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Structure theorems for vector valued Siegel modular forms of degree 2 and weight $\det^k \otimes \text{Sym}(10)$

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We prove the explicit structure theorems of modules $\bigoplus_k M_{\det^k \otimes \text{Sym}(10)}(\text{Sp}_2(\mathbb{Z}_2))$ of vector valued Siegel modular forms of degree 2, where k runs over the set of even integers or odd integers. We also check the conjecture given by Ibukiyama [7] for modules of vector valued Siegel modular forms of degree 2 of weights $\det^* \otimes \text{Sym}(8)$ and $\det^* \otimes \text{Sym}(10)$.

Keywords: structure theorems of modular forms, vector valued Siegel modular forms, determinant of Siegel modular forms

Mathematics Subject Classification 2010: 11F46, 11F11

1. Introduction

For $k, j \in \mathbb{Z}_{\geq 0}$, we denote by $M_{k,j}(\Gamma^{(2)})$ the space of vector valued Siegel modular forms of weight $\det^k \otimes \text{Sym}(j)$. Here $\text{Sym}(j)$ is the symmetric tensor representation of degree j of $\text{GL}_2(\mathbb{C})$. Let A_{ev} be the ring of scalar valued Siegel modular forms of degree 2, level 1 and even weights:

$$A_{\text{ev}} = \bigoplus_{k \in 2\mathbb{Z}} M_k(\Gamma^{(2)}),$$

where $M_k(\Gamma^{(2)}) = M_{k,0}(\Gamma^{(2)})$. For $j \in \mathbb{Z}_{\geq 0}$ and $\epsilon = 0, 1$, we define the graded A_{ev} module of vector valued Siegel modular forms as

$$M_{\text{Sym}(j)}^{\epsilon}(\Gamma^{(2)}) = \bigoplus_{k \equiv \epsilon \pmod{2}} M_{k,j}(\Gamma^{(2)}).$$

For a graded module $M = \sum_{k \in \mathbb{Z}} M_k$ and an integer a define $M(a)$ by $M(a)_k = M_{a+k}$. Roughly speaking, our first main theorem is stated as follows.

Theorem 1.1. *As a graded A_{ev} module, $M_{\text{Sym}(10)}^0(\Gamma^{(2)})$ has the following free resolution.*

$$\begin{aligned} 0 \rightarrow A_{\text{ev}}(-24) \oplus A_{\text{ev}}(-26) \rightarrow \\ A_{\text{ev}}(-6) \oplus A_{\text{ev}}(-8) \oplus A_{\text{ev}}(-10)^2 \oplus A_{\text{ev}}(-12)^2 \oplus A_{\text{ev}}(-14)^3 \\ \oplus A_{\text{ev}}(-16)^2 \oplus A_{\text{ev}}(-18) \oplus A_{\text{ev}}(-20) \rightarrow M_{\text{Sym}(10)}^0(\Gamma^{(2)}) \rightarrow 0. \end{aligned} \quad (1.1)$$

As a graded A_{ev} module, $M_{\text{Sym}(10)}^1(\Gamma^{(2)})$ has the following free resolution.

$$\begin{aligned} 0 \rightarrow A_{\text{ev}}(-27) \oplus A_{\text{ev}}(-29) \rightarrow \\ A_{\text{ev}}(-9) \oplus A_{\text{ev}}(-11) \oplus A_{\text{ev}}(-13) \oplus A_{\text{ev}}(-15)^3 \oplus A_{\text{ev}}(-17)^3 \\ \oplus A_{\text{ev}}(-19)^2 \oplus A_{\text{ev}}(-21) \oplus A_{\text{ev}}(-23) \rightarrow M_{\text{Sym}(10)}^1(\Gamma^{(2)}) \rightarrow 0. \end{aligned} \quad (1.2)$$

Moreover maps in the free resolutions can be given explicitly. In other words, $M_{\text{Sym}(10)}^0(\Gamma^{(2)})$ is generated by 13 modular forms of determinant weights 6, 8, 10, 10, 12, 12, 14, 14, 14, 16, 16, 18, 20 and they satisfy two fundamental relations. $M_{\text{Sym}(10)}^1(\Gamma^{(2)})$ is generated by 13 modular forms of determinant weights 9, 11, 13, 15, 15, 15, 17, 17, 17, 19, 19, 21, 23 and they satisfy two fundamental relations.

In §6, we give relations and generators explicitly.

By Igusa [10], there exists a cusp form $\chi_{35} \in M_{35}(\Gamma^{(2)})$ of weight 35 and we have

$$M_{\text{Sym}(0)}^1(\Gamma^{(2)}) = A_{\text{ev}}\chi_{35}.$$

And by Igusa [9], A_{ev} is isomorphic to the polynomial ring of 4 variables over $\mathbb{C}$ and is generated by $\phi_4, \phi_6, \chi_{10}$ and χ_{12} . Here ϕ_4 (resp. ϕ_6) is the Siegel Eisenstein series of weight 4 (resp. 6). And χ_{10} (resp. χ_{12}) is the cusp form of weight 10 (resp. 12). We put $K_{\text{ev}} = \text{Frac}(A_{\text{ev}})$. Let $j \in 2\mathbb{Z}_{\geq 0}$ and $\epsilon = 0$ or 1. By [20], we have

$$\dim_{K_{\text{ev}}} M_{\text{Sym}(j)}^\epsilon(\Gamma^{(2)}) \otimes_{A_{\text{ev}}} K_{\text{ev}} = j + 1. \quad (1.3)$$

We put

$$\det(M_{\text{Sym}(j)}^\epsilon(\Gamma^{(2)})) = \wedge^{j+1} M_{\text{Sym}(j)}^\epsilon(\Gamma^{(2)}),$$

then $\det(M_{\text{Sym}(j)}^\epsilon(\Gamma^{(2)})) \neq 0$. We fix an isomorphism $\wedge^{j+1} \text{Sym}(j) \cong \det^{j(j+1)/2}$. Then for $f_i \in M_{k_i, j}(\Gamma^{(2)})$ ($1 \leq i \leq j+1$), we have $f_1 \wedge \cdots \wedge f_{j+1} \in M_k(\Gamma^{(2)})$, where $k = \sum_i k_i + j(j+1)/2$. Since j is even, we have

$$\det(M_{\text{Sym}(j)}^\epsilon(\Gamma^{(2)})) \subset \begin{cases} A_{\text{ev}} & \text{if } j/2 + \epsilon \text{ is even,} \\ A_{\text{ev}}\chi_{35} & \text{if } j/2 + \epsilon \text{ is odd.} \end{cases}$$

Thus there exists a non-zero ideal $I_{j, \epsilon}$ of A_{ev} such that

$$\det(M_{\text{Sym}(j)}^\epsilon(\Gamma^{(2)})) = \begin{cases} I_{j, \epsilon} & \text{if } j/2 + \epsilon \text{ is even,} \\ I_{j, \epsilon}\chi_{35} & \text{if } j/2 + \epsilon \text{ is odd.} \end{cases}$$

Definition 1.1. For $j \in 2\mathbb{Z}_{\geq 0}$ and $\epsilon = 0, 1$, we denote by $\text{gcd}(I_{j, \epsilon})$ the greatest common divisor of generators of the ideal $I_{j, \epsilon}$ above. We define $f_{j, \epsilon} \in A_{\text{ev}} \oplus A_{\text{ev}}\chi_{35}$ by

$$f_{j, \epsilon} = \begin{cases} \text{gcd}(I_{j, \epsilon}) & \text{if } j/2 + \epsilon \text{ is even,} \\ \text{gcd}(I_{j, \epsilon})\chi_{35} & \text{if } j/2 + \epsilon \text{ is odd,} \end{cases}$$

The element $f_{j,\epsilon}$ is defined up to a non-zero constant.

Remark 1.1.

- (1) In §4, we construct $f_{j,\epsilon}$ by the alternating product of minors of matrices appearing in a free resolution of $M_{\text{Sym}(j)}^\epsilon(\Gamma^{(2)})$.
- (2) If $M_{\text{Sym}(j)}^\epsilon(\Gamma^{(2)})$ is a free A_{ev} module and $\{f_1, \dots, f_{j+1}\}$ is its basis, we can take $f_{j,\epsilon}$ as $f_1 \wedge \dots \wedge f_{j+1}$.
- (3) By structure theorems of $M_{\text{Sym}(2)}^\epsilon(\Gamma^{(2)})$ (cf. [17], [7]) and the result on a Wronskian of scalar valued Siegel modular forms (cf. [1]), we have

$$\begin{aligned}\det\left(M_{\text{Sym}(2)}^0(\Gamma^{(2)})\right) &= (\phi_4, \phi_6, \chi_{10}, \chi_{12})^2 \chi_{35}, \\ \det\left(M_{\text{Sym}(2)}^1(\Gamma^{(2)})\right) &= (\phi_4, \phi_6, \chi_{10}, \chi_{12}) \chi_{35}^2.\end{aligned}$$

Thus we have $f_{2,0} = \chi_{35}$ and $f_{2,1} = \chi_{35}^2$ up to non-zero constants.

In §4, we shall prove

Proposition 1.1. *For $j \in 2\mathbb{Z}_{\geq 0}$ and $\epsilon = 0, 1$, we have*

$$f_{j,\epsilon} \in M_{35(j/2+\epsilon)}(\Gamma^{(2)}).$$

The second main result of this paper is the following theorem.

Theorem 1.2. *Let j be an even number and assume $0 \leq j \leq 10$. For $\epsilon = 0, 1$, we have*

$$f_{j,\epsilon} = \chi_{35}^{j/2+\epsilon},$$

up to a non-zero constant.

Remark 1.2. The statement of the theorem was conjectured by Ibukiyama [7] if $j = 4, 6, 8$.

We recall preceding results on the structure theorems of the graded A_{ev} module $M_{\text{Sym}(j)}^\epsilon(\Gamma^{(2)})$. First, Igusa [9], [10] gave generators of $A_{\text{ev}} = M_{\text{Sym}(0)}^0(\Gamma^{(2)})$ and proved that $M_{\text{Sym}(0)}^1(\Gamma^{(2)})$ is a free A_{ev} module of rank 1. T. Satoh [17] proved the structure theorem of $M_{\text{Sym}(2)}^0(\Gamma^{(2)})$. The module $M_{\text{Sym}(2)}^0(\Gamma^{(2)})$ is generated by six modular forms, they satisfy four relations and the four relations satisfy one relation. Ibukiyama [7] proved structure theorems of $M_{\text{Sym}(2)}^1(\Gamma^{(2)})$, $M_{\text{Sym}(4)}^0(\Gamma^{(2)})$, $M_{\text{Sym}(4)}^1(\Gamma^{(2)})$ and $M_{\text{Sym}(6)}^0(\Gamma^{(2)})$. The module $M_{\text{Sym}(2)}^1(\Gamma^{(2)})$ is generated by four modular forms and they satisfy one relation. $M_{\text{Sym}(j)}^\epsilon(\Gamma^{(2)})$ is free of rank $j+1$ for $(j, \epsilon) = (4, 0), (4, 1), (6, 0)$. Van Dorp [22] proved $M_{\text{Sym}(6)}^1(\Gamma^{(2)})$ is free of rank 7. Kiyuna [13] proved $M_{\text{Sym}(8)}^\epsilon(\Gamma^{(2)})$ is free of rank 9 for $\epsilon = 0, 1$. And they all gave generators and relations explicitly.

By Theorem 1.1 and §6, we can compute a basis of $M_{k,10}(\Gamma^{(2)})$ explicitly. This is one of motivations of this paper. In §8, we give some examples of Hecke eigenforms. A program for computing $M_{k,10}(\Gamma^{(2)})$ can be found at [19].

2. Definition and notation

In this section, we define Siegel modular forms of degree two and introduce notation used throughout in this paper. We define Siegel upper half space $\mathbb{H}_2$ of degree two by

$$\mathbb{H}_2 = \{x + iy \mid x, y \in \text{Sym}_2(\mathbb{R}), y \text{ is positive definite}\}.$$

Here $\text{Sym}_2(\mathbb{R})$ is the space of symmetric matrices with entries in $\mathbb{R}$. For a commutative ring R , we define the symplectic group of degree 2 by

$$\text{Sp}_2(R) = \{g \in \text{GL}_4(R) \mid {}^t g w_2 g = w_2\},$$

where $w_2 = \begin{pmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{pmatrix}$. Then $\text{Sp}_2(\mathbb{R})$ acts on $\mathbb{H}_2$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = (az + b)(cz + d)^{-1}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}_2(\mathbb{R}), \quad z \in \mathbb{H}_2.$$

Here $a, b, c, d \in M_2(\mathbb{R})$. We put $\Gamma^{(2)} = \text{Sp}_2(\mathbb{Z})$.

Let $\rho : \text{GL}_2(\mathbb{C}) \rightarrow \text{Aut}_{\mathbb{C}}(V)$ be an irreducible polynomial representation of $\text{GL}_2(\mathbb{C})$ and $\chi : \text{Sp}_2(\mathbb{Z}) \rightarrow \mathbb{C}^\times$ a character. A V -valued holomorphic function $f : \mathbb{H}_2 \rightarrow V$ is said to be a (holomorphic) Siegel modular form of degree two, weight ρ and character χ if

$$f(g \cdot z) = \rho(cz + d) \chi(g) f(z), \quad \text{for all } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}_2(\mathbb{Z}).$$

We denote the space of Siegel modular forms of degree two, weight ρ and character χ by $M_\rho(\Gamma^{(2)}, \chi)$. If χ is the trivial character, we simply denote $M_\rho(\Gamma^{(2)}, \chi)$ by $M_\rho(\Gamma^{(2)})$.

For a non-negative integer j , we denote by V_j the space of homogeneous polynomials of u_1 and u_2 of degree j , where u_1 and u_2 are variables. We define the symmetric representation $\text{Sym}(j)$ of degree j by

$$(\text{Sym}(j)(\alpha)P)(u_1 \ u_2) = P((u_1 \ u_2)\alpha),$$

for $P \in V_j$ and $\alpha \in \text{GL}_2(\mathbb{C})$. We denote by $\det$ the determinant representation of $\text{GL}_2(\mathbb{C})$. When $\rho = \det^k \otimes \text{Sym}(j)$, we put $M_{k,j}(\Gamma^{(2)}) = M_\rho(\Gamma^{(2)})$. For $F \in M_{k,j}(\Gamma^{(2)})$, we define the Siegel operator by

$$\Phi(F)(z) = \lim_{t \rightarrow +\infty} F \left(\begin{pmatrix} z & 0 \\ 0 & it \end{pmatrix} \right).$$

Then by Arakawa [2], we have

$$\Phi(F) \in S_{k+j}(\text{SL}_2(\mathbb{Z}))u_1^j,$$

if $j > 0$. Here $S_{k+j}(\text{SL}_2(\mathbb{Z}))$ is the space of elliptic cusp forms of weight $k + j$, level 1. For $F \in M_{k,j}(\Gamma^{(2)})$, we call F a cusp form if $\Phi(F) = 0$ and denote by $S_{k,j}(\Gamma^{(2)})$ the space of cusp forms.

Let $F \in M_{k,j}(\Gamma^{(2)})$. Then by K ocherer principle, we have the following Fourier expansion:

$$F \left(\begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} \right) = \sum_{n,r,m} a((n,r,m); F) \exp(2\pi(nz_{11} + rz_{12} + mz_{22})).$$

Here $a((n,r,m); F) \in V_j$ and the summation index runs over the following set:

$$\{(n,r,m) \mid n,r,m \in \mathbb{Z}, n, m, 4nm - r^2 \geq 0\}.$$

Let q_{11}, q_{12}, q_{22} be variables and $\mathbb{C}[q_{12}, q_{12}^{-1}][q_{11}, q_{22}]$ a ring of formal power series. We embed $M_{k,j}(\Gamma^{(2)})$ to $\mathbb{C}[q_{12}, q_{12}^{-1}][q_{11}, q_{22}] \otimes_{\mathbb{C}} V_j$ by

$$F \mapsto \sum_{i=0}^j \left(\sum_{n,r,m} a((n,r,m); F)_i q_{11}^n q_{12}^r q_{22}^m \right) u_1^{j-i} u_2^i,$$

where for $v \in V_j$, we denote by v_i the coefficient of $u_1^{j-i} u_2^i$ of v .

3. Generators of the ring of scalar valued modular forms

Later, we construct vector valued Siegel modular forms by scalar valued Siegel modular forms. We recall well-known results on the generator of the ring of scalar valued modular forms of degree two.

For an even integer $k \geq 4$, we denote by ϕ_k the Siegel Eisenstein series of degree two and weight k . We normalize ϕ_k so that $a((0,0,0); \phi_k) = 1$. We put

$$\begin{aligned} \chi_{10} &= \frac{2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 53}{43867} (\phi_4 \phi_6 - \phi_{10}), \\ \chi_{12} &= \frac{2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 337}{131 \cdot 593} (3^2 \cdot 7^2 \phi_4^3 + 2 \cdot 5^3 \phi_6^2 - 691 \phi_{12}). \end{aligned}$$

Then we have $\chi_k \in S_k(\Gamma^{(2)})$ and $a((1,1,1); \chi_k) = 1$ for $k = 10, 12$. We can also construct χ_{10} and χ_{12} by the Saito-Kurokawa lift.

Theorem 3.1 (Igusa [9], [10]). *The modular forms ϕ_4 , ϕ_6 , χ_{10} and χ_{12} are algebraically independent over $\mathbb{C}$ and we have*

$$A_{\text{ev}} = \mathbb{C}[\phi_4, \phi_6, \chi_{10}, \chi_{12}], \quad M_{\text{Sym}(0)}^1(\Gamma^{(2)}) = \chi_{35} A_{\text{ev}},$$

where $\chi_{35} \in S_{35}(\Gamma^{(2)})$ is a cusp form of weight 35.

Let sgn be the unique nontrivial character of $\Gamma^{(2)}$ (see [14]). Then sgn is quadratic. There exists a square root $\chi_5 \in S_5(\Gamma^{(2)}, \text{sgn})$ of χ_{10} . The cusp form χ_5 has the following Fourier expansion:

$$\chi_5(Z) = \sum_{\substack{n, m, 4nm-r^2 \geq 0 \\ n, r, m \in 1/2 + \mathbb{Z}}} a((n,r,m); \chi_5) \mathbf{e}(nz_{11} + rz_{12} + mz_{22}).$$

We normalize χ_5 so that $a((1/2, 1/2, 1/2); \chi_5) = 1$.

Fourier coefficients of Siegel-Eisenstein series ϕ_k are explicitly known and the computation is not difficult. We can compute Fourier coefficients of χ_{35} by the Wronskian given in [1]. Since χ_5 is the Saito-Kurokawa lift of a Jacobi theta series (see [15], [6]), we can easily compute Fourier coefficients of χ_5 .

4. The determinant of the module of vector valued Siegel modular forms

In this section, we give another interpretation of $f_{j,\epsilon}$ in Definition 1.1 and prove Proposition 1.1.

4.1. The determinant of a based exact sequence

We recall the definition and basic property of the determinant of a based exact sequence over a field. We follow appendix A of [5].

Let K be a field and W a vector space of dimension $n < \infty$ over K . We put $\det(W) = \wedge^n W$. For a one dimensional vector space V , we put $V^{-1} = \text{Hom}_K(V, K)$. For a non-zero element $v \in V$, we denote by v^{-1} the element of V^{-1} such that $v^{-1}(v) = 1$.

For $i \in \mathbb{Z}$, let W_i be a finite dimensional vector space over a field K and $d_i : W_i \rightarrow W_{i+1}$ a linear map. We put $d = \{d_i \mid i \in \mathbb{Z}\}$ and $W_\bullet = \bigoplus_{i \in \mathbb{Z}} W_i$. We assume $(W_\bullet, d)$ forms a finite exact sequence, that is there exists $n \in \mathbb{Z}_{\geq 0}$ such that $W_i = 0$ for $|i| \geq n$ and $\ker(d_i) = \text{Im}(d_{i-1})$ for all i . We define the determinant of $W_\bullet$ by

$$\det(W_\bullet) = \bigotimes_i \det(W_i)^{(-1)^i}.$$

Then by [5], there is a natural isomorphism called the Euler isomorphism

$$\text{Eu}_d : \det(W_\bullet) \rightarrow K.$$

For a three term exact sequence

$$0 \rightarrow W_0 \xrightarrow{d_0} W_1 \xrightarrow{d_1} W_2 \rightarrow 0,$$

the Euler isomorphism $\det(W_0) \otimes \det(W_2) \otimes (\det(W_1))^{-1} \rightarrow K$ is given as follows. Suppose $a_1 \wedge \cdots \wedge a_n \in \det(W_0)$, $c_1 \wedge \cdots \wedge c_m \in \det(W_2)$ and $f \in (\det(W_1))^{-1}$. For $c_j \in W_2$, we take $\tilde{c}_j \in W_1$ such that $d_1(\tilde{c}_j) = c_j$. Then we define $\text{Eu}_d(a_1 \wedge \cdots \wedge a_n \wedge c_1 \wedge \cdots \wedge c_m \wedge f)$ by

$$f(d_0(a_1) \wedge \cdots \wedge d_0(a_n) \wedge \tilde{c}_1 \wedge \cdots \wedge \tilde{c}_m).$$

It is easy to see that this is well-defined. The definition for general exact sequences is reduced to this case.

Let W be a vector space and $e = \{e_\alpha \mid 1 \leq \alpha \leq \dim_K(W)\}$ a basis of W . Denote by $\det(e)$ by the wedge product $e_1 \wedge \cdots \wedge e_{\dim_K(W)}$. Let $(W_\bullet, d)$ be a finite exact

sequence and suppose that $e = \{e(i)\}$ is a system of bases in all W_i so that each $e(i)$ is a basis in W_i . We define $\det(e) \in \det(W_\bullet)$ by

$$\det(e) = \bigotimes_i \det(e(i))^{(-1)^i}.$$

Here we put $\det(e(i)) = 1$ if $W_i = \{0\}$. We call $(W_\bullet, d, e)$ a based exact sequence.

Definition 4.1. For a based exact sequence $(W_\bullet, d, e)$, we define its determinant by

$$\det(W_\bullet, d, e) = \text{Eu}_d(\det(e)) \in K^\times.$$

Next we explain how to compute determinants of based exact sequences. Let $(W_\bullet, d, e)$ be a based exact sequence. For $i \in \mathbb{Z}$, let $B_i = \{1 \leq \alpha \leq \dim_K W_i\}$ be an ordered set of indices of basis. Assume $d_i \neq 0$. We denote by D_i the matrix representation of d_i with respect to $e(i)$ and $e(i+1)$. We put $D_i = \left(m_{\alpha, \beta}^{(i)}\right)_{\alpha \in B_{i+1}, \beta \in B_i}$. For subsets $X \subset B_{i+1}$ and $Y \subset B_i$, we define $(D_i)_{X,Y}$ by the sub-matrix $\left(m_{\alpha, \beta}^{(i)}\right)_{\alpha \in X, \beta \in Y}$.

Definition 4.2. A collection of subsets $I_i \subset B_i$ ($i \in \mathbb{Z}$) is called admissible if

$$|I_i| = \sum_{j \geq 0} (-1)^j \dim_K W_{i-1-j}$$

and $(D_i)_{I_{i+1}, B_i \setminus I_i}$ is invertible if $d_i \neq 0$.

Remark 4.1. It can be easily proved that an admissible collection exists (see [5]).

We can compute the determinant of a based exact sequence by the alternating product of minors. For the proof of the following theorem, see [5].

Theorem 4.1. Let (I_i) be an admissible collection for a based exact sequence $(W_\bullet, d, e)$. We put $\Delta_i = \det\left((D_i)_{I_{i+1}, B_i \setminus I_i}\right)$. Then we have

$$\det(W_\bullet, d, e) = \prod_{i \in \mathbb{Z}} \Delta_i^{(-1)^{i-1}}.$$

Here we understand $\Delta_i = 1$ if $d_i = 0$.

Example 4.1. Let $n \in \mathbb{Z}_{\geq 1}$. We define a three term based exact sequence as follows.

$$0 \rightarrow W_{-2} \xrightarrow{d_{-2}} W_{-1} \xrightarrow{d_{-1}} W_0 \rightarrow 0 \quad (4.1)$$

where $W_{-2} = K$, $W_{-1} = K^{n+1}$, $W_0 = K^n$. We take a basis $e(i)$ of W_i by the standard basis of W_i . Let ${}^t(a_1, a_2, \dots, a_{n+1})$ be the matrix representation of d_{-2} with respect to the standard bases, $e_1, e_2, \dots, e_{n+1}$ the standard basis of W_{-1} and

$f_1, f_2, \dots, f_n$ the standard basis of W_0 . Let $B_i = \{1 \leq \alpha \leq \dim_K W_i\}$ be the ordered set of indices of basis. We assume $a_{n+1} \neq 0$. We define a subset $I_i \subset B_i$ by

$$I_i = \begin{cases} B_i & \text{if } i \geq 0, \\ \{n+1\} & \text{if } i = -1, \\ \emptyset & \text{if } i \leq -2. \end{cases}$$

Then I_i is an admissible collection. Let Δ_i be the determinant defined in Theorem 4.1. Then we have

$$\begin{aligned} \Delta_{-1} f_1 \wedge f_2 \wedge \dots \wedge f_n &= d_{-1}(e_1) \wedge d_{-1}(e_2) \wedge \dots \wedge d_{-1}(e_n), \\ \Delta_{-2} &= a_{n+1}. \end{aligned}$$

Thus by Theorem 4.1,

$$\det(W_\bullet, d, e) f_1 \wedge f_2 \wedge \dots \wedge f_n = a_{n+1}^{-1} d_{-1}(e_1) \wedge d_{-1}(e_2) \wedge \dots \wedge d_{-1}(e_n),$$

where $\det(W_\bullet, d, e)$ is the determinant of (4.1).

4.2. The determinant of a free resolution of $M_{\text{Sym}(j)}^\epsilon(\Gamma^{(2)})$

Next, we define the determinant of a given free resolution of $M_{\text{Sym}(j)}^\epsilon(\Gamma^{(2)})$ for $\epsilon = 0, 1$. And we shall prove that this determinant satisfies the property given in Definition 1.1.

Definition 4.3. Let $\epsilon = 0, 1$ and

$$0 \rightarrow F_{-r} \xrightarrow{\psi_{-r}} F_{-r+1} \xrightarrow{\psi_{-r+1}} \dots \xrightarrow{\psi_{-2}} F_{-1} \xrightarrow{\psi_{-1}} M \rightarrow 0 \quad (\text{Rsl})$$

a free resolution of $M = M_{\text{Sym}(j)}^\epsilon(\Gamma^{(2)})$ as a graded A_{ev} module. Here

$$F_i = \bigoplus_{1 \leq \nu \leq n_i} A_{\text{ev}}(-a_\nu^{(i)}), \quad \text{for } -r \leq i \leq -1,$$

be a graded free module, where $a_\nu^{(i)} \in \mathbb{Z}$. We take a basis $e(0) = \{f_1, \dots, f_{j+1}\}$ of M . Define a based exact sequence $(W_\bullet, d, e)$ associated with the free resolution Rsl and $e(0)$ as follows. We put $W_0 = M \otimes_{A_{\text{ev}}} K_{\text{ev}}$ and $W_i = F_i \otimes_{A_{\text{ev}}} K_{\text{ev}}$ for $-r \leq i \leq -1$. For $i > 0$ or $i < -r$, we put $W_i = 0$. We take a basis $e(i)$ of W_i as the basis obtained by the standard basis of F_i . We define $d_i : W_i \rightarrow W_{i+1}$ by $d_i = \psi_i \otimes_{A_{\text{ev}}} \text{id}_{K_{\text{ev}}}$. Then we define the determinant of the free resolution Rsl by

$$\det(\text{Rsl}) = \det(W_\bullet, d, e) f_1 \wedge \dots \wedge f_{j+1}.$$

By [5], $\det(\text{Rsl})$ does not depend on the choice of the basis $f_1, \dots, f_{j+1}$.

Proposition 4.1. For $j \in 2\mathbb{Z}_{\geq 0}$ and $\epsilon = 0, 1$, let $\det(\text{Rsl})$ be the determinant given in Definition 4.3 and $f_{j,\epsilon}$ an element of $A_{\text{ev}} \oplus A_{\text{ev}}\chi_{35}$ given in Definition 1.1. Then we have

$$\det(\text{Rsl}) = f_{j,\epsilon},$$

up to a non-zero constant.

Proof. By definition of $f_{j,\epsilon}$, it is enough to prove

$$A_P \det(\text{Rsl}) = \det(M) \otimes_A A_P,$$

for any height one prime P of A . Here $M = M_{\text{Sym}(j)}^\epsilon(\Gamma^{(2)})$ and $A = A_{\text{ev}}$. In the proof, we use the same notation in Definition 4.3. Let P be a height one prime of A . Since A_P is a discrete valuation ring, we can take a basis $\{f_1, \dots, f_{j+1}\}$ of M so that $\det(M) \otimes_A A_P = A_P f_1 \wedge \dots \wedge f_{j+1}$. Put $\kappa = A_P / P A_P$. Since M and F_i ($-r \leq i \leq -1$) are torsion free A modules, the following sequence $\text{Rsl} \otimes_A \kappa$ is exact.

$$0 \rightarrow F_{-r} \otimes_A \kappa \rightarrow F_{-r+1} \otimes_A \kappa \cdots \rightarrow F_{-1} \otimes_A \kappa \rightarrow M \otimes_A \kappa \rightarrow 0$$

We define a based exact sequence $(V_\bullet, \bar{d}, \bar{e})$ over κ as follows. We put $V_0 = M \otimes_A \kappa$ and $V_i = F_i \otimes_A \kappa$ for $-r \leq i \leq -1$. Otherwise we put $V_i = 0$. Define $\bar{d}_i : V_i \rightarrow V_{i+1}$ by $\bar{d}_i = \psi_i \otimes_A \text{id}_\kappa$. Let $\bar{e}(0) = \{f_1 \otimes 1_\kappa, \dots, f_{j+1} \otimes 1_\kappa\}$ and $\bar{e}(i)$ a basis obtained by the standard basis of F_i for $-r \leq i \leq -1$. Denote by B_i the ordered index set of $\bar{e}(i)$. We take an admissible collection $\{I_i\}$ for a based exact sequence $(V_\bullet, \bar{d}, \bar{e})$. Then $\{I_i\}$ is an admissible collection for $(W_\bullet, d, e)$ in Definition 4.3. Then by Theorem 4.1, we have $\det(W_\bullet, d, e) \in A_P^\times$. Therefore we have $A_P \det(\text{Rsl}) = A_P f_1 \wedge \dots \wedge f_{j+1} = \det(M) \otimes_A A_P$. $\square$

Since Rsl is a free resolution as a graded module, there exist $f \in M_k(\Gamma^{(2)})$ and $g \in M_l(\Gamma^{(2)})$ for some k and l such that $\det(\text{Rsl})$ is equal to f/g . By Proposition 4.1, we can take $g = 1$. We can compute the weight of f .

Proposition 4.2. *Let $\det(\text{Rsl})$ be the determinant given in Definition 4.3. Then the weight of $\det(\text{Rsl})$ is equal to $35(j/2 + \epsilon)$.*

Proof. Let $(W_\bullet, d, e)$ be a based exact sequence given in Definition 4.3, D_i the matrix representation of d_i with respect to $e(i)$ and $e(i+1)$ and B_i the ordered index set of $e(i)$. We take an admissible collection $\{I_i\}$ for $(W_\bullet, d, e)$. For $f \in M_k(\Gamma^{(2)})$, we denote by $\text{wt}(f)$ the weight of f . For $-r \leq i \leq -2$ and $\alpha \in B_{i+1}$, $\beta \in B_i$, the weight of (α, β) -entry of D_i is equal to $a_\beta^{(i)} - a_\alpha^{(i+1)}$. Therefore we have

$$\text{wt}\left((\Delta_i)_{I_{i+1}, B_i \setminus I_i}\right) = \sum_{\alpha \in B_i \setminus I_i} a_\alpha^{(i)} - \sum_{\alpha \in I_{i+1}} a_\alpha^{(i+1)},$$

for $-r \leq i \leq -2$. By Theorem 4.1, we have

$$\text{wt}(\det((W_\bullet, d, e))) = \sum_{-r \leq i \leq -2} (-1)^{i-1} \sum_{\alpha \in B_i} a_\alpha^{(i)} + \sum_{\alpha \in I_{-1}} a_\alpha^{(-1)}.$$

By

$$\text{wt}(f_1 \wedge \dots \wedge f_{j+1}) = \sum_{\alpha \in B_{-1} \setminus I_{-1}} a_\alpha^{(-1)} + \frac{1}{2}j(j+1),$$

we have

$$\text{wt}(\det(\text{Rsl})) = \sum_{-r \leq i \leq -1} (-1)^{i-1} \sum_{\alpha \in B_i} a_\alpha^{(i)} + \frac{1}{2}j(j+1).$$

Therefore we obtain the assertion of our Proposition by [20]. $\square$

Corollary 4.1. *Proposition 1.1 is true.*

Proof. This follows from Proposition 4.1 and Proposition 4.2. $\square$

Next we compute $f_{j,\epsilon}$ given in Definition 1.1 for $j \leq 8$.

Proposition 4.3. *For $j = 0, 2, 4, 6, 8$ and $\epsilon = 0, 1$, let $f_{j,\epsilon}$ be a Siegel modular form given in Definition 1.1. Then we have*

$$f_{j,\epsilon} = \chi_{35}^{j/2+\epsilon},$$

up to a non-zero constant.

Proof. If $j = 0$, the assertion follows from the structure theorem proved by Igusa [9], [10]. If $j = 4, 6$, the assertion follows from [20], since $M_{\text{Sym}(j)}^\epsilon(\Gamma^{(2)})$ is free in this case. We can check the assertion in the case when $j = 8$ by a similar computation to [20]. Source code, for checking this, can be found at https://github.com/stakemori/det_vec_vald_SMFs. Suppose $j = 2$. For simplicity, we assume $\epsilon = 1$. By Ibukiyama [7], $M = M_{\text{Sym}(2)}^1(\Gamma^{(2)})$ has the following free resolution.

$$0 \rightarrow A(-33) \rightarrow A(-21) \oplus A(-23) \oplus A(-27) \oplus A(-29) \rightarrow M \rightarrow 0 \quad (\text{Rsl-1})$$

Here $A = A_{\text{ev}}$. In the matrix form, the second map is given by

$${}^t(-12\chi_{12}, 10\chi_{10}, -6\phi_6, 4\phi_4).$$

The third map sends the standard basis to elements defined in [7] of determinant weights 21, 23, 27, 29 respectively. We write them as g_{21} , g_{23} , g_{27} and g_{29} . We use the same notation in Definition 4.3 and put $f_1 = g_{21}$, $f_2 = g_{23}$, $f_3 = g_{27}$. We define $I_i \subset B_i$ by

$$I_i = \begin{cases} B_i & \text{if } i \geq 0, \\ \{4\} & \text{if } i = -1, \\ \emptyset & \text{if } i \leq -2. \end{cases}$$

Then $\{I_i\}$ is an admissible collection. By Theorem 4.1, we have

$$\det((W_\bullet, d, e)) = (4\phi_4)^{-1}.$$

By the proof of [7], we have

$$g_{21} \wedge g_{23} \wedge g_{27} = \phi_4 \chi_{35}^2,$$

up to a non-zero constant. By definition, the determinant of the free resolution is equal to $4^{-1}\chi_{35}^2$. By Proposition 4.1, we have our assertion. $\square$

5. Rankin-Cohen-Ibukiyama type differential operators

We use Rankin-Cohen-Ibukiyama type differential operators to construct generators of $M_{\text{Sym}(10)}^0(\Gamma^{(2)})$ and $M_{\text{Sym}(10)}^1(\Gamma^{(2)})$. We recall differential operators defined by Eholzer-Ibukiyama [3] and van Dorp [22]. We use the same notation used in [11].

Let k, l be positive integers and χ, ψ characters of $\Gamma^{(2)}$. We take scalar valued Siegel modular forms $f \in M_k(\Gamma^{(2)}, \chi)$ and $g \in M_l(\Gamma^{(2)}, \psi)$. For $j \in \mathbb{Z}_{\geq 0}$, Eholzer-Ibukiyama [3] constructed vector valued Siegel modular forms

$$\begin{aligned}\{f, g\}_{\text{Sym}(j)} &\in M_{k+l, j}(\Gamma^{(2)}, \chi\psi), \\ \{f, g\}_{\det^2 \text{Sym}(j)} &\in M_{k+l+2, j}(\Gamma^{(2)}, \chi\psi),\end{aligned}$$

by Rankin-Cohen-Ibukiyama type differential operators.

Next we review the differential operator defined by van Dorp [22]. As before, let k, l be positive integers and χ, ψ characters of $\Gamma^{(2)}$. Let $F \in M_{k, j}(\Gamma^{(2)}, \chi)$ and $g \in M_l(\Gamma^{(2)}, \psi)$ be a vector valued Siegel modular form and a scalar valued Siegel modular form respectively. Then van Dorp [22] constructed

$$\{F, g\}_{\det \text{Sym}(j)} \in M_{k+l+1, j}(\Gamma^{(2)}, \chi\psi),$$

by a differential operator. Though he proved the proposition only when both χ and ψ are trivial characters, the same proof works for this case.

Finally, we define differential operators on three scalar valued Siegel modular forms. We use these differential operators in order to construct vector valued Siegel modular forms of odd determinant weights from scalar valued Siegel modular forms. For $i = 1, 2, 3$, let k_i be a positive integer and χ_i a character of $\Gamma^{(2)}$. For $f_i \in M_{k_i}(\Gamma^{(2)}, \chi_i)$ ($i = 1, 2, 3$), we define differential operators as follows:

$$\begin{aligned}\{f_1, f_2, f_3\}_{\det \text{Sym}(j)} &= \left\{ \{f_1, f_2\}_{\text{Sym}(j)}, f_3 \right\}_{\det \text{Sym}(j)} \\ &\in M_{k_1+k_2+k_3+1, j}(\Gamma^{(2)}, \chi_1\chi_2\chi_3), \\ \{f_1, f_2, f_3\}_{\det^3 \text{Sym}(j)} &= \left\{ \{f_1, f_2\}_{\det^2 \text{Sym}(j)}, f_3 \right\}_{\det \text{Sym}(j)} \\ &\in M_{k_1+k_2+k_3+3, j}(\Gamma^{(2)}, \chi_1\chi_2\chi_3),\end{aligned}$$

6. Precise statement of the structure theorems

In this section, we give generators and relations of $M_{\text{Sym}(10)}^0(\Gamma^{(2)})$ and $M_{\text{Sym}(10)}^1(\Gamma^{(2)})$ explicitly.

6.1. Generators of $M_{\text{Sym}(10)}^0(\Gamma^{(2)})$

In this subsection, we define thirteen generators $M_{\text{Sym}(10)}^0(\Gamma^{(2)})$. We define ten of them as follows.

$$\begin{aligned}
F_{10} &= \{\varphi_4, \varphi_6\}_{\text{Sym}(10)}, & F_{12} &= \{\varphi_4, \varphi_4^2\}_{\text{Sym}(10)}, \\
G_{12} &= \{\varphi_4, \varphi_6\}_{\det^2 \text{Sym}(10)}, & F_{14} &= \{\varphi_4, \varphi_4 \varphi_6\}_{\text{Sym}(10)}, \\
G_{14} &= \{\varphi_4, \chi_{10}\}_{\text{Sym}(10)}, & H_{14} &= \{\chi_5, \varphi_4 \chi_5\}_{\text{Sym}(10)}, \\
F_{16} &= \{\varphi_4, \varphi_6^2\}_{\text{Sym}(10)}, & G_{16} &= \{\varphi_4, \chi_{12}\}_{\text{Sym}(10)}, \\
F_{18} &= \{\varphi_4, \varphi_4 \chi_{10}\}_{\text{Sym}(10)}, & F_{20} &= \{\varphi_4, \varphi_4 \chi_{12}\}_{\text{Sym}(10)}.
\end{aligned}$$

We define remaining three modular forms as follows.

$$\begin{aligned}
F_6 = \frac{1}{667705262220781424640000\chi_{12}} & \left(299617786417098240F_{18} + 1789501343175B_{18}^{(0)} \right. \\
& - 2697884306920857600B_{18}^{(1)} + 4499692750175B_{18}^{(2)} - 3530479328750B_{18}^{(3)} \\
& + 1270685788932672000B_{18}^{(4)} + 310661133307499520B_{18}^{(5)} + 140785854300B_{18}^{(6)} \\
& + 142952698392B_{18}^{(7)} - 153516996000B_{18}^{(8)} + 14151466506240000B_{18}^{(9)} \\
& \left. - 553525841756160000B_{18}^{(10)} \right).
\end{aligned}$$

$$\begin{aligned}
F_8 = \frac{1}{5007789466655860684800000\chi_{10}} & \left(-8434553345905639680F_{18} - 66731865311475B_{18}^{(0)} \right. \\
& + 16180689061303603200B_{18}^{(1)} - 75624543557850B_{18}^{(2)} + 5670763346875B_{18}^{(3)} \\
& - 14212628597653920000B_{18}^{(4)} + 4104176746346127360B_{18}^{(5)} \\
& - 27222227916975B_{18}^{(6)} - 2824331333544B_{18}^{(7)} + 305313705000B_{18}^{(8)} \\
& \left. - 1559002738648320000B_{18}^{(9)} - 3590159128657920000B_{18}^{(10)} \right).
\end{aligned}$$

$$\begin{aligned}
G_{10} = \frac{1}{13800125293914960000\varphi_4^2} & \left(-15916405775169019760640F_{18} - 132162848896253175B_{18}^{(0)} \right. \\
& - 36964508172629003366400B_{18}^{(1)} - 61881543692326050B_{18}^{(2)} \\
& + 4575977962943750B_{18}^{(3)} - 20639189761094963520000B_{18}^{(4)} \\
& + 516230561016664865280B_{18}^{(5)} - 23611666345271550B_{18}^{(6)} \\
& - 2865001595829912B_{18}^{(7)} + 215990508690000B_{18}^{(8)} \\
& - 2792613466587732480000B_{18}^{(9)} - 9147741644136775680000B_{18}^{(10)} \\
& \left. - 39157855521483699000000B_{18}^{(11)} \right).
\end{aligned}$$

Here $B_{18}^{(0)}, \dots, B_{18}^{(11)}$ are given as follows.

$$\begin{aligned}
B_{18}^{(0)} &= \{\varphi_4, \varphi_4^2 \varphi_6\}_{\text{Sym}(10)}, & B_{18}^{(1)} &= \{\chi_5, \varphi_4^2 \chi_5\}_{\text{Sym}(10)}, \\
B_{18}^{(2)} &= \{\varphi_6, \varphi_4^3\}_{\text{Sym}(10)}, & B_{18}^{(3)} &= \{\varphi_6, \varphi_6^2\}_{\text{Sym}(10)}, \\
B_{18}^{(4)} &= \{\varphi_6, \chi_{12}\}_{\text{Sym}(10)}, & B_{18}^{(5)} &= \{\varphi_4^2, \chi_{10}\}_{\text{Sym}(10)}, \\
B_{18}^{(6)} &= \{\varphi_4^2, \varphi_4 \varphi_6\}_{\text{Sym}(10)}, & B_{18}^{(7)} &= \{\varphi_4, \varphi_4^3\}_{\det^2 \text{Sym}(10)}, \\
B_{18}^{(8)} &= \{\varphi_4, \varphi_6^2\}_{\det^2 \text{Sym}(10)}, & B_{18}^{(9)} &= \{\varphi_4, \chi_{12}\}_{\det^2 \text{Sym}(10)}, \\
B_{18}^{(10)} &= \{\chi_5, \varphi_6 \chi_5\}_{\det^2 \text{Sym}(10)}, & B_{18}^{(11)} &= \varphi_4 \varphi_6 F_8.
\end{aligned}$$

We note that the set $\{F_{18}, B_{18}^{(0)}, \dots, B_{18}^{(11)}\}$ forms a basis of the space $M_{18, 10}(\Gamma^{(2)})$.

In §7.3, we shall explain F_6 , F_8 and F_{10} are holomorphic Siegel modular forms.

6.2. Generators of $M_{\text{Sym}(10)}^1(\Gamma^{(2)})$

We define seven of thirteen generators of $M_{\text{Sym}(10)}^1(\Gamma^{(2)})$ as follows.

$$\begin{aligned}
F_{17} &= \{\varphi_4, \varphi_6, \varphi_6\}_{\det \text{Sym}(10)}, & G_{17} &= \{\varphi_4, \varphi_4^2, \varphi_4\}_{\det \text{Sym}(10)}, \\
H_{17} &= \{\chi_5, \varphi_6, \chi_5\}_{\det \text{Sym}(10)}, & F_{19} &= \{\varphi_4, \chi_5, \varphi_4 \chi_5\}_{\det \text{Sym}(10)}, \\
G_{19} &= \{\varphi_4, \varphi_4^2, \varphi_6\}_{\det \text{Sym}(10)}, & F_{21} &= \{\varphi_4, \varphi_6, \chi_{10}\}_{\det \text{Sym}(10)}, \\
F_{23} &= \{\varphi_4, \varphi_6, \chi_{12}\}_{\det \text{Sym}(10)}.
\end{aligned}$$

We define remaining six modular forms as follows.

$$\begin{aligned}
F_9 &= \frac{1}{4636508397286125012516864000\chi_{10}} \left(-6725180154573619200F_{19} - 7089754342125G_{19} \right. \\
&\quad + 22117267502616B_{19}^{(0)} + 4205094643659571200B_{19}^{(1)} - 7138425156720B_{19}^{(2)} \\
&\quad - 5140411073357414400B_{19}^{(3)} - 28473519757610188800B_{19}^{(4)} \\
&\quad \left. - 78089462464B_{19}^{(5)} \right).
\end{aligned}$$

$$\begin{aligned}
F_{11} &= \frac{1}{670791145440700956672000\varphi_4^2} \left(-29326410727288012800F_{19} - 154720948329675G_{19} \right. \\
&\quad + 296435594930904B_{19}^{(0)} + 47067973368230707200B_{19}^{(1)} \\
&\quad - 112872064939920B_{19}^{(2)} - 81197866538488627200B_{19}^{(3)} \\
&\quad - 357184189630601625600B_{19}^{(4)} - 20661646909696B_{19}^{(5)} \\
&\quad \left. - 670791145440700956672000B_{19}^{(6)} \right).
\end{aligned}$$

$$F_{13} = \frac{1}{17600933845382823936000\varphi_6} \left(-5870720280512102400F_{19} + 3571610126175G_{19} \right. \\ \left. - 17576872544088B_{19}^{(0)} - 3898171979838259200B_{19}^{(1)} + 4328987691600B_{19}^{(2)} \right. \\ \left. + 3418705164194611200B_{19}^{(3)} + 18831546114903244800B_{19}^{(4)} \right. \\ \left. - 286545870208B_{19}^{(5)} - 17600933845382823936000B_{19}^{(6)} \right).$$

$$F_{15} = \frac{1}{13165054156800\varphi_4} \left(258020594380800F_{19} - 12669930B_{19}^{(2)} + 37752728832000B_{19}^{(3)} \right. \\ \left. - 143718507878400B_{19}^{(4)} - 37349B_{19}^{(5)} \right).$$

$$G_{15} = \frac{1}{19508428800\varphi_4} B_{19}^{(0)}.$$

$$H_{15} = \frac{1}{92155379097600\varphi_4} \left(594965135278080B_{19}^{(1)} - 944735220B_{19}^{(2)} + 97485935339520B_{19}^{(3)} \right. \\ \left. - 4267924896614400B_{19}^{(4)} + 550172207B_{19}^{(5)} \right).$$

Here $B_{19}^{(0)}, \dots, B_{19}^{(6)}$ are given by

$$\begin{aligned} B_{19}^{(0)} &= \{\varphi_4, \varphi_6, \varphi_4^2\}_{\det \text{Sym}(10)}, & B_{19}^{(1)} &= \{\varphi_4, \varphi_4\chi_5, \chi_5\}_{\det \text{Sym}(10)}, \\ B_{19}^{(2)} &= \{\varphi_4, \varphi_4\varphi_6, \varphi_4\}_{\det \text{Sym}(10)}, & B_{19}^{(3)} &= \{\chi_5, \varphi_4^2, \chi_5\}_{\det \text{Sym}(10)}, \\ B_{19}^{(4)} &= \{\chi_5, \varphi_4\chi_5, \varphi_4\}_{\det \text{Sym}(10)}, & B_{19}^{(5)} &= \{\varphi_6, \varphi_4^2, \varphi_4\}_{\det \text{Sym}(10)}, \\ B_{19}^{(6)} &= \varphi_4\varphi_6F_9. \end{aligned}$$

We note that the set $\{F_{19}, G_{19}, B_{19}^{(0)}, \dots, B_{19}^{(6)}\}$ forms a basis of the space $M_{19, 10}(\Gamma^{(2)})$. In §7.3, we shall explain that $F_9, F_{11}, F_{13}, F_{15}, G_{15}$ and H_{15} are holomorphic modular forms.

6.3. Relations of $M_{\text{Sym}(10)}^0(\Gamma^{(2)})$

Lemma 6.1. *Generators of even determinant weights satisfy the following two relations.*

$$\begin{aligned} a_{18}F_6 + a_{16}F_8 + a_{14}F_{10} + b_{14}G_{10} + a_{12}F_{12} + b_{12}G_{12} + a_{10}F_{14} + b_{10}G_{14} & \quad (6.1) \\ + c_{10}H_{14} + a_8F_{16} + b_8G_{16} + a_6F_{18} + a_4F_{20} &= 0, \end{aligned}$$

and

$$\begin{aligned} a_{20}F_6 + \alpha_{18}F_8 + \alpha_{16}F_{10} + \beta_{16}G_{10} + \alpha_{14}F_{12} + \beta_{14}G_{12} + \alpha_{12}F_{14} + \beta_{12}G_{14} & \quad (6.2) \\ + \gamma_{12}H_{14} + \alpha_{10}F_{16} + \beta_{10}G_{16} + \alpha_8F_{18} + \alpha_6F_{20} &= 0. \end{aligned}$$

Here a_{18} and a_{16} are defined by

$$a_{18} = 15643660032(67331642279\varphi_4^3\varphi_6 - 6689228000\varphi_6^3 + 2370923189526528\varphi_4^2\chi_{10} + 4612035407616000\varphi_6\chi_{12}),$$

and

$$a_{16} = 5105916816000(387254979\varphi_4^4 - 20240608\varphi_4\varphi_6^2 + 6135186530304\varphi_6\chi_{10} + 4114923337728\varphi_4\chi_{12}).$$

Coefficients $a_{14}, \dots, a_4$ are defined by

$$\begin{aligned} a_{14} &= -11824384\varphi_4(-392419241\varphi_4\varphi_6 + 145847127883968\chi_{10}), \\ b_{14} &= -1986496512\varphi_4(-31914599\varphi_4\varphi_6 + 10054388155392\chi_{10}), \\ a_{12} &= 830289075(23924401\varphi_4^3 - 1654848\varphi_6^2 + 1331133253632\chi_{12}), \\ b_{12} &= 76257199200(23313\varphi_4^3 - 2080\varphi_6^2 + 1434101760\chi_{12}), \\ a_{10} &= -135273640320(5849\varphi_4\varphi_6 + 412304256\chi_{10}), \\ b_{10} &= 41009272012800(-909943\varphi_4\varphi_6 + 346335575040\chi_{10}), \\ c_{10} &= 261554147777367244800(-\varphi_4\varphi_6 + 3468906\chi_{10}), \\ a_8 &= 312420856840400\varphi_4^2, \\ b_8 &= 70552675506957557760\varphi_4^2, \\ a_6 &= -89694864607315968000\varphi_6, \\ a_4 &= 165176542976857251840\varphi_4. \end{aligned}$$

Coefficients α_{20} , α_{18} , α_{16} and β_{16} are defined by

$$\begin{aligned} \alpha_{20} &= -5214553344(425048543296\phi_4^5 + 484587670889\phi_4^2\phi_6^2 + 15696098336318400\phi_4\phi_6\chi_{10} \\ &\quad + 62924853322985472\phi_4^2\chi_{12} - 890719685310873600000\chi_{10}^2), \end{aligned}$$

$$\alpha_{18} = 5105916816000(-1811050991\phi_4^3\phi_6 - 24020864\phi_6^3 - 145283115931200\phi_4^2\chi_{10} + 87510477938688\phi_6\chi_{12}),$$

$$\alpha_{16} = 5912192(19495410909\phi_4^4 - 23419603319\phi_4\phi_6^2 - 1950560293550400\phi_6\chi_{10} + 2818879454297088\phi_4\chi_{12}),$$

and

$$\beta_{16} = 1986496512(649847848\phi_4^4 - 809420843\phi_4\phi_6^2 - 86364658900800\phi_6\chi_{10} + 87573208859136\phi_4\chi_{12}).$$

Coefficients $\alpha_{14}, \dots, \alpha_6$ are defined by

$$\begin{aligned} \alpha_{14} &= -4151445375\phi_4(22269553\phi_4\phi_6 + 1622175242688\chi_{10}), \\ \beta_{14} &= -381285996000\phi_4(21233\phi_4\phi_6 + 1927074240\chi_{10}), \\ \alpha_{12} &= -135273640320(14911\phi_4^3 - 44156\phi_6^2 + 3872074752\chi_{12}), \\ \beta_{12} &= -205046360064000(2855021\phi_4^3 - 3764964\phi_6^2 + 718771802112\chi_{12}), \\ \gamma_{12} &= -16347134236085452800(1649\phi_4^3 - 1729\phi_6^2 + 211323168\chi_{12}), \\ \alpha_{10} &= 11238160318000(-139\phi_4\phi_6 + 11793600\chi_{10}), \\ \beta_{10} &= -59317870781030400(5947\phi_4\phi_6 + 1014249600\chi_{10}), \\ \alpha_8 &= 448474323036579840000\phi_4^2, \\ \alpha_6 &= -825882714884286259200\phi_6. \end{aligned}$$

We shall prove Lemma 6.1 in §7.4.

6.4. Relations of $M_{\text{Sym}(10)}^1(\Gamma^{(2)})$

Lemma 6.2. *Generators of odd determinant weights satisfy the following two relations.*

$$\begin{aligned} d_{18}F_9 + d_{16}F_{11} + d_{14}F_{13} + d_{12}F_{15} + e_{12}G_{15} + f_{12}H_{15} + d_{10}F_{17} + e_{10}G_{17} \\ + f_{10}H_{17} + d_8F_{19} + e_8G_{19} + d_6F_{21} + d_4F_{23} = 0, \end{aligned} \quad (6.3)$$

and

$$\begin{aligned} \delta_{20}F_9 + \delta_{18}F_{11} + \delta_{16}F_{13} + \delta_{14}F_{15} + \epsilon_{14}G_{15} + \zeta_{14}H_{15} + \delta_{12}F_{17} + \epsilon_{12}G_{17} \\ + \zeta_{12}H_{17} + \delta_{10}F_{19} + \epsilon_{10}G_{19} + \delta_8F_{21} + \delta_6F_{23} = 0. \end{aligned} \quad (6.4)$$

Here d_{18} and d_{16} are defined by

$$d_{18} = -195284202946560(-199029244391\phi_4^3\phi_6 + 2697950850992640\phi_4^2\chi_{10} + 3549133607846400\phi_6\chi_{12}),$$

and

$$\begin{aligned} d_{16} = -341747355156480(6216500122\phi_4^4 - 25056024021\phi_4\phi_6^2 - 471257790566400\phi_6\chi_{10} \\ + 2428645469322240\phi_4\chi_{12}). \end{aligned}$$

Coefficients $d_{14}, \dots, d_4$ are defined by

$$\begin{aligned} d_{14} &= -74893556325778260295680\phi_4(-433\phi_4\phi_6 + 14131200\chi_{10}), \\ d_{12} &= -102195265536(-685230911\phi_4^3 + 35309376921600\chi_{12}), \\ e_{12} &= -1093591536500736(39827745779\phi_4^3 + 468263910297600\chi_{12}), \\ f_{12} &= -255488163840(193718497\phi_4^3 + 4258403020800\chi_{12}), \\ d_{10} &= 955493351615296(-49\phi_4\phi_6 + 3686400\chi_{10}), \\ e_{10} &= 387290963292597(7\phi_4\phi_6 + 115200\chi_{10}), \\ f_{10} &= 5177826116350101946368(-7\phi_4\phi_6 + 460800\chi_{10}), \\ d_8 &= -2133719553440976076800\phi_4^2, \\ e_8 &= 0, \\ d_6 &= -396262202781895557120\phi_6, \\ d_4 &= -66043700463649259520\phi_4. \end{aligned}$$

Coefficients δ_{20} , δ_{18} and δ_{16} are defined by

$$\begin{aligned} \delta_{20} &= -27897743278080(-1439954360064\phi_4^5 + 46749649327\phi_4^2\phi_6^2 + 30085945775101440\phi_4\phi_6\chi_{10} \\ &\quad + 34225097039884800\phi_4^2\chi_{12} - 7541542192496423731200\chi_{10}^2), \end{aligned}$$

$$\begin{aligned} \delta_{18} &= 341747355156480(16282786754\phi_4^3\phi_6 + 2556737145\phi_6^3 - 5088306946498560\phi_4^2\chi_{10} \\ &\quad + 895389802076160\phi_6\chi_{12}), \end{aligned}$$

and

$$\delta_{16} = -10699079475111180042240(-3036\phi_4^4 + 5\phi_4\phi_6^2 - 49459200\phi_6\chi_{10} + 130636800\phi_4\chi_{12}).$$

Coefficients $\delta_{14}, \dots, \delta_6$ are defined by

$$\begin{aligned}\delta_{14} &= -8073425977344\phi_4(-8673809\phi_4\phi_6 + 260218327680\chi_{10}), \\ \epsilon_{14} &= -1093591536500736\phi_4(39827745779\phi_4\phi_6 + 659445588817920\chi_{10}), \\ \zeta_{14} &= -255488163840\phi_4(193718497\phi_4\phi_6 + 11030709116160\chi_{10}), \\ \delta_{12} &= 955493351615296(-44\phi_4^3 - 5\phi_6^2 + 4838400\chi_{12}), \\ \epsilon_{12} &= -55327280470371(-44\phi_4^3 - 5\phi_6^2 + 967680\chi_{12}), \\ \zeta_{12} &= 739689445192871706624(-44\phi_4^3 - 5\phi_6^2 + 4193280\chi_{12}), \\ \delta_{10} &= 2133719553440976076800(-\phi_4\phi_6 + 64512\chi_{10}), \\ \epsilon_{10} &= -35692735177045739520\chi_{10}, \\ \delta_8 &= -396262202781895557120\phi_4^2, \\ \delta_6 &= -66043700463649259520\phi_6.\end{aligned}$$

We shall prove Lemma 6.2 in §7.4.

7. Proof of the main results

In this section, we prove Theorem 1.1 and Theorem 1.2. Also we prove generators given in §6 are holomorphic modular forms. The actual computation is done by a computer algebra system. We use SageMath [18] and a SageMath package for Siegel modular forms degree two [19].

7.1. Hilbert series

For $j \in \mathbb{Z}_{\geq 0}$ and $\epsilon \in \{0, 1\}$, we define the Hilbert series $h_{j,\epsilon}(t)$ of $M_{\text{Sym}(j)}(\Gamma^{(2)})$ by

$$h_{j,\epsilon}(t) = \sum_{k=0}^{\infty} \dim_{\mathbb{C}} M_{k,j}(\Gamma^{(2)}) t^k.$$

Then $h_{10,\epsilon}(t)$ is given as follows.

Lemma 7.1.

$$\begin{aligned}h_{10,0}(t) &= \frac{t^6 + t^8 + 2t^{10} + 2t^{12} + 3t^{14} + 2t^{16} + t^{18} + t^{20} - t^{24} - t^{26}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}, \\ h_{10,1}(t) &= \frac{t^9 + t^{11} + t^{13} + 3t^{15} + 3t^{17} + 2t^{19} + t^{21} + t^{23} - t^{27} - t^{29}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.\end{aligned}$$

In particular, expansions of these Hilbert series are given by

$$\begin{aligned}h_{10,0}(t) &= t^6 + t^8 + 3t^{10} + 4t^{12} + 7t^{14} + 9t^{16} + 13t^{18} \\ &\quad + 17t^{20} + 22t^{22} + 27t^{24} + 35t^{26} + 41t^{28} + O(t^{30})\end{aligned}$$

and

$$\begin{aligned}h_{10,1}(t) &= t^9 + t^{11} + 2t^{13} + 5t^{15} + 6t^{17} + 9t^{19} \\ &\quad + 13t^{21} + 16t^{23} + 21t^{25} + 28t^{27} + 33t^{29} + O(t^{31}).\end{aligned}$$

Proof. $h_{10,0}(t)$ is given in [7]. But for the sake of completeness, we give a proof. By Tsushima [21], the dimension of $S_{k,j}(\Gamma^{(2)})$ is known if $k > 4$. By Arakawa [2], we can compute $\dim_{\mathbb{C}} M_{k,j}(\Gamma^{(2)}) - \dim_{\mathbb{C}} S_{k,j}(\Gamma^{(2)})$ if $k > 4$. Therefore we have to prove $M_{k,10}(\Gamma^{(2)}) = 0$ if $k \leq 4$. The vanishing of $M_{k,10}(\Gamma^{(2)})$ for $k \leq 0$ follows from Freitag's work [4]. We prove $M_{k,10}(\Gamma^{(2)}) = S_{k,10}(\Gamma^{(2)})$ for $1 \leq k \leq 4$. If k is odd, this follows from the definition. By Ibukiyama [7], $M_{2,10}(\Gamma^{(2)}) = S_{2,10}(\Gamma^{(2)})$. Since $S_{14}(\mathrm{SL}_2(\mathbb{Z})) = 0$, we have $M_{4,10}(\Gamma^{(2)}) = S_{4,10}(\Gamma^{(2)})$. Thus we have $M_{k,10}(\Gamma^{(2)}) = S_{k,10}(\Gamma^{(2)})$ for $1 \leq k \leq 4$. Again, by Ibukiyama [7], we have $S_{k,10}(\Gamma^{(2)}) = 0$ if $0 \leq k \leq 4$. Therefore we have $M_{k,10}(\Gamma^{(2)}) = 0$ if $0 \leq k \leq 4$. $\square$

7.2. Precision

We explain to what extent we compute Fourier coefficients of the generators. For $f \in M_k(\Gamma^{(2)})$, we consider f as an element of $\mathbb{C}[q_{12}, q_{12}^{-1}][q_{11}, q_{22}]$ as in §2.

We recall Sturm type theorem for scalar valued Siegel modular forms of degree 2 for our need. We introduce some notation. For $f \in \mathbb{C}[q_{12}, q_{12}^{-1}][q_{11}, q_{22}]$, we denote coefficients of f as follows:

$$f = \sum_{(n,r,m)} a((n,r,m); f) q_{11}^n q_{12}^r q_{22}^m.$$

For $a \in \mathbb{R}_{\geq 0}$ and a subring $B \subset \mathbb{C}$, we define a ring by

$$R_a(B) = R'(B)/(q_{11}^{[a]+1}, q_{22}^{[a]+1}). \quad (7.1)$$

Here $[\cdot]$ is the Gauss symbol and $R'(B)$ is defined by

$$R'(B) = \{f \in B[q_{12}, q_{12}^{-1}][q_{11}, q_{22}] \mid a((n,r,m); f) = 0 \text{ if } 4nm - r^2 < 0\}.$$

Theorem 7.1 (Kikuta-Takemori [12]). *Let $k \in \mathbb{Z}_{\geq 0}$ and p a prime number. Suppose $f \in M_k(\Gamma^{(2)})$ has $\mathbb{Z}_{(p)}$ integral Fourier coefficients. We put*

$$\bar{f} = \text{The image of } f \text{ in } \mathbb{Z}/p\mathbb{Z}[q_{12}, q_{12}^{-1}][q_{11}, q_{22}]$$

and

$$b_k = \begin{cases} k/10 & \text{if } k \text{ is even,} \\ (k-5)/10 & \text{if } k \text{ is odd.} \end{cases}$$

If the image of $\bar{f}$ in $R_{b_k}(\mathbb{Z}/p\mathbb{Z})$ vanishes, then we have $\bar{f} = 0$.

By the theorem above and the fact that $M_k(\Gamma^{(2)})$ has a basis of modular forms with integral Fourier coefficients, we have the following lemma.

Lemma 7.2. *Let $f \in M_k(\Gamma^{(2)})$ and assume the image of f in $R_{b_k}(\mathbb{C})$ vanishes, where b_k is as in Theorem 7.1. Then we have $f = 0$.*

Remark 7.1. The statement of [20] is true, but the proof is not sufficient.

In Lemma 7.5, we shall prove the determinant of the first 11 generators (ordered by the determinant weight) of $M_{\text{Sym}(10)}^0(\Gamma^{(2)})$ (resp. $M_{\text{Sym}(10)}^1(\Gamma^{(2)})$) is equal to $\chi_{35}^5(\phi_4^3 - \phi_6^2)$ (resp. $\chi_{35}^6(\phi_4^3 - \phi_6^2)$) up to a non-zero constant. By Lemma 7.2, it is enough to compute generators of $M_{\text{Sym}(10)}^0(\Gamma^{(2)})$ (resp. $M_{\text{Sym}(10)}^1(\Gamma^{(2)})$) in $R_{18}(\mathbb{C}) \otimes_{\mathbb{C}} V_{10}$ (resp. $R_{22}(\mathbb{C}) \otimes_{\mathbb{C}} V_{10}$).

7.3. Holomorphy of generators

In §6, we constructed generators of small determinant weights by dividing modular forms. Since dividing by a cusp form is subtle, we introduce the following lemma.

Lemma 7.3. *For $a \in \mathbb{Z}_{\geq 1}$, let $R_a(\mathbb{C})$ be the ring defined by (7.1). Let*

$$\begin{aligned} f &= \sum_{(n,r,m)} a((n,r,m); f) q_{11}^n q_{12}^r q_{22}^m \bmod (q_{11}^{a+1}, q_{22}^{a+1}), \\ g &= \sum_{(n,r,m)} a((n,r,m); g) q_{11}^n q_{12}^r q_{22}^m \bmod (q_{11}^{a+1}, q_{22}^{a+1}), \end{aligned}$$

be elements of $R_a(\mathbb{C})$. Assume $a((1, -1, 1); g) = 1$ and g is cuspidal, that is $a((n, r, m); g) = 0$ if $4nm - r^2 = 0$ and $n, m \leq a$. Then there uniquely exists $h \in R_{a-1}(\mathbb{C})$ such that $\tilde{f} = \tilde{g}h$. Here $\tilde{f}$ (resp. $\tilde{g}$) is the image of f (resp. g) in $R_{a-1}(\mathbb{C})$. Moreover, we have the following recursive equation.

$$\begin{aligned} a((n, r, m); h) + a((1, 1, 1); g) a((n, r-2, m); h) \\ = a((n+1, r-1, m+1); f), \end{aligned} \quad (7.2)$$

for $(n, r, m) \in \mathbb{Z}^3$ with $0 \leq n, m \leq a-1$ and $4nm - r^2 \geq 0$.

Proof. We put $S = \{(n, r, m) \in \mathbb{Z}^3 \mid n, m, 4nm - r^2 \geq 0\}$. We define an order $\leq$ of S so that $(n, r, m) \leq (n', r', m')$ if and only if one of following conditions holds.

- (1) $n + m \leq n' + m'$.
- (2) $n + m = n' + m'$ and $n \leq n'$.
- (3) $n = n'$, $m = m'$ and $r \leq r'$.

Let $h = \sum_{(n,r,m) \geq 0} a((n, r, m); h) q_{11}^n q_{12}^r q_{22}^m + \bmod(q_{11}^a, q_{22}^a)$ be an element of $R_{a-1}(\mathbb{C})$. By the condition $\tilde{f} = \tilde{g}h$, we have

$$\sum_{\substack{T_1+T_2=(n+1, r-1, m+1) \\ T_1, T_2 \in S}} a(T_1; g) a(T_2; h) = a((n+1, r-1, m+1); f), \quad (7.3)$$

for $(n, r, m) \in S$ with $n, m \leq a-1$. Here addition of S is defined by entry-wise. By (7.3) and the cuspidality of g , we obtain (7.2). Thus we can recursively define $a((n, r, m); h)$ for $(n, r, m) \in S$ with $n, m \leq a-1$ by the order of S . Uniqueness follows from the recursive equation above. $\square$

Next, we prove holomorphy of generators.

Lemma 7.4. *Meromorphic modular forms $F_6, F_8, G_{10}, F_9, F_{11}, F_{13}, F_{15}$ and H_{15} given in §6.1 and §6.2 are holomorphic.*

Proof. For simplicity, we only prove F_9 is holomorphic. We can prove other cases in a similar way. Let N_{19} be the subspace of $M_{19,10}(\Gamma^{(2)})$ spanned by $F_{19}, G_{19}, B_{19}^{(0)}, \dots, B_{19}^{(5)}$. Then N_{19} is a subspace of $M_{19,10}(\Gamma^{(2)})$ of codimension 1. And we can calculate a basis of N_{19} explicitly from Fourier coefficients of scalar valued Siegel modular forms. Put $C_{19} = F_{19}|T(2)$, where $T(2)$ is the Hecke operator. For the explicit action of $T(2)$ on $M_{k,j}(\Gamma^{(2)})$, see [2]. Then we have $N_{19} + \mathbb{C}C_{19} = M_{19,10}(\Gamma^{(2)})$. We can check this equality by computing vector valued modular forms in $R_6(\mathbb{Q})^{11}$. By Lemma 7.1, there uniquely exists $F'_9 \in M_{9,10}(\Gamma^{(2)})$ such that $F'_9 \neq 0$ up to a non-zero constant. By a similar argument in [11] we can explicitly find rational numbers $\alpha, \beta, \gamma, \delta, c^{(i)}, d^{(i)}$ for $-1 \leq i \leq 5$ such that

$$\begin{aligned}\phi_4\phi_6F'_9 &= \alpha F_{19} + \beta C_{19} + c^{(-1)}G_{19} + \sum_{i=0}^5 c^{(i)}B_{19}^{(i)}, \\ \phi_{10}F'_9 &= \gamma F_{19} + \delta C_{19} + d^{(-1)}G_{19} + \sum_{i=0}^5 d^{(i)}B_{19}^{(i)}.\end{aligned}$$

Then we have $\beta = \delta \neq 0$ and $\alpha \neq \gamma$. Since C_{19} is computationally expensive, we remove C_{19} . Then we have

$$\chi_{10}F'_9 = \alpha' F_{19} + \beta' G_{19} + \sum_{i=0}^5 e^{(i)}B_{19}^{(i)},$$

for some rational numbers $\alpha', \beta', e^{(i)}$ ($0 \leq i \leq 5$). By explicit computation of these rational numbers, we have $F'_9 = F_9$ up to a non-zero constant. Thus F_9 is holomorphic. $\square$

7.4. Relations

In this subsection, we prove Lemma 6.1 and Lemma 6.2. For simplicity, we only prove relation (6.1).

Proof. [Proof of relation (6.1)] For $k \in \mathbb{Z}_{\geq 0}$, let X_k be the following set of monomial of $\phi_4, \phi_6, \chi_{10}$ and χ_{12} :

$$X_k = \{\phi_4^a \phi_6^b \chi_{10}^c \chi_{12}^d \mid a, b, c, d \in \mathbb{Z}_{\geq 0}, 4a + 6b + 10c + 12d = k\}.$$

We define a finite set S by

$$\begin{aligned}X_{18}F_6 \cup X_{16}F_8 \cup X_{14}F_{10} \cup X_{14}G_{10} \\ \cup X_{12}F_{12} \cup X_{12}G_{12} \cup X_{10}F_{14} \cup X_{10}G_{14} \\ \cup X_{10}H_{14} \cup X_8F_{16} \cup X_8G_{16} \cup X_6F_{18}.\end{aligned}$$

Then we have $|S| = 27$. By computing elements of S in $R_6(\mathbb{C}) \otimes_{\mathbb{C}} V_{10}$, we see that S is linearly independent over $\mathbb{C}$. By Lemma 7.1, S is a basis of $M_{24,10}(\Gamma^{(2)})$. By numerical computation, we can check that the image of the left hand side of (6.1) in $R_6(\mathbb{C}) \otimes_{\mathbb{C}} V_{10}$ is equal to 0. Therefore the left hand side of (6.1) is equal to 0. $\square$

7.5. Determinants of generators and structure theorems

Determinants of modular forms given in §6.1 and 6.2 are given as follows.

Lemma 7.5.

$$F_6 \wedge F_8 \wedge F_{10} \wedge G_{10} \wedge F_{12} \wedge G_{12} \wedge F_{14} \wedge G_{14} \wedge H_{14} \wedge F_{16} \wedge G_{16} = (\phi_4^3 - \phi_6^2) \chi_{35}^5, \quad (7.4)$$

up to a non-zero constant and

$$F_9 \wedge F_{11} \wedge F_{13} \wedge F_{15} \wedge G_{15} \wedge H_{15} \wedge F_{17} \wedge G_{17} \wedge H_{17} \wedge F_{19} \wedge G_{19} = (\phi_4^3 - \phi_6^2) \chi_{35}^6, \quad (7.5)$$

up to a non-zero constant.

Proof. As is explained in §7.2, we have to compute both sides of (7.4) (resp. (7.5)) in $R_{18}(\mathbb{Q})$ (resp. $R_{22}(\mathbb{Q})$). Computation is done by using SageMath [18] and a package [19] for Siegel modular forms of degree 2. See https://github.com/stakemori/det_vec_vald_SMFs for a script to check this. $\square$

Next we prove Theorem 1.1.

Theorem 7.2. *Theorem 1.1 is true, that is the sequences (1.1) and (1.2) are exact. Here the second linear map of the sequence (1.1) is given by sending the standard basis to $(a_{18}, a_{16}, \dots, a_4)$ and $(\alpha_{20}, \alpha_{18}, \dots, \alpha_6)$, where $a_{18}, a_{16}, \dots, a_4; \alpha_{20}, \alpha_{18}, \dots, \alpha_6$ are give in (6.1) and (6.2). The third map of the sequence (1.1) is given by sending the standard basis to $F_6, F_8, \dots, F_{20}$. Linear maps of the sequence (1.2) is given in a similar way by using $F_9, F_{11}, \dots, H_{15}$, (6.3) and (6.4).*

Proof. For simplicity, we only prove that (1.1) is exact. We denote by ψ_{-2} the second map of (1.1) and by ψ_{-1} the third map of (1.1). Because

$$\det \begin{pmatrix} a_6 & a_4 \\ \alpha_8 & \alpha_6 \end{pmatrix} = \phi_4^3 - \phi_6^2, \quad (7.6)$$

up to a non-zero constant, ψ_{-2} is injective. By Lemma 6.1, we have $\text{Im}(\psi_{-2}) \subset \ker(\psi_{-1})$. By (1.3) and Lemma 7.5,

$$\{F_6, F_8, F_{10}, G_{10}, F_{12}, G_{12}, F_{14}, G_{14}, H_{14}, F_{16}, G_{16}\}$$

is a basis of $M_{\text{Sym}(10)}^0(\Gamma^{(2)}) \otimes_{A_{\text{ev}}} K_{\text{ev}}$. Therefore we have

$$(\ker(\psi_{-1})/\text{Im}(\psi_{-2})) \otimes_{A_{\text{ev}}} K_{\text{ev}} = \{0\}.$$

Since $\det \begin{pmatrix} a_6 & a_4 \\ \alpha_8 & \alpha_6 \end{pmatrix}$ and $\det \begin{pmatrix} b_8 & a_4 \\ \beta_{10} & \alpha_6 \end{pmatrix}$ are co-prime in A_{ev} , we have $\ker(\psi_{-1}) = \text{Im}(\psi_{-2})$. Therefore the Hilbert series of $\text{Im}(\psi_{-1})$ is equal to

$$\frac{t^6 + t^8 + 2t^{10} + 2t^{12} + 3t^{14} + 2t^{16} + t^{18} + t^{20} - t^{24} - t^{26}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})},$$

which is equal to that of $M_{\text{Sym}(10)}^0(\Gamma^{(2)})$ by Lemma 7.1. Therefore ψ_{-1} is surjective. This completes the proof. $\square$

Finally, we prove Theorem 1.2.

Proof. [Proof of Theorem 1.2] By Proposition 4.3, it is enough to prove the assertion when $j = 10$. We can prove the assertion of the theorem by a similar argument to the proof of Proposition 4.3 by using Theorem 4.1, Lemma 7.5 and (7.6). $\square$

8. Examples and a table of Fourier coefficients

Since constructions of modular forms given in §6.1 and 6.2 are complicated, the author has tested them in several ways. By the explicit structure theorems, we can construct a basis of the space $M_{k,10}(\Gamma^{(2)})$ and compute the action of Hecke operators on the space. In the case when Hecke eigenvalues are known, we test if eigenforms have correct eigenvalues. We also test the generalized Ramanujan conjecture for a non CAP cuspidal eigenform.

For tests, we introduce the following notation. Let $F \in M_{k,j}(\Gamma^{(2)})$ be a Hecke eigenform and p a prime number. We denote by $Q_p(F; X)$ the polynomial of degree 4 of X so that $\prod_{p:\text{prime}} Q_p(F; p^{-s})^{-1}$ is the spinor L -function of F and denote by $R_p(F; X)$ the polynomial of degree 5 of X so that $\prod_{p:\text{prime}} R_p(F; p^{-s})^{-1}$ is the standard L -function of F . That is, $Q_p(F; X)$ and $R_p(F; X)$ are defined as follows.

$$\begin{aligned} Q_p(F; X) &= (1 - \alpha_0 X)(1 - \alpha_0 \alpha_1 X)(1 - \alpha_0 \alpha_2 X)(1 - \alpha_0 \alpha_1 \alpha_2 X), \\ R_p(F; X) &= (1 - X)(1 - \alpha_1 X)(1 - \alpha_1^{-1} X)(1 - \alpha_2 X)(1 - \alpha_2^{-1} X). \end{aligned}$$

Here $\alpha_0, \alpha_1, \alpha_2$ are the p -Satake parameters of F .

8.1. Test for Klingen-Eisenstein series and the K - R - S lift

Since Hecke eigenvalues of Eisenstein series and lifts are known, we test them. Let $k, j \in 2\mathbb{Z}_{\geq 0}$. For a normalized eigenform $f = \sum_{n=1}^{\infty} a(n; f)q^n \in S_{k+j}(\text{SL}_2(\mathbb{Z}))$ with $k > 4$, Arakawa [2] defined the Klingen-Eisenstein series $E(f) \in M_{k,j}(\Gamma^{(2)})$. The Klingen-Eisenstein series $E(f)$ satisfies the following identities:

$$\begin{aligned} \Phi(E(f)) &= fu_1^j, \\ Q_p(E(f); X) &= Q_p^{(1)}(f; X)Q_p^{(1)}(f; p^{(k-2)}X). \end{aligned} \tag{8.1}$$

Here p is a prime number and $Q_p^{(1)}(f; X) = 1 - a(p; f)X + p^{k+j-1}X^2$.

For a normalized eigenform $f = \sum_{n=1}^{\infty} a(n; f)q^n \in S_k(\mathrm{SL}_2(\mathbb{Z}))$, Ramakrishnan and Shahidi [16] constructed a lift $\mathrm{KS}(f) \in M_{k+1, k-2}(\mathrm{SL}_2(\mathbb{Z}))$ called the Kim-Ramakrishnan-Shahidi lift (K-R-S lift in short). The K-R-S lift $\mathrm{KS}(f)$ satisfies the following identity:

$$Q_p(\mathrm{KS}(f); X) = \prod_{i=0}^3 (1 - \alpha_p^i \beta_p^{3-i} X), \quad (8.2)$$

where p is a prime number. α_p and β_p are defined by $1 - a(p; f)X + p^{k-1}X^2 = (1 - \alpha_p X)(1 - \beta_p X)$.

Let

$$\begin{aligned} \Delta &= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + O(q^6) \in S_{12}(\mathrm{SL}_2(\mathbb{Z})), \\ f_{16} &= q + 216q^2 - 3348q^3 + 13888q^4 + 52110q^5 + O(q^6) \in S_{16}(\mathrm{SL}_2(\mathbb{Z})) \end{aligned}$$

be normalized eigenforms of weight 12 and 16 respectively. Then we have

$$F_6 = E(f_{16}), \quad F_{13} = \mathrm{KS}(\Delta). \quad (8.3)$$

Fourier coefficients of F_6 and F_{13} are given in Table 1. We test (8.3). By Table 1 and the explicit action of Hecke operators given in [2], we have the following identities.

$$\begin{aligned} Q_2(F_6; X) &= (1 - 2^3 \cdot 3^3 X + 2^{15} X^2)(1 - 2^7 \cdot 3^3 X + 2^{23} X^2), \\ Q_2(F_{13}; X) &= (1 - 2^9 \cdot 3^2 \cdot 29 X + 2^{33} X^2)(1 + 2^{14} \cdot 3 X + 2^{33} X^2). \end{aligned}$$

Thus, we can check (8.1) and (8.2) for $p = 2$. We note that Fourier coefficients of the K-R-S lift $\mathrm{KS}(\Delta)$ and the non-lift eigenform of weight $\det^{13} \mathrm{Sym}(10)$ were given in [8].

8.2. Test for the generalized Ramanujan conjecture

Next, we test the generalized Ramanujan conjecture for the cusp form F_9 . By Table 1, we have

$$R_2(F_9; X) = 1 + \frac{667}{512}X + \frac{845}{2048}X^2 - \frac{845}{2048}X^3 - \frac{667}{512}X^4 - X^5.$$

Roots of $R_2(F_9; X)/(X - 1)$ in the complex field with 53 bits precision are given by $\alpha, \bar{\alpha}, \beta, \bar{\beta}$. Here α and β are given as

$$\begin{aligned} \alpha &= -0.966296691713208 - 0.257430968580139i, \\ \beta &= -0.185070495786792 - 0.982725247253387i. \end{aligned}$$

Then we have $|\alpha| = |\beta| = 1.00000000000000$. Thus we can check that the absolute values of 2-Satake parameters of F_9 are equal to 1.

8.3. Table of Fourier coefficients

For $i, j \in \mathbb{Z}_{\geq 0}$ and $v \in V_j$, we denote by v_i the coefficient $u_1^{j-i} u_2^i$. Then Fourier coefficients of F_6, F_{13} and F_9 are given as follows.

$(n, r, m), i$	$a((n, r, m); F_6)_i$	$a((n, r, m); F_{13})_i$	$a((n, r, m); F_9)_i$
(1, 0, 1), 0	-330	0	0
(1, 0, 1), 1	0	4	-4
(1, 0, 1), 2	1350	0	0
(1, 0, 1), 3	0	-16	72
(1, 0, 1), 4	-2100	0	0
(1, 0, 1), 5	0	0	0
(1, 0, 1), 6	-2100	0	0
(1, 0, 1), 7	0	16	-72
(1, 0, 1), 8	1350	0	0
(1, 0, 1), 9	0	-4	4
(1, 0, 1), 10	-330	0	0
(1, 1, 1), 0	-88	0	0
(1, 1, 1), 1	-440	-2	2
(1, 1, 1), 2	-720	-9	9
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Table 1: Fourier coefficients of some of generators

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