E=10^2$, thus the relative error in energy is around the order of 10^{-11} .

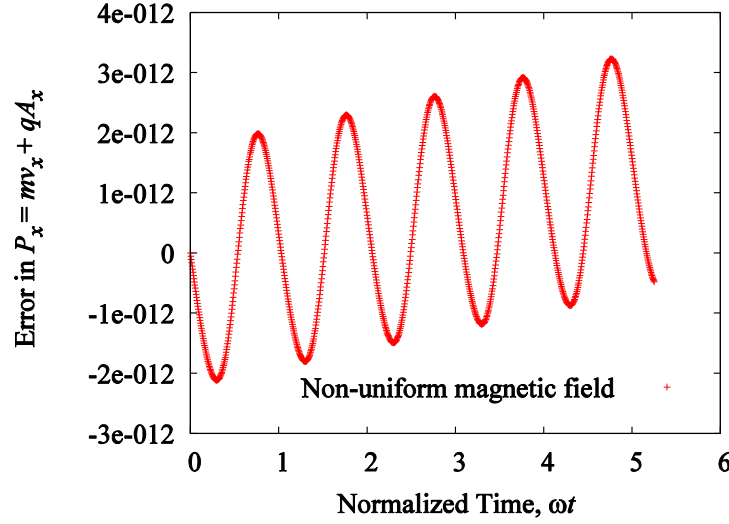


Fig. 1.2 Numerical error in momentum in x – direction for non-uniform magnetic field $L_B = 5.219 \times 10^{-4} \text{ m}$.

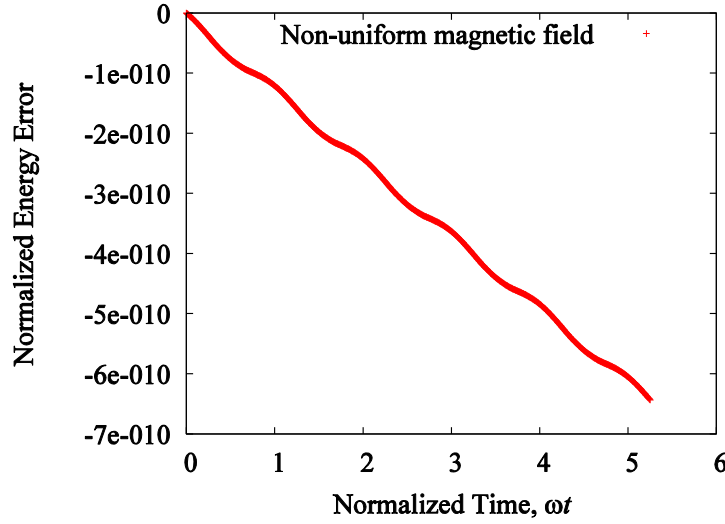


Fig. 1.3 Numerical error in energy of non-uniform magnetic field with $L_B = 5.219 \times 10^{-4} \text{ m}$.

1.4 Exact wavefunction in a uniform magnetic field

The exact solution $\psi(\mathbf{r}, t)$ for the two-dimensional Schrödinger Eq. (1.1) with a uniform magnetic field with a Landau gauge [6], of $A_x = -By, A_y = 0, A_z = 0$, is shown as

$$\psi(\mathbf{r}, t) = \frac{e^{ikx}}{\sqrt{\sqrt{\pi}\ell_B}} \exp\left(-\frac{1}{2\ell_B^2}\left(y - \frac{u(t)}{\omega}\right)^2\right) \times \exp\left[i\left(\frac{y_0^2 \sin 2\omega t}{4\ell_B^2} - \frac{yy_0 \sin \omega t}{\ell_B^2} - \frac{\omega t}{2}\right)\right], \quad (1.7)$$

where $\ell_B \equiv \sqrt{\hbar/qB}$ is the magnetic length [6], the $\omega \equiv qB/m$ is the cyclotron frequency, $y_0 = k\ell_B^2$, and $u(t)$ is the classical velocity of the particle in x – direction. By referring to Eq. (1.7) above, it is obvious that the standard deviation, variance, or uncertainty, in position remain constant throughout the time. Therefore, in the case of uniform magnetic field, $L_B = \infty$, $\sigma_r^2(t) = \ell_B^2 = \text{const}$.

1.5 Numerical calculation normalization

In the numerical calculation, the following parameters as listed in Table 1.1 are the normalized parameters use in this study. The lengths are normalized by a cyclotron radius of a proton with a speed of 10 m/s in a magnetic field of 10 T. The cyclotron frequency in such a case is used for normalization of the time.

Throughout the calculation, we use the normalized grid size of $\Delta x = \Delta y = 0.02$ and the normalized time step of $\Delta t = 2\pi \times 10^{-5}$. This normalized grid size is sufficiently small to use as noted in Ref. [25].

Table 1.1 Normalized parameter for mass of the particle, charge, magnetic flux density, velocity, length and time.

Parameters	Normalization
Mass of the particle	$m = 1.6722 \times 10^{-27} \text{ kg}$
Charge	$q = 1.602 \times 10^{-19} \text{ C}$
Magnetic flux density	$B = 10 \text{ T}$
Velocity	$\bar{v} = 10 \text{ m/s}$
Length	$\bar{\rho} = 1.04382 \times 10^{-8} \text{ m}$
Time	$\bar{t} = 1.04382 \times 10^{-9} \text{ s}$

Figure 1.4 and 1.5 show the grid size and the timestep dependence of the increment of peak variance. The normalized grid size of $\Delta x = \Delta y \leq 0.1$ is sufficiently small to use. The smaller the grid size, the more time is consumed for calculation without much improvement on the accuracy. The same go for the timestep, the smaller timestep we use, and more time consumed for calculation. These will result in higher cost of calculating. Thus, we use $\Delta x = \Delta y = 0.02$ and $\Delta t = 2\pi \times 10^{-5}$ throughout our calculation for optimal accuracy and calculation cost.

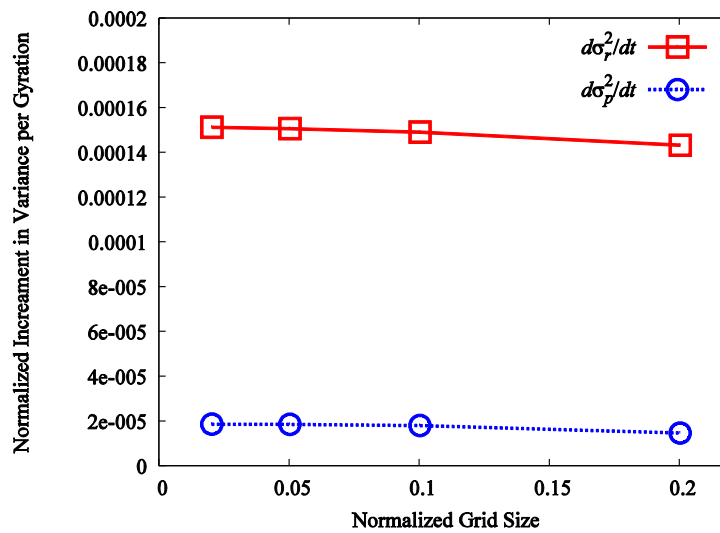


Fig. 1.4 Normalized increment variance per gyration against normalized grid size, $\Delta x = \Delta y$ with $\Delta t = 2\pi \times 10^{-5}$ for a speed of 10 m/s and non-uniform magnetic field $L_B = 5.219 \times 10^{-4} \text{ m}$.

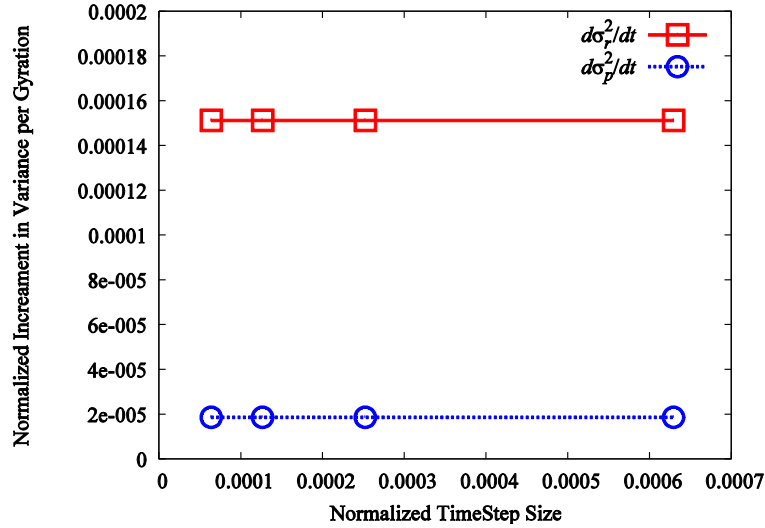


Fig. 1.5 Normalized increment variance per gyration against normalized timestep size, $\Delta x = \Delta y = 0.02$ with timestep for a speed of 10 m/s and non-uniform magnetic field $L_B = 5.219 \times 10^{-4} \text{ m}$.

1.6 Numerical error subtraction

The variance, or the uncertainty, in position of a particle is shown in Fig. 1.6. In the numerical calculation, we have numerically calculated the particle for 5 gyrations. Both uniform magnetic field $L_B = \infty$, and the non-uniform magnetic field $L_B = 5.219 \times 10^{-4} \text{ m}$ variance in position, σ_r^2 , have small difference in value. Each maximum or top peak of the normalized variance for both uniform magnetic field $L_B = \infty$, and non-uniform magnetic field $L_B = 5.219 \times 10^{-4} \text{ m}$ are recoded. Thus, we have 5 maximum or top peak values as shown in Fig. 1.7.

Referring to the exact wavefunction in a uniform magnetic field $L_B = \infty$, the variance in position should remain constant: $\sigma_r^2(t) = \ell_B^2 = \text{const}$. However, there is slight increment in the variance as shown in Fig. 1.8. The difference between the numerical calculation and the theoretical value here is attributed to the numerical errors. The numerical errors are due to the inevitable non-zero grid size and the time-step as well as the finite bit calculation.

In the numerical calculation, both the non-uniform magnetic field and the uniform magnetic fields' increments of variance in position are assumed to consist of the same numerical errors. In this case, both the non-uniform magnetic field variance and the uniform magnetic field variances behave non-linearly, as shown in Fig. 1.8. Since this numerical errors are undesirable in the calculation, we subtract the increment in variance for the non-uniform magnetic field with the non-uniform magnetic field.

After the subtraction of the non-uniform magnetic field's peak variance in position with the uniform magnetic field's peaks variance, we get a linear relationship for the increment of variance in position as shown in Fig. 1.9. This method is used for the following Chap. 3-5, however we measure the expansion rates of variance using the average value.

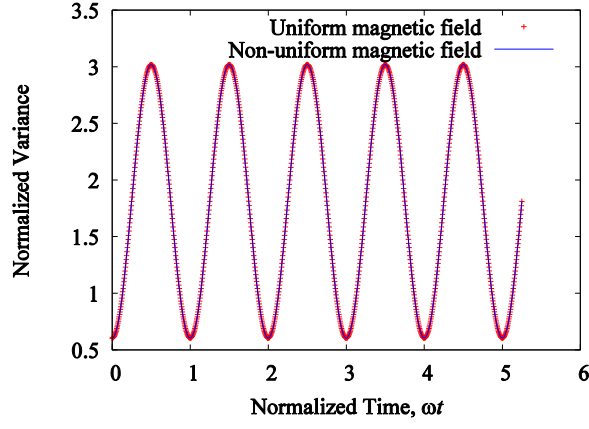


Fig. 1.6 Comparison of σ_r^2 between the uniform magnetic field $L_B = \infty$, in red plot, and the non-uniform magnetic field $L_B = 5.219 \times 10^{-4} \text{ m}$, in blue line.

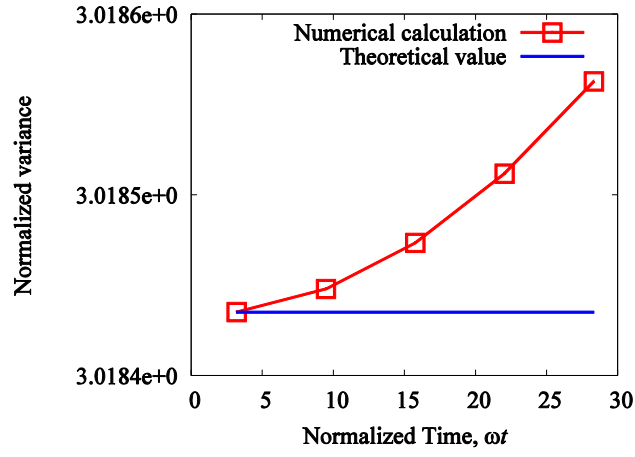


Fig. 1.7 Comparison of σ_r^2 normalized peak variance in position of uniform magnetic field $L_B = \infty$, between numerical calculation, in red and theoretical σ_r^2 , in blue dotted line.

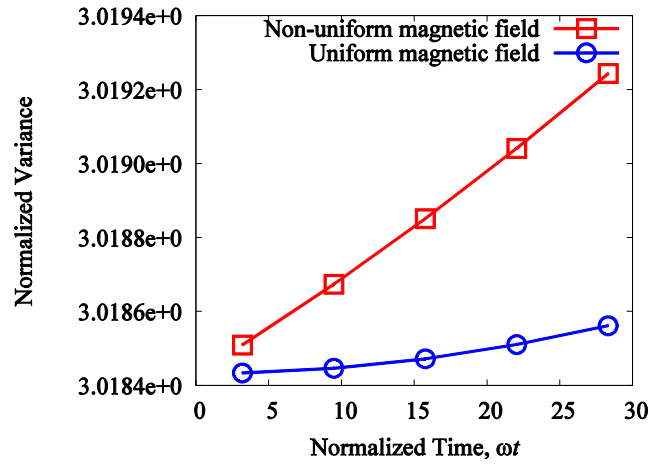


Fig. 1.8 Comparison of σ_r^2 normalized increment of peak variance in position σ_r^2 between uniform magnetic field $L_B = \infty$, in blue circle, compare with $L_B = 5.219 \times 10^{-4} \text{ m}$, in red square.

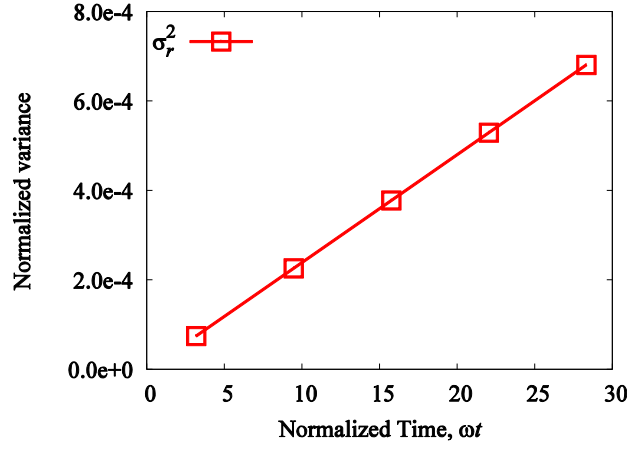


Fig. 1.9 Normalized increment of peak variance in position σ_r^2 for non-uniform magnetic field $L_B = 5.219 \times 10^{-4}$ m after subtraction with uniform magnetic field $L_B = \infty$ peak variance in position σ_r^2 .

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Chapter 2

Quantum mechanical uncertainty energy

2.1 Initial condition: Electrostatic potential due to a field particle

Here the field particle is a quantum-mechanical particle, whose center is assumed to be at the origin with the wavefunction ψ_f similar to that given in Eq. (1.5), but is fixed in the space and the time, as

$$\psi_f(r) = \frac{\exp\left(-\frac{x^2 + y^2}{2\sigma_B^2}\right)}{\sqrt{\pi\sigma_B^2}} \times \frac{\exp\left(-\frac{z^2}{2\sigma_z^2}\right)}{\sqrt{\pi\sigma_z^2}}, \quad (2.1)$$

where σ_z^2 is the variance in position in z -direction, i.e., along the magnetic field. In the magnetically confined fusion plasmas, $\sigma_z \sim h/mv_0 \ll \sigma_B$ holds, so that the square of the second factor can be approximately the same as a Dirac delta function $\delta(z)$ centered at $z = 0$. Thus, the electrostatic potential V_f in the $x-y$ plane, due to the distributed charge is given by

$$V_f(R) \simeq \frac{q_f}{4\pi\epsilon_0\sigma_B^2} \frac{4}{\pi} \int_0^\infty \frac{R' e^{-(R'/\sigma_B)^2}}{R + R'} K(M) dR', \quad (2.2)$$

where $R = \sqrt{x^2 + y^2}$, q_f is an electric charge of the field particle, ϵ_0 is the vacuum permittivity, and $K(M)$ is the complete elliptic integral of the first kind with the parameter M being defined as $M \equiv 4RR'/(R + R')^2$ [1]. In the case of large distance $R \gg \sigma_B$ from the field particle, the electrostatic potential in Eq. (2.2) will become a classical electrostatic potential, $V_f(R) = q_f/(4\pi\epsilon_0 R)$ by a point charge q_f .

2.2 Errors in numerical calculations

In these numerical calculations, the normalized initial speeds in the range of $1 \leq v_0 \leq 10$, which are much slower than the thermal speed of a fusion plasmas, used here due to the numerical reason. The restriction on numerical grid sizes Δx and Δy has demanded a lot of computer memory for a fast particle. The required grid sizes that need to be much smaller than the de Broglie wavelength and inversely proportional to the particle speed. In fusion plasmas, these combinations of an initial speed range v_0 and electric charge $q_f = 2 \times 10^{-5} e$ have the similar potential-to-kinetic-energy ratio.

The accuracy of the calculations are shown in Fig. 2.1, in which the energy is conserved as much as 9 digits by comparing the errors to the normalized initial energy $E = 50.254$ for $v_0 = 10$, $E = 13.458$ for $v_0 = 5$, and $E = 3.892$ for $v_0 = 1$.

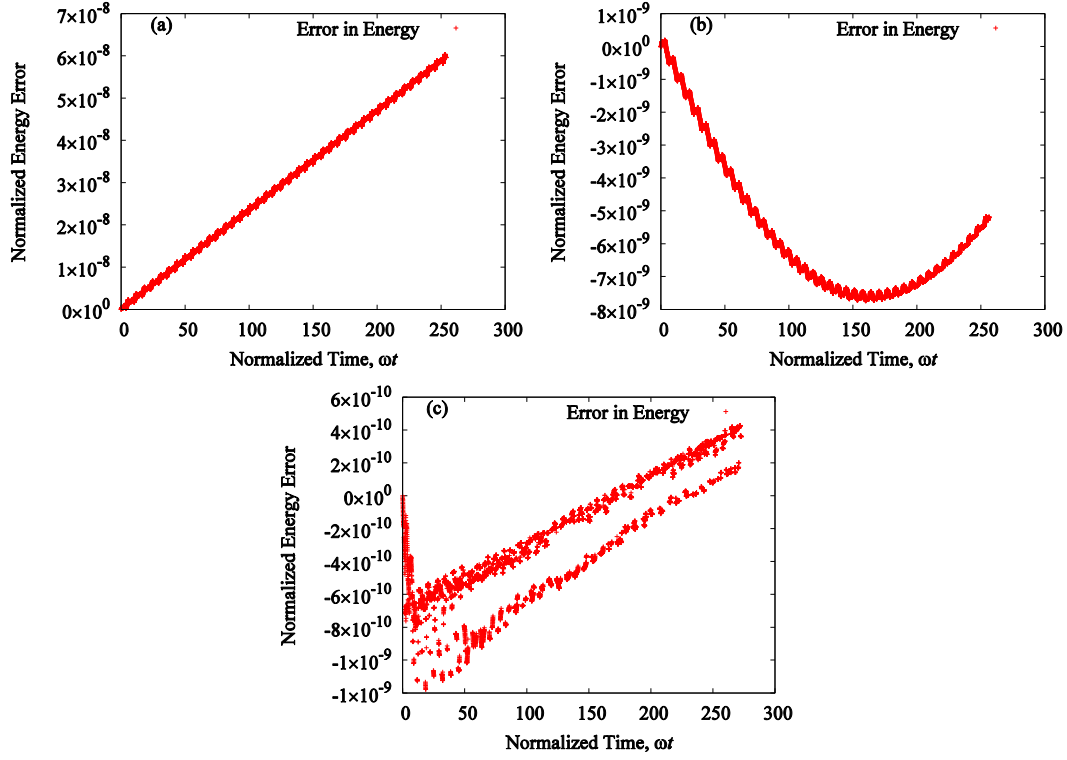


Fig. 2.1 Time evolution of relative errors in energy, in the presence of a fixed field particle at the origin. (a) Normalized initial speed $v_0 = 10$. (b) Normalized initial speed $v_0 = 5$. (c) Normalized initial speed $v_0 = 1$.

2.2 Case 1: Repulsive force

In these numerical calculations, the parameters are normalized as; the mass of a particle $m = 1.6722 \times 10^{-27}$ kg, charge $q = 1.602 \times 10^{-19}$ C, magnetic flux density $B = 10$ T, velocity $\bar{v} = 10$ m/s, length $\bar{\rho} = 1.04382 \times 10^{-8}$ m and time $\bar{t} = 1.04382 \times 10^{-9}$ s. The lengths are normalized by cyclotron radius of a proton with a speed of 10 m/s in a magnetic field of 10 T. The cyclotron frequency in such a case is used for normalization of the time.

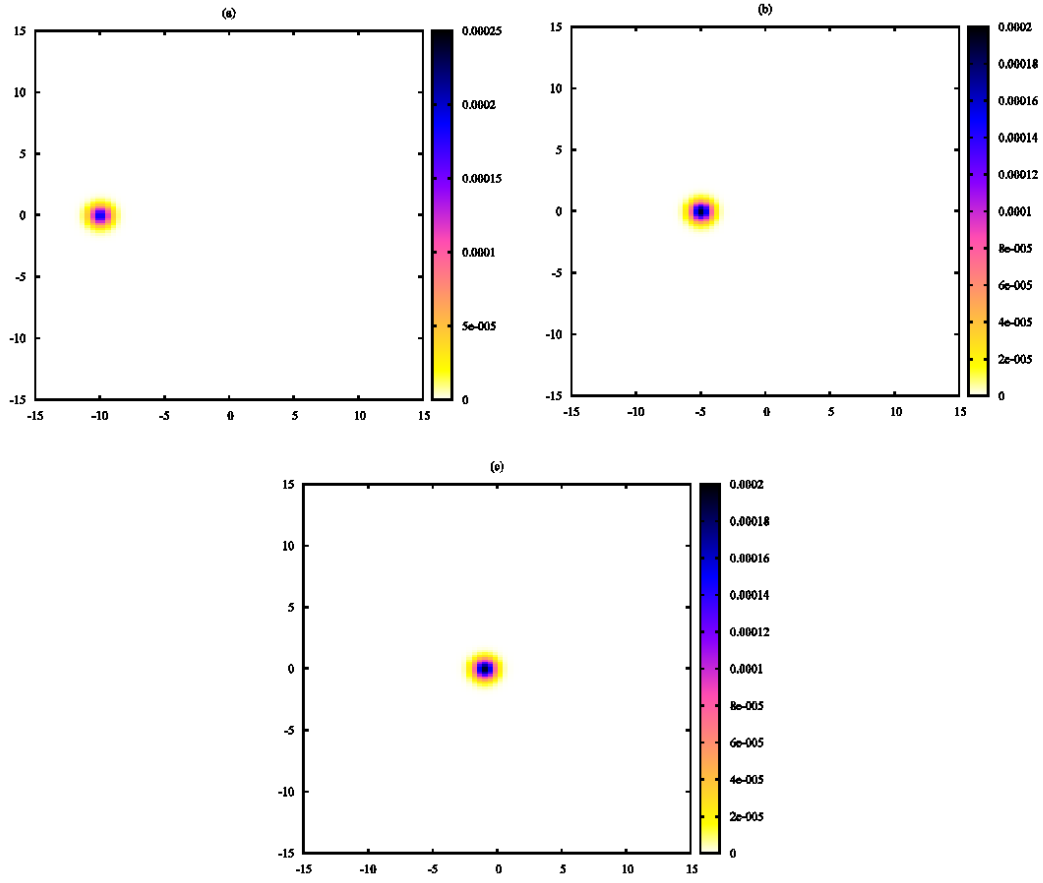


Fig. 2.2 Initial condition $t = 0$ of probability density function (PDF) of a single charged particle, in the presence of a fixed field particle at the origin. (a) Normalized initial speed $v_0 = 10$. (b) Normalized initial speed $v_0 = 5$. (c) Normalized initial speed $v_0 = 1$.

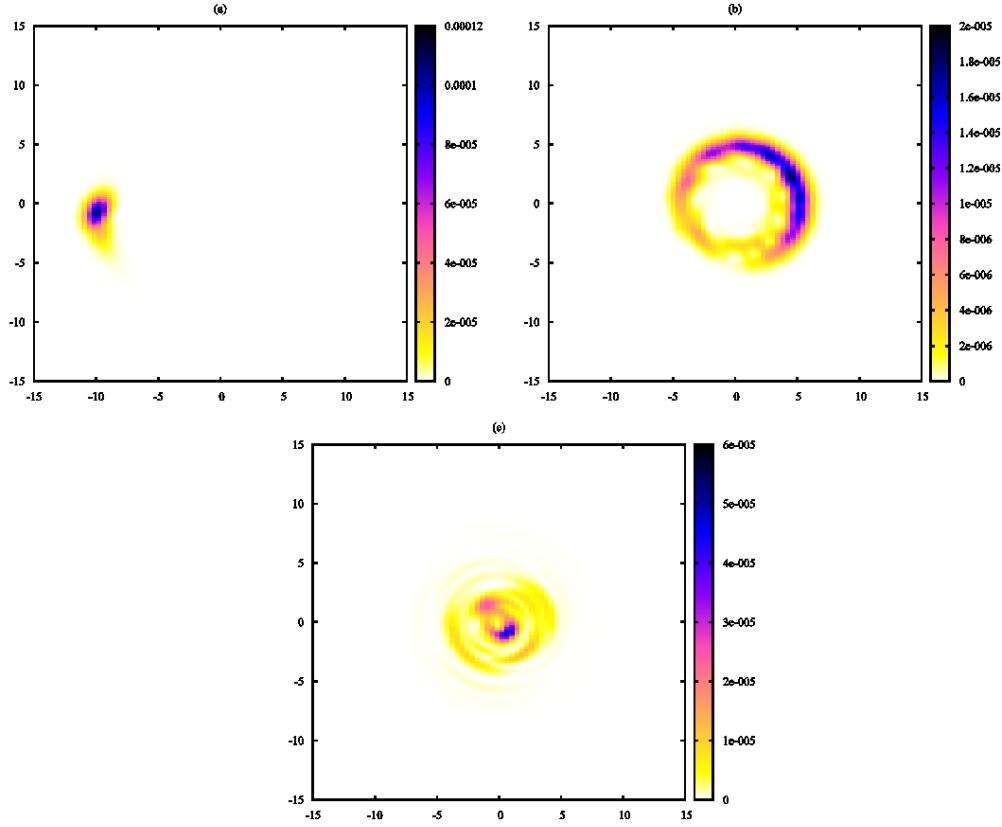


Fig. 2.3 The PDF of a single charged particle, in the presence of a fixed field particle at the origin, after 40 gyrations.

(a) Normalized initial speed $v_0 = 10$. (b) Normalized initial speed $v_0 = 5$. (c) Normalized initial speed $v_0 = 1$.

The magnetic length for a proton in $B = 10$ T, $\sigma_B \equiv \sqrt{\hbar / eB} \sim 10^{-8}$ m is a measurement for the spread of a wave function in the plane perpendicular to the magnetic field. With these normalization, the reduced Planck constant $\hbar \sim 0.60$, the initial uncertainty in position $\sigma_B \equiv \sqrt{\hbar / eB} \sim \sqrt{0.60}$ and the initial uncertainty in kinetic momentum $(3/2)\hbar eB \sim 0.91$ are the order of unity. Note that the kinetic energy of a classical proton speed $v_0 \sim 27$ m/s in $B = 10$ T is corresponding to the uncertainty of the momentum. In the numerical results to be presented in the following subsections, the Schrödinger Equation is solved numerically for the time duration of forty cyclotron rotations by a proton.

For the low-speed case, $v_0 = 5$ and $v_0 = 1$, initially the probability distribution function (PDF) is a circular shape and then changes to elongated shape at second gyration. This tendency grows further for each gyration. Eventually at the 40th gyration, the PDF of the particle tends to have almost uniformly distributed along the classical cyclotron orbit, as shown in Fig. 2.3(b), in which the width of the distribution is nearly the magnetic length of $\sigma_B = \sqrt{\hbar / qB}$.

2.3 Expectation values and variances

The time evolution of total energy $E = K + U$, kinetic energy $K = \langle (mv)^2 \rangle / 2m$ and the potential energy $U = \langle qV \rangle$ is shown in Fig. 2.4 for the initial speed of $v_0 = 10$, $v_0 = 5$ and $v_0 = 1$. In

quantum mechanical point of view, kinetic energy is the sum of directional energy and uncertainty energy. Referring to Fig. 2.4(a), for a high-speed case, $v_0 = 10$ the particle shows the similar behavior to the classical one. Only small amount of the directional energy is converted to the uncertainty energy. However, for the slow particle-cases, $v_0 = 5$ and $v_0 = 1$ it is shown in Fig. 2.4(b,c) that the directional energy is converted to uncertainty energy after some time. Comparing with the same amount of gyrations, this phenomenon is significant for slower particle-cases.

Figure 2.5 shows the time evolution of normalized variance of momentum $\sigma_p^2 = \langle (mv)^2 \rangle - \langle mv \rangle^2$, and that of position $\sigma_r^2 = \langle r^2 \rangle - \langle r \rangle^2$, for the high and low speed particle-cases. Figure 2.6 compares the expectation value of mechanical momentum $\langle mv \rangle$ and Fig. 2.7 is the comparison of expectation value of position $\langle r \rangle$ for the high and low speed particle-cases.

In the case of high speed particle, $v_0 = 10$, the trajectory in the phase $\langle r, p \rangle$ is similar to the classical one and the variance in momentum and position oscillate with almost constant amplitudes [2]. However, the trajectories in the both momentum space and position in space are gradually decreasing in respective to the cyclotron radii. In the meantime, the variance oscillations show significant increase in the variance $\Delta\sigma^2$.

In the longer period of evolutions, the expansion phenomenon shows the similar outcome to the lower speed case, $v_0 = 5$. It is shown that a significant decrement in radii with time for both trajectories in momentum space and the configuration space. At the same time, the variances in momentum and configuration space grow with time and reach saturation. The saturation stage reached when all the directional energy is being converted to the uncertainty energy.

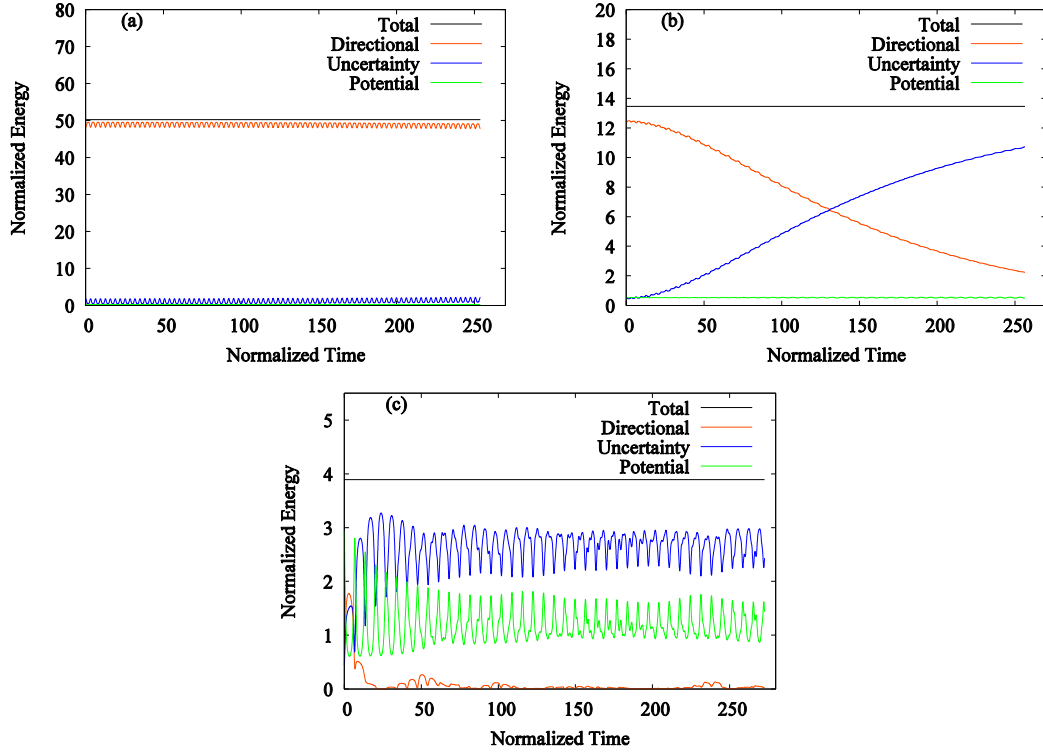


Fig. 2.4 Time evolution of normalized energies for 40 gyrations. (a) Normalized initial speed $v_0 = 10$ and normalized initial energy, $E = 50.254$. (b) Normalized initial speed $v_0 = 5$ and normalized initial energy, $E = 13.458$. (c) Normalized initial speed $v_0 = 1$ and normalized initial energy, $E = 3.892$.

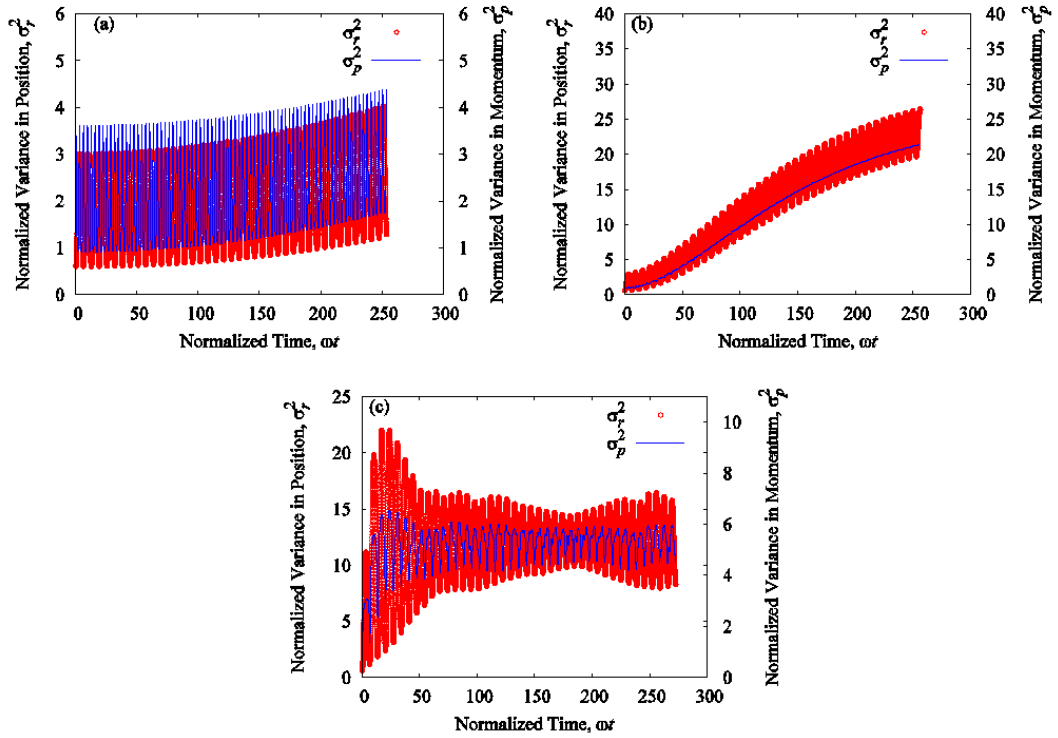


Fig. 2.5 Normalized variance in position σ_r^2 and total momentum σ_p^2 , for 40 gyrations. (a) Normalized initial speed $v_0 = 10$. (b) Normalized initial speed $v_0 = 5$. (c) Normalized initial speed $v_0 = 1$.

In the low speed case, $v_0 = 1$, most of the directional energy is converted to the uncertainty energy in the first 10 gyrations. Corresponding to this phenomenon, the variance in momentum and position reach saturation at the same time. However, due to repulsion by the field particle, there are large amplitude oscillation in the kinetic and the potential energy. The particle gyration is not in a fixed orbit as shown in Fig. 2.7(c).

The decrements in the magnitude of momentum $\langle mv \rangle$ as shown in Fig. 2.6 are correspond to the increment in the variance of momentum

$$\Delta\sigma_p^2 = \langle (mv)^2 \rangle - \langle mv \rangle^2 - (3/2)\hbar qB. \quad (2.3)$$

As the results of conservation of energy, part of the energy is transferred from the directional energy to the uncertainty $\Delta\sigma_p^2/2m$ or the zero-point energy [3].

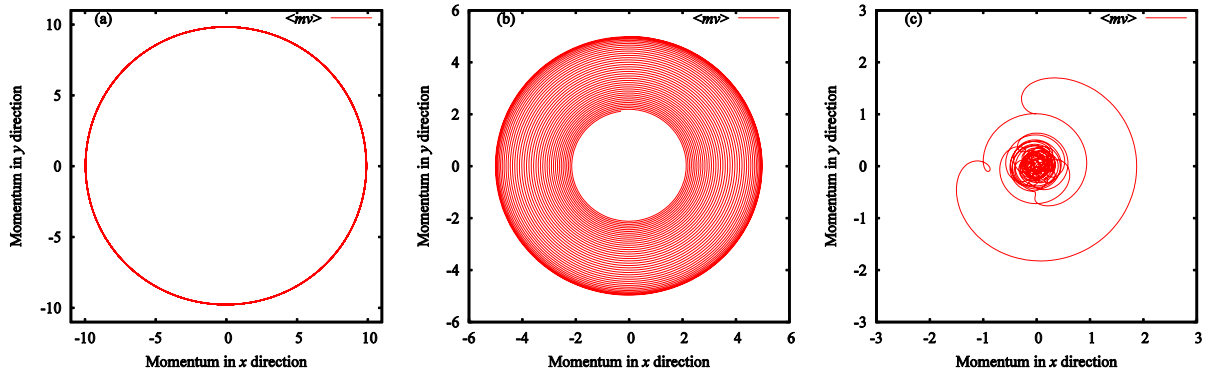


Fig. 2.6 Normalized expectation values of momentum $\mathbf{p} = \langle mv \rangle$ after 40 gyrations. (a) Normalized initial speed $v_0 = 10$ and normalized initial momentum $(mu_0, mv_0) = (0, 10)$. (b) Normalized initial speed $v_0 = 5$ and normalized initial momentum $(mu_0, mv_0) = (0, 5)$. (c) Normalized initial speed $v_0 = 1$ and normalized initial momentum $(mu_0, mv_0) = (0, 1)$.

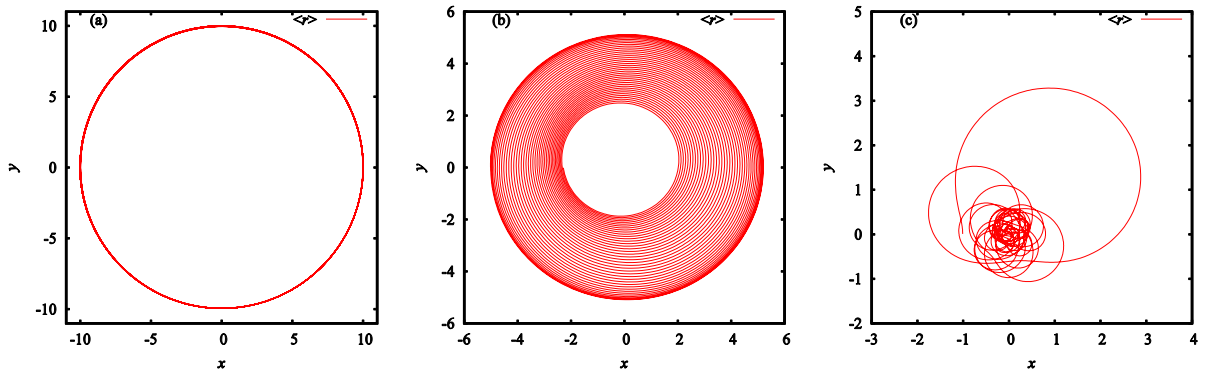


Fig. 2.7 Normalized expectation position $\langle \mathbf{r} \rangle$ after 40 gyrations. (a) Normalized initial speed $v_0 = 10$ and initial position $(10, 0)$. (b) Normalized initial speed $v_0 = 5$ and initial position $(-5, 0)$. (c) Normalized initial speed $v_0 = 1$ and initial position $(-1, 0)$.

2.4 Case 2: Attractive force

In the previous section, the study on quantum mechanical expansion for a proton which gyrates around a fixed positive charge is discussed. It is interesting to note that for a repulsive force the radii of gyration decrease with the gyration cycle due to the directional energy conversion to the uncertainty energy. In this section, further study is conducted for a proton which gyrates around a fixed negative charge which creating an attractive force between the proton and the field particle.

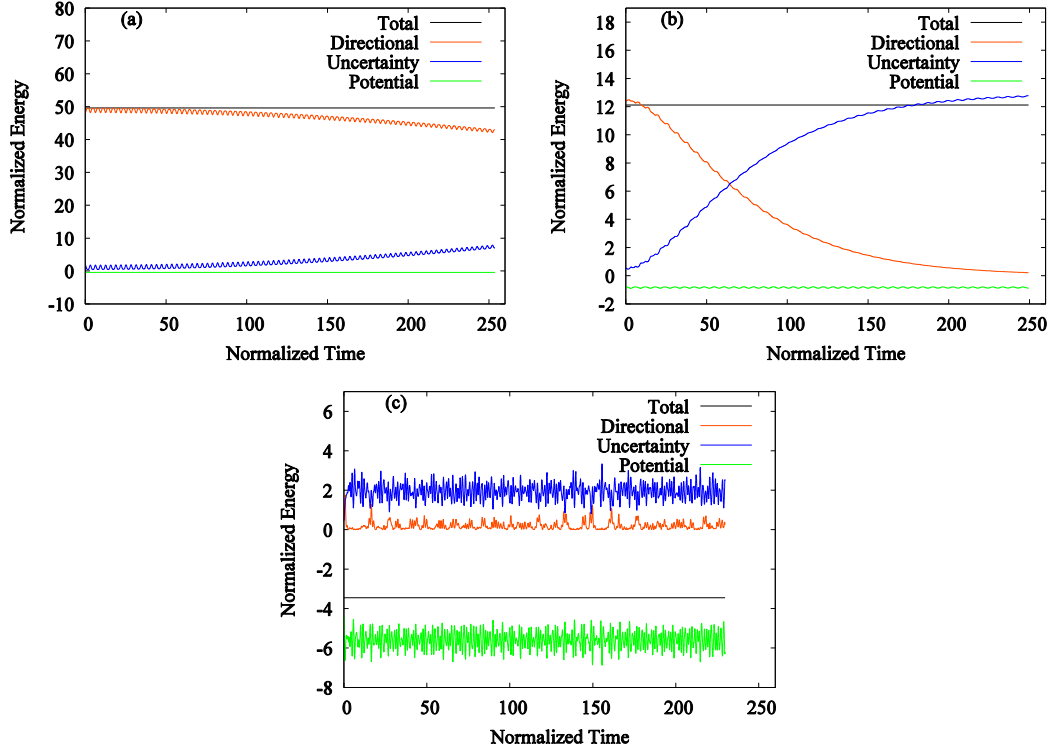


Fig. 2.8 Time evolution of normalized energy for 40 gyrations. (a) Normalized initial speed $v_0 = 10$ and initial normalized energy, $E = 49.592$. (b) $v_0 = 5$ with $E = 12.119$. (c) $v_0 = 1$ with $E = -3.457$.

In Fig. 2.8, the time evolution of normalized energy for 40 gyrations are shown. The energy conversion due to the negative field particle (attractive force) is shown in Fig. 2.8, while the positive charged field particle-case (repulsive force) was shown in Fig. 2.4. It is shown that the quantum mechanical expansion of a particle has similar behaviors for the both attractive and repulsive force cases. However, in the comparison with the same 40 gyrations cycle time between the attractive force case and the repulsive force case, the attractive force case shows faster conversion of the directional energy to the uncertainty energy.

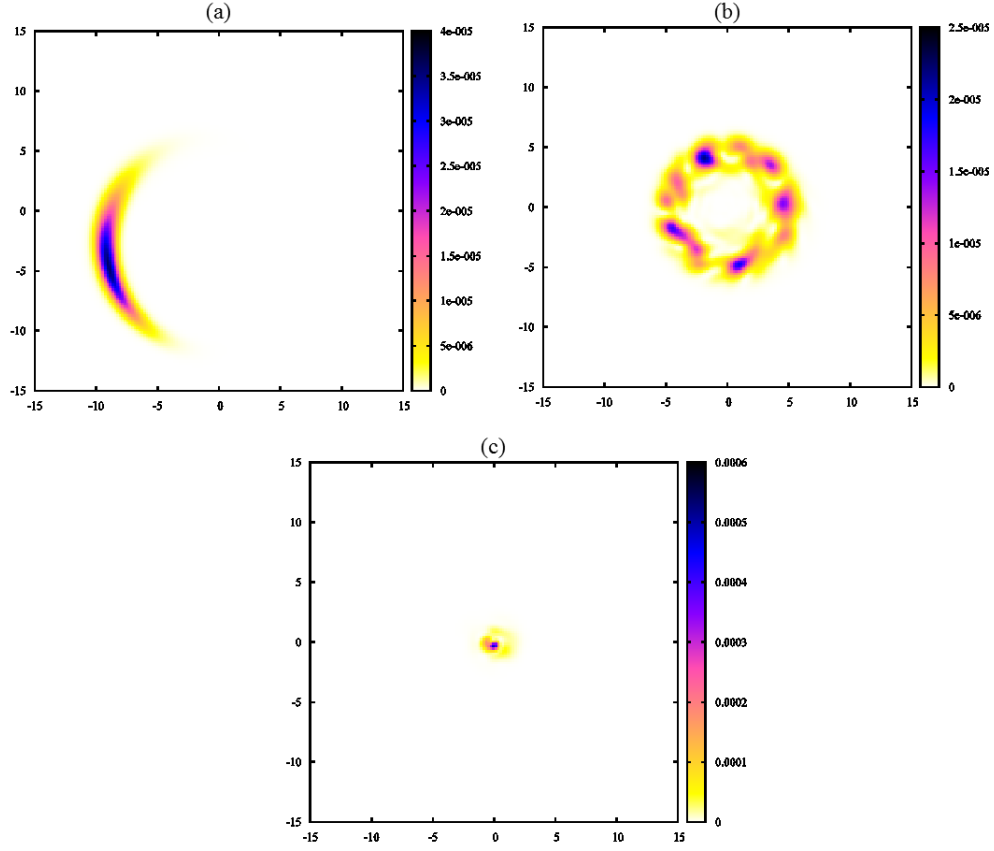


Fig. 2.9 The PDF of a single charged particle, in the presence of a fixed field particle at the origin, after 40 gyrations.

(a) Normalized initial speed $v_0 = -10$. (b) Normalized initial speed $v_0 = -5$. (c) Normalized initial speed $v_0 = -1$.

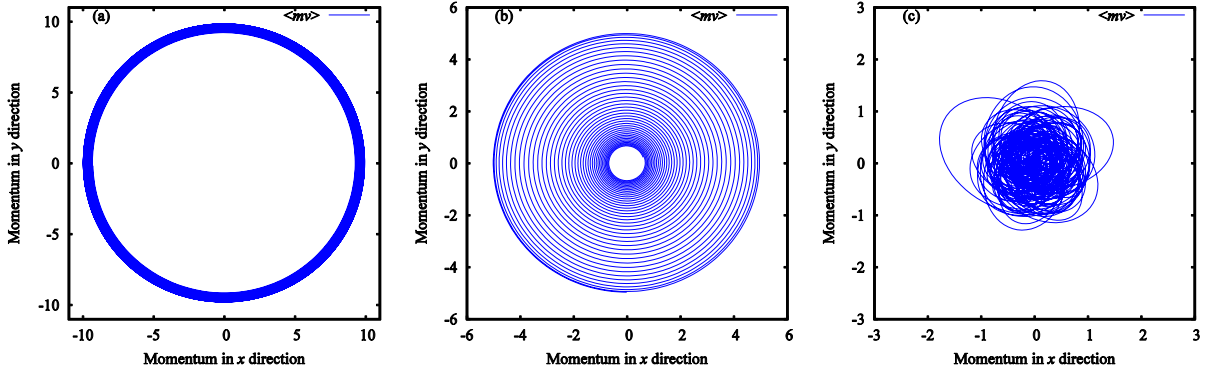


Fig. 2.10 Normalized expectation values of momentum $\mathbf{p} = \langle m\mathbf{v} \rangle$ after 40 gyrations. (a) Normalized initial speed $v_0 = -10$

and normalized initial momentum $(mu_0, mv_0) = (0, -10)$. (b) Normalized initial speed $v_0 = -5$ and normalized initial

momentum $(mu_0, mv_0) = (0, -5)$. (c) Normalized initial speed $v_0 = -1$ and normalized initial momentum

$(mu_0, mv_0) = (0, -1)$.

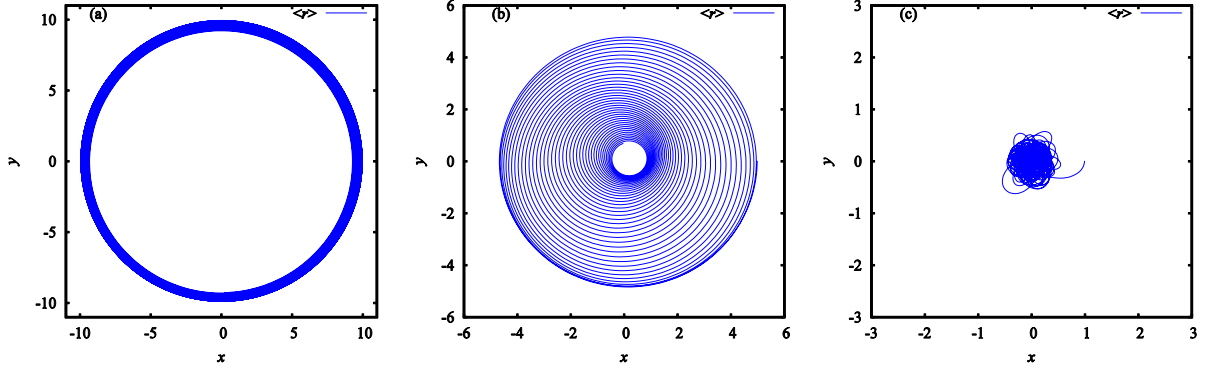


Fig. 2.11 Normalized expectation values of position $\langle \mathbf{r} \rangle$ after 40 gyrations. (a) Normalized initial speed $v_0 = -10$ and initial position (10,0). (b) Normalized initial speed $v_0 = -5$ and initial position (5,0). (c) Normalized initial speed $v_0 = -1$ and initial position (1,0).

2.5 Summary

The two-dimensional time-dependent Schrödinger equation for a magnetized proton in the presence of a fixed positive or negative field particle in the presence a uniform magnetic field is solved in this chapter. In the relatively high-speed case of $v_0 = 10$, the behaviors are similar to those of the classical ones. However, in the extension of time, even the fast particle behaves like the low-speed case of $v_0 = 5$, where the magnitudes in the both momentum $mv = |mv|$ and position $r = |\mathbf{r}|$ are appreciably decreasing with time. The kinetic energy $K = m\langle v^2 \rangle / 2$ and the potential energy $U = \langle qV \rangle$ do not show appreciable changes except for a small amplitude oscillation in high speed case, because of the increasing variances, i.e. uncertainty, in the both momentum and position.

The increment in variance of momentum $\Delta\sigma_p^2$ corresponds to the decrement in the magnitude of momentum $\langle mv \rangle$: Part of energy is transferred from the directional (classical kinetic) energy to the uncertainty (quantum mechanical zero-point) energy $\Delta\sigma_p^2 / 2m$. Such an energy conversion is faster for the attractive force case, e.g. electron-proton interaction, than the repulsive force case. For example, the quantum mechanical particle energy conversion in magnetically confined plasmas reduces the electron kinetic or mechanical speed, leading to the reduction in the relative speeds to ions including protons, deuterons and tritons. This should result in the enhanced recombination rate of an electron with a proton or its isotopes to form a hydrogen atom which is electrically neutral and is free to escape from the confining magnetic field. In summary, quantum-mechanical analyses are necessary even for fast charged particles as long as their long-time behavior is concerned in the presence of a magnetic field.

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Chapter 3

Quantum mechanical grad- B drift

3.1 Initial condition

The two-dimensional Schrödinger equation is solved numerically and theoretically for a wavefunction ψ at position $\mathbf{r}$ and time t ,

$$i\hbar \frac{\partial \psi}{\partial t} = \left\{ \frac{1}{2m} (-i\hbar \nabla - q\mathbf{A})^2 + qV \right\} \psi, \quad (3.1)$$

where V and $\mathbf{A}$ stand for the scalar and vector potentials, m and q the mass and electric charge of the particle under consideration, $i \equiv \sqrt{-1}$ the imaginary unit. In the presence of a uniform magnetic field, with a Landau gauge [1], of $A_x = -By, A_y = 0, A_z = 0$, and by solving the Heisenberg equation of motion,

$$i\hbar \frac{d\hat{X}(t)}{dt} = [\hat{X}(t), \hat{H}], \quad (3.2)$$

where the square bracket $[\cdot, \cdot]$ is the commutator, $\hat{H}$ is the Hamiltonian, $\hat{X}$ can be any operator and its time development $\hat{X}(t)$, it is well known that the time dependent operators for position $\hat{x}(t)$ and $\hat{y}(t)$ are obtained analytically as,

$$\begin{aligned} \hat{x}(t) &= \hat{x} + \frac{\hat{P}_y}{qB} - \frac{\hat{P}_y}{qB} \cos \omega t + \left(\hat{y} + \frac{\hat{P}_x}{qB} \right) \sin \omega t, \\ \hat{y}(t) &= -\frac{\hat{P}_x}{qB} + \left(\hat{y} + \frac{\hat{P}_x}{qB} \right) \cos \omega t + \frac{\hat{P}_y}{qB} \sin \omega t, \end{aligned}$$

where $\omega \equiv qB/m$ is the cyclotron frequency, $\hat{P}_y = -i\hbar \partial_y$ is the y -component of the momentum operator and $\hat{P}_x = -i\hbar \partial_x$ is the x -component of the momentum operator. The expectation value of the position $\hat{x}(t)$ and $\hat{y}(t)$ are obtained by implementing the initial condition for wavefunction at $\mathbf{r} = \mathbf{r}_0$ with $\mathbf{r}_0$ being the initial center of wavefunction ψ is given by

$$\psi(\mathbf{r}, 0) = \frac{1}{\sqrt{\pi}\sigma_B} \exp \left\{ -\frac{(\mathbf{r} - \mathbf{r}_0)^2}{2\sigma_B^2} + i\mathbf{k}_0 \cdot \mathbf{r} \right\}, \quad (3.3)$$

where the magnetic length σ_B is the initial standard deviation, and $\hbar\mathbf{k}_0$ is the initial canonical momentum.

A particle in a uniform magnetic field can be solved straight forwardly, however for the case of non-uniform magnetic case, the theoretical derivations become long and complicated. Spitzer [2], pointed out that for a general classical equation of drift velocity, its can only be solved analytical by an approximate theory. Hence, in this chapter, we solved the Heisenberg equation of motion analytically with the presence of a weakly non-uniform magnetic field $\mathbf{B} = B(1 - \hat{y}/L_B)\mathbf{e}_z$, a Landau gauge-like quadratic vector potential is given as

$$\mathbf{A} = -B\hat{y}\left(1 - \frac{\hat{y}}{2L_B}\right)\mathbf{e}_x. \quad (3.4)$$

The Landau gauge we use here is given as $\mathbf{A} = A_x(y)\mathbf{e}_x$.

The grad- B drift velocity analytical solution is compared with the numerical calculation. For the numerical calculation, a two-dimensional Schrödinger equation code is developed and the calculation is done on a GPU (Nvidia GTX-980: 2048cores/4GB @1.126GHz), using CUDA [3]. Furthermore, the numerical errors had removed from the numerical calculation by subtracting the variances in uniform magnetic field to non-uniform magnetic field [4, 5].

3.2 Time dependent operator for weakly non-uniform magnetic field

Substituting the vector potential in Eq. (3.4) into the two-dimensional Schrödinger equation in Eq. (3.1), the Hamiltonian $\hat{H}$ for a charge particle with a mass m and a charge q in the absence of an electrostatic potential for the non-relativistic charge particle, is given to first order in L_B^{-1} as,

$$\hat{H} = \frac{1}{2m}\left(\hat{P}_y^2 + \hat{\Pi}_x^2 - \frac{\hat{\Pi}_x^3}{qBL_B}\right), \quad (3.5)$$

where

$$\hat{\Pi}_x = \left(1 + \frac{\hat{P}_x}{qBL_B}\right)qB\hat{y} + \left(1 + \frac{\hat{P}_x}{2qBL_B}\right)\hat{P}_x. \quad (3.6)$$

The *exact* mechanical momentum operator $m\hat{v} = \hat{\mathbf{P}} - q\hat{\mathbf{A}} = (m\hat{u}, m\hat{v})$ is given as,

$$m\hat{u} = \hat{P}_x + qB\hat{y}\left(1 - \frac{\hat{y}}{2L_B}\right), \quad (3.7)$$

$$m\hat{v} = \hat{P}_y, \quad (3.8)$$

where $\hat{v}$ is the velocity operator. The mechanical momentum operator $m\hat{u}$ to first order in L_B^{-1} is given as,

$$m\hat{u} = \hat{\Pi}_x - \frac{\hat{\Pi}_x^2}{2qBL_B}. \quad (3.9)$$

From the Heisenberg equation of motion, the time derivative of $\hat{\Pi}_x$ is given as,

$$\frac{d\hat{\Pi}_x}{dt} = \left(1 + \frac{\hat{P}_x}{qBL_B}\right)\omega\hat{P}_y = \hat{\Omega}\hat{P}_y, \quad (3.10)$$

which leads to the definition of the angular frequency operator as,

$$\hat{\Omega} = \left(1 + \frac{\hat{P}_x}{qBL_B}\right)\omega. \quad (3.11)$$

On the other hand, for the momentum operator in the y -direction $\hat{P}_y$ we have,

$$\frac{d\hat{P}_y}{dt} = -\hat{\Omega}\left(\hat{\Pi}_x - \frac{3\hat{\Pi}_x^2}{2qBL_B}\right). \quad (3.12)$$

To derive the time dependent momentum operators $\hat{P}_x(t)$ and $\hat{\Pi}_x(t)$ with Heisenberg equation of motion, Eq. (3.10) and Eq. (3.12) are expanded using the Heisenberg picture,

$$\hat{X}(t) = \exp\left(-\frac{t}{i\hbar} \hat{H}\right) \hat{X} \exp\left(\frac{t}{i\hbar} \hat{H}\right).$$

Let us choose the operator $\hat{X}$ as

$$\hat{X} = \hat{P}_x, \quad (3.13)$$

and

$$\hat{X} = \hat{\Pi}_x + \frac{6\hat{P}_y^2 - 9\hat{\Pi}_x^2}{2qBL_B}. \quad (3.14)$$

Using Heisenberg equation of motion, the time derivative of the operator $\hat{X}$ given in Eq. (3.16) can be obtain as,

$$\begin{aligned} \frac{d\hat{X}}{dt} &= +\hat{\Omega}^1 \left(\hat{P}_y + \frac{1}{2} \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{qBL_B} \right) - 4(2\hat{\Omega})^1 \frac{\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x}{qBL_B}, \\ \frac{d^2\hat{X}}{dt^2} &= -\hat{\Omega}^2 \left(\hat{\Pi}_x - \frac{3}{2} \frac{\hat{\Pi}_x^2}{qBL_B} - \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{qBL_B} \right) - 4(2\hat{\Omega})^2 \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{qBL_B}, \\ \frac{d^3\hat{X}}{dt^3} &= -\hat{\Omega}^3 \left(\hat{P}_y + \frac{1}{2} \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{qBL_B} \right) + 4(2\hat{\Omega})^3 \frac{\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x}{qBL_B}, \\ \frac{d^4\hat{X}}{dt^4} &= +\hat{\Omega}^4 \left(\hat{\Pi}_x - \frac{3}{2} \frac{\hat{\Pi}_x^2}{qBL_B} - \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{qBL_B} \right) + 4(2\hat{\Omega})^4 \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{qBL_B}, \end{aligned}$$

and so on. The derivations of time derivative of for operator $\hat{X}$ of the higher orders are obtained and combined using Heisenberg picture. Later, the expressions are simplified by using Taylor expansion and time derivative for operator $\hat{X}(t)$ obtained as

$$\begin{aligned} \hat{X}(t) &= \left(\hat{\Pi}_x - \frac{3}{2} \frac{\hat{\Pi}_x^2}{qBL_B} - \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{qBL_B} \right) \cos \hat{\Omega}t + \left(\hat{P}_y + \frac{1}{2} \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{qBL_B} \right) \sin \hat{\Omega}t \\ &\quad + 4 \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{qBL_B} \cos 2\hat{\Omega}t - 4 \frac{\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x}{qBL_B} \sin 2\hat{\Omega}t, \end{aligned} \quad (3.15)$$

Substituting back the expression for time derivative for operator $\hat{X}(t)$ in to Eq. (3.14), and the expression for time dependent operators $\hat{P}_x(t)$ and $\hat{\Pi}_x(t)$, to first order in L_B^{-1} , are

$$\hat{P}_x(t) = \hat{P}_x, \quad (3.16)$$

$$\begin{aligned} \hat{\Pi}_x(t) &= \frac{3\hat{P}_y^2 + 3\hat{\Pi}_x^2}{4qBL_B} + \left(\hat{\Pi}_x - \frac{2\hat{P}_y^2 + \hat{\Pi}_x^2}{2qBL_B} \right) \cos \hat{\Omega}t + \left(\hat{P}_y + \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2qBL_B} \right) \sin \hat{\Omega}t \\ &\quad + \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{4qBL_B} \cos 2\hat{\Omega}t - \frac{\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x}{4qBL_B} \sin 2\hat{\Omega}t. \end{aligned} \quad (3.17)$$

Note that the operator $\hat{P}_x(t) = \hat{P}_x$ does not change with time, since the Hamiltonian $\hat{H}$ in Eq. (3.5) does not include the position operator $\hat{x}$.

To derive the remaining time dependent momentum operator $\hat{P}_y(t)$, let us introduce an operator $\hat{X}$ as

$$\hat{X} = \hat{P}_y + \frac{1}{2} \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{qBL_B} - 2 \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{qBL_B},$$

Using Heisenberg equation of motion, one can show the following relations

$$\frac{d^{2n} \hat{X}}{dt^{2n}} = +(-1)^n \hat{\Omega}^{2n} \left(\hat{P}_y + \frac{1}{2} \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{qBL_B} \right) - 2(-1)^n (2\hat{\Omega})^{2n} \frac{\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x}{qBL_B} \quad \text{for } n \geq 0,$$

$$\frac{d^{2n-1} \hat{X}}{dt^{2n-1}} = -(-1)^{n-1} \hat{\Omega}^{2n-1} \left(\hat{\Pi}_x - \frac{3}{2} \frac{\hat{\Pi}_x^2}{qBL_B} - \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{qBL_B} \right) - 2(-1)^{n-1} (2\hat{\Omega})^{2n-1} \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{qBL_B} \quad \text{for } n \geq 1,$$

which leads to

$$\begin{aligned} \hat{X}(t) = & \left(\hat{P}_y + \frac{1}{2} \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{qBL_B} \right) \cos \hat{\Omega}t - 2 \frac{\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x}{qBL_B} \cos 2\hat{\Omega}t \\ & - \left(\hat{\Pi}_x - \frac{3}{2} \frac{\hat{\Pi}_x^2}{qBL_B} - \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{qBL_B} \right) \sin \hat{\Omega}t - 2 \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{qBL_B} \sin 2\hat{\Omega}t. \end{aligned}$$

Combining operator $\hat{\Pi}_x(t)$ with operator $\hat{X}(t)$ above, the time dependent momentum operator $\hat{P}_y(t)$ is found, to first order in L_B^{-1} , as

$$\begin{aligned} \hat{P}_y(t) = & \left(\hat{P}_y + \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2qBL_B} \right) \cos \hat{\Omega}t - \frac{\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x}{2qBL_B} \cos 2\hat{\Omega}t \\ & - \left(\hat{\Pi}_x - \frac{2\hat{P}_y^2 + \hat{\Pi}_x^2}{2qBL_B} \right) \sin \hat{\Omega}t - \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{2qBL_B} \sin 2\hat{\Omega}t. \end{aligned} \quad (3.18)$$

Using the results above, the time dependent operator of position $\hat{x}(t)$ and $\hat{y}(t)$ are obtained. From the operator $\hat{\Pi}_x$ Eq. (3.6), the time dependent operator $\hat{y}(t)$ is found and shown, to first order in L_B^{-1} as,

$$qB\hat{y}(t) = \left(1 - \frac{\hat{P}_x}{qBL_B} \right) \hat{\Pi}_x(t) - \left(1 - \frac{\hat{P}_x}{2qBL_B} \right) \hat{P}_x, \quad (3.19)$$

which leads to

$$\begin{aligned} y(t) = & - \left(\frac{1}{qB} - \frac{\hat{P}_x}{2q^2 B^2 L_B} \right) \hat{P}_x + \frac{3\hat{P}_y^2 + 3\hat{\Pi}_x^2}{4q^2 B^2 L_B} + \left(\frac{\hat{\Pi}_x}{qB} - \frac{2\hat{P}_y^2 + \hat{\Pi}_x^2}{2q^2 B^2 L_B} - \frac{\hat{P}_x \hat{\Pi}_x}{q^2 B^2 L_B} \right) \cos \hat{\Omega}t \\ & + \left(\frac{\hat{P}_y}{qB} + \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2q^2 B^2 L_B} - \frac{\hat{P}_x \hat{P}_y}{q^2 B^2 L_B} \right) \sin \hat{\Omega}t + \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{4q^2 B^2 L_B} \cos 2\hat{\Omega}t - \frac{\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x}{4q^2 B^2 L_B} \sin 2\hat{\Omega}t. \end{aligned} \quad (3.20)$$

On the other hand, the time dependent operator $\hat{x}(t)$ is obtained by integrating $\hat{u}(t)$ form with time t ,

$$\hat{x}(t) = \hat{x} + \int_0^t \hat{u}(t) dt, \quad (3.21)$$

which leads to,

$$\begin{aligned} \hat{x}(t) = \hat{x} + & \left(\frac{1}{qB} - \frac{\hat{P}_x}{q^2 B^2 L_B} \right) \hat{P}_y + \frac{\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x}{4q^2 B^2 L_B} + \frac{\hat{P}_y^2 + \hat{\Pi}_x^2}{2mqBL_B} t - \left(\frac{\hat{P}_y}{qB} + \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2q^2 B^2 L_B} \right) \cos \hat{\Omega} t \\ & + \left(\frac{\hat{\Pi}_x}{qB} - \frac{2\hat{P}_y^2 + \hat{\Pi}_x^2}{2q^2 B^2 L_B} \right) \sin \hat{\Omega} t + \frac{\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x}{4q^2 B^2 L_B} \cos 2\hat{\Omega} t + \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{4q^2 B^2 L_B} \sin 2\hat{\Omega} t. \end{aligned} \quad (3.22)$$

From the operator $m\hat{u}$, together with the operator $\hat{\Pi}_x(t)$, the time dependent momentum operator $m\hat{u}(t)$ along x-axis is obtained as

$$\begin{aligned} m\hat{u}(t) = & \frac{\hat{P}_y^2 + \hat{\Pi}_x^2}{2qBL_B} + \left(\hat{\Pi}_x - \frac{2\hat{P}_y^2 + \hat{\Pi}_x^2}{2qBL_B} \right) \cos \hat{\Omega} t + \left(\hat{P}_y + \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2qBL_B} \right) \sin \hat{\Omega} t \\ & + \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{2qBL_B} \cos 2\hat{\Omega} t - \frac{\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x}{2qBL_B} \sin 2\hat{\Omega} t. \end{aligned} \quad (3.23)$$

The time dependent variance of any operator $\hat{X}(t)$ is defined as

$$\sigma_x^2(t) \equiv \langle \hat{X}^2(t) \rangle - \langle \hat{X}(t) \rangle^2.$$

When the square of the operator includes the product of the two operators, $\hat{X}^2(t) = \hat{Y}(t)\hat{Z}(t) + \dots$, its contribution to the variance is

$$\sigma_x^2(t) = \langle \hat{Y}(t)\hat{Z}(t) \rangle - \langle \hat{Y}(t) \rangle \langle \hat{Z}(t) \rangle + \dots,$$

which is the covariance between $\hat{Y}(t)$ and $\hat{Z}(t)$. Using this relation, the time evolution of variance in position and in momenta are formally given as

$$\begin{aligned} \frac{d\hat{\sigma}_r^2(t)}{dt} &= \langle \hat{\mathbf{r}}(t) \cdot \hat{\mathbf{v}}(t) + \hat{\mathbf{v}}(t) \cdot \hat{\mathbf{r}}(t) \rangle - 2\langle \hat{\mathbf{r}}(t) \rangle \cdot \langle \hat{\mathbf{v}}(t) \rangle, \\ \frac{d\hat{\sigma}_{mv}^2(t)}{dt} &= \left\langle m\hat{\mathbf{v}}(t) \cdot \frac{d\hat{\mathbf{m}}\hat{\mathbf{v}}(t)}{dt} + \frac{d\hat{\mathbf{m}}\hat{\mathbf{v}}(t)}{dt} \cdot m\hat{\mathbf{v}}(t) \right\rangle - 2\left\langle \frac{d\hat{\mathbf{m}}\hat{\mathbf{v}}(t)}{dt} \right\rangle \cdot \langle m\hat{\mathbf{v}}(t) \rangle, \\ \frac{d\hat{\sigma}_p^2(t)}{dt} &= \left\langle \hat{\mathbf{P}}(t) \cdot \frac{d\hat{\mathbf{P}}(t)}{dt} + \frac{d\hat{\mathbf{P}}(t)}{dt} \cdot \hat{\mathbf{P}}(t) \right\rangle - 2\left\langle \frac{d\hat{\mathbf{P}}(t)}{dt} \right\rangle \cdot \langle \hat{\mathbf{P}}(t) \rangle. \end{aligned}$$

In order to evaluate the formal solution above, it is noted that the solutions consist of multiple combination of operators such as $\hat{x}$, $\hat{y}$, $\hat{P}_x$, $\hat{P}_y$, $\hat{\Pi}_x$, and $\hat{\Omega}_x$. Here are some parts of the combinations of expectation values of the operators as

$$\begin{aligned} \langle \hat{\Pi}_x \hat{P}_x \hat{P}_y \rangle &= \langle \hat{\Pi}_x \rangle \langle \hat{P}_x \rangle \langle \hat{P}_y \rangle + \frac{i\hbar qB}{2} \left\langle \left(1 + \frac{\hat{P}_x}{qBL_B} \right) \hat{P}_x \right\rangle, \\ \langle \hat{P}_y \hat{\Pi}_x \hat{P}_x \rangle &= \langle \hat{\Pi}_x \rangle \langle \hat{P}_x \rangle \langle \hat{P}_y \rangle - \frac{i\hbar qB}{2} \left\langle \left(1 + \frac{\hat{P}_x}{qBL_B} \right) \hat{P}_x \right\rangle, \end{aligned}$$

$$\begin{aligned}\langle \hat{\Pi}_x \hat{P}_y^2 \rangle &= \langle \hat{P}_y^2 \rangle \langle \Pi_x \rangle + i\hbar qB \langle \hat{P}_y \rangle \left\langle 1 + \frac{\hat{P}_x}{qBL_B} \right\rangle, \\ \langle \hat{P}_y^2 \hat{\Pi}_x \rangle &= \langle \hat{P}_y^2 \rangle \langle \Pi_x \rangle - i\hbar qB \langle \hat{P}_y \rangle \left\langle 1 + \frac{\hat{P}_x}{qBL_B} \right\rangle.\end{aligned}$$

With further evaluations of combinations of operators and the combinations of solutions given above, it will be shown in the following section that our analytical results agree with the numerical ones in the expansion rates of variance in position, mechanical momentum, and total momentum.

3.3.1 Time average of variance

By solving Heisenberg equation of motion, the time averages over cyclotron period of variance in position $\overline{\sigma_r^2(t)}$, mechanical momentum $\overline{\sigma_{mv}^2(t)}$, and total momentum $\overline{\sigma_p^2(t)}$, to zeroth order in L_B^{-1} , are given as

$$\overline{\sigma_r^2(t)} = \overline{\sigma_x^2(t)} + \overline{\sigma_y^2(t)} = \frac{7}{4} \frac{\hbar}{qB} + \frac{5}{4} \frac{\hbar}{qB} = 3 \frac{\hbar}{qB}, \quad (3.24)$$

$$\overline{\sigma_{mv}^2(t)} = \overline{\sigma_{mu}^2(t)} + \overline{\sigma_{mv}^2(t)} = \frac{3}{4} \hbar qB + \frac{3}{4} \hbar qB = \frac{3}{2} \hbar qB, \quad (3.25)$$

$$\overline{\sigma_p^2(t)} = \overline{\sigma_{p_x}^2(t)} + \overline{\sigma_{p_y}^2(t)} = \frac{1}{2} \hbar qB + \frac{3}{4} \hbar qB = \frac{5}{4} \hbar qB, \quad (3.26)$$

using $\sigma_x^2 = \sigma_y^2 = \hbar / 2qB$, $\sigma_{p_x}^2 = \sigma_{p_y}^2 = \sigma_{mv}^2 = \hbar qB / 2$, and $\sigma_{\Pi_x}^2 = \sigma_{mv}^2 = \sigma_{p_x}^2 + (qB)^2 \sigma_y^2 = \hbar qB$ for the initial wavefunction when gradient scale length $L_B = \infty$. It is noted that $\sigma_{p_y}^2(t) = \sigma_{mv}^2(t)$ since $\hat{P}_y = m\hat{v}$ is due to the Landau gauge $\mathbf{A} = A_x(y)\mathbf{e}_y$ adopted in this study.

3.3.2 Numerical time average of variance

A two-dimensional time-dependent Schrödinger equation code is developed and the numerical calculation is done on a GPU (Nvidia GTX-980: 2048cores/4GB @1.126GHz), using CUDA [3]. Numerical calculation for the case under considerations is made in order to confirm the expressions for the time average of variances. Referring to Fig. 3.1, our numerical results, the time average of variance in position $\overline{\sigma_r^2(t)}$, variance in mechanical momentum $\overline{\sigma_{mv}^2(t)}$, and total momentum $\overline{\sigma_p^2(t)}$ over cyclotron period are,

$$\overline{\sigma_r^2(t)} = \overline{\sigma_x^2(t)} + \overline{\sigma_y^2(t)} = 1.747 \frac{\hbar}{qB} + 1.248 \frac{\hbar}{qB} = 2.996 \frac{\hbar}{qB}, \quad (3.27)$$

$$\overline{\sigma_{mv}^2(t)} = \overline{\sigma_{mu}^2(t)} + \overline{\sigma_{mv}^2(t)} = 0.750 \hbar qB + 0.750 \hbar qB = 1.500 \hbar qB, \quad (3.28)$$

$$\overline{\sigma_p^2(t)} = \overline{\sigma_{p_x}^2(t)} + \overline{\sigma_{p_y}^2(t)} = 0.500 \hbar qB + 0.750 \hbar qB = 1.249 \hbar qB. \quad (3.29)$$

Comparing the analytical results with the numerical results, the analytical solutions given in Eqs. (3.24-3.26) are generally consistent with the numerical results in Eqs. (3.27-3.29) on the time averaged variances in position, mechanical momentum, and total momentum.

In the case of a uniform magnetic field, $L_B = \infty$, the variance in position $\sigma_r^2(t) = \sigma_B^2 = \text{const}$, therefore the increment in uniform magnetic field variance value should remain constant. However, in the numerical calculation, there is a small increment of variance in position, which due to the numerical error. Corresponding to this fact, the numerical error had removed from the numerical calculation by subtracting the variances in the uniform magnetic field from the non-uniform magnetic field [4, 5].

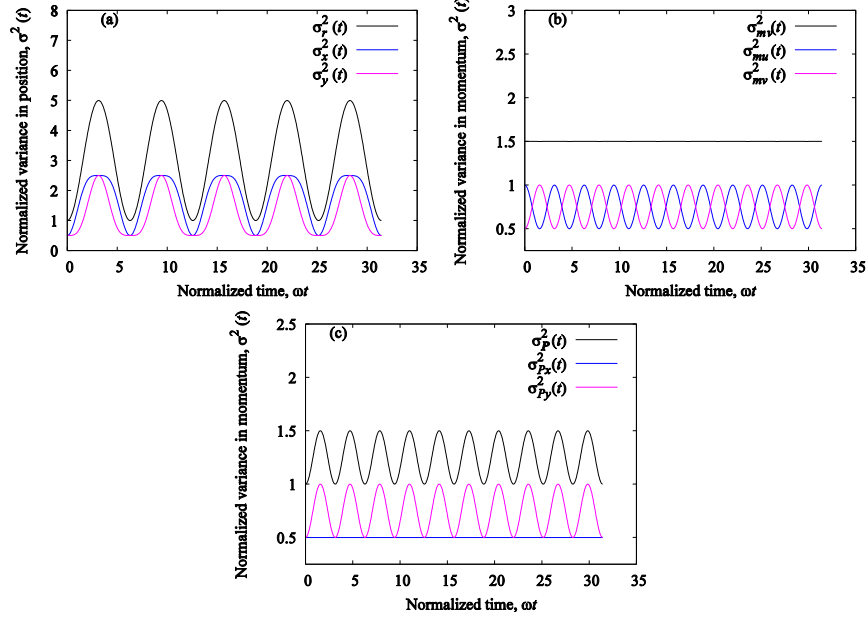


Fig. 3.1 Numerical time evolution of increment in normalized variance for 5 gyrations. (a) Variance in position σ_r^2 normalized by $\hbar/qB$. (b) Variance in mechanical momentum σ_{mv}^2 normalized by $\hbar qB$. (c) Variance in total momentum σ_p^2 normalized by $\hbar qB$.

3.4.1 Expansion rates of variance

By solving Heisenberg equation of motion, the analytical solution of expansion rates of variance in position, mechanical momentum, and total momentum to the first order in L_B^{-1} , are given as

$$\frac{d\sigma_r^2(t)}{dt} = \frac{d\sigma_x^2(t)}{dt} + \frac{d\sigma_y^2(t)}{dt} = \frac{3}{2} \frac{\hbar}{qB} \frac{v_0}{L_B} + \frac{1}{2} \frac{\hbar}{qB} \frac{v_0}{L_B} = 2 \frac{\hbar}{qB} \frac{v_0}{L_B}, \quad (3.30)$$

$$\frac{d\sigma_{mv}^2(t)}{dt} = \frac{d\sigma_{mu}^2(t)}{dt} + \frac{d\sigma_{mv}^2(t)}{dt} = \frac{1}{2} \hbar qB \frac{v_0}{L_B} + \frac{1}{2} \hbar qB \frac{v_0}{L_B} = \hbar qB \frac{v_0}{L_B}, \quad (3.31)$$

$$\frac{d\sigma_p^2(t)}{dt} = \frac{d\sigma_{px}^2(t)}{dt} + \frac{d\sigma_{py}^2(t)}{dt} = 0 + \frac{1}{2} \hbar qB \frac{v_0}{L_B} = \frac{1}{2} \hbar qB \frac{v_0}{L_B}. \quad (3.32)$$

3.4.2 Expansion time

The characteristic time τ_r for the variance in position $\sigma_r^2(t)$ to reaches the square of interparticle separation of $n^{-2/3}$ is estimated using analytical solution of expansion rate of variance in position in Eq. (3.30), as

$$\tau_r = \frac{n^{-2/3}}{2\hbar v_0 / qBL_B}. \quad (3.33)$$

The variance in momentum $\sigma_{mv}^2(t)$ reaches the square of initial mechanical momentum mv_0 at τ_{mv} as

$$\tau_{mv} \equiv \frac{(mv_0)^2}{\hbar qBv_0 / L_B}. \quad (3.34)$$

These finding can be applied to a broad scientific research fields, for example, in the magnetically confined fusion plasmas.

For a torus plasma with a temperature T , the gradient scale length of the non-uniformity of the field L_B can be replaced by the major radius R_0 of the torus, and the initial velocity v_0 by the thermal speed $v_{th} = \sqrt{2T/m}$ [1], in Eq. (3.33), which is in proportion to $\sqrt{m/TR_0}$, thus the isotope effect of $\tau \propto \sqrt{m}$ appears. This leads to the characteristic time of $\tau_r = 0.25$ ms for a typical plasmas of $T \sim 10$ keV and $n \sim 10^{20} \text{ m}^{-3}$ with $R_0 \sim 3\text{m}$, which is much shorter than the proton collision time of $\tau_{ii} \sim 20$ ms. Since the ions and the electrons in fusion plasma are considered to be at the same temperature, and the velocities of electrons are much faster than protons. So, the time for electrons to reach interparticle separation is much faster than for ions. The electrons should be treated as a uniform background for the ions after the electrons expansion time.

The expansion time in position given in Eq. (3.33) implies that the probability density function (PDF) of such energetic charged particles expand fast in the plane perpendicular to the magnetic field. The broad distribution of individual particle in space means that their Coulomb interactions with other particles of distance less than $\sigma_r(t)$ becomes weaker than that expected in classical mechanics. This could make effective electronic charge smaller, $e_{\text{eff}} < e$, and thus make larger Debye shielding length than the classical one given by $\lambda_D = \sqrt{\epsilon_0 k_B T / ne^2}$. Such a Debye screening modification due to quantum mechanical effect was pointed out based on a quantum hydrodynamic (QHD) model [6], in which the Fermi pressure and Bohm potential were included in the fluid equation.

3.4.3 Numerical expansion rates of variance

In order to confirm the expressions for the analytical solution of expansion rates variance, we have made the numerical calculation with using multiple parameter combinations. The numerical expansion rates are calculated by averaging the increment of variance over a cyclotron period. For the numerical calculation, let us divide the time-dependent variance $\sigma_r^2(t)$ into its initial value $\sigma_r^2(0)$ and the increment $\Delta\sigma_r^2(t)$ as $\sigma_r^2(t) \equiv \sigma_r^2(0) + \Delta\sigma_r^2(t)$. Referring to Fig. 3.2, Fig. 3.3, and Fig. 3.4 which are the numerical results, the average increment throughout the cyclotron gyrations, the expansion rates of variance in position, mechanical momentum, and total momentum are calculated as,

$$\frac{d\sigma_r^2(t)}{dt} = \frac{d\sigma_x^2(t)}{dt} + \frac{d\sigma_y^2(t)}{dt} = 1.495 \frac{\hbar}{qB} \frac{v_0}{L_B} + 0.492 \frac{\hbar}{qB} \frac{v_0}{L_B} = 1.987 \frac{\hbar}{qB} \frac{v_0}{L_B}, \quad (3.35)$$

$$\frac{d\sigma_{mv}^2(t)}{dt} = \frac{d\sigma_{mu}^2(t)}{dt} + \frac{d\sigma_{mv}^2(t)}{dt} = 0.503\hbar qB \frac{v_0}{L_B} + 0.502\hbar qB \frac{v_0}{L_B} = 1.005\hbar qB \frac{v_0}{L_B}, \quad (3.36)$$

$$\frac{d\sigma_P^2(t)}{dt} = \frac{d\sigma_{P_x}^2(t)}{dt} + \frac{d\sigma_{P_y}^2(t)}{dt} = 0.000\hbar qB \frac{v_0}{L_B} + 0.502\hbar qB \frac{v_0}{L_B} = 0.502\hbar qB \frac{v_0}{L_B}. \quad (3.37)$$

Comparing the analytical results in Eqs. (3.30-3.32) and the numerical results Eqs. (3.35-3.37), the analytical results are generally consistent with the numerical results on the expansion rates in position, mechanical momentum, and total momentum. The analytical results cannot be perfectly exact as numerical calculations as the facts due to the analytical results being solved to the first order in magnetic gradient scale length L_B^{-1} .

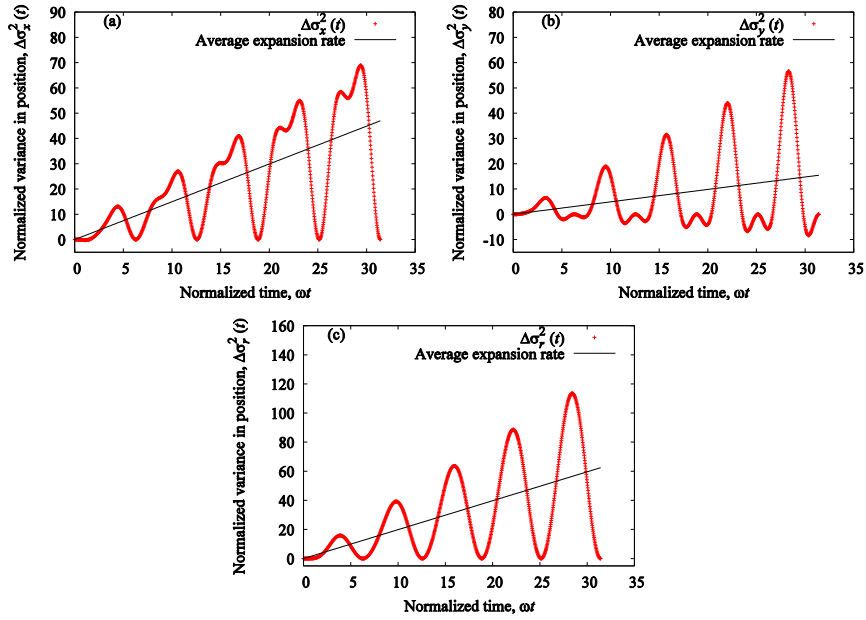


Fig. 3.2 Numerical time evolution of increment $\Delta\sigma^2(t)$ in variance of position normalized by $\hbar v_0 / qBL_B$ for 5 gyrations. (a) Incremental variances $\Delta\sigma_x^2$ with an average expansion rate of 1.495. (b) Incremental variances $\Delta\sigma_y^2$ with an average expansion rate of 0.492. (c) Incremental variances $\Delta\sigma_r^2$ with an average expansion rate of 1.987.

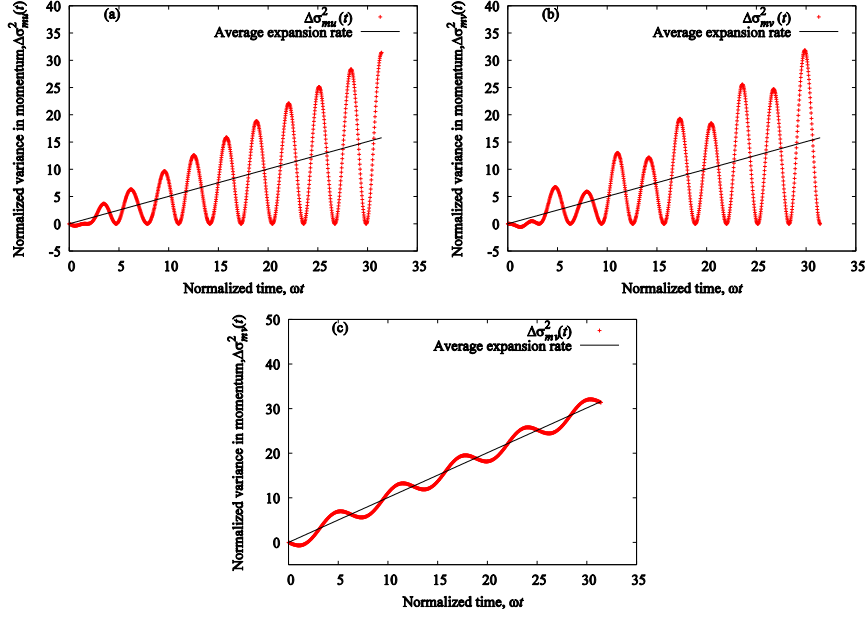


Fig. 3.3 Numerical time evolution of increment $\Delta\sigma^2(t)$ in variance of mechanical momentum normalized by $\hbar q B v_0 / L_B$ for 5 gyrations. (a) Incremental variances $\Delta\sigma^2_{mu}$ with an average expansion rate of 0.503. (b) Incremental variances $\Delta\sigma^2_{mv}$ with an average expansion rate of 0.502. (c) Incremental variances $\Delta\sigma^2_{mv}$ with an average expansion rate of 1.005.

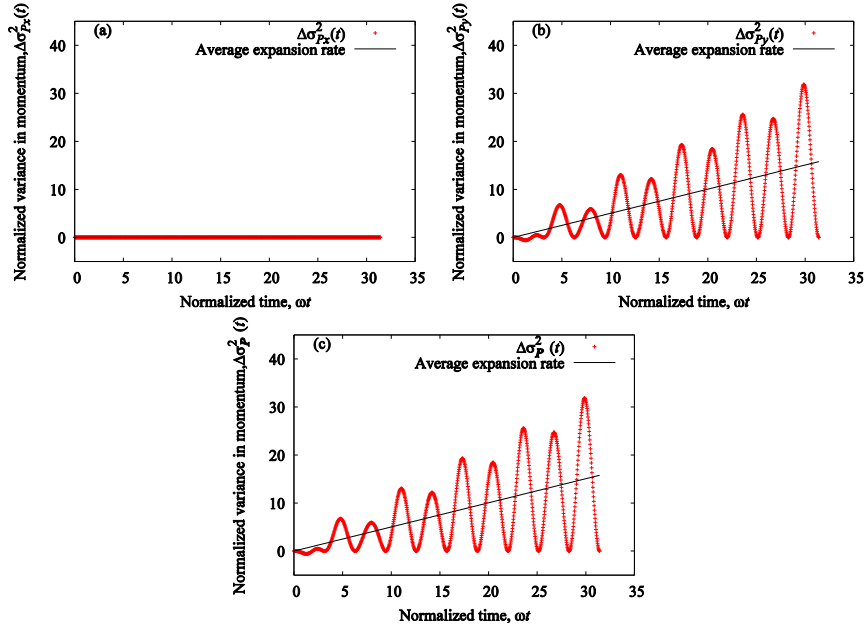


Fig. 3.4 Numerical time evolution of increment $\Delta\sigma^2(t)$ in variance of total momentum normalized by $\hbar q B v_0 / L_B$ for 5 gyrations. (a) Incremental variances $\Delta\sigma^2_{Px}$ with an average expansion rate of 0. (b) Incremental variances $\Delta\sigma^2_{Py}$ with an average expansion rate of 0.502. (c) Incremental variances $\Delta\sigma^2_P$ with an average expansion rate of 0.502.

It is interesting to note that the analytical results of expansion rates of variance in Eqs. (3.30-3.32), are y -component initial velocity v_0 dependent. To confirm the reliability of the y -direction velocity $\mathbf{v}_0 = (0, v_0)$ in our analytical solution, the numerical calculations for particle with initial

velocity in the opposite direction are conducted. Figure 3.5(a) is the numerical result of $\Delta\sigma^2(t)$ for initial velocity in opposite direction $v_0 = (0, -1)$, which shows the numerical factor of the particle expansion rate at -1.987 . However, after the normalization of the initial velocity v_0 , again it is affirmed that the numerical calculation results are again consistent with the analytical results Eq. (3.30).

In addition, numerical calculation with a normalized initial velocity of $(u_0, v_0) = (1, 0)$ and $(u_0, v_0) = (-1, 0)$ are performed to confirm the analytical solution. From the numerical calculation, the numerical factor of the expansion rate is -0.082 for $u_0 = 1$, that as shown in Fig. 3.5(b), and $+0.083$ is for $u_0 = -1$ that as shown in Fig. 3.5(c), both of which are close to zero or null. These again show that the numerical calculations are consistent with the analytical results given in Eq. (3.30).

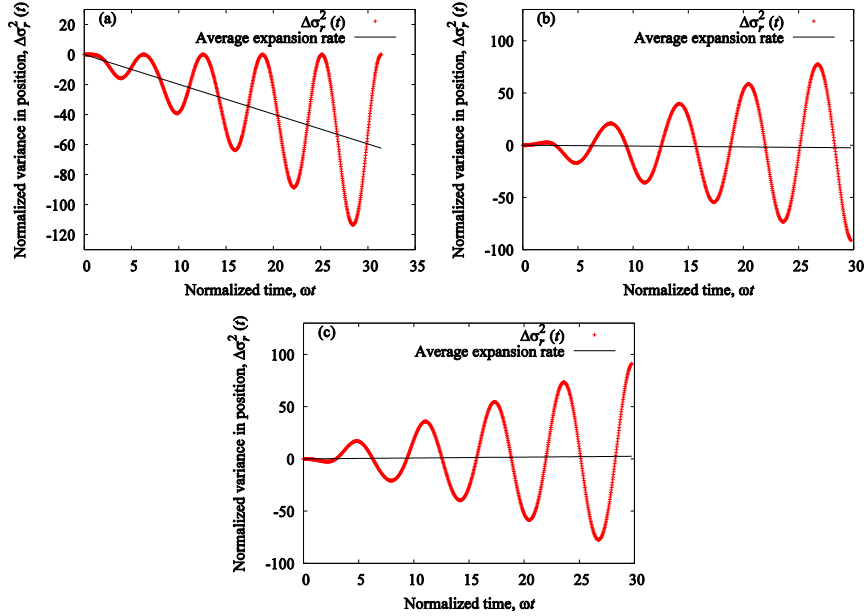


Fig. 3.5 Numerical time evolution of increment in position $\Delta\sigma_r^2$ for 5 gyrations after normalized by $\hbar|v_0|/qBL_B$.

- (a) Normalized initial velocity $(u_0, v_0) = (0, -1)$ with an average expansion rate of -1.987 . (b) Normalized initial velocity of $(u_0, v_0) = (1, 0)$ with an average expansion rate of -0.082 . (c) Normalized initial velocity of $(u_0, v_0) = (-1, 0)$ with an average expansion rate of $+0.083$.

3.5 Analytical grad-B drift velocity

Since the time dependent momentum operator $m\hat{u}(t)$ is obtained, the grad-B drift operator is obtained straight forwardly as

$$\hat{u}_{\nabla B} \equiv \overline{\hat{u}(t)} = \frac{\hat{p}_y^2 + \hat{\Pi}_x^2}{2mqBL_B} = \frac{\hat{H}}{qBL_B} + O(L_B^{-2}). \quad (3.38)$$

The expectation value of the grad-B drift velocity operator $u_{\nabla B} \equiv \langle \hat{u}_{\nabla B} \rangle$ is given as follows

$$\begin{aligned}
u_{\nabla B} &= \frac{1}{\pi\sigma_B^2} \int_{\Sigma} e^{-\frac{(r-r_0)^2}{2\sigma_B^2} - ik_0 \cdot r} \hat{u}_{\nabla B} e^{-\frac{(r-r_0)^2}{2\sigma_B^2} + ik_0 \cdot r} d^2\mathbf{r} \\
&= \frac{mv_0^2}{2qBL_B} + \frac{1}{2qBL_B} \left(\frac{\hbar^2}{m\sigma_B^2} + \frac{m\omega^2\sigma_B^2}{2} \right),
\end{aligned} \tag{3.39}$$

where $mv_0 \equiv \langle m\hat{v} \rangle = \langle \hat{\mathbf{P}} - q\hat{\mathbf{A}} \rangle$. When we use the magnetic length of $\sqrt{\hbar/qB}$ as σ_B [26], then we have

$$u_{\nabla B} = \frac{mv_0^2}{2qBL_B} + \frac{3\hbar}{4mL_B}. \tag{3.40}$$

Note that the first term of $u_{\nabla B}$ coincides with the classical formula for the grad- B drift and the second term represents the quantum mechanical drift due to the uncertainty [7].

3.6 Numerical grad- B drift velocity

To confirm the theoretical grad- B drift velocity Eq. (3.40), the numerical calculation is performed and the results are shown in Fig. 3.6. The Numerical grad- B drift velocity is obtained by subtracting the position of the particle in x -direction of non-uniform magnetic field case with uniform magnetic case. The subtracted value is the drifted position of the particle in the presence of a non-uniform magnetic field. Note that the constant $(1 + \varepsilon)$ from Eq. (3.41) is the gradient of the increment of the graph in Fig. 3.6 with ε being the numerical error.

In these numerical calculations, the parameters are normalized as; mass of the particle $m = 1.6722 \times 10^{-27}$ kg, charge $q = 1.602 \times 10^{-19}$ C, magnetic flux density $B = 10$ T, velocity $\bar{v} = 10$ m/s, length $\bar{\rho} = 1.04382 \times 10^{-8}$ m, and time $\bar{t} = 1.04382 \times 10^{-9}$ s. Lengths are normalized by cyclotron radius of a proton with a speed of 10 m/s in a magnetic field of 10 T. The cyclotron frequency in such a case is used for normalization of the time. This normalization leads to the normalized initial standard deviation given as $\sigma_B = \sqrt{\hbar/eB} \cong 0.777$. Let us make the affirmation between theoretical derivation and numerical results by using the deviation analysis below,

$$u_{\nabla B} = (1 + \varepsilon) \left(\frac{mv_0^2}{2qBL_B} + \frac{3\hbar}{4mL_B} \right), \tag{3.41}$$

where ε is the numerical error. For analytical solution, since it is exact, the numerical error $\varepsilon = 0$ and we will get the exact solution as in Eq. (3.40). Comparisons between analytical solutions with numerical calculation are conducted. With various combinations of physical parameters, such as m , q , v_0 , B , and L_B , the numerical results of grad- B drift velocity is shown in Fig. 3.7 and the expression is shown in Eq. (3.42) below

$$u_{\nabla B} = 1.0044 \left(\frac{mv_0^2}{2qBL_B} + \frac{3\hbar}{4mL_B} \right). \tag{3.42}$$

Note that Fig. 3.7 is the combination of 8 sets of data in Fig. 3.8.

Comparing between the numerical results Eq. (3.42) and analytical solutions Eq. (3.40), it is concluded that the analytical solutions have good agreement with theoretical analysis, with 0.44%

error. This error is due to the limitation of the analytical grad- B drift velocity operator being solved to the first order in magnetic gradient scale length L_B^{-1} .

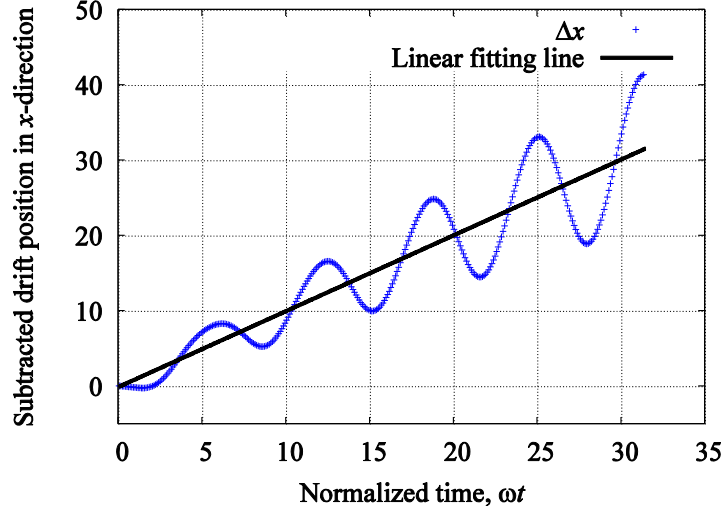


Fig. 3.6 Numerical time evolution of particle drift in x -direction Δx for 5 gyrations.

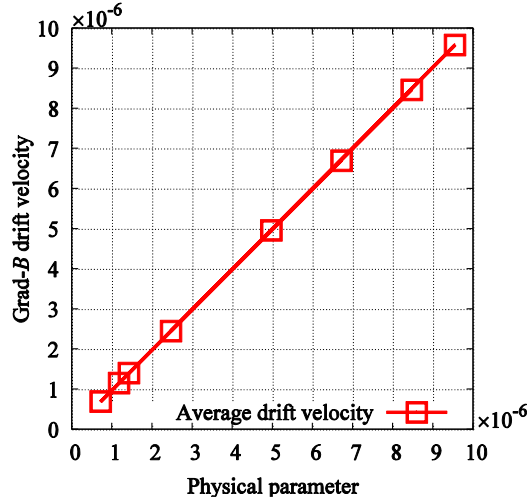


Fig. 3.7 Grad- B drift velocity $\langle \hat{u}_{vB} \rangle$ against physical parameter $(mv_0^2 / 2qBL_B + 3\hbar / 4mL_B)$ with different sets of parameters of charge q , magnetic field B , mass m , initial velocity v_0 , and gradient scale length L_B .

3.7 Summary

In the presence of a weakly non-uniform magnetic field or Landau gauge-like quadratic vector potential of $\mathbf{A} = -B\hat{y}(1 - \hat{y}/2L_B)\mathbf{e}_x$, a charged particle, i.e. proton will encounter drift effect. In this chapter, the grad- B drifts velocity of a charged particle is solved with considering quantum mechanical effect by using the Heisenberg equation of motion. It is shown that the grad- B drift velocity operator obtained in this study agrees with the classical counterpart, when the uncertainty is ignored. The time

evolution of the position and momentum operators are also analytically obtained for the non-relativistic spinless charged particle.

The theoretical derivation solutions do agree with the numerical results on the grad- B drift velocity in the presence of a non-uniform magnetic field. The quantum mechanical grad- B drift velocity formulations clearly show the drift velocity dependence on mass m and gradient scale length L_B . The result implies that light particles with low energy would drift faster than classical drift theory predicts.

It is analytically shown that the variance in position reaches the square of the interparticle separation, which is the characteristic time much shorter than the proton collision time of plasma fusion. After this time the wavefunctions of the neighboring particles would overlap, as a result the conventional classical analysis may lose its validity. It is also shown that for the laser-plasma interaction, the quantum effect could make effective electronic charge become smaller, $e_{\text{eff}} < e$, and thus make larger Debye shielding length than the classical one.

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Chapter 4

Quantum mechanical $E \times B$ drift

4.1 Initial condition

The unsteady Schrödinger equation for a wavefunction $\psi(\mathbf{r}, t)$ at position $\mathbf{r}$ and time t is given as,

$$i\hbar \frac{\partial \psi}{\partial t} = \left\{ \frac{1}{2m} (-i\hbar \nabla - q\mathbf{A})^2 + qV \right\} \psi, \quad (4.1)$$

where $V = V(\mathbf{r})$ and $\mathbf{A} = \mathbf{A}(\mathbf{r})$ stand for the scalar and vector potentials, m and q stand for the mass and the electric charge of the particle under consideration, $i \equiv \sqrt{-1}$ the imaginary unit.

The initial condition for wavefunction at $\mathbf{r} = \mathbf{r}_0$ and $t = 0$ which the initial center of wavefunction $\psi(\mathbf{r}, 0)$ which is given as

$$\psi(\mathbf{r}, 0) = \frac{1}{\sqrt{\pi}\sigma_B} \exp\left(-\frac{(\mathbf{r} - \mathbf{r}_0)^2}{2\sigma_B^2} + i\mathbf{k}_0 \cdot \mathbf{r}\right), \quad (4.2)$$

where the magnetic length $\sigma_B = \sqrt{\hbar/qB}$ is the initial standard deviation, and $\hbar\mathbf{k}_0$ is the initial canonical momentum.

In the presence of a weakly non-uniform magnetic field $\mathbf{B} = B(1 - y/L_B)\mathbf{e}_z$ [1, 2], a Landau gauge-like quadratic vector potential is given as

$$\mathbf{A} = -By\left(1 - \frac{y}{2L_B}\right)\mathbf{e}_x, \quad (4.3)$$

and with the presence of a weakly non-uniform electric field $\mathbf{E} = E(1 - y/L_E)\mathbf{e}_y$ [2], a scalar potential is given as

$$V = -Ey\left(1 - \frac{y}{2L_E}\right). \quad (4.4)$$

The Landau gauge we use here is given as $\mathbf{A} = A_x(y)\mathbf{e}_x$.

4.2 Time dependent operators for the non-uniform electric and magnetic field

Substituting the vector potential in Eq. (4.3) and the electrostatic potential in Eq. (4.4) into the two-dimensional Schrödinger equation in Eq. (4.1), the Hamiltonian $\hat{H}$ for a charge particle with a mass m and a charge q for a non-relativistic charge particle, to first order in L_B^{-1} and L_E^{-1} , is given as

$$\hat{H} = \frac{1}{2m} \left(\hat{P}_y^2 + \hat{\Pi}_x^2 - \frac{\hat{\Pi}_x^3}{qBL_B} \right) + \Delta\hat{H}, \quad (4.5)$$

where $\hat{P}_y = -i\hbar\partial_y$ is the y -component of the momentum operator and $\hat{P}_x = -i\hbar\partial_x$ is the x -component of the momentum operator defined in operator $\hat{\Pi}_x$ as,

$$\hat{\Pi}_x = \left(1 + \frac{\hat{P}_x}{qBL_B} - \frac{3}{2qBL_B} \frac{mE}{B} + \frac{1}{qBL_E} \frac{mE}{B} \right) qB\hat{y} + \left(1 + \frac{\hat{P}_x}{2qBL_B} - \frac{1}{2qBL_E} \frac{mE}{B} \right) \left(\hat{P}_x - \frac{mE}{B} \right), \quad (4.6)$$

and $\Delta\hat{H}$ is given as,

$$\Delta\hat{H} = \left[\left(\hat{P}_x - \frac{1}{2} \frac{mE}{B} \right) + \frac{1}{2} \left(\frac{1}{qBL_E} - \frac{1}{qBL_B} \right) \left(\hat{P}_x - \frac{mE}{B} \right)^2 \right] \frac{E}{B},$$

It is assumed that L_B^{-1} is of the same order as L_E^{-1} . The mechanical momentum operators $m\hat{v} = \hat{\mathbf{P}} - q\hat{\mathbf{A}} = (m\hat{u}, m\hat{v})$ to first order in L_B^{-1} and L_E^{-1} are given as,

$$m\hat{u} = \left[1 + \left(\frac{1}{qBL_B} - \frac{1}{qBL_E} \right) \frac{mE}{2B} - \frac{\hat{\Pi}_x}{2qBL_B} \right] \hat{\Pi}_x + \left[1 - \left(\frac{1}{qBL_B} - \frac{1}{qBL_E} \right) \left(\hat{P}_x - \frac{mE}{B} \right) \right] \frac{mE}{B}, \quad (4.7)$$

$$m\hat{v} = \hat{P}_y. \quad (4.8)$$

For any operator $\hat{X}$, its time development $\hat{X}(t)$ is given by the following Heisenberg equation of motion,

$$i\hbar \frac{d\hat{X}(t)}{dt} = [\hat{X}(t), \hat{H}], \quad (4.9)$$

where the square bracket $[\cdot, \cdot]$ is the commutator. We have solved the Heisenberg equation of motion with the electric and magnetic gradient length scale to first order in L_E^{-1} and L_B^{-1} in this study, with using the approximation of L_B^{-1} is of the same order as L_E^{-1} . From the Heisenberg equation of motion, the time derivative of $\hat{\Pi}_x$ and $\hat{P}_y$ are

$$\frac{d\hat{\Pi}_x}{dt} = +\hat{\Omega}\hat{P}_y, \quad (4.10)$$

$$\frac{d\hat{P}_y}{dt} = -\hat{\Omega} \left(\hat{\Pi}_x - \frac{3\hat{\Pi}_x^2}{2qBL_B} \right), \quad (4.11)$$

where the angular frequency operator $\hat{\Omega}$ is defined, using the usual cyclotron angular frequency $\omega = qB/m$, as

$$\hat{\Omega} = \left[1 + \frac{\hat{P}_x}{qBL_B} - \left(\frac{3}{2qBL_B} - \frac{1}{qBL_E} \right) \frac{mE}{B} \right] \omega. \quad (4.12)$$

It should be noted that the Heisenberg equations of motion, Eqs. (4.10) and (4.11), are exactly the same as those solved as in Chap. 3, where there is a weakly non-uniform magnetic field, but no electric field. The only difference between this study and Chap. 3 is that the operator $\hat{\Omega}$ in Eq. (4.12) includes a constant L_E term which is not an operator. Thus, the time dependent operators $\hat{P}_x(t)$ and $\hat{\Pi}_x(t)$ are the same as those in Chap. 3 as

$$\hat{P}_x(t) = \hat{P}_x, \quad (4.13)$$

$$\begin{aligned}\hat{\Pi}_x(t) = & \frac{3\hat{P}_y^2 + 3\hat{\Pi}_x^2}{4qBL_B} + \left(\hat{\Pi}_x - \frac{2\hat{P}_y^2 + \hat{\Pi}_x^2}{2qBL_B} \right) \cos \hat{\Omega}t + \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{4qBL_B} \cos 2\hat{\Omega}t \\ & + \left(\hat{P}_y + \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2qBL_B} \right) \sin \hat{\Omega}t - \frac{\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x}{4qBL_B} \sin 2\hat{\Omega}t,\end{aligned}\quad (4.14)$$

To derive the remaining time dependent momentum operator $\hat{P}_y(t)$, let us introduce an operator $\hat{X}$ as

$$\hat{X} = \hat{P}_y + \frac{1}{2} \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{qBL_B} - 2 \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{qBL_B}$$

Using Heisenberg equation of motion Eq. (4.9), one can show the following relations

$$\begin{aligned}\frac{d^{2n} \hat{X}}{dt^{2n}} = & +(-1)^n \hat{\Omega}^{2n} \left(\hat{P}_y + \frac{1}{2} \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{qBL_B} \right) - 2(-1)^n (2\hat{\Omega})^{2n} \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{qBL_B} \quad \text{for } n \geq 0, \\ \frac{d^{2n-1} \hat{X}}{dt^{2n-1}} = & -(-1)^{n-1} \hat{\Omega}^{2n-1} \left(\hat{\Pi}_x - \frac{3}{2} \frac{\hat{\Pi}_x^2}{qBL_B} - \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{qBL_B} \right) - 2(-1)^{n-1} (2\hat{\Omega})^{2n-1} \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{qBL_B} \quad \text{for } n \geq 1,\end{aligned}$$

which leads to

$$\begin{aligned}\hat{X}(t) = & \left(\hat{P}_y + \frac{1}{2} \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{qBL_B} \right) \cos \hat{\Omega}t - 2 \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{qBL_B} \cos 2\hat{\Omega}t \\ & - \left(\hat{\Pi}_x - \frac{3}{2} \frac{\hat{\Pi}_x^2}{qBL_B} - \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{qBL_B} \right) \sin \hat{\Omega}t - 2 \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{qBL_B} \sin 2\hat{\Omega}t.\end{aligned}$$

Combining Eq. (4.14) with $\hat{X}(t)$ above, the time dependent momentum operator $\hat{P}_y(t)$ is found, to first order in L_B^{-1} , as

$$\begin{aligned}\hat{P}_y(t) = & \left(\hat{P}_y + \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2qBL_B} \right) \cos \hat{\Omega}t - \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2qBL_B} \cos 2\hat{\Omega}t \\ & - \left(\hat{\Pi}_x - \frac{2\hat{P}_y^2 + \hat{\Pi}_x^2}{2qBL_B} \right) \sin \hat{\Omega}t - \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{2qBL_B} \sin 2\hat{\Omega}t.\end{aligned}\quad (4.15)$$

Referring to the Eqs. (4.14) and (4.15), it is known that, $\hat{P}_y^2(t) + \hat{\Pi}_x^2(t) = 2m\hat{H}$ and the time dependent Hamiltonian $\hat{H}(t) = \text{const} = \hat{H}$.

Using the results above, we have obtained the time dependent operator of position $\hat{x}(t)$ and $\hat{y}(t)$. From the definition of the mechanical momentum operator $m\hat{u} = \hat{P}_x + qB\hat{y}(1 - \hat{y}/2L_B)$, the time dependent operator $\hat{y}(t)$ is found to first order in L_B^{-1} and L_E^{-1} as,

$$\begin{aligned}qB\hat{y}(t) = & \left(1 - \frac{\hat{P}_x}{qBL_B} - \frac{1}{qBL_E} \frac{mE}{B} + \frac{3}{2qBL_B} \frac{mE}{B} \right) \hat{\Pi}_x(t) - \hat{P}_x + \frac{\hat{P}_x^2}{2qBL_B} \\ & + \left(1 + \frac{\hat{P}_x}{qBL_E} - \frac{2\hat{P}_x}{qBL_B} - \frac{1}{qBL_E} \frac{mE}{B} + \frac{3}{2qBL_B} \frac{mE}{B} \right) \frac{mE}{B},\end{aligned}\quad (4.16)$$

which leads to

$$\begin{aligned}
qB\hat{y}(t) = & \frac{3\hat{P}_y^2 + 3\hat{\Pi}_x^2}{4qBL_B} - \hat{P}_x + \frac{\hat{P}_x^2}{2qBL_B} + \left(1 + \frac{\hat{P}_x}{qBL_E} - \frac{2\hat{P}_x}{qBL_B} - \frac{1}{qBL_E} \frac{mE}{B} + \frac{3}{2qBL_B} \frac{mE}{B}\right) \frac{mE}{B} \\
& + \left(\hat{\Pi}_x - \frac{\hat{\Pi}_x \hat{P}_x}{qBL_B} - \frac{\hat{\Pi}_x}{2qBL_E} \frac{mE}{B} + \frac{3\hat{\Pi}_x}{2qBL_B} \frac{mE}{B} - \frac{2\hat{P}_y^2 + \hat{\Pi}_x^2}{2qBL_B}\right) \cos \hat{\Omega}t + \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{4qBL_B} \cos 2\hat{\Omega}t \\
& + \left(\hat{P}_y - \frac{\hat{P}_y \hat{P}_x}{qBL_B} - \frac{\hat{P}_y}{2qBL_E} \frac{mE}{B} + \frac{3\hat{\Pi}_x}{2qBL_B} \frac{mE}{B} + \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2qBL_B}\right) \sin \hat{\Omega}t - \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{4qBL_B} \sin 2\hat{\Omega}t.
\end{aligned} \tag{4.17}$$

From the operator $m\hat{u} = \hat{P}_x + qB\hat{y}(1 - \hat{y}/2L_B)$, the time dependent operator $m\hat{u}(t)$ is found as,

$$\begin{aligned}
m\hat{u}(t) = & \frac{\hat{P}_y^2 + \hat{\Pi}_x^2}{2qBL_B} + \left[1 - \left(\frac{1}{qBL_B} - \frac{1}{qBL_E}\right) \hat{P}_x + \left(\frac{1}{qBL_B} - \frac{1}{qBL_E}\right) \frac{mE}{B}\right] \frac{mE}{B} \\
& + \left\{ \left[1 + \left(\frac{1}{qBL_B} - \frac{1}{qBL_E}\right) \frac{mE}{2B}\right] \hat{\Pi}_x - \frac{2\hat{P}_y^2 + \hat{\Pi}_x^2}{2qBL_B} \right\} \cos \hat{\Omega}t + \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{2qBL_B} \cos 2\hat{\Omega}t \\
& + \left\{ \left[1 + \left(\frac{1}{qBL_B} - \frac{1}{qBL_E}\right) \frac{mE}{2B}\right] \hat{P}_y + \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2qBL_B} \right\} \sin \hat{\Omega}t - \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2qBL_B} \sin 2\hat{\Omega}t.
\end{aligned} \tag{4.18}$$

From the time dependent momentum operator $m\hat{u}(t)$, the drift velocity operator $\hat{u}_{\text{drift}}$ along x-axis due to the non-uniform electric and magnetic field to first order in L_B^{-1} and L_E^{-1} is obtained as

$$\hat{u}_{\text{drift}} = \frac{\hat{P}_y^2 + \hat{\Pi}_x^2}{2qmBL_B} + \left[1 - \left(\frac{1}{qBL_B} - \frac{1}{qBL_E}\right) \left(\hat{P}_x - \frac{mE}{B}\right)\right] \frac{E}{B}.$$

The expectation value of any operator $\hat{X}$ is obtained by implementing the initial condition for the wavefunction at $\mathbf{r} = \mathbf{r}_0$ with is $\mathbf{r}_0$ being the initial center of wavefunction ψ is given by

$$\psi(\mathbf{r}, 0) = \frac{1}{\sqrt{\pi}\sigma_B} \exp\left[-\frac{(\mathbf{r} - \mathbf{r}_0)^2}{2\sigma_B^2} + i\mathbf{k}_0 \cdot \mathbf{r}\right],$$

where the magnetic length $\sigma_B = \sqrt{\hbar/qB}$ is the initial standard deviation, and $\hbar\mathbf{k}_0$ is the initial canonical momentum. Thus, the expectation value of drift velocity is obtained as,

$$\begin{aligned}
\langle \hat{u}_{\text{drift}} \rangle = & \frac{mv_0^2}{2qBL_B} + \frac{1}{2qBL_B} \left(\frac{\hbar^2}{m\sigma_B^2} + \frac{m\omega^2\sigma_B^2}{2} \right) \\
& + \left(1 + \frac{mu_0 - qBy_0 - mE/B}{qBL_E} - \frac{2mu_0 - qBy_0 - 3mE/2B}{qBL_B} \right) \frac{E}{B},
\end{aligned} \tag{4.19}$$

where $mv_0 \equiv \langle m\hat{v} \rangle = \langle \hat{P} - q\hat{A} \rangle$. When we use the magnetic length of $\sqrt{\hbar/qB}$ as σ_B [3], then we have

$$u_{\text{VB}} = \frac{mv_0^2}{2qBL_B} + \frac{3\hbar}{4mL_B}, \tag{4.20}$$

$$u_{\text{E} \times B} = \left(1 + \frac{mu_0 - qBy_0 - mE/B}{qBL_E} - \frac{2mu_0 - qBy_0 - 3mE/2B}{qBL_B} \right) \frac{E}{B}. \tag{4.21}$$

The drift velocity as shown in Eq. (4.20) is due solely to the magnetic field non-uniformity. The drift velocity due to the electric and magnetic field non-uniformity, $u_{E \times B}$ in Eq. (4.21) is the same as the classical drift velocity to first order in L_B^{-1} and L_E^{-1} , which can be easily derived from the classical equation of motion.

The time dependent operator $\hat{x}(t)$ is obtained by integrating $\hat{u}(t)$ given in Eq. (4.18) with time t ,

$$\hat{x}(t) = \hat{x} + \int_0^t \hat{u}(t) dt,$$

which leads to,

$$\begin{aligned} m\hat{x}(t) = m\hat{x} &+ \left[1 + \left(\frac{1}{qBL_B} - \frac{1}{qBL_E} \right) \frac{mE}{B} \right] \frac{\hat{P}_y}{\hat{\Omega}} + \left\{ \left[1 - \left(\frac{1}{qBL_B} - \frac{1}{qBL_E} \right) \left(\hat{P}_x - \frac{mE}{B} \right) \right] \frac{mE}{B} + \frac{\hat{P}_y^2 + \hat{\Pi}_x^2}{2qBL_B} \right\} t \\ &+ \frac{\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x}{4qBL_B \omega} + \left\{ \left[1 + \left(\frac{1}{qBL_B} - \frac{1}{qBL_E} \right) \frac{mE}{B} \right] \hat{\Pi}_x - \frac{2\hat{P}_y^2 + \hat{\Pi}_x^2}{2qBL_B} \right\} \frac{\sin \hat{\Omega}t}{\hat{\Omega}} + \frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{2qBL_B} \frac{\sin 2\hat{\Omega}t}{2\hat{\Omega}} \\ &- \left\{ \left[1 + \left(\frac{1}{qBL_B} - \frac{1}{qBL_E} \right) \frac{mE}{B} \right] \hat{P}_y + \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2qBL_B} \right\} \frac{\cos \hat{\Omega}t}{\hat{\Omega}} + \frac{\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x}{2qBL_B} \frac{\cos 2\hat{\Omega}t}{2\hat{\Omega}}, \end{aligned} \quad (4.22)$$

The time evolution of variance in position and momenta are formally given as

$$\begin{aligned} \frac{d\hat{\sigma}_r^2(t)}{dt} &= \langle \hat{r}(t) \cdot \hat{v}(t) + \hat{v}(t) \cdot \hat{r}(t) \rangle - 2\langle \hat{r}(t) \rangle \cdot \langle \hat{v}(t) \rangle, \\ \frac{d\hat{\sigma}_{mv}^2(t)}{dt} &= \left\langle m\hat{v}(t) \cdot \frac{dm\hat{v}(t)}{dt} + \frac{dm\hat{v}(t)}{dt} \cdot m\hat{v}(t) \right\rangle - 2\left\langle \frac{dm\hat{v}(t)}{dt} \right\rangle \cdot \langle m\hat{v}(t) \rangle, \\ \frac{d\hat{\sigma}_p^2(t)}{dt} &= \left\langle \hat{P}(t) \cdot \frac{d\hat{P}(t)}{dt} + \frac{d\hat{P}(t)}{dt} \cdot \hat{P}(t) \right\rangle - 2\left\langle \frac{d\hat{P}(t)}{dt} \right\rangle \cdot \langle \hat{P}(t) \rangle. \end{aligned}$$

In order to evaluate the formal solution above, it is noted that the solutions consist of multiple combination of the operators such as $\hat{x}$, $\hat{y}$, $\hat{P}_x$, $\hat{P}_y$, $\hat{\Pi}_x$, and $\hat{\Omega}_x$. Here are some parts of the combinations of expectation values as

$$\begin{aligned} \langle \hat{\Pi}_x \hat{P}_x \hat{P}_y \rangle &= \langle \hat{\Pi}_x \rangle \langle \hat{P}_x \rangle \langle \hat{P}_y \rangle + \frac{i\hbar qB}{2} \left\langle \hat{P}_x + \frac{\hat{P}_x^2}{qBL_B} + \frac{\hat{P}_x}{2qBL_E} \frac{mE}{B} - \frac{3\hat{P}_x}{2qBL_B} \frac{mE}{B} \right\rangle, \\ \langle \hat{P}_y \hat{\Pi}_x \hat{P}_x \rangle &= \langle \hat{\Pi}_x \rangle \langle \hat{P}_x \rangle \langle \hat{P}_y \rangle - \frac{i\hbar qB}{2} \left\langle \hat{P}_x + \frac{\hat{P}_x^2}{qBL_B} + \frac{\hat{P}_x}{2qBL_E} \frac{mE}{B} - \frac{3\hat{P}_x}{2qBL_B} \frac{mE}{B} \right\rangle, \\ \langle \hat{\Pi}_x \hat{P}_y^2 \rangle &= \langle \hat{\Pi}_x \rangle \langle \hat{P}_y^2 \rangle + i\hbar qB \langle \hat{P}_y \rangle \left\langle 1 + \frac{\hat{P}_x}{qBL_B} + \frac{1}{2qBL_E} \frac{mE}{B} - \frac{3}{2qBL_B} \frac{mE}{B} \right\rangle, \\ \langle \hat{P}_y^2 \hat{\Pi}_x \rangle &= \langle \hat{\Pi}_x \rangle \langle \hat{P}_y^2 \rangle - i\hbar qB \langle \hat{P}_y \rangle \left\langle 1 + \frac{\hat{P}_x}{qBL_B} + \frac{1}{2qBL_E} \frac{mE}{B} - \frac{3}{2qBL_B} \frac{mE}{B} \right\rangle. \end{aligned}$$

4.2.1 Time dependent operator of expansion of variance

With further evaluations of different combinations of operators and the combinations of solutions given above, finally, it is shown that the expansion rate of variance in position, to first order in L_B^{-1} and L_E^{-1} ,

$$\frac{d\sigma_r^2(t)}{dt} = \frac{d\sigma_x^2(t)}{dt} + \frac{d\sigma_y^2(t)}{dt} = \frac{3}{2} \frac{\hbar v_0}{qBL_B} + \frac{1}{2} \frac{\hbar v_0}{qBL_B} = 2 \frac{\hbar v_0}{qBL_B}. \quad (4.23)$$

Further on, the expansion rate of variance in mechanical momentum to first order in L_B^{-1} and L_E^{-1} ,

$$\frac{d\sigma_{mv}^2(t)}{dt} = \frac{d\sigma_{mu}^2(t)}{dt} + \frac{d\sigma_{mv}^2(t)}{dt} = \frac{1}{2} \frac{\hbar qBv_0}{L_B} + \frac{1}{2} \frac{\hbar qBv_0}{L_B} = \frac{\hbar qBv_0}{L_B}, \quad (4.24)$$

and the expansion rate of variance in total momentum to first order in L_B^{-1} and L_E^{-1} ,

$$\frac{d\sigma_p^2(t)}{dt} = \frac{d\sigma_{p_x}^2(t)}{dt} + \frac{d\sigma_{p_y}^2(t)}{dt} = 0 + \frac{1}{2} \frac{\hbar qBv_0}{L_B} = \frac{1}{2} \frac{\hbar qBv_0}{L_B}. \quad (4.25)$$

It is interesting to note that most of the operators derived are dependent on the both magnetic gradient scale length L_B and electric gradient scale length L_E . However, upon operation of the expectation, the expansion rates depend only on the magnetic gradient scale length L_B , but not on the electric gradient scale length L_E .

4.2.2 Expansion times

From the analytical results Eqs. (4.23-4.25), the variance in position $\sigma_r^2(t)$ reaches the square of interparticle separation of $n^{-2/3}$ at τ_r as

$$\tau_r \equiv \frac{n^{-2/3}}{2\hbar v_0/qBL_B}. \quad (4.26)$$

The variance in momentum $\sigma_{mv}^2(t)$ reaches the square of initial mechanical momentum mv_0 as

$$\tau_{mv} \equiv \frac{(mv_0)^2}{\hbar qBv_0/L_B}. \quad (4.27)$$

These finding can be applied to a broad scientific research fields, for example, in the magnetically confined fusion plasmas.

For a torus plasma with a temperature T , the gradient scale length of the non-uniformity of the field L_B can be replaced by the major radius R_0 of the torus, and the initial velocity v_0 by the thermal speed $v_{th} = \sqrt{2T/m}$ [4], in Eq. (4.26), which is in proportion to $\sqrt{m/TK_0}$, thus the isotope effect of $\tau \propto \sqrt{m}$ appears. This leads to the characteristic time of $\tau_r = 0.25$ ms for a typical fusion plasmas of $T \sim 10$ keV and $n \sim 10^{20} \text{ m}^{-3}$ with $R_0 \sim 3\text{m}$, which is much shorter than the proton collision time of $\tau_{ii} \sim 20$ ms. Since the ions and the electrons in fusion plasma are considered to be at the same temperature, and the velocities of electrons are much faster than protons. So, the time for electrons to reach interparticle separation is much faster than for ions. The electrons should be treated as a uniform background for the ions after the electrons expansion time.

The laser-plasma interaction produced plasmas have multi-MeV protons and electrons [5, 6]. The expansion time in position given in Eq. (4.26) implies that the probability density function (PDF) of such energetic charged particles expand fast in the plane perpendicular to the magnetic field. The broad distribution of individual particle in space means that their Coulomb interactions with other particles of distance less than $\sigma_r(t)$ becomes weaker than that expected in classical mechanics. This could make effective electronic charge smaller, $e_{\text{eff}} < e$, and thus make larger Debye shielding length than the classical one given by $\lambda_D = \sqrt{\epsilon_0 k_B T / ne^2}$. Such a Debye screening modification due to quantum mechanical effect was pointed out based on a quantum hydrodynamic (QHD) model [7], in which the Fermi pressure and Bohm potential were included in the fluid equation.

Plasmas are known to satisfy the condition $n\lambda_D^3 \gg 1$, thus the average interparticle distance $n^{-1/3}$ is much smaller than the Debye length, irrespective of the density and the energy (or temperature). In dense plasmas with the density of $n > (qB/\hbar)^{3/2} = \sigma_r^{-3}(t=0)$, the expansion time τ_r is zero, since $n^{-1/3} < \sqrt{\hbar/qB}$ at $t=0$. In the opposite case of $n < (qB/\hbar)^{3/2}$, high energy particles have short $\tau_r \propto v_0^{-1}$. Thus, the interparticle interaction, characterized by e_{eff} , in high density and high energy plasmas are always smaller than e if $n > (eB/\hbar)^{3/2}$, otherwise becomes smaller in a short time of the order of the expansion time τ_r .

4.3 Numerical confirmation

A two-dimensional time-dependent Schrödinger equation code is developed and the numerical calculation is done on a GPU (Nvidia GTX-980: 2048cores/4GB @1.126GHz), using CUDA [8]. In the numerical calculations, the parameters are normalized by those of a proton with a speed of 10 [m/s] in the presence of a magnetic field of 10 [T]. The lengths and times are normalized by its cyclotron radius and cyclotron frequency, respectively. These leads to the normalized electric field in unit of 100 [V/m], and to the normalized initial standard deviation of $\sigma_B = \sqrt{\hbar/eB} \cong 0.777$.

In the presence of a scalar potential $V = -Ey(1 - y/2L_E)$ and the vector potential $\mathbf{A} = -By(1 - y/2L_B)\mathbf{e}_x$, we calculate the numerical expansion rates by averaging the variances over five cyclotron periods. We have divided the time-dependent variance $\sigma_r^2(t) \equiv \sigma_r^2(0) + \Delta\sigma_r^2(t)$ into its initial value $\sigma_r^2(0)$ and the increment $\Delta\sigma_r^2(t)$. From Fig. 4.1, the numerical expansion rate of variance in position is shown as,

$$\frac{d\sigma_r^2(t)}{dt} = \frac{d\sigma_x^2(t)}{dt} + \frac{d\sigma_y^2(t)}{dt} = 1.494 \frac{\hbar v_0}{qBL_B} + 0.492 \frac{\hbar v_0}{qBL_B} = 1.986 \frac{\hbar v_0}{qBL_B}. \quad (4.28)$$

Figure 4.2 shows the expansion rate of variance in mechanical momentum as,

$$\frac{d\sigma_{mv}^2(t)}{dt} = \frac{d\sigma_{mu}^2(t)}{dt} + \frac{d\sigma_{mv}^2(t)}{dt} = 0.502 \frac{\hbar qBv_0}{L_B} + 0.502 \frac{\hbar qBv_0}{L_B} = 1.004 \frac{\hbar qBv_0}{L_B}, \quad (4.29)$$

and the numerical expansion rate of variance in total momentum is shown in Fig. 4.3, as

$$\frac{d\overline{\sigma_p^2(t)}}{dt} = \frac{d\overline{\sigma_{p_x}^2(t)}}{dt} + \frac{d\overline{\sigma_{p_y}^2(t)}}{dt} = 0.000 \frac{\hbar q B v_0}{L_B} + 0.502 \frac{\hbar q B v_0}{L_B} = 0.502 \frac{\hbar q B v_0}{L_B}. \quad (4.30)$$

Comparing the analytical results with the numerical calculation, the analytical results given are generally consistent with the numerical expansion rates in position, mechanical momentum and total momentum. For the numerical analysis of expansion rates of variance in position Eq. (4.28), mechanical momentum Eq. (4.29), and the total momentum Eq. (4.30), it is noted that, to first order in L_B^{-1} and L_E^{-1} , the expressions are independent of the electric gradient scale length L_E [9].

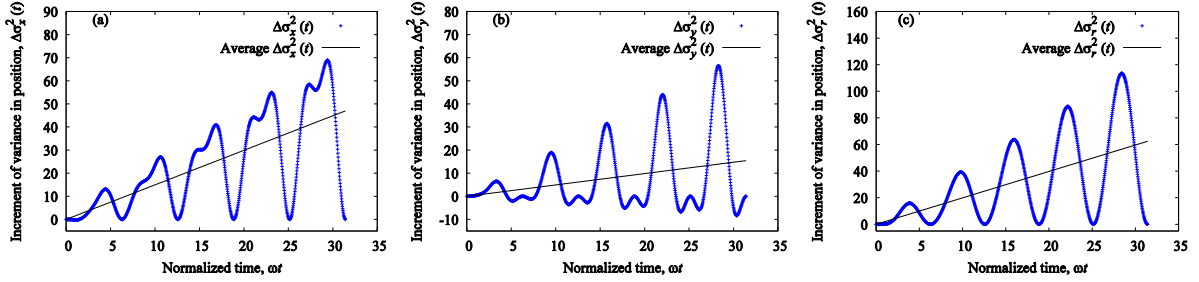


Fig. 4.1 Numerical time evolution of increment $\Delta\sigma^2(t)$ in variance of position normalized by $\hbar v_0 / qBL_B$ for 5 gyrations. (a) Incremental variances $\Delta\sigma_x^2(t)$ with an average expansion rate of 1.494. (b) Incremental variances $\Delta\sigma_y^2(t)$ with an average expansion rate of 0.492. (c) Incremental variances $\Delta\sigma_r^2(t)$ with an average expansion rate of 1.986.

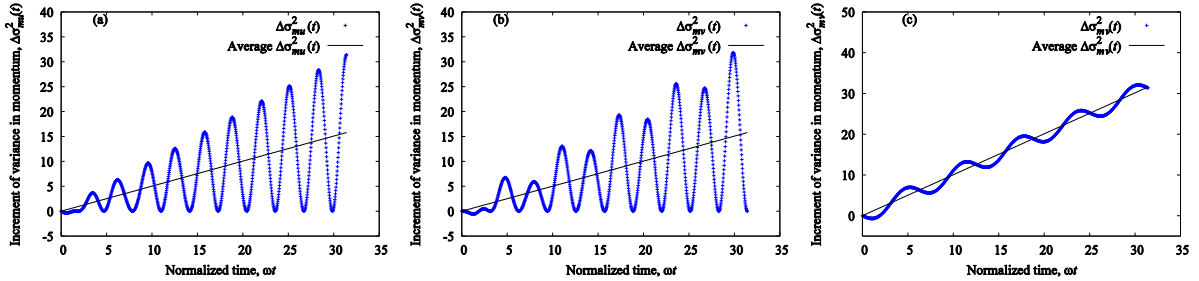


Fig. 4.2 Numerical time evolution of increment $\Delta\sigma^2(t)$ in variance of mechanical momentum normalized by $\hbar q B v_0 / L_B$ for 5 gyrations. (a) Incremental variances $\Delta\sigma_{mu}^2(t)$ with an average expansion rate of 0.502. (b) Incremental variances $\Delta\sigma_{mv}^2(t)$ with an average expansion rate of 0.502. (c) Incremental variances $\Delta\sigma_m^2(t)$ with an average expansion rate of 1.004.

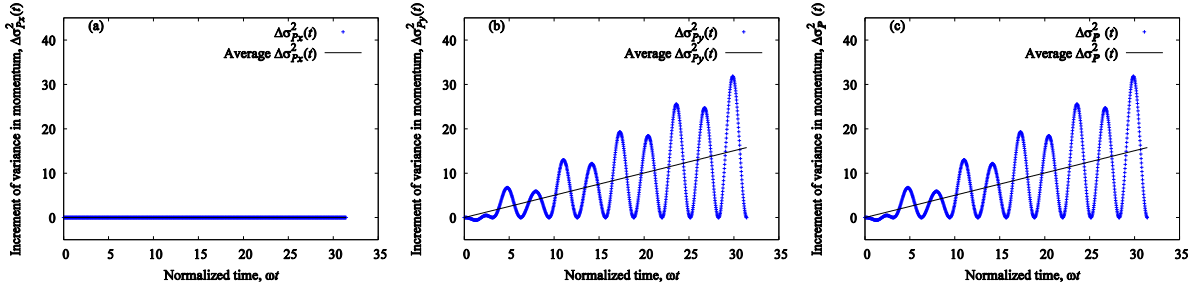


Fig. 4.3 Numerical time evolution of increment $\Delta\sigma^2(t)$ in variance of total momentum normalized by $\hbar q B v_0 / L_B$ for 5 gyrations. (a) Incremental variances $\Delta\sigma^2_{Px}(t)$ with an average expansion rate of 0. (b) Incremental variances $\Delta\sigma^2_{Py}(t)$ with an average expansion rate of 0.502. (c) Incremental variances $\Delta\sigma^2_{Pz}(t)$ with an average expansion rate of 0.502.

4.4 Summary

By solving the Heisenberg equation of motion, the time evolution of position and momentum operators for a non-relativistic spinless charged particle in the presence of a weakly non-uniform electric field or a scalar potential $V = -Ey(1 - y/2L_E)$, and a weakly non-uniform magnetic field or the Landau gauge-like quadratic vector potential $\mathbf{A} = -By(1 - y/2L_B)\mathbf{e}_x$ are theoretically derived. Using the time dependent operators, it is shown how the variances in position and momentum grow with time. The theoretical solutions are generally consistent with the numerical results on the expansion rates in position and momenta in the presence of a weakly non-uniform electric and magnetic field. It is important to note that the expansion rates do not depend on the strength of the electric field E and its gradient scale length L_E to first order in L_E^{-1} . It is also shown that the drift velocity operator agrees with the classical counterpart.

It is analytically shown that the variance in position $\sigma_r^2(t)$ reaches the square of interparticle separation of $n^{-2/3}$ at $\tau_r = qBL_B n^{-2/3}/2\hbar v_0$ and the variance in momentum $\sigma_{mv}^2(t)$ reaches the square of initial mechanical momentum mv_0 at $\tau_{mv} = L_B(mv_0)^2/\hbar q B v_0$. These finding can be applied to a broad scientific research fields in the plasma physics for example, in the magnetically confined fusion plasmas, where the expansion time plays an important role. It is also shown that for the laser-plasma interaction, the quantum effect could make the effective electronic charge become smaller, $e_{\text{eff}} < e$, and thus make larger Debye shielding length than the classical one.

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Chapter 5

Quantum mechanical $E \times B$ drift at higher order of electromagnetic field inhomogeneity

5.1 Initial condition

The unsteady Schrödinger equation for a wavefunction $\psi(\mathbf{r}, t)$ at position $\mathbf{r}$ and time t is given as,

$$i\hbar \frac{\partial \psi}{\partial t} = \left\{ \frac{1}{2m} (-i\hbar \nabla - q\mathbf{A})^2 + qV \right\} \psi, \quad (5.1)$$

where $V = V(\mathbf{r})$ and $\mathbf{A} = \mathbf{A}(\mathbf{r})$ stand for the scalar and vector potentials, m and q the mass and the electric charge of the particle under consideration, $i \equiv \sqrt{-1}$ the imaginary unit.

The initial condition for wavefunction at $\mathbf{r} = \mathbf{r}_0$ and $t = 0$ which the initial center of wavefunction $\psi(\mathbf{r}, 0)$ which given by

$$\psi(\mathbf{r}, 0) = \frac{1}{\sqrt{\pi}\sigma_B} \exp\left(-\frac{(\mathbf{r} - \mathbf{r}_0)^2}{2\sigma_B^2} + i\mathbf{k}_0 \cdot \mathbf{r}\right), \quad (5.2)$$

where the magnetic length $\sigma_B = \sqrt{\hbar/qB}$ is the initial standard deviation, and $\hbar\mathbf{k}_0$ is the initial canonical momentum.

The two-dimensional Schrödinger equation is solved in the presence of an inhomogeneous magnetic field $\mathbf{B} = B_0(1 - y/L_B)\mathbf{e}_z$ [1-3], in which a Landau gauge-like quadratic vector potential is given as

$$\mathbf{A} = -B_0 y \left(1 - \frac{y}{2L_B}\right) \mathbf{e}_x, \quad (5.3)$$

and with the presence of an inhomogeneous electric field $\mathbf{E} = E_0(1 - y/L_E + y^2/\ell_E^2)\mathbf{e}_y$ [1], a quadratic scalar potential is given as

$$V = -E_0 y \left(1 - \frac{y}{2L_E} + \frac{y^2}{3\ell_E^2}\right). \quad (5.4)$$

The Landau gauge we use here is given as $\mathbf{A} = A_x(y)\mathbf{e}_x$.

5.2 Time dependent operators for the inhomogeneous electric and magnetic field

Substituting the Landau gauge-like vector potential in Eq. (5.3) and the electrostatic potential in Eq. (5.4) into the two-dimensional Schrödinger equation in Eq. (5.1), the Hamiltonian $\hat{H}$ for a charge particle with a mass m and a charge q without the electrostatic potential for the non-relativistic charge particle, to first order in L_B^{-1} , L_E^{-1} , and ℓ_E^{-2} is given as

$$2m\hat{H} = \hat{p}_y^2 + \hat{\Pi}_x^2 - \hat{\Pi}_x^3 \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) + \Delta\hat{H}, \quad (5.5)$$

and

$$\Delta\hat{H} = \left[\hat{P}_x - \frac{mE}{2B} - \left(\frac{1}{2qBL_B} - \frac{1}{2qBL_E} \right) \left(\hat{P}_x - \frac{mE}{B} \right)^2 + \frac{1}{3q^2B^2\ell_E^2} \left(\hat{P}_x - \frac{mE}{B} \right)^3 \frac{E}{B} \right] \frac{E}{B},$$

where $\hat{P}_y = -i\hbar\partial_y$ is the y -component of the momentum operator and $\hat{P}_x = -i\hbar\partial_x$ is the x -component of the momentum operator defined in operator $\hat{\Pi}_x$ as,

$$\begin{aligned} \hat{\Pi}_x = & \left[1 + \left(\frac{3}{2qBL_B} + \frac{1}{q^2B^2\ell_E^2} \frac{mE}{B} \right) \left(\hat{P}_x - \frac{mE}{B} \right) - \frac{\hat{P}_x}{2qBL_B} + \frac{1}{2BL_E} \frac{mE}{B} \right] (qBy) \\ & + \left(1 + \frac{\hat{P}_x}{2qBL_B} - \frac{1}{2BL_E} \frac{mE}{B} \right) \left(\hat{P}_x - \frac{mE}{B} \right). \end{aligned} \quad (5.6)$$

It is note that L_B^{-1} is the same order as L_E^{-1} . The mechanical momentum operators $m\hat{v} = \hat{\mathbf{P}} - q\hat{\mathbf{A}} = (m\hat{u}, m\hat{v})$ to first order in L_B^{-1} , L_E^{-1} , and ℓ_E^{-2} are given as,

$$\begin{aligned} m\hat{u} = & \left[1 - \left(\frac{3}{2qBL_B} + \frac{1}{q^2B^2\ell_E^2} \frac{mE}{B} \right) \left(\hat{P}_x - \frac{mE}{B} \right) - \left(\frac{1}{qBL_B} - \frac{1}{2BL_E} \right) \frac{mE}{B} + \frac{3\hat{P}_x}{2qBL_B} - \frac{\hat{\Pi}_x}{2qBL_B} \right] \hat{\Pi}_x \\ & - \left(1 + \frac{\hat{P}_x}{qBL_B} - \frac{1}{qBL_E} \frac{mE}{B} \right) \left(\hat{P}_x - \frac{mE}{B} \right) + \left(\frac{3}{2qBL_B} + \frac{1}{q^2B^2\ell_E^2} \frac{mE}{B} \right) \left(\hat{P}_x - \frac{mE}{B} \right)^2 \\ & + \left(\frac{\hat{P}_x}{qBL_B} - \frac{1}{2qBL_B} \frac{mE}{B} \right) \frac{mE}{B} + \left(1 - \frac{\hat{P}_x}{2qBL_B} \right) \hat{P}_x, \end{aligned} \quad (5.7)$$

$$m\hat{v} = \hat{P}_y. \quad (5.8)$$

For any operator $\hat{X}$, its time development $\hat{X}(t)$ is given by the following Heisenberg equation of motion,

$$i\hbar \frac{d\hat{X}(t)}{dt} = [\hat{X}(t), \hat{H}], \quad (5.9)$$

where the square bracket $[\cdot, \cdot]$ is the commutator. From the Heisenberg equation of motion, the time derivative $\frac{d\hat{\Pi}_x}{dt}$ is shown as,

$$\begin{aligned} \frac{d\hat{\Pi}_x}{dt} = & \left[1 + \left(\frac{1}{qBL_B} + \frac{1}{q^2B^2\ell_E^2} \frac{mE}{B} \right) \hat{P}_x - \left(\frac{3}{2qBL_B} + \frac{1}{2BL_E} - \frac{1}{q^2B^2\ell_E^2} \frac{mE}{B} \right) \frac{mE}{B} \right] \omega \hat{P}_y \\ = & \hat{\Omega} \hat{P}_y, \end{aligned} \quad (5.10)$$

which leads to the definition of the angular frequency operator as,

$$\hat{\Omega} = 1 + \left(\frac{1}{qBL_B} + \frac{1}{q^2B^2\ell_E^2} \frac{mE}{B} \right) \hat{P}_x - \left(\frac{3}{2qBL_B} + \frac{1}{2BL_E} - \frac{1}{q^2B^2\ell_E^2} \frac{mE}{B} \right) \frac{mE}{B}, \quad (5.11)$$

where $\omega = qB/m$ is the usual cyclotron angular frequency. On the other hand, for the momentum operator in the y -direction $\hat{P}_y$ we have,

$$\frac{d\hat{P}_y}{dt} = - \left[\hat{\Pi}_x - \left(\frac{3}{2qBL_B} + \frac{mE}{q^2B^3L_E^2} \right) \hat{\Pi}_x^2 \right] \hat{\Omega}, \quad (5.12)$$

Further on, in order to derive the time dependent momentum operators $\hat{P}_x(t)$, $\hat{P}_y(t)$, and $\hat{\Pi}_x(t)$ with Heisenberg equation of motion operation, we expand the Eq. (5.10) and Eq. (5.12) with Heisenberg picture, $\hat{X}(t) = \exp(-\hat{H}t/\hbar) \hat{X} \exp(\hat{H}t/\hbar)$ and simplified the equations by using Taylor expansion. Let us choose the operators $\hat{\pi}_x$ and $\hat{\pi}_y$ as

$$\hat{\pi}_x = \hat{\Pi}_x - \frac{3}{2} \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) (2\hat{P}_y^2 - \hat{\Pi}_x^2), \quad (5.13)$$

and

$$\hat{\pi}_y = \hat{P}_y - \frac{3}{2} \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) (\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x). \quad (5.14)$$

Using Heisenberg equation of motion, the time derivative of the operator $\hat{\pi}_x$ given in Eq. (5.13) can be obtain as,

$$\begin{aligned} \frac{d^{2n-1} \hat{\pi}_x}{dt^{2n-1}} &= +(-1)^{n-1} \hat{\Omega}^{2n-1} \left[\hat{P}_y + \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2} \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) \right] \text{ for } n \geq 1 \\ &\quad - 4(-1)^{n-1} (2\hat{\Omega})^{2n-1} \left(\frac{\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x}{qBL_B} \right) \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right), \\ \frac{d^{2n} \hat{\pi}_x}{dt^{2n}} &= +(-1)^n \hat{\Omega}^{2n} \left[\hat{\Pi}_x - \frac{3}{2} \hat{\Pi}_x^2 \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) \right. \\ &\quad \left. - (\hat{P}_y^2 - \hat{\Pi}_x^2) \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) \right] \text{ for } n \geq 0 \\ &\quad + 4(-1)^n (2\hat{\Omega})^{2n} (\hat{P}_y^2 - \hat{\Pi}_x^2) \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right), \end{aligned}$$

which leads to

$$\begin{aligned} \hat{\pi}_x(t) &= \left[\hat{\Pi}_x - \frac{3}{2} \hat{\Pi}_x^2 \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) - (\hat{P}_y^2 - \hat{\Pi}_x^2) \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) \right] \cos \hat{\Omega}t \\ &\quad + \left[\hat{P}_y + \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2} \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) \right] \sin \hat{\Omega}t \\ &\quad + 4(\hat{P}_y^2 - \hat{\Pi}_x^2) \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) \cos 2\hat{\Omega}t \\ &\quad - 4(\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x) \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) \sin 2\hat{\Omega}t, \end{aligned}$$

Combining the operator $\hat{P}_y(t)$ with the operator $\hat{\pi}_x(t)$ above, the time dependent momentum operator $\hat{\Pi}_x(t)$ is found, to first order in L_B^{-1} , L_E^{-1} , and ℓ_E^{-2} as

$$\begin{aligned}
\hat{\Pi}_x(t) = & \left(\frac{3}{4qBL_B} + \frac{1}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) (\hat{P}_y^2 + \hat{\Pi}_x^2) + \left[\hat{\Pi}_x - \left(\frac{1}{2qBL_B} + \frac{1}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) (2\hat{P}_y^2 + \hat{\Pi}_x^2) \right] \cos \hat{\Omega}t \\
& + \left[\hat{P}_y + \left(\frac{1}{2qBL_B} + \frac{1}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) (\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y) \right] \sin \hat{\Omega}t \\
& + \left[\left(\frac{1}{4qBL_B} + \frac{1}{6q^2B^2\ell_E^2} \frac{mE}{B} \right) (\hat{P}_y^2 - \hat{\Pi}_x^2) \right] \cos 2\hat{\Omega}t \\
& - \left[\left(\frac{1}{4qBL_B} + \frac{1}{6q^2B^2\ell_E^2} \frac{mE}{B} \right) (\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x) \right] \sin 2\hat{\Omega}t,
\end{aligned} \tag{5.15}$$

Next, to obtain the time dependent operator $\hat{P}_y(t)$, we use the Heisenberg equation of motion on to the operator $\hat{\pi}_y$ as given in Eq. (5.14). Thus, the time derivative $\hat{\pi}_y$ given can be obtain as,

$$\begin{aligned}
\frac{d^{2n-1} \hat{\pi}_y}{dt^{2n-1}} = & -(-1)^{n-1} \hat{\Omega}^{2n-1} \left[\hat{\Pi}_x - \frac{3}{2} \hat{\Pi}_x^2 \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) \right. \\
& \left. - (\hat{P}_y^2 - \hat{\Pi}_x^2) \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) \right] \quad \text{for } n \geq 1 \\
& - 2(-1)^{n-1} (2\hat{\Omega})^{2n-1} (\hat{P}_y^2 - \hat{\Pi}_x^2) \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right), \\
\frac{d^{2n} \hat{\pi}_y}{dt^{2n}} = & +(-1)^n \hat{\Omega}^{2n} \left[\hat{P}_y + \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2} \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) \right] \quad \text{for } n \geq 0 \\
& - 2(-1)^n (2\hat{\Omega})^{2n} (\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x) \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right),
\end{aligned}$$

which leads to

$$\begin{aligned}
\hat{\pi}_y(t) = & \left[\hat{P}_y + \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2} \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) \right] \cos \hat{\Omega}t \\
& - \left[\hat{\Pi}_x - \frac{3}{2} \hat{\Pi}_x^2 \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) - (\hat{P}_y^2 - \hat{\Pi}_x^2) \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) \right] \sin \hat{\Omega}t \\
& - 2(\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x) \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) \cos 2\hat{\Omega}t \\
& - 2(\hat{P}_y^2 - \hat{\Pi}_x^2) \left(\frac{1}{qBL_B} + \frac{2}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) \sin 2\hat{\Omega}t.
\end{aligned}$$

Combining the operator $\hat{\Pi}_x(t)$ with the operator $\hat{\pi}_y(t)$ above, the time dependent momentum operator $\hat{P}_y(t)$ is found, to first order in L_B^{-1} , L_E^{-1} , and ℓ_E^{-2} as

$$\begin{aligned}
\hat{P}_y(t) = & \left[\hat{P}_y + \left(\frac{1}{2qBL_B} + \frac{1}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) (\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y) \right] \cos \hat{\Omega}t \\
& - \left[\hat{\Pi}_x - \left(\frac{1}{2qBL_B} + \frac{1}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) (2\hat{P}_y^2 + \hat{\Pi}_x^2) \right] \sin \hat{\Omega}t \\
& - \left(\frac{1}{2qBL_B} + \frac{1}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) (\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x) \cos 2\hat{\Omega}t \\
& - \left(\frac{1}{2qBL_B} + \frac{1}{3q^2B^2\ell_E^2} \frac{mE}{B} \right) (\hat{P}_y^2 - \hat{\Pi}_x^2) \sin 2\hat{\Omega}t.
\end{aligned} \tag{5.16}$$

From the solution above, we obtain the time dependent momenta operators for $\hat{\Pi}_x(t)$ and $\hat{P}_y(t)$. It is known that, $\hat{P}_y^2(t) + \hat{\Pi}_x^2(t) = 2m\hat{H}$ and the time dependent Hamiltonian $\hat{H}(t) = \text{const} = \hat{H}$. For the remaining time dependent momentum operator, $\hat{P}_x(t) = \hat{P}_x$ does not change with time, since the Hamiltonian $\hat{H}$ in Eq. (5.5) does not include the position operator $\hat{x}$.

Using the results above, we derive the remaining time dependent operator $\hat{x}(t)$ and $\hat{y}(t)$ as below. Using the operator of $\hat{\Pi}_x$ given in Eq. (5.6), the time dependent operator $\hat{y}(t)$ is found and shown to first order in L_B^{-1} , L_E^{-1} , and ℓ_E^{-2} as,

$$\begin{aligned}
qB\hat{y}(t) = & \hat{\Pi}_x(t) \left[1 - \left(\frac{3}{2qBL_B} + \frac{1}{q^2B^2\ell_E^2} \frac{mE}{B} \right) \left(\hat{P}_x - \frac{mE}{B} \right) + \frac{\hat{P}_x}{2qBL_B} - \frac{1}{2BL_E} \frac{mE}{B} \right] \\
& - \left(1 + \frac{\hat{P}_x}{qBL_B} - \frac{1}{qBL_E} \frac{mE}{B} \right) \left(\hat{P}_x - \frac{mE}{B} \right) + \left(\frac{3}{2qBL_B} + \frac{1}{q^2B^2\ell_E^2} \frac{mE}{B} \right) \left(\hat{P}_x - \frac{mE}{B} \right)^2,
\end{aligned} \tag{5.17}$$

which leads to

$$\begin{aligned}
qB\hat{y}(t) = & \left(\frac{1}{qBL_B} + \frac{2}{3} \frac{1}{q^2B^2\ell_E^2} \frac{mE}{B} \right) \left(\frac{3\hat{P}_y^2 + 3\hat{\Pi}_x^2}{4} \right) - \left(1 + \frac{\hat{P}_x}{qBL_B} - \frac{1}{qBL_E} \frac{mE}{B} \right) \left(\hat{P}_x - \frac{mE}{B} \right) \\
& + \left(\frac{3}{2qBL_B} + \frac{1}{q^2B^2\ell_E^2} \frac{mE}{B} \right) \left(\hat{P}_x - \frac{mE}{B} \right)^2 + \left(\frac{1}{qBL_B} + \frac{2}{3} \frac{1}{q^2B^2\ell_E^2} \frac{mE}{B} \right) \left(\frac{\hat{P}_y^2 - \hat{\Pi}_x^2}{4} \right) \cos 2\hat{\Omega}t \\
& - \left(\frac{1}{qBL_B} + \frac{2}{3} \frac{1}{q^2B^2\ell_E^2} \frac{mE}{B} \right) \frac{(\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x)}{4} \sin 2\hat{\Omega}t \\
& + \left[\hat{\Pi}_x \hat{\eta}_y - \left(\frac{1}{qBL_B} + \frac{2}{3} \frac{1}{q^2B^2\ell_E^2} \frac{mE}{B} \right) \frac{(2\hat{P}_y^2 + \hat{\Pi}_x^2)}{2} \right] \cos \hat{\Omega}t \\
& + \left[\hat{P}_y \hat{\eta}_y + \left(\frac{1}{qBL_B} + \frac{2}{3} \frac{1}{q^2B^2\ell_E^2} \frac{mE}{B} \right) \frac{(\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y)}{2} \right] \sin \hat{\Omega}t,
\end{aligned} \tag{5.18}$$

where

$$\hat{\eta}_y = 1 - \left(\frac{3}{2qBL_B} + \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \left(\hat{P}_x - \frac{mE}{B} \right) + \frac{\hat{P}_x}{2qBL_B} - \frac{1}{2BL_E} \frac{mE}{B}.$$

From the operator $m\hat{u}$ in Eq. (5.7), together with the operator $\hat{\Pi}_x(t)$ in Eq. (5.15), the time dependent mechanical momentum operators along x-axis due to inhomogeneous electric and magnetic field is obtained as

$$\begin{aligned} m\hat{u}(t) = & \frac{1}{2} \left(\frac{1}{qBL_B} + \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \left(\hat{P}_y^2 + \hat{\Pi}_x^2 \right) + \frac{mE}{B} + \frac{2\hat{P}_x}{qBL_B} \frac{mE}{B} - \frac{1}{2qBL_B} \left(\frac{mE}{B} \right)^2 \\ & + \frac{1}{qBL_E} \frac{mE}{B} \left(\hat{P}_x - \frac{mE}{B} \right) - \frac{3\hat{P}_x^2}{2qBL_B} + \left(\frac{3}{2qBL_B} + \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \left(\hat{P}_x - \frac{mE}{B} \right)^2 \\ & + \left(\frac{1}{2qBL_B} + \frac{1}{6} \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \left(\hat{P}_y^2 - \hat{\Pi}_x^2 \right) \cos 2\hat{\Omega}t \\ & - \left(\frac{1}{2} \frac{1}{qBL_B} + \frac{1}{6} \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \left(\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x \right) \sin 2\hat{\Omega}t \\ & + \left(\hat{\eta}_{mu} \hat{\Pi}_x - \left(\frac{1}{qBL_B} + \frac{2}{3} \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \frac{(2\hat{P}_y^2 + \hat{\Pi}_x^2)}{2} \right) \cos \hat{\Omega}t \\ & + \left(\hat{\eta}_{mu} \hat{P}_y + \left(\frac{1}{qBL_B} + \frac{2}{3} \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \frac{(\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y)}{2} \right) \sin \hat{\Omega}t, \end{aligned} \quad (5.19a)$$

where

$$\begin{aligned} \hat{\eta}_{mu} = & 1 - \left(\frac{3}{2qBL_B} + \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \left(\hat{P}_x - \frac{mE}{B} \right) \\ & + \frac{3\hat{P}_x}{2qBL_B} - \frac{1}{qBL_B} \frac{mE}{B} - \frac{1}{2BL_E} \frac{mE}{B}. \end{aligned} \quad (5.19b)$$

Next, the time dependent operator $\hat{x}(t)$ is obtained by integrating $\hat{u}(t)$ from Eq. (5.8) with the time t ,

$$\hat{x}(t) = \hat{x} + \int_0^t \hat{u}(t) dt,$$

which leads to,

$$\begin{aligned}
m\hat{x}(t) = & m\hat{x} + \frac{1}{2} \left(\frac{1}{qBL_B} + \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) (\hat{P}_y^2 + \hat{\Pi}_x^2) t + \left(\frac{3}{2qBL_B} + \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \left(\hat{P}_x - \frac{mE}{B} \right)^2 t \\
& + \frac{mE}{B} t + \frac{2\hat{P}_x}{qBL_B} \frac{mE}{B} t - \frac{1}{2qBL_B} \left(\frac{mE}{B} \right)^2 t + \frac{1}{qBL_E} \frac{mE}{B} \left(\hat{P}_x - \frac{mE}{B} \right) t - \frac{3\hat{P}_x^2}{2qBL_B} t \\
& + \frac{\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x}{2\hat{\Omega}} \left(\frac{1}{2} \frac{1}{qBL_B} + \frac{1}{6} \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) + (\hat{P}_y^2 - \hat{\Pi}_x^2) \left(\frac{1}{2qBL_B} + \frac{1}{6} \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \frac{\sin 2\hat{\Omega}t}{2\hat{\Omega}} \\
& - \left[\frac{\hat{P}_y \hat{\eta}_{mu}}{\hat{\Omega}} + \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2\hat{\Omega}} \left(\frac{1}{qBL_B} + \frac{2}{3} \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \right] \\
& + (\hat{\Pi}_x \hat{P}_y + \hat{P}_y \hat{\Pi}_x) \left(\frac{1}{2} \frac{1}{qBL_B} + \frac{1}{6} \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \frac{\cos 2\hat{\Omega}t}{2\hat{\Omega}} \\
& + \left(\hat{\Pi}_x \hat{\eta}_{mu} - \frac{2\hat{P}_y^2 + \hat{\Pi}_x^2}{2qBL_B} - \frac{2\hat{P}_y^2 + \hat{\Pi}_x^2}{3q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \frac{\sin \hat{\Omega}t}{\hat{\Omega}} \\
& - \left(\hat{P}_y \hat{\eta}_{mu} + \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{2qBL_B} + \frac{\hat{P}_y \hat{\Pi}_x + \hat{\Pi}_x \hat{P}_y}{3q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \frac{\cos \hat{\Omega}t}{\hat{\Omega}}.
\end{aligned} \tag{5.20}$$

With the derivations above, we finally obtained all the operators for $\hat{x}(t)$, $\hat{y}(t)$, $\hat{P}_x(t)$, $\hat{P}_y(t)$, $\hat{\Pi}_x(t)$, and $\hat{\Omega}_x(t)$.

Next, we derive the uncertainty-driven expansion of variance in position and momenta. The time dependent variance of any operator $\hat{X}(t)$ is defined as [4, 5]

$$\sigma_x^2(t) \equiv \langle \hat{X}^2(t) \rangle - \langle \hat{X}(t) \rangle^2,$$

When the square of the operator includes the product of the two operators, $\hat{X}^2(t) = \hat{Y}(t)\hat{Z}(t) + \dots$, its contribution to the variance is $\sigma_x^2(t) = \langle \hat{Y}(t)\hat{Z}(t) \rangle - \langle \hat{Y}(t) \rangle \langle \hat{Z}(t) \rangle + \dots$, which is the covariance between $\hat{Y}(t)$ and $\hat{Z}(t)$. In order to evaluate the time evolution of variance in position $d\hat{\sigma}_r^2(t)/dt$, mechanical momentum $d\hat{\sigma}_{mv}^2(t)/dt$, and total momentum $d\hat{\sigma}_p^2(t)/dt$, it is necessary to make multiple combination of operators such as $\hat{x}$, $\hat{y}$, $\hat{P}_x$, $\hat{P}_y$, $\hat{\Pi}_x$, and $\hat{\Omega}_x$. Some parts of the combinations of expectation values are shown as,

$$\begin{aligned}
\langle \hat{\Pi}_x \hat{P}_x \hat{P}_y \rangle &= \langle \hat{\Pi}_x \rangle \langle \hat{P}_x \rangle \langle \hat{P}_y \rangle + \frac{i\hbar qB}{2} \left\langle \left(\frac{1}{qBL_B} + \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \hat{P}_x^2 \right. \\
&\quad \left. + \left(1 - \frac{3}{2qBL_B} + \frac{1}{2BL_E} - \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \frac{mE}{B} \hat{P}_x \right\rangle, \\
\langle \hat{P}_y \hat{\Pi}_x \hat{P}_x \rangle &= \langle \hat{\Pi}_x \rangle \langle \hat{P}_x \rangle \langle \hat{P}_y \rangle - \frac{i\hbar qB}{2} \left\langle \left(\frac{1}{qBL_B} + \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \hat{P}_x^2 \right. \\
&\quad \left. + \left(1 - \frac{3}{2qBL_B} + \frac{1}{2BL_E} - \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \frac{mE}{B} \hat{P}_x \right\rangle,
\end{aligned}$$

$$\begin{aligned}
\langle \hat{\Pi}_x \hat{P}_y^2 \rangle &= \langle \hat{P}_y^2 \rangle \langle \Pi_x \rangle + i\hbar qB \langle \hat{P}_y \rangle \left\langle \left(\frac{1}{qBL_B} + \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \hat{P}_x \right. \\
&\quad \left. + \left(1 - \frac{3}{2qBL_B} + \frac{1}{2BL_E} - \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \frac{mE}{B} \right\rangle, \\
\langle \hat{P}_y^2 \hat{\Pi}_x \rangle &= \langle \hat{P}_y^2 \rangle \langle \Pi_x \rangle - i\hbar qB \langle \hat{P}_y \rangle \left\langle \left(\frac{1}{qBL_B} + \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \hat{P}_x \right. \\
&\quad \left. + \left(1 - \frac{3}{2qBL_B} + \frac{1}{2BL_E} - \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \frac{mE}{B} \right\rangle.
\end{aligned}$$

Further different set combinations of operators are obtained to solve the expansion rates of variance.

Finally, by solving equations the equations above, the analytical expansion rate of variance in position to first order in L_B^{-1} , L_E^{-1} , and ℓ_E^{-2} is shown as,

$$\begin{aligned}
\frac{d\sigma_r^2(t)}{dt} &= \frac{d\sigma_x^2(t)}{dt} + \frac{d\sigma_y^2(t)}{dt} \\
&= \frac{3}{2} \left(\frac{\hbar v_0}{qBL_B} + \frac{\hbar v_0}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) + \frac{1}{2} \left(\frac{\hbar v_0}{qBL_B} + \frac{\hbar v_0}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \\
&= 2 \left(\frac{\hbar v_0}{qBL_B} + \frac{\hbar v_0}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right).
\end{aligned} \tag{5.21}$$

The analytical expansion rate of variance in mechanical momentum to first order in L_B^{-1} , L_E^{-1} , and ℓ_E^{-2} ,

$$\begin{aligned}
\frac{d\sigma_{mv}^2(t)}{dt} &= \frac{d\sigma_{mu}^2(t)}{dt} + \frac{d\sigma_{mv}^2(t)}{dt} \\
&= \frac{1}{2} \left(\frac{\hbar v_0 qB}{L_B} + \frac{\hbar v_0}{\ell_E^2} \frac{mE}{B} \right) + \frac{1}{2} \left(\frac{\hbar v_0 qB}{L_B} + \frac{\hbar v_0}{\ell_E^2} \frac{mE}{B} \right) \\
&= \frac{\hbar v_0 qB}{L_B} + \frac{\hbar v_0}{\ell_E^2} \frac{mE}{B}.
\end{aligned} \tag{5.22}$$

The analytical expansion rate of variance in total momentum to first order in L_B^{-1} , L_E^{-1} , and ℓ_E^{-2} ,

$$\begin{aligned}
\frac{d\sigma_P^2(t)}{dt} &= \frac{d\sigma_{P_x}^2(t)}{dt} + \frac{d\sigma_{P_y}^2(t)}{dt} \\
&= 0 + \frac{1}{2} \left(\frac{\hbar v_0 qB}{L_B} + \frac{\hbar v_0}{\ell_E^2} \frac{mE}{B} \right) \\
&= \frac{1}{2} \left(\frac{\hbar v_0 qB}{L_B} + \frac{\hbar v_0}{\ell_E^2} \frac{mE}{B} \right).
\end{aligned} \tag{5.23}$$

It is interesting to note that most of the operators derived are dependent on gradient scale lengths L_B^{-1} , L_E^{-1} and ℓ_E^{-2} . However, the expansion rates of variance are independent from the operator L_E^{-1} .

Further on, a code to solve the two-dimensional time-dependent Schrödinger for a magnetized proton in the presence of the inhomogeneous electric and magnetic field have developed. We solved the time-dependent Schrödinger Eq. (5.1) and the initial center of wavefunction Eq. (5.2) using the

finite difference method (FDM) in space with the Crank-Nicolson scheme. We adopt the successive over relaxation (SOR) scheme for time integration.

The size of spatial discretization for the FDM is small enough to satisfy $\Delta x \sim \Delta y \ll 1/k_0 = \lambda_0/2\pi$ where λ_0 is the de Broglie wavelength. This restriction on Δx and Δy demands a lot of computer memory for fast particle, thus the calculations are executed in parallel on a GPU (Nvidia GTX-980: 2048cores/4GB @1.126GHz), using CUDA [6]. The numerical error had removed from the numerical calculation by subtracting the variances in the inhomogeneous electric and magnetic field, from the homogeneous electric and magnetic field.

Figure 5.1 shows the expansion rate in position $\Delta\sigma_x^2 + \Delta\sigma_y^2 = \Delta\sigma_r^2$ with 4 different initial velocity directions. Figure 5.2 shows the expansion rate in mechanical momentum $\Delta\sigma_{mu}^2 + \Delta\sigma_{mv}^2 = \Delta\sigma_{mv}^2$ with 4 different initial velocity directions. Each of the sub figures in Fig. 5.1 will contribute for one data for Fig. 3 and each of the sub figures in Fig. 5.2 contribute in one data for Fig. 5.4. Throughout the cyclotron gyrations, the average increment is obtained.

Using multiple parameters combination, the numerical the numerical results are shown in Fig. 5.3, Fig. 5.4, and Fig. 5.5, for the expansion rates of variance in position, mechanical momentum and total momentum. We calculate the numerical expansion rates by averaging the increment of variance over a cyclotron period. Let us divide the time-dependent variance $\sigma_r^2(t)$ into its initial value $\sigma_r^2(0)$ and the increment $\Delta\sigma_r^2(t)$ as $\sigma_r^2(t) \equiv \sigma_r^2(0) + \Delta\sigma_r^2(t)$. The numerical expansion rates of variance in position, mechanical momentum, and total momentum are shown as

$$\begin{aligned} \frac{d\sigma_r^2(t)}{dt} &= \frac{d\sigma_x^2(t)}{dt} + \frac{d\sigma_y^2(t)}{dt} \\ &= 1.566 \left(\frac{\hbar v_0}{qBL_B} + \frac{\hbar v_0}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) + 0.508 \left(\frac{\hbar v_0}{qBL_B} + \frac{\hbar v_0}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \\ &= 2.074 \left(\frac{\hbar v_0}{qBL_B} + \frac{\hbar v_0}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right), \end{aligned} \quad (5.24)$$

$$\begin{aligned} \frac{d\sigma_{mv}^2(t)}{dt} &= \frac{d\sigma_{mu}^2(t)}{dt} + \frac{d\sigma_{mv}^2(t)}{dt} \\ &= 0.497 \left(\frac{\hbar v_0 qB}{L_B} + \frac{\hbar v_0}{\ell_E^2} \frac{mE}{B} \right) + 0.512 \left(\frac{\hbar v_0 qB}{L_B} + \frac{\hbar v_0}{\ell_E^2} \frac{mE}{B} \right) \\ &= 1.009 \left(\frac{\hbar v_0 qB}{L_B} + \frac{\hbar v_0}{\ell_E^2} \frac{mE}{B} \right), \end{aligned} \quad (5.25)$$

$$\begin{aligned}
\overline{\frac{d\sigma_p^2(t)}{dt}} &= \overline{\frac{d\sigma_{p_x}^2(t)}{dt}} + \overline{\frac{d\sigma_{p_y}^2(t)}{dt}} \\
&= 0.000 + 0.512 \left(\frac{\hbar v_0 q B}{L_B} + \frac{\hbar v_0}{\ell_E^2} \frac{mE}{B} \right) \\
&= 0.512 \left(\frac{\hbar v_0 q B}{L_B} + \frac{\hbar v_0}{\ell_E^2} \frac{mE}{B} \right).
\end{aligned} \tag{5.26}$$

Comparing the theoretical solution with the numerical calculation, the theoretical solutions given are generally consistent with the numerical results on the expansion rates in position, mechanical momentum and total momentum. There are slight discrepancies between the numerical factor and the analytical results. This is due to the limitation of the analytical solution is being solved to the first order in gradient scale length L_B^{-1} , L_E^{-1} , and ℓ_E^{-2} .

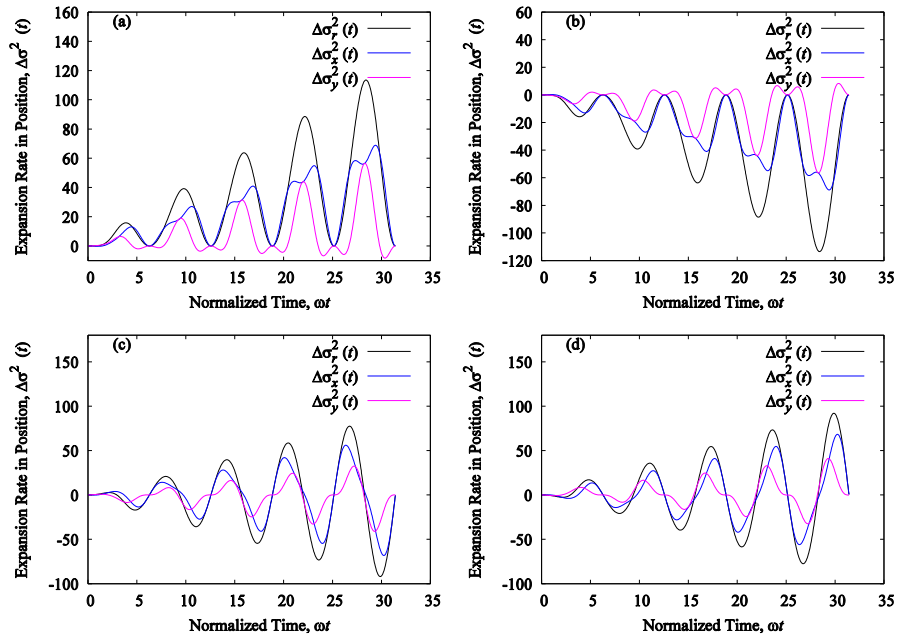


Fig. 5.1 Numerical time evolution of increment $\Delta\sigma^2(t)$ in position for 5 gyrations after normalized by $\left(\hbar|v_0|/qBL_B + \hbar mE|v_0|/q^2B^3\ell_E^2\right)$. (a) Normalized initial velocity $(u_0, v_0) = (0, 1)$. (b) Normalized initial velocity $(u_0, v_0) = (0, -1)$ (c) Normalized initial velocity of $(u_0, v_0) = (1, 0)$ (d) Normalized initial velocity of $(u_0, v_0) = (-1, 0)$.

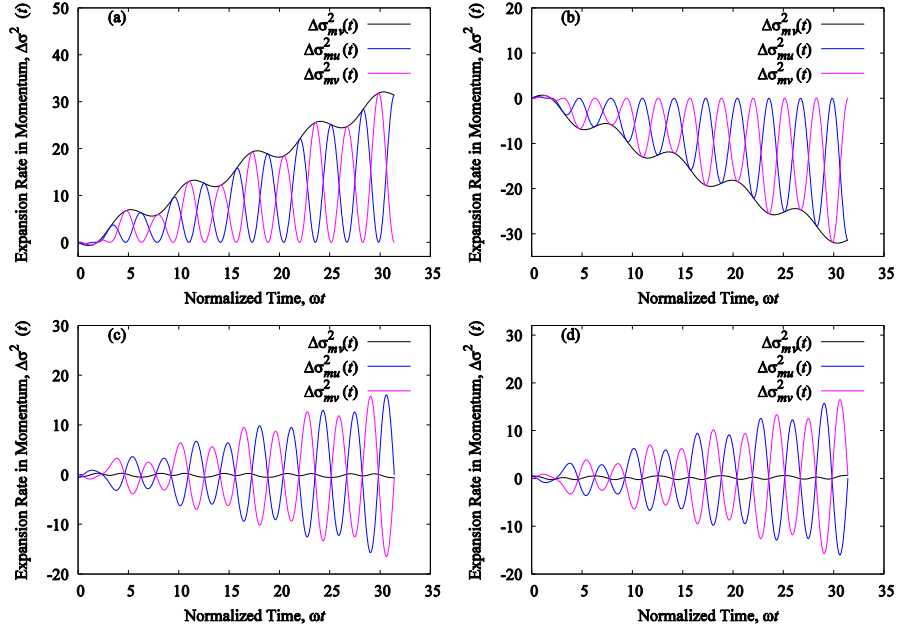


Fig. 5.2 Numerical time evolution of increment $\Delta\sigma^2(t)$ in mechanical momentum for 5 gyrations after normalized by $(\hbar q B |v_0|/L_B + \hbar m E |v_0|/B \ell_E^2)$. (a) Normalized initial velocity $(u_0, v_0) = (0, 1)$. (b) Normalized initial velocity $(u_0, v_0) = (0, -1)$ (c) Normalized initial velocity of $(u_0, v_0) = (1, 0)$ (d) Normalized initial velocity of $(u_0, v_0) = (-1, 0)$.

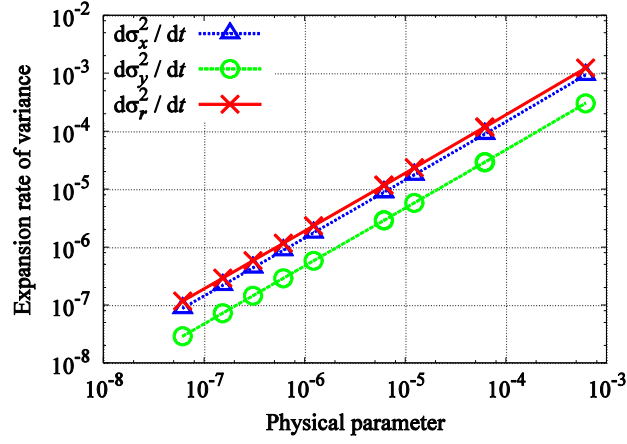


Fig 5.3 Expansion rate of variance in position against physical parameters $(\hbar v_0/qBL_B + \hbar v_0mE/q^2B^3\ell_E^2)$ with different sets of parameters of charge q , magnetic field B , electric field E , mass m , initial velocity v_0 , and gradient scale length

L_B , L_E , and ℓ_E .

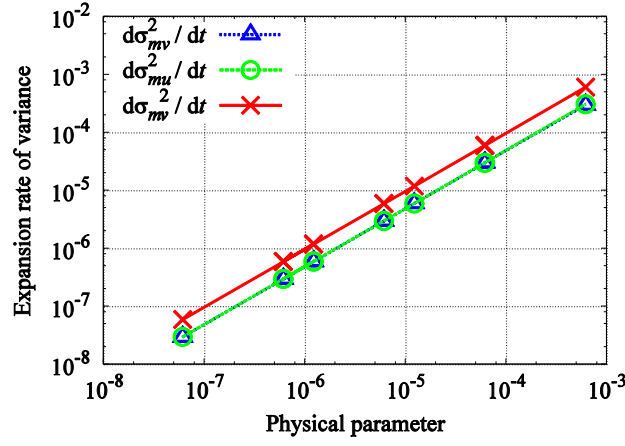


Fig. 5.4 Expansion rate of variance in mechanical momentum against physical parameters $\left(\hbar q B v_0 / L_B + \hbar m E v_0 / B \ell_E^2\right)$ with different sets of parameters of charge q , magnetic field B , electric field E , mass m , initial velocity v_0 , and gradient scale length L_B , L_E , and ℓ_E .

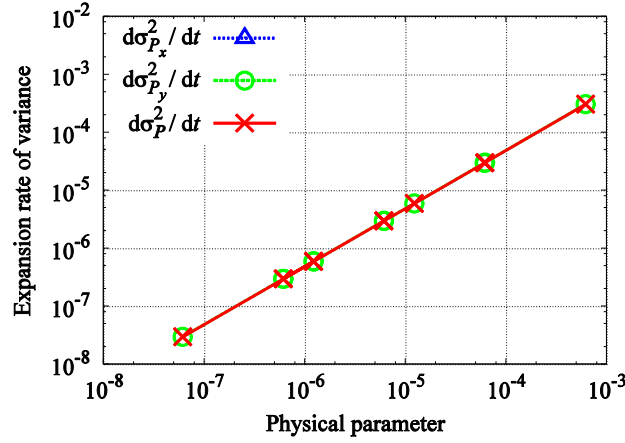


Fig. 5.5 Expansion rate of variance in total momentum against physical parameters $\left(\hbar q B v_0 / L_B + \hbar m E v_0 / B \ell_E^2\right)$ with different sets of parameters of charge q , magnetic field B , electric field E , mass m , initial velocity v_0 , and gradient length L_B , L_E , and ℓ_E .

5.3 Expansion times

The characteristic time τ_r for the variance in position $\Delta\sigma_r^2$ to reach the square of interparticle separation of $n^{-2/3}$ is estimated using Eq. (5.21), as

$$\tau_r \equiv \frac{1}{2} \frac{n^{-2/3}}{\hbar v_0 / q B L_B + \hbar v_0 m E / q^2 B^3 L_E^2}. \quad (5.27)$$

The variance in momentum $\sigma_{mv}^2(t)$ reaches the square of initial mechanical momentum $m v_0$ at τ_{mv} as

$$\tau_{mv} \equiv \frac{(m v_0)^2}{\hbar q B v_0 / L_B + m E \hbar v_0 / B \ell_E^2}. \quad (5.28)$$

This finding can be applied many field, i.e. fusion plasmas. For a typical torus plasma with a temperature T , the gradient scale length of the inhomogeneity of the field L_B can be replaced by the major radius R_0 of the torus, and the initial velocity v_0 can be replaced by the thermal speed $v_{th} = \sqrt{2T/m}$, in Eq. (5.28), we have

$$\tau_r = \frac{1}{2\hbar\sqrt{2k_B T/m}} \frac{n^{-2/3}}{\left(1/qBR_0 + mE/q^2 B^3 L_E^2\right)}, \quad (5.29)$$

which is in proportion to $\sqrt{m/TR_0}$, thus the isotope effect of $\tau \propto \sqrt{m}$ appears.

Let us apply the characteristic of the typical fusion plasma with a temperature of $T=10\text{keV}$, number density $n=10^{20}\text{m}^{-3}$, a magnetic field $B=3\text{T}$, radius $L_B \sim R_0 \sim 3\text{m}$, $\mathbf{E} \times \mathbf{B}$ drift velocity $E/B=10^4\text{ms}^{-1}$, and a gradient scale length of electric field $\ell_E \sim 1\text{m}$. This leads to the characteristic time τ_r for typical fusion plasma to be $\tau_r = 0.25\text{ms}$, which is much shorter than the proton-proton collision time of $\tau_{pp} \sim 20\text{ms}$. However, the term with ℓ_E has insignificant effect on the characteristic time τ_r , for a typical fusion plasma.

Since the ions and the electrons in fusion plasma are considered to be at the same temperature, and the velocities of electrons are much faster than protons. So, the time for electrons to reach interparticle separation is much faster than for ions. The electrons should be treated as a uniform background for the ions after the electrons expansion time. The laser-plasma interaction produced plasmas have multi-MeV protons and electrons. The expansion time in position given in Eq. (5.26) implies that the probability density function (PDF) of such energetic charged particles expand fast in the plane perpendicular to the magnetic field. The broad distribution of individual particle in space means that their Coulomb interactions with other particles of distance less than $\sigma_r(t)$ becomes weaker than that expected in classical mechanics. This could make effective electronic charge smaller, $e_{\text{eff}} < e$, and thus make larger Debye shielding length than the classical one given by $\lambda_D = \sqrt{\epsilon_0 k_B T / ne^2}$. Such a Debye screening modification due to quantum mechanical effect was pointed out based on a quantum hydrodynamic (QHD) model [7], in which the Fermi pressure and Bohm potential were included in the fluid equation.

5.4 Quantum mechanical $\mathbf{E} \times \mathbf{B}$ drift velocity

Finally, using the time dependent mechanical momentum operators along x-axis, $m\hat{u}(t)$ in Eqs. (5.19), the drift velocity is obtained as,

$$\begin{aligned}
u_{\text{drift}} = & \left(\frac{mv_0^2}{2} + \frac{3\hbar qB}{4m} \right) \left(\frac{1}{qBL_B} + \frac{1}{q^2 B^2 \ell_E^2} \frac{mE}{B} \right) \\
& + \left[1 - \frac{2mu_0 - qBy_0 - 1.5(mE/B)}{qBL_B} + \frac{mu_0 - qBy_0 - (mE/B)}{qBL_E} + \frac{\hbar qB}{2q^2 B^2 \ell_E^2} \right. \\
& \left. + \frac{(mu_0)^2 + (qBy_0)^2 + 1.5(mE/B)^2 - 2(mu_0)(qBy_0) - 3mu_0(mE/B) + 2(mE/B)qBy_0}{q^2 B^2 \ell_E^2} \right] \frac{E}{B},
\end{aligned} \tag{5.27}$$

where the grad- B drift velocity and the $\mathbf{E} \times \mathbf{B}$ drift velocity [1] are shown as,

$$u_{\nabla B} = \frac{mv_0^2}{2qBL_B} + \frac{3\hbar}{4mL_B}, \tag{5.28}$$

$$\begin{aligned}
u_{\mathbf{E} \times \mathbf{B}} = & \left[1 - \frac{2mu_0 - qBy_0 - 1.5(mE/B)}{qBL_B} + \frac{mu_0 - qBy_0 - (mE/B)}{qBL_E} \right. \\
& \left. + \frac{\left\{ (mv_0)^2 + 2(mu_0)^2 + 2(qBy_0)^2 + 3(mE/B)^2 \right.}{2q^2 B^2 \ell_E^2} \right. \\
& \left. \left. - 4(mu_0)(qBy_0) - 6mu_0(mE/B) + 4(mE/B)qBy_0 \right\}}{2q^2 B^2 \ell_E^2} \right] \frac{E}{B} \\
& + \frac{5}{4} \frac{\hbar}{qB\ell_E^2} \frac{E}{B}.
\end{aligned} \tag{5.29}$$

Due to the inhomogeneity of magnetic field, the drift velocity $u_{\nabla B}$ is found in Eq. (5.28). The first term of the drift velocity is the classical grad- B drift velocity, which this coincides with the classical drift velocity, and the second term $3\hbar / 4mL_B$ represents the quantum mechanical grad- B drift velocity due to the uncertainty. This drift velocity is the same as shown in Chap. 3 however, in the absence of the inhomogeneous electric field.

In the presence of the both inhomogeneous electric and magnetic field, $u_{\mathbf{E} \times \mathbf{B}}$ is found in Eq. (5.29). The first term of the drift velocity is the classical $\mathbf{E} \times \mathbf{B}$ drift velocity due to the homogeneous electric field, which this coincides with the classical drift velocity and can be easily derived from the equation of motion. The second term $5\hbar E / 4qB^2 \ell_E^2$ represents the quantum mechanical $\mathbf{E} \times \mathbf{B}$ drift velocity due to the uncertainty.

The quantum mechanical $\mathbf{E} \times \mathbf{B}$ drift velocity formulations clearly show the drift velocity dependence on the electric field E , magnetic field B , charge q , and gradient scale length ℓ_E^{-2} . However, the drift velocity is independent of gradient scale length L_E^{-1} .

5.5. Summary

To summarize, the time evolution of position and momentum operators for a charged particle in the presence of a weakly inhomogeneous electric and magnetic field are obtained by solving the Heisenberg equation of motion using a perturbation method. Using the time dependent operators, it

is shown how the variances or uncertainty in position and momenta grow with time. The uncertainty-driven expansion rates of variance in a weakly inhomogeneous electric and magnetic field are obtained. The analytical results are highly reliable with evidence of the numerical results on the expansion rates in position and momenta. The result clearly shows that, with the effect of the inhomogeneous electric, the expansions rates in position and momenta would be higher.

In the presence of a weakly inhomogeneous electric and magnetic field, the $\mathbf{E} \times \mathbf{B}$ drift velocity operator is obtained. It is also shown that the drift velocity operator agrees with the classical counterpart. The quantum mechanical part of the drift velocity formula, $u_{\text{drift}}^{\text{QM}} = 3\hbar / 4mL_B + 5\hbar E / 4qB^2 \ell_E^2$, clearly shows the dependence on mass m , charge q , magnetic field B , electric field E , and gradient scale lengths both L_B and ℓ_E . However, the drift velocity $u_{\text{drift}}^{\text{QM}}$ is independent of the gradient scale length L_E . The result implies that, with the effect of an inhomogeneous electric and magnetic field, low energy light particles would drift faster than the classical drift theory predicts.

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Chapter 6

Conclusion

It is analytically shown that the variance in position reaches the square of the interparticle separation, which is the characteristic time much shorter than the proton collision time of a plasma fusion. After this time the wavefunctions of the neighboring particles would overlap, thus the conventional classical analysis may lose its validity. The expansion time in position implies that the probability density function of such energetic charged particles expands fast in the plane perpendicular to the magnetic field and their Coulomb interaction with other particles becomes weaker than that expected in the classical mechanics.

Previously, quantum mechanical approach in plasma studies are mainly concentrated in the low density and temperature cases. Therefore, here we would like to highlight that the quantum-mechanical analyses are necessary even for fast charged particles as long as their long-time behavior is concerned in the presence of a electromagnetic field.

Next, we are advancing on the investigation in studying the quantum effects in the presence of a sinusoidal electric field; which the sinusoidal electric field happen in practice, such as a charge distribution can arise in a plasma during a wave motion.

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