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Proceedings of the 41st Sapporo Symposium on Partial Differntial Equations

Edited by
S.-I. Ei, Y. Giga, S. Jimbo, H. Kubo, T. Ozawa, T. Sakajo,
H. Takaoka, Y. Tonegawa and K. Tsutaya

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Edited by
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Sapporo, 2016

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PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 8 through August 10 in 2016 at Conference Hall, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Professor Taira Shirota started the symposium more than 40 years ago. Professor Kôji Kubota and late Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

S.-I. Ei, Y. Giga, S. Jimbo, H. Kubo, T. Ozawa, T. Sakajo,
H. Takaoka, Y. Tonegawa, and K. Tsutaya

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The 41st Sapporo Symposium on Partial Differential Equations
（第４１回偏微分方程式論札幌シンポジウム）

Period（期間） August 8, 2016 - August 10, 2016

Venue（場所） The 1st Floor, Conference Hall, Hokkaido University
北海道大学 学術交流会館 1 階小講堂

URL http://www.math.sci.hokudai.ac.jp/sympo/sapporo/program160808.html

Organizers（組織委員）： Shin-Ichiro Ei（栄伸一郎）
Hideo Kubo（久保英夫）

August 8, 2016 (Monday)

09:50-10:00 Opening

10:00-11:00 James P. Keener (University of Utah)
The PDE's of cell biology

11:00-11:30 *

11:30-12:30 柴田徹太郎 (広島大学) Tetsutaro Shibata (Hiroshima University)
Asymptotic properties of bifurcation curves related to inverse bifurcation problems

12:30-14:10 お昼休み (Lunch Break)

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14:40-15:10 横田智巳（東京理科大学）Tomomi Yokota (Tokyo University of Science)
Mathematical analysis of a chemotaxis model via maximal regularity

15:10-15:40 Hung Vinh Tran (University of Wisconsin Madison)
Some inverse problems in periodic homogenization of nonlinear partial differential equations

15:40-16:10 *

16:10-16:40 可香谷隆 (東京工業大学) Takashi Kagaya (Tokyo Institute of Technology)
Exponential stability of a traveling wave for an area preserving curvature motion

16:40-17:10 Borislav Yordanov (Hokkaido University)
On low frequency asymptotics for dissipative wave equations

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10:00-11:00 Chiun-Chuan Chen (National Taiwan University)
A new maximum principle for diffusive Lotka-Volterra systems of competing species

11:00-11:30 *

11:30-12:30 Michael Ruzhansky (Imperial College London)
Very weak solutions to hyperbolic equations

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Traveling waves composed of convex closed curves in anisotropic curve shortening flow with a driving force

15:10-15:40 Chris Jeavons (Waseda University)
Remark on the stability of the Hardy-Littlewood-Sobolev inequality and related problems

15:40-16:10 *

16:10-16:40 眞崎聡（大阪大学）Satoshi Masaki (Osaka University)
On scattering problem on nonlinear Schrödinger equations in mass-subcritical case

18:00-20:00 懇親会（エンレイソウ）Reception at Faculty House Trillium (Enreiso)

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Evolution of dispersal in a mutation selection model

11:00-11:30 *

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The PDE's of Cell Biology

J. P. Keener
University of Utah

July 1, 2016

1 Introduction

Diffusion is the enemy of life. This is because diffusion causes small particles to spread out, and for aggregates of particles to dissipate. Thus, in order to be alive and maintain its structure, an organism must have ways to counteract the constant tendency of things to spread out. And indeed they do. Plants, for example, are able to harness the energy of the sun to convert carbon dioxide and water into high energy compounds such as carbohydrates. These high energy compounds are then carefully deconstructed by living things to do work moving things around and building and repairing their structures. In this way, living things are able to combat the tendency of structures to dissipate and fall apart.

However, living organisms do much more than simply counteract diffusion; they actually exploit it for specific purposes. That is, they expend energy to concentrate molecules and then use the fact that molecules move by diffusion down their concentration gradient to do useful things. How do they actually do this? The short answer is that they couple diffusion with appropriate chemical reactions that are able to exploit the inherent diffusive motion.

In this talk, I give several examples of the ways that cells use diffusion to their advantage, and describe the partial differential equations that model these processes. In particular, here I describe how molecular diffusion and reaction are used to signal, to create functional aggregates (via cell polarization), and to make length measurements. In this way, I hope to convince you that living organisms have made diffusion their friend, not their enemy.

2 About Diffusion

Most molecules move by a random walk, well described by brownian motion. Thus, the position x of a single particle is a random variable and the probability density function of this random variable $p(x,t)$ evolves according to the diffusion equation

$$\frac{\partial p}{\partial t} = D\nabla^2 p. \tag{1}$$

In this context, the diffusion equation (1) is also known as a Fokker-Planck equation or Chapman-Kolmogorov equation for this stochastic process.

If, however, there are a large number of independently diffusing particles with concentration C, then the average flux of particles is well-described by Fick's law,

$$J = -D\nabla C, \tag{2}$$

and this, when coupled with the conservation law $\frac{\partial C}{\partial t} = -\nabla \cdot J$, gives the diffusion equation

$$\frac{\partial C}{\partial t} = D\nabla^2 C, \tag{3}$$

which is, of course, the same as (1). Thus, the diffusion equation has (at least) two different interpretations.

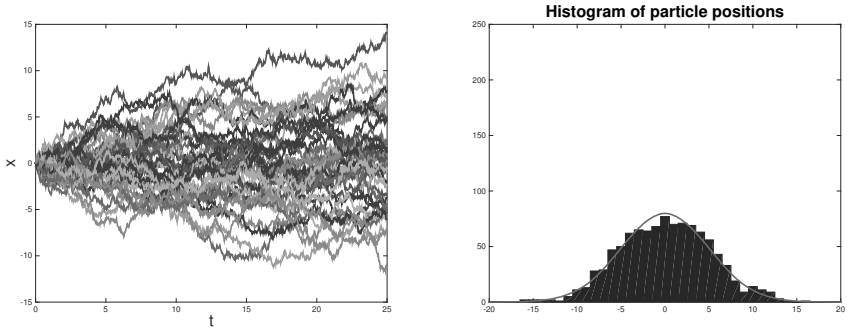


Figure 1: Left: Space-time trajectories of 40 diffusing particles. Right: Space-histogram of 1000 particle positions compared with the gaussian distribution solution of the diffusion equation (solid curve).

Solutions of the diffusion equation are well known to smooth out spatial inhomogeneities, corresponding to this well-known fact of diffusion.

For diffusion of molecules across a permeable membrane, Fick's law reduces to the statement

$$J = \frac{AD}{L}(C_1 - C_2), \tag{4}$$

where C_1 and C_2 are the concentrations on either side of the membrane (see Fig. 2), L is the distance between compartments, and A is the crossectional area of the membrane surface. In words, this says that flux J is proportional to the concentration difference across the membrane and inversely proportional to the length of the pore across the membrane. This leads to the following important observations:

- Flux is from high to low concentrations;
- Flux is decreased when the concentration difference is small;
- Flux is decreased when length is large.

Thus, the rate of flux contains quantitative information about concentrations and lengths.

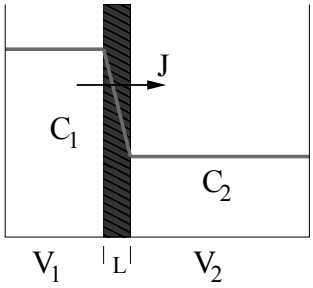


Figure 2: Diffusion across a membrane is proportional to the concentration difference and inversely proportional to the distance between compartments.

Figure 3: Alan Hodgkin 1914-1998, Andrew Huxley 1917-2012

2.1 Signalling

The study of signalling was given its monumental kickstart by Hodgkin and Huxley with their study of action potential generation and propagation in squid giant axon [2].

It is now known that the basic principles underlying axonal signalling are the same for signalling in many other contexts, including cardiac and skeletal muscle contraction, calcium signalling in oocytes, and cAMP signalling in dycteostelium discoideum.

The basic features of how nerve cells work can be understood by thinking of the cell membrane as an electric circuit with the cell membrane acting as a capacitor that can store charge, and various ion-selective channels acting like resistors for current flow. (See Fig. 4). The Hodgkin-Huxley equations invoke Kirckhoff's laws to track the accumulated charge Q across a cell membrane, namely

$$\frac{dQ}{dt} \equiv C_m \frac{dV}{dt} = -I_{\mathrm{Na}} - I_{\mathrm{K}} - I_{\mathrm{l}}, \tag{5}$$

where I_{Na} is sodium current, I_{K} is potassium current, and I_{l} chloride leak current. Cells expend energy to create concentration differences between inside and outside the cell, with high sodium outside the cell and high potassium inside the cell, with the additional consequence that there is a voltage difference between inside and outside, called the transmembrane potential.

Movement of ions is governed by both diffusion and electrotaxis via the Nernst Planck equation

$$J = -D(\nabla C + \frac{zF}{RT}\nabla\psi), \tag{6}$$

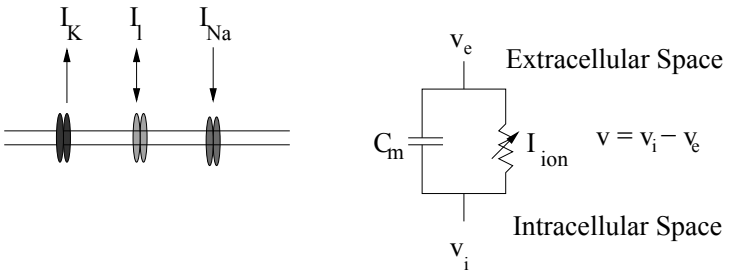


Figure 4: Circuit diagram for nerve cell membrane.

i.e., down their concentration gradient and down the electropotential gradient, thereby decreasing the gradient of concentration and decreasing the gradient of the electric potential. Consequently, sodium "wants" to move into and potassium "wants" to move out of the cell. What they actually do is determined by their gating, i.e., if they are open or closed.

Here is the "trick" that makes it all work. Sodium and potassium channels are regulated by the transmembrane potential in a time dependent fashion, with voltage and time dependent currents

$$I_{Na} = g_{Na} m^3 h (v - v_{Na}), \qquad I_K = g_K h^4 (v - v_K), \tag{7}$$

where m, n, and h are governed by differential equations of the form

$$\frac{dw}{dt} = \alpha_w(v) w + \beta_w(v)(1 - w), \qquad w = m, n, h, \tag{8}$$

and are called gating variables. Specifically, ion channels open when the potential increases, and close when the potential decreases. Since sodium current is inward (down its gradient), its current increases the potential, giving a positive feedback (see Fig. 5). On the other hand, potassium current is outward, decreasing the potential and thereby giving a negative feedback.

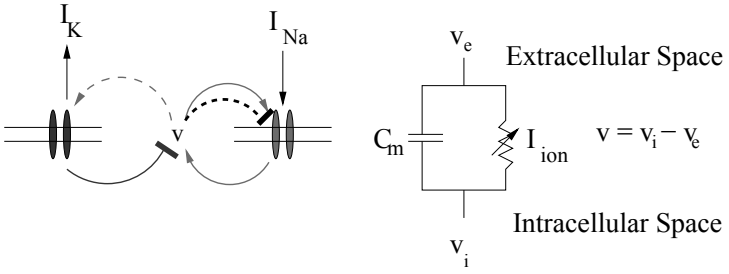


Figure 5: Transmembrane currents affect the potential v which in turn causes ion channels to open or close, leading to the above feedback diagram.

This combination of flow down concentration gradients coupled with positive and negative feedbacks gives rise to the action potential, a transient increase and decrease of transmembrane potential, depicted in Fig. 6.

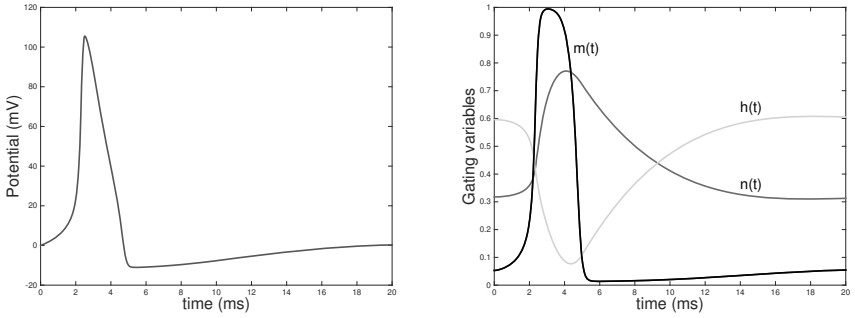


Figure 6: Left: Action potential and Right: corresponding ion channel gating variable dynamics for the Hodgkin-Huxley equations.

Nerve axons are spatially extended structures, and the dynamics of the transmembrane potential for a one dimensional axon is given by the partial differential equation system

$$C_m \frac{\partial v}{\partial t} + I_{ion}(v, w) = \frac{\partial}{\partial x}\left(\frac{1}{r_c}\frac{\partial v}{\partial x}\right) \quad \text{where } \frac{dw}{dt} = g(v, w), \qquad w \in R^3. \tag{9}$$

This equation is referred to as the cable equation, and is a diffusion-reaction equation.

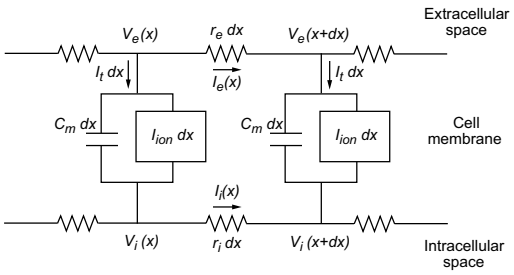


Figure 7: Discretization of a nerve axon into small membrane patches of length dx.

Hodgkin and Huxley calculated that their equations had propagating pulse solutions (travelling waves), a breakthrough discovery. Since their initial discovery, there have been thousands of papers written describing the wave behavior of these reaction-diffusion systems.

For higher dimensional media, such as cardiac tissue, one finds the bidomain equations[5]

$$C_m \frac{\partial v}{\partial t} + I_{ion}(v, w) = \nabla \cdot (\sigma_i \nabla \phi_i) \quad \text{where } \frac{dw}{dt} = g(v, w), \qquad w \in R^n, \tag{10}$$

$$\nabla \cdot (\sigma_i \nabla \phi_i + \sigma_e \nabla \phi_e) = 0, \tag{11}$$

where $v = \phi_i - \phi_e$, ϕ_i and ϕ_e are the intracellular and extracellular potentials, respectively.

In addition to having traveling wave solutions, these equations are known to have self-sustained "reentrant" wave pattern solutions, otherwise known as spiral (in 2-d) and scroll (in 3-d) waves. An illustration of such a reentrant wave in a very rapidly growing forest is shown in Fig. 8.

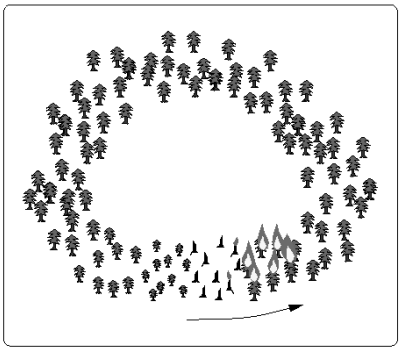


Figure 8: A reentrant wave pattern in an excitable medium with rapid recovery.

Figure 9: Alan Turing 1912-1954

2.2 Cell Polarization

Another landmark paper, also published in 1952, was that of Turing [6]. In this paper, Turing showed that if two chemical species interacted as activator and inhibitor, and if these two chemical species had sufficiently different diffusion coefficients, then a spatially homogeneous steady state solution could be unstable and there could arise periodic, spatially inhomogeneous, steady, chemical patterns. Turing argued that this could be the chemical basis for the origin of patterns in biology, i.e., morphogenesis.

There have been thousands of papers written on Turing patterns with claims of relevance to a diverse range of patterns including zebra stripes, stripes on zebra fish, and patterns on shells. However, most of these claims as well as the general biological relevance of Turing's precise mechanism remain controversial.

Figure 10: Zebra fish, Zebra stripes, and Shell patterns

In this section, I will show that something akin to Turing's mechanism may be responsible for cell polarization. Cell polarization is the phenomenon by which a cell polarizes itself, i.e., places its front at a particular location in response to a chemical signal gradient, enabling

it to move in the direction toward the source of the signal. A specific example is the ability of a macrophage to move in the direction of a bacterium in order to consume it. (Movies of this are readily observable on the internet, search for neutrophil chases bacterium.) Features of cell polarization:

- There is a response to localized stimuli or small gradients to establish a pole.
- Polarity is maintained when the stimulus is decreased;
- Pole is relocated with a sufficient change of stimulus, and can follow a stimulus of sufficient size.

The chemical basis of cell polarization is known. Small GTPases (here denoted as A) are regulators of actin nucleation and growth in eukaryotic cells that are activated by a membrane-bound signal cascade. When they are recruited to the cell membrane, they are activated and in activated form stimulate actin polymerization. (see Fig. 11.)

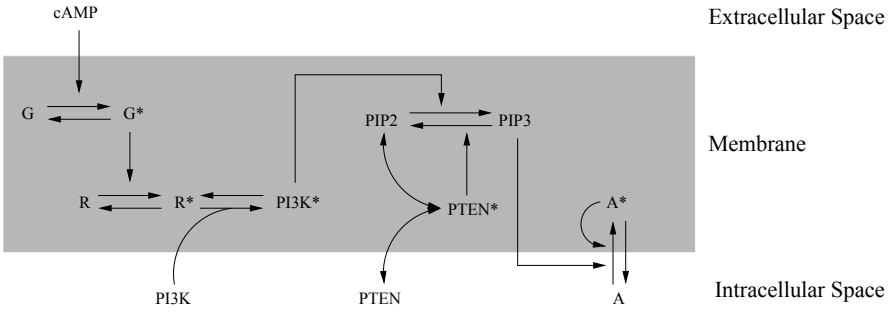


Figure 11: Signal transduction pathway for GTPase activation.

To build a model of this process, we assume that the chemical species A

- Is activated by a signalling cascade;
- In active form (A^*) is membrane bound, diffuses slowly on the membrane, and regulates actin polymerization;
- In inactive form (A) is in the cytosol, and diffuses freely.
- The active form acts to activate the inactive form (positive feedback).

These features are depicted in Fig. 12.

We assume that the cell membrane is a circle of radius R and that activated A is bound to the membrane while the inactivated form is free to diffuse in the cytoplasm. For simplicity, we assume that the cytoplasmic domain is also a circle of radius R, rather than a circular disc. (This model is adapted from [3].) With $u = [A^*]$, $v = [A]$, the partial differential equations describing this process are

$$\frac{\partial u}{\partial t} = \frac{D_u}{R^2}\frac{\partial^2 u}{\partial \theta^2} + f(u, v), \tag{12}$$

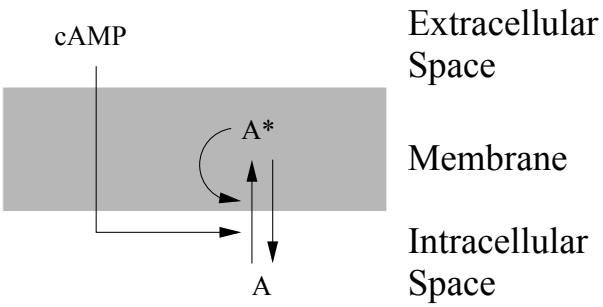


Figure 12: Simplified model of the activation of chemical species A.

$$\frac{\partial v}{\partial t} = \frac{D_v}{R^2}\frac{\partial^2 v}{\partial \theta^2} - f(u,v), \tag{13}$$

where

$$f(u,v) = (S(\theta,t) + \frac{\gamma u^2}{K^2+u^2})v - \delta u, \tag{14}$$

and θ is the angular variable, $D_u \ll D_v$, and u and v are 2π-periodic in θ. Here, S is the stimulus, and A^* is assumed to act as an enzyme with a Hill function rate of order 2. Since u and v are merely different forms of the same molecule, $u+v$ is a conserved quantity,

$$\int_0^{2\pi} (u+v)d\theta = w, \tag{15}$$

with w a constant independent of time. Plots of the function $f(u, v = w-u)$ are shown in Fig. 13.

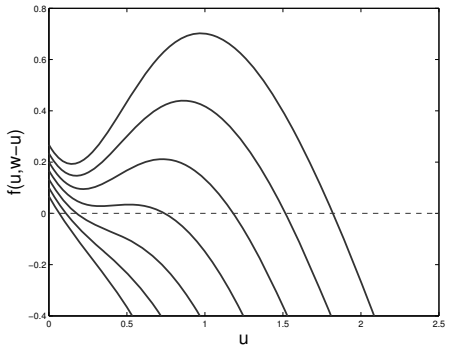


Figure 13: Plots of the function $f(u, w-u)$ for several values of w.

Solutions of this system of equations has features that are important for polarization:

- There is a localized response to stimuli (see Fig. 14 (left));
- The localized response is hysteretic; (compare Fig. 14, left and right, which show different steady state profiles with the same stimulus.)
- The localized response can track a moving stimulus (not shown here).

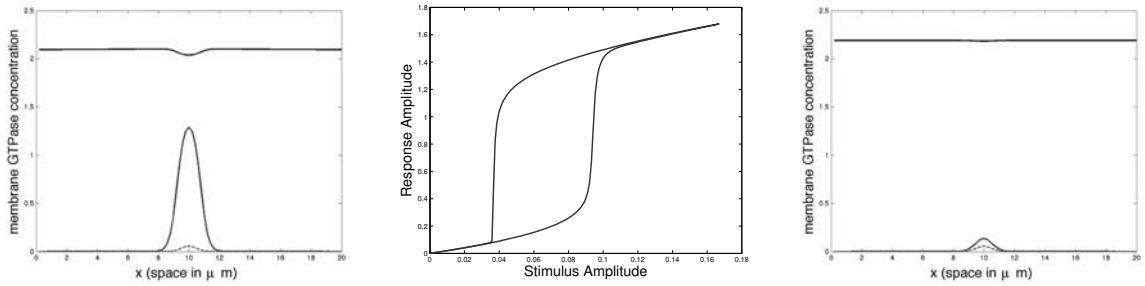


Figure 14: Localized response to stimulus profile (shown dashed) shows hysteretic behavior.

What we have learned from this example is that a reaction that converts a chemical from one that diffuses rapidly into one that diffuses slowly can lead to recruitment and large localized concentrations, in spite of the fact that diffusing molecules always move down their concentration gradients. In fact, it is this feature of diffusion that is exploited by this chemical reaction, since the conversion of the fast diffusing chemical to a slowly diffusing molecule creates a concentration deficit into which the fast diffusing molecule can diffuse. It needs to be mentioned, however, that energetically, there is no free lunch, as this localization process requires activation of the chemical species, involving phosphorylation, an energy consuming reaction.

3 Length Measurement

The bacterial flagellar rotary motor is a carefully constructed nanomachine, built in a precise step-by-step fashion, first the basal body, then the hook, and finally the flagellum (See Fig. 15). Questions concernings the construction of this machine include how the switches between steps are coordinated and how the hook length is regulated.

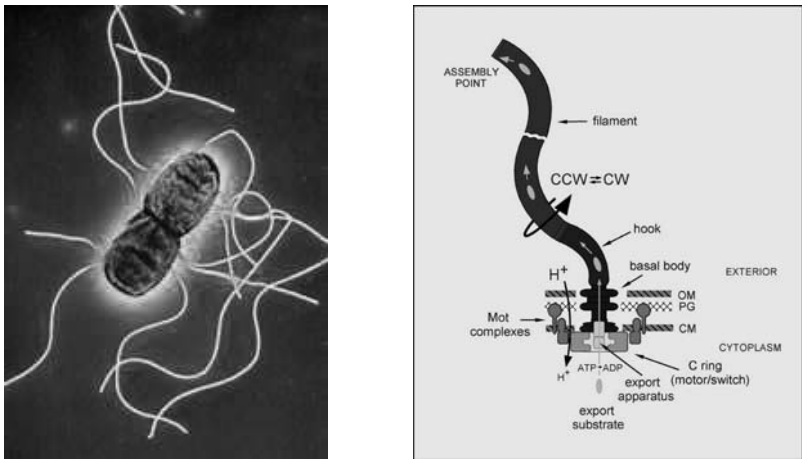


Figure 15: Left: Salmonella with multiple flagellar motors, and Right: Diagram of a flagellar rotary motor showing basal body, hook, and filament.

Hook is constructed by secretion of the monomer FlgE and its length is known to be

regulated by hook regulatory protein FliK. Hook length data for different variants of FliK are shown in Fig. 16.

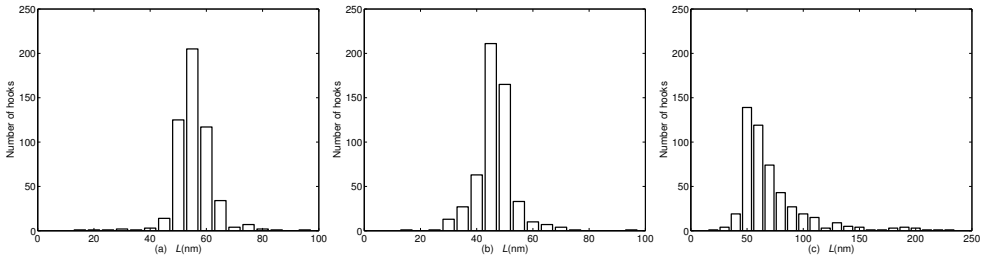
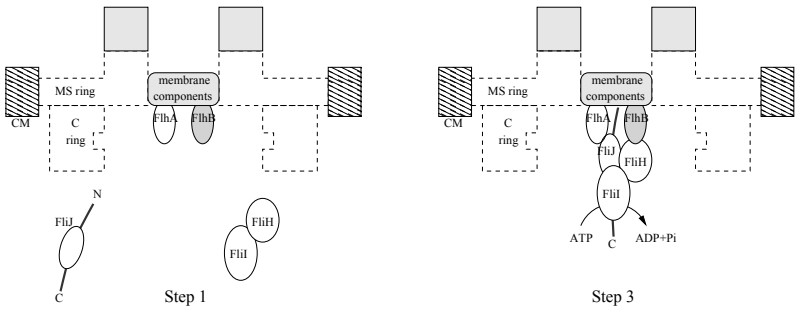


Figure 16: Hook length histogram data for Left: Wild type (M = 55nm), Center: FliK overexpressed (M = 47nm), and Right: FliK underexpressed (M = 75nm)

FliK is known as the hook length regulatory protein because

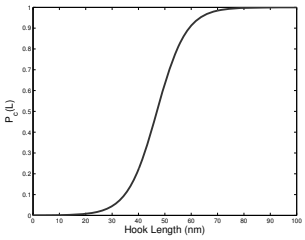
- FliK is secreted only during hook production.
- Mutants of FliK produce long hooks; overproduction of FliK gives shorter hooks. (see Fig. 16)
- Lengthening FliK by amino acid insertion gives longer hooks.
- 5-10 molecules of FliK are secreted per hook (115-120 molecules of FlgE).

Polymerization of the hook and filament are accomplished through secretion of unfolded monomers. Because the hook/filament is a hollow tube of inner diameter 2 nm, secreted molecules must be chaperoned to the entry site to prevent them from folding. Once at the entry site, the N terminus is recognized by the secretion gatekeeper, FlhB. Only those molecules with correct N terminus sequence are secreted.



Secretion is regulated by FlhB. During hook formation, only FlgE and FliK can be secreted. After the hook is complete, FlgE and FliK are no longer secreted, but other molecules can be secreted (those needed for filament growth.) The switch occurs when the C-terminus of FlhB is cleaved by FliK.

Question: Why is the switch of the FlhB recognition sequence length dependent? The answer is that during hook formation, when FlgE is being secreted, molecules of FliK are secreted once in a while to test the length of the hook. FliK causes a switch in the recognition sequence of FlhB with some probability that is a function of the length, $P_c(L)$. A sketch of the function $P_c(L)$ derived from hook length data is shown to the right [1].



What is the mechanism that determines $P_c(L)$? Our hypothesis is that FliK binds to FlhB during translocation to cause switching of secretion target by cleaving a recognition sequence. FliK molecules move through the growing tube by diffusion. They remain unfolded before and during secretion, but begin to fold as they exit the tube. Because folding is the energetically preferred configuration, folding on exit prevents back diffusion, giving a brownian ratchet effect. For short hooks, folding speeds the molecular movement and prevents FlhB cleavage, whereas for long hooks, movement is solely by diffusion and allows more time for cleavage (see Fig. 17).

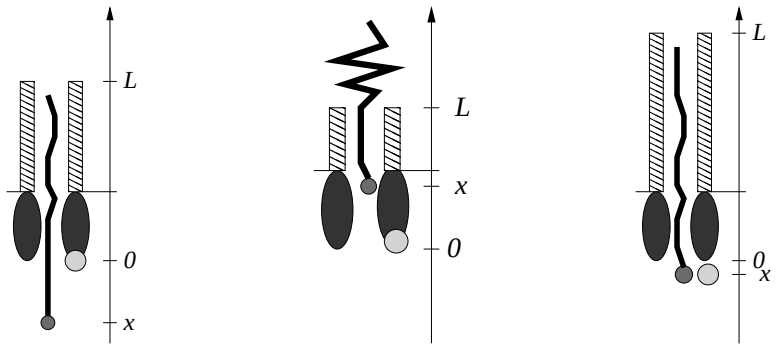


Figure 17: Unfolded, secreted molecules move through the hollow tube by diffusion, but when their ends leave the tube, they fold, preventing backward movement, a biased diffusion effect.

We model this stochastic process as follows [4]. The position $x(t)$ of the C-terminus is governed by the stochastic langevin differential equation

$$\nu dx = F(x)dt + \sqrt{2k_bT\nu}dW, \tag{16}$$

where $F(x) = F_0H(x + l - L)$ represents the folding force acting on the unfolded FliK molecule, l is the length of the unfolded FliK monomer, $W(t)$ is brownian white noise. Let $P(x,t)$ be the probability density of being at position x at time t with FlhB uncleaved, and $Q(t)$ be the probability of being cleaved by time t. Then

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}(F(x)P) + D\frac{\partial^2 P}{\partial x^2} - g(x)P, \tag{17}$$

$$\frac{dQ}{dt} = \int_a^b g(x)P(x,t)dx. \tag{18}$$

where $g(x)$ is the position-dependent rate of FlhB cleavage, taken to be a gaussian function (Fig. 18). This is the Chapman-Kolmogorov equation for the cleaving process.

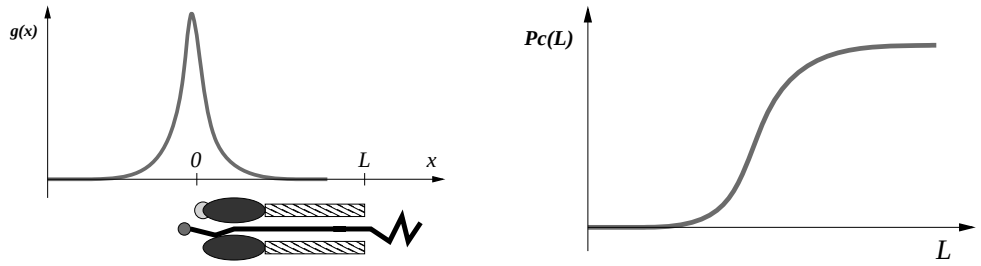


Figure 18: Left: The binding probability function $g(x)$, and Right: The solution of the splitting probability problem $P_c(L)$.

To determine the probability of cleavage $\pi_c(x)$ starting from position x, we must solve the splitting probability problem,

$$D\frac{d^2\pi_c}{dx^2} + F(x)\frac{d\pi_c}{dx} - g(x)\pi_c = -g(x), \tag{19}$$

subject to $\pi_c'(a) = 0$ and $\pi_c(b) = 0$. The function we seek is $P_c(L) = \pi_c(a)$, and for this choice of $g(x)$ is shown in Fig. 18. This function is nearly identical to the function found from hook length data shown in Fig. 3.

Conclusion: The diffusive movement and folding dynamics of FliK molecules contains information that enable these molecules to make measurements of length.

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Asymptotic properties of bifurcation curves related to inverse bifurcation problems

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Abstract

Recently, several authors have been investigating some nonlinear inverse and direct bifurcation problem in several frameworks. In this talk, we first introduce a typical inverse bifurcation problem for semilinear elliptic eigenvalue problems from a view point of L^2-approach. Next, we show the recent results of inverse and direct bifurcation problems in the case of ODE with oscillating nonlinear terms by using time-map method.

Keywords: Inverse bifurcation problems, asymptotic length of bifurcation curves.

1 Introduction

In this section, we introduce a concept of inverse bifurcation problems. We consider

$$\begin{aligned} -\Delta u + f(u) &= \lambda u \quad \text{in } \Omega, \\ u &> 0, \quad \text{in } \Omega, \\ u(0) &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

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Here, $\Omega \subset \mathbf{R}^N$ is an appropriately smooth bounded domain, and $\lambda > 0$ is a bifurcation parameter. We call (1.1) *the direct problem if $f(u)$ is a given nonlinear term*, and qualitative properties of bifurcation diagrams have been investigated by many authors. We refer to [3,4,5,6,10,11,12] and the references therein.

In this section, we assume that $f(u)$ is *unknown to* satisfy the conditions (A.1)–(A.3):

(**A.1**) $f(u)$ is a function of C^1 for $u \geq 0$ satisfying $f(0) = f'(0) = 0$.

(**A.2**) $f(u)/u$ is strictly increasing for $u \geq 0$.

(**A.3**) $f(u)/u \to \infty$ as $u \to \infty$.

The conditions (A.1)–(A.3) have been introduced by [1] and the examples of $f(u)$ which satisfy (A.1)–(A.3) are:

$$\begin{aligned} f(u) &= u^p \quad (p > 1), \\ f(u) &= u^p + u^q + u^m \quad (p > q > m > 1). \end{aligned}$$

Under the conditions (A.1)–(A.3), the shape of the bifurcation diagram is like the Fig. 1 below. The inverse bifurcation problems imply that we determine the unknown term $f(u)$ from some information about the bifurcation curve λ.

Our approach to the inverse bifurcation problems in this section is considering the problems in L^2-framework. Since (1.1) is regarded as an eigenvalue problem, it is quite natural to deal with it in L^2-framework. We note that it is also important to study it in L^1-framework, since if $f(u) = u^2$, then (1.1) is called diffusive logistic equation, and $\|u\|_1$ imply the total number of some species.

For $1 \leq q \leq \infty$, the L^q-bifurcation curves of (1.1) are defined as follows. It is well known from [1] that the following fundamental properties of bifurcation curves hold.

(1) We denote by $\|u\|_q$ the L^q-norm of u. Let $\alpha > 0$ be any given constant. Then there exists a unique solution pair $(\lambda, u) = (\lambda(q, \alpha), u_\alpha) \in \mathbf{R}_+ \times C^2(\bar{\Omega})$ satisfying $\|u_\alpha\|_q = \alpha$.

(2) The following set gives all the solutions of (1.1):

$$\{(\lambda(q, \alpha), u_\alpha) : \alpha > 0\} \subset \mathbf{R}_+ \times C^2(\bar{\Omega}).$$

(3) Let λ_1 be the first eigenvalue of $-\Delta_D$. Then $\lambda(q, \alpha) \to \lambda_1$ as $\alpha \to 0$. Furthermore, $\lambda(q, \alpha)$ is strictly increasing for $\alpha > 0$ and tends to infinity as $\alpha \to \infty$.

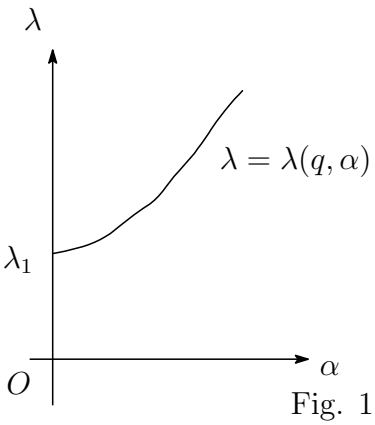

Fig. 1: bifurcation diagram

Now we state our result. Let $f(u) = f_1(u)$ and $f(u) = f_2(u)$ be unknown to satisfy (A.1)–(A.3). Furthermore, let

$$F_j(u) := \int_0^u f_j(s)ds \qquad (j = 1, 2).$$

Assume that F_1 and F_2 satisfy the following condition (B.1).

(B.1) Let $W := \{u \geq 0 : F_1(u) = F_2(u)\}$. Then W consists, at most, of the (finite or infinite numbers of) intervals and the points $\{u_n\}_{n=1}^{\infty}$ whose accumulation point is only ∞.

Theorem 1.1 ([13]). *Assume that f_1 and f_2 are unknown to satisfy (A.1)–(A.3) and (B.1). Furthermore, if $N \geq 2$, then assume that f_1 and f_2 satisfy the following (A.4).*

(A.4) For $u, v \geq 0$,

$$F_j(u + v) \leq C(F_j(u) + F_j(v)) \qquad (j = 1, 2).$$

Suppose $\lambda_1(2, \alpha) = \lambda_2(2, \alpha)$ for any $\alpha > 0$. Here, $\lambda_j(2, \alpha)$ is the L^2-bifurcation curve associated with $f(u) = f_j(u)$ $(j = 1, 2)$. Then $f_1(u) \equiv f_2(u)$ for $u \geq 0$.

2 Sketch of the Proof of Theorem 1.1

We give the sketch of the proof of Theorem 1.1 for the case $N = 1$ for simplicity. Since we consider (1.1) in L^2-framework, the variational approach is useful in this case. We put $\Omega = I = (0, 1)$. For $w \in H_0^1(I)$ and $\alpha > 0$, we set

$$\Psi_j(w) := \frac{1}{2}\|w'\|_2^2 + \int_0^1 F_j(w(t))dt \quad (j = 1, 2), \tag{2.1}$$

$$Q_\alpha := \{v \in H_0^1(I) : \|w\|_2 = \alpha\}. \tag{2.2}$$

Let $j = 1, 2$ and $\alpha > 0$. Then the critical value $C_j(\alpha)$ is defined by

$$C_j(\alpha) := \min\{\Psi_j(w) : w \in Q_\alpha\}. \tag{2.3}$$

By the standard argument of variational method, we see that there exists a Lagrange multiplier $\lambda_j(2,\alpha)$ and a unique minimizer $u_{j,\alpha} \in Q_\alpha$ which is the solution to (1.1) with $f = f_j$. Then by direct calculation, we find that the following equality for the critical values holds for $\alpha \geq 0$.

$$C_1(\alpha) = C_2(\alpha). \tag{2.4}$$

Proof of Theorem 1.1. We see that $0 \in W$, and we have only to consider the case where $0 \in W$ is contained in the interval $[0,\epsilon] \subset W$ for a constant $0 < \epsilon \ll 1$, since the other case can be treated by the similar argument. In this case, we see that for $0 \leq u \leq \epsilon$,

$$F_1(u) = F_2(u).$$

Let $K = [0, u_1)$ be a connected component of W satisfying $[0,\epsilon] \subset K$. Then if $u_1 < \infty$, then by (B.1), we can choose a small constant $0 < \delta \ll 1$ such that

$$\begin{aligned} F_1(u) &= F_2(u) \quad (0 \leq u \leq u_1), \\ F_1(u) &< F_2(u), \quad (u_1 < u < u_1 + \delta). \end{aligned}$$

Now we choose $\alpha > 0$ satisfying $\|u_{2,\alpha}\|_\infty = u_1 + \delta$. Then

$$\begin{aligned} C_1(\alpha) &\leq \Psi_1(u_{2,\alpha}) = \frac{1}{2}\|u'_{2,\alpha}\|_2^2 + \int_0^1 F_1(u_{2,\alpha}(t))dt \\ &< \frac{1}{2}\|u'_{2,\alpha}\|_2^2 + \int_0^1 F_2(u_{2,\alpha}(t))dt \\ &= \Psi_2(u_{2,\alpha}) = C_2(\alpha). \end{aligned}$$

This is a contradiction, since we have (2.4). This implies that $u_1 = \infty$. Namely, $F_1(u) \equiv F_2(u)$, which implies $f_1(u) \equiv f_2(u)$. Thus the proof is complete. ∎

3 New approach to inverse bifurcation problems

We next consider the following nonlinear eigenvalue problems

$$-u''(t) = \lambda\left(u(t) + g(u(t))\right), \quad t \in I =: (-1,1), \tag{3.1}$$

$$u(t) > 0, \quad t \in I, \tag{3.2}$$

$$u(1) = u(-1) = 0, \tag{3.3}$$

where $g(u) \in C^2(\bar{\mathbf{R}}_+)$ and $\lambda > 0$ is a parameter. It is well known (cf. [9]) that under some appropriate conditions on $g(u)$, λ is parameterized by $\alpha = \|u\|_\infty$. Namely, $\lambda = \lambda(\alpha)$ and $\lambda(\alpha)$ is continuous for $\alpha > 0$. Here, λ also depends on g. So we use the notation $\lambda = \lambda(g,\alpha)$.

The typical $g(u)$ we consider here is $g_1(u) = \frac{1}{2}u\sin u$. In this case, we know from [8] that

$$\lambda(g_1, \alpha) \to \frac{\pi^2}{4} \qquad (\alpha \to 0,\ \ \alpha \to \infty) \tag{3.4}$$

and oscillates infinitely many times as $\alpha \to \infty$.
We are also interested in $g_2(u) = \frac{1}{2}\sin u$. It seems clear that the asymptotic behavior of $\lambda(g_2, \alpha)$ is as follows.

$$\lambda(g_2, \alpha) \to \frac{\pi^2}{6} \quad (\alpha \to 0), \quad \lambda(g_2, \alpha) \to \frac{\pi^2}{4} \ (\alpha \to \infty). \tag{3.5}$$

In this section, we study the new concept called *asymptotic length* of $\lambda(g, \alpha)$ $(\alpha \gg 1)$ defined by

$$L(g, \alpha) := \int_{\alpha}^{2\alpha} \sqrt{1 + (\lambda'(g, s))^2}ds. \tag{3.6}$$

To propose a new idea for inverse bifurcation problem, we restrict our attention to $g(u)$, which satisfies

$$L(g, \alpha) = \alpha + o(\alpha) \quad (\alpha \to \infty), \tag{3.7}$$

because $g_1(u)$ and $g_2(u)$ seem to satisfy (3.7). We expect from (3.7) that the second term of $L(g, \alpha)$ contains an important information about g.

Without some conditions on g, it seems difficult to treat an inverse bifurcation problem. Therefore, by the idea mentioned above, we consider the set

$$g \in \Lambda := \{g \in C^2(\bar{\mathbf{R}}_+) : \lambda(g, \alpha) \to \pi^2/4 \ \text{ as } \ \alpha \to \infty\},$$

and propose the following new inverse bifurcation problem.

Inverse problem A. Suppose that $g \in \Lambda$ satisfies (3.7). Then can we distinguish g from g_i $(i = 1, 2)$ by using the information about the second term of $L(g, \alpha)$?

To study the Inverse problem A, we need the asymptotic formulas for $\lambda(g_i, \alpha)$ $(i = 1, 2)$ as $\alpha \to \infty$.

Theorem 3.1 ([14]). *Let $g(u) = g_1(u) = \frac{1}{2}u\sin u$. Then as $\alpha \to \infty$*

$$L(g_1, \alpha) = \alpha + \frac{\pi^3}{32}\log 2 + o(1), \tag{3.8}$$

$$\lambda(g_1, \alpha) = \frac{\pi^2}{4} - \frac{\pi}{2}\sqrt{\frac{\pi}{2\alpha}}\sin\left(\alpha - \frac{\pi}{4}\right) + o\left(\frac{1}{\sqrt{\alpha}}\right), \tag{3.9}$$

$$\lambda'(g_1, \alpha) = -\frac{\pi}{2}\sqrt{\frac{\pi}{2\alpha}}\cos\left(\alpha - \frac{\pi}{4}\right) + o(\alpha^{-1/2}). \tag{3.10}$$

Theorem 3.2 ([14]). *Let $g(u) = g_2(u) = (1/2)\sin u$. Then as $\alpha \to \infty$*

$$L(g_2, \alpha) = \alpha + \frac{3\pi^3}{256}\alpha^{-2} + o(\alpha^{-2}), \tag{3.11}$$

$$\lambda(g_2, \alpha) = \frac{\pi^2}{4} - \frac{\pi}{2\alpha}\sqrt{\frac{\pi}{2\alpha}}\sin\left(\alpha - \frac{1}{4}\pi\right) + o(\alpha^{-3/2}), \tag{3.12}$$

$$\lambda'(g_2, \alpha) = -\frac{\pi}{2\alpha}\sqrt{\frac{\pi}{2\alpha}}\cos\left(\alpha - \frac{\pi}{4}\right) + o(\alpha^{-3/2}). \tag{3.13}$$

Remark. The following asymptotic formula (3.14) was shown in [8] by using the argument of complex analysis.

$$\left(\frac{\pi^2}{4} - \lambda(g_1, \alpha)\right) = \frac{1}{2}\lambda(g_1, \alpha)\sin\left(\alpha - \frac{\pi}{4}\right)\sqrt{\frac{2\pi}{\alpha|v_\lambda'(0)|}} + O(\alpha^{-1}), \tag{3.14}$$

where $v_\lambda(x) := u_\lambda(x)/\alpha$ and $v_\lambda \to c\cos(\pi x/2)$ as $\alpha \to \infty$, where $c > 0$ is some constant. We see that (3.9) improves (3.14). We also mention that we prove (3.9) by the time-map method and a careful calculation.

Although we obtain Theorems 3.1 and 3.2, Inverse problem A is still difficult to treat. So we consider it in a simpler situation.

Question A. Let $g(u) \in \Lambda$ satisfy the assumption (C.1).

(**C.1**) $g(0) = g'(0) = 0$, $g'(u) \geq 0$ for $u > 0$ and $g(u) = Cu^m$ for $u \geq 1$, where $C > 0$ and $0 < m < 1$ are constants.

Then can we distinguish $g(u)$ from $g_i(u)$ by $L(g, \alpha)$?

Theorem 3.3 ([14]). *Let $g(u)$ satisfy (**C.1**). Then as $\alpha \to \infty$,*

$$L(g, \alpha) = \alpha + \frac{2^{2m-3} - 1}{2(2m-3)}A(m)^2\alpha^{2m-3} + o(\alpha^{2m-3}), \tag{3.15}$$

$$\lambda(g, \alpha) = \frac{\pi^2}{4} - \frac{\pi}{m+1}CC(m)\alpha^{m-1} + o(\alpha^{m-1}), \tag{3.16}$$

$$\lambda'(g, \alpha) = -\frac{m-1}{m+1}\pi CC(m)\alpha^{m-2} + o(\alpha^{m-2}), \tag{3.17}$$

where

$$A(m) := \frac{(1-m)\pi CC(m)}{1+m}, \quad C(m) = \int_0^1 \frac{1 - s^{m+1}}{(1-s^2)^{3/2}}ds. \tag{3.18}$$

In particular, let $m = 1/2$. If we choose $C > 0$ appropriately, then $L(g, \alpha)$ in (3.15) is equal to (3.11) up to the second term.

It seems from (3.8) and (3.15) that, since the height of the oscillation of $g_1(u)$ is rather large, the positive answer to the Question A has been obtained. On the contrary, since the height of the oscillation of $g_2(u)$ is smaller than that of $g_1(u)$, the negative answer to the Question A has been obtained. Therefore, the asymptotic length of the bifurcation curve seems to be one of the concept to distinguish some oscillatory and monotone nonlinear terms.

All the proofs of the theorems in this section depend on the time-map method. After careful and complicated calculation, we obtain these theorems.

4 Idea of the proof of Theorem 3.1

In this section, let $g(u) = g_1(u) = (u/2)\sin u$. For simplicity, we write $\lambda = \lambda(\alpha) = \lambda(g_1, \alpha)$ and

$$f(u) := u + \frac{1}{2}u\sin u, \tag{4.1}$$

$$F(u) := \int_0^u f(s)ds = \frac{1}{2}u^2 - \frac{1}{2}u\cos u + \frac{1}{2}\sin u. \tag{4.2}$$

In this case, the well known time map (cf. [2,9]) is represented as follows.

$$\sqrt{\lambda} = \frac{\alpha}{\sqrt{2}}\int_0^1 \frac{1}{\sqrt{F(\alpha) - F(\alpha s)}}ds := \frac{\alpha}{\sqrt{2}}B(\alpha). \tag{4.3}$$

By this and direct calculation, we have

$$\lambda(\alpha) = \frac{\alpha^2}{2}B(\alpha)^2, \tag{4.4}$$

$$\lambda'(\alpha) = \alpha B(\alpha)^2 - \frac{\alpha^2}{2}B(\alpha)R(\alpha), \tag{4.5}$$

where

$$R(\alpha) := \int_0^1 \frac{f(\alpha) - sf(\alpha s)}{(F(\alpha) - F(\alpha s))^{3/2}}ds. \tag{4.6}$$

We put

$$A_\alpha(s) := \frac{-\cos\alpha + s\cos(\alpha s)}{\alpha} + \frac{\sin\alpha - \sin(\alpha s)}{\alpha^2} := A_{1,\alpha}(s) + A_{2,\alpha}(s). \tag{4.7}$$

Then for $0 \le s \le 1$ and $\alpha \gg 1$,

$$F(\alpha) - F(\alpha s) = \frac{1}{2}\alpha^2\{1 - s^2 + A_\alpha(s)\}. \tag{4.8}$$

Lemma 4.1. *For $\alpha \gg 1$*

$$B(\alpha) \;=\; \frac{\sqrt{2}}{\alpha}\left\{\frac{\pi}{2} - \frac{1}{2}K_1(\alpha)(1+o(1))\right\}, \tag{4.9}$$

where

$$K_1(\alpha) \;:=\; \int_0^1 \frac{1}{(1-s^2)^{3/2}} A_\alpha(s)ds. \tag{4.10}$$

Lemma 4.2. *As $\alpha \to \infty$,*

$$K_1(\alpha) = \sqrt{\frac{\pi}{2\alpha}}\sin\left(\alpha - \frac{1}{4}\pi\right) + O(\alpha^{-1}). \tag{4.11}$$

We show (4.11) by using the asymptotic formulas for Struve functions $\mathbf{H}_\nu(\alpha)$ and Neumann's functions $N_\nu(\alpha)$ as $\alpha \to \infty$ (cf. [7, p. 972, p. 997]):

$$\mathbf{H}_\nu(\alpha) \;=\; N_\nu(\alpha) + \frac{1}{\pi}\sum_{m=0}^{p-1} \frac{\Gamma\left(m+\frac{1}{2}\right)\left(\frac{\alpha}{2}\right)^{-2m+\nu-1}}{\Gamma\left(\nu+\frac{1}{2}-m\right)} + O(\alpha^{\nu-2p-1}), \tag{4.12}$$

$$N_\nu(\alpha) \;=\; \sqrt{\frac{2}{\pi\alpha}}\sin\left(\alpha - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + O(\alpha^{-3/2}). \tag{4.13}$$

By (4.4), (4.12), (4.13) Lemmas 4.1 and 4.2, we obtain (3.9). ∎

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Mathematical analysis of a chemotaxis model via maximal regularity

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1. Problem

The purpose of this work is to establish global-in-time existence and uniform-in-time boundedness of solutions to a chemotaxis model via maximal regularity for parabolic equations. More specifically, we consider the following Cauchy problem for quasilinear "degenerate" Keller–Segel systems of parabolic-parabolic type:

$$\text{(KS)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (\nabla u^m - u^{q-1}\nabla v) & \text{in } \mathbb{R}^N \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = \Delta v - v + u & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $N \in \mathbb{N}$, $m \geq 1$, $q \geq 2$ and (u_0, v_0) is the initial data satisfying

$$u_0 \geq 0, \quad u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \tag{1.1}$$

$$v_0 \geq 0, \quad v_0 \in L^1(\mathbb{R}^N) \cap W^{1,r}(\mathbb{R}^N) \ (\exists\, r > N). \tag{1.2}$$

The Keller–Segel system (KS) with the simplest choices $m = 1$ and $q = 2$ is proposed by Keller and Segel [**13**] in 1970. The system describes a part of the life cycle of cellular slime molds with *chemotaxis*. In more detail, slime molds move towards higher concentration of the chemical substance when they plunge into hunger. Such chemotactic effect is originally described by $-\nabla \cdot (u\nabla v)$, while the diffusion effect of slime molds is represented by Δu, where we denote by $u(x,t)$ the density of the cell population and by $v(x,t)$ the concentration of the signal substance at place x and time t. A number of variations of the original Keller–Segel system are proposed and studied (see Hillen–Painter [**4**]). A quasilinear system such as (KS) was proposed by Painter and Hillen [**18**]. In particular, we emphasize that the diffusion term Δu^m in (KS) is allowed to be "degenerate". This means that the mathematical analysis of (KS) will be more delicate because of the lack of regularity of solutions.

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2. Known results

From a mathematical point of view, it is a meaningful question whether solutions *remain bounded* or *blow up*. This depends on the relation between the powers m and q in the diffusion term Δu^m and the chemotaxis term $-\nabla \cdot (u^{q-1}\nabla v)$ in (KS).

In the case of *non-degenerate* diffusion such as $\Delta(u+1)^m$, Tao and Winkler [**21**] proved that there exists a unique global-in-time bounded classical solution of the Neumann problem on bounded convex domains under the condition

$$q < m + \frac{2}{N}.$$

The result was extended to the Neumann problem on bounded non-convex domains by Ishida, Seki and Yokota [**8**]. As to the case $q = 2$, Senba and Suzuki [**19**] proved boundedness of solutions to (KS) with non-degenerate diffusion under some additional conditions for the initial data. In contrast, Winkler [**22**] showed that the Neumann problem with non-degenerate diffusion such as $\Delta(u+1)^m$ admits a smooth solution which blows up either in finite or infinite time if $q > m + \frac{2}{N}$. Moreover, Winkler [**23**] and Ciéslak and Stinner [**2**] asserted that the solution blows up in finite time. This means that the case $q = m + \frac{2}{N}$ is critical.

In the case of *degenerate* diffusion, the Cauchy problem (KS) was firstly studied by Sugiyama and Kunii [**20**] in which existence of global-in-time weak solutions was shown when

$$q \le m.$$

After that, the condition $q \le m$ for global existence was extended to

$$q < m + \frac{2}{N}$$

by [**9**]. Unfortunately, both [**20**] and [**9**] assert only global existence. As to the Neumann problem on *bounded* domains, boundedness of solutions was proved by [**8**]. However, boundedness in the Cauchy problem on *the whole space* $\mathbb{R}^N$ is still open. On the other hand, when $q \ge m + \frac{2}{N}$ and the initial data (u_0, v_0) is small in some sense, existence of global-in-time weak solutions was proved by [**10, 11**] and boundedness was established by [**5**]; whereas, if $q > m + \frac{2}{N}$, then blow-up (either in finite or infinite) was studied by [**7, 12**].

In summary, boundedness and blow-up in finite time in the Cauchy problem (KS) are still open. In this work we establish boundedness by a simple way via maximal regularity in parabolic equations.

3. Main result

As stated above, there is a lack of regularity of solutions to (KS) because the diffusion is of degenerate type. So we need the notion of weak solutions to (KS).

Definition 3.1. Let $T > 0$. A pair (u, v) of nonnegative functions is called a *weak solution* of (KS) on $[0, T)$ if

(a) $u \in L^\infty(0, T; L^p(\mathbb{R}^N))$ $(\forall p \in [1, \infty])$, $u^m \in L^2(0, T; H^1(\mathbb{R}^N))$;
(b) $v \in L^\infty(0, T; H^1(\mathbb{R}^N))$;
(c) for every $\varphi \in C_{\rm c}^\infty(\mathbb{R}^N \times [0, T))$,

$$\int_0^T \int_{\mathbb{R}^N} (\nabla u^m \cdot \nabla \varphi - u^{q-1} \nabla v \cdot \nabla \varphi - u \varphi_t) dx dt = \int_{\mathbb{R}^N} u_0(x) \varphi(x, 0)\, dx,$$
$$\int_0^T \int_{\mathbb{R}^N} (\nabla v \cdot \nabla \varphi + v \varphi - u \varphi - v \varphi_t)\, dx dt = \int_{\mathbb{R}^N} v_0(x) \varphi(x, 0)\, dx.$$

If $T > 0$ can be taken arbitrarily, then (u, v) is called a *global weak solution* of (KS).

We now state the main result.

Theorem 3.1. *Let $N \in \mathbb{N}$, $m \geq 1$ and $q \geq 2$. Assume that*

$$q < m + \frac{2}{N}.$$

Then for any initial data (u_0, v_0) fulfilling (1.1) *and* (1.2)*, there exists a global weak solution (u, v) of* (KS) *such that u is bounded in the following sense*:

$$\operatorname*{ess\text{-}sup}_{t \in (0, \infty)} \|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq c_\infty, \tag{3.1}$$

where c_∞ is a positive constant.

Remark 3.2. Uniqueness of weak solutions to (KS) was proved by Miura and Sugiyama [**16**] and Kim and Lee [**14**].

4. Outline of the proof

(1) Construction of local-in-time approximate solutions.

By standard technique for nondegenerate parabolic systems we obtain local-in-time solutions $(u_\varepsilon, v_\varepsilon)$ of (KS) with Δu^m (of degenerate type) replaced with $\Delta (u + \varepsilon)^m$, $\varepsilon > 0$ (of nondegenerate type).

(2) Estimates for approximate solutions.

In order to extend local-in-time approximate solutions $(u_\varepsilon, v_\varepsilon)$ globally in time we need the L^∞-estimate for u_ε which will be derived from the L^r-estimate:

$$\|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)} \leq M_r < \infty \quad \text{for each } r \in [1, \infty).$$

(3) Passage to the limit.

Letting $\varepsilon \to 0$, we can show that $(u, v) := \lim_{\varepsilon \to 0} (u_\varepsilon, v_\varepsilon)$ is the desired solution.

The key to the proof is the L^r-estimate for u_ε in the above step (2). We derive the L^r-estimate from standard testing arguments for the first equation in (KS). One of the difficulties is how to estimate the chemotaxis term $-\nabla\cdot(u_\varepsilon^{q-1}\nabla v_\varepsilon)$, in particular, Δv_ε. To overcome the difficulty we turn our eyes to maximal regularity for parabolic equations as follows.

Lemma 4.1 (see Ladyženskaja–Solonnikov–Ural'ceva [**15**], Hieber–Prüss [**3**])**.** *Let $1 < p < \infty$ and $a > 0$. For $f \in L^p(0,T;L^p(\mathbb{R}^N))$ put*

$$G_a[f](t) := \int_0^t e^{-a(t-s)}e^{(t-s)\Delta}f(s)\,ds, \quad t \in (0,T).$$

Then there exists a constant $C_p > 0$ such that for all $f \in L^p(0,T;L^p(\mathbb{R}^N))$,

$$\|\Delta(G_a[f])\|_{L^p(0,T;\,L^p(\mathbb{R}^N))} \le C_p\|f\|_{L^p(0,T;L^p(\mathbb{R}^N))}.$$

In terms of G_a we can rewrite the second equation $\frac{\partial v_\varepsilon}{\partial t} = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon$ in the following form:

$$\begin{aligned} v_\varepsilon(t) &= e^{-t}e^{t\Delta}v_0 + \int_0^t e^{-(t-s)}e^{(t-s)\Delta}u_\varepsilon(s)\,ds \\ &= e^{-t}e^{t\Delta}v_0 + G_1[u_\varepsilon](t). \end{aligned}$$

Thus Δv_ε is represented by $\Delta(G_1[v_\varepsilon])$ and we can apply Lemma 4.1. Once Δv_ε is estimated by u_ε in some sense, we can establish the L^r-estimates for u_ε in the same way as in the single parabolic equation

$$\frac{\partial u}{\partial t} = \Delta u^m + u^q \quad \text{in } \mathbb{R}^N \times (0,\infty)$$

by using some inequalities, e.g., Hölder's inequality, Young's inequality, the Gagliardo–Nirenberg inequality and the mass conservation law

$$\|u_\varepsilon(t)\|_{L^1(\mathbb{R}^N)} = \|u_0\|_{L^1(\mathbb{R}^N)}, \quad t \ge 0.$$

For the rigorous proof of the mass conservation law see Ishida–Maeda–Yokota [**6**].

Remark 4.2. The above idea via maximal regularity was first used by Senba and Suzuki [**19**]. We also employed it in the previous works [**9**], [**10**] and [**11**]. However, we have not yet used effectively some decaying effects in the equations. We also note that Cao [**1**] developed this idea in a three-dimensional chemotaxis-haptotaxis model.

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SOME INVERSE PROBLEMS IN PERIODIC HOMOGENIZATION OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. This is mostly based on the work of Luo, Yu and the author [20]. We look at the effective Hamiltonian $\overline{H}$ associated with the Hamiltonian $H(p,x) = H(p) + V(x)$ in the periodic homogenization theory. Our central goal is to understand the relation between V and $\overline{H}$. We formulate some inverse problems concerning this relation. Such type of inverse problems are in general very challenging. In the paper, we discuss several special cases in both convex and non-convex settings.

Homogenization theory aims to identify and study macroscopic behavior of PDEs which typically have high oscillations in the space (or time-space) variables (cf. e.g (1.1)) for instance. Basic problems include (I) well-posedness: obtaining the existence of limiting effective equations (cf. e.g. (1.1)) as $\varepsilon \to 0$; (II) understanding finer properties of the limiting process and the effective equation. PDEs are usually set in self-averaging (periodic, almost periodic or random) environments. In the periodic setting, (I) is quite well established for some nonlinear PDEs such as first-order and second-order Hamilton-Jacobi equations, fully nonlinear elliptic equations. However, very little is known about (II) because of the nonlinear nature in these equations. We propose to develop some new tools and directions to study (II) in various aspects systematically in the periodic setting. One of the direction is through some inverse problems.

1. Setting of the inverse problem

For each $\varepsilon > 0$, let $u^\varepsilon \in C(\mathbb{R}^n \times [0,\infty))$ be the viscosity solution to the following Hamilton-Jacobi equation

$$\begin{cases} u_t + H\left(Du^\varepsilon, \frac{x}{\varepsilon}\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1.1}$$

The Hamiltonian $H = H(p,x) \in C(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies

(H1) $x \mapsto H(p,x)$ is $\mathbb{Z}^n$-periodic,

(H2) $p \mapsto H(p,x)$ is coercive uniformly in x, i.e.,

$$\lim_{|p|\to+\infty} H(p,x) = +\infty \quad \text{uniformly for } x \in \mathbb{R}^n,$$

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and the initial data $g \in \mathrm{BUC}(\mathbb{R}^n)$, the set of bounded, uniformly continuous functions on $\mathbb{R}^n$.

It was proven by Lions, Papanicolaou and Varadhan [19] that u^ε, as $\varepsilon \to 0$, converges locally uniformly to u, the solution of the effective equation,

$$\begin{cases} u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1.2}$$

The effective Hamiltonian $\overline{H} : \mathbb{R}^n \to \mathbb{R}$ is determined by the cell problems as follows. For any $p \in \mathbb{R}^n$, we consider the following cell problem

$$H(p + Dv, x) = c \quad \text{in } \mathbb{T}^n, \tag{1.3}$$

where $\mathbb{T}^n$ is the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$. We here seek for a pair of unknowns $(v, c) \in C(\mathbb{T}^n) \times \mathbb{R}$ in the viscosity sense. It was established in [19] that there exists a unique constant $c \in \mathbb{R}$ such that (1.3) has a solution $v \in C(\mathbb{T}^n)$. We then denote by $\overline{H}(p) = c$.

In this paper, we always consider the Hamiltonian H of the form $H(p,x) = H(p) + V(x)$. Our main goal is to study the relation between the potential energy V and the effective Hamiltonian $\overline{H}$. In the case where H is uniformly convex, Concordel [8, 9] provided some first general results on the properties of $\overline{H}$, which is convex in this case. In particular, she achieved some representation formulas of $\overline{H}$ by using optimal control theory and showed that $\overline{H}$ has a flat part under some appropriate conditions on V. The connection between properties of $\overline{H}$ and weak KAM theory can be found in E [10], Evans and Gomes [11], Fathi [13] and the references therein. We refer the readers to Evans [12, Section 5] for a list of interesting viewpoints and open questions. To date, deep properties of $\overline{H}$ are still not yet well understood.

In the case where H is not convex, there have been not so many results on qualitative and quantitative properties of $\overline{H}$. Very recently, Armstrong, Tran and Yu [1, 2] studied nonconvex stochastic homogenization and derived qualitative properties of $\overline{H}$ in the general one dimensional case, and in some special cases in multi-dimensional spaces. The general case in multi-dimensional spaces is still out of reach.

We present here a different question concerning the relation between V and $\overline{H}$. In its simplest way, the question can be thought of as: how much can we recover the potential energy V provided that we know H and $\overline{H}$? More precisely, the main question that Luo, Yu and I [20] pose is the following:

Question 1.1. *Let $H \in C(\mathbb{R}^n)$ be a given coercive function, and $V_1, V_2 \in C(\mathbb{R}^n)$ be two given potential energy functions which are $\mathbb{Z}^n$-periodic. Set $H_1(p,x) = H(p) + V_1(x)$ and $H_2(p,x) = H(p) + V_2(x)$ for $(p,x) \in \mathbb{R}^n \times \mathbb{R}^n$. Suppose that $\overline{H}_1$ and $\overline{H}_2$ are two effective Hamiltonians corresponding to the two Hamiltonians H_1 and H_2 respectively. If*

$$\overline{H}_1 \equiv \overline{H}_2,$$

then what can we conclude about the relations between V_1 and V_2? Especially, can we identify some common "computable" properties shared by V_1 and V_2?

To the best of our knowledge, such kind of questions have never been explicitly stated and studied before. This is closely related to the exciting projects of going beyond the well-posedness of the homogenization and understanding deep properties of the effective Hamiltonian, which are in general very hard. In [17], Jing, Yu and I studied some other inverse problems for the effective burning velocity from an inviscid quadratic Hamilton-Jacobi model.

In this extended abstract, I will just point out some results from [20] and discuss some further questions.

2. Main Results

2.1. Dimension $n \geq 1$.

Theorem 2.1. *Assume $V_2 \equiv 0$. Suppose that there exists $p_0 \in \mathbb{R}^n$ such that $H \in C(\mathbb{R}^n)$ is differentiable at p_0 and $DH(p_0)$ is an irrational vector, i.e.,*

$$DH(p_0) \cdot m \neq 0 \quad \text{for all } m \in \mathbb{Z}^n \setminus \{0\}.$$

Then

$$\overline{H}_1(p_0) = \overline{H}_2(p_0) \quad \text{and} \quad \min_{\mathbb{R}^n} \overline{H}_1 = \min_{\mathbb{R}^n} \overline{H}_2 \quad \Rightarrow \quad V_1 \equiv 0.$$

In particular,

$$\overline{H}_1 \equiv \overline{H}_2 \qquad \Rightarrow \qquad V_1 \equiv 0. \tag{2.4}$$

Note that we do not assume H is convex in the above theorem. As $V_2 \equiv 0$, it is clear that $\overline{H}_2 = H$. The theorem infers that if $\overline{H}_1(p_0) = H(p_0)$, $\min_{\mathbb{R}^n} \overline{H}_1 = \min_{\mathbb{R}^n} H$ and $DH(p_0)$ is an irrational vector, then in fact $V_1 \equiv 0$. The requirement on $DH(p_0)$ seems technical on the first hand, but it is, in fact, optimal.

Remark 2.1. *If the set*

$$G = \{DH(p) \,:\, H \text{ is differentiable at } p \text{ for } p \in \mathbb{R}^n\}$$

only contains rational vectors, the conclusion of Theorem 2.1 might fail. Below is a simple example.

Let $n = 2$. Suppose that $V \in C^\infty(\mathbb{T}^2)$ and $V \leq 0$. Denote $Q = [0,1]^2$. We can think of V as a function defined on Q with periodic boundary condition. Assume further that $\partial Q \subset \{V = 0\}$. Let

$$H(p) = \max\{K_1(p_1),\ K_2(p_2)\} \quad \text{for all } p = (p_1, p_2) \in \mathbb{R}^2.$$

Here $K_i \in C(\mathbb{R})$ is coercive for $i = 1, 2$. Then it is not hard to verify that

$$\overline{H}(p) = H(p) = \max\{K_1(p_1),\ K_2(p_2)\} \quad \text{for all } p \in \mathbb{R}^2.$$

If neither V_1 nor V_2 is constant, the situation usually involves complicated dynamics and becomes much harder to analyze. In this paper, we establish some preliminary results. A vector $Q \in \mathbb{R}^n$ satisfies a Diophantine condition if there exist C, $\alpha > 0$ such that

$$|Q \cdot k| \geq \frac{C}{|k|^\alpha} \quad \text{for any } k \in \mathbb{Z}^n \setminus \{0\}.$$

Theorem 2.2. *Assume that $V_1, V_2 \in C^\infty(\mathbb{T}^n)$.*

(1) *Suppose that* $H \in C^2(\mathbb{R}^n)$, $\sup_{\mathbb{R}^n} \|D^2 H\| < +\infty$ *and* H *is superlinear. Then for* $i = 1, 2$ *and any vector* $Q \in \mathbb{R}^n$ *satisfying a Diophantine condition,*

$$\int_{\mathbb{T}^n} V_i \, dx = \lim_{\lambda \to +\infty} \left(\overline{H}_i(\lambda P_\lambda) - H(\lambda P_\lambda) \right). \tag{2.5}$$

Here $P_\lambda \in \mathbb{R}^n$ *is choosen such that* $DH(\lambda P_\lambda) = \lambda Q$. *In particular,*

$$\overline{H}_1 \equiv \overline{H}_2 \quad \Rightarrow \quad \int_{\mathbb{T}^n} V_1 \, dx = \int_{\mathbb{T}^n} V_2 \, dx.$$

(2) *Suppose that* $H(p) = \frac{1}{2}|p|^2$. *We have that, for* $i = 1, 2$ *and any irrational vector* $Q \in \mathbb{R}^n$,

$$\int_{\mathbb{T}^n} V_i \, dx = \lim_{\lambda \to +\infty} \left(\overline{H}_i(\lambda Q) - \frac{1}{2}\lambda^2 |Q|^2 \right) \tag{2.6}$$

and

$$\lim_{\lambda \to +\infty} \left(\lambda^2 |Q|^2 - \max_{q \in \partial \overline{H}_i(\lambda Q)} q \cdot \lambda Q \right) = 0 \tag{2.7}$$

(3) *Suppose that* $H(p) = \frac{1}{2}|p|^2$. *If there exits* $\tau > 0$ *such that*

$$\sum_{k \in \mathbb{Z}^n} (|\lambda_{k1}|^2 + |\lambda_{k2}|^2) e^{|k|^{n+\tau}} < +\infty, \tag{2.8}$$

then

$$\overline{H}_1 \equiv \overline{H}_2 \quad \Rightarrow \quad \int_{\mathbb{T}^n} |V_1|^2 \, dx = \int_{\mathbb{T}^n} |V_2|^2 \, dx.$$

Here $\{\lambda_{ki}\}_{k \in \mathbb{Z}^n}$ *are Fourier coefficients of* V_i.

Remark 2.2. *Due to the stability of the effective Hamiltonian, (2.5) and (2.6) still hold when* $V_1, V_2 \in C(\mathbb{T}^n)$. *The equality* (2.6) *is essentially known in case* Q *satisfies a Diophantine condition. The average of the potential function is the constant term in the asymptotic expansion. See* [3, 15, 16] *for instance.*

Moreover, when $H(p) = \frac{1}{2}|p|^2$, *if* V_1 *and* V_2 *are both smooth, through direct computations of the asymptotic expansions,* $\overline{H}_1 \equiv \overline{H}_2$ *leads to a series of identical quantities associated with* V_1 *and* V_2, *which involve complicated combinations of Fourier coefficients. It is very difficult to calculate those quantities and our goal is to to extract some new computable quantitites from those almost uncheckable ones. The above theorem says that the average and the* L^2 *norm of the potential can be recovered. The fast decay condition (2.8) is a bit restrictive at this moment. It can be slightly relaxed if we transform the problem into the classical moment problem and apply Carleman's condition.*

In fact, we conjecture that the distribution of the potential function should be determined by the effective Hamiltonian under reasonable assumptions. When $n = 1$, *this is proved in Theorem 2.3 for much more general Hamiltonians. High dimensions will be studied in a future work.*

2.2. **One dimensional case.** When $n = 1$, we have a much clearer understanding of this inverse problem. Let us first define some terminologies.

Definition 2.1. *We say that V_1 and V_2 have the same distribution if*

$$\int_0^1 f(V_1(x))\,dx = \int_0^1 f(V_2(x))\,dx$$

for any $f \in C(\mathbb{R})$.

Definition 2.2. *$H : \mathbb{R} \to \mathbb{R}$ is called strongly superlinear if there exists $a \in \mathbb{R}$ such that the restriction of H to $[a, +\infty)$ ($H|_{[a,+\infty)} : [a, +\infty) \to \mathbb{R}$) is smooth, strictly increasing, and*

$$\lim_{x\to+\infty} \frac{\psi^{(k)}(x)}{\psi^{(k-1)}(x)} = 0 \quad \text{for all } k \in \mathbb{N}. \tag{2.9}$$

Here $\psi = \psi^{(0)} = \left(H|_{[a,+\infty)}\right)^{-1} : [H(a), +\infty) \to [a, +\infty)$ and $\psi^{(k)}$ is the k-th derivative of ψ for $k \in \mathbb{N}$.

Note that condition (2.9) is only about the asymptotic behavior at $+\infty$. There is a large class of functions satisfying the above condition, e.g. $H(p) = e^p$, $H(p) = (c + |p|)^\gamma$ for $p \in [a, +\infty)$ for any $a \in \mathbb{R}$, $\gamma > 1$ and $c \geq 0$. As nothing is required for the behavior of H in $(-\infty, a)$ (except coercivity at $-\infty$), H clearly can be nonconvex.

Theorem 2.3. *Assume $n = 1$ and V_1, $V_2 \in C(\mathbb{T})$. Then the followings hold:*

(1) *If H is quasi-convex, then*

$$V_1 \text{ and } V_2 \text{ have the same distribution} \quad \Rightarrow \quad \overline{H}_1 \equiv \overline{H}_2.$$

(2) *If H is strongly superlinear, then*

$$\overline{H}_1 \equiv \overline{H}_2 \quad \Rightarrow \quad V_1 \text{ and } V_2 \text{ have the same distribution.}$$

3. Some further questions

The inverse problem pointed out above is just one among many important others. I want to single out some other related problems here. In term of front propagation, Yu and I propose the following one, which is extremely interesting and has deep connections to geometry.

Question 3.1. *Assume that $H_i(p, x) = a_i(x)|p|$ for $a_i \in C(\mathbb{T}^n, (0, \infty))$ for $i = 1, 2$. Let $\overline{H}_i$ be the effective Hamiltonian corresponding to H_i. Assume that $\overline{H}_1 = \overline{H}_2$. What are the relations between a_1 and a_2?*

In particular, if $\overline{H}_1(p) = |p|$ for all $p \in \mathbb{R}^n$, can we deduce that $a_1 \equiv 1$ for $n \geq 3$? When $n = 2$, this follows from a stronger result of Bangert [6] *which relies on clear understanding of the structure of Aubry-Mather sets in two dimensional space.*

Remark 3.1. *One important point is that homogenization is equivalent to large time average. In light of this, the latter part of Question 3.1 can be reformulated*

in term of geometry as following. Let $d(\cdot,\cdot):\mathbb{R}^n\times\mathbb{R}^n\to[0,\infty)$ *be a Riemannian distance defined as: For* $x,y\in\mathbb{R}^n$,

$$d(x,y):=\min_{\xi\in C_{x,y}}\int_0^1\frac{|\dot\xi(s)|}{a_1(\xi(s))}\,ds,$$

where $C_{x,y}$ *is the set of all Lipschitz continuous curves* $\xi:[0,1]\to\mathbb{R}^n$ *such that* $\xi(0)=x$ *and* $\xi(1)=y$. *Then the stable norm corresponding to* d *is given by*

$$\|x\|_S:=\lim_{\varepsilon\to0}\varepsilon d\left(0,\frac{x}{\varepsilon}\right)=\lim_{t\to\infty}\frac{d(0,tx)}{t}\qquad\text{for } x\in\mathbb{R}^n.$$

If we know that the stable norm $\|\cdot\|_S$ *is same as the Euclidean norm in* $\mathbb{R}^n$, *then can we conclude that* $a_1\equiv1$*?*

Burago, Ivanov and Kleiner [7] *addressed a similar inverse problem and studied the differentiability of the stable norm* $\|\cdot\|_S$. *Auer and Bangert* [4] *showed that the stable norm is differentiable at all "totally irrational" directions for* $n\geq2$.

As of now, the second-order equation case with the presence of a possibly degenerate diffusion term is completely open. Here, one needs to take into account the interaction between the Hamiltonian and the diffusion, and clearly this is the critical case since the scalings of two terms are the same (of order 2). I expect that a much complicated picture appears in this question.

Question 3.2. *Assume that* $H_i(p,x)=|p|^2/2+V_i(x)$, *where* $V_i\in C(\mathbb{T}^n)$ *are given potential energies for* $i=1,2$, *and* $A=A(x)$ *is a given* $n\times n$ *matrix, which is* $\mathbb{Z}^n$*-periodic and non-negative definite. Let* $\overline{H}_i$ *be the corresponding effective Hamiltonian of*

$$\frac{1}{2}|P+Dv_i|^2+V_i=\overline{H}_i(P)+\operatorname{tr}(A(x)D^2v_i)\qquad\text{in }\mathbb{T}^n.$$

Assume that $\overline{H}_1=\overline{H}_2$. *What are the relations between* V_1 *and* V_2*?*

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Exponential stability of a traveling wave for an area preserving curvature motion

Takashi Kagaya *

1 Introduction

In this talk, we deal with the motion by area preserving curvature flow of planar curves having two moving end points on the x-axis with fixed interior contact angles to this axis. The problem is formulated as follows. For any curve Γ lying on the upper half-plane having two end points, we shall denote the left and right interior angles by $\mathrm{Ang}_-(\Gamma)$ and $\mathrm{Ang}_+(\Gamma)$. For any given such curve $\Gamma^0 = \Gamma(0)$, our problem is to look for a family of curves $\{\Gamma(t)\}_{t\geq 0}$ having two end points on the x-axis with the same fixed contact angles as above evolve by the area preserving curvature flow equation:

$$V = \kappa - \frac{\int_{\Gamma(t)} \kappa\, ds}{\int_{\Gamma(t)} ds} = \kappa + \frac{\psi_+ + \psi_-}{L(t)}, \qquad \mathrm{Ang}_\pm(\Gamma(t)) = \psi_\pm. \tag{1.1}$$

Here V is the outward normal velocity of the curve, $L(t)$ is the length of $\Gamma(t)$, κ is the (signed) curvature of $\Gamma(t)$, s is the arc-length parameter of $\Gamma(t)$ and $\psi_\pm \in (0, \pi/2)$ are constants. If the solution curve is represented as a graph, i.e., $\Gamma(t) = \{(x, u(x,t)); l_-(t) \leq x \leq l_+(t)\}$ for some functions u and $l_\pm$, the problem is reduced to the following free boundary problem:

$$\begin{cases} u_t = \dfrac{u_{xx}}{1+u_x^2} + \dfrac{\psi_+ + \psi_-}{L(t)}\sqrt{1+u_x^2}, & x \in (l_-(t), l_+(t)),\ t > 0, \\ u(l_\pm(t), t) = 0, & t > 0, \\ u_x(l_\pm(t), t) = \mp \tan\psi_\pm, & t > 0, \\ u(x,0) = u^0(x),\ x \in [l_-^0, l_+^0], \quad l_\pm(0) = l_\pm^0, \end{cases} \tag{1.2}$$

where $L(t) = \int_{l_-(t)}^{l_+(t)} \sqrt{1+u_x^2(x,t)}\, dx$ and we assume that

$$\psi_\pm \in (0, \pi/2), \qquad -\infty < l_-^0 < l_+^0 < \infty. \tag{1.3}$$

The purpose of this talk is to study the asymptotic behavior of the solution as $t \to \infty$ for the problem (1.2). Throughout this talk, we also assume

$$\begin{aligned} &u^0 \in C^2([l_-^0, l_+^0]), \qquad u^0(l_\pm^0) = 0, \\ &u_x^0(l_\pm^0) = \mp \tan\psi_\pm, \qquad (u^0)_{xx} < 0 \ \text{ on } \ [l_-^0, l_+^0]. \end{aligned} \tag{1.4}$$

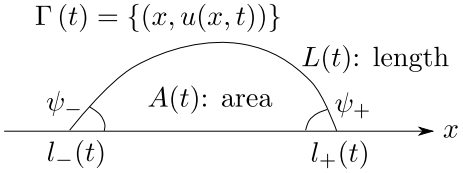

Figure 1: The graph of $u(\cdot, t)$

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As is easily seen, a direct calculation yields

$$A'(t) = 0, \quad E'(t) \le 0, \tag{1.5}$$

where $A(t) = \int_{l_-(t)}^{l_+(t)} u(x,t)\,dx$ (the area enclosed by the graph $\Gamma(t)$ and x-axis) and

$$E(t) = L(t) - l_+(t)\cos\psi_+ + l_-(t)\cos\psi_-. \tag{1.6}$$

The reason is that the problem (1.1) is the gradient flow of $\hat{E}(\Gamma)$ on the space of smooth plane curves Γ under the restriction

$$A(\Gamma) = m \tag{1.7}$$

with fixed constant $m > 0$ as in Gage's paper [2], where Γ is a curve lying on the upper half-plane having two end points $(l_-(\Gamma), 0)$ and $(l_+(\Gamma), 0)$, $A(\Gamma)$ is the area enclosed by Γ and x-axis,

$$\hat{E}(\Gamma) = L(\Gamma) - l_+(\Gamma)\cos\psi_+ + l_-(\Gamma)\cos\psi_- \tag{1.8}$$

with fixed constants $\psi_\pm \in (0, \pi/2)$ and $L(\Gamma)$ is the length of the curve Γ. We remark that we don't assume $\mathrm{Ang}_\pm(\Gamma) = \psi_\pm$, however the contact angle condition $\mathrm{Ang}_\pm(\Gamma(t)) = \psi_\pm$ is required to derive the gradient flow. When the constants $\psi_\pm$ are equal, by calculating the first variation of $\hat{E}(\Gamma)$, we see the set of minimizers for $\hat{E}(\Gamma)$ is consists of arcs whose the left and right interior angles are $\psi := \psi_\pm$ (cf. [4]). Hence we may expect the minimizers are stationary solutions of (1.1) and minimizers have some stability in the case of $\psi_+ = \psi_-$. On the other hand, in the case of $\psi_+ \neq \psi_-$, if a minimizer of $\hat{E}(\Gamma)$ exists the first variation of $\hat{E}(\Gamma)$ implies that the minimizer should be an arc whose the left and right interior angles are ψ_- and ψ_+, respectively. However, the two angles formed by a circle and a line at two intersection points should be equal, thus we have no existence of minimizers of $\hat{E}(\Gamma)$ and we are interested in what happens in the case of $\psi_+ \neq \psi_-$.

In order to study the asymptotic behavior of solutions for (1.2), we state the following global existence and boundedness of the solutions.

Theorem 1.1 (Globally existence [3]) *Assume $(u_0, l_\pm^0)$ satisfies* (1.3) *and* (1.4). *Then the smooth solution of* (1.2) *exists uniquely and globally in time. Furthermore, $u(\cdot, t)$ preserve the concavity and $L(t)$ remains bounded from above and below by two positive constants as $t \to \infty$.*

2 Traveling waves

A traveling wave of the problem (1.2) is a solution that has the form $u^*(x,t) = \mathcal{U}(x - ct - a)$, where c denotes the wave speed, $\mathcal{U}(\xi)$ is called the profile of the wave and a is an arbitrary constant that adjusts the phase. Substituting this form into (1.2), we obtain

$$\begin{cases} \dfrac{\mathcal{U}_{\xi\xi}}{1+\mathcal{U}_\xi^2} + c\mathcal{U}_\xi + \dfrac{\psi_+ + \psi_-}{L^*}\sqrt{1+\mathcal{U}_\xi^2} = 0 \quad \text{in } (-b, b), \\ \mathcal{U}(\pm b) = 0, \quad \mathcal{U}_\xi(\pm b) = \mp\tan\psi_\pm, \end{cases} \tag{2.1}$$

where $L^* = \int_{-b}^{b} \sqrt{1+\mathcal{U}_\xi^2(\xi)}\,d\xi$. Here $b > 0$ is some constant that shift the center of the support of $\mathcal{U}$ to the origin.

The following theorem is the existence of traveling waves of the problem (1.2).

Theorem 2.1 (Existence of traveling waves [3]) *For any given $\psi_\pm \in (0, \pi/2)$ and $L^* > 0$, there exist unique constants $b > 0$, $c \in \mathbb{R}$ and a unique concave function $\mathcal{U}(\xi)$ that satisfy (2.1). Furthermore, c satisfies*

$$c \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0 \quad \textit{if and only if} \quad \psi_- \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} \psi_+,$$

and the correspondence between A^* and L^* is one-to-one, where A^* is the area enclosed by the graph of the solution $\mathcal{U}$ and x-axis.

For the proof of Theorem 2.1, a key tool is the equation for the curvature of the curves $\Gamma(t) = \{(x, u(x,t)); l_-(t) \le x \le l_+(t)\}$. Define a new parameter $\theta(x,t) = \arctan u_x(x,t)$ and the curvature of $\Gamma(t)$ is represented by $\kappa(\theta, t) = u_{xx}/(1+u_x^2)^{3/2}$. We note that the parameter $\theta(x,t)$ is well-difined since $u(\cdot, t)$ preserve the concavity, and we have the following equation:

$$\begin{cases} \kappa_t = \kappa^2 \left(\kappa_{\theta\theta} + \kappa + \dfrac{\psi_+ + \psi_-}{L(t)} \right), & -\psi_+ < \theta < \psi_+, \quad t > 0, \\ \kappa_\theta = \cot\theta \left(\kappa + \dfrac{\psi_+ + \psi_-}{L(t)} \right), & \theta = \mp\psi_\pm, \quad t > 0, \\ \kappa(\theta, 0) = \kappa_0(\theta), & -\psi_+ < \theta < \psi_-, \end{cases} \tag{2.2}$$

where

$$L(t) = \int_{-\psi_+}^{\psi_-} -\frac{d\theta}{\kappa(\theta, t)}\, d\theta \tag{2.3}$$

and the initial data κ_0 satisfies the following additional condition:

$$\int_{-\psi_+}^{\psi_-} \frac{\sin\theta}{\kappa_0(\theta)}\, d\theta = 0. \tag{2.4}$$

On the other hand, if $\kappa(\theta, t)$ is a solution of (2.2) with the initial condition (2.4), the functions

$$l_\pm(t) = l_\pm^0 \pm \int_0^t \frac{1}{\sin\psi_\pm} \left(\kappa(\mp\psi_\pm, \tilde{t}) + \frac{\psi_+ + \psi_-}{L(\tilde{t})} \right) d\tilde{t}, \quad u(x(\theta,t),t) = -\int_\theta^{\psi_-} \frac{\sin\tilde{\theta}}{\kappa(\tilde{\theta}, t)}\, d\tilde{\theta} \tag{2.5}$$

are the solution of (1.2) with the following initial conditions and variable x:

$$x(\theta,t) = l_-(t) - \int_\theta^{\psi_-} \frac{\cos\tilde{\theta}}{\kappa(\tilde{\theta}, t)}\, d\tilde{\theta}, \quad u_0(x) = u(x(\theta,0),0), \quad l_+^0 = x(\psi_+, 0) \tag{2.6}$$

and l_-^0 is an arbitrary number. From the composition of u in (2.5), we may see that the initial condition (2.4) is equivalent to the boundary condition $u(l_\pm^0, 0) = 0$. We also remark that the condition $u(l_\pm(t), t) = 0$ is preserved since the condition $\int_{-\psi_+}^{\psi_-} \sin\theta/\kappa(\theta,t)\, d\theta = 0$ is preserved for any solution κ of (2.2) with the initial condition (2.4). Now, we consider the curvature κ^* of the graph of $u^*(x,t) = \mathcal{U}(x - ct - a)$. By a simple calculation we can see the curvature κ^* is independent of the time t, hence κ^* should be a stationary solution of (2.2) and satisfies the condition (2.4). Thus our aim is to obtain stationary solutions of (2.2) and we can prove the unique stationary solution is

$$\kappa^*(\theta; L^*) = -c(L^*)\sin\theta - \frac{\psi_+ + \psi_-}{L^*} \tag{2.7}$$

for any $L^* > 0$, where $c(L^*)$ is an unique constant which depends on L^* and the stationary solution κ^* satisfies $L^* = -\int_{-\psi_+}^{\psi_-} 1/\kappa^*(\theta)\ d\theta$. Furthermore, can see $c(L^*)$ satisfies (2.1) and κ^* satisfies $\kappa^* > 0$ for any $\theta \in [-\psi_+, \psi_-]$. By applying the composition (2.5) and (2.6), we obtain the existence of the traveling wave $\mathcal{U}(\cdot; L^*)$. Next, we consider the correspondence between L^* and A^*. Let $A(\mathcal{U}(\cdot; L^*))$ be the area enclosed by the graph of $\mathcal{U}(\cdot; L^*)$. Then we see $A(\mathcal{U}(\cdot; \lambda L^*)) = \lambda^2 A(\mathcal{U}(\cdot; L^*))$ for $\lambda > 0$ and hence Theorem 2.1 holds.

3 Exponential stability of the traveling wave

We proved the existence of the traveling waves in Section 2. In this section, we study the asymptotic behavior for the solution of (1.2). In suitable conditions, the solution converges to a traveling wave. The following theorem state the details.

Theorem 3.1 (Asymptotic shape [3]) *Let $(u, l_\pm)$ be a solution of* (1.2) *with the initial conditions* (1.3) *and* (1.4)*, and let $\mathcal{U}(\xi)$ be a solution of* (2.1) *with*

$$\int_{l_-^0}^{l_+^0} u_0(x)\,dx = \int_{-b}^{b} \mathcal{U}(\xi)\,d\xi.$$

Let $\kappa(\theta, t) := u_{xx}/(1+u_x^2)^{3/2}$ be the curvature of the solution curve, where $\theta(x,t) = \arctan u_x(x,t)$. Let $\kappa^*(\theta) := \mathcal{U}_{\xi\xi}/(1+\mathcal{U}_\xi^2)^{3/2}$ with $\theta(\xi) = \arctan \mathcal{U}_\xi(\xi)$. Then there exists an absolute constant $\varepsilon_0 > 0$ such that if u satisfies the condition

$$\|\kappa(\cdot, t) - \kappa^*\|_{L^\infty([-\psi_+, \psi_-])} < \varepsilon_0, \tag{3.1}$$

there exists a constant $a \in \mathbb{R}^n$ such that

$$l_\pm(t) - ct \to a \pm b, \quad \|u(\cdot, t) - \mathcal{U}(\cdot - ct - a)\|_{L^\infty(\mathbb{R}^n)} \to 0 \tag{3.2}$$

exponentially as $t \to \infty$. Here we regard that $u \equiv 0$ outside the interval $[l_-(t), l_+(t)]$ and $\mathcal{U} \equiv 0$ outside the interval $[-b, b]$.

Hereafter, we fix the curvature κ^* defined in Theorem 3.1. According to the composition (2.5) and (2.6), if we prove a convergence $\kappa \to \kappa^*$ exponentially and uniformly as $t \to \infty$, then Theorem 3.1 follows from the exponential convergence of κ. In order to prove the exponential convergence for the curvature, we want to linearize (2.2) around the stationary solution κ^* on some space with the restriction (2.4), however the condition (2.4) is nonlinear condition. There we expand the equation (2.2) as

$$\begin{cases} \kappa_t = \kappa^2 \left(\kappa_{\theta\theta} + \kappa + \dfrac{\psi_+ + \psi_-}{L(t)} - \dfrac{l_+'(t)}{L(t)} H \right), & -\psi_+ < \theta < \psi_+, \quad t > 0, \\ \kappa_\theta = \cot\theta \left(\kappa + \dfrac{\psi_+ + \psi_-}{L(t)} - \dfrac{l_+'(t)}{L(t)} H \right), & \theta = \mp\psi_\pm, \quad t > 0, \\ \kappa(\theta, 0) = \kappa_0(\theta), & -\psi_+ < \theta < \psi_-, \end{cases} \tag{3.3}$$

where $L(t)$ and $l_+(t)$ are defined by (2.3) and (2.5), respective, and H is defined by

$$H = H(0), \quad H(t) = -\int_{-\psi_+}^{\psi_-} \frac{\sin\theta}{\kappa(\theta, t)}\,dt.$$

We remark that $H(t)$ is equivalent to the hight of the graph of $u(x,t)$ at the right end point (see Figure 2), where $u(x,t)$ is a function composed by the formula (2.5) and (2.6). If κ_0 satisfies (2.4), the equation (3.3) coincide with the equation (2.2), thus (3.3) is expanded equation of (2.2) for any initial condition of κ_0 and it is obviously that (2.7) is a stationary solution of (3.3). Furthermore, the solutions of (3.3) have the two preserving value $A(t)$ and $H(t)$, where $A(t)$ is the area enclosed by the graph of the function $u(x,t)$ composed by (2.5) and (2.6), x-axis and the line $\{x = l_+(t)\}$. By using such expansion, we will linearize the equation (3.3) around the stationary solution κ^* to prove the exponentially convergence of κ.

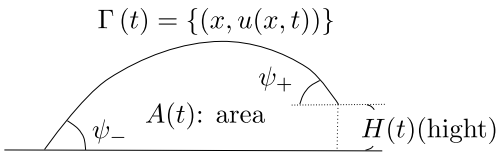

Figure 2: The image of $H(t)$

In order to linearize (3.3), we define the following weighted L^2 space:

$$L^2_* := \{v \in L^2[-\psi_+, \psi_-]; \|v\|_* < \infty\}$$

equipped with the inner product

$$\langle v_1, v_2\rangle_* := \int_{-\psi_+}^{\psi_-} v_1(\theta)v_2(\theta)\,\frac{d\theta}{(\kappa^*(\theta))^2}.$$

The linearized operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset L^2_* \to L^2_*$ of (3.3) around the stationary solution κ^* on the space L^2_* is formed by

$$\mathcal{L}(v) = (\kappa^*)^2\left(v_{\theta\theta} + v + \frac{1}{L^*}\int_{-\psi_+}^{\psi_-}\frac{v}{\kappa^*}\,d\theta\right),$$
$$\mathcal{D}(\mathcal{L}) = \left\{v \in H^2_*; v_\theta = \cot\theta\left(v + \frac{1}{L^*}\int_{-\psi_+}^{\psi_-}\frac{v}{\kappa^*}\,d\theta\right) \text{ at } \theta = \mp\psi_\pm\right\},$$

where $L^* = -\int_{-\psi_+}^{\psi_-} 1/\kappa^*(\theta)\,d\theta$ and H^2_* is the associated Sobolev space of L^2_*. By using the similar argument as [1, Lemma 2], we can show the operator $\mathcal{L}$ is sectorial and the spectrum consists of infinite countably many eigenvalues with finite multiplicities. According to Prüss, Simonett and Zacher [5, Theorem2.1], if the conditions

(i) the set of all stationary solution $\mathcal{E}$ of (3.3) is a finite dimensional smooth manifold in H^2_*,

(ii) the tangent space $\mathrm{Tan}_{\kappa^*}\mathcal{E}$ of $\mathcal{E}$ at κ^* is $\mathrm{Ker}(\mathcal{L})$, and the eigenvalue 0 is semisimple,

(iii) $\sigma(\mathcal{L}) \setminus \{0\} \subset \{z \in \mathbb{R} : z < 0\}$, where $\sigma(\mathcal{L})$ is the spectrum of $\mathcal{L}$

hold, then there exists a small constant ε_0 such that the solution κ of (3.3) converges some element of $\mathcal{E}$ exponentially and uniformly as $t \to \infty$ whenever the condition (3.1) holds. The unique stationary solution $\kappa^*(\cdot; \tilde{L}, \tilde{H})$ of (3.3) exists depending on $\tilde{L} > 0$ and $\tilde{H} \in \mathbb{R}$, namely, for any $\tilde{L} > 0$ and $\tilde{H} \in \mathbb{R}$ there exists a unique stationary solution $\kappa^*(\cdot; \tilde{L}, \tilde{H})$ of (3.3) such that

$$\kappa^*(\theta; \tilde{L}, \tilde{H}) = -c(\tilde{L}, \tilde{H})\sin\theta - \frac{\psi_+ + \psi_-}{\tilde{L}} + \frac{c(\tilde{L}, \tilde{H})}{\tilde{L}}\tilde{H},$$

where $c(\tilde{L}, \tilde{H})$ is a unique constant depending only on $\tilde{L}$ and $\tilde{H}$ and the stationary solution $\kappa^*(\cdot; \tilde{L}, \tilde{H})$ satisfies

$$\tilde{L} = -\int_{-\psi_+}^{\psi_-}\frac{1}{\kappa^*(\theta; \tilde{L}, \tilde{H})}\,d\theta, \quad \tilde{H} = -\int_{-\psi_+}^{\psi_-}\frac{\sin\theta}{\kappa^*(\theta; \tilde{L}, \tilde{H})}\,d\theta.$$

Hence we see the condition (i) holds since if we regard $c(\tilde{L}, \tilde{H})$ is function with respect to $\tilde{L} > 0$ and $\tilde{H} \in \mathbb{R}$, we can prove the function c is smooth. By a simple calculation, we can prove the tangent space $\mathrm{Tan}_{\kappa^*}\mathcal{E}$ is

$$\mathrm{Tan}_{\kappa^*}\mathcal{E} = \mathrm{span}\{1, \sin\theta\}. \tag{3.4}$$

Here we consider the adjoint operator $\mathcal{L}^* : \mathcal{D}(\mathcal{L}^*) \subset L^2_* \to L^2_*$ of the operator $\mathcal{L}$ to analyze the spectrum of $\mathcal{L}$. The adjoint operator $\mathcal{L}^*$ is formed by

$$\mathcal{L}^*(w)(\theta) = (\kappa^*)^2(\theta)\left(w_{\theta\theta}(\theta) + w(\theta)\right) + \frac{\kappa^*}{L^*}\left(\int_{-\psi_+}^{\psi_-} w(\theta)\,d\theta + w(\psi_-)\cot\psi_- + w(-\psi_+)\cot\psi_+\right),$$
$$\mathcal{D}(\mathcal{L}^*) = \left\{w \in H^2_*; v_\theta = w(\theta)\cot\theta \text{ at } \theta = \mp\psi_\pm\right\}.$$

This operator $\mathcal{L}^*$ seems to be easier to handle than the operator $\mathcal{L}$, since the boundary condition is not nonlocal. By regarding the operator $\mathcal{L}^*$ as a perturbation of the self adjoint operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2_* \to L^2_*$,

$$\mathcal{A}(\phi) = (\kappa^*)^2(\phi_{\theta\theta} + \phi), \quad \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{L}^*),$$

we can prove the spectrum $\sigma(\mathcal{L}^*)$ of the operator $\mathcal{L}^*$ consists of infinite countably many eigenvalues with finite multiplicities and

$$\sigma(\mathcal{L}^*) = \{\lambda_i \in \mathbb{R}; 0 = \lambda_1 < \lambda_2 < \cdots\}, \quad \mathrm{Ker}(\mathcal{L}^*) = \mathrm{span}\{S^*, \sin\theta\},$$

where

$$S^*(\theta) = -\sin\theta\int_{\psi_-}^{\theta}\frac{\cos\theta}{\kappa^*}\,d\theta + \cos\theta\int_{\psi_-}^{\theta}\frac{\sin\theta}{\kappa^*}\,d\theta$$

is the support function of the traveling wave whose curvature is κ^* and whose center is the left end point of the traveling wave. Since the operator $\mathcal{L}$ is the adjoint operator of $\mathcal{L}^*$, we obtain

$$\sigma(\mathcal{L}) = \{\lambda_i \in \mathbb{R}; 0 = \lambda_1 < \lambda_2 < \cdots\}, \quad \mathrm{Ker}(\mathcal{L}) = \mathrm{span}\{1, \sin\theta\}$$

and the eigenvalue 0 is semisimple, hence we conclude that the conditions (ii) and (iii) hold by combining (3.4). Applying [5, Theorem2.1], for the solution $(u, l_\pm)$ in Theorem 3.1 satisfying the condition (3.1) there exists a stationary solution $\kappa^*(\cdot; \tilde{L}, \tilde{H}) \in \mathcal{E}$ such that

$$\|\kappa(\cdot, t) - \kappa^*(\cdot; \tilde{L}, \tilde{H})\|_{L^\infty([-\psi_+, \psi_-])} \to 0$$

exponentially as $t \to \infty$, where κ is defined as in Theorem 3.1. Since the correspondence between the area A^* and the length L^* is one-to-one as in Theorem 2.1 and the equation (3.3) has the two preserving value $A(t)$ and $H(t)$, we can prove $\tilde{L} = L^*$ and $\tilde{H} = 0$, and it yields $\kappa^*(\cdot; \tilde{L}, \tilde{H}) = \kappa^*$. Hence, combining the property $\kappa^* > 0$ on $[-\psi_+, \psi_-]$ and representation (2.5) and (2.6), we have

$$l'_\pm(t) - c = \frac{\pm 1}{\sin\psi_\pm}\left\{(\kappa(\theta, t) - \kappa^*(\theta)) + \left(\frac{\psi_+ + \psi_-}{\int_{-\psi_+}^{\psi_-}\frac{d\theta}{\kappa^*(\theta)}} - \frac{\psi_+ + \psi_-}{\int_{-\psi_+}^{\psi_-}\frac{d\theta}{\kappa(\theta,t)}}\right)\right\} \to 0$$

exponentially as $t \to \infty$. Thus we can define a constant a as

$$a = l_-(0) + \int_0^\infty l'_-(t) - c\,dt + b$$

and we have the exponential convergence (3.2) for this constant a by combining the property $\kappa^* > 0$ on $[-\psi_+, \psi_-]$ and representation (2.5) and (2.6) again.

4 Remarks

We remark the following relation between traveling wave and the energies E and $\hat{E}$ defined by (1.6) and (1.8). Theorem 2.1 state $c = 0$ when $\psi_+ = \psi_-$, and it yields the graph $\Gamma^* = \{(x, \mathcal{U}(x)); -b \leq x \leq b\}$ is a stationary solution of (1.1). Furthermore, the curvature of Γ^* is a constant from the expression of the curvature (2.7). This implies the plane curve Γ^* is an arc and $\mathrm{Ang}_+(\Gamma^*) = \mathrm{Ang}_-(\Gamma^*) = \psi_\pm$, which is a minimizer of $\hat{E}(\Gamma)$ (see Section 1). Hence Theorem 3.1 state the expected conclusion in Section 1, the exponential stability of the minimizers for $\hat{E}(\Gamma)$. On the other hand, we showed no existence of minimizers for $\hat{E}(\Gamma)$ in Section 1 when $\psi_+ \neq \psi_-$. In this case, we showed the existence of the traveling waves of (1.2) and the exponential stability of the traveling waves. Let $(u^*, l^*_\pm)$ be a solution such that $u^*(x,t) = \mathcal{U}(x - ct - a)$ and $l^*_\pm(t) = \pm b + ct + a$ for some $a \in \mathbb{R}$, and let $E^*(t)$ be the energy associated to the solution $(u^*, l^*_\pm)$ as in (1.6). Then we have the strictly decreasing of $E^*(t)$ with respect to $t > 0$ and $E^*(t) \to -\infty$ as $t \to \infty$ by a simple calculation. Hence the energy (1.6) is not bounded below.

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ON LOW FREQUENCY ASYMPTOTICS FOR DISSIPATIVE WAVE EQUATIONS

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Abstract. We discuss recent results on the asymptotic behavior of dissipative wave equations obtained in collaboration with P. Radu and G. Todorova and published in "The generalized diffusion phenomenon and applications," SIAM J. Math. Analysis 48.1 (2016): 174–203. The interesting phenomenon is that solutions of such equations approach certain solutions of diffusion equations as time goes to infinity. This observation allows us to transfer sharp estimates from the latter to the former relying on the Markov property and Gagliardo-Nirenberg or Nash-type inequalities available for diffusion equations.

1. Introduction

The propagation of waves in dissipative elastic media is commonly described by hyperbolic equations with damping and suitable boundary conditions, such as

$$\begin{aligned} c(x)\partial_t^2 u - \nabla\cdot b(x)\nabla u + a(x)\partial_t u = 0 \qquad & x\in\Omega, \quad t>0, \\ u = 0 \qquad & x\in\ \partial\Omega, \quad t\geq 0, \\ u = u_0, \quad \partial_t u = u_1 \qquad & x\in\Omega, \quad t=0, \end{aligned} \tag{1.1}$$

where $u(x,t)$ is the unknown displacement at position x and time t. In problems of geophysics [16] and acoustics [17], the equation can be studied in an exterior domain $\Omega\subseteq\mathbb{R}^n$ and the above Dirichlet boundary condition is set on its smooth boundary $\partial\Omega$. Typical assumptions are also $a,\ c\in C(\Omega)$ and $b_{ij}\in C^1(\Omega)$, where

$$\begin{gathered} a_0\leq a(x)\leq a_1, \\ \forall\xi\in\mathbb{R}^n \quad b_0|\xi|^2\leq b(x)\xi\cdot\xi\leq b_1|\xi|^2, \\ c_0\leq c(x)\leq c_1, \end{gathered} \tag{1.2}$$

with positive $a_i,\ b_i,\ c_i$ for $i=0,1$. The initial values $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$.

We are interested in the long time behavior of solutions and, in particular, the decay rates of

$$\|u(t)\|_2^2 \ \text{ and } \ E(t;u)=\frac{1}{2}\left(\|c^{1/2}\partial_t u(t)\|_2^2+\|(b^{1/2}\nabla u(t)\|_2^2\right),$$

as $t\to\infty$. This problem has been studied extensively by the multiplier method in Mochizuki and Nakazawa [8], Mochizuki and Nakao [7], Nakao [9, 10], Ikehata [4, 5] and the references therein. The energy decay rate has been found to be of order $(1+t)^{-1}$, except for [5] where R. Ikehata has established the faster rate $(1+t)^{-2}$ for some types of initial data in weighted spaces. It is important that our assumptions on the elliptic operator are very general, so we do not require it to stabilize to the Laplacian at large $|x|$. In the latter case, classical arguments can be used; see Racke [11] where the generalized Fourier transform was used, and Dan and Shibata [3] for a low-frequency resolvent expansion. More references for results concerning decay

rates of solutions of damped wave equations can be found in the introductions of the papers [12, 13, 14, 15].

Let us introduce $\mathcal{H} = L^2(\Omega, a(x)dx)$ and consider the self-adjoint operator B defined by

$$B = -\frac{1}{a(x)}\nabla \cdot b(x)\nabla, \quad \mathcal{D}(B) = H^2(\Omega, a(x)dx) \cap H_0^1(\Omega, a(x)dx). \tag{1.3}$$

Of course, condition (1.2) on a implies that these weighted spaces coincide with the usual Sobolev spaces.

Theorem 1.1. *Assume that (1.2) hold and $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ are compactly supported. Let $(u, \partial_t u) \in C(\mathbb{R}_+, H_0^1(\Omega) \times L^2(\Omega))$ be the unique weak solution to problem (1.1) and B be the self-adjoint operator defined through (1.3) and the Dirichlet boundary condition. If $q \in (1, 2]$, then*

$$\begin{aligned} &\|u(t) - e^{-tB}(u_0 + cu_1/a)\|_2 \\ &\lesssim (\|u_0\|_2 + \|\nabla u_0\|_2 + \|u_1\|_2 + \|u_0\|_q + \|u_1\|_q)(t+1)^{-n/2(1/q-1/2)-1}. \end{aligned} \tag{1.4}$$

As a consequence, u satisfies the Matsumura [6] *decay estimate*

$$\begin{aligned} \|u(t)\|_2 \lesssim &(\|u_0\|_2 + \|\nabla u_0\|_2 + \|u_1\|_2 + \|u_0\|_q + \|u_1\|_q)(t+1)^{-n/2(1/q-1/2)-1} \\ &+ (\|u_0\|_1 + \|u_1\|_1 + \|u_0\|_2 + \|u_1\|_2)(t+1)^{-n/4}. \end{aligned} \tag{1.5}$$

We will derive estimates for $u(x,t)$ from the "diffusion phenomenon" which means that we will first show the difference between u and its diffusion approximation $e^{-tB}(u_0 + cu_1/a)$ to be relatively small in $\mathcal{H}$. Such result hold in any Hilbert spaces, so an abstract framework is more convenient.

2. Abstract Setting

Let us restate (1.1) as the following abstract initial value problem on $(0, \infty)$:

$$C\partial_t^2 u + \partial_t u + Bu = 0, \quad u(0) = u_0, \quad \partial_t u(0) = u_1, \tag{2.1}$$

where $(u_0, u_1) \in \mathcal{D}(\sqrt{B}) \times \mathcal{H}$. The self-adjoint operators B and C satisfy

- $(H1)$ $\mathcal{D}(B)$ is dense in $\mathcal{H}$ and C is a bounded operator on $\mathcal{H}$;
- $(H2)$ $\langle Bu, u\rangle > 0$ for $u \in \mathcal{D}(B)$ and $u \neq 0$;
- $(H3)$ $c_1\|u\|^2 \geq \langle Cu, u\rangle \geq c_0\|u\|^2$ for $u \in \mathcal{H}$, where $c_1 \geq c_0 > 0$.

We work in a real Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. The above assumptions not only guarantee the existence and uniqueness of mild solutions $(u, \partial_t u) \in C(\mathbb{R}_+, \mathcal{D}(\sqrt{B}) \times \mathcal{H})$, but also imply the estimate

$$\|\sqrt{C}\partial_t u(t)\|^2 + \|\sqrt{B}u(t)\|^2 \leq \|\sqrt{C}u_1\|^2 + \|\sqrt{B}u_0\|^2, \quad t \geq 0.$$

In fact, the proof is a simple application of the Lumer-Phillips theorem.

3. Diffusion Approximation

The common idea behind all proofs of diffusion phenomenon is to treat $C\partial_t^2 u$ as a decaying source in $\partial_t u + Bu = -C\partial_t^2 u$. Solving for u,

$$u(t) = e^{-tB}u_0 - \int_0^t e^{-(t-s)B}C\partial_s^2 u(s)\, ds. \tag{3.1}$$

However, the suggested approximation $u(t) \approx e^{-tB}u_0$ is incorrect. There are also losses of regularity due to estimating u by its second-order derivative $\partial_t^2 u$.

It turns out that the correct approximation procedure is far from simple.

3.1. **Mollified problem.** Let $\chi \in C_0^\infty([0,1])$ and assume $\partial_t^i \chi(0) = 0$ for all i,

$$\chi \geq 0, \quad \|\chi\|_{L^1} = 1.$$

We define the convolution of χ and solutions u of (2.1) as

$$\tilde{u}(t) = \int_0^t \chi(s) u(t-s)\, ds, \quad t \geq 0. \tag{3.2}$$

Clearly $u \mapsto \tilde{u}$ is a linear operator $L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathcal{H}) \to C^\infty(\mathbb{R}_+, \mathcal{H})$ and $\partial_t^i \tilde{u}(0) = 0$ for all i. These two conditions are crucial for the diffusion approximation which is constructed inductively and requires C^∞ regularity and zero data at each step.

Then the initial value problem (2.1) for $u(t)$ transforms into

$$C\partial_t^2 \tilde{u} + \partial_t \tilde{u} + B\tilde{u} = f, \quad \partial_t^j \tilde{u}(0) = 0, \quad j = 0, 1, \dots \tag{3.3}$$

with

$$f(t) = \chi(t) C u_1 + \partial_t \chi(t) C u_0 + \chi(t) u_0. \tag{3.4}$$

Here $f \in C_0^\infty([0,1], \mathcal{H})$, so $u(t)$ and $\tilde{u}(t)$ satisfy homogeneous equations at $t > 1$ but different initial conditions at $t = 0$. Of course, it remains to be seen that $\{u(t), \partial_t u(t), \sqrt{B}u(t)\}$ and $\{\tilde{u}(t), \partial_t \tilde{u}(t), \sqrt{B}\tilde{u}(t)\}$ have similar long time behaviors.

We represent $\tilde{u}(t)$ as the sum of $k+1$ solutions of diffusion equations and a remainder: $\tilde{u}(t) = u^0(t) + u^1(t) + \cdots + u^k(t) + r^k(t)$, where the sequence of functions is determined from

$$\begin{aligned}
\partial_t u^0 + B u^0 &= f, \\
\partial_t u^1 + B u^1 &= -C\partial_t^2 u^0, \\
&\vdots \\
\partial_t u^k + B u^k &= -C\partial_t^2 u^{k-1}, \\
C\partial_t^2 r^k + \partial_t r^k + B r^k &= -C\partial_t^2 u^k,
\end{aligned} \tag{3.5}$$

with f defined in (3.4). All functions vanish at $t = 0$ together with all derivatives:

$$\partial_t^i u^j(0) = \partial_t^i r^k(0) = 0 \quad \text{for} \quad j = 0, 1, \dots, k \quad \text{and} \quad i = 0, 1, \dots$$

Adding together equations (3.5), we have the decomposition

$$\tilde{u}(t) = u^k_{\mathrm{dif}}(t) + r^k(t), \quad u^k_{\mathrm{dif}}(t) = u^0(t) + u^1(t) + \cdots + u^k(t).$$

3.2. **Resolvents.** Assume that $g \in C^\infty(\mathbb{R}_+, \mathcal{H})$ and $g(t) = 0$ for $t \leq 0$. The resolvents $g \mapsto R_w g$ and $g \mapsto R_d g$ are given by the unique mild solutions of

$$\text{(wave)} \qquad C\partial_t^2 v + \partial_t v + Bv = g, \quad v(t) = 0 \ \text{ for } t \leq 0, \tag{3.6}$$

$$\text{(diffusion)} \qquad \partial_t v + Bv = g, \quad v(t) = 0 \ \text{ for } t \leq 0, \tag{3.7}$$

respectively. The two resolvents commute with all derivatives,

$$\partial_t^i R_w = R_w \partial_t^i, \quad \partial_t^i R_d = R_d \partial_t^i, \quad i = 1, 2, \dots$$

when they act on functions vanishing together with all derivatives at $t \leq 0$.

We can readily solve the chain of equations (3.5) in terms of R_w and R_d:

$$\begin{aligned} u^0(t) &= R_d f(t), \\ u^j(t) &= (-\partial_t)^j (\partial_t R_d C)^j u^0(t), \quad j = 1, \dots, k, \\ r^k(t) &= (-\partial_t)^{k+1} (\partial_t R_w C)(\partial_t R_d C)^k u^0(t). \end{aligned} \tag{3.8}$$

The crucial next step is to prove the fast decay of high-order time derivatives for both R_w and R_d, in order to verify the smallness of the remainder $r^k(t)$.

3.3. **Remainder estimates.** Recall that the remainder $r^k(t) = \tilde{u}(t) - u^k_{\rm dif}(t)$ is

$$r^k(t) = (-\partial_t)^{k+1} (\partial_t R_w C)(\partial_t R_d C)^k u^0(t), \tag{3.9}$$

where $u^0(t) = R_d f(t)$. We need to show that $r^k(t)$ decays fast in average as $t \to \infty$.

The next two results, although easy to verify, are very useful.

Lemma 3.1. *Let $\theta \geq 2l$, $L \geq 1$ and assume that $g(t) = 0$ for $t \leq 0$. Then*

$$\begin{aligned} \int_0^t (s+L)^\theta \|\partial_s^{l+1} R_w g(s)\|^2 ds \quad &\lesssim \quad \sum_{j=1}^{l} \int_0^t (s+L)^{\theta-2l+2j} \|\partial_s^j g(s)\|^2 ds \\ &\quad + \int_0^t (s+L)^{\theta-2l} \|\partial_s R_w g(s)\|^2 ds. \end{aligned}$$

Lemma 3.2. *Let $\theta \geq 2l$, $L \geq 1$ and assume that $g(t) = 0$ for $t \leq 0$. Then*

$$\begin{aligned} \int_0^t (s+L)^\theta \|\partial_s^{l+1} R_d g(s)\|^2 ds \quad &\lesssim \quad \sum_{j=1}^{l} \int_0^t (s+L)^{\theta-2l+2j} \|\partial_s^j g(s)\|^2 ds \\ &\quad + \int_0^t (s+L)^{\theta-2l} \|\partial_s R_d g(s)\|^2 ds. \end{aligned}$$

Applying these lemmas, we can show that the remainder can be made arbitrarily small; this completes the proof of diffusion phenomenon.

Proposition 3.3. *Let $k \geq 1$ and $r^k(t) = \tilde{u}(t) - u^k_{\rm dif}(t)$ be defined by (3.9). Then*

$$(i) \qquad \int_0^t (s+1)^{2(k-1)} \|r^k(s)\|^2 ds \lesssim \|u_0\|^2 + \|u_1\|^2,$$

$$(ii) \qquad \int_0^t (s+1)^{2k} \|\partial_s r^k(s)\|^2 ds \lesssim \|u_0\|^2 + \|u_1\|^2,$$

for all $t \geq 0$. (Here u_0 and u_1 are the initial values in problem (2.1).)

4. Estimates for Markov Semigroups

We study the diffusion approximation $u^k_{\rm dif}(t)$ in L^q spaces, where various embedding and interpolation inequalities are available. Thus, we make the following additional assumptions on the Hilbert space and operators: $\mathcal{H} = L^2(\Omega, \mu)$, where (Ω, μ) is a σ-finite measure space, and

$(H4)$ $-B$ generates Markov semigroups $\{e^{-tB}\}_{t \geq 0}$ on $L^q(\Omega, \mu)$, $q \in [1, 2]$.

$(H5)$ $\exists m > 0$ such that $\|e^{-tB} g\|_2 \leq c_q t^{-m/2(1/q - 1/2)} (\|g\|_q + \|g\|_2)$ for $g \in L^q(\Omega, \mu) \cap L^2(\Omega, \mu)$, $t > 0$, $q \in [1, 2]$.

$(H6)$ C is a bounded operator $L^q(\Omega, \mu) \to L^q(\Omega, \mu)$ for $q \in [1, 2]$.

Under these conditions, we can prove that $u^k_{\rm dif}(t) \approx R_d f(t) \approx e^{-tB}(u_0 + Cu_1)$.

Theorem 4.1. *Assume that (H1)-(H6) hold and let* $(u, \partial_t u) \in C(\mathbb{R}_+, \mathcal{D}(\sqrt{B}) \times \mathcal{H})$ *be the unique mild solution of (2.1). If* $q \in (1, 2]$*, then*

$$(i) \qquad \|u(t) - e^{-tB}(u_0 + Cu_1)\|_2 \lesssim (\|u_0\|_2 + \|\sqrt{B}u_0\|_2 + \|u_1\|_2 + \|u_0\|_q + \|u_1\|_q)(t+1)^{-m/2(1/q-1/2)-1}.$$

(The diffusion semigroup decays slower: $\|e^{-tB}g\|_2 \lesssim t^{-m/2(1/q-1/2)}(\|g\|_q + \|g\|_2)$*.)*

$$(ii) \qquad \|u(t)\|_2 \lesssim (\|u_0\|_2 + \|u_1\|_2 + \|u_0\|_1 + \|u_1\|_1)(t+1)^{-m/4} + (\|u_0\|_2 + \|\sqrt{B}u_0\|_2 + \|u_1\|_2 + \|u_0\|_q + \|u_1\|_q)(t+1)^{-m/2(1/q-1/2)-1},$$

$$(iii) \qquad E^{1/2}(t;u) \lesssim (\|u_0\|_2 + \|\sqrt{B}u_0\|_2 + \|u_1\|_2 + \|u_0\|_1 + \|u_1\|_1)(t+1)^{-m/4-1} + (\|u_0\|_2 + \|\sqrt{B}u_0\|_2 + \|u_1\|_2 + \|u_0\|_q + \|u_1\|_q)(t+1)^{-m/2(1/q-1/2)-3/2}.$$

(Hence, $\|u(t)\|_2$ *and* $E^{1/2}(t;u)$ *have the decay rates of the diffusion semigroup if the exponent* q *is sufficiently close to* 1*.)*

It is clear that Theorem 1.1 is a consequence of this abstract result and well known estimates for the diffusion semigroup generated by B in (1.3); see [1], [2].

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A new maximum principle for diffusive Lotka-Volterra systems of competing species

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1 Introduction

In ecology, one central issue is to understand coexistence of many species. Among the interactions between species, two basic and important mechanisms are

(1) competition: the act or process of trying to gain or win something by defeating or establishing superiority over others; and

(2) cooperation: mutually beneficial interaction among organisms.

In this talk, we focus on competition and assume there is no "direct cooperation" in our models. We are concerned with the relation between competition and coexistence, and investigate the following questions.

(1) Does competition always reduce the chance of coexistence? Or sometimes helps coexistence?

(2) Can species co-exist under very strong competition?

We consider the two-species Lotka-Volterra competition-diffusion system first and then turn to the corresponding three-species system.

2 Two species system

The diffusive Lotka-Volterra system of two competing species can be written as follows

$$\begin{cases} u_t = d_1 \triangle u + u\,(\sigma_1 - c_{11}\,u - c_{12}\,v), & y \in \Omega, \quad t > 0, \\ \\ v_t = d_2 \triangle v + v\,(\sigma_2 - c_{21}\,u - c_{22}\,v), & y \in \Omega, \quad t > 0 \end{cases} \tag{2.1}$$

with the Neumann boundary condition. Here $u(y,t)$ and $v(y,t)$ stand for the density of the two species u and v, respectively; d_i, σ_i, c_{ii} $(i = 1, 2)$, and c_{ij} $(i, j = 1, 2$ with $i \neq j)$ are the respective diffusion rates, intrinsic growth rates, intra-specific competition rates,

and inter-specific competition rates, all of which are assumed to be positive. To study (2.1), we first consider the corresponding ODE system

$$\begin{cases} u_t = u\,(\sigma_1 - c_{11}\,u - c_{12}\,v), \quad t > 0, \\ \\ v_t = v\,(\sigma_2 - c_{21}\,u - c_{22}\,v), \quad t > 0. \end{cases} \tag{2.2}$$

According to different parameters in the non-linear terms, the dynamics of the ODE are classified into four types. Two of them are the strong competition case with

- $\frac{\sigma_1}{c_{11}} > \frac{\sigma_2}{c_{21}}$, $\frac{\sigma_2}{c_{22}} > \frac{\sigma_1}{c_{12}}$,

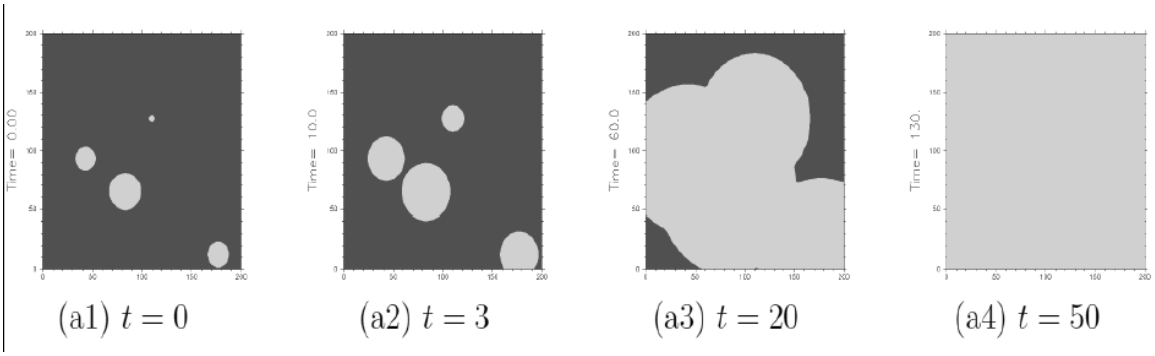
and the weak competition case with

- $\frac{\sigma_1}{c_{11}} < \frac{\sigma_2}{c_{21}}$, $\frac{\sigma_2}{c_{22}} < \frac{\sigma_1}{c_{12}}$.

For the weak competition case, the ODE (2.2) and PDE (2.1) often show coexistence of the two species. For the strong competition case, the ODE (2.2) and PDE (2.1) (say, in a convex domain) often show the behavior of competition exclusion, i.e., only one species can survive.

- In this talk, we are especially interested in the **strong competition case**.

The following series of pictures is a simulation made by Mimura and Tohma (see Ecological Complexity, 21, 215-232 (2015)), where two species are in strong competition.



(a1) $t = 0$ (a2) $t = 3$ (a3) $t = 20$ (a4) $t = 50$

The problem as to which species will survive in a strongly competitive system is of importance in ecology. In order to tackle this problem, we consider traveling wave solutions on $\mathbb{R}$, which are solutions of the form

$$(u(y,t), v(y,t)) = (u(x), v(x)), \quad x = y - \theta\,t, \tag{2.3}$$

where θ is the propagation speed of the traveling wave. In general, the sign of θ indicates which species is stronger and can survive.

We note that by using a suitable scaling, the two-species system (2.1) on $\mathbb{R}$ can be rewritten as

$$\begin{cases} u_t = \quad u_{xx} + \quad u\,(1 - u - a_1\,v), \quad x \in \mathbb{R}, \quad t > 0, \\ \\ v_t = d\,v_{xx} + k\,v\,(1 - a_2\,u - v), \quad x \in \mathbb{R}, \quad t > 0, \end{cases} \tag{2.4}$$

where d, k, a_1 and a_2 are positive parameters.

Substituting (2.3)into (2.4), we obtain the equations for a traveling wave which connecting the equilibria $(1,0)$ and $(0,1)$

$$\begin{cases} u_{xx} + \theta\, u_x + \ \ u\,(1 - u - a_1\, v) = 0, \\ d\, v_{xx} + \theta\, v_x + k\, v\,(1 - a_2\, u - v) = 0, \quad x \in \mathbb{R}, \\ (u,v)(-\infty) = (1,0), \quad (u,v)(+\infty) = (0,1). \end{cases} \tag{2.5}$$

The existence of a unique traveling wave (up to a translation) was proved by Gardner, Conley-Gardner, and Kan-on (see, for example, Kan-on [6]).

For the traveling wave solution of (2.5), u will occupy the whole domain eventually if $\theta > 0$ while v will occupy the whole domain eventually if $\theta < 0$. From the viewpoint of ecology, we conclude that the sign of θ determines which species is stronger, i.e. u is stronger if $\theta > 0$ and v is stronger if $\theta < 0$. For (2.4) on a higher dimensional domain, the traveling wave solution of (2.5) can also be used to approximate the dynamics of the boundary between the two species.

3 Three species system

Since it is not easy to coexist for two species under strong competition, we turn to the three species system

$$\begin{cases} u_t = d_1\, u_{xx} + u\,(\,\sigma_1 - c_{11}\, u - c_{12}\, v - c_{13}\, w\,), \\ v_t = d_2\, v_{xx} + v\,(\,\sigma_2 - c_{21}\, u - c_{22}\, v - c_{23}\, w\,), \quad x \in \Omega \\ w_t = d_3\, w_{xx} + w\,(\,\sigma_3 - c_{31}\, u - c_{32}\, v - c_{33}\, w\,). \end{cases} \tag{3.6}$$

We are interested to know if there is more chance to co-exist for a three species system even they are under competition with one another. Again we focus on the traveling wave solutions first. It is easy to find a monotone 3-species waves. Miller first constructed a non-monotone wave and H. Ikeda constructed a symmetric pulse solution to (3.6). Besides these very interesting 3-species waves, people are curious if there are other types of waves? For example, whether there is a wave with v decreasing from σ_2/c_{22} to 0, u increasing from 0 to σ_1/c_{11}, and w being a pulse like in the middle (let's call it the type 4 wave)? In [2], for suitable parameters of the nonlinear terms, we found exact type 4 waves which can be represented in terms of the hyperbolic tangent. For example, we have the following.

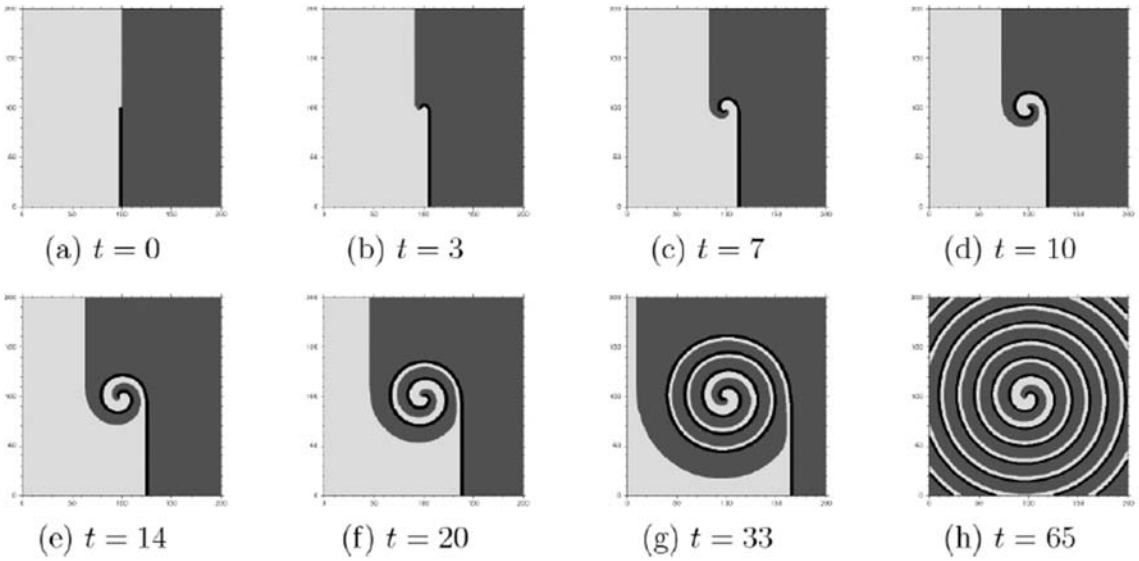
$$\begin{cases} u'' + su' + u(1 - u - \frac{8}{5}v - 2w) = 0, \\ v'' + sv' + v(1 - \frac{19}{25}u - v - \frac{1}{16}w) = 0, \\ w'' + sw' + w(1 - \frac{18}{25}u - \frac{8}{5}v - w) = 0, \quad z \in \mathbb{R} \end{cases}$$

has an exact TW solution

$$\begin{cases} u(z) = \frac{1}{2}(1 + \tanh(\frac{z}{5})) \\ v(z) = \frac{1}{4}(1 - \tanh(\frac{z}{5}))^2 \\ w(z) = \frac{4}{25}(1 - \tanh^2(\frac{z}{5})) \end{cases} \text{ with speed } s = \frac{11}{50}.$$

In recent work, we obtained more existence results for type 4 waves.

One interesting application of type 4 waves is to use it to produce a spiral wave on a 2-dimensional domain, as shown in the work by Mimura-Tohma. See "Dynamic coexistence in a three species competition diffusion system", Ecological Complexity, 21, 215-232 (2015) and the pictures below. Also there are more new 2-dimensional waves and patterns of 3-species found in "Two dimensional traveling waves arising from planar front interaction in a three species competition diffusion system", L. Contento, M. Mimura and M. Tohma, to appear in Japan Journal of Industrial and Applied Mathematics. An important conclusion of their numerical study is that under competition, a 3-species system has more chance to coexist, at least as a dynamic pattern, than a 2-species system. In these co-existence patterns, type 4 waves play a crucial rule. The following series of pictures is a simulation made by Mimura and Tohma (see Ecological Complexity, 21, 215-232 (2015)).



(a) $t = 0$ (b) $t = 3$ (c) $t = 7$ (d) $t = 10$

(e) $t = 14$ (f) $t = 20$ (g) $t = 33$ (h) $t = 65$

4 Total mass

Since type 4 waves provide an important mechanism for the 3 species to coexist, we are interested in finding more quantitative criteria for existence and non-existence of type 4 waves. Intuitively, w has no chance to survive if $c_{31}u + c_{32}v$ in the third equation of (3.6) is too large and a type 4 wave cannot exist in this case. Therefore it is natural to consider the problem

- Question: How to estimates $\alpha u + \beta v$?

When α and β are the average weights of the species u and v respectively, $\alpha u + \beta v$ denotes the total biomass of them. Similarly we can also consider $\alpha u + \beta v + \gamma w$ for the total mass of the three species to understand the total capacity of the 3-species system.

We consider this problem for (2.5), i.e., the two species traveling wave solutions. By adding the two equations in (2.5), we obtain an equation involving $p(x) = \alpha\, u + \beta\, v$ and

$q(x) = \alpha\, u + d\,\beta\, v,$

$$\begin{aligned} 0 &= \alpha\left(u_{xx} + \theta\, u_x + u\,(1 - u - a_1\, v)\right) + \beta\left(d\, v_{xx} + \theta\, v_x + kv\,(1 - a_2\, u - v)\right) \\ &= q''(x) + \theta\, p'(x) + \alpha\, u\,(1 - u - a_1\, v) + \beta k\, v\,(1 - a_2\, u - v). \end{aligned} \tag{4.7}$$

The case where $d = 1$ has been considered in [1]. Obviously, difficulties arise when $d \neq 1$. We consider the strong competition case $a_1 > 1$ and $a_2 > 1$ here.

Theorem 4.1 (Estimates for $q(x)$)**.** *Suppose that $a_1 > 1, a_2 > 1$ and $(u(x), v(x))$ is a nonnegative solution to* (2.5)*. Then*

$$\min\left[\frac{\alpha}{a_2\, d}, \frac{\beta}{a_1}\right] \min[1, d^2] \leq q(x) \leq \max\left[\frac{\alpha}{d}, \beta\right] \max[1, d^2] \quad \text{for } x \in \mathbb{R}, \tag{4.8}$$

where $q(x) = \alpha\, u(x) + d\,\beta\, v(x)$ and α, β are arbitrary positive constants.

In particular, we notice that the estimate of q in Theorem 4.1 does not depend on the propagating speed θ and the constant k.

Using the properties of the nonlinear terms of (2.5) more delicately, one can obtain better but complicated estimates for $u + v$. In the following, we just state an improved result for $d = k = 1$ since the form of the lower bound obtained is simple in this case.

Theorem 4.2. *Suppose $d = k = 1$, $a_1 > 1, a_2 > 1$, and $(u(x), v(x))$ is a nonnegative solution to* (2.5)*. Then for $x \in \mathbb{R}$*

$$\frac{4}{a_1 + a_2 + 2} \leq u(x) + v(x) \leq 1. \tag{4.9}$$

The lower bound for $u+v$ obtained by Theorem 4.1 is $\min[1/a_1, 1/a_2]$, which is smaller than or equal to $\frac{4}{a_1+a_2+2}$ and is less sharp when $a_1, a_2 > 1$. Note that the lower bound in (4.9) approaches 1 as (a_1, a_2) approaches $(1, 1)$.

As an application of Theorem 4.1, we establish nonexistence of traveling waves solutions for the Lotka-Volterra system of three competing species, i.e., nonexistence of solutions of

$$\begin{cases} d_1\, u_{xx} + \theta\, u_x + u\,(\,\sigma_1 - c_{11}\, u - c_{12}\, v - c_{13}\, w\,) = 0, & x \in \mathbb{R}, \\ d_2\, v_{xx} + \theta\, v_x + v\,(\,\sigma_2 - c_{21}\, u - c_{22}\, v - c_{23}\, w\,) = 0, & x \in \mathbb{R}, \\ d_3\, w_{xx} + \theta\, w_x + w\,(\,\sigma_3 - c_{31}\, u - c_{32}\, v - c_{33}\, w\,) = 0, & x \in \mathbb{R}, \end{cases} \tag{4.10}$$

where $u(x,t)$, $v(x,t)$ and $w(x,t)$ represent the density of the three species u, v and w respectively; d_i, σ_i, c_{ii} $(i = 1, 2, 3)$, and c_{ij} $(i, j = 1, 2, 3, i \neq j)$ are the diffusion rates, the intrinsic growth rates, the intra-specific competition rates, and the inter-specific competition rates, respectively. These constants are all assumed to be positive.

For (4.10), a type 4 wave satisfies the boundary conditions

$$(u, v, w)(-\infty) = \left(\frac{\sigma_1}{c_{11}}, 0, 0\right), \quad (u, v, w)(\infty) = \left(0, \frac{\sigma_2}{c_{22}}, 0\right) \tag{4.11}$$

As mentioned, such a wave is investigated in [2].

On the other hand, nonexistence of solutions for (4.10) and (4.11) is studied in [1] when the diffusion rates d_1, d_2, and d_3 are assumed to be identical. In [1], a subtle structure of the competing system, which heavily relies on equal diffusivity, is employed. With the aid of Theorem 4.1, we give a much more general nonexistence of solutions for (4.10) and (4.11) when the diffusion rates of the species are no longer the same.

Theorem 4.3 (**Nonexistence of 3-species wave**). *Let* $\phi_1 = \sigma_1 c_{33} - \sigma_3 c_{13}$ *and* $\phi_2 = \sigma_2 c_{33} - \sigma_3 c_{23}$. *Assume that the following hypotheses hold:*

[**H1**] $\phi_1, \phi_2 > 0$;

[**H2**] $c_{21}\,\phi_1 > c_{11}\,\phi_2, c_{12}\,\phi_2 > c_{22}\,\phi_1$;

[**H3**] $\min\left[\dfrac{c_{31}\,\phi_2}{c_{21}\,d_2}, \dfrac{c_{32}\,\phi_1}{c_{12}\,d_1}\right] \min\left[d_1^2, d_2^2\right] \geq \sigma_3\, c_{33}$.

Then (4.10) *and* (4.11) *has no positive solution* $(u(x), v(x), w(x))$.

We remark that when σ_3 is small, the conditions (H1)-(H3) hold if u and v are in strong competition.

The proof of Theorem 4.1 is elementary. We describe it as follows.

Lemma 4.4. *Under the bistable condition* $a_1 > 1$ *and* $a_2 > 1$, *the quadratic curve* $\alpha\, u\,(1 - u - a_1\, v) + \beta\, k\, v\,(1 - a_2\, u - v) = 0$ *is a hyperbola for* $\alpha > 0$ *and* $\beta > 0$.

Proof. The discriminant of the quadratic curve $\alpha\, u\,(1-u-a_1\, v)+\beta\, k\, v\,(1-a_2\, u-v) = 0$ is $(\alpha\, a_1+\beta\, k\, a_2)^2-4\,\alpha\,\beta\, k$. Since $a_1, a_2 > 1$, we have $(\alpha\, a_1+\beta\, k\, a_2)^2 \geq 4\,\alpha\,\beta\, k\, a_1\, a_2 > 4\,\alpha\,\beta\, k$. The positivity of the discriminant gives the desired result. □

The lemma indicates that the quadratic curve

$$F(u, v) := \alpha\, u\,(1 - u - a_1\, v) + \beta\, k\, v\,(1 - a_2\, u - v) = 0 \tag{4.12}$$

cannot either be an ellipse or a parabola under the bistable condition $a_1, a_2 > 1$.

In Propositions 4.5 and 4.6 below, we give a lower bound and an upper bound for $q(x)$, respectively. Combining the results in Propositions 4.5 and 4.6, we immediately obtain Theorem 4.1.

Proposition 4.5 (**Lower bound for** $q(x)$). *Let* $a_1 > 1$ *and* $a_2 > 1$. *Suppose that* $(u(x), v(x))$ *is* C^2, *nonnegative, and satisfies the following differential inequalities and asymptotic behaviour:*

$$\begin{cases} u_{xx} + \theta\, u_x + \ \ u\,(1 - u - a_1\, v) \leq 0, \quad x \in \mathbb{R}, \\ d\, v_{xx} + \theta\, v_x + k\, v\,(1 - a_2\, u - v) \leq 0, \quad x \in \mathbb{R}, \\ (u, v)(-\infty) = (1, 0), \quad (u, v)(+\infty) = (0, 1). \end{cases} \tag{4.13}$$

Then we have for $x \in \mathbb{R}$,

$$q(x) \geq \min\left[\frac{\alpha}{a_2\, d}, \frac{\beta}{a_1}\right] \min[1, d^2]. \tag{4.14}$$

Proof. Let $\mathcal{R} = \{(u.v) \,|\, 1-u-a_1\,v \geq 0, 1-a_2\,u-v \geq 0, u \geq 0, v \geq 0\}$. First we construct an appropriate *N-barrier* consisting of three lines $\alpha\,u + d\,\beta\,v = \lambda_2$, $\alpha\,u + \beta\,v = \eta$ and $\alpha\,u + d\,\beta\,v = \lambda_1$, and chose λ_1, λ_2 and η as large as possible such that $Q_{\lambda_1} \subset P_\eta \subset Q_{\lambda_2} \subset \mathcal{R}$, where $Q_\lambda = \{(u,v) \,|\, \alpha\,u + d\,\beta\,v \leq \lambda, u \geq 0, v \geq 0\}$ and $P_\eta = \{(u,v) \,|\, \alpha\,u + \beta\,v \leq \eta, u \geq 0, v \geq 0\}$. Then we show that λ_1 can be taken to equal the value on the right hand side of (4.14) and $q(x) \geq \lambda_1$ can be verified via the structure of the N-barrier.

Now we illustrate how to construct *the N-barrier* in detail. For the case of $d \geq 1$ and $\beta\,a_2\,d \geq \alpha\,a_1$, the N-barrier is constructed in the following three steps (see Figure 2.1(a)):

(1) *The construction of the upper pink line*: we draw on the uv-plane the upper pink line $\alpha\,u + d\,\beta\,v = \lambda_2$ which passes through $(\frac{1}{a_2}, 0)$. This gives $\lambda_2 = \frac{\alpha}{a_2}$, and hence the upper pink line is represented by the equation $\alpha\,u + d\,\beta\,v = \frac{\alpha}{a_2}$. The v-coordinate of the v-intercept of $\alpha\,u + d\,\beta\,v = \frac{\alpha}{a_2}$ is $\frac{\alpha}{\beta\,a_2\,d}$, which is less than or equal to $\frac{1}{a_1}$ by the assumption $\beta\,a_2\,d \geq \alpha\,a_1$. This means that the v-coordinate of the v-intercept of $\alpha\,u + d\,\beta\,v = \frac{\alpha}{a_2}$ is below the v-coordinate of v-intercept of the $1 - u - a_1\,v = 0$.

(2) *The construction of the yellow line*: we let the the yellow line $\alpha\,u + \beta\,v = \eta$ start from $(0, \frac{\alpha}{\beta\,a_2\,d})$. This leads to $\eta = \frac{\alpha}{a_2\,d}$ and hence the yellow line is represented by the equation $\alpha\,u + \beta\,v = \frac{\alpha}{a_2\,d}$. The u-coordinate of the u-intercept of $\alpha\,u + \beta\,v = \frac{\alpha}{a_2\,d}$ is $\frac{1}{a_2\,d}$, which is less than or equal to $\frac{1}{a_2}$ by the assumption $d \geq 1$. This means that the u-coordinate of the u-intercept of $\alpha\,u + \beta\,v = \frac{\alpha}{a_2\,d}$ is less than or equal to the u-coordinate of u-intercept of $\alpha\,u + d\,\beta\,v = \frac{\alpha}{a_2}$.

(3) *The construction of the lower pink line*: we draw the lower pink line $\alpha\,u + d\,\beta\,v = \lambda_1$ passing through $(\frac{1}{a_2\,d}, 0)$. This gives $\lambda_1 = \frac{\alpha}{a_2\,d}$.

There are three other cases, each of which can be treated in a similar manner for the construction of the corresponding N-barrier (see Figures 2.1(b), 2.1(c), and 2.1(d)). More precisely, we have the following four cases and for each case, we take different λ_1, λ_2 and η, and show that $q(x)$ has the lower bound λ_1 for $x \in \mathbb{R}$:

- If $d \geq 1$,

  (i) when $\beta\,a_2\,d \geq \alpha\,a_1$, we take $(\lambda_1, \lambda_2, \eta) := (\dfrac{\alpha}{a_2\,d}, \dfrac{\alpha}{a_2}, \dfrac{\alpha}{a_2\,d})$;

  (ii) when $\beta\,a_2\,d < \alpha\,a_1$, we take $(\lambda_1, \lambda_2, \eta) := (\dfrac{\beta}{a_1}, \dfrac{\beta\,d}{a_1}, \dfrac{\beta}{a_1})$.

- If $d < 1$,

  (iii) when $\beta\,a_2\,d \geq \alpha\,a_1$, $(\lambda_1, \lambda_2, \eta) := (\dfrac{\alpha\,d}{a_2}, \dfrac{\alpha}{a_2}, \dfrac{\alpha}{a_2})$;

  (iv) when $\beta\,a_2\,d < \alpha\,a_1$, $(\lambda_1, \lambda_2, \eta) := (\dfrac{\beta\,d^2}{a_1}, \dfrac{\beta\,d}{a_1}, \dfrac{\beta\,d}{a_1})$.

  We note that case (i) corresponds to Figure 2.1(a), in which the N-barrier has been constructed in the above three steps. The other cases (ii), (iii), and (iv) correspond to Figures 2.1(b), 2.1(c), and 2.1(d), respectively.

We first observe that the property $q(x) \geq \lambda_1$ in the four cases can be reduced to the following two cases:

- for $\beta\, a_2\, d \geq \alpha\, a_1$, $q(x) \geq \frac{\alpha}{a_2} \min[d, 1/d]$ for all $x \in \mathbb{R}$;
- for $\beta\, a_2\, d < \alpha\, a_1$, $q(x) \geq \frac{\beta}{a_1} \min[1, d^2]$ for all $x \in \mathbb{R}$.

Combining the two cases above leads to $q(x) \geq \min[\frac{\alpha}{a_2\, d}, \frac{\beta}{a_1}] \min[1, d^2]$ for all $x \in \mathbb{R}$, which is the desired result.

Now we show $q(x) \geq \lambda_1$ in $(i) \sim (iv)$. The two inequalities in (4.13) and (4.12) give

$$q''(x) + \theta\, p'(x) + F(u(x), v(x)) \leq 0. \tag{4.15}$$

For $d > 1$, we first prove (i) by contradiction. Suppose that, contrary to our claim, there exists $z \in \mathbb{R}$ such that $q(z) < \lambda_1$. Since $u, v \in C^2(\mathbb{R})$, by $(u, v)(-\infty) = (1, 0)$ and $(u, v)(+\infty) = (0, 1)$, we may assume $\min_{x\in\mathbb{R}} q(x) = q(z)$. We denote respectively by z_2 and z_1 the first points at which the solution $(u(x), v(x))$ intersects the line $\alpha\, u + d\, \beta\, v = \lambda_2$ in the uv-plane when x moves from z towards ∞ and $-\infty$ (as shown in Figure 2.1(a)). For the case where $\theta \leq 0$, we integrate (4.15) with respect to x from z_1 to z and obtain

$$q'(z) - q'(z_1) + \theta\, (p(z) - p(z_1)) + \int_{z_1}^{z} F(u(x), v(x))\, dx \leq 0. \tag{4.16}$$

On the other hand we have:

- since $\min_{x\in\mathbb{R}} q(x) = q(z)$, $q'(z) = \alpha\, u'(z) + d\, \beta\, v'(z) = 0$;
- $q(z_1) = \lambda_2$ follows from the fact that z_1 is on the line $\alpha\, u + d\, \beta\, v = \lambda_2$. Since z_1 is the first point for $q(x)$ taking the value λ_2 when x moves from z to $-\infty$, we conclude that $q(z_1 + \delta) \leq \lambda_2$ for $z - z_1 > \delta > 0$ and $q'(z_1) \leq 0$;
- $p(z) < \eta$ since z is below the line $\alpha\, u + \beta\, v = \eta$; $p(z_1) > \eta$ since z is above the line $\alpha\, u + \beta\, v = \eta$;
- it is readily seen that the quadratic curve $F(u, v) = 0$ passes through the points $(0, 0)$, $(1, 0)$, $(0, 1)$, and (u^*, v^*) in the uv-plane. Let $A_+ = \{(u, v) \,|\, F(u, v) \geq 0, u \geq 0, v \geq 0\}$. By Lemma 4.4 and the property that $F(u, v) < 0$ for large u and v, it follows that A_+ is the region bounded by a hyperbola, u-axis and v-axis. Moreover, $\{(u(x), v(x)) \,|\, z_1 \leq x \leq z\} \subset \mathcal{R} \subset A_+$. Therefore we have $\int_{z_1}^{z} F(u(x), v(x))\, dx > 0$.

Summarizing the above arguments, we obtain

$$q'(z) - q'(z_1) + \theta\, (p(z) - p(z_1)) + \int_{z_1}^{z} F(u(x), v(x))\, dx > 0, \tag{4.17}$$

which contradicts (4.16). Therefore when $\theta \leq 0$, $q(x) \geq \lambda_1$ for $x \in \mathbb{R}$. For the case where $\theta \geq 0$, integrating (4.15) with respect to x from z to z_2 yields

$$q'(z_2) - q'(z) + \theta\, (p(z_2) - p(z)) + \int_{z}^{z_2} F(u(x), v(x))\, dx \leq 0. \tag{4.18}$$

In a similar manner, it can be shown that $q'(z_2) \geq 0$, $q'(z) = 0$, $p(z_2) > \eta$, $p(z) < \eta$, and $\int_z^{z_2} F(u(x), v(x))\, dx > 0$. These together contradict (4.18). Consequently, (i) is proved for $d > 1$. For $d = 1$, we have $q = p$ and (4.15) becomes

$$p''(x) + \theta\, p'(x) + F(u(x), v(x)) \leq 0, \quad x \in \mathbb{R}. \tag{4.19}$$

Moreover, when $d = 1$ we take $\lambda_1 = \lambda_2 = \eta = \frac{\alpha}{a_2}$, i.e. the three lines $\alpha\, u + d\, \beta\, v = \lambda_1$, $\alpha\, u + d\, \beta\, v = \lambda_2$, and $\alpha\, u + \beta\, v = \eta$ coincide. Analogously to the case of $d > 1$, we assume that there exists $\hat{z} \in \mathbb{R}$ such that $p(\hat{z}) < \lambda_1$ and $\min_{x\in\mathbb{R}} p(x) = p(\hat{z})$. Due to $\min_{x\in\mathbb{R}} p(x) = p(\hat{z})$, we have $p'(\hat{z}) = 0$ and $p''(\hat{z}) \geq 0$. Since $(u(\hat{z}), v(\hat{z}))$ is in the interior of $\mathcal{R}$, which is contained in the interior of A_+, we have $F(u(\hat{z}), v(\hat{z})) > 0$. These together give $p''(\hat{z}) + \theta\, p'(\hat{z}) + F(u(\hat{z}), v(\hat{z})) > 0$, which contradicts (4.19). Thus, $p(x) \geq \lambda_1$ for all $x \in \mathbb{R}$ when $d = 1$. As a result, the proof of (i) is completed.

The proofs for cases (ii), (iii), and (iv) are similar (see Figures 2.1(b), 2.1(c), and 2.1(d)). This completes the proof of Proposition 4.5. □

Proposition 4.6 (**Upper bound for** $q(x)$)**.** *Assume that $a_1 > 1$, $a_2 > 1$, and that $(u(x), v(x))$ is C^2, nonnegative, and satisfies the following differential inequalities:*

$$\begin{cases} u_{xx} + \theta\, u_x + \ u\,(1 - u - a_1\, v) \geq 0, \quad x \in \mathbb{R}, \\ d\, v_{xx} + \theta\, v_x + k\, v\,(1 - a_2\, u - v) \geq 0, \quad x \in \mathbb{R}, \\ (u, v)(-\infty) = \mathbf{e}_2, \quad (u, v)(+\infty) = \mathbf{e}_3. \end{cases} \tag{4.20}$$

Then for $x \in \mathbb{R}$, we have

$$q(x) \leq \max\left[\frac{\alpha}{d}, \beta\right] \max[1, d^2]. \tag{4.21}$$

Proof. As in the proof of Proposition 4.5, there are also four cases and for each case, we can construct the N-barrier and prove that $q(x) \leq \lambda_1$ for $x \in \mathbb{R}$:

- If $d \geq 1$,
  - (i) when $\beta\, d \geq \alpha$, we take $(\lambda_1, \lambda_2, \eta) := (\beta\, d^2, \beta\, d, \beta\, d)$;
  - (ii) when $\beta\, d < \alpha$, $(\lambda_1, \lambda_2, \eta) := (\alpha\, d, \alpha, \alpha)$.
- If $d < 1$,
  - (iii) when $\beta\, d \geq \alpha$, $(\lambda_1, \lambda_2, \eta) := (\beta, \beta\, d, \beta)$;
  - (iv) when $\beta\, d < \alpha$, $(\lambda_1, \lambda_2, \eta) := (\frac{\alpha}{d}, \alpha, \frac{\alpha}{d})$.

Combining the four cases above, it follows that

- for $\beta\, d \geq \alpha$, $q(x) \leq \beta \max(1, d^2)$ for all $x \in \mathbb{R}$;
- for $\beta\, d < \alpha$, $q(x) \leq \alpha \max(d, 1/d)$ for all $x \in \mathbb{R}$,

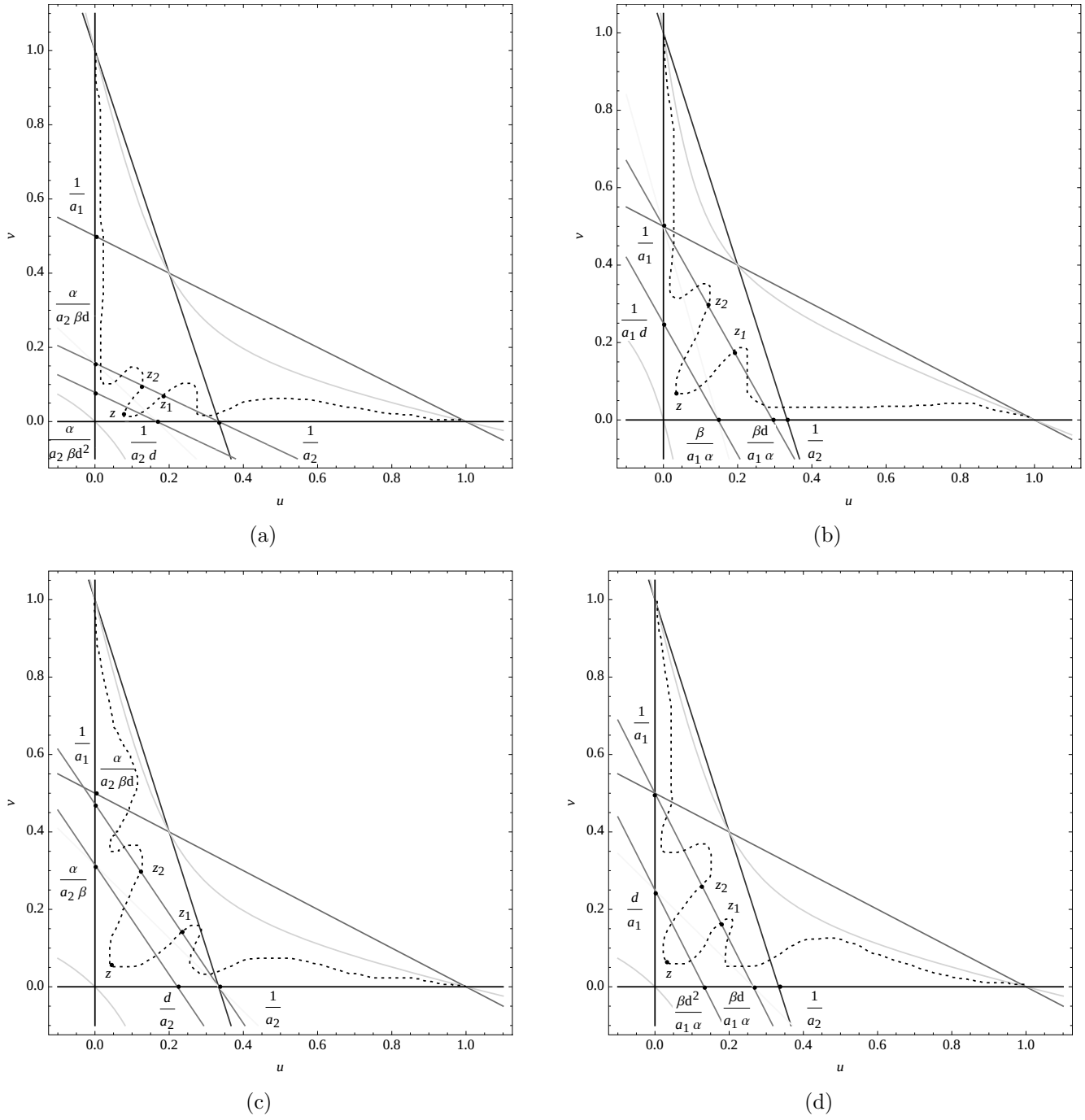


Figure 4.1: Red line: $1 - u - a_1\, v = 0$; blue line: $1 - a_2\, u - v = 0$; green curve: $\alpha\, u\,(1 - u - a_1\, v) + \beta\, k\, v\,(1 - a_2\, u - v) = 0$; magenta line (above): $\alpha\, u + d\,\beta\, v = \lambda_2$; magenta line (below): $\alpha\, u + d\,\beta\, v = \lambda_1$; yellow line: $\alpha\, u + \beta\, v = \eta$; dashed curve: $(u(x), v(x))$. (a) $a_1 = 2$, $a_2 = 3$, $\alpha = 17$, $\beta = 18$, and $d = 2$ give $\lambda_1 = \frac{17}{6}$, $\lambda_2 = \frac{17}{3}$, and $\eta = \frac{17}{6}$. (b) $a_1 = 2$, $a_2 = 3$, $\alpha = 17$, $\beta = 5$, and $d = 2$ give $\lambda_1 = \frac{5}{2}$, $\lambda_2 = 5$, and $\eta = \frac{5}{2}$. (c) $a_1 = 2$, $a_2 = 3$, $\alpha = 17$, $\beta = 18$, and $d = \frac{2}{3}$ give $\lambda_1 = \frac{34}{9}$, $\lambda_2 = \frac{17}{3}$, and $\eta = \frac{17}{3}$. (d) $a_1 = 2$, $a_2 = 3$, $\alpha = 17$, $\beta = 18$, and $d = \frac{1}{2}$ give $\lambda_1 = \frac{9}{4}$, $\lambda_2 = \frac{9}{2}$, and $\eta = \frac{9}{2}$.

which implies $q(x) \le \max[\frac{\alpha}{d}, \beta] \max[1, d^2]$ for all $x \in \mathbb{R}$. The rest part of the proof is similar to that of Proposition 4.5 and is hence omitted.

□

We prove Theorem 4.3 by contradiction.

Proof of Theorem 4.3. Suppose to the contrary that there exists a solution $(u(x), v(x), w(x))$ to (4.10),(4.11). Due to the fact that $w(x) > 0$ for $x \in \mathbb{R}$ and $w(\pm\infty) = 0$, we can find $x_0 \in \mathbb{R}$ such that $\max_{x\in\mathbb{R}} w(x) = w(x_0) > 0$, $w''(x_0) \le 0$, and $w'(x_0) = 0$. Since $w(x)$ satisfies $d_3\, w_{xx} + \theta\, w_x + w(\sigma_3 - c_{31}\, u - c_{32}\, v - c_{33}\, w) = 0$, we obtain

$$\sigma_3 - c_{31}\, u(x_0) - c_{32}\, v(x_0) - c_{33}\, w(x_0) \ge 0, \tag{4.22}$$

which gives

$$w(x) \le w(x_0) \le \frac{1}{c_{33}}\big(\sigma_3 - c_{31}\, u(x_0) - c_{32}\, v(x_0)\big) < \frac{\sigma_3}{c_{33}},\ x \in \mathbb{R}. \tag{4.23}$$

As a consequence, we have

$$\begin{cases} d_1\, u_{xx} + \theta\, u_x + u(\sigma_1 - c_{13}\, \sigma_3\, c_{33}^{-1} - c_{11}\, u - c_{12}\, v) \le 0, & x \in \mathbb{R}, \\ d_2\, v_{xx} + \theta\, v_x + v(\sigma_2 - c_{23}\, \sigma_3\, c_{33}^{-1} - c_{21}\, u - c_{22}\, v) \le 0, & x \in \mathbb{R}. \end{cases} \tag{4.24}$$

[**H1**] assures the positivity of $\sigma_1 - c_{13}\, \sigma_3\, c_{33}^{-1}$ and $\sigma_2 - c_{23}\, \sigma_3\, c_{33}^{-1}$, while the strong competition condition of u and v follows from [**H2**]. Consequently, we obtain a lower bound of $c_{31}\, u(x) + c_{32}\, v(x)$, i.e.

$$c_{31}\, u(x) + c_{32}\, v(x) \ge c_{33}^{-1}\, \min\left[\frac{c_{31}\, \phi_2}{c_{21}\, d_2}, \frac{c_{32}\, \phi_1}{c_{12}\, d_1}\right] \min\left[d_1^2, d_2^2\right],\ x \in \mathbb{R}. \tag{4.25}$$

The condition [**H3**] then yields

$$c_{31}\, u(x) + c_{32}\, v(x) \ge \sigma_3,\ x \in \mathbb{R}, \tag{4.26}$$

which contradicts (4.22). This completes the proof.

□

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Very weak solutions to hyperbolic equations

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The talk will be based on the joint works with Claudia Garetto (University of Loughborough) and Niyaz Tokmagambetov (Kazakhstan, and currently Imperial College London), and this abstract is based on the joint work with the latter.

In this talk we will discuss hyperbolic equations with time-dependent singular coefficients. The singularity here means that the coefficients are allowed to be distributions.

Part of the talk will be devoted to the setting of the wave equation in $\mathbb{R}^n$, but to motivate part of the appearing constructions let us briefly describe an example of the Landau Hamiltonian. In this description we closely follow our recent preprint [RT16a] so there is an overlap in the presentation.

Consider a non-relativistic particle with mass m and electric charge e moving in a given electromagnetic field. We concentrate for simplicity on the 2D version which can be easily extended to higher dimensions. To describe the electromagnetic field in the plane one usually uses the electromagnetic scalar and vector potentials $q, \mathbb{A}$. The dynamics of a particle with mass m and charge e on the Euclidean xy–plane, while interacting with a perpendicular homogeneous electromagnetic field, is determined by the Hamiltonian (see [LL77])

$$\mathbb{H}_0 := \frac{1}{2m}\Big(ih\nabla - \frac{e}{c}\mathbb{A}\Big)^2 + eq, \tag{1}$$

where h denotes Planck's constant, c is the speed of light and i the imaginary unit. Denote by $2B > 0$ the strength of the magnetic field and select the symmetric gauge

$$\mathbb{A} = -\frac{\mathrm{r}}{2} \times 2\mathrm{B} = (-By, Bx),$$

where r $= (x, y) \in \mathbb{R}^2$. For simplicity, we set $m = e = c = h = 1$ in (1), leading to the Landau Hamiltonian

$$\mathbb{H}_1 := \mathbb{H} + q$$

where

$$\mathbb{H} := \frac{1}{2}\left(\left(i\frac{\partial}{\partial x} - By\right)^2 + \left(i\frac{\partial}{\partial y} + Bx\right)^2\right), \tag{2}$$

acting on the Hilbert space $L^2(\mathbb{R}^2)$. It is a classical result (see [F28, L30]) that the spectrum of the operator $\mathbb{H}$ consists of infinite number of eigenvalues with infinite multiplicity of the form

$$\lambda_n = (2n+1)B, \;\; n = 0, 1, 2, \ldots . \tag{3}$$

These eigenvalues are called the Euclidean Landau levels. Denote the eigenspace of $\mathbb{H}$ corresponding to the eigenvalue λ_n in (3) by

$$\mathcal{A}_n(\mathbb{R}^2) = \{\varphi \in L^2(\mathbb{R}^2), \; \mathbb{H}\varphi = \lambda_n\varphi\}. \tag{4}$$

The following functions form an orthogonal basis for $\mathcal{A}_n(\mathbb{R}^2)$ (see [ABGM15, HH13]):

(5)

$$\begin{cases} e^1_{k,n}(x,y) = \sqrt{\dfrac{n!}{(n-k)!}}B^{\frac{k+1}{2}} \exp\Big(-\dfrac{B(x^2+y^2)}{2}\Big)(x+iy)^k L_n^{(k)}(B(x^2+y^2)), \;\; 0 \le k, \\ e^2_{j,n}(x,y) = \sqrt{\dfrac{j!}{(j+n)!}}B^{\frac{n-1}{2}} \exp\Big(-\dfrac{B(x^2+y^2)}{2}\Big)(x-iy)^n L_j^{(n)}(B(x^2+y^2)), \;\; 0 \le j, \end{cases}$$

where $L_n^{(\alpha)}$ is the Laguerre polynomial defined as

$$L_n^{(\alpha)}(t) = \sum_{k=0}^{n}(-1)^k C_{n+\alpha}^{n-k}\frac{t^k}{k!}, \;\; \alpha > -1.$$

To simplify the notation further we denote

$$e^k_\xi := e^k_{j,n} \;\text{ for }\; \xi = (j,n), \;\; j,n = 0,1,2,...; \;\; k = 1,2. \tag{6}$$

In his book ([P86], p. 35), Perelomov points out that the basis (5) had been used by Feynman and Schwinger in a somewhat different form in order to obtain an explicit expression for the matrix elements of the displacement operator. The functions (5) are also related to the complex Hermite polynomials [I16]. They occur naturally in several problems and different representations are used. For instance, they have recently found applications in quantization [ABG12, BG14, CGG10], time-frequency analysis [A10], partial differential equations [G08] and planar point processes [HH13].

We can also mention papers [K16, N96], where the authors investigated properties of eigenfunctions of perturbed Hamiltonians, and in [S14, KP04, M91, PR07, PRV13, LR14, RT08] asymptotics of the eigenvalues for perturbed Landau Hamiltonians were described.

In this abstract we consider the wave equation for the Landau Hamiltonian with time-dependent irregular electric potential and varying in time electromagnetic field. More precisely, for a distributional propagation speed function $a = a(t) \geq 0$ and for the distributional electromagnetic scalar potential $q = q(t)$, we consider the Cauchy problem for the Landau Hamiltonian $\mathbb{H}$ in the form

$$\begin{cases} \partial_t^2 u(t,x) + a(t)[\mathbb{H} + q(t)]u(t,x) = 0, \ (t,x) \in [0,T] \times \mathbb{R}^2, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^2, \\ \partial_t u(0,x) = u_1(x), \ x \in \mathbb{R}^2. \end{cases} \tag{7}$$

A special feature of our analysis is that we want to allow a and q to be distributions. For instance, if the electric potential produces shocks these can be modelled with δ-distributions, for example by taking $q = \delta_1$, the δ-distribution at time $t = 1$. Moreover, if the velocity $a(t)$ also contained δ-type terms, as an example of such an equation we could consider

$$\begin{cases} \partial_t^2 u(t,x) + \delta_1[\mathbb{H} + \delta_1]u(t,x) = 0, \ (t,x) \in [0,T] \times \mathbb{R}^2, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^2, \\ \partial_t u(0,x) = u_1(x), \ x \in \mathbb{R}^2. \end{cases} \tag{8}$$

The problem that we are interested in is as follows:

How to understand the Cauchy problems (7)-(8) *and their well-posedness?*

There are several difficulties already at the fundamental level for such problems, first of all in general impossibility of multiplying distributions due to the famous Schwartz' impossibility result [S54]. Second, even if we could somehow make sense of the product aq being a distribution by e.g. imposing wave front conditions, we would still have to multiply it with $u(t,x)$ which, a-priori, may also have singularities in t, thus leading to another multiplication problem. Moreover, another difficulty (for the global in space analysis of (7)) is that the coefficients of $\mathbb{H}$ increase in space thus leading to potential problems at infinity if we treat the problem only locally.

In our analysis we assume that a is a positive distribution so that the Cauchy problem (7) is of hyperbolic type, at least when a and q are regular. More precisely, we will assume that there exists a constant $a_0 > 0$ such that

$$a \geq a_0 > 0,$$

where $a \geq a_0$ means that $a - a_0 \geq 0$, i.e. $\langle a - a_0, \psi \rangle \geq 0$ for all $\psi \in C_0^\infty(\mathbb{R})$, $\psi \geq 0$. Incidentally, the structure theory of distributions implies that a is a Radon measure but this does not remove the multiplication problems or problem with understanding the meaning of the well-posedness of the Cauchy problem (7).

Nevertheless, we are able to study the well-posedness of (7) using an adaptation of the notion of very weak solutions introduced in [GR15b] in the context of hyperbolic problems with distributional coefficients in $\mathbb{R}^n$.

As noted, the equation (7) can not be, in general, understood distributionally, so we are forced to weaken the notion of solutions. However, we want to do it in a way so that we can recapture classical solutions should they exist. Thus, in this talk we will present the following facts:

- The Cauchy problem (7) admits a *very weak solution* even for distributional Cauchy data u_0 and u_1. The very weak solution is unique in an appropriate sense.
- If the coefficients a and q are regular so that the Cauchy problem (7) has a 'classical' solution, the very weak solution recaptures this classical solution in the limit of the regularising parameter. This shows that the introduced notion of a very weak solution is consistent with classical solutions should the latter exist.
- When the classical solution does not exist, the very weak solution comes with an explicit numerical scheme modelling the limiting behaviour of regularised solutions.

We also note that at the same time our analysis will yield results for the modified problem

$$(9)\quad \begin{cases} \partial_t^2 u(t,x) + a(t)\mathbb{H}u(t,x) + q(t)u(t,x) = 0,\ (t,x) \in [0,T] \times \mathbb{R}^2, \\ u(0,x) = u_0(x),\ x \in \mathbb{R}^2, \\ \partial_t u(0,x) = u_1(x),\ x \in \mathbb{R}^2, \end{cases}$$

for distributions a, q with $a \geq a_0 > 0$ for some constant a_0.

For second order operators $\mathbb{H}$ independent of x the Cauchy problems of this type have been intensively studied, however for more regular (starting from Hölder) coefficients, see for example [CC13, CDGS79, CDSK02, CDSR03, DS98] and references therein. For the setting of distributional coefficients see [GR15b].

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Traveling waves composed of convex closed curves in anisotropic curve shortening flow with a driving force

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1 Introduction

In this talk, we investigate the existence of traveling waves composed of closed curves $\Gamma(t)$ in $\mathbb{R}^2$, which moves by

$$\beta(\theta)V = \gamma(\theta) - \alpha(\theta)\kappa, \tag{1}$$

where $\Gamma(t)$ is a closed curve depending on the time t in $\mathbb{R}^2$, κ is the curvature of $\Gamma(t)$, V is the normal velocity of $\Gamma(t)$, θ is the angle between x-axis and normal vector $\mathbf{n}$ on $\Gamma(t)$, α, β and γ are 2π-periodic functions.

The interface (1) appears in various equations. The most simple example of (1) is *the curve shortening flow*, describing the motion of grain boundaries, which corresponds to the case $\alpha \equiv \beta \equiv 1$ and $\gamma \equiv 0$ in (1):

$$V = -\kappa. \tag{2}$$

Grayson[5] showed that any closed curves without self-intersection in $\mathbb{R}^2$ eventually become to the convex shape in a finite time even though the initial shape of $\Gamma(0)$ is non-convex. According to Gage-Hamilton[4], any convex closed curve shrinks to a single point in a finite time. In addition, it was confirmed that (1) has a traveling wave called "Grim Reaper" defined on the whole space (cf.[1]). As the another typical example of (1), there is *the curvature-eikonal flow*, related to wave propagation, which corresponds to the case $\alpha \equiv \beta \equiv 1$ and $\gamma \equiv A$ in (1):

$$V = A - \kappa, \tag{3}$$

where A is a positive constant. For instance, Ninomiya-Taniguchi [9] showed the existence of traveling waves defined on the whole space, called *V-shaped traveling front*.

On the other hand, (1) also appears in the singular limit of some reaction-diffusion equations. For instance, Ei-Yanagida [3] investigated the behaviour of the motion of the competitive Lotka-Volterra equations with a small parameter $\varepsilon > 0$ as follows :

$$\begin{cases} u_t = \varepsilon^2 \Delta u + (1 - u - cv)u & \text{in } Q_T, \\ v_t = d\varepsilon^2 \Delta v + (a - bu - v)v & \text{in } Q_T, \end{cases} \tag{4}$$

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Here Q_T stands for the parabolic domain, that is, $Q_T := \Omega \times (0,T)$, where Ω is a bounded domain in $\mathbb{R}^N$, T is a positive constant. They showed that, under $d > 0$ and $0 < 1/c < a/b$, as ε is sufficiently small, the region Ω is occupied by $\Omega_1(t) := \{x \in \Omega : (u,v) \sim (r_1/a_1, 0)\}$ and $\Omega_2(t) := \{x \in \Omega : (u,v) \sim (0, r_2/a_2)\}$. In addition, these two regions are approximately separated by a hypersurface $\Gamma(t)$ which moved by

$$V = c - \varepsilon L(d)(N-1)\kappa, \tag{5}$$

where $L(d)$ is a positive constant depending on the diffusion coefficient d, c is the speed of traveling wave for (4) in one-dimensional case. Note that (5) corresponds to the case $\alpha \equiv \varepsilon L(d)(N-1)$, $\beta \equiv 1$ and $\gamma \equiv c$ in (1). Similarly, there are some reaction-diffusion systems such that mean curvature flows (1) appear in the singular limit of the systems when the diffusion rate is very small (see [8]).

Moreover (1) has a concern in some free boundary problems. The authors [6] analyzed a free boundary problem describing cell locomotion as follows:

$$\begin{cases} u_t = d\Delta u + k_1 U - u + k_2 & \text{in } Q := \bigcup_{t>0} \Omega(t) \times \{t\}, \\ u = 1 + A\kappa & \text{on } \Gamma := \bigcup_{t>0} \Gamma(t) \times \{t\}, \\ V = \eta U - 1 - A\kappa & \text{on } \Gamma, \end{cases} \tag{6}$$

where $\Omega(t)$ is a domain depending on the time t in $\mathbf{R}^2$, $\Gamma(t)$ is the boundary of $\Omega(t)$. The function U is written by

$$U = C_0 - \int_{\Omega(t)} u\, dx,$$

d, k_1, k_2, A, C_0 are positive constants, η is a given function defined on $\Gamma(t)$. As seen in (6), the free boundary condition

$$V = \eta U - 1 - A\kappa \tag{7}$$

is similar to (1) with $\alpha \equiv A$, $\beta \equiv 1$ and $\gamma \equiv \eta U - 1$. Note that γ is a given function in (1) and U is a unknown function in (7). In this model, u stands for the density of Actinfilament, U stands for the concentration of G-actin, Ω is the cytoskelton of cell and η represents the activity of the Polymerization of Actinfilament (see [6] for details).

Actually, when we consider the existence of traveling wave of (6), it is essentially important to study the existence of (1)(see [6]). In these days, Choi-Lui [2] treated another free boundary problem related to cell locomotion. They also succeeded to show the existence of traveling waves for the problem. In the proof, it was confirmed that the analysis of traveling waves to (1) plays an important role.

As mentioned above, we verify that (1) appears in some interface equations, free boundary problems and the singular limit of reaction-diffusion systems. From this consideration, we can expect that the following question is important to show the existence of traveling waves in some partial differential equations:

> **Q.** Under what condition of α, β and γ doe the interface equation (1) have traveling waves ? What shape is it ?

In this talk, we introduce a partial answer to this question. From now on, we only treat traveling waves composed of simple closed curves.

2 Main results

Throughout this talk, we impose the following assumption on α and β:

Assumption 1. *We assume that $\alpha \equiv 1$ and β is a positive Lipschitz function and 2π-periodic.*

Definition 1. *Let $\Gamma(t)$ be a smooth simple closed curve satisfying (1). Then we call $\Gamma(t)$ traveling wave of (1) if there exists a pair $(c, \mathbf{n}_0)$ such that*

$$\Gamma(t) = \{\mathbf{x} := (x, y) \in \mathbb{R}^2 \mid \mathbf{x} - ct\,\mathbf{n}_0 \in \Gamma(0)\},$$

where c is a positive constant and $\mathbf{n}_0$ is a unit vector with an angle θ_0 in $[0, 2\pi)$, i.e., $\mathbf{n}_0 = (\cos\theta_0, \sin\theta_0)$.

By simple calculations, we immediately know the convexity of traveling waves.

Lemma 1. *Let $\Gamma(t)$ be a traveling wave with C^3 in (1). Then the traveling wave is strictly convex.*

Our main results are stated as follows:

Theorem 1. *Assume that γ is a positive Lipschitz function and 2π-periodic. Then there exists a unique traveling wave of (1).*

Theorem 2. *There exists at most one traveling wave of (1).*

If β has a symmetry, our statement will be more sharp. To avoid the complexity, we only stated the result for the simple case.

Theorem 3. *Assume that β satisfies $\beta(\theta) = \beta(-\theta)$ in $[0, \pi]$. Then there exists a unique traveling wave of (1) if and only if γ satisfies the following three conditions :*

(a) *γ is a Lipschitz function and 2π-periodic.*

(b) *$\gamma(\theta) = \gamma(-\theta)$ and $\gamma > 0$ in $[0, \pi/2]$.*

(c) *If $\gamma > 0$ in $(\pi/2, \pi]$, then it holds*

$$\int_0^\pi \frac{\cos\theta}{\gamma(\theta)}\, d\theta \le 0,$$

where the equality is true for $c = 0$. Otherwise it holds

$$\sup_{\pi/2<\theta<\pi} \frac{\gamma}{\beta\cos\theta} < \inf_{0<\theta<\pi/2} \frac{\gamma}{\beta\cos\theta}.$$

Remark 1. *In Theorem 3, if we assume that $\beta(\theta - \theta_*) = \beta(-\theta + \theta_*)$ for a constant $\theta_* \in [0, 2\pi)$ instead of $\beta(\theta) = \beta(-\theta)$, the similar result is obtained.*

3 Application

In this section, we introduce the application to some interface equations and free boundary problems. By Theorem 2 and 3, we immediately obtain the following result:

Theorem 4. (2) *does not have any traveling waves.* (3) *and* (5) *only have a disk-shaped traveling wave with* $c = 0$.

Next we investigate the existence of traveling waves to a free boundary problem (6) as shown in the above example. Before we state the reuslt, we give the definition of traveling wave to (6).

Definition 2. *Let* $(u, \Omega(t))$ *be a classical solution of* (6). *Then we call* $(u, \Omega(t))$ *traveling wave of* (6) *if there exists a pair* $(c, \mathbf{n}_0)$ *such that* $(u, \Omega(t))$ *satisfy* (6) *and*

$$\Omega(t) = \{\mathbf{x} = (x, y) \in \mathbb{R}^2 \mid \mathbf{x} - ct\mathbf{n}_0 \in \Gamma(0)\}$$
$$u(\mathbf{x}, t) = u(\mathbf{x} - ct\mathbf{n}_0, 0)$$

where c *and* $\mathbf{n}_0$ *are the same notation as in Definition 1.*

Remark 2. *The classcial solution of* (6) *implies that* $(u, \Omega(t)) \in C^{2,1}(Q) \times C^{4,2}$.

Theorem 5 (M-Ninomiya [7])**.** *Let* η *be a* 2π*-periodic function in* $C^2(\mathbb{R})$ *satisfying*

$$\eta(\theta) = \eta(-\theta) \text{ in } [0, \pi], \quad \eta(\theta) > \eta(\pi - \theta) > 0 \text{ in } [0, \pi/2). \tag{8}$$

Then, if k_1, k_2 *and* C_0 *satisfy*

$$k_1 C_0 - 1 + k_2 < 0, \quad \eta_{min} C_0 - 1 > 0,$$

there exists a positive constant A_0 *for any* $A \in (0, A_0)$*, there exist a traveling wave of* (6) *with the following properties:*

(P_1) Ω *is convex and* $c > 0$.

(P_2) *For any* $(x, y) \in \Omega$*, it holds that*

$$k_2 \le u < 1 + \max_{\partial\Omega} \theta_s, \quad U > 0.$$

Remark 3. *We can extend this result to more general one if we replace* (8) *by*

$$\eta(\theta - \theta_*) = \eta(-\theta + \theta_*) \text{ in } [0, \pi], \quad \eta(\theta - \theta_*) > \eta(\pi - \theta + \theta_*) > 0 \text{ in } [0, \pi/2).$$

for a constant $\theta_* \in [0, 2\pi)$.

4 Conclusion

In this talk, we focused on the driving force γ and investigated under what condition of γ (1) has a traveling wave. In conclusion, we obtained some results related to the existence, non-existence and shape of traveling waves to interface equations and a free boundary problem in $\mathbb{R}^2$. However we do not have any results in $\mathbb{R}^3$. It is interesting for us to investigate this problem. In addition, there does not exist a non-convex traveling wave under our assumption for γ. Therefore it is also interesting to investigate under what condition of γ there exists a non-convex shaped traveling waves to (1).

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REMARK ON THE STABILITY OF THE HARDY–LITTLEWOOD–SOBOLEV INEQUALITY AND RELATED PROBLEMS

CHRIS JEAVONS

Abstract. This talk will be based upon recent work [5] joint with Neal Bez (Saitama Univ.) and Tohru Ozawa (Waseda Univ.).

For $1 < p, q < \infty$ and measure spaces X and Y, assume that a linear operator T satisfies the inequality

$$\|Tf\|_{L^q(Y)} \leq \|T\|\|f\|_{L^p(X)}. \tag{1}$$

Recall that the operator norm $\|T\|$ may be interpreted as the best (i.e smallest) constant in the inequality (1), or in other words,

$$\|T\| = \inf_{f \in L^p \setminus \{0\}} \frac{\|Tf\|_{L^q(Y)}}{\|f\|_{L^p(X)}}.$$

The problem of computing optimal constants and determining cases of equality for estimates of the form (1) has attracted considerable attention in recent years and has found a number of applications in the nonlinear theory; see [15] for a particularly striking example.

It is well-known that the usual adjoint operator T^* satisfies

$$\|T^*G\|_{L^{p'}(Y)} \leq \|T\|\|G\|_{L^{q'}(X)} \tag{2}$$

and that the constant is optimal, where $r' := \frac{r}{r-1}$ for $1 < r < \infty$. Less well-known, however, is the fact that a relation exists between nontrivial $f \in L^p$ and $G \in L^{q'}$ such that one has equality in (1) and (2), respectively. An important example is contained in the paper [18] of Lieb, where a characterisation of all extremisers (i.e. cases of equality) for the fractional Sobolev inequality

$$\|G\|_{L^{p'}(\mathbb{R}^d)} \leq C_{\mathrm{FS}} \|(-\Delta)^{\frac{d-\lambda}{4}} G\|_{L^2(\mathbb{R}^d)} \tag{3}$$

is established as a consequence of the analogous result for its dual, equivalent to the diagonal Hardy–Littlewood–Sobolev inequality

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(x)\overline{g(y)}}{|x-y|^\lambda} \, \mathrm{d}x\mathrm{d}y \right| \leq C_{\mathrm{HLS}} \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{R}^d)}. \tag{4}$$

Here, $p = \frac{2d}{2d-\lambda}$ and $0 < \lambda < d$, and we use the notation C_{FS} and C_{HLS} for the optimal constants in (3) and (4), respectively (for convenience, we suppress the dependence on λ and d of these quantities). The sharp inequality (4) and associated characterisation of extremisers is also proved in [18]. In the case $\lambda = d - 2$ (i.e. the

classical Sobolev inequality) the value of C_{FS} and characterisation of extremisers for (3) was obtained earlier by Aubin [1] and Talenti [23] independently.

A more recent example concerns the trace theorem on the sphere. Specifically, for the operator $\mathcal{R}$ defined by $\mathcal{R}f := f|_{\mathbb{S}^{d-1}}$, consider the inequality

$$\|\mathcal{R}f\|_{L^q(\mathbb{S}^{d-1})} \leq C_{\mathrm{T}} \|(-\Delta)^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^d)} \tag{5}$$

for $s \in (\frac{1}{2}, \frac{d}{2})$ and $q := \frac{2(d-1)}{d-2s}$, and where the constant C_{T} is taken to be optimal. By Hölder's inequality, this is a refinement of the classical trace theorem on the sphere

$$\|\mathcal{R}f\|_{L^2(\mathbb{S}^{d-1})} \leq |\mathbb{S}^{d-1}|^{\frac{1}{2}-\frac{1}{q}} C_{\mathrm{T}} \|(-\Delta)^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^d)}, \tag{6}$$

and this constant can be shown to be optimal. The sharp form of inequality (6) is related to a number of recent results concerning sharp smoothing estimates for solutions to linear dispersive equations, see for example [7]. The optimal constant for (6) was first found in [22], and the optimal constant in (5) and the characterisation of extremisers for both (5) and (6) was proved in [6] and independently in [4]; the characterisation is proved by first establishing the same thing for an appropriate dual inequality.

In what follows we will consider the following closely related problem in view of the above discussion: suppose one has a sharp inequality such as (1), which has nontrivial extremisers. If the inequality is close to being attained (in some sense) on some nontrivial input, then is (and in what sense is) that input close to an extremiser? Such results are known as *stability estimates*; the classical example is Bonnesen's refinement of the isoperimetric inequality (see [21] for discussion), but this problem has also been studied for important analytic inequalities connected to the study of PDEs. For example, for the inequality (3) it is known that there exists $\alpha > 0$ such that

$$\alpha \inf_{G_* \in M_{\mathrm{FS}}} \|(-\Delta)^{\frac{d-\lambda}{4}} (G - G_*)\|_2^2 \leq C_{\mathrm{FS}}^2 \|(-\Delta)^{\frac{d-\lambda}{4}} G\|_2^2 - \|G\|_{p'}^2 \tag{7}$$

for any G such that $(-\Delta)^{\frac{d-\lambda}{4}} G \in L^2(\mathbb{R}^d)$, where we use the notation M_{FS} for the set of functions for which equality holds in (3); we choose to suppress the dependence of this set on d and λ. The inequality (7) goes back to Brezis–Lieb who in [8] conjectured that it should hold in the case $\lambda = d - 2$, i.e. for the classical Sobolev inequality. In this case, Bianchi–Egnell [9] proved that (7) holds, and their approach was generalised to further values of λ in [3] and [20], and to the full range of λ by Chen–Frank–Weth in [13]. For further examples of stability estimates and applications in related contexts, see [12] for the $W^{1,p}$ Sobolev inequality, as well as [10] and [11] for some Gagliardo–Nirenberg–Sobolev inequalities.

Our main result is the following in the context of the general estimate (1). In order to state it, we define the class of non-trivial extremisers for the inequality (1) by

$$M(T) := \{f \in L^p(X) \setminus \{0\} : \|Tf\|_{L^q(Y)} = \|T\| \|f\|_{L^p(X)}\};$$

the class $M(T^*)$ for (2) is defined similarly. We also define, for $1 < r < \infty$, the distance between $f \in L^r$ and $S \subset L^r$

$$d_r(f, S) := \inf_{f_* \in S} \|f - f_*\|_r,$$

and we use $A \lesssim B$ ($A \gtrsim B$) to denote $A \leq CB$ ($A \geq CB$) for some constant $C > 0$ which may depend on p, q or $\|T\|$ but never on f or G. The value of C may change from line to line.

Theorem 1. *Let $T : L^p(X) \to L^q(Y)$ for $1 < p, q < \infty$. Assume that*

$$\|T\|^\eta \|G\|_{q'}^\eta - \|T^* G\|_{p'}^\eta \gtrsim d_{q'}(G, M(T^*))^{\eta'} \tag{8}$$

holds for any $G \in L^{q'} \setminus \{0\}$, where $\eta := \max\{p, 2\}$. If $d_p(f, M(T)) < \frac{\|f\|_p}{4}$ then

$$\|T\|^\eta \|f\|_p^\eta - \|Tf\|_q^\eta \gtrsim d_p(f, M(T))^\eta. \tag{9}$$

Further, if for any $(f_n) \subset L^p$, $\frac{\|Tf_n\|_q}{\|f_n\|_p} \to \|T\|$ implies that (up to a subsequence) $\frac{d_p(f_n, M(T))}{\|f_n\|_p} \to 0$, then (9) holds for any $f \neq 0$.

Remark. The value of the exponent η in Theorem 1 depends on whether the domain of T is L^p with $p < 2$ or $p > 2$, and is optimal in the sense that one cannot expect anything better than (9) to hold under the assumption (8). This is connected to the optimal exponents in the well-known uniform convexity inequalities for L^p space; in particular, L^p is no better than 2-uniformly convex when $p < 2$, and no better than p-uniformly convex when $p > 2$, see [2].

Our first application is to the inequality (4): for $0 < \lambda < d$ and $p = \frac{2d}{2d-\lambda}$ consider the inequality

$$\|(-\Delta)^{\frac{\lambda-d}{4}} f\|_{L^2(\mathbb{R}^d)} \leq C_{\mathrm{FS}} \|f\|_{L^p(\mathbb{R}^d)}; \tag{10}$$

note that by duality the constant C_{FS} is optimal for this inequality. Recently, in [19] the inequality

$$d_p(f, M_{\mathrm{HLS}})^2 \lesssim C_{\mathrm{FS}}^2 \|f\|_{L^p(\mathbb{R}^d)}^2 - \|(-\Delta)^{\frac{\lambda-d}{4}} f\|_{L^2(\mathbb{R}^d)}^2 \tag{11}$$

is proved for $f \neq 0$. Here and throughout, M_{HLS} is defined to be the set of extremisers for (10) (again, suppressing the dependence on d and λ); one has equality in (4) if and only if $f = cg \in M_{\mathrm{HLS}}$, see [18]. The inequality (11) is in fact proved in the more general setting of the Heisenberg group, and an extension of (7) to this setting is also established in [19]; the underlying sharp versions of (3) and (4) in this case were proved in [16]. We note that (11) implies the following stability estimate for (4)

$$d_p(f, M_{\mathrm{HLS}}) d_p(g, M_{\mathrm{HLS}}) \lesssim C_{\mathrm{HLS}} \|f\|_p \|g\|_p - \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(x)\overline{g(y)}}{|x-y|^\lambda} \, \mathrm{d}x\mathrm{d}y \right| \tag{12}$$

for any nontrivial f and g, as a simple corollary. Indeed, by standard arguments (see e.g. [18]) the right hand side of (11) equals a constant multiple of the right hand side of (12) with $f = g$, so in this case we are done. If $f \neq g$ then note that by dividing (12) through by $\frac{\|g\|_p}{\|f\|_p}$ and using the fact that M_{HLS} is preserved under multiplication by scalars, we may assume that $\|f\|_p = \|g\|_p$. Define

$$I_\lambda(f, g) = \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(x)\overline{g(y)}}{|x-y|^\lambda} \, \mathrm{d}x\mathrm{d}y \right|,$$

then since the kernel $|x-y|^{-\lambda}$ is positive definite one has

$$I_\lambda(f, g) \leq I_\lambda(f, f)^{\frac{1}{2}} I_\lambda(g, g)^{\frac{1}{2}} \leq \frac{1}{2}(I_\lambda(f, f) + I_\lambda(g, g)),$$

from which it follows that

$$\begin{aligned} C_{\mathrm{HLS}}\|f\|_p\|g\|_p - I_\lambda(f,g) &\geq \frac{1}{2}\left(C_{\mathrm{HLS}}\|f\|_p^2 - I_\lambda(f,f) + C_{\mathrm{HLS}}\|g\|_p^2 - I_\lambda(g,g)\right) \\ &\gtrsim d_p(f, M_{\mathrm{HLS}})^2 + d_p(g, M_{\mathrm{HLS}})^2 \\ &\gtrsim d_p(f, M_{\mathrm{HLS}}) d_p(g, M_{\mathrm{HLS}}), \end{aligned}$$

as claimed.

The strategy used in [19] to prove (11) is broadly similar to the one used to prove (7) in [13] but with a key difference: since $p < 2$, the Lebesgue exponents in (7) are at least two and so one may consider a classical Taylor expansion to second order of the right hand side, but in (11) one encounters an L^p-norm with $p < 2$, so such an expansion is not possible. To remedy this, it is observed in [19] that a deep technical result developed by Christ in [14] may be used to substitute for this failure of Taylor expansion; using this fact and solving an appropriate generalisation of the variational problem used in [9] and [13], the inequality (11) is obtained.

By combining Theorem 1 with some known compactness properties of the inequalities (3) and (4), we can obtain the following result. We state it here on $\mathbb{R}^d$ for the sake of exposition but believe it to be valid within the more general framework of [19].

Corollary 1. *Suppose that $p = \frac{2d}{2d-\lambda}$ for $0 < \lambda < d$. Then the inequalities* (7) *and* (11) *are equivalent.*

As a consequence we obtain an alternative, simpler proof of (11) by using either (7) or its generalisation from [19] in a black-box fashion, thereby avoiding the use of the result from [14]; as far as we know, the fact that one may use duality to obtain stability estimates in this manner has not been noted before.

Remark. The properties of (3) and (4) used in the proof of Corollary 1 are quite generic; in particular we do not use the precise characterisation of M_{FS} or M_{HLS}. When establishing and making use of remainder term inequalities, it is not always necessary to have such a precise characterisation (see [11] for an example) and as such, we hope that our approach may prove useful in the study of related problems.

Remark. One drawback of our result is that it does not allow one to transfer bounds on the implicit constants between the global versions of (7) and (11); it remains an open problem to obtain a proof of either of these inequalities which gives a bound on the implicit constant. For some results in this direction for (3) and (4), see [17] and the references contained there.

Our second application is to the L^q trace inequality. For convenience, we work with the equivalent formulation of (5)

$$\|\mathcal{S}f\|_{L^q(\mathbb{S}^{d-1})} \leq \|\mathcal{S}\|_{L^2\to L^q}\|f\|_{L^2(\mathbb{R}^d)}, \tag{13}$$

where $\mathcal{S}f := \mathcal{R}(-\Delta)^{-\frac{s}{2}}f$. By duality, (13) is equivalent to

$$\|\mathcal{S}^*G\|_{L^2(\mathbb{R}^d)} \leq \|\mathcal{S}\|_{L^2\to L^q}\|G\|_{L^{q'}(\mathbb{S}^{d-1})}, \tag{14}$$

and it is proved in [6] that this is equivalent to the following diagonal sharp Hardy–Littlewood–Sobolev inequality on the sphere:

$$\left| \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} G(\omega)\overline{G(\eta)}(1-\omega\cdot\eta)^{s-\frac{d}{2}}\,\mathrm{d}\sigma(\omega)\mathrm{d}\sigma(\eta) \right| \leq C\|G\|^2_{L^{q'}(\mathbb{S}^{d-1})}.$$

By applying the stereographic projection from [18], it may be shown that this is equivalent to the inequality (4) on $\mathbb{R}^{d-1}$ (with $\lambda = d-2s$) in the case $f = g$. As a consequence, we can use Theorem 1 and an appropriate version of (12) to obtain the following new stability theorem for (13); as usual we use $M(\mathcal{S})$ to denote the set of functions for which equality holds in the estimate (13).

Corollary 2. *Suppose that $s \in (\frac{1}{2}, \frac{d}{2})$, and $q = \frac{2(d-1)}{d-2s}$. Then the inequality*

$$d_2(f, M(\mathcal{S}))^2 \lesssim \|\mathcal{S}\|^2_{L^2\to L^q}\|f\|^2_{L^2(\mathbb{R}^d)} - \|\mathcal{S}f\|^2_{L^q(\mathbb{S}^{d-1})} \tag{15}$$

holds for any $f \in L^2(\mathbb{R}^d)\setminus\{0\}$.

Crucial to our proof of Theorem 1 is the following lemma, which makes precise our discussion of a relation between the sets of extremisers for (3)-(6), above.

Lemma 1. *Suppose that $T : L^p(X) \to L^q(Y)$. Then $M(T) \neq \emptyset$ if and only if $M(T^*) \neq \emptyset$. In this case,*

$$M(T) = |T^*M(T^*)|^{p'-2}T^*M(T^*).$$

Although it does not seem to appear in the literature in this generality this result is likely to be folk-lore, being implicit in the papers [6] and [18] already mentioned, as well as in [14] in work on a stability estimate for the Hausdorff–Young inequality.

In my talk I will discuss the proofs of the above results and describe some possible extensions of Theorem 1.

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ON SCATTERING PROBLEM ON NONLINEAR SCHRÖDINGER EQUATIONS IN MASS-SUBCRITICAL CASE

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1. Introduction

This talk is based on a joint work with R. Killip, J. Murphy, and M. Visan [3]. We consider the following nonlinear Schrödinger equation:

$$\text{(NLS)} \qquad \begin{cases} i\partial_t u + \Delta_x u = \mu|u|^p u, & t \in \mathbb{R}, x \in \mathbb{R}^d, \\ u(t_0, x) = u_0(x) \in e^{it_0\Delta}\mathcal{F}\dot{H}^{s_c}, \end{cases}$$

where, $d \geqslant 1$, $\mu = \pm 1$, and $t_0 \in \mathbb{R}$. Our aim is to investigate time global behavior of solutions to (NLS) in *mass-subcritical case* $p < 4/d$. More specifically, we consider the case

$$\max\left(\frac{2}{d}, \frac{4}{d+2}\right) < p < \frac{4}{d}. \tag{1}$$

The function space $\mathcal{F}\dot{H}^s$ $(0 \leqslant s < d/2)$ is a homogeneous weighted L^2 space defined by the norm $\|f\|_{\mathcal{F}\dot{H}^s} = \||x|^s f\|_{L^2}$. $f \in e^{it\Delta}\mathcal{F}\dot{H}^s$ implies $e^{-it\Delta}f \in \mathcal{F}\dot{H}^s$. s_c is a scaling critical exponent defined by $s_c := \frac{2}{p} - \frac{d}{2} \in (0, \min(1, d/2))$. Remark that (NLS) has a scaling property: If $u(t,x)$ is a solution then

$$u_{[\lambda]}(t,x) := \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x) \tag{2}$$

is also a solution. Then, $\mathcal{F}\dot{H}^{s_c}$ is scale invariant in such a sense that the $\mathcal{F}\dot{H}^{s_c}$-norm is invariant under

$$f_{\{\lambda\}}(x) := \lambda^{\frac{2}{p}} f(\lambda x). \tag{3}$$

Namely $\|f_{\{\lambda\}}\|_{\mathcal{F}\dot{H}^{s_c}} = \|f\|_{\mathcal{F}\dot{H}^{s_c}}$ for any $\lambda > 0$. Local well-posedness of (NLS) in $\mathcal{F}\dot{H}^{s_c}$ is given in [5] (see also [11]). For any given data $u_0 \in e^{it_0\Delta}\mathcal{F}\dot{H}^{s_c}$, there exist an interval $I \ni t_0$ and unique solution $u \in C(I, e^{i\cdot\Delta}\mathcal{F}\dot{H}^{s_c})$. We call a solution in this class as an $\mathcal{F}\dot{H}^{s_c}$-solution. If data belongs to L^2 then the solution is global thanks to mass-conservation.

The equation (NLS) does not have time translation symmetry. Since $e^{it_1\Delta}\mathcal{F}\dot{H}^{s_c} \neq e^{it_2\Delta}\mathcal{F}\dot{H}^{s_c}$ for $t_1 \neq t_2$, $u(t) \in C(I, e^{i\cdot\Delta}\mathcal{F}\dot{H}^{s_c})$ does not imply $u(t+\tau) \in C(I+\tau, e^{i\cdot\Delta}\mathcal{F}\dot{H}^{s_c})$ for $\tau \neq 0$. Remark that, by means of the scaling property of *linear Schrödinger equation*, we have

$$f \in e^{it_0\Delta}\mathcal{F}\dot{H}^{s_c} \Longleftrightarrow f_{\{\lambda\}} \in e^{i(t_0/\lambda^2)\Delta}\mathcal{F}\dot{H}^{s_c}. \tag{4}$$

We consider time global behavior of solution. Let $I_{\max} = I_{\max}(u) = (T_{\min}, T_{\max})$ be a maximal interval of an $\mathcal{F}\dot{H}^{s_c}$ solution $u(t)$. We say a solution $u(t)$ to (NLS) scatters forward (resp. backward) in time if $T_{\max} =$

∞ and $\lim_{t\to\infty} e^{-it\Delta}u(t)$ exists (resp. $T_{\min} = -\infty$ and $\lim_{t\to-\infty} e^{-it\Delta}u(t)$ exists)

There are at least two strategies to prove scattering. The first one is making use of smallness assumption in a suitable topology. Intuitively, smallness of data implies that of a corresponding solution, and then smallness of nonlinearity compared with the linear part. The other one, which is used in large data case, is by a priori estimate due to conserved quantities such as mass (or charge)

$$M[u] := \|u(t)\|_{L^2}^2$$

and energy

$$E[u] := \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{\mu}{p+2}\|u(t)\|_{L^{p+2}}^{p+2}.$$

In our case, small data scattering holds in $\mathcal{F}\dot{H}^{s_c}$ (see [5]). However, as for large data case, the mass-subcritical assumption makes it difficult to obtain a criteria for scattering in terms of mass and energy. Indeed, if $\mu = -1$ then there exists a standing wave solution of the form

$$u(t,x) = e^{it}Q(x)$$

where Q is a solution to $-\Delta Q + Q = |Q|^p Q$. Remark that Q is smooth and decays fast so that the above conserved quantities make sense. By scaling (2), $e^{it\lambda^2}Q_{\{\lambda\}}(x)$ is solution. Remark that

$$M[Q_{\{\lambda\}}] = \lambda^{2s_c}M[Q], \quad E[Q_{\{\lambda\}}] = \lambda^{2+2s_c}E[Q].$$

Then, mass-subcritical assumption $p < 4/d$ yields $s_c > 0$ and so they both become small if $\lambda \to 0$ and large if $\lambda \to \infty$. This shows that there exists a non-scattering solution which has arbitrary small (or large) conserved quantities.

A third strategy to show scattering is that to assume uniform in time boundedness of a solution with respect to scale critical norm (such as homogeneous Sobolev norm). This strategy is well suit to the situation that no conserved quantity is available. For this kind of result in defocusing mass-supercritical settings, see [1, 2, 4, 7, 8, 9, 10, 12]. This kind of results also reveal that scattering of a solution is equivalent to uniform boundedness in time.

We extend this kind of scattering result to the mass-subcritical case $p < 4/d$. The main result is as follows

Theorem 1 ([3]). *Under* (1) *with* $\mu = \pm 1$, *if an* $\mathcal{F}\dot{H}^{s_c}$*-solution* $u(t)$ *satisfies*

$$\sup_{t\in[t_0,T_{\max})} \left\|e^{-it\Delta}u(t)\right\|_{\mathcal{F}\dot{H}^{s_c}} < \infty \tag{5}$$

then the solution scatters forward in time.

Remark 2. It is worth mentioning that Theorems 1 holds also for focusing case $\mu = -1$. This reflects the fact that boundedness assumption (5) is a very strong one. Indeed, the norm of standing waves $e^{it}Q(x)$ are order $O(|t|^{s_c})$ as $|t| \to \infty$ and so they are immediately excluded by the assumption. This suggests that the framework of weighted space is not so suitable to analysis of non-scattering solutions. It would be appropriate to say that we can not

see the difference between defocusing case $\mu = +1$ and focusing case $\mu = -1$ in this framework.

2. Reduction to a minimization problem

We restate the above main result in terms of a minimization problem with respect to non-scattering solutions to (NLS). Recall that time translation symmetry is broken. Therefore, the validity of the theorem may depends on t_0, the time when a data is given. We now make some reductions.

Step 1 (reduction to $t_0 = -1, 0, 1$)

By scaling property (2), the case $0 < t_0 < \infty$ is reduced to the case $t_0 = 1$. Similarly, the case $-\infty < t_0 < 0$ follows from the case $t_0 = -1$. As a result, we only have to consider the case $t_0 = -1, 0, 1$.

Step 2 ($t_0 = 0$ follows from $t_0 = 1$)

The case $t_0 = 0$ follows from $t_0 = 1$. Indeed, by local well-posedness, if data is given at $t_0 = 0$ then a corresponding solution satisfies $T_{\max} > 0$. Then, by scaling, we may assume that $T_{\max} > 1$.

Step 3 (continuation-to-zero problem reduces $t_0 = -1$ to $t_0 = 1$)

The case $t_0 = -1$ also follows from the $t_0 = 1$ case if the following *continuation-to-zero* problem is affirmative: For any data $u_0 \in e^{-i\Delta}\mathcal{F}\dot{H}^{s_c}$, a solution $u(t)$ satisfies (5) can be extended to $t = 0$, namely, $T_{\max} > 0$.

Step 4 (unifying $t_0 = 1$ and continuation-to-zero problem)

By time reversible symmetry $u(t, x) \mapsto \overline{u(-t, x)}$, continuation from $t_0 = -1$ to $t_0 = 0$ is equivalent to continuation from $t_0 = 1$ to $t_0 = 0$.

According to the above reduction, we introduce following two quantities. The first quantity is

$$E_\infty := \inf \left\{ \overline{\lim_{t \uparrow T_{\max}}} \left\| e^{-it\Delta} u(t) \right\|_{\mathcal{F}\dot{H}^{s_c}} \,\middle|\, \begin{array}{l} u \in C(I_{\max}, e^{i\cdot\Delta}\mathcal{F}\dot{H}^{s_c}) : \text{sol. to (NLS)}, \\ u(t) \text{ does not scatter forward in time}, \\ 1 \in I_{\max} \end{array} \right\}.$$

Remark that main theorem in the case $t_0 = 1$ is equivalent to $E_\infty = \infty$. The second quantity is

$$E_0 := \inf \left\{ \overline{\lim_{t \downarrow T_{\min}}} \left\| e^{-it\Delta} u(t) \right\|_{\mathcal{F}\dot{H}^{s_c}} \,\middle|\, \begin{array}{l} u \in C(I_{\max}, e^{i\cdot\Delta}\mathcal{F}\dot{H}^{s_c}) : \text{sol. to (NLS)}, \\ 0 \leqslant T_{\min} < 1 \end{array} \right\}.$$

The affirmative answer to the continuation-to-zero problem is equivalent to $E_0 = \infty$. Now, Theorem 1 is rephrased as

Theorem 3. *Suppose that* (1) *holds and* $\mu = \pm 1$. *Then,* $E_0 = E_\infty = \infty$.

3. Outline of the proof

The proof is based on a concentration/rigidity type argument. Assume for contradiction that $E_\infty < \infty$ or $E_0 < \infty$. Then, we obtain a special solution which is so-called *minimal blowup solution*.

To this end, we introduce a third value E_c. In view of scale symmetry, E_0 and E_∞ are closely related each other. In our proof, we see that $E_\infty < \infty$

and $E_0 < \infty$ is equivalent, and that if one of these is finite then these values coincide. Introduction of a third value E_c enables us to handle these problems simultaneously.

For an interval I, $S_I(u)$ denotes a scattering norm

$$S_I(u) := \|u\|_{L_t^{q,2}(I,L^r)}$$

where (q,r) is a suitable pair satisfying $\frac{2}{q} + \frac{d}{r} = \frac{2}{p}$. Scattering of a solution is characterized as boundedness with respect to this norm:

$$u(t) \text{ scatters forward in time} \Longleftrightarrow S_{[t_0,T_{\max})}(u) < \infty.$$

We first introduce

$$L(E) := \sup\left\{ S_I(u) \,\middle|\, \begin{array}{l} u \in C(I, e^{i\cdot\Delta}\mathcal{F}\dot{H}^{s_c}) : \text{sol. to (NLS)}, \\ I \subset\subset (0,\infty),\ \sup_{t\in I} \left\|e^{-it\Delta}u(t)\right\|_{\mathcal{F}\dot{H}^{s_c}} \leqslant E \end{array} \right\}.$$

In the above definition, it is essential that I is *not* a maximal interval. By small data scattering $L(E) \leqslant CE$ for small E. By a stability estimate, $L(E)$ is continuous $[0,\infty) \to [0,\infty]$. We now let

$$E_c := \sup\{E \mid L(E) < \infty\} = \inf\{E \mid L(E) = \infty\}.$$

The following proposition describes a relation among E_0, E_∞, and E_c.

Proposition 4. *If* $\min(E_0, E_\infty) < \infty$ *then* $E_c \leqslant \min(E_0, E_\infty)$.

By this proposition, to obtain Theorem 3, it suffices to show that $E_c = \infty$. The key step for the proof is the following.

Proposition 5 (Existence of a minimal blowup solution)**.** *If* $E_c < \infty$ *then* $E_\infty = E_c$ *holds and there exists a solution* $u_c(t)$ *with* $1 \in I_{\max}(u_c)$ *and the following properties:*

(1) u_c *does not scatter forward in time.*
(2) $\sup_{t\geqslant 1} \left\|e^{-it\Delta}u_c(t)\right\|_{\mathcal{F}\dot{H}^{s_c}} = \overline{\lim_{t\uparrow T_{\max}}} \left\|e^{-it\Delta}u_c(t)\right\|_{\mathcal{F}\dot{H}^{s_c}} = E_c$.
(3) $u_c(t)$ *is* almost periodic modulo symmetry *for* $t \geqslant 1$. *Namely, there exist a compact set* $K \subset \mathcal{F}\dot{H}^{s_c}$, *a scale function* $h(t) : [1, T_{\max}) \to \mathbb{R}_+$, *a frequency center function* $\xi(t) : [1, T_{\max}) \to \mathbb{R}^d$, *and* $\phi : (0,\infty) \to K$ *such that* u_c *is represented as*

$$e^{-it\Delta}u_c(t) = e^{-ix\cdot\xi(t)}(\phi(t))_{\{1/h(t)\}}(x) = e^{-ix\cdot\xi(t)}h(t)^{-\frac{2}{p}}\phi\left(t, \frac{x}{h(t)}\right)$$

for $1 \leqslant t < T_{\max}$. *Further* $h(t) \lesssim t^{-1/2}$.

Remark 6. Because of boundedness assumption (5), freedom of translation in space is removed, and $h(t)$ is at most a self-similar order. Even in mass-subcritical setting, a weaker boundedness gives us an almost-periodic-modulo-symmetry solution without these properties [6].

To obtain a contradiction from $E_c < \infty$, we make further analysis on the almost-periodic-modulo-symmetry solution and construct a better almost-periodic-modulo-symmetry solution, a self-similar solution.

Definition 7. *An almost periodic solution $u(t)$ is said to be self-similar if $I_{\max} = (0, \infty)$ and $h(t) = t^{-1/2}$ and $\xi(t) \equiv 0$, that is it is represented as*

$$u(t) = (\psi(t))_{\{1/\sqrt{t}\}}(x) = t^{-\frac{1}{p}} \psi\left(t, \frac{x}{\sqrt{t}}\right)$$

for $t \in (0, \infty)$, with a compact set $\tilde{K} \subset e^{i\Delta}\mathcal{F}\dot{H}^{s_c}$ and a function $\psi : (0, \infty) \to \tilde{K}$.

Remark 8. By scaling property (4), $\psi \in e^{i\Delta}\mathcal{F}\dot{H}^{s_c}$ implies that $\psi_{\{1/\sqrt{t}\}} \in e^{it\Delta}\mathcal{F}\dot{H}^{s_c}$.

By a successful renormalization argument on $u_c(t_n)$ ($t_n \to \infty$), we can construct a self-similar solution from the solution given in Proposition 5.

Proposition 9 (Reduction to self-similar scenario)**.** *If $E_c < \infty$ then there exists a self-similar solution $v_c(t)$ with*

$$\overline{\lim_{t\uparrow\infty}} \left\|e^{-it\Delta}v_c(t)\right\|_{\mathcal{F}\dot{H}^{s_c}} = \overline{\lim_{t\downarrow 0}} \left\|e^{-it\Delta}v_c(t)\right\|_{\mathcal{F}\dot{H}^{s_c}} = E_c.$$

In particular, $E_0 = E_c$ holds.

Remark 10. A combination of above propositions implies that "$E_\infty < \infty \Rightarrow E_0 < \infty$". This shows that the continuation-to-zero problem $E_0 = \infty$ is not easier than the scattering criterion $E_\infty = \infty$. Similarly, we also have the opposite direction "$E_0 < \infty \Rightarrow E_\infty < \infty$". These two problems are equivalent in this sense.

Remark 11. This kind of reduction is known in mass-critical and -supercritical cases. In those cases, there appear some other possibilities such as soliton-like scenario. In our case, boundedness (5) is so strong that no other cases takes place.

To complete the proof of Theorem 3, we will deny existence of a self-similar solution.

Proposition 12 (Preclusion of self-similar scenario)**.** *Let $u(t)$ be a self-similar solution in the sense of Definition 7. Then, $u(t) \in L^2$.*

This proposition shows that a self-simmilar solution $u(t)$ is global. This contracts with $I_{\max} = (0, \infty)$. The proof of the proposition is based on a reduced Duhamel formula.

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EVOLUTION OF DISPERSAL IN A MUTATION SELECTION MODEL

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Organisms disperse. They disperse to mate, to search for food, to avoid predators, and to look for better environments in general. It is thus of interest to ask which patterns of dispersal can confer some selective or ecological advantage. There are at least two kinds of dispersal: unconditional and conditional. Unconditional dispersal does not depend on habitat quality or population density, while conditional dispersal does depend on some or all of such factors. Hastings [2] showed that in both patch and diffusion models that for unconditional dispersal in spatially varying but temporally constant environments slower dispersal rate is selected. To describe Hastings"s result, consider

$$\begin{aligned} &\frac{\partial u}{\partial t} = \mu\Delta u + u[m(x) - u] && \text{in } D \times (0,\infty), \\ &\frac{\partial u}{\partial n} = 0 && \text{on } \partial D \times (0,\infty), \end{aligned} \tag{0.1}$$

where $u(x,t)$ represents the population densities of a single species with dispersal rate μ, the function $m(x)$ represents the growth rate. Here D is a bounded open domain in $\mathbb{R}^N$ with smooth boundary, denoted by ∂D, n denotes the outward unit normal vector on ∂D, and $\frac{\partial}{\partial n} = n \cdot \nabla$. Throughout this abstract, we assume

(A) $m(x)$ is a positive, non-constant function in $C(\bar{D})$.

We assume that $m(x)$ is non-constant to reflect the spatial heterogeneity of the environment. It is well known that for any non-negative and not identically zero initial data, $u(x,t) \to \theta = \theta(x,\mu)$, where $\theta(\cdot,\mu)$ is the unique positive solution of

$$\begin{aligned} &\mu\Delta u + u[m(x) - u] = 0 && \text{in } D, \\ &\frac{\partial u}{\partial n} = 0 && \text{on } \partial D. \end{aligned} \tag{0.2}$$

Suppose that species u reaches equilibrium. Hastings [2] asked: if a mutant is introduced into the population, when can it grow when rare? Mathematically, he considered

$$\begin{aligned} &\frac{\partial v}{\partial t} = \nu\Delta v + v[m(x) - \theta(x;\mu) - v] && \text{in } D \times (0,\infty), \\ &\frac{\partial v}{\partial n} = 0 && \text{on } \partial D \times (0,\infty). \end{aligned} \tag{0.3}$$

Clearly, $v = 0$ is an equilibrium of (0.3). Hastings' question can be rephrased as: when is $v = 0$ locally stable? It is shown in [2] that $v = 0$ is unstable if $\nu < \mu$ and

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stable if $\mu > \nu$. Biologically this means that a mutant can invade when rare if and only if it is the slower (interesting!) diffuser.

Dockery et al [1] further considered the semilinear parabolic system

$$\begin{aligned} &\frac{\partial u}{\partial t} = \mu\Delta u + u[m(x) - u - v] && \text{in } D \times (0,\infty), \\ &\frac{\partial v}{\partial t} = \nu\Delta v + v[m(x) - u - v] && \text{in } D \times (0,\infty), \\ &\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 && \text{on } \partial D \times (0,\infty), \end{aligned} \tag{0.4}$$

which models two species that are competing for the same resources, where $u(x,t)$ and $v(x,t)$ represent the population densities of competing species 1 and 2 with respective dispersal rates μ and ν, the function $m(x)$ represents their common intrinsic growth rate. We shall assume that μ and ν are positive constants.

If we assume that the initial data $u(x,0)$ and $v(x,0)$ are non-negative and not identically zero, then by maximum principle $u(x,t) > 0$ and $v(x,t) > 0$ for every $x \in \overline{\Omega}$ and every $t > 0$. Moreover, $u(x,t)$ and $v(x,t)$ are classical solutions of (0.4) and exist for all time $t > 0$. Of particular interest are the dynamics of (0.4).

Under assumption (A) (0.4) has two semi-trivial states, denoted by $(\theta(\cdot,\mu),0)$ and $(0,\theta(\cdot,\nu))$ for every $\mu > 0$ and every $\nu > 0$. It is shown in [1] that if $\mu < \nu$, then $(\theta(\cdot,\mu),0)$ is globally asymptotically stable among all non-negative non-trivial initial data. In other words, *the slower diffuser wins*, independent of the initial data. Note that this global convergence result covers Hastings"s earlier local stability result. By symmetry, a similar conclusion holds when $\mu > \nu$. In particular, (0.4) has no coexistence states (i.e., steady states with both components u and v being positive) if $\mu \neq \nu$.

Why is the slower diffuser always the winner? Such a phenomenon is a little surprising at the first look: if $\mu = \nu$, it is clear that neither species will die out; in fact, there are a continuum of positive equilibria and the two species will coexist since they are identical (the coexistence state depends on the initial data). However, if the diffusion rates are slightly different, the slower diffuser becomes the eventual winner as time evolves. One possible explanation is that as time evolves, the effective growth rate $a(x,t) := m(x) - u(x,t) - v(x,t)$ for both species will eventually change sign in D. The slower diffuser keeps relatively low density in the region where $a(x,t)$ is negative, which might help it gain some competitive advantage. An open problem is whether the slowest diffuser still wins the competition in the context of three or more competing species. The emerging mathematical difficulty is that competition models for three or more species are not monotone systems anymore.

Dockery et al. [1] actually considered another level of complexity in the following model:

$$\begin{cases} \frac{\partial}{\partial t}u_i = \alpha_i\Delta u_i + \left[m(x) - \sum_{j=1}^k u_j\right] u_i + \epsilon^2 \sum_{j=1}^k M_{ij}u_j \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } D \times (0,\infty), i = 1,...,k, \\ \frac{\partial}{\partial n}u_i = 0 \qquad\qquad\qquad\qquad \text{on } \partial D \times (0,\infty), i = 1,...,k, \\ u_i(x,0) = u_{i,0}(x) \qquad\qquad\qquad\qquad \text{in } D, i = 1,...,k, \end{cases} \tag{0.5}$$

where $\alpha_1 < \alpha_2 < ... < \alpha_k$ are positive constants, , M_{ij} is an irreducible real $k \times k$ matrix that models the mutation process so that $M_{ii} < 0$ for all i, and $M_{ij} \geq 0$ for $i \neq j$ and $\epsilon^2 \geq 0$ is called mutation rate.

When $\epsilon = 0$, Dockery et al. [1] proved that no two species can coexist at equilibrium, i.e. the set of non-trivial, non-negative steady states of the system (0.5) is given by

$$\{(\theta_{\alpha_1}, 0, ..., 0), (0, \theta_{\alpha_2}, 0, ..., 0), ..., (0, ..., \theta_{\alpha_k})\},$$

where θ_α is the unique positive solution of

$$\alpha\Delta\theta + \theta(m - \theta) = 0 \quad \text{in } D, \qquad \frac{\partial\theta}{\partial n} = 0 \quad \text{on } \partial D.$$

Moreover, among the non-trivial steady states, only $(\theta_{\alpha_1}, ..., 0)$, the steady state where the slowest diffuser survives, is stable and the rest of the steady states are all unstable.

Dockery et al. [1] further studied the effect of small mutation. When $0 < \epsilon \ll 1$, They prove that (0.5) has a unique steady state $\tilde{U} = (\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_N)$ in the space of non-trivial, non-negative functions. Furthermore, $\tilde{u}_i > 0$ for all i, and $\tilde{U} \to (\theta_{\alpha_1}, 0, ..., 0)$ as $\epsilon \to 0$; i.e. the system (0.5) equilibrates only when the slowest species is dominant and all other species remain at low densities. Note that when $0 < \epsilon \ll 1$, $(\theta_{\alpha_1}, ..., 0)$ is perturbed to become $\tilde{U}$ and other non-negative steady states of the system (0.5) is perturbed outside the positive cone so that at least one component becomes negative (and thus not biologically relevant as density must be non-negative). One can also think backward: when $\epsilon = 0$ the system has many non-negative and non-trivial steady states, and only one of them is stable. When ϵ is positive and small, there is only one such steady state. This means that mutation "picks up" the right candidate.

We are interested whether the situation in the discrete (in trait) framework can be extended to the continuum framework. More specifically, we consider a population structured by spatial variable $x \in D$ and the motility trait $\alpha \in \mathcal{A}$. Here $\mathcal{A} = [\underline{\alpha}, \overline{\alpha}]$, with $\overline{\alpha} > \underline{\alpha} > 0$, denotes a set of continuum phenotypic traits. The diffusion rate is parameterized by the variable α, and mutation is modeled by a diffusion process with constant rate ϵ^2. Each individual is in competition for resources with all other individuals at the same spatial location. Denoting by $u(x, t, \alpha)$ the population density of the species with trait $\alpha \in \mathcal{A}$ at location $x \in D$ and time $t > 0$, the model is given by

$$\text{(0.6)} \quad \begin{cases} u_t = \alpha\Delta u + [m(x) - \hat{u}(x,t))]\, u + \epsilon^2 u_{\alpha\alpha}, & x \in D, \alpha \in (\underline{\alpha}, \bar{\alpha}), t > 0, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial D, \alpha \in (\underline{\alpha}, \bar{\alpha}), t > 0, \\ u_\alpha = 0, & x \in D, \alpha \in \{\underline{\alpha}, \overline{\alpha}\}, t > 0, \\ u(0, x, \alpha) = u_0(x, \alpha), & x \in D, \alpha \in (\underline{\alpha}, \bar{\alpha}). \end{cases}$$

Here $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ denotes the Laplace operator in the spatial variables,

$$\hat{u}(x, t) := \int_{\underline{\alpha}}^{\overline{\alpha}} u(t, x, \alpha)\, d\alpha.$$

It is recently shown by Lam [3] that when ϵ is positive and small, (0.6) has a unique positive state, denoted by u_ϵ, which is also locally stable. Note that when $\epsilon = 0$, formally (0.6) has a continuum of equilibrium given by $\delta(\cdot - \alpha)\theta_\alpha(x)$. The goal of this note is to show that, as $\epsilon \to 0$,

$$u_\epsilon(x, \alpha) \to \delta(\alpha - \underline{\alpha})\theta_{\underline{\alpha}}(x),$$

i.e. u_ϵ converges to a Dirac mass supported at the lowest possible trait value $\underline{\alpha}$. This is consistent with the findings of Dockery et al. [1] for discrete traits.

In the rest of this note we consider the asymptotic behavior of the unique positive steady states of (0.6), denoted by u_ϵ, which satisfies

$$\begin{cases} \alpha\Delta u_\epsilon + \epsilon^2 (u_\epsilon)_{\alpha\alpha} + [m(x) - \hat{u}_\epsilon(x)]\, u_\epsilon = 0 & \text{in } \Omega := D \times (\underline{\alpha}, \bar{\alpha}), \\ \frac{\partial u_\epsilon}{\partial n} = 0 & \text{on } \partial D \times (\underline{\alpha}, \overline{\alpha}), \\ (u_\epsilon)_\alpha = 0 & \text{in } D \times \{\underline{\alpha}, \overline{\alpha}\}, \end{cases} \tag{0.7}$$

where

$$\hat{u}_\epsilon(x) = \int_{\underline{\alpha}}^{\overline{\alpha}} u_\epsilon(x, \alpha)\, d\alpha. \tag{0.8}$$

The existence of positive solutions to (0.7) can be stated as follows:

Theorem 0.1. *Suppose* (A) *holds, then* (0.7) *has at least one positive solution for all* $\epsilon > 0$.

If ϵ is positive and small, Lam [3] recently showed that (0.7) has a unique positive solution and it is stable. The global stability of this steady state is a challenging problem.

For the rest of this abstract we will focus on the asymptotic behavior of positive solutions of (0.7) as $\epsilon \to 0$. To this end, we define the following quantities:

Definition 0.2. (i) Let $\theta_{\underline{\alpha}}(x)$ be the unique positive solution of

$$\begin{cases} \underline{\alpha}\Delta\theta + \theta(m(x) - \theta) = 0 & \text{in } D, \\ \frac{\partial \theta}{\partial n} = 0 & \text{on } \partial D. \end{cases} \tag{0.9}$$

(ii) For each $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, we denote the principal eigenvalue and principal positive eigenfunction of the following problem by $\sigma^*(\alpha)$ and $\psi^*(x, \alpha)$, respectively:

$$\begin{cases} \alpha\Delta\psi + (m(x) - \theta_{\underline{\alpha}}(x))\psi + \sigma\psi = 0 & \text{in } D, \\ \frac{\partial \psi}{\partial n} = 0 \quad \text{on } \partial D, \quad \text{and} \quad \int_D \psi^2\, dx = \int_D \theta_{\underline{\alpha}}^2\, dx. \end{cases} \tag{0.10}$$

(Note that by (i), $\theta_{\underline{\alpha}}(x)$ is a positive eigenfunction for (0.10) when $\alpha = \underline{\alpha}$. By uniqueness of the (normalized) principal eigenfunction, we have $\sigma^*(\underline{\alpha}) = 0$, and $\psi^*(x, \underline{\alpha}) = \theta_{\underline{\alpha}}(x)$ for $x \in D$.)

(iii) Denote by $\eta^*(s)$ the unique positive solution to

$$\begin{cases} \eta'' + (a_0 - a_1 s)\eta = 0 & \text{for } s > 0, \\ \eta'(0) = 0 = \eta(+\infty) & \text{and } \int_0^\infty \eta(s)\, ds = 1, \end{cases} \tag{0.11}$$

where a_0, a_1 are positive constants determined by $a_1 = \frac{\partial \sigma^*}{\partial \alpha}(\underline{\alpha})$ and $a_0 = (a_1)^{2/3} A_0$, where A_0 is the absolute value of the first negative zero of the derivative of the Airy function.

When $m(x) \equiv 1$, $u_\epsilon \equiv 1/(\overline{\alpha} - \underline{\alpha})$ is the unique positive steady state and globally asymptotically stable [3]. The outcome changes drastically when $m(x)$ is non-constant. u_ϵ concentrates at the lowest value in the trait variable, as $\epsilon \to 0$. The following result concerns the asymptotic behavior of solution u_ϵ as $\epsilon \to 0$.

Theorem 0.3. *For any* $\beta > 0$, *there exists* $C > 0$ *independent of* $\epsilon > 0$ *such that*

$$u_\epsilon(x, \alpha) \le C\epsilon^{-2/3} \exp\left(-\beta(\alpha - \underline{\alpha})\epsilon^{-2/3}\right) \tag{0.12}$$

in $\Omega = D \times (\underline{\alpha}, \overline{\alpha})$. *Moreover, as* $\epsilon \to 0$

$$\left\| \epsilon^{2/3} u_\epsilon(x,\alpha) - \theta_{\underline{\alpha}}(x) \eta^* \left(\frac{\alpha - \underline{\alpha}}{\epsilon^{2/3}} \right) \right\|_{L^\infty(\Omega)} \to 0 \tag{0.13}$$

where $\theta_{\underline{\alpha}}(x)$ *and* $\eta^*(s)$ *are given as above. In particular,*

$$\hat{u}_\epsilon(x) = \int_{\underline{\alpha}}^{\overline{\alpha}} u_\epsilon(x,\alpha)\, d\alpha \to \theta_{\underline{\alpha}}(x) \quad \text{as } \epsilon \to 0.$$

Finally, we state a Liouville-type theorem for positive harmonic functions in cylinder domains, which plays an important role in our analysis.

Proposition 0.4. *Let* $k \in \mathbb{N}$, D *be a bounded smooth domain in* $\mathbb{R}^N$ *and* u *be a non-negative harmonic function on* $\Omega := D \times \mathbb{R}^k \subset \mathbb{R}^{N+k}$, *so that* $\frac{\partial u}{\partial n} = 0$ *on* $\partial D \times \mathbb{R}^k$. *Then* u *is necessarily a constant.*

Remark 0.5. This note is based on the joint work with Professor King-Yeung Lam of Ohio State University [4]. A closely related result, under a slightly different formulation, is independently proved by B. Perthame and P.E. Souganidis in [5] under a different approach, where an intermediate trait attains the minimum diffusion rate and an interior Dirac mass is found when the mutation rate tends to zero.

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Supersolutions of nonlinear parabolic systems and their applications

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This talk is concerned with nonnegative solutions of the nonlinear parabolic system

$$\begin{cases} \partial_t u = \Delta u + v^p, & x \in \Omega, \ \ t > 0, \\ \partial_t v = \Delta v + u^q, & x \in \Omega, \ \ t > 0, \\ u = v = 0, & x \in \partial\Omega, \ \ t > 0, \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), & x \in \Omega, \end{cases} \tag{1}$$

where $p > 0$, $q > 0$ with $pq > 1$, Ω is a (possibly unbounded) smooth domain in $\mathbf{R}^N$ ($N \geq 1$) and both u_0 and v_0 are nonnegative and locally integrable functions in Ω. Problem (1) is an example of a simple reaction-diffusion system that can be used as a model to describe heat propagation in a two component combustible mixture. See [21] for a survey.

The first part of this talk is based on the joint work with with T. Kawakami and M. Sierżęga (see [17]). By using scalar nonlinear parabolic equations, we construct supersolutions of (1) and obtain optimal sufficient conditions for the existence of local-in-time solutions and global-in-time solutions. Our arguments are simple and applicable to various nonlinear parabolic systems without complicated calculations due to combinations of power and exponential nonlinearities.

There are several results on the existence of solutions of (1). Here we recall the following well-known results, which were proved in [1, 2, 3, 20] (see also [21, Section 32]).

(A) Let $p, q \geq 1$ and $r_1, r_2 \in (1, \infty)$. Assume

$$\max\{P(r_1, r_2), Q(r_1, r_2)\} \leq 2,$$

where

$$P(r_1, r_2) := N\left(\frac{p}{r_2} - \frac{1}{r_1}\right), \qquad Q(r_1, r_2) := N\left(\frac{q}{r_1} - \frac{1}{r_2}\right).$$

Then, for any $(u_0, v_0) \in L^{r_1}(\Omega) \times L^{r_2}(\Omega)$, problem (1) possesses a local-in-time solution.

(B) Let $\Omega = \mathbf{R}^N$ and $pq > 1$. If

$$\frac{\max\{p, q\} + 1}{pq - 1} < \frac{N}{2}, \tag{2}$$

then problem (1) possesses a global-in-time positive solution provided that $(u_0, v_0) \not\equiv (0,0)$ and both $\|u_0\|_{L^{r_1^*}(\Omega)}$ and $\|v_0\|_{L^{r_2^*}(\Omega)}$ are sufficiently small, where

$$r_1^* := \frac{N}{2}\frac{pq-1}{p+1}, \qquad r_2^* := \frac{N}{2}\frac{pq-1}{q+1}. \tag{3}$$

(C) Let $\Omega = \mathbf{R}^N$ and $pq > 1$. If $(u_0, v_0) \not\equiv (0,0)$ and

$$\frac{\max\{p,q\}+1}{pq-1} \geq \frac{N}{2},$$

then problem (1) admits no global-in-time positive solution.

The optimality of assumption (2) in (B) follows from (C). We will give an easy proof of (A) and (B) by using supersolutions of (1).

Let us now outline the construction of supersolutions. Given (u, v) a positive (classical) solution of (1), we begin by setting $U := u^\alpha$ and $V := v^\beta$, where $\alpha \geq 1$ and $\beta \geq 1$. Then (U, V) satisfies

$$\begin{cases} \partial_t U = \Delta U + \alpha U^{1-\frac{1}{\alpha}} V^{\frac{p}{\beta}} - \dfrac{\alpha-1}{\alpha}\dfrac{|\nabla U|^2}{U}, & x \in \Omega,\ t > 0, \\ \partial_t V = \Delta V + \beta V^{1-\frac{1}{\beta}} U^{\frac{q}{\alpha}} - \dfrac{\beta-1}{\beta}\dfrac{|\nabla V|^2}{V}, & x \in \Omega,\ t > 0, \\ U = V = 0, & x \in \partial\Omega,\ t > 0, \\ (U(x,0), V(x,0)) = (u_0(x)^\alpha, v_0(x)^\beta), & x \in \Omega. \end{cases} \tag{4}$$

Let $(\tilde{U}, \tilde{V})$ be a positive solution of

$$\begin{cases} \partial_t \tilde{U} = \Delta \tilde{U} + \alpha \tilde{U}^{1-\frac{1}{\alpha}} \tilde{V}^{\frac{p}{\beta}}, & x \in \Omega,\ t > 0, \\ \partial_t \tilde{V} = \Delta \tilde{V} + \beta \tilde{V}^{1-\frac{1}{\beta}} \tilde{U}^{\frac{q}{\alpha}}, & x \in \Omega,\ t > 0, \\ \tilde{U} = \tilde{V} = 0, & x \in \partial\Omega,\ t > 0, \\ (\tilde{U}(x,0), \tilde{V}(x,0)) = (u_0(x)^\alpha, v_0(x)^\beta), & x \in \Omega. \end{cases} \tag{5}$$

Since $\alpha \geq 1$ and $\beta \geq 1$, by (4) and (5) we see that (U, V) is a subsolution of (5). It follows from the comparison principle that

$$\tilde{U}(x,t) \geq u(x,t)^\alpha, \qquad \tilde{V}(x,t) \geq v(x,t)^\beta, \qquad x \in \Omega,\ t > 0.$$

Let $\overline{w}$ be a solution of

$$\begin{cases} \partial_t w = \Delta w + \alpha w^A + \beta w^B, & x \in \Omega,\ t > 0, \\ w = 0, & x \in \partial\Omega,\ t > 0, \\ w(x,0) = u_0(x)^\alpha + v_0(x)^\beta, & x \in \Omega, \end{cases}$$

where

$$A := 1 - \frac{1}{\alpha} + \frac{p}{\beta}, \qquad B := 1 - \frac{1}{\beta} + \frac{q}{\alpha}.$$

Then $(\overline{w}, \overline{w})$ is a supersolution of (5). This implies that $(\overline{w}^{\frac{1}{\alpha}}, \overline{w}^{\frac{1}{\beta}})$ is a supersolution of (1). Therefore, by the comparison principle we obtain

$$0 \le u(x,t)^\alpha \le \overline{w}(x,t), \qquad 0 \le v(x,t)^\beta \le \overline{w}(x,t), \qquad x \in \Omega, \ \ t > 0.$$

This supersolution with a suitable choice of α and β enables us to obtain sufficient conditions for the existence of local-in-time solutions and global-in-time solutions of problem (1). In particular, we easily obtain (A) and (B).

The latter part of this talk is based on the joint work with with Y. Fujishima and H. Maekawa (see [13]). We study qualitative properties of the blow-up set B for problem (1). Let the solution (u, v) of (1) blow up at $t = T$, that is

$$\limsup_{t \to T} \left[\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} \right] = \infty.$$

We define type I blow-up for the solution (u, v). Let

$$\xi(t) := \kappa_1 (T-t)^{-A}, \qquad \eta(t) := \kappa_2 T - t)^{-B},$$

where

$$\kappa_1 = (AB^p)^{1/D}, \quad \kappa_2 = (A^q B)^{1/D}, \quad D = pq - 1, \quad A = \frac{p+1}{D}, \quad B = \frac{q+1}{D}.$$

Then (ξ, η) satisfies

$$\xi'(t) = \eta(t)^p, \quad \eta'(t) = \xi(t)^q, \quad \text{in} \quad (0,T), \qquad \lim_{t \to T} \xi(t) = \lim_{t \to T} \eta(t) = \infty.$$

Motivated by this, we say that the blow-up of the solution (u, v) is type I if

$$\limsup_{t \to T} \left[(T-t)^A \|u(t)\|_{L^\infty(\Omega)} + (T-t)^B \|v(t)\|_{L^\infty(\Omega)} \right] < \infty.$$

We define the blow-up set B of the solution (u, v) by

$$B := \big\{ x \in \overline{\Omega} : \text{there exists a sequence } \{(x_n, t_n)\} \subset \overline{\Omega} \times (0,T) \text{ such that } \lim_{n \to \infty} (x_n, t_n) = (x, T), \ \lim_{n \to \infty} [u(x_n, t_n) + v(x_n, t_n)] = +\infty \big\}.$$

We develop the arguments in [12] and establish a blow-up criterion for problem (1) and give sufficient conditions for no boundary blow-up.

Let us briefly recall some results on the blow-up problem for the scalar semilinear heat equation

$$\begin{cases} \partial_t w = \Delta w + w^\gamma, & x \in \Omega, \ \ t > 0, \\ w = 0, & x \in \partial\Omega, \ \ t > 0, \\ w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{6}$$

where $\gamma > 1$ and w_0 is a nonnegative, continuous and bounded function on $\overline{\Omega}$. The blow-up set for problem (6) has been investigated by many mathematicians from various points of view since the pioneering work due to Weissler [23] (see [21, Chapter 24]). Among others,

Friedman and McLeod [5] proved that the solution w of (6) does not blow up on $\partial\Omega$ if Ω is bounded and convex. Furthermore, Giga and Kohn [14, 15, 16] obtained a blow-up criterion and studied the blow-up behavior and the blow-up set for problem (6). In particular, they showed in [16] that the blow-up set is bounded if Ω is convex, $(N-2)\gamma < N+2$ and $w_0 \in H^1(\Omega)$. Subsequently, Ishige and Mizoguchi [18] modified the argument in [16] and proved that type I blowing up solutions of (6) do not blow up on $\partial\Omega$ provided that Ω is a bounded smooth domain in $\mathbf{R}^N$ and $(N-2)\gamma \le N+2$. On the other hand, in [8, 11, 12], Ishige and Fujishima developed the argument in [24] and established another blow-up criterion for type I blowing up solutions of a semilinear heat equation with small diffusion. Their blow-up criterion enables us to study the location of the blow-up sets for type I blowing up solutions of (6) without any assumptions such as $(N-2)\gamma < N+2$ and the convexity of the domain Ω (see also [6, 7, 9, 10]). Furthermore, in [12], they improved their blow-up criterion and gave sufficient conditions for the boundedness of the blow-up set and no boundary blow-up.

On the other hand, system (1) is one of the simplest nonlinear parabolic systems. However, little is known concerning the blow-up set for system (3). As far as we know, the only available results are in [4, 19, 22], which gave sufficient conditions for radially symmetric solutions of (3) in a ball to blow up at the center of the ball.

Now we are ready to state the main results of this paper. The first theorem is concerned with the boundedness of the blow-up set.

Theorem 1 *Assume*

$$(u_0, v_0) \in (L^\infty(\Omega) \cap L^{r_1}(\Omega)) \times (L^\infty(\Omega) \cap L^{r_2}(\Omega))$$

for some r_1, $r_2 \in [1, \infty)$. Let (u, v) be a type I blowing up solution of (1). Then the blow-up set for the solution (u, v) is bounded.

We give a sufficient condition for no boundary blow-up for problem (1).

Theorem 2 *Let (u, v) be a solution of (1) which exhibits type I blow-up at $t = T$, where $0 < T < \infty$. Then, for any $\eta \in (0, 1)$, there exist $T' \in (0, T)$ and $\sigma > 0$ such that, if*

$$\limsup_{t \to T} \left[(T-t)^{A+\frac{1}{2}} \|\nabla u(t)\|_{L^\infty(\Omega)} + (T-t)^{B+\frac{1}{2}} \|\nabla v(t)\|_{L^\infty(\Omega)} \right] \le \sigma, \tag{7}$$

then

$$B \subset \bigcap_{T' < t < T} \left\{ x \in \overline{\Omega} \,:\, \max\left\{ \frac{(T-t)^A}{\kappa_1} u(x,t), \frac{(T-t)^B}{\kappa_2} v(x,t) \right\} > \eta \right\} \tag{8}$$

and $B \cap \partial\Omega = \emptyset$.

Next we give a sufficient condition for (7) and (8).

Theorem 3 *Let Ω be a uniformly $C^{2,\theta}$ domain in $\mathbf{R}^N$ $(0 < \theta < 1)$. Let (u, v) be a solution of (1) and blow up at $t = T$, where $0 < T < \infty$. Let*

$$1 < r < r_* := \begin{cases} \infty, & N = 1, 2, \\ N(N+2)/(N-1)^2, & N \ge 3. \end{cases}$$

Then, for any $R > 0$, there exists $\delta > 0$ such that, if

$$|p - r| + |q - r| < \delta, \qquad \|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)} \le R,$$

then (u, v) satisfies (7) *and* (8).

We prove Theorems 1 and 2 by studying the blow-up set for supersolutions of (1), which are constructed by the refinement of the technique developed in [8]–[12] with the aid of our supersolutions of (1). Let (u, v) be a solution of (1) and blow up at $t = T$, where $0 < T < \infty$. For any sufficiently small $\epsilon \in (0, T)$, set

$$u_\epsilon(x,t) := \epsilon^A u(x, T - \epsilon + \epsilon t), \qquad u_{0,\epsilon}(x) := \epsilon^A u(x, T - \epsilon),$$
$$v_\epsilon(x,t) := \epsilon^B v(x, T - \epsilon + \epsilon t), \qquad v_{0,\epsilon}(x) := \epsilon^B v(x, T - \epsilon).$$

Then (u_ϵ, v_ϵ) satisfies

$$\begin{cases} \partial_t u_\epsilon = \epsilon \Delta u_\epsilon + v_\epsilon^p & \text{in} \quad \Omega \times (1 - T/\epsilon, 1), \\ \partial_t v_\epsilon = \epsilon \Delta v_\epsilon + u_\epsilon^q & \text{in} \quad \Omega \times (1 - T/\epsilon, 1), \\ u_\epsilon = v_\epsilon = 0 & \text{on} \quad \partial\Omega \times (0, 1), \end{cases}$$

and (u_ϵ, v_ϵ) blows up at $t = 1$. Set

$$U_\epsilon := (\lambda_1 u_\epsilon)^a \quad \text{with} \quad a := K \cdot \frac{\max\{q, p\} + 1}{p + 1},$$
$$V_\epsilon := (\lambda_2 v_\epsilon)^b \quad \text{with} \quad b := K \cdot \frac{\max\{q, p\} + 1}{q + 1},$$

where $\lambda_1 := (ab^p)^{1/D}$, $\lambda_2 := (a^q b)^{1/D}$. and $K > 0$ is chosen to satisfy $a \ge 1$ and $b \ge 1$. Then

$$\begin{cases} \partial_t U_\epsilon \le \epsilon \Delta U_\epsilon + U_\epsilon^{1 - \frac{1}{a}} V_\epsilon^{\frac{p}{b}}, & x \in \Omega, \ t > 0, \\ \partial_t V_\epsilon \le \epsilon \Delta V_\epsilon + U_\epsilon^{\frac{q}{a}} V_\epsilon^{1 - \frac{1}{b}}, & x \in \Omega, \ t > 0, \\ U_\epsilon = V_\epsilon = 0, & x \in \partial\Omega, \ t > 0. \end{cases}$$

Then we apply the arguments in [8]–[12] to construct a supersolution $\overline{W}_\epsilon$ of the problem

$$\begin{cases} \partial_t W_\epsilon = \epsilon \Delta W_\epsilon + W_\epsilon^\gamma & \text{in} \quad \mathbf{R}^N \times (0, 1), \\ W_\epsilon(x, 0) = \max\{(\lambda_1 u_{0,\epsilon}(x))^a, (\lambda_2 v_{0,\epsilon}(x))^b\} & \text{in} \quad \mathbf{R}^N, \end{cases}$$

where

$$\gamma := 1 - \frac{1}{a} + \frac{p}{b} = 1 + \frac{q}{a} - \frac{1}{b}.$$

Then it follows from the comparison principle that

$$\max\{U_\epsilon(x,t), V_\epsilon(x,t)\} \le \overline{W}_\epsilon(x,t), \qquad x \in \Omega, \ t > 0,$$

which implies

$$0 \le u_\epsilon(x,t) \le \lambda_1^{-1} \overline{W}_\epsilon(x,t)^{1/a}, \quad 0 \le v_\epsilon(x,t) \le \lambda_2^{-1} \overline{W}_\epsilon(x,t)^{1/b},$$

for $x \in \Omega$ and $t > 0$. Then we can study the location of the blow-up set of the solution (u, v) by the use of $\overline{W}_\epsilon$ and obtain Theorems 1 and 2. Theorem 3 is proved by the use of the Liouville type theorems for scalar semilinear heat equations.

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