Geometric Theory of Systems of Ordinary Differential Equations I

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Geometric Theory of Systems of Ordinary Differential Equations

Noboru Tanaka

Contents

Explanatory Notes

In the latter part of his life Tanaka devoted himself to the investigation of geometry and integration of the systems of differential equations which are involutive and of finite type, especially those of ordinary differential equations.

Intending to publish a book on these subjects, he had been accumulating manuscripts. In 1989 when he delivered a report to JSPS for the scientific grant that he had received, he included to it the manuscript which would form the first part of the book with the title "Geometric Theory of Systems of Ordinary Differential Equations, I ". He thereby wrote a detailed introduction which explains the results so far obtained and showed the scope of the book to be accomplished. The endeavor for the book was continued and especially accelerated in his last years. When he passed away on the 4 March 2011, there was left a volume of manuscript on his desk.

The present issue is to publish his manuscript unifying the first half already published as the report to JSPS and the second half left on his desk, thus might realize what he first intended. The second half of the manuscript gives detailed proofs of almost all what he summarized in the introduction of the report. The subjects which were reported but are not included in the present issue are (III3) and (IV) of the introduction.

Our project to publish the manuscript actually started in the summer of 2012, when we visited Tanaka's house and were introduced by Mrs. Tanaka to the study where Tanaka had used to work to find a great volume of manuscript left on the desk. There agreed Mrs. Tanaka and us all to publish the manuscript in an appropriate form. Since a few parts of the manuscript were incomplete for forming a book, we decided to put the manuscript on a web site after reforming it in TEXformat. The work of making TEX-version was performed by the member listed below. The work, however, needed an unexpectedly long time, and we finally finished it in October of 2016. In February 2017 we talked with staffs of Department of Mathematics, Hokkaido University, and on their suggestion we decided to publish this work as one of *Hokkaido University Technical Report Series in Mathematics.*

This work is published in two volumes:

1. TEX-version of manuscripts
2. Copy of the original manuscripts

The TEX-version is edited to make a single-book containing two parts: Part I treats geometry of the systems of differential equations which are involutive and of finite type, in particular geometry of the systems of ordinary differential equations, and Part II treats the integration problems derived from two remarkable classes of systems of ordinary differential equations of the second order. The materials that were already included in the report to JSPS were Introduction, Chapter I, Chapter II , Appendix A of Part I, and Bibliography. The other part of the book has not been published in any form.

In making the TEX-version we intended that it should be identical with the original manuscript as far as possible. However, some differences were inevitable. The following is the list of the changes that we made.

- Correction of apparent mistakes
- Adjustment of unclearly assigned section-numbers (see below)
- Some change of formats due to technical restriction of TEXstyle.

On the other hand, the original manuscripts contain both sections written in English and in Japanese, and we transformed them as they were, without translation. Also, there were some sections where only notes were left. Those sections seem to be completed with Tanaka's published papers (see below).

We now would like to describe more detailed remarks (for Part 1 only). In the following, I, II, etc. represents chapter number; for example, III.2 means Chapter III, Section 2 and 2.5 means Section 2, Subsection 5, etc..

1. The contents of Introduction, Chapters I and II, Appendix A and Bibliography were first published as the report for Grant-in-aid for Scientific Research No. 62540001 (1988) "Geometric Theory of Systems of Ordinary Differential Equations".
2. Section III.1 contains only notes; see [20], [31].
3. In Chapter III, the subsections 2.5, 2.6, 3.3, 3.4, 3.5 are written in Japanese.
4. In Chapter III, there are two subsections in the manuscripts numbered 1.3. So, considering the contents, we moved one to 2.6.
5. There are two subsections numbered 3.4 in the manuscripts. We renumbered them 3.4 and 3.5.
6. Section IV.1 contains only notes; see [20].
7. The most part of Chapter V is written in Japanese.
8. For Chapter V two series of manuscripts have been left; one contains four sections V.1 to V.4 and the other consists of V.1 only. Considering the contents of them, we put the latter as the second section of the former. The first three lines, however, were moved to the beginning of the chapter. Thus, Chapter V in TEX-version consists of five sections.
9. In V.3, the section numbers III.8 to III.11 are quoted, but they do not exist. It is likely that they correspond to V.2 to V.5, and we add footnotes there. We guess that the numbering of chapters and sections were changed during the preparation of manuscripts.
10. Chapter VII contains only one section and Subsection 1.4 has no contents. The original manuscript contains several pages of formulas after Subsection 1.5, but they were omitted in TEX-version. This chapter is written in Japanese.
11. Originally there was only a bibliography attached to the report to JSPS and no bibliography corresponding to the citations referred in the manuscript often only by author names. We therefore added References by inferring the corresponding papers.

This project was achieved through the collaboration of the following people:

Y. Agaoka, E. Kaneda, K. Kiso, K. Kiyohara, T. Morimoto, Y. Nakanishi, K. Sugahara, C. Tsukamoto, K. Yamaguchi, T. Yatsui.

The major part of technical matters in TEXformat was covered by T. Yatsui.

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15, March, 2017.

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Introduction

For around ten years we have worked on geometric theory of systems of ordinary differential equations, and obtained many results on the geometry as well as the geometric integration of these systems, which will be published as a book in the near future.

This report just corresponds to the first two chapters of the book, and is concerned with the following geometries:

(1) The geometry of pseudo-projective structures of order 2 of bidegree (n, r), which formulates the geometry of involutive systems of partial differential equations of the second order

$$\frac{\partial^2 y^a}{\partial x^i \partial x^j} = f^a_{ij}\left((x^k), (y^b), \left(\frac{\partial y^b}{\partial x^k}\right)\right).$$

Here, (n, r) is a given pair of integers with $1 \leqq r \leqq n-1$, and the indices i, j, k range over the integers $1, \ldots, r$, and the indices a, b over the integers $1, \ldots, n-r$.

(2) The geometry of generalized pseudo-projective structures of bidegree (n, r), which formulates the geometry of involutive and "generic" systems of partial differential equations of the second order

$$\begin{cases} \dfrac{\partial^2 y^a}{\partial x^i \partial x^j} = f^a_{ij}\left((x^k), (y^b), \left(\dfrac{\partial y^b}{\partial x^k}\right)\right), \\ f^\lambda\left((x^k), (y^b), \left(\dfrac{\partial y^b}{\partial x^k}\right)\right) = 0. \end{cases}$$

For rigorous formulations of these structures or rather differential equations see §2 in Chapter I. Here we only remark the following : A generalized pseudo-projective structure $\mathcal{X}$ of bidegree (n, r) on an n-dimensional manifold M is a pair formed by a fibred submanifold R of $J(M, r)$ and a differential system E of rank r on R which satisfy certain conditions, where $J(M, r)$ is the Grassmann bundle of r-dimensional contact elements to M. Then a pseudo-projective structure of order 2 of bidegree (n, r) on M simply means a generalized pseudo-projective structure $\mathcal{X}$ of bidegree (n, r) on M such that R is an open fibred submanifold of $J(M, r)$. It is shown that there is naturally associated to every projective structure $\mathfrak{D}$ on an n-dimensional manifold M a pseudo-projective structure $\mathcal{X}(\mathfrak{D})$ of bidegree $(n, 1)$.

Remark. Let (k, n, r) be a triplet of integers with $k \geqq 3$ and $1 \leqq r \leqq n-1$. Let us consider the geometry of involutive systems of partial differential equations of the k-th order

$$\frac{\partial^k y^a}{\partial x^{i_1} \ldots \partial x^{i_k}} = f^a_{i_1 \ldots i_k}\left((x^j), (y^b), \ldots, \left(\frac{\partial^l y^b}{\partial x^{j_1} \ldots \partial x^{j_l}}\right), \ldots\right),$$

where the indices $i_1, \ldots, j, j_1, \ldots,$ range over the integers $1, \ldots, r$, the indices a, b over the integers $1, \ldots, n-r$, and the index l over the integers $1, \ldots, k-1$. In the classical fashion the isomorphism in the geometry is considered with respect to point transformations of the space of the variables x, y^b, if $k = 2$ or $k \geqq 3$, $r \leqq n-2$, and with respect to contact transformations of the same space otherwise. Then we remark that the geometry can be formulated as the geometry of pseudo-projective structures of order k of bidegree (n, r), being a kind of generalized pseudo-projective structure. Thus we have seen that the generalized pseudo-projective structures constitute a considerably large class of differential equations.

In §3 of Chapter I we introduce the notion of a (generalized) pseudo-projective system. First of all a generalized pseudo-projective system is defined to be a triplet formed by a manifold R and two differential systems, E and F, on it which satisfy the following conditions:

(i) (R, E, F) is a pseudo-product manifold. Namely $E \cap F = 0$, and both E and F are completely integrable.
(ii) The system $D(= E + F)$ is not reduced to the zero system, and is non-degenerate.
(iii) The full derived system of D coincides with the tangent bundle $T(R)$.

Next, a pseudo-projective system (by which we here mean that of order 2 of bidegree (n, r)) is defined to be a generalized pseudo-projective system (R, E, F) such that $\operatorname{rank}(T(R)/D) = n - r$, $\operatorname{rank} E = r$, $\operatorname{rank} F = r(n - r)$.

Let us consider a (generalized) pseudo-projective structure $\mathcal{X}$ on a manifold M which is expressed by the pair (R, E). Let F be the vertical tangent bundle of R. Then the triplet (R, E, F) turns out to be a (generalized) pseudo-projective system, which is called *associated*. Furthermore it is shown that (from the local view-point) the geometry of (generalized) pseudo-projective structures may be represented by the geometry of (generalized) pseudo-projective systems.

Now, a pseudo-product manifold is an abstraction of a submanifold R of a product manifold $M \times N$ equipped with the induced structure (E, F) from the product structure. Since the conditions (ii) and (iii) above are genericness conditions on the system D, we therefore see that a generalized pseudo-projective system essentially means a generic submanifold or a generic relation in a product manifold. This reminds us of the axiomatic projective geometry, i.e., the geometry of relations R in product sets $M \times N$ satisfying the so-called axioms of projective geometry. Accordingly we have known that the geometry of generalized pseudo-projective systems well inherits the basic thought in the axiomatic projective geometry.

Let us now consider a generalized pseudo-projective triplet (M, N, R), which is a rigorous formulation of a generic relation R in a product manifold $M \times N$. As we have seen, there is associated to the relation R the generalized pseudo-projective system (R, E, F). By the non-degeneracy assumption on the system $D(= E + F)$ it follows that there is naturally associated to the relation R two kinds of generalized pseudo-projective structure, $\mathcal{X}$ on M and $\mathcal{X}^*$ on N, which are called the *dual* of each other. This fact assures the existence of the dual $\mathcal{X}^*$ of any generalized pseudo-projective structure $\mathcal{X}$.

§§1 and 2 in Chapter II are concerned with the prolongations of generalized pseudo-projective systems. Let (R, E, F) be a generalized pseudo-projective system, and assume that the system $D(= E + F)$ is regular in the sense that the k-th derived system D^{-k-1} of D is a subbundle of $T(R)$ for each $k \geqq 1$. Then there is associated to every point $x \in R$ the symbol algebra $\mathfrak{T}(x)$ of D at x, which means the graded Lie algebra $\mathfrak{t}(x) = \sum_{p<0} \mathfrak{g}_p(x)$ with the natural bracket operation, where $\mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x)$. We have $\mathfrak{g}_{-1}(x) = D(x) = E(x) + F(x)$ (direct sum), and both $E(x)$ and $F(x)$ are abelian subalgebras of $\mathfrak{t}(x)$. Then the triplet $\mathfrak{L}(x)$ formed by $\mathfrak{T}(x)$, $E(x)$ and $F(x)$ is called the *symbol algebra* of (R, E, F) at x.

The symbol algebra $\mathfrak{L}(x)$ can be easily abstracted to yield the notion of an abstract symbol algebra of generalized pseudo-projective system or a non-degenerate pseudo-product FGLA. Let $\mathfrak{L}$ be a model (abstract) symbol algebra of generalized pseudo-projective system. Then it is prolonged to a transitive graded Lie algebra $\mathfrak{G}$, and this is shown to be necessarily of finite dimension. Now, a generalized pseudo-projective system (R, E, F) is called of type $\mathfrak{L}$, if the symbol algebra $\mathfrak{L}(x)$ of the system at each $x \in R$ is isomorphic to $\mathfrak{L}$. Then it is shown that through a prolongation process the geometry of generalized pseudo-projective system of type $\mathfrak{L}$ may be represented by the geometry of a class of manifolds with absolute parallelism (cf. Theorem 1.7 in Chapter II).

Besides these we study some global problems on generalized pseudo-projective structures, etc. In §2 of Chapter I we show that there is naturally associated to every generalized pseudo-projective structure $\mathcal{X}$ on a manifold M the "arithmetic distance" $\rho(p, q)$ in M. In paragraph §3.7 of Chapter I we moreover propose some global problems on compact pseudo-projective triplets of order 2.

For the rest of the introduction we shall explain our main results which have been hitherto obtained and are not included in the text of this report.

(I) Let $\mathfrak{L}$ and $\mathfrak{G}$ be as above, and let $\mathfrak{g} = \sum_p \mathfrak{g}_p$ be the expression of $\mathfrak{G}$. Assume that $\mathfrak{g}$ is simple. Then there is naturally associated to $\mathfrak{G}$ a homogeneous space $G/G^{(0)}$ such that the Lie algebra of G (resp. of $G^{(0)}$) is $\mathfrak{g}$ (resp. $\mathfrak{g}^{(0)} = \sum_{p \geqq 0} \mathfrak{g}_p$). Then we have the following

Theorem 1. *There is naturally associated to every generalized pseudo-projective system (R, E, F) of type $\mathfrak{L}$ a normal Cartan connection Γ of model space $G/G^{(0)}$ on R. Furthermore Γ can be constructed by elementary operations, i.e., rational operations, differentiations and the inverse function theorem regarded as operations.*

Note that the connection Γ is expressed by the pair of a principal fibre bundle P over the base space R with structure group $G^{(0)}$ and a $\mathfrak{g}$-valued 1-form ω on P called the connection form. For the normality condition see [20].

Theorem 1 can be derived from the main theorem there. It should be noted that the same existence theorem, a variant of Theorem 1, can be also obtained in the case where $\mathfrak{g}$ is not semi-simple and $\mathfrak{G}$ satisfies certain additional conditions. These conditions include a condition assuring the existence of harmonic theory in the complex $\{C^q(\mathfrak{G}), \partial\}$ associated with $\mathfrak{G}$, the notation being as in [20].

(II) First of all we remark that the geometry of pseudo-projective structure of order k of bidegree (n, r) may be represented by the geometry of pseudo-projective systems of order k of bidegree (n, r), being a kind of generalized pseudo-projective system. (This has been already remarked in the case where $k = 2$.)

Let us consider a pseudo-projective system (R, E, F) of order k of bidegree (n, r). Then it is shown that the symbol algebras $\mathfrak{L}(x)$ of the system at any points $x \in R$ are mutually isomorphic, and essentially depend on the triplet (k, n, r). Thus we may speak of the model symbol algebra $\mathfrak{L}$ of pseudo-projective system of order k of bidegree (n, r). Then it is shown that the prolongation $\mathfrak{G}$ of $\mathfrak{L}$ is as follows:

(a) If $k = 2$, then the underlying Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$ is isomorphic to the simple Lie algebra $\mathfrak{sl}(n+1, \mathbb{R})$, and $\mathfrak{G}$ takes the following form: $\mathfrak{g} = \sum_{p=-2}^{2} \mathfrak{g}_p$.

(b) If $k = 3$ and $r = n - 1$, then $\mathfrak{g}$ is isomorphic to the simple Lie algebra $\mathfrak{sp}(n, \mathbb{R})$, and $\mathfrak{G}$ takes the following form: $\mathfrak{g} = \sum_{p=-3}^{3} \mathfrak{g}_p$.

(c) Otherwise $\mathfrak{g}$ is not semi-simple, and $\mathfrak{G}$ takes the following form $\mathfrak{g} = \sum_{p=-k}^{1} \mathfrak{g}_p$. Furthermore $\mathfrak{G}$ satisfies the additional conditions mentioned above.

By Theorem 1 and its variant we therefore obtain the following

Theorem 2. *There is naturally associated to every pseudo-projective system (R, E, F) of order k of bidegree (n, r) a normal connection Γ of model space $G/G^{(0)}$ on R. Furthermore Γ can be constructed by rational operations and differentiations.*

Now, there are naturally associated to $\mathfrak{G}$ two kinds of homogeneous spaces, $G/A^{(0)}$ and $G/B^{(0)}$, together with $G/G^{(0)}$ such that $G^{(0)} = A^{(0)} \cap B^{(0)}$. Hence $G/G^{(0)}$ may be identified with a submanifold of $G/A^{(0)} \times G/B^{(0)}$. Then it is shown that the triplet $(G/A^{(0)}, G/B^{(0)}, G/G^{(0)})$ gives a generalized pseudo-projective triplet, and that the generalized pseudo-projective system associated with the relation $G/G^{(0)}$ is a pseudo-projective system of order k of bidegree (n, r), which is called *standard*. It follows that the generalized pseudo-projective structure $\mathcal{X}_0$ on $G/A^{(0)}$ associated with the relation $G/G^{(0)}$ is a pseudo-projective structure of order k of bidegree (n, r), which is also called standard. Then we notice that the standard structure $\mathcal{X}_0$ can be locally expressed by the system $\partial^k y^a/\partial x^{i_1} \dots \partial x^{i_k} = 0$ or precisely by the associated system of the second order, and that the standard structure $\mathcal{X}_0$ together with the triplet $(G/A^{(0)}, G/B^{(0)}, G/G^{(0)})$ may be characterized in terms of the projective geometry. In more detail we have the following.

(a) The case where $k = 2$. Let $\mathbb{P}^n$ be the n-dimensional projective space. Then $\mathcal{X}_0$ may be characterized as the differential equation defining the r-dimensional projective subspaces of $\mathbb{P}^n$. Incidentally we denote by $\Sigma(\mathbb{P}^n, r)$ the Grassmann manifold of r-dimensional projective subspaces of $\mathbb{P}^n$, and defined a relation in the product manifold $\mathbb{P}^n \times \Sigma(\mathbb{P}^n, r)$ by

$$R = \{ (p, \alpha) \in \mathbb{P}^n \times \Sigma(\mathbb{P}^n, r) \mid p \subset \alpha \}.$$

We also denote by $\hat{G}$ the group of projective transformations of $\mathbb{P}^n$. Then we have the following natural identifications:

$$G = \hat{G}, \quad G/A^{(0)} = \mathbb{P}^n, \quad G/B^{(0)} = \Sigma(\mathbb{P}^n, r),$$
$$G/G^{(0)} = R = J(\mathbb{P}^n, r).$$

(b) The case where $k = 3$ and $r = n - 1$. Let us consider the $(2n - 1)$-dimensional projective space $\mathbb{P}^{2n-1}$ with the standard contact structure. Then $\mathcal{X}_0$ may be characterized as the differential equation defining the Lagrangean projective subspaces of $\mathbb{P}^{2n-1}$.

(c) The case where $k = 3$, $r \leqq n-2$ or $k \geqq 4$. Let L be the universal line bundle over the r-dimensional projective space $\mathbb{P}^r$, and set $V = \mathbb{R}^{r+1}$ and $W = \mathbb{R}^{n-r}$. Let us consider the vector bundle $H = W \otimes L^{-(k-1)}$ over $\mathbb{P}^r$, where $L^{-(k-1)}$ denotes the $(k-1)$-th power of the dual L^{-1} of L. Set $S = W \otimes S^{k-1}(V^*)$, where $S^{k-1}(V^*)$ stands for the $(k-1)$-th symmetric product of the dual space V^* of V. As is well known, there is a unique natural injective linear map θ of S into the space of cross sections of H. These being prepared, $\mathcal{X}_0$ may be characterized as the differential equation defining the cross sections $\theta(\alpha)$ $(\alpha \in S)$ of H. Incidentally the group G is naturally isomorphic to a certain subgroup of the affine transformation group $AF(S)$ of S containing the translations, and this group may be characterized as the group of fibre-preserving transformations of H which naturally induce transformations on the cross sections $\theta(\alpha)$.

(III) A profound study has been made of systems of ordinary differential equations of the second order.

(III.1) Let $\mathcal{X}$ be a pseudo-projective structure of order 2 of bidegree $(n, 1)$, and let (R, E, F) be the associated pseudo-projective system of order 2 of bidegree $(n, 1)$. Furthermore let Γ be the normal connection on R associated with (R, E, F). In the following we assume that $n \geqq 3$. For the graded Lie algebra $\mathfrak{G}$ and the connection Γ we shall use the notations as in [20].

The curvature K of the connection Γ is decomposed as follows: $K = \sum_{p=0}^{5} K^p$, and hence the harmonic part HK of K as follows: $HK = \sum_{p=0}^{5} HK^p$. Calculating the spaces $H^{p,2}(\mathfrak{G})$ of harmonic forms, we see that $K^0 = 0$, $K^1 = HK^1$, and $HK = K^1 + HK^2$. Since HK gives a fundamental system of invariants, it follows that both the invariants K^1 and HK^2 vanish, if and only if the equation $\mathcal{X}$ is equivalent to the standard equation $\mathcal{X}_0$.

Now, $\mathcal{X}$ is said to be *of the first restricted type*, if the invariant HK^2 vanishes, and *of the second restricted type*, if the invariant K^1 vanishes. Then we have the following

Theorem 3. (1) *$\mathcal{X}$ is of the first restricted type, if and only if $\mathcal{X}$ is equivalent to the differential equation $\mathcal{X}(\mathfrak{D})$ of geodesics associated with a projective structure $\mathfrak{D}$ on an n-dimensional manifold M.*

(2) *$\mathcal{X}$ is of the second restricted type, if and only if the dual $\mathcal{X}^*$ of $\mathcal{X}$ is equivalent to the differential equation $\mathcal{Y}(\Phi)$ associated with an M_2-structure Φ on a $2(n-1)$-dimensional manifold N.*

Here, M_2 means the algebra $M_2(\mathbb{R})$ of square matrices of degree 2, and an M_2-structure is a kind of geometric structure analogous to a quaternion structure. More precisely, let N be a manifold, and consider the algebra $\mathrm{End}(T_\alpha(N))$ of endomorphisms of the tangent vector space $T_\alpha(N)$ to N at each point $\alpha \in N$. Then an *almost M_2-structure* on N is an assignment Φ which assigns to every point $\alpha \in N$ a subalgebra Φ_α of $\mathrm{End}(T_\alpha(N))$ isomorphic to $M_2(\mathbb{R})$ and which is differentiable in a suitable sense. It can be shown that if N admits an almost[1] M_2-structure, N is necessarily of even dimension. Clearly an almost[1] M_2-structure on a 2-dimensional manifold is a trivial structure. Now, an almost M_2-structure Φ on N is called an *M_2-structure*, either if $\dim N \geqq 6$, and there is an affine connection ∇ (without torsion)[1] on N which leaves Φ parallel or if $\dim N = 4$, and there is an affine connection ∇ on N which leaves Φ parallel and of which the curvature satisfies certain conditions. It should be noted that the complexification N^c of a real analytic quaternion manifold N is endowed with an M_2-structure in the complex analytic category, provided $\dim N \geqq 8$.

Example. Let $f(t^1, \dots, t^{n-2})$ be any function of the $n-2$ variables $t^1, \dots, t^{n-2}$, where $n \geqq 3$. Let us consider the following system $\mathcal{X}$ of ordinary differential equations of the second order:

$$\begin{cases} \dfrac{d^2y^a}{dx^2} = 0 & (1 \leqq a \leqq n-2), \\ \dfrac{d^2y^{n-1}}{dx^2} = f\left(\dfrac{dy^1}{dx}, \dots, \dfrac{dy^{n-2}}{dx}\right). \end{cases}$$

Then $\mathcal{X}$ is of the second restricted type, and it is equivalent to the standard system $d^2y^i/dx^2 = 0$ $(1 \leqq i \leqq n-1)$, if and only if $f(t^1, \dots, t^{n-2})$ is a polynomial of the variables $t^1, \dots, t^{n-2}$ of degree at most 2.

[1] Editor's note: Added by the editor.

Finally let (R, E, F) be a pseudo-projective system of order 2 of bidegree $(n, 1)$, and let P be the principal fibre bundle of the associated connection Γ. Then we remark that the curvature K of Γ may be represented by $n+1$ homogeneous polynomials of $n-1$ variables of degree 3 with functions on P as coefficients. In particular it follows that if $n = 3$, the invariant HK^2 may be represented by a homogeneous polynomial $\mathcal{F}$ of two variables of degree 4, and the curvature K by $\mathcal{F}$ and two homogeneous polynomials, $\mathcal{G}$ and $\mathcal{H}$, of two variables of degree 3. This fact plays an important role in the classification problem for pseudo-projective systems of order 2 of bidegree $(3, 1)$ which will be explained in (III.3).

(III.2) Let $\mathcal{X}$, (R, E, F) and Γ be as in (III.1). We preserve the assumption that $n \geqq 3$. Now, let $\mathfrak{a}$ be the sheaf of germs of local infinitesimal automorphisms of (R, E, F), being a sheaf of Lie algebras. Since (R, E, F) admits the connection Γ, it follows that each stalk $\mathfrak{a}(x)$ of $\mathfrak{a}$ is of finite dimension or more precisely $\dim \mathfrak{a}(x) \leqq \dim \mathfrak{g} = n^2 + 2n$. From now on we assume that $\dim \mathfrak{a}(x)$ is locally constant. Note that this assumption is automatically satisfied in the real or complex analytic category.

Theorem 4. *Assume that $\mathcal{X}$ is of the second restricted type. Then the integration of $\mathcal{X}$ at any point $x \in R$ can be carried out by quadratures and the integrations of Lie's differential equations associated with the simple components of the Lie algebra $\mathfrak{a}(x)$.*

For example Lie's differential equation associated with the simple Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is the Riccati equation

$$\frac{dy}{dx} = a_0(x) + a_1(x)y + a_2(x)y^2.$$

As an immediate consequence of the theorem we have the following

Corollary. *If $\mathfrak{a}(x) = \{0\}$, then the integration of $\mathcal{X}$ at x can be carried out by elementary operations.*

Theorem 4 compares with Élie Cartan's theorems for certain classes of single ordinary differential equations of the second and third order. (See [4].)

Clearly the integration of $\mathcal{X}$ is reduced to the integration of the system E of rank 1. Let us consider the triplet $(G/A^{(0)}, G/B^{(0)}, G/G^{(0)})$. Then the proof of Theorem 4 is based on the following geometric facts:

(i) There is naturally associated to the normal connection Γ of model space $G/G^{(0)}$ a transverse M_2-structure $\tilde{\Phi}$ on the foliated manifold (R, E), and in turn there is associated to $\tilde{\Phi}$ a normal transverse connection $\tilde{\Gamma}$ of model space $G/B^{(0)}$ on (R, E). Furthermore both $\tilde{\Phi}$ and $\tilde{\Gamma}$ can be constructed by rational operations and differentiations, from which follows that $\tilde{\Gamma}$ can be constructed from $\mathcal{X}$ by the same operations.

(ii) The stalk $\mathfrak{a}(x)$ of $\mathfrak{a}$ is naturally isomorphic to the stalk $\hat{\mathfrak{a}}(x)$ of the reduced sheaf $\hat{\mathfrak{a}}$ of germs of local infinitesimal automorphisms of $\tilde{\Gamma}$. Here, $\hat{\mathfrak{a}}$ means the quotient sheaf $\tilde{\mathfrak{a}}/\mathcal{E}$, where $\tilde{\mathfrak{a}}$ is the sheaf of germs of local infinitesimal automorphisms of $\tilde{\Gamma}$, and $\mathcal{E}$ is the sheaf of germs of local cross sections of E, being a subsheaf of ideals of $\tilde{\mathfrak{a}}$.

Note that these facts have been inspired by (2) of Theorem 3.

Finally we add the following

Theorem 4′. *Assume that $\mathfrak{a}$ is transitive at a point $x \in R$, i.e., $T_x(R) = \{\, X_x \mid X \in \mathfrak{a}(x) \,\}$. Then the integration of $\mathcal{X}$ at x can be carried out by quadratures and the integrations of Lie's differential equations associated with the simple components of $\mathfrak{a}(x)$.*

This fact is a consequence of a general theorem on the equivalence of linear group structures of finite type. Accordingly the proofs of Theorems 4 and 4′ are essentially different.

(III.3) The final explanation in (III) is concerned with the (local) classification problem for the systems of ordinary differential equations of the second order for two independent variables. Clearly the problem is reduced (locally) to classify the pseudo-projective systems of order 2 of bidegree (3,1). In the following these systems will be simply called *pseudo-projective systems*. Moreover everything will be considered in the complex analytic category.

Let $\mathcal{R}$ be a pseudo-projective system, and let us consider the invariants K^1 and HK^2. From now on the underlying manifold R of $\mathcal{R}$ is assumed to be connected. Then $\mathcal{R}$ is called of class (A), if $K^1 = 0$, of class (B), if $HK^2 = 0$, and of class (C), if $K^1 \neq 0$ and $HK^2 \neq 0$.

Let $\mathfrak{a}$ be the sheaf of germs of local infinitesimal automorphisms of $\mathcal{R}$. As we have remarked, the dimension of each stalk $\mathfrak{a}(x)$ of $\mathfrak{a}$ is constant, whence the stalks $\mathfrak{a}(x)$ are mutually isomorphic as Lie algebras. This being said, $\mathfrak{a}(\mathcal{R})$ denotes a model Lie algebra isomorphic to $\mathfrak{a}(x)$. Now, $\mathcal{R}$ is said to be *transitive*, if $\mathfrak{a}$ is transitive at each $x \in R$, and to be *intransitive*, otherwise.

For each $x \in R$ we denote by $\mathfrak{a}^0(x)$ the isotropy subalgebra of $\mathfrak{a}(x)$: $\mathfrak{a}^0(x) = \{ X \in \mathfrak{a}(x) \mid X_x = 0 \}$. We have

$$\dim(\mathfrak{a}(x)/\mathfrak{a}^0(x)) \leqq \dim R = 5.$$

Then we consider the following conditions on the system $\mathcal{R}$:

(i) $\dim \mathfrak{a}^0(x) \geqq 1$ at each $x \in R$,
(ii) $\dim \mathfrak{a}^0(x)$ is constant, and $\dim(\mathfrak{a}(x)/\mathfrak{a}^0(x)) = 5$ or 4. In other words, either $\mathfrak{a}$ is transitive or all the orbits of $\mathfrak{a}$ are of codimension 1.

(A) We have completely classified the pseudo-projective systems of class (A) satisfying conditions (i) and (ii). In more detail we have obtained the following:

(1) These systems are classified into 7 transitive systems $\mathcal{R}'$ and 8 intransitive systems $\mathcal{R}''$.

(2) The dimensions of the Lie algebras $\mathfrak{a}(\mathcal{R}')$ for the transitive systems $\mathcal{R}'$ are 15, 9, 8, 7, 6, 6, 6, and the dimensions of the Lie algebras $\mathfrak{a}(\mathcal{R}'')$ for the intransitive systems $\mathcal{R}''$ are 6, 5, 5, 5, 5, 5, 5, 5.

(3) The transitive system $\mathcal{R}'$ with $\dim \mathfrak{a}(\mathcal{R}') = 7$ is parameterized by an arbitrary constant, the intransitive system $\mathcal{R}''$ with $\dim \mathfrak{a}(\mathcal{R}'') = 6$ by an arbitrary function $f(t)$ ($\not\equiv 0$) of one variable t, and three of the intransitive systems $\mathcal{R}''$ with $\dim \mathfrak{a}(\mathcal{R}'') = 5$ by arbitrary constants. Furthermore one of the intransitive systems $\mathcal{R}''$ with $\dim \mathfrak{a}(\mathcal{R}'') = 5$ is parameterized by the solutions of the single ordinary differential equation

$$4ff'' - f'^2 + 20f + 9t^2 + 36 = 0,$$

and another of the same systems by the solutions of the ordinary differential equation

$$f'^3 + 27f(f' + 3t) = 0 \quad \text{with} \quad f \neq 0, -\frac{9}{4}t^2.$$

Now, let $\mathcal{R}$ be a pseudo-projective system of class (A), and assume that $\mathcal{R}$ is not locally equivalent to any one of the 15(=7+8) classified systems. Then it is shown that $\dim \mathfrak{a}(\mathcal{R}) \leqq 5$.

(B) Let $\mathcal{R}$ be a pseudo-projective system of class (B). Then we proved that $\dim \mathfrak{a}(\mathcal{R}) \leqq 8$, and classified the system $\mathcal{R}$ with $\dim \mathfrak{a}(\mathcal{R}) = 8$ or 7. The result shows that there is a unique system in either case, and it is transitive. Furthermore the system $\mathcal{R}'$ with $\dim \mathfrak{a}(\mathcal{R}') = 7$ is parameterized by an arbitrary constant.

(C) Let $\mathcal{R}$ be a pseudo-projective system of class (C). Then we proved that $\dim \mathfrak{a}(\mathcal{R}) \leqq 7$, and that there is a unique system $\mathcal{R}'$ with $\dim \mathfrak{a}(\mathcal{R}') = 7$, and it is transitive.

Let us now consider a pseudo-projective structure $\mathcal{X}$, and let $\mathcal{R}$ be the associated pseudo-projective system. By Theorems 4 and 4′ combined with the results above we have the following:

(1) If $\dim \mathfrak{a}(\mathcal{R}) = 15$, then $\mathcal{X}$ is of class (A), and can be locally expressed by the system

$$\frac{d^2y_1}{dx^2} = \frac{d^2y_2}{dx^2} = 0.$$

Furthermore the Lie algebra $\mathfrak{a}(\mathcal{R})$ is isomorphic to the simple Lie algebra $\mathfrak{sl}(4, \mathbb{C})$, and hence the integration of $\mathcal{X}$ can be carried out by the integration of the generalized Riccati equation of degree 4, i.e., Lie's differential equation associated with $\mathfrak{sl}(4, \mathbb{C})$.

(2) If $\dim \mathfrak{a}(\mathcal{R}) < 15$, we have $\dim \mathfrak{a}(\mathcal{R}) \leqq 9$. If $\dim \mathfrak{a}(\mathcal{R}) = 9$, then $\mathcal{X}$ is of class (A), and can be locally expressed by the system

$$\frac{d^2y_1}{dx^2} = 0, \quad \frac{d^2y_2}{dx^2} = \frac{1}{3}\left(\frac{dy_1}{dx}\right)^3.$$

(Compare the system given in Example, (III.1).) Furthermore the semi-simple part of $\mathfrak{a}(\mathcal{R})$ is isomorphic to the simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, and hence the integration of $\mathcal{X}$ can be carried out by quadratures and the integration of the Riccati equation.

(3) If $\dim \mathfrak{a}(\mathcal{R}) = 8$, there occur the following two cases: (i) $\mathcal{X}$ is of class (A), and can be locally expressed by the system

$$\frac{d^2y_1}{dx^2} = 0, \quad \frac{d^2y_2}{dx^2} = 2\,\frac{dy_1}{dx}\left(\frac{dy_2}{dx}\right)^2 \Big/ \left(y_2\frac{dy_1}{dx} - 1\right).$$

Furthermore $\mathfrak{a}(\mathcal{R})$ is isomorphic to the simple Lie algebra $\mathfrak{sl}(3, \mathbb{C})$, and hence the integration of $\mathcal{X}$ can be carried out by the integration of the generalized Riccati equation of degree 3.

(ii) $\mathcal{X}$ is of class (B), and can be locally expressed by the system

$$\frac{d^2y_1}{dx^2} = 0, \quad \frac{d^2y_2}{dx^2} = \frac{2}{3}\left(y_1 - x\frac{dy_1}{dx}\right).$$

Furthermore $\mathfrak{a}(\mathcal{R})$ is solvable, and hence the integration of $\mathcal{X}$ can be carried out by quadratures.

And so forth.

(IV) In physics it is known that the differential equation of the orbits of rays in the space-time or generally in an indefinite Riemannian space is a conformal invariant, and hence the equation is naturally associated with the conformal structure on the space. In our framework of the geometry of generalized pseudo-projective systems, we have obtained a thorough generalization of an indefinite conformal structure as well as the associated equation of the orbits of rays. Indeed we have introduced the notions of a pseudo-conformal structure R of degree n and a co-pseudo-conformal structure $\bar{R}$ of degree n, both defined on n-dimensional manifolds M ($n \geqq 3$), which have the following properties (1) $\sim$ (5).

(1) A pseudo-conformal structure R of degree n on M is a fibred submanifold of codimension 1 of $J(M, 1)$, the Grassmann bundle of 1-dimensional contact elements to M. Hence it may be regarded as a differential equation for 1-dimensional submanifolds of M, and can be locally expressed by a single ordinary differential equation of the first order:

$$f\left(x, (y^j), \left(\frac{dy^j}{dx}\right)\right) = 0.$$

Here and in the following the indices i and j range over the integers $1, \ldots, n-1$. Note that a pseudo-conformal structure may be regarded as a generalization of the light cone or the

Monge cone in an indefinite conformal structure, while a co-pseudo-conformal structure as a generalization of the co-light cone.

(2) A co-pseudo-conformal structure $\bar{R}$ of degree n on M is a fibred submanifold of codimension 1 of $J(M, n-1)$, the Grassmann bundle of contact elements of codimension 1 to M. Hence it may be regarded as a differential equation for submanifolds of codimension 1 of M, and can be expressed by a single partial differential equation of the first order:

$$g\left((x^j), y, \left(\frac{\partial y}{\partial x^j}\right)\right) = 0.$$

It should be noted that a co-pseudo-conformal structure $\bar{R}$ is "generic" among the fibred submanifolds of codimension 1 of $J(M, n-1)$, while a pseudo-conformal structure is not generic at all.

(3) There is a natural one-to-one correspondence between the pseudo-conformal structures R of degree n on M and the co-pseudo-conformal structures $\bar{R}$ of degree n on M, and the correspondences $R \to \bar{R}$ and $\bar{R} \to R$ are kinds of Bäcklund transformation. Furthermore if $\bar{R}$ corresponds to R, then there is a natural fibre-preserving diffeomorphism δ_R of R onto $\bar{R}$ which induces the identity transformation of M.

(4) Let R be a pseudo-conformal structure of degree n on M, and $\bar{R}$ the associated co-pseudo-conformal structure of degree n on M. Let $\bar{D}'$ be the contact system of $\bar{R}$, and let $\bar{E}$ be the Cauchy characteristic system of $\bar{D}'$, which is a subbundle of $\bar{D}'$ of rank 1. We now denote by E the pull-back of $\bar{E}$ to R by δ_R^{-1}. Then the pair $\mathcal{X}$ formed by R and E gives a generalized pseudo-projective structure on M, and hence may be regarded as a differential equation for 1-dimensional submanifolds of M, which can be expressed by a system of ordinary differential equations of the following form:

$$\begin{cases} \dfrac{d^2y^i}{dx^2} = f^i\left(x, (y^j), \left(\dfrac{dy^j}{dx}\right)\right), \\ f\left(x, (y^j), \left(\dfrac{dy^j}{dx}\right)\right) = 0. \end{cases}$$

The equation $\mathcal{X}$ just generalizes the equation of the orbits of rays associated with an indefinite conformal structure.

(5) Let F be the vertical tangent bundle of the fibred manifold R. Then the system (R, E, F) gives a generalized pseudo-projective system, which can be abstracted to yield the notion of a pseudo-conformal system of degree n. Thus the geometry of the equations $\mathcal{X}$ may be represented by the geometry of pseudo-conformal systems of degree n. Then it follows from Theorem 1 that there is naturally associated to every pseudo-conformal system (R, E, F) of degree n a normal Cartan connection Γ of R. We notice that the model symbol algebra $\mathfrak{L}$ admits the index r $(2 \leqq 2r \leqq n)$ as an invariant, and that the prolongation $\mathfrak{G}$ of $\mathfrak{L}$ takes the form $\mathfrak{g} = \sum_{p=-3}^{3} \mathfrak{g}_p$, and $\mathfrak{g}$ is isomorphic to the simple Lie algebra $\mathfrak{so}(r+1, n-r+1)$.

At this point we mention that the geometry of the equations $\mathcal{X}$, to a considerable extent, resembles the geometry of systems of ordinary differential equations of the second order. Therefore most results in the latter geometry are expected to find their counterparts in the former geometry. For example we shall have the notion of a contact M_2-structure in connection with the dual $\mathcal{X}^*$ of $\mathcal{X}$, which will correspond to the notion of an M_2-structure. Finally we add that the dual (R, F, E) of a pseudo-conformal system (R, E, F) of degree 3 means a pseudo-projective system of order 3 of bidegree (2,1), which implies that the geometry of pseudo-conformal structures of degree 3 is closely related to the geometry of single ordinary differential equations of the third order.

Part I

CHAPTER I

On certain classes of systems of differential equations of the second order and pseudo-product manifolds

§1. Preliminaries — Differential systems and the Grassmann bundles —

1.1. General remarks on terminologies and notations.

(a) As usual $\mathbb{R}$ denotes the field of real numbers, and $\mathbb{C}$ the field of complex numbers.

(b) Let M be a manifold. As usual $T_x(M)$ denotes the tangent vector space to M at $x \in M$, and $T(M)$ the tangent vector bundle of M: $T(M) = \bigcup_{x \in M} T_x(M)$. $T^*(M)$ denotes the cotangent bundle of M: $T^*(M) = \bigcup_{x \in M} T_x^*(M)$, $T_x^*(M)$ being the dual space of $T_x(M)$. Let φ be a map of a manifold M to a manifold M'. φ_* denotes the differential of φ, which is a bundle homomorphism of $T(M)$ to $T(M')$. Given a differential form ω' on M', $\varphi^*\omega'$ denotes the pull-back of ω' by φ.

(c) Let E be a vector bundle over a manifold M. For $x \in M$, $E(x)$ or E_x denotes the fibre of E at x. $\operatorname{rank} E$ denotes the fibre dimension of E: $\operatorname{rank} E = \dim E(x)$. Given an open set U of M, $\Gamma(U, E)$ denotes the space of all local cross sections of E defined on U.

(d) Let $\mathcal{E}$ be a sheaf over a manifold M. For $x \in M$, $\mathcal{E}(x)$ or $\mathcal{E}_x$ denotes the stalk of $\mathcal{E}$ at x. Given an open set U of M, $\Gamma(U, \mathcal{E})$ denotes the space of local cross sections of $\mathcal{E}$ defined on U. Let E be a vector bundle over a manifold M, and $\mathcal{E}$ the sheaf of germs of local cross sections of E. Then a germ of a stalk $\mathcal{E}(x)$ will be often confounded with a local cross section s of E defined on a neighborhood of x which represents the germ.

(e) An imbedding of a manifold M to a manifold M' is an injective immersion φ of M to M' which gives a homeomorphism of M onto the image $\varphi(M)$. As usual a submanifold of a manifold M' is a manifold M such that M is a subset of M', and the inclusion map ι of M to M' is an immersion. An imbedded submanifold of M' is a submanifold M of M' such that ι is an imbedding, i.e., the topology of M coincides with the relative topology.

(f) Let π be a map of a manifold R to a manifold M. Then R is called a *fibred manifold* over M with projection π, if the following conditions are satisfied: (i) π is surjective, and (ii) π is a submersion, i.e., the differential π_* of π maps $T_x(R)$ onto $T_{\pi(x)}(M)$ for each $x \in R$. Let R be a fibred manifold over M with projection π. Let R' be a submanifold of R, and π' the restriction of π to R'. Then R' is called a *fibred submanifold* of R, if R' becomes a fibred manifold over M with projection π'.

(g) A local diffeomorphism of a manifold M to a manifold M' is a diffeomorphism φ of an open set of M onto an open set of M'. Given a point (x_0, x_0') of $M \times M'$, a local diffeomorphism of M to M' at (x_0, x_0') is a local diffeomorphism φ of M to M' such that x_0 is in the domain of φ and $\varphi(x_0) = x_0'$.

1.2. Differential systems.

(A) Differential systems. A differential system on a manifold R is a subbundle D of the tangent vector bundle $T(R)$ of R.

Let D be a differential system on a manifold R. Set $\sigma = \operatorname{rank}(T(R)/D)$, and let x_0 be any point of R. Then there are 1-forms θ^a $(1 \leqq a \leqq \sigma)$ defined on a neighborhood O of x_0

such that D is defined there by the system of Pfaffian equations:

$$\theta^a = 0,$$

that is, the restriction $D|O$ of D to O consists of all $X \in T(O)$ such that $\theta^a(X) = 0$. Note that θ^a are automatically linearly independent at each point of O.

Let R and R' be two manifolds, and let $(D_i)_{i\in I}$ and $(D'_i)_{i\in I}$ be finite families of differential systems on them respectively. Then an isomorphism of $(R,(D_i))$ to $(R',(D'_i))$ is a diffeomorphism φ of R onto R' such that its differential φ_* maps D_i onto D'_i for each i. We remark that the notion of isomorphism naturally gives rise to the notions of local isomorphism, of (local) automorphism, and of (local) infinitesimal automorphism:

A local isomorphism of $(R,(D_i))$ to $(R',(D'_i))$ is a diffeomorphism φ of an open set O of R onto an open set O' of R' which gives an isomorphism of $(O,(D_i|O))$ to $(O',(D'_i|O'))$. Let (x_0, x'_0) be a point of $R \times R'$. A local isomorphism of $(R,(D_i))$ to $(R',(D'_i))$ at (x_0,x'_0) is a local diffeomorphism φ of R to R' at (x_0,x'_0) which gives a local isomorphism of $(R,(D_i))$ to $(R',(D'_i))$. A (local) automorphism of $(R,(D_i))$ is a (local) isomorphism of $(R,(D_i))$ to itself. A (local) infinitesimal automorphism of $(R,(D_i))$ is a (local) vector field X on R such that for any local 1-parameter group (ϕ_t) of local transformations of R generated by X, each ϕ_t gives a local automorphism of $(R,(D_i))$. It is easy to see that a local vector field X is a (local) infinitesimal automorphism of $(R,(D_i))$, if and only if for any i and any local cross section Y of D_i the bracket $[X,Y]$ is a local cross section of D_i.

In this book we shall be concerned with various kinds of manifolds with family of differential systems, e.g., product manifolds, pseudo-product manifolds and (generalized) pseudo-projective systems.

Finally let φ be a map of a manifold R to a manifold R', and let D' be a differential system on R'. Let us consider the inverse image $\varphi_*^{-1}(D')$ of D' by the differential φ_*. Set $D = \varphi_*^{-1}(D')$, and $D(x) = D \cap T_x(R)$ for each $x \in R$. Then $D(x)$ is a subspace of $T_x(R)$, and $D = \bigcup_x D(x)$. Now let x_0 be any point of R, and suppose that D' is defined, on a neighborhood O' of $\varphi(x_0)$, by a system of Pfaffian equations: $\theta'^a = 0$ $(1 \leqq a \leqq \sigma)$, where $\sigma = \operatorname{rank}(T(R')/D')$. Then D is defined, on $\varphi^{-1}(O')$, by the system of Pfaffian equations: $\varphi^*\theta'^a = 0$.

Thus D is, so to speak, a differential system possibly with singularities, which is called the *pull-back* of D' by φ. Especially the pull-back of the zero differential system, 0, on R' by φ means the kernel of the differential φ_*, and we prefer to use this last terminology. Furthermore, if R is a submanifold of R', and φ is the inclusion map of R to R', D is particularly called the pull-back of D' to R.

(B) The torsion of a differential system. Let R be a manifold, and let D be a differential system on it. Let x be any point of R. As is easily verified, there is a unique skew-symmetric bilinear map γ_x of $D(x) \times D(x)$ to $T(x)/D(x)$ such that

$$\gamma_x(X_x, Y_x) = \varpi([X,Y]_x),$$

where $T = T(R)$, ϖ is the projection of T onto T/D, and X and Y are any local cross sections of D defined on a neighborhood of x. The bilinear map γ_x thus obtained is called the *torsion* of D at x.

Now let x_0 be any point of R, and suppose that D is defined, on a neighborhood of x_0, by a system of Pfaffian equations: $\theta^a = 0$ $(1 \leqq a \leqq \sigma)$, where $\sigma = \operatorname{rank}(T(R)/D)$. Let O be a sufficiently small neighborhood of x_0, and take 1-forms ω^i $(1 \leqq i \leqq \rho)$ on O such that the 1-forms θ^a and ω^i are linearly independent at each point $x \in O$, where $\rho = \operatorname{rank} D$. Then there are unique functions T^a_{ij} $(1 \leqq i, j \leqq \rho)$ on O such that $T^a_{ij} = -T^a_{ji}$, and

$$d\theta^a \equiv \frac{1}{2} \sum_{i,j} T^a_{ij}\, \omega^i \wedge \omega^j \qquad (\mathrm{mod}\ \theta^1, \ldots, \theta^\sigma).$$

It is easy to see that

$$\theta^a(\gamma_x(X, Y)) = -d\theta^a(X, Y) = -\sum_{i,j} T^a_{ij}(x)\, \omega^i(X)\, \omega^j(Y),$$

where x is any point of O, and X and Y are any vectors of $D(x)$.

By definition the system D is completely integrable, if for any local cross sections X and Y of D (with the same domain) the bracket $[X, Y]$ is likewise a cross section of D. Clearly D is completely integrable, if and only if the torsion γ_x vanishes at each $x \in R$.

The system D is called *non-degenerate*, if the torsion γ_x is non-degenerate at each $x \in R$, i.e., the condition "$x \in R$, $X \in D(x)$, and $\gamma_x(X, Y) = 0$ for all $Y \in D(x)$" implies $X = 0$. In particular D is called a *contact structure* on R, if D is non-degenerate, $\mathrm{rank}(T(R)/D) = 1$, and $\dim R > 1$. A manifold R equipped with a contact structure D is called a *contact manifold*. Note that a contact manifold is necessarily odd dimensional.

(C) Integral manifolds and integral elements of a differential system. Let R be a manifold, and let D be a differential system on it. An integral manifold of D is a connected submanifold N of R such that $T_x(N) \subset D(x)$ at each $x \in N$. In particular if D is completely integrable, an integral manifold of D will always mean that of the maximal dimension, unless otherwise stated.

If N is an integral manifold of D, and x is a point of N, the torsion γ_x of D at x clearly vanishes on the tangent space $T_x(N)$. Now, let z be a contact element to R at a point x. Then z is called an *integral element* of D, if z is a subspace of $D(x)$, and the torsion γ_x vanishes on z.

(D) Completely integrable systems. Let R be a manifold, and let E be a completely integrable (differential) system on it. A first integral of E is a function f defined on an open set O of R which satisfies the following differential equation:

$$Xf = 0 \quad \text{for any } X \in E(x) \text{ and any } x \in O.$$

Let x_0 be any point of R. Then the famous Frobenius' theorem states that there are first integrals f^a $(1 \leqq a \leqq \sigma)$ of E defined on a neighborhood of x_0 such that the differentials df^a are linearly independent at x_0, where $\sigma = \mathrm{rank}(T(R)/E)$. The system (f^a) is called a *fundamental system* of first integrals of E at x_0.

Let R be a fibred manifold over a manifold M with projection π. Let V denote the kernel of the differential $\pi_* : V = \pi_*^{-1}(0)$. Clearly V is a subbundle of $T(R)$, and is completely integrable, which is called the *vertical tangent bundle* of the fibred manifold R.

Let R and E be as in the outset. We denote by R/E the set of all (maximal) integral manifolds of E, and by π the projection of R onto R/E. Now, R is said to be regular with respect to E, if the set R/E becomes naturally a manifold or more precisely if R/E is equipped with a differentiable structure so that (i) R becomes a fibred manifold over R/E with projection π, and (ii) the vertical tangent bundle of this fibred manifold coincides with E. Note that there is at most one differentiable structure on R/E satisfying the first condition, and that if R satisfies the second axiom of countability the first condition implies the second.

Finally we add the following

Lemma 1.1. *Let R be a manifold, and let (E_i) be a finite family of completely integrable systems on it. Then every point x_0 of R has a neighborhood O which is regular with respect to E_i for each i.*

This fact follows easily from Frobenius' theorem and Lemma 1.2 below.

For any positive number ε we denote by $S^{n-1}(\varepsilon)$ the sphere of the standard n-dimensional Euclidean space $\mathbb{R}^n$ of radius ε centered at the origin o.

Lemma 1.2. *Let φ be any local transformation of $\mathbb{R}^n$ at (o, o), i.e., such that o is in the domain of φ and $\varphi(o) = o$. If ε is sufficiently small, then the image $\varphi(S^{n-1}(\varepsilon))$ of $S^{n-1}(\varepsilon)$ by φ becomes a convex hypersurface of $\mathbb{R}^n$.*

Indeed it can be shown that if ε is sufficiently small, the second fundamental form of the hypersurface $\varphi(S^{n-1}(\varepsilon))$ of $\mathbb{R}^n$ becomes definite everywhere, meaning that $\varphi(S^{n-1}(\varepsilon))$ is a convex hypersurface of $\mathbb{R}^n$.

(E) The Cauchy characteristic system of a differential system. Let R be a manifold, and let D be a differential system on it. For each $x \in R$ we denote by $E(x)$ the null space of the torsion γ_x of D at x:

$$E(x) = \{\, X \in D(x) \mid \gamma_x(X, Y) = 0 \quad \text{for all } Y \in D(x) \,\},$$

and set $E = \bigcup_x E(x)$. It is clear that E is a subbundle of $T(R)$, if and only if $\dim E(x)$ is constant. We now assert that E is completely integrable, provided E is a subbundle of $T(R)$. Indeed let x be any point of R. Let X and Y be any local cross sections of E defined on a neighborhood of x, and Z any local cross section of D defined on a neighborhood of x. Clearly $[X, Y]_x$, $[X, Z]_x$ and $[Y, Z]_x$ are all in $D(x)$. Since

$$[[X, Y], Z] = [[X, Z], Y] + [X, [Y, Z]],$$

it follows that

$$\gamma_x([X, Y]_x, Z_x) = \gamma_x([X, Z]_x, Y_x) + \gamma_x(X_x, [Y, Z]_x) = 0,$$

showing that $[X, Y]_x \in E(x)$. This proves our assertion.

Thus E is, so to speak, a completely integrable system possibly with singularities, which is denoted by $\mathrm{ch}(D)$, and is called the *Cauchy characteristic system* of D.

Now, assume that E is a subbundle of $T(R)$, and that R is regular with respect to E. Set $M = R/E$, and let π be the projection of R onto M.

Lemma 1.3 (É. Cartan)**.** *There is a unique differential system $\bar{D}$ on M such that the given system D is the pull-back of $\bar{D}$ by π: $D = \pi_*^{-1}(\bar{D})$. Furthermore the system $\bar{D}$ is non-degenerate.*

Proof. We first recall that a local vector field X on R is a local infinitesimal automorphism of (R, D), if and only if for any local cross section Y of D, $[X, Y]$ is a local cross section of D. Then we see that a local vector field X on R is a local cross section of E, if and only if it is at once a local cross section of D and a local infinitesimal automorphism of (R, D). Now, let $\mathcal{E}$ be the sheaf of all germs of local cross sections X of E, which becomes naturally a sheaf of Lie algebras. Clearly each fibre N of the fibred manifold R over M is an orbit of $\mathcal{E}$: $T_x(N) = E(x) = \{\, X_x \mid X \in \mathcal{E}(x) \,\}$ for each $x \in N$.

Let Γ be the pseudo-group of local transformations of R generated by $\mathcal{E}$ (cf. [15]). For any local cross section X of $\mathcal{E}$ let (ϕ_t) be any local 1-parameter group of local transformations of R generated by X. Then Γ is generated by the local transformations of the form ϕ_t. From the observations above we easily obtain the following: (i) Every $\varphi \in \Gamma$ is a local automorphism of (R, D), (ii) Γ leaves each fibre of R invariant, i.e., if $\varphi \in \Gamma$, and x is a point in the domain of φ, then the two points x and $\varphi(x)$ are in the same fibre of R, and (iii) Γ acts transitively on each fibre of R, i.e., if x and x' are two points in the same fibre of R, then there is $\varphi \in \Gamma$ such that $x' = \varphi(x)$. It is now easy to see that there is a unique differential system $\bar{D}$ on M such that $D = \pi_*^{-1}(\bar{D})$, proving the first assertion.

Clearly the differential $\pi_* : T(R) \to T(M)$ naturally induces a bundle homomorphism, say π_*, of $T(R)/D$ onto $T(M)/\bar{D}$, which gives an isomorphism of $T_x(R)/D(x)$ onto $T_{\pi(x)}(M)/\bar{D}(\pi(x))$ for each $x \in R$. Now, let x be any point of R, and let us consider the torsion γ_x of D at x as well as the torsion $\bar{\gamma}_{\pi(x)}$ of $\bar{D}$ at $\pi(x)$. Then we can easily show that

$$\bar{\gamma}_{\pi(x)}(\pi_*(X), \pi_*(Y)) = \pi_*(\gamma_x(X, Y)), \quad X, Y \in D(x),$$

from which follows immediately that $\bar{\gamma}_{\pi(x)}$ is non-degenerate. Hence the system $\bar{D}$ is non-degenerate, proving the second assertion. □

(F) The derived systems of a differential system. Let R be a manifold. For simplicity we denote by T the tangent bundle $T(R)$ of R. We also denote by $\mathcal{O}$ the sheaf of germs of local functions on R, which is a sheaf of rings. Furthermore we denote by $\mathcal{T}$ the sheaf of germs of local vector fields on R or of local cross sections of T, which is a sheaf of Lie algebras, and is a module over $\mathcal{O}$ or an $\mathcal{O}$-module.

Now, let D be a differential system on R. We denote by $\mathcal{D}$ the sheaf of germs of local cross sections of D, which is an $\mathcal{O}$-submodule of the $\mathcal{O}$-module $\mathcal{T}$. Let x be any point of R. For any integer $p < 0$ we define a subspace $\mathcal{D}^p(x)$ of $\mathcal{T}(x)$ inductively as follows:

$$\begin{aligned} &\mathcal{D}^{-1}(x) = \mathcal{D}(x), \\ &\mathcal{D}^p(x) = [\mathcal{D}^{p+1}(x), \mathcal{D}(x)] + \mathcal{D}^{p+1}(x) \quad \text{for } p \leqq -2. \end{aligned}$$

We then set $\mathcal{D}^p = \bigcup_x \mathcal{D}^p(x)$. Clearly $\mathcal{D}^p$ are $\mathcal{O}$-submodules of $\mathcal{T}$, and

$$\mathcal{T} \supset \cdots \supset \mathcal{D}^p \supset \cdots \supset \mathcal{D}^{-2} \supset \mathcal{D}^{-1} = \mathcal{D}.$$

We can easily prove the following

Lemma 1.4. *$[\mathcal{D}^p(x), \mathcal{D}^q(x)] \subset \mathcal{D}^{p+q}(x)$.*

For each $x \in R$ let us now consider the natural map of $\mathcal{T}(x)$ onto $T(x)$, i.e., the map which maps every $X \in \mathcal{T}(x)$ to $X_x \in T(x)$. Then we denote by $D^p(x)$ the image of $\mathcal{D}^p(x)$ by the natural map, and set $D^p = \bigcup_x D^p(x)$. Clearly we have

$$T \supset \cdots \supset D^p \supset \cdots \supset D^{-2} \supset D^{-1} = D.$$

Moreover it is clear that D^p becomes a subbundle of T, if and only if $\dim D^p(x)$ is constant. Thus D^p is a differential system possibly with singularities. For any positive integer k the system D^{-k-1} will be called the k-th *derived system* of D, and the sheaf $\mathcal{D}^{-k-1}$ the k-th *derived sheaf* of $\mathcal{D}$.

Let us now consider the torsion γ_x of D at any point $x \in R$. Then we remark that the factor space $D^{-2}(x)/D(x)$ is the subspace of $T(x)/D(x)$ spanned by the vectors $\gamma_x(X, Y)$, where $X, Y \in D(x)$.

Now, we denote by $\mathcal{D}^{-\infty}$ the union $\bigcup_{p<0} \mathcal{D}^p$, which is an $\mathcal{O}$-submodule of $\mathcal{T}$. For each $x \in R$ we also denote by $D^{-\infty}(x)$ the image of $\mathcal{D}^{-\infty}(x)$ by the natural map of $\mathcal{T}(x)$ onto $T(x)$, and set $D^{-\infty} = \bigcup_x D^{-\infty}(x)$. By Lemma 1.4 we see that $\mathcal{D}^{-\infty}(x)$ is a subalgebra of the Lie algebra $\mathcal{T}(x)$. Moreover it is clear that $D^{-\infty}$ becomes a subbundle of $T(R)$, if and only if $\dim D^{-\infty}(x)$ is constant. Thus $D^{-\infty}$ is a completely integrable system possibly with singularities. The system $D^{-\infty}$ will be called the *full derived system* of D, and the sheaf $\mathcal{D}^{-\infty}$ the *full derived sheaf* of $\mathcal{D}$.

For example consider the case where D is a contact structure. Then it is clear that the first derived system of D coincides with T: $T = D^{-\infty} = D^{-2}$.

1.3. The Grassmann bundles. Let V be an n-dimensional vector space. Then $\mathrm{Gr}(V, r)$ will denote the Grassmann manifold of r-dimensional subspaces of V, which is a compact, connected manifold of dimension $r(n-r)$.

Now, let (n, r) be a pair of integers with $1 \leqq r \leqq n-1$. Let M be an n-dimensional manifold. We denote by $J(M, r)$ the subset of all r-dimensional contact elements to M or in other words,

$$J(M, r) = \bigcup_p \mathrm{Gr}(T_p(M), r).$$

We then denote by π the natural map of $J(M, r)$ onto M: $\pi(z) = p$, if $z \in \mathrm{Gr}(T_p(M), r)$. As is easily seen, $J(M, r)$ is a fibre bundle over the base space M with projection π (associated with the frame bundle of M), which is called the *Grassmann bundle* of r-dimensional contact elements to M. Note that $J(M, 1)$ and $J(M, n-1)$ may be regarded as the projective bundles associated with the vector bundles $T(M)$ and $T^*(M)$ respectively.

In the following the indices i, j range over the integers $1, \dots, r$, and the indices a, b over the integers $1, \dots, n-r$.

Let z_0 be any point of $J(M, r)$, and let (x^i, y^a) be any coordinate system of M at $\pi(z_0)$ such that the differentials dx^i are linearly independent, restricted to the contact element z_0. Such a coordinate system will be called *adapted* to z_0. Now, let U be the domain of the coordinate system. We denote by $\tilde{U}$ the subset of $J(M, r)$ consisting of all $z \in J(M, r)$ such that $\pi(z) \in U$, and the coordinate system is adapted to z. Clearly $\tilde{U}$ is an open set of $J(M, r)$, $z_0 \in \tilde{U}$, and $\pi(\tilde{U}) = U$. Moreover it is clear that there are unique functions p_i^a on $\tilde{U}$ such that

$$dy^a(X) = \sum_i p_i^a(z) dx^i(X),$$

where z is any point of $\tilde{U}$, and X is any vector in the contact element z. Then we see that the functions $x^i (= x^i \circ \pi)$, $y^a (= y^a \circ \pi)$ and p_j^b together form a coordinate system of $J(M, r)$, which is defined on $\tilde{U}$. The coordinate system, thus obtained, is called the *canonical coordinate system* of $J(M, r)$ at z_0 corresponding to the coordinate system (x^i, y^a) of M or simply a canonical coordinate system of $J(M, r)$ at z_0.

Consider the following three special cases: (A) $r = 1$; (B) $r = n-1$; (C) $n = 2$, and hence $r = 1$. In case (A) the canonical coordinate system will be described as $(x, y^1, \cdots, y^{n-1}, p^1, \cdots, p^{n-1})$ by setting $x = x^1$ and $p^a = p_1^a$, and in case (B) as $(x^1, \cdots, x^{n-1}, y, p_1, \cdots, p_{n-1})$ by setting $y = y^1$ and $p_i = p_i^1$, and in case (C) as (x, y, p) by setting $x = x^1$, $y = y^1$ and $p = p_1^1$.

For any $z \in J(M, r)$ we denote by $C(z)$ the subspace of $T_z(J(M, r))$ consisting of all vectors $X \in T_z(J(M, r))$ such that $\pi_*(X)$ is in the contact element z to M:

$$C(z) = \{\, X \in T_z(J(M, r)) \mid \pi_*(X) \in z \,\}.$$

We then set $C = \bigcup_z C(z)$, which is easily shown to become a subbundle of $T(J(M, r))$. The system C is called the *contact system* of $J(M, r)$. Let V be the vertical tangent bundle of $J(M, r)$. Then it is clear that $V \subset C$, and $\mathrm{rank}(C/V) = r$.

Let us consider any canonical coordinate system (x^i, y^a, p^b_j) of $J(M,r)$ at any point z_0. We define 1-forms θ^a on the domain $\tilde{U}$ of the coordinate system by

$$\theta^a = dy^a - \sum_i p^a_i dx^i.$$

Then it is clear that V is defined, on $\tilde{U}$, by the system of Pfaffian equations: $dx^i = dy^a = 0$, and C by the system of Pfaffian equations: $\theta^a = 0$. We have

$$d\theta^a = \sum_i dx^i \wedge dp^a_i,$$

from which follows immediately that C is non-degenerate, and the first derived system of C coincides with $T(J(M,r))$.

Summarizing the results above on the systems C and V, we have

Lemma 1.5. (1) *V is completely integrable, and $V \subset C$.*

(2) *C is non-degenerate, and the first derived system of C coincides with $T(J(M,r))$.*

(3) *$\operatorname{rank}(T(J(M,r))/C) = n - r$, $\operatorname{rank}(C/V) = r$, and $\operatorname{rank} V = r(n-r)$.*

Now, let α be any r-dimensional submanifold of M. For each point $p \in \alpha$ the tangent space $T_p(\alpha)$ gives an r-dimensional contact element to M and hence a point of $J(M,r)$. This being said, we denote by $p(\alpha)$ the image of the map $p \mapsto T_p(\alpha)$ of α to $J(M,r)$. Clearly $p(\alpha)$ is an r-dimensional submanifold of $J(M,r)$, and the projection π maps $p(\alpha)$ diffeomorphically onto the submanifold α of M. The submanifold $p(\alpha)$ thus obtained is called the *prolongation* of α. We now assert that $p(\alpha)$ is an integral manifold of the contact system C. Indeed take any point z of $p(\alpha)$. If we put $p = \pi(z)$, we have

$$\pi_*(T_z(p(\alpha))) = T_p(\alpha) = z,$$

whence $T_z(p(\alpha)) \subset C(z)$. This proves our assertion.

Lemma 1.6. *Let $\tilde{\alpha}$ be an r-dimensional integral manifold of C. If π maps $\tilde{\alpha}$ diffeomorphically onto a submanifold, say α, of M, then $\tilde{\alpha} = p(\alpha)$.*

Proof. Take any point z of $\tilde{\alpha}$. If we put $p = \pi(z)$, we have

$$T_p(\alpha) = \pi_*(T_z(\tilde{\alpha})) \subset \pi_*(C(z)) = z.$$

It follows that $T_p(\alpha) = z$, proving the lemma. □

Let us now consider any canonical coordinate system (x^i, y^a, p^b_j) of $J(M,r)$ at any point z_0. Let α be an r-dimensional submanifold of M. Suppose that $p(\alpha)$ is in the domain $\tilde{U}$ of the coordinate system, and that α is defined by a system of equations of the following form: $y^a = u^a(x^1, \dots, x^r)$. Then, in virtue of Lemma 1.6 we can easily see that $p(\alpha)$ is defined by the following system of equations:

$$\begin{cases} y^a = u^a(x^1, \dots, x^r), \\ p^a_j = \dfrac{\partial u^a}{\partial x^j}(x^1, \dots, x^r). \end{cases}$$

Finally let us consider another n-dimensional manifold M' as well as the Grassmann bundle $J(M',r)$. Let π' be the projection of $J(M',r)$ onto M', and let C' and V' be the contact system and the vertical tangent bundle of $J(M',r)$ respectively. Now, suppose that there is given a diffeomorphism φ of M onto M'. If z is an r-dimensional contact element to M, $\varphi_*(z)$ is an r-dimensional contact element to M', and the map $z \mapsto \varphi_*(z)$ gives a diffeomorphism of $J(M,r)$ onto $J(M',r)$, which we denote by $p(\varphi)$, and call the *prolongation* of φ. Clearly

$$\pi' \circ p(\varphi) = \varphi \circ \pi,$$

and $p(\varphi)$ gives an isomorphism of $(J(M,r), C, V)$ to $(J(M',r), C', V')$.

1.4. Fibred submanifolds of the Grassmann bundles, and Realization lemma. Let M be an n-dimensional manifold, and let R be a fibred submanifold of the Grassmann bundle $J(M,r)$. The notations being as in the previous paragraph, we denote by D and F the pull-backs of C and V to R respectively: $D = \iota_*^{-1}(C)$, and $F = \iota_*^{-1}(V)$, ι being the inclusion map of R to $J(M,r)$. It is clear that F is nothing but the vertical tangent bundle of the fibred manifold R over M, and that for each $z \in R$

$$D(z) = \{\, X \in T_z(R) \mid \pi_*(X) \in z \,\}.$$

In particular it follows that D is a subbundle of $T(R)$. The system D thus obtained is called the *contact system* of R.

Lemma 1.7. (1) *F is completely integrable, and $F \subset D$.*
(2) $\mathrm{ch}(D) \cap F = 0$.
(3) $\mathrm{rank}(T(R)/D) = n - r$, *and* $\mathrm{rank}(D/F) = r$.

Proof. (1) is clear, and (3) follows immediately from the expression of $D(z)$ given above. (2) follows from Lemma 1.9 below, proving the lemma. □

Let us now consider another n-dimensional manifold M', and let R' be a fibred submanifold of $J(M',r)$. Let π' be the projection of $J(M',r)$ onto M', and let D' and F' be the contact system and the vertical tangent bundle of R' respectively.

Lemma 1.8. *Let (z_0, z_0') be a point of $R \times R'$. If $\tilde\varphi$ is a local isomorphism of (R, D, F) to (R', D', F') at (z_0, z_0'), then there is a unique local diffeomorphism φ of M to M' at $(\pi(z_0), \pi'(z_0'))$ such that $\tilde\varphi = p(\varphi)$ on a neighborhood of z_0 in R.*

Proof. Since $\tilde\varphi$ gives a local isomorphism of (R, F) to (R', F') at (z_0, z_0'), we easily see that there is a unique local diffeomorphism φ of M to M' at $(\pi(z_0), \pi'(z_0'))$ such that $\pi' \circ \tilde\varphi = \varphi \circ \pi$ on a neighborhood of z_0 in R. Now, let O be a sufficiently small neighborhood of z_0 in R, and take any point z of O. Since $\tilde\varphi$ gives a local isomorphism of (R, D) to (R', D') at (z_0, z_0'), it follows that

$$\tilde\varphi(z) = \pi'_*(D'(\tilde\varphi(z))) = \pi'_*(\tilde\varphi_*(D(z))) = \varphi_*(\pi_*(D(z))) = \varphi_*(z) = p(\varphi)(z),$$

whence $\tilde\varphi = p(\varphi)$ on O. This proves the lemma. □

Let R be a fibred manifold over a manifold M with projection μ, and let D be a differential system on R. Let F be the vertical tangent bundle of R, and assume that $F \subset D$, and $1 \leqq r \leqq n-1$, where $n = \dim M$, and $r = \mathrm{rank}(D/F)$. As for the Grassmann bundle $J(M,r)$, we shall use the notations as in the previous paragraph.

Lemma 1.9 (Realization lemma)**.** *There is a unique map φ of R to $J(M,r)$ satisfying the following conditions:*

(i) $\pi \circ \varphi = \mu$,
(ii) *D is the pull-back of C by φ: $D = \varphi_*^{-1}(C)$.*

Furthermore the kernel $\varphi_*^{-1}(0)$ *of the differential* φ_* *coincides with the intersection* $\mathrm{ch}(D) \cap F$.

In particular it follows that if D *is non-degenerate, i.e.,* $\mathrm{ch}(D) = 0$*, there is a unique immersion* φ *of* R *to* $J(M,r)$ *satisfying conditions* (i) *and* (ii).

Proof of Lemma 1.9. We first remark that for each $x \in R$, $\mu_*(D(x))$ gives an r-dimensional contact element to M, because $F(x) \subset D(x)$, and $r = \dim(D(x)/F(x))$. This being said, we define a map φ of R to $J(M,r)$ by

$$\varphi(x) = \mu_*(D(x)), \qquad x \in R.$$

Clearly $\pi \circ \varphi = \mu$. For $X \in T_x(R)$ we have: $X \in \varphi_*^{-1}(C) \iff \varphi_*(X) \in C(\varphi(x)) \iff \pi_*(\varphi_*(X)) \in \varphi(x) \iff \mu_*(X) \in \mu_*(D(x)) \iff X \in D(x)$. Hence we obtain $\varphi_*^{-1}(C) = D$. Now, let φ' be any map of R to $J(M,r)$ such that $\pi \circ \varphi' = \mu$, and $\varphi_*'^{-1}(C) = D$. For $X \in T_x(R)$ we have: $\mu_*(X) \in \varphi'(x) \iff \pi_*(\varphi'_*(X)) \in \varphi'(x) \iff \varphi'_*(X) \in C(\varphi'(x)) \iff X \in D(x) \iff \mu_*(X) \in \mu_*(D(x)) = \varphi(x)$. Therefore we obtain $\varphi'(x) = \varphi(x)$, and hence $\varphi' = \varphi$, proving the first assertion.

Let x be any point of R. We take a coordinate system (x^i, y^a) of M adapted to the contact element $\varphi(x)$ and consider the corresponding canonical coordinate system $(x^i \circ \pi, y^a \circ \pi, p^b_j)$ of $J(M,r)$ at $\varphi(x)$. Let X be a vector of $T_x(R)$. Clearly $\varphi_*(X) = 0$, if and only if $X(x^i \circ \pi \circ \varphi) = X(y^a \circ \pi \circ \varphi) = X(p^a_i \circ \varphi) = 0$. Since $\pi \circ \varphi = \mu$, it follows that $\varphi_*(X) = 0$, if and only if $X \in F(x)$, and $X(p^a_i \circ \varphi) = 0$. Let us now consider the 1-forms $\theta^a = d(y^a \circ \pi) - \sum_i p^a_i d(x^i \circ \pi)$, and recall that C is defined, on a neighborhood of $\varphi(x)$ by the system of Pfaffian equations: $\theta^a = 0$. Set $\eta^a = \varphi^* \theta^a$. Since $D = \varphi_*^{-1}(C)$, we see that D is defined, on a neighborhood of x, by the system of Pfaffian equations $\eta^a = 0$. We have $\eta^a = d(y^a \circ \mu) - \sum_i (p^a_i \circ \varphi) d(x^i \circ \mu)$, and hence $d\eta^a = \sum_i d(x^i \circ \mu) \wedge d(p^a_i \circ \varphi)$. Therefore we obtain

$$d\eta^a(X,Y) = -\sum_i X(p^a_i \circ \varphi) \cdot Y(x^i \circ \mu),$$

where $X \in F(x)$, and $Y \in D(x)$. We also recall that $\mathrm{ch}(D)(x)$ consists of all $X \in D(x)$ such that $d\eta^a(X,Y) = 0$ for all $Y \in D(x)$. Furthermore we see that the linear map $Y \mapsto (Y(x^i \circ \mu))$ maps $D(x)$ onto $\mathbb{R}^r$, because $F(x) \subset D(x)$. Therefore it follows that the intersection $\mathrm{ch}(D)(x) \cap F(x)$ consists of all $X \in F(x)$ with $X(p^a_i \circ \varphi) = 0$. As we have seen, this set consists of all $X \in T_x(R)$ with $\varphi_*(X) = 0$. We have thus shown that $\varphi_*^{-1}(0) \cap T_x(R) = \mathrm{ch}(D)(x) \cap F(x)$, and hence $\varphi_*^{-1}(0) = \mathrm{ch}(D) \cap F$, proving the second assertion. □

1.5. The formal derivations d/dx^i. Let (n, r) be a pair of integers with $1 \leqq r \leqq n-1$. In the following the indices $i, j, i_1, i_2, \dots$ will range over the integers $1, \dots, r$, and the indices a, b over the integers $1, \dots, n-r$.

Let (x^i) and (y^a) be the canonical coordinate systems of $\mathbb{R}^r$ and $\mathbb{R}^{n-r}$ respectively. Then (x^i, y^a) gives the canonical coordinate system of $\mathbb{R}^n = \mathbb{R}^r \times \mathbb{R}^{n-r}$. Now, let x be any point of $\mathbb{R}^r$, and let u be any map of a neighborhood of x to $\mathbb{R}^{n-r}$. For any integer $k \geqq 0$ we may consider the k-jet $j^k_x(u)$ of u at x (cf. [22]). Then we denote by J^k the set of all the k-jets $j^k_x(u)$. Clearly J^0 may be identified with $\mathbb{R}^n$. For any $0 \leqq \ell \leqq k$ we also denote by ρ^k_ℓ the projection of J^k onto J^ℓ: $\rho^k_\ell(j^k_x(u)) = j^\ell_x(u)$. We now define functions $p^a_{i_1 \dots i_\ell}$ $(1 \leqq \ell \leqq k)$ on J^k by

$$p^a_{i_1 \dots i_\ell}(j^k_x(u)) = \frac{\partial^\ell u^a}{\partial x^{i_1} \cdots \partial x^{i_\ell}}(x),$$

where $u^a = y^a \circ u$: The function $p^a_{i_1 \ldots i_\ell}$ will be sometimes denoted by $\partial^\ell y^a / \partial x^{i_1} \cdots \partial x^{i_\ell}$. Now, J^k becomes a manifold so that the functions $x^i (= x^i \circ \rho^k_0)$, $y^a (= y^a \circ \rho^k_0)$ and $p^b_{i_1 \cdots i_\ell}$ $(i_1 \leqq \cdots \leqq i_\ell,\ 1 \leqq \ell \leqq k)$ together form a coordinate system of J^k globally defined on it. The coordinate system of J^k thus defined is called the *canonical coordinate system* of J^k.

Let f be any function defined on an open set O of J^k. Then we define functions df/dx^i on $(\rho^{k+1}_k)^{-1}(O)$ in the following manner: Take any point $j^{k+1}_x(u)$ of $(\rho^{k+1}_k)^{-1}(O)$, and denote by $j^k(u)$ the map $x' \mapsto j^k_{x'}(u)$ of a neighborhood of x to J^k. Then df/dx^i are defined by

$$\frac{df}{dx^i}(j^{k+1}_x(u)) = \frac{\partial f \circ j^k(u)}{\partial x^i}(x).$$

In terms of the canonical coordinate systems of J^k and J^{k+1} the functions df/dx^i may be expressed as follows:

$$\frac{df}{dx^i} = \frac{\partial f}{\partial x^i} + \sum_a \frac{\partial f}{\partial y^a} p^a_i + \sum_{\ell=1}^{k} \sum{}' \frac{\partial f}{\partial p^a_{i_1 \cdots i_\ell}} p^a_{i i_1 \cdots i_\ell},$$

where $\sum'$ means a summation with respect to the index a and the system $(i_1, \ldots, i_k)$ of indices with $i_1 \leqq \cdots \leqq i_k$.

The differential operators d/dx^i thus obtained are called the *formal derivations* corresponding to the coordinate system (x^i, y^a), x^i being considered as independent variables, and y^a as dependent variables. Clearly we have

$$\frac{d}{dx^i}\left(\frac{df}{dx^j}\right) = \frac{d}{dx^j}\left(\frac{df}{dx^i}\right).$$

§2. Pseudo-projective and generalized pseudo-projective structures

2.1. Projective geometry. Let (n, r) be a pair of integers with $1 \leqq r \leqq n-1$. Let $\mathbb{P}^n$ be the n-dimensional projective space over $\mathbb{R}$. In this paragraph we shall give a global formulation of the differential equation defining the r-dimensional projective subspaces of $\mathbb{P}^n$, and explain some related facts.

Let $\Sigma(\mathbb{P}^n, r)$ be the set of all r-dimensional projective subspaces of $\mathbb{P}^n$, which may be naturally identified with the Grassmann manifold $\mathrm{Gr}(\mathbb{R}^{n+1}, r+1)$. Hence $\Sigma(\mathbb{P}^n, r)$ is a compact, connected manifold of dimension $(n+1)(n-r)$. We define a relation R in the product manifold $\mathbb{P}^n \times \Sigma(\mathbb{P}^n, r)$ by

$$R = \{\, (p, \alpha) \in \mathbb{P}^n \times \Sigma(\mathbb{P}^n, r) \mid p \subset \alpha \,\},$$

where "$p \subset \alpha$" means the point p lies on the projective subspace α. This relation will be called the *fundamental relation* in the product manifold. It is easy to see that R is a compact, connected submanifold of the product manifold, and $\dim R = n + r(n-r)$.

Let μ and ν be the projections of R to $\mathbb{P}^n$ and $\Sigma(\mathbb{P}^n, r)$ respectively: $\mu(x) = p$ and $\nu(x) = \alpha$, where $x = (p, \alpha) \in R$. We easily see that R is a fibred manifold over $\mathbb{P}^n$ with projection μ, and at the same time is a fibred manifold over $\Sigma(\mathbb{P}^n, r)$ with projection ν. We also see that the fibres of these fibred manifolds are all connected. We denote by F and E the vertical tangent bundles of the fibred manifolds respectively: $F = \mu_*^{-1}(0)$ and $E = \nu_*^{-1}(0)$.

Let us now consider the Grassmann bundle $J(\mathbb{P}^n, r)$. If $(p, \alpha) \in R$, the tangent space $T_p(\alpha)$ to the projective subspace α at p is an r-dimensional contact element to $\mathbb{P}^n$, and hence gives a point of $J(\mathbb{P}^n, r)$. We easily see that the correspondence $(p, \alpha) \mapsto T_p(\alpha)$ gives a diffeomorphism, say ι, of R onto $J(\mathbb{P}^n, r)$. Clearly we have $\hat{\mu} \circ \iota = \mu$, $\hat{\mu}$ being the projection of $J(\mathbb{P}^n, r)$ onto $\mathbb{P}^n$.

Now, consider the differential system $\iota_*(E)$ on $J(\mathbb{P}^n, r)$, which is completely integrable and of rank r.

Lemma 2.1. *The maximal integral manifolds of $\iota_*(E)$ consist of all the prolongation $p(\alpha)$ of r-dimensional projective subspaces α.*

Proof. If α is an r-dimensional projective subspace, we clearly have $p(\alpha) = \iota(\nu^{-1}(\alpha))$, where α in the right hand side should be considered as a point of $\Sigma(\mathbb{P}^n, r)$. Moreover it is clear that the maximal integral manifolds of E consist of all the fibres $\nu^{-1}(\alpha)$ of the fibred manifold R over $\Sigma(\mathbb{P}^n, r)$. Therefore the lemma follows. □

Let β be an r-dimensional connected submanifold of $\mathbb{P}^n$. By Lemma 2.1 it follows that the prolongation $p(\beta)$ of β is an integral manifold of $\iota_*(E)$, if and only if β is an open submanifold of some r-dimensional projective subspace α. Consequently the pair $(J(\mathbb{P}^n, r), \iota_*(E))$ may be regarded as a differential equation for r-dimensional submanifolds of $\mathbb{P}^n$, which defines the r-dimensional projective subspaces.

Let $(u^0, \ldots, u^n)$ be any homogeneous coordinate system of $\mathbb{P}^n$. As usual the ratios $u^1/u^0, \ldots, u^n/u^0$ define a coordinate system of $\mathbb{P}^n$, which we denote by $(x^1, \ldots, x^r, y^1, \ldots, y^{n-r})$. Then we remark the obvious fact: With respect to this coordinate system the equation $(J(\mathbb{P}^n, r), \iota_*(E))$ may be expressed as the system of partial differential equations of the second order:

$$\frac{\partial^2 y^a}{\partial x^i \partial x^j} = 0 \quad (1 \leqq i, j \leqq r,\ 1 \leqq a \leqq n - r).$$

Let C and V be the contact system and the vertical tangent bundle of $J(\mathbb{P}^n, r)$ respectively.

Lemma 2.2. *$V = \iota_*(F)$, and $C = \iota_*(E) + V$* (direct sum).

Proof. Since $\hat{\mu} \circ \iota = \mu$, we have $V = \iota_*(F)$. Since $E \cap F = 0$, it follows that $\iota_*(E) \cap V = 0$. By Lemma 2.1 we have $\iota_*(E) \subset C$. Since $\operatorname{rank}(C/V) = \operatorname{rank} \iota_*(E) = r$, we have therefore seen that $C = \iota_*(E) + V$ (direct sum), proving the lemma. □

Finally we write down the fundamental properties of the triplet (R, E, F):

(1) $E \cap F = 0$.
(2) Both E and F are completely integrable.
(3) The system $E + F$ is non-degenerate.
(4) $\dim R = n + r(n - r)$, $\operatorname{rank} E = r$, and $\operatorname{rank} F = r(n - r)$.

Indeed (1), (2) and (4) are clear, and (3) follows immediately from Lemmas 1.4 and 2.2.

It should be noted that the triplet $(\mathbb{P}^n, \Sigma(\mathbb{P}^n, r), R)$, the triplet (R, E, F) and the equation $(J(\mathbb{P}^n, r), \iota_*(E))$ are closely interrelated.

2.2. Pseudo-projective structures. Let (n, r) be a pair of integers with $1 \leqq r \leqq n - 1$. Let M be an n-dimensional manifold, and let R be an *open* fibred submanifold of $J(M, r)$. Let us consider the contact system D and the vertical tangent bundle F of R, which are nothing but the restrictions of the contact system C and the vertical tangent bundle V of $J(M, r)$ to R respectively. By Lemma 1.5 we know that (i) F is completely integrable, and $F \subset D$, (ii) D is non-degenerate, and the first derived system of D coincides with $T(R)$, and (iii) $\operatorname{rank}(T(R)/D) = n - r$, $\operatorname{rank}(D/F) = r$, and $\operatorname{rank} F = r(n - r)$. Now, let E be a differential system on R. Then the pair (R, E) is called a *pseudo-projective structure* (of order 2) of bidegree (n, r) on M, if the following conditions are satisfied:

(P.1) $D = E + F$ (direct sum),

(P.2) E is completely integrable,
(P.3) Each fibre of R is connected.

A pseudo-projective structure will be denoted in principle by a capital script, e.g., $\mathcal{X}$, and then the pair (R, E) will be called the *expression* of $\mathcal{X}$; R itself will be called the underlying fibred manifold of $\mathcal{X}$. The same notation and terminology will be also used for generalized pseudo-projective structures, which will be defined in §2.4. Incidentally we shall introduce, in that paragraph, the notion of (local) isomorphism for the generalized pseudo-projective structures, which of course holds good for the pseudo-projective structures.

The notations being as in the previous paragraph, let us consider the completely integrable system $\iota_*(E)$ on $J(\mathbb{P}^n, r)$. By Lemma 2.2 we see that the pair $(J(\mathbb{P}^n, r),\ \iota_*(E))$ defines a pseudo-projective structure of bidegree (n, r) on the projective space $\mathbb{P}^n$, which we denote by $\mathcal{X}_0$, and call the *standard pseudo-projective structure* of bidegree (n, r). We recall that $\mathcal{X}_0$ may be regarded as the differential equation defining the r-dimensional projective subspaces of $\mathbb{P}^n$.

In general let $\mathcal{X}$ be a pseudo-projective structure of bidegree (n, r) on a manifold M, and (R, E) its expression. Then an r-dimensional submanifold α of M is said to be a *solution* of $\mathcal{X}$, if the prolongation $p(\alpha)$ of α is in R, and is an integral manifold of E.

In this fashion $\mathcal{X}$ may be regarded as a differential equation for r-dimensional submanifolds of M. From the global view-point the notion of a solution of $\mathcal{X}$ should be generalized as follows: A solution of $\mathcal{X}$ in the wide sense is the image α of some integral manifold $\tilde{\alpha}$ of E by the projection μ of R onto M. (Note that the restriction of μ to $\tilde{\alpha}$ is an immersion, and that $\tilde{\alpha}$ is uniquely determined by α.) However, in our later arguments a solution of $\mathcal{X}$ will always mean a solution in the former sense, unless otherwise stated.

We shall now give a local expression of $\mathcal{X}$ in terms of a canonical coordinate system. In the following the indices i, j, k, ℓ will range over the integers $1, \dots, r$, and the indices a, b over the integers $1, \dots, n - r$.

Let z_0 be any point of R, and let (x^i, y^a, p^b_j) be any canonical coordinate system of R or of $J(M, r)$ at z_0. Let $\tilde{U}$ be the domain of the coordinate system. Then F is defined, on the intersection $R \cap \tilde{U}$, by the system of Pfaffian equations: $dx^i = dy^a = 0$, and D by the system of Pfaffian equations: $dy^a - \sum_j p^a_j dx^j = 0$. Since $D = E + F$ (direct sum), it follows that there are unique functions f^a_{ij} on the intersection $R \cap \tilde{U}$ such that E is defined there by the system of Pfaffian equations:

$$\begin{cases} dy^a - \sum_j p^a_j dx^j = 0, \\ dp^a_i - \sum_j f^a_{ij} dx^j = 0. \end{cases}$$

Now, f^a_{ij} may be expressed as functions of (x^k), (y^b) and (p^b_k). Let α be an r-dimensional connected submanifold of M. Suppose that $p(\alpha)$ is in $\tilde{U}$, and that α is defined by a system of equations of the following form: $y^a = u^a(x^1, \dots, x^r)$. Then we easily see that α is a solution of $\mathcal{X}$, if and only if (u^a) gives a solution of the following system, say $(\mathcal{X})$, of partial differential equations of the second order:

$$\frac{\partial^2 y^a}{\partial x^i \partial x^j} = f^a_{ij}\left((x^k), (y^b), \left(\frac{\partial y^b}{\partial x^k}\right)\right).$$

In the special case where $r = 1$, this system simply means the following system of ordinary differential equations of the second order:

$$\frac{d^2y^a}{dx^2} = f^a\left(x, (y^b), \left(\frac{dy^b}{dx}\right)\right),$$

where the indices a, b range over the integers $1, \dots, n-1$, and we set $x = x^1$ and $f^a = f^a_{11}$.

Using the functions f^a_{ij}, we now define functions f^a_{ijk} by

$$f^a_{ijk} = \frac{\partial f^a_{ij}}{\partial x^k} + \sum_b \frac{\partial f^a_{ij}}{\partial y^b} \cdot p^b_k + \sum_{b,\ell} \frac{\partial f^a_{ij}}{\partial p^b_\ell} \cdot f^b_{\ell k}.$$

Then it can be shown that the system $(\mathcal{X})$ is involutive or it satisfies the integrability conditions:

$$f^a_{ij} = f^a_{ji}, \quad f^a_{ijk} = f^a_{ikj},$$

which describe that the system E, restricted to $R \cap \tilde{U}$, is completely integrable. For the definition of an involutive system, see [7].

System $(\mathcal{X})$, thus obtained, is called the *local expression* of $\mathcal{X}$ at z_0 corresponding to the canonical coordinate system (x^i, y^a, p^b_j) of $J(M, r)$ or corresponding to the coordinate system (x^i, y^a) of M. Conversely it is clear that any involutive system of the form $(\mathcal{X})$ can be obtained as a local expression of some pseudo-projective structure $\mathcal{X}$ of bidegree (n, r).

2.3. Projective structures and the associated pseudo-projective structures of bidegree $(n, 1)$. First of all we recall the definition of an affine connection together with some related definitions. For the details see [9] and [10].

Let M be a manifold. We denote by $\mathcal{T}$ and $\mathcal{T}^{(1,1)}$ the sheaves of germs of local cross sections of T and $T^* \otimes T$ respectively, where $T = T(M)$. Let ∇ be a sheaf homomorphism of $\mathcal{T}$ to $\mathcal{T}^{(1,1)}$, which naturally induces a linear map, denoted by the same letter ∇, of $\Gamma(U, T)$ to $\Gamma(U, T^* \otimes T)$ for any open set U of M. Then ∇ is called an affine connection on M, if it satisfies the following condition:

$$\nabla(fY) = f\nabla Y + df \otimes Y,$$

where Y is any vector field defined on any open set U of M, and f is any function defined on U.

Let ∇ be an affine connection on M. Then ∇ is said to be *without torsion*, if $\nabla_X Y - \nabla_Y X = [X, Y]$ for any local vector fields X, Y on M. (Note that $\nabla_X Y$ means the contraction of ∇Y by X.) Now, let $\sigma(t)$ $(t \in I)$ be a curve of M, I being an open interval. Then σ is called a *geodesic* of (M, ∇), if the tangent vector field $\dfrac{d\sigma}{dt}(t)$ $(t \in I)$ to the curve σ is parallel with respect to the connection ∇. We remark that for any $p \in M$, $X \in T_p(M)$ and $t_0 \in \mathbb{R}$ there is a unique maximal geodesic $\sigma(t)$ $(t \in I)$ of (M, ∇) such that $t_0 \in I$, $\sigma(t_0) = p$, and $\dfrac{d\sigma}{dt}(t_0) = X$.

We shall now recall the definition of a projective structure. In the following, affine connections will be always assumed to be without torsion.

Let ∇ and ∇' be two affine connections on a manifold M. Then ∇' is said to be *projectively equivalent* to ∇, if the geodesics of (M, ∇') coincide with the geodesics of (M, ∇). Here, each geodesic of (M, ∇) or of (M, ∇') should be confounded with its image in M. Clearly this relation becomes an equivalence relation. As is well known, ∇' is projectively equivalent to ∇, if and only if there is a 1-form ζ on M such that

$$\nabla'_X Y = \nabla_X Y + \zeta(X)Y + \zeta(Y)X,$$

where X and Y are any local vector fields on M.

Let M be a manifold, and let $\mathfrak{D}(M)$ be the set of all affine connections ∇ defined on open sets, say $U(\nabla)$, of M. Then a *projective structure* on M is a subset $\mathfrak{D}$ of $\mathfrak{D}(M)$ which satisfies the following conditions:

(i) $M = \bigcup_{\nabla \in \mathfrak{D}} U(\nabla)$,

(ii) If $\nabla, \nabla' \in \mathfrak{D}$, and $U(\nabla) \cap U(\nabla') \neq \emptyset$, then ∇' is projectively equivalent to ∇ on the intersection $U(\nabla) \cap U(\nabla')$,

(iii) $\mathfrak{D}$ is maximal in the following sense: Let $\mathfrak{D}'$ be another subset of $\mathfrak{D}(M)$ satisfying conditions (i) and (ii). If $\mathfrak{D} \subset \mathfrak{D}'$, then $\mathfrak{D}'$ coincides with $\mathfrak{D}$.

This definition is particularly valid in the real or complex analytic category, while in the C^∞ category the projective structure $\mathfrak{D}$ can be reasonably replaced by its subset

$$\mathfrak{D}_0 = \{ \nabla \in \mathfrak{D} \mid U(\nabla) = M \},$$

which is nothing but a class of mutually projectively equivalent affine connections on M.

We shall show that to every projective structure there is naturally associated a pseudo-projective structure of bidegree $(n, 1)$.

Let M be an n-dimensional manifold, and let us consider its tangent bundle $T(M)$. Let $\overset{\circ}{T}(M)$ be the complement of the zero cross section of $T(M)$, which is an open fibred submanifold of $T(M)$. We denote by ϖ the projection of $\overset{\circ}{T}(M)$ onto M. For each $\lambda \in \mathbb{R} - \{0\}$ we define a transformation τ_λ of $\overset{\circ}{T}(M)$ by $\tau_\lambda(y) = \lambda y$ for all $y \in \overset{\circ}{T}(M)$. Now, let ∇ be an affine connection on M. As is well known, there is a unique vector field, say $X(\nabla)$, on $\overset{\circ}{T}(M)$ having the following property: Let $\sigma(t)$ $(t \in I)$ be any geodesic of (M, ∇) with $\dot{\sigma}(t) \neq 0$, where $\dot{\sigma}(t) = \dfrac{d\sigma}{dt}(t)$. Then $\dot{\sigma}(t)$ $(t \in I)$ is an integral curve of $X(\nabla)$. We can easily prove the following

Lemma 2.3. (1) $\varpi_*(X(\nabla)_y) = y, \quad y \in \overset{\circ}{T}(M).$

(2) $(\tau_\lambda)_*(X(\nabla)) = \dfrac{1}{\lambda} \cdot X(\nabla), \quad \lambda \in \mathbb{R} - \{0\}.$

The vector field $X(\nabla)$ is usually called the *spray* associated with (M, ∇).

Let us now consider the Grassmann bundle $J(M,1)$, and let C and V be the contact system and the vertical tangent bundle of $J(M,1)$ respectively. We denote by π the projection of $J(M,1)$ onto M, and by ρ the natural map of $\overset{\circ}{T}(M)$ onto $J(M,1)$: For each $y \in \overset{\circ}{T}(M)$, $\rho(y)$ is the 1-dimensional contact element to M spanned by the vector y . Clearly we have $\pi \circ \rho = \varpi$. We also note that $\overset{\circ}{T}(M)$ is a fibred manifold over $J(M,1)$ with projection ρ. Now, let z be any point of $J(M,1)$. Take any point y of $\overset{\circ}{T}(M)$ with $z = \rho(y)$, and set $Y = \rho_*(X(\nabla)_y)$. By (1) of Lemma 2.3 we have $\pi_*(Y) = y$ $(\in z)$, from which follows that $Y \in C(z)$, and $Y \notin V(z)$. This being said, we denote by $E(z)$ the 1-dimensional subspace of $C(z)$ spanned by Y. By (2) of Lemma 2.3 we then find that $E(z)$ does not depend on the choice of y. Now, set $E(\nabla) = \bigcup_z E(z)$. Then we see that $E(\nabla)$ is a subbundle of C, and $C = E(\nabla) + V$ (direct sum). Hence the pair $(J(M,1), E(\nabla))$ defines a pseudo-projective structure of bidegree $(n, 1)$, which we denote by $\mathcal{X}(\nabla)$.

Let $\sigma(t)$ $(t \in I)$ be any geodesic of (M, ∇). We define a curve $\hat{\sigma}(t)$ $(t \in I)$ of $J(M,1)$ by $\hat{\sigma}(t) = \rho(\dot{\sigma}(t))$ for all $t \in I$. $\dot{\sigma}(t)$ being an integral curve of $X(\nabla)$, we can easily prove the following

Lemma 2.4. *The image $\hat{\sigma}(I)$ of I by $\hat{\sigma}$ is an integral manifold of $E(\nabla)$, and hence the image $\sigma(I)$ of I by σ is a solution of $\mathcal{X}(\nabla)$ in the wide sense. Furthermore the integral manifolds of $E(\nabla)$ are exhausted by the submanifolds $\hat{\sigma}(I)$ of $J(M,1)$ corresponding to geodesics $\sigma(t)$ $(t \in I)$ of (M,∇).*

Now, let $\mathfrak{D}$ be a projective structure on an n-dimensional manifold M. For any $\nabla \in \mathfrak{D}$ let us consider the pseudo-projective structure $\mathcal{X}(\nabla)$ or $(J(U(\nabla),1), E(\nabla))$ of bidegree $(n,1)$ on the domain $U(\nabla)$ of ∇. Note that $J(U(\nabla),1)$ is an open submanifold of $J(M,1)$, and $J(M,1) = \bigcup_{\nabla\in\mathfrak{D}} J(U(\nabla),1)$. By Lemma 2.4 it follows that if $\nabla, \nabla' \in \mathfrak{D}$, and $U(\nabla)\cap U(\nabla') \neq \emptyset$, the systems $E(\nabla)$ and $E(\nabla')$ coincide on the intersection $J(U(\nabla),1)\cap J(U(\nabla'),1)$. Therefore the system $E(\nabla)$ can be pieced together to yield a differential system, say $E(\mathfrak{D})$, defined on the whole of $J(M,1)$, and the pair $(J(M,1), E(\mathfrak{D}))$ defines a pseudo-projective structure of bidegree $(n,1)$ on M, which we denote by $\mathcal{X}(\mathfrak{D})$. Clearly a solution (in the wide sense) of $\mathcal{X}(\mathfrak{D})$ means a geodesic of the projective structure $\mathfrak{D}$. Thus $\mathcal{X}(\mathfrak{D})$ will be sometimes called the *differential equation of geodesics* of $\mathfrak{D}$.

Let $\mathcal{X}$ be a pseudo-projective structure of bidegree $(n,1)$ on a manifold M, and (R,E) its expression. In terms of a local expression of $\mathcal{X}$ we shall discuss conditions that $\mathcal{X}$ can be obtained as a restriction of the pseudo-projective structure $\mathcal{X}(\mathfrak{D})$ associated with some projective structure $\mathfrak{D}$ on M. In the following the indices i,j,k will range over the integers $1,\dots,n-1$, and the indices a,b,c over the integers $0,1,\dots,n-1$.

Let (x,y,p) be any canonical coordinate system of R at any point z_0, where $y=(y^i)$ and $p=(p^i)$, and let us consider the corresponding local expression of $\mathcal{X}$ at z_0:

$$\frac{d^2y^i}{dx^2} = f^i\left(x,y,\frac{dy}{dx}\right).$$

Proposition 2.5 (cf. [3])**.** *The following two statements are mutually equivalent:*

(1) *There is a projective structure $\mathfrak{D}$ on M such that E is the restriction of $E(\mathfrak{D})$ to R.*

(2) *The functions $f^i(x,y,p)$ take the following forms:*

$$f^i(x,y,p) = p^i g^0(x,y,p) + g^i(x,y,p).$$

Here, $g^a(x,y,p)$ are polynomials of the variable p of order at most 2 with functions of the variables x,y as coefficients.

In particular we consider the case where $n=2$. Let (x,y,p) be any canonical coordinate system of R at any point z_0, and let us consider the corresponding local expression of $\mathcal{X}$ at z_0:

$$\frac{d^2y}{dx^2} = f\left(x,y,\frac{dy}{dx}\right).$$

Then we remark that the second statement in the proposition means the following: The function $f(x,y,p)$ is a polynomial of the variable p of order at most 3 with functions of the variables x,y as coefficients, that is, $f(x,y,p)$ takes the following form:

$$f(x,y,p) = A(x,y) + 3B(x,y)p + 3C(x,y)p^2 + D(x,y)p^3.$$

Proof of Proposition 2.5. The implication (1) $\Rightarrow$ (2). Clearly we may assume that there is an affine connection ∇ defined on the whole of M such that $\mathcal{X} = \mathcal{X}(\nabla)$, i.e., $R = J(M,1)$ and $E = E(\nabla)$. Let (x,y,p) be any canonical coordinate system of R at any point z_0, and let $\tilde{U}$ be its domain. Denote by $(x^0,\dots,x^{n-1})$ the coordinate system $(x,y^1,\dots,y^{n-1})$ of M, and consider the coefficients Γ^a_{bc} of ∇ with respect to (x^a): $\nabla_{X_b}X_c = \sum_a \Gamma^a_{bc}X_a$, where

$X_a = \partial/\partial x^a$. Note that $\Gamma^a_{bc} = \Gamma^a_{cb}$, because ∇ is without torsion. Now, let $\sigma(t)$ $(t \in I)$ be any geodesic of (M, ∇) with $\hat{\sigma}(I) = \rho(\sigma(I)) \subset \tilde{U}$. If we put $x^a(t) = x^a(\sigma(t))$, the functions $x^a(t)$ satisfy the following system of ordinary differential equations:

$$\frac{d^2x^a}{dt^2} + \sum_{b,c} \Gamma^a_{bc} \cdot \frac{dx^b}{dt} \cdot \frac{dx^c}{dt} = 0,$$

where Γ^a_{bc} should be preferably replaced by $\Gamma^a_{bc}(x^0, \ldots, x^{n-1})$. By Lemma 2.4 $\sigma(I)$ is a solution of $\mathcal{X}$, and is defined by a system of equations of the following form: $x^i = x^i(x^0)$. Then our task is to seek the system of ordinary differential equations satisfied by the functions $x^i(x^0)$. Since $x^i(t) = x^i(x^0(t))$, we have

$$\frac{dx^i}{dt} = \frac{dx^i}{dx^0} \cdot \frac{dx^0}{dt}, \quad \frac{d^2x^i}{dt^2} = \frac{d^2x^i}{(dx^0)^2} \cdot \left(\frac{dx^0}{dt}\right)^2 + \frac{dx^i}{dx^0} \cdot \frac{d^2x^0}{dt^2}.$$

Hence we easily find that the desired system is

$$\frac{d^2x^i}{(dx^0)^2} = \frac{dx^i}{dx^0} \cdot P^0 - P^i,$$

where

$$P^a = \Gamma^a_{00} + 2\sum_k \Gamma^a_{0k} \cdot \frac{dx^k}{dx^0} + \sum_{j,k} \Gamma^a_{jk} \cdot \frac{dx^j}{dx^0} \cdot \frac{dx^k}{dx^0}.$$

We have therefore seen that the functions f^i corresponding to the canonical coordinate system (x, y, p) take the forms as stated in the proposition.

The implication (2) $\Rightarrow$ (1). From the discussion above we deduce that for each $z_0 \in R$ there is an affine connection $\nabla \in \mathfrak{D}(M)$ such that $z_0 \in J(U(\nabla), 1)$, and $E = E(\nabla)$ on some neighborhood of z_0. Now, let $\mathfrak{D}$ be the set of all affine connections $\nabla \in \mathfrak{D}(M)$ such that $E = E(\nabla)$ on the intersection $R \cap J(U(\nabla), 1)$. Then, in view of the remark above we can easily verify that $\mathfrak{D}$ is a projective structure on M, and E is the restriction of $E(\mathfrak{D})$ to R. (Note that each fibre of R is assumed to be connected.) We have thus completed the proof of the proposition. □

2.4. Generalized pseudo-projective structures. Let (n, r) be a pair of integers with $1 \leqq r \leqq n-1$. Let M be an n-dimensional manifold, and let R be an imbedded fibred submanifold of $J(M, r)$. Let us consider the contact system D and the vertical tangent bundle F of R. Concerning these systems, recall Lemma 1.7. Now, let E be a differential system on R. Then the pair (R, E) is called a *generalized pseudo-projective structure* of bidegree (n, r) on M, if the following conditions are satisfied:

(GP.1) $D = E + F$ (direct sum),
(GP.2) E is completely integrable,
(GP.3) Each fibre of R is connected,
(GP.4) D is non-degenerate,
(GP.5) The full derived system of D coincides with $T(R)$.

We notice that under conditions (GP.1) and (GP.2), condition (GP.4) is equivalent to the condition that $\mathrm{ch}(D) \cap E = 0$, $\mathrm{ch}(D)$ being the Cauchy characteristic system of D. Indeed, we have $\mathrm{ch}(D) \cap F = 0$ by Lemma 1.7, and $\mathrm{ch}(D) = \mathrm{ch}(D) \cap E + \mathrm{ch}(D) \cap F$ by Lemma 3.3 in the next section.

Let $\mathcal{X}$ be a generalized pseudo-projective structure of bidegree (n, r) on M, and (R, E) its expression. In the same fashion as in the case of a pseudo-projective structure $\mathcal{X}$ may be regarded as a differential equation: A solution of $\mathcal{X}$ is an r-dimensional submanifold α of M

of which the prolongation $p(\alpha)$ is in R, and is an integral manifold of E. It should be noted that if $\dim R < \dim J(M,r)$, R itself may be regarded as a differential equation. Furthermore a solution of $\mathcal{X}$ in the wide sense is the image $\mu(\tilde{\alpha})$ of an integral manifold $\tilde{\alpha}$ of E by the projection μ of R onto M.

Let $\mathcal{X}$ and $\mathcal{X}'$ be two generalized pseudo-projective structures of bidegree (n,r) on manifolds M and M', and let (R,E) and (R',E') be their expressions, respectively. By an isomorphism of $\mathcal{X}$ to $\mathcal{X}'$ we mean a diffeomorphism φ of M onto M' such that the prolongation $p(\varphi)$ naturally induces an isomorphism of (R,E) to (R',E') as manifolds with differential system. Clearly a diffeomorphism φ of M onto M' is an isomorphism of $\mathcal{X}$ to $\mathcal{X}'$, if and only if it naturally induces a one-to-one correspondence between the solutions of $\mathcal{X}$ and the solutions of $\mathcal{X}'$. Now, let μ and μ' be the projections of R and R' onto M and M' respectively, and let (z_0,z_0') be a point of $R\times R'$. By a local isomorphism of $\mathcal{X}$ to $\mathcal{X}'$ at (z_0,z_0') we mean a local diffeomorphism φ of M to M' at $(\mu(z_0),\mu'(z_0'))$ such that the prolongation $p(\varphi)$ of φ naturally induces a local isomorphism of (R,E) to (R',E') at (z_0,z_0') as manifolds with differential system.

We shall now make some remarks on the geometry of generalized pseudo-projective structures $\mathcal{X}$.

(1) In the next paragraph we shall introduce the notion of the arithmetic distance ρ in M associated with $\mathcal{X}$, and in the next section the notion of the dual $\mathcal{X}^*$ of $\mathcal{X}$ under some additional conditions on $\mathcal{X}$, which is a generalized pseudo-projective structure on the manifold of maximal solutions of $\mathcal{X}$. We shall find that conditions (GP.5) and (GP.4) play crucial roles in the definitions of these notions respectively.

(2) Let us consider the sheaf $\mathfrak{a}$ of germs of local infinitesimal automorphisms of $\mathcal{X}$ or precisely of the associated generalized pseudo-projective system $\mathcal{R}$. (For the definition of this notion see the next section.) In §2 of Chapter II we shall show that under certain natural conditions the structure $\mathcal{X}$ is of finite type in the sense that each stalk of $\mathfrak{a}$ is of finite dimension (see Proposition 2.9 in Chapter II). We then remark that conditions (GP.4) and (GP.5) likewise play crucial roles in the proof of this fact.

(3) In Chapter VI we shall introduce the notion of pseudo-projective structures of order $k \geqq 3$, being special kinds of generalized pseudo-projective structure. Then we shall see that the geometry of systems of ordinary differential equations of order $k \geqq 3$ may be represented by the geometry of pseudo-projective structures of order k.

Let $\mathcal{X}$ be a generalized pseudo-projective structure of bidegree (n,r) on a manifold M, and (R,E) its expression. We shall give a local expression of $\mathcal{X}$ in terms of a canonical coordinate system of $J(M,r)$. In the following C and V will denote the contact system and the vertical tangent bundle of $J(M,r)$ respectively. The indices i,j,k will range over the integers $1,\dots,r$, the indices a,b over the integers $1,\dots,n-r$, and the index λ over the integers $1,\dots,\delta$, where $\delta = \dim J(M,r) - \dim R$.

Let z_0 be any point of R, and let (x^i,y^a,p^b_j) be any canonical coordinate system of $J(M,r)$ at z_0. Let us consider any regular system of local equations $f^\lambda = 0$ for the submanifold R of $J(M,r)$ at z_0. Namely f^λ are functions defined on a neighborhood O_1 of z_0 in $J(M,r)$ such that $R\cap O_1 = \{\, z\in O_1 \mid f^\lambda(z)=0 \,\}$, and the differentials df^λ are linearly independent at z_0. Let us also take any differential system $\tilde{E}$ defined on a neighborhood O_2 of z_0 in $J(M,r)$ such that $\tilde{E}\subset C$, and $\tilde{E}=E$ on the intersection $R\cap O_2$. Clearly we have $C=\tilde{E}+V$ (direct sum) on a sufficiently small neighborhood O of z_0 in $J(M,r)$. Therefore it follows that there are unique functions f^a_{ij} on the neighborhood O such that $\tilde{E}$ is defined there by the system of Pfaffian equations:

$$\begin{cases} dy^a - \sum_j p^a_j dx^j = 0, \\ dp^a_i - \sum_j f^a_{ij} dx^j = 0. \end{cases}$$

Now, f^λ and f^a_{ij} may be expressed as functions of (x^k), (y^b) and (p^b_k). Let α be an r-dimensional connected submanifold of M. Suppose that the prolongation $p(\alpha)$ of α is in O, and that α is defined by a system of equations of the following form: $y^a = u^a(x^1, \ldots, x^r)$. We first see that α is a solution of the equation R, if and only if (u^a) gives a solution of the following system, say (R), of partial differential equations of the first order:

$$f^\lambda\left((x^k), (y^b), \left(\frac{\partial y^b}{\partial x^k}\right)\right) = 0.$$

We next see that α is a solution of the equation $\mathcal{X}$, if and only if (u^a) gives a solution of the following system, say $(\mathcal{X})$, of partial differential equations of the second order:

$$\begin{cases} f^\lambda\left((x^k), (y^b), \left(\dfrac{\partial y^b}{\partial x^k}\right)\right) = 0, \\ \dfrac{\partial^2 y^a}{\partial x^i \partial x^j} = f^a_{ij}\left((x^k), (y^b), \left(\dfrac{\partial y^b}{\partial x^k}\right)\right). \end{cases}$$

We note that the system $(\mathcal{X})$ is involutive, because the system E is completely integrable.

System $(\mathcal{X})$ (resp. (R)), thus obtained, will be called a *local expression* of $\mathcal{X}$ (resp. of (R)) at z_0 corresponding to the canonical coordinate system (x^i, y^a, p^b_j) of $J(M, r)$ or corresponding to the coordinate system (x^i, y^a) of M.

2.5. The arithmetic distance associated with a generalized pseudo-projective structure. Let D be a differential system on a manifold R, and let us consider the full derived system $D^{-\infty}$ of D. By an integral curve of D we mean a curve $\sigma(t)$ $(a \leqq t \leqq b)$ of R such that the tangent vector $\dfrac{d\sigma}{dt}(t)$ to the curve is in $D(\sigma(t))$ for each t. It is well known that if R is connected, and $D^{-\infty} = T(R)$, then any two points x and y of R can be joined by a piece-wise integral curve of D (cf. Chow [6] and Tanaka [19]). Now, let (E_i) be a family of differential systems on R such that $D = \sum_i E_i$, i.e., $D(x) = \sum_i E_i(x)$ for each $x \in R$. Then we have the following stronger result.

Lemma 2.6. *If R is connected, and $D^{-\infty} = T(R)$, then any two points x and y of R can be joined by a composite of integral curves of E_i.*

Before proceeding to the proof of this fact we shall apply it to the study of generalized pseudo-projective structures.

Let $\mathcal{X}$ be a generalized pseudo-projective structure of bidegree (n, r) on a manifold M, and (R, E) its expression. By a *chain* in $\mathcal{X}$ we mean a system Λ formed by a finite number of points p_i $(0 \leqq i \leqq k)$ of M and k maximal solutions α_i $(1 \leqq i \leqq k)$ of $\mathcal{X}$ in the wide sense such that p_{i-1}, $p_i \in \alpha_i$ for each $1 \leqq i \leqq k$. The non-negative integer k will be called the *length* of the chain Λ and will be denoted by $k(\Lambda)$. Furthermore the points p_0 and p_k will be denoted respectively by $p(\Lambda)$ and $q(\Lambda)$.

From now on we assume that R or equivalently M is connected.

Lemma 2.7. *For any two points p and q of M there is a chain Λ in $\mathcal{X}$ joining p and q: $p = p(\Lambda)$ and $q = q(\Lambda)$.*

Proof. Let us consider the contact system D of R. Then $D = E + F$, F being the vertical tangent bundle of R, and the full derived system $D^{-\infty}$ of D coincides with $T(R)$. Therefore we know from Lemma 2.6 that any two points z and w of R can be joined by a composite of integral curves of E and F. Furthermore an integral curve of E means a curve in some maximal integral manifold of E, and an integral curve of F a curve in some fibre of R. Now, the lemma follows immediately from these facts. □

Let p and q be any two points of M. Owing to Lemma 2.7 we may define a non-negative integer $\rho(p, q)$ to be the minimum of the lengths $k(\Lambda)$ of chains Λ in $\mathcal{X}$ joining p and q. It is easy to see that the function $(p, q) \mapsto \rho(p, q)$ gives a distance (function) in M, which we call the *arithmetic distance* associated with $\mathcal{X}$. Clearly the arithmetic distance or the assignment $\mathcal{X} \mapsto \rho$ is a global invariant in the geometry of generalized pseudo-projective structures. In particular we know that the arithmetic distance ρ is invariant under the automorphism group $\mathrm{Aut}(\mathcal{X})$ of $\mathcal{X}$: $\rho(\varphi(p), \varphi(q)) = \rho(p, q)$ for all $\varphi \in \mathrm{Aut}(\mathcal{X})$ and $p, q \in M$.

Remark . Let M be an irreducible symmetric R-space, which is a compact and connected symmetric space. Then a submanifold α of M is called a *Helgason sphere*, if the following conditions are satisfied: (i) α is a totally geodesic sphere with minimum radius, and (ii) α is of maximum dimension among the submanifolds satisfying condition (i). Among others it is known that Helgason spheres are congruent to each other under the largest connected group of isometries of M, and that any two points p and q of M can be joined by a chain of Helgason spheres. For any $p, q \in M$ one may therefore define a non-negative integer $\rho(p, q)$ to be the minimum of the lengths of chains of Helgason spheres joining the two points. Then the function $(p, q) \mapsto \rho(p, q)$ gives a distance in M, which is called the *arithmetic distance* in the symmetric R-space M. (A detailed account of these can be founded in Takeuchi [17].) In §7[1] we shall make some studies on the arithmetic distances associated with generalized pseudo-projective structures. Especially we shall see that the arithmetic distance in a Grassmann manifold, being an irreducible symmetric R-space, is obtained as the arithmetic distance associated with a suitable generalized pseudo-projective structure on the manifold. It is reasonably expected that the same fact exists concerning an arbitrary irreducible symmetric R-space.

We shall now prove Lemma 2.6.

First of all we prepare some general notations. Given a differential system H on R, $\Gamma(H)$ denotes the space of all cross sections of H, and $\Gamma_0(H)$ the subspace of $\Gamma(H)$ consisting of all cross sections of H with compact support. We set $T = T(R)$. For any $X \in \Gamma_0(T)$ let $(\phi_t)_{t\in\mathbb{R}}$ be the one-parameter group of transformations of R generated by X. Then e^X denotes the transformation ϕ_1, which is usually called the exponential map corresponding to X. Note that $\phi_t = e^{tX}$.

Now, we denote by G the group of all the transformations of R of the following form: $e^{X_1} \cdots e^{X_k}$, where each X_λ is a vector field in $\Gamma_0(E_i)$ for some i. We then denote by $\tilde{G}$ the set of all the one-parameter families $(\phi_t)_{t\in\mathbb{R}}$ of transformations of R such that $\phi_0 = 1$, the identity transformation of R, and $\phi_t \in G$ for all t. Furthermore, for each $(\phi_t) \in \tilde{G}$ we denote by $V((\phi_t))$ the vector field on R induced by (ϕ_t):

$$V((\phi_t))_x = \frac{\partial}{\partial t}\phi_t(x)|_{t=0}, \quad x \in R.$$

[1]Editor's note: It is uncertain where §7 is.

We then denote by $\mathfrak{g}$ the set of all the vector fields on R of the form: $V((\phi_t))$.

Lemma 2.8. *$\mathfrak{g}$ is a subspace of $\Gamma(T)$, and if $\varphi \in G$ and $X \in \mathfrak{g}$, then $\varphi_*(X) \in \mathfrak{g}$.*

Proof. Let (ϕ_t) and (ψ_t) be any two families in $\tilde{G}$. Then we have $(\phi_t\psi_t) \in \tilde{G}$, and $V((\phi_t\psi_t)) = V((\phi_t)) + V((\psi_t))$. If $\lambda \in \mathbb{R}$, we have $(\phi_{\lambda t}) \in \tilde{G}$, and $V((\phi_{\lambda t})) = \lambda V((\phi_t))$. Furthermore if $\varphi \in G$, we have $(\varphi \circ \phi_t \circ \varphi^{-1}) \in G$, and $V((\varphi \circ \phi_t \circ \varphi^{-1})) = \varphi_* V((\phi_t))$, proving the lemma. □

For each $x \in R$ we define a subspace $\mathfrak{g}(x)$ of $T(x)$ by $\mathfrak{g}(x) = \{ X_x \mid X \in \mathfrak{g} \}$.

Lemma 2.9. *If $X_1, \dots, X_\ell \in \mathfrak{g}$ and $x \in R$, then $(\mathrm{ad}(X_1) \cdots \mathrm{ad}(X_{\ell-1})X_\ell)_x \in \mathfrak{g}(x)$.*

Proof. We prove this by induction on the integer ℓ. k being a positive integer, assume that the lemma is true for any $\ell < k$. Let $X_1, \dots, X_k$ be any k vector fields in $\mathfrak{g}$, and take $(\phi_t) \in \tilde{G}$ with $X_k = V((\phi_t))$. By Lemma 2.8 we have $\phi_{t*}(X_{k-1}) \in \mathfrak{g}$. By the assumption of induction we therefore obtain $(\mathrm{ad}(X_1) \cdots \mathrm{ad}(X_{k-2})\phi_{t*}(X_{k-1}))_x \in \mathfrak{g}(x)$. Since

$$\frac{\partial}{\partial t}(\phi_{t*}(X_{k-1}))_y|_{t=0} = [X_{k-1}, X_k]_y, \quad y \in R,$$

it follows that

$$(\mathrm{ad}(X_1) \cdots \mathrm{ad}(X_{k-1})X_k)_x = \frac{\partial}{\partial t}(\mathrm{ad}(X_1) \cdots \mathrm{ad}(X_{k-1})\phi_{t*}(X_{k-1}))_x|_{t=0} \in \mathfrak{g}(x),$$

proving the lemma. □

If $X \in \Gamma_0(E_i)$, we have $e^{tX} \in G$, and hence $X = V((e^{tX})) \in \mathfrak{g}$. Therefore we obtain $\Gamma_0(E_i) \subset \mathfrak{g}$. Since $D = \sum_i E_i$, and $D^{-\infty} = T$, it follows from Lemma 2.9 that $\mathfrak{g}(x) = T(x)$ for each $x \in R$.

We show that the group G acts transitively on R. Indeed, let x_0 be any point of R. Set $n = \dim R$. Since $\mathfrak{g}(x_0) = T(x_0)$, we can find n vector fields $X_1, \dots, X_n$ in $\mathfrak{g}$ such that the n vectors $(X_1)_{x_0}, \dots, (X_n)_{x_0}$ form a basis of $T(x_0)$. For each $1 \leqq a \leqq n$ take $(\phi^a_t) \in \tilde{G}$ with $X_a = V((\phi^a_t))$, and set $\phi_t = \phi^1_{t^1} \cdots \phi^n_{t^n}$ for any $t = (t^1, \dots, t^n) \in \mathbb{R}^n$. Clearly we have $\phi_0 = 1$, and $(\partial/\partial t^a)\phi_t(x_0)|_{t=0} = (X_a)_{x_0}$. Therefore the map $t \mapsto \phi_t(x_0)$ gives a diffeomorphism of a neighborhood of 0 onto a neighborhood of x_0. Since R is connected, we have thus shown that G acts transitively on R.

Now, let x_0 and x be any two points of R. Since G acts transitively on R, we can find $\varphi \in G$ such that $x = \varphi(x_0)$. By the very definition of the group G, φ may be written in the form: $\varphi = e^{Y_N} \cdots e^{Y_1}$, where for each $1 \leqq A \leqq N$, Y_A is in $\Gamma_0(E_i)$ for some i. If we set $x_A = e^{Y_A} \cdots e^{Y_1}(x_0)$, we have $x = x_N$ and $x_A = e^{Y_A}(x_{A-1})$. We also see that the curve $e^{tY_A}(x_{A-1})$ $(0 \leqq t \leqq 1)$ of R joins the two points p_{A-1} and p_A, and is an integral curve of a suitable system E_i.

We have thus completed the proof of Lemma 2.6. □

Remark. We recall that $D^{-\infty}$ is a completely integrable system possibly with singularities. By an integral manifold of $D^{-\infty}$ we mean a connected submanifold N of R such that $T_x(N) = D^{-\infty}(x)$ at each $x \in N$. Now, assume that D together with R is real analytic. Then we know that through every point $x_0 \in R$ there passes a unique (real analytic) maximal integral manifold, say R', of $D^{-\infty}$ (see Appendix A in Chapter II). Let R'_0 be the set of all the points $x \in R$ which can be joined by composites of (differentiable) integral curves of E_i. Then we assert that $R'_0 = R'$. Indeed we easily have $R'_0 \subset R'$. Let D' and E'_i denote the restrictions of D and E_i to R' respectively, which are differential systems on R'. Clearly $D' = \sum_i E'_i$, and the full derived system of D' coincides with the restriction, say $D'^{-\infty}$, of $D^{-\infty}$ to R'. Since $D'^{-\infty} = T(R')$, we see from Lemma 2.6 that $R'_0 = R'$, proving our assertion.

In particular it follows that if $R'_0 = R$, then R is connected, and $D^{-\infty} = T(R)$. This shows that the converse of Lemma 2.6 is true, provided D together with R is real analytic. Accordingly consider a generalized pseudo-projective structure $\mathcal{X}$. Then we have seen that under the real analyticity assumption on $\mathcal{X}$ condition (GP.5) is the very condition assuring the existence of the arithmetic distance associated with $\mathcal{X}$.

§3. Generalized pseudo-projective systems and the duality in the generalized pseudo-projective structures

3.1. Product and pseudo-product manifolds. Let us consider the product $M \times N$ of two manifolds M and N, which we simply denote by R. Then R becomes naturally fibred manifolds at once over M and N, and we denote by F and E their vertical tangent bundles respectively: $F = \mu_*^{-1}(0)$ and $E = \nu_*^{-1}(0)$, μ and ν being the projections of R onto M and N respectively. Clearly the pair (E, F) satisfies the following conditions:

(1) $T(R) = E + F$ (direct sum),
(2) Both E and F are completely integrable.

This being said, a *product structure* on a manifold R is a pair formed by differential systems E and F on R satisfying conditions (1) and (2). Then a *product manifold* is a manifold R equipped with a product structure (E, F).

Now, let R' be a product manifold, and (E', F') its product structure. Let R be a submanifold of R'. Then we denote by E and F the pull-backs of E' and F' to R respectively: $E = \iota_*^{-1}(E')$ and $F = \iota_*^{-1}(F')$, ι being the inclusion map of R to R'. These are differential systems possibly with singularities on R, and let us thus consider the following regularity condition on R:

Both E and F are subbundles of $T(R)$ or equivalently both $\dim E(x)$ and $\dim F(x)$ $(x \in R)$ are constant.

If R satisfies the regularity condition, the pair (E, F) clearly satisfies the following conditions:

(i) $E \cap F = 0$,
(ii) Both E and F are completely integrable.

This being said, a *pseudo-product structure* on a manifold R is a pair formed by differential systems E and F on R satisfying conditions (i) and (ii). Then a *pseudo-product manifold* is a manifold R equipped with a pseudo-product structure (E, F) or the triplet (R, E, F). It will be denoted in principle by a capital script, e.g., $\mathcal{R}$, and then the triplet (R, E, F) will be called the *expression* of $\mathcal{R}$. The same notation and terminology will be also used for almost pseudo-product manifolds which will be defined in §2.1 of Chapter II.

Here, we remark that we already have the notion of (local) isomorphism for the pseudo-product manifolds, because they form a special class of manifolds with family of differential systems (see (A) in §1.2.). The same remark holds good for the almost pseudo-product manifolds.

We shall now explain some fundamental facts on pseudo-product manifolds.

As we have seen, a submanifold R of a product manifold R' becomes naturally a pseudo-product manifold, provided R satisfies the regularity condition. Conversely let R be a pseudo-product manifold, and (E, F) its pseudo-product structure. Assume that R is regular with respect to E and F, and set $M = R/F$, $N = R/E$, and $R' = M \times N$. Denote by μ and ν the projections of R onto M and N respectively, and define a map $\mu * \nu$ of R to R' by

$$(\mu * \nu)(x) = (\mu(x), \nu(x)), \quad (x \in R).$$

Since $E \cap F = 0$, we see that $\mu * \nu$ is an immersion. Moreover it is clear that the pseudo-product structure (E, F) of R is induced from the product structure (E', F') of R' by $\mu * \nu$, that is, E and F are the pull-backs of E' and F' by $\mu * \nu$ respectively.

In view of Lemma 1.1 we have therefore proved the following.

Proposition 3.1. *Any pseudo-product manifold R can be locally realized as a submanifold of a product manifold. More precisely, let x_0 be any point of R. Then there are a product manifold R' and an imbedding ι of a neighborhood of x_0 to R' such that the pseudo-product structure (E, F) of R, restricted to the neighborhood, is induced from the product structure (E', F') of R' by ι, and such that*

$$2 \dim R = \dim R' + \operatorname{rank} E + \operatorname{rank} F.$$

In connection with this proposition, we also add the following proposition, of which the proof is left to the readers as an exercise.

Proposition 3.2. *For i ($= 1$ or 2) let R'_i be a product manifold, and R_i its submanifold. Assume that R_i satisfies the regularity condition, and $2 \dim R_i = \dim R'_i + \operatorname{rank} E_i + \operatorname{rank} F_i$, where (E_i, F_i) denotes the pseudo-product structure of R_i. Then a diffeomorphism φ of R_1 onto R_2 is an isomorphism of R_1 onto R_2 as pseudo-product manifolds, if and only if it is extended to a local isomorphism, say $\tilde{\varphi}$, of R'_1 to R'_2 as product manifolds. Furthermore $\tilde{\varphi}$ is uniquely determined as a germ.*

Finally, let (R, E, F) be a pseudo-product manifold. Since $E \cap F = 0$, the sum $E + F$ gives a subbundle of $T(R)$, which we denote by D. Let us consider the Cauchy characteristic system $\operatorname{ch}(D)$ of D (see (E) in §1.2.). Then we have the following

Lemma 3.3. $\operatorname{ch}(D) = \operatorname{ch}(D) \cap E + \operatorname{ch}(D) \cap F$.

Proof. Consider the torsion γ_x of D at $x \in R$. Since both E and F are completely integrable, we see that $\gamma_x(X, Y) = 0$ for any vectors X and Y in $E(x)$ or in $F(x)$. The lemma follows easily from this fact. □

3.2. Generalized pseudo-projective structures and generalized pseudo-projective systems. Let $\mathcal{R}$ be a pseudo-product manifold, and (R, E, F) its expression. Then $\mathcal{R}$ is called a *generalized pseudo-projective system*, if the following conditions are satisfied:

((GP.1)) The system $D(= E + F)$ is not reduced to the zero system, and is non-degenerate,
((GP.2)) The full derived system of D coincides with $T(R)$.

Our discussions from now on will be mainly concerned with condition ((GP.1)). Incidentally we know from Lemma 3.3 that this condition is equivalent to the following condition: $D \neq 0$, and $\operatorname{ch}(D) \cap E = \operatorname{ch}(D) \cap F = 0$.

We shall now give a few definitions on a generalized pseudo-projective system $\mathcal{R}$.

Clearly the triplet (R, F, E) defines a generalized pseudo-projective system, which we denote by $\mathcal{R}^*$, and call the *dual* of $\mathcal{R}$.

We easily have $1 \leqq \operatorname{rank} E \leqq \dim R - \operatorname{rank} F - 1$. This being said, let (n, r) be a pair of integers with $1 \leqq r \leqq n - 1$. Then $\mathcal{R}$ is said to be of bidegree (n, r), if $n = \dim R - \operatorname{rank} F$, and $r = \operatorname{rank} E$.

Set $r = \operatorname{rank} E$. Then an r-dimensional submanifold of R is said to be a *solution* of $\mathcal{R}$, if it is an integral manifold of E. In this fashion $\mathcal{R}$ may be regarded as a differential equation for r-dimensional submanifolds of R.

The next proposition indicates that the correspondence $\mathcal{X} \mapsto \mathcal{R}$ is compatible with the respective local isomorphisms.

Proposition 3.4. *Let $\mathcal{X}$ and $\mathcal{X}'$ be two generalized pseudo-projective structures of bidegree (n, r) on manifolds M and M', and let $\mathcal{R}$ and $\mathcal{R}'$ be the generalized pseudo-projective systems of bidegree (n, r), respectively. Let (z_0, z_0') be a point of $R \times R'$, where R and R' are the underlying fibred manifolds of $\mathcal{X}$ and $\mathcal{X}'$ respectively. If φ is a local isomorphism of $\mathcal{X}$ to $\mathcal{X}'$ at (z_0, z_0'), then the prolongation $p(\varphi)$ of φ, restricted to R, gives a local isomorphism $\tilde{\varphi}$ of $\mathcal{R}$ to $\mathcal{R}'$ at (z_0, z_0'). Conversely if $\tilde{\varphi}$ is a local isomorphism of $\mathcal{R}$ to $\mathcal{R}'$ at (z_0, z_0'), then there is a unique local isomorphism φ of $\mathcal{X}$ to $\mathcal{X}'$ at (z_0, z_0') such that $\tilde{\varphi}$ coincides with $p(\varphi)$, restricted to R.*

Indeed, the first assertion is clear, and the second follows immediately from Lemma 1.8.

We also remark that the correspondence $\mathcal{X} \mapsto \mathcal{R}$ is compatible with the respective isomorphisms.

Conversely, let $\mathcal{R}$ be a generalized pseudo-projective system of bidegree (n, r), and (R, E, F) its expression. First of all let us consider the following condition on $\mathcal{R}$:

(C.1) R is regular with respect to F.

Assume this condition, and set $M = R/F$. Since $\operatorname{ch}(D) \cap F = 0$, we see from Lemma 1.9 that there is a unique immersion, say ι, of R to $J(M, r)$ such that $\mu = \hat{\mu} \circ \iota$, and $D = \iota_*^{-1}(C)$, where μ and $\hat{\mu}$ are the projections of R and $J(M, r)$ onto M respectively, and C is the contact system of $J(M, r)$. Recall that ι is concretely given by

$$\iota(x) = \mu_*(D(x)) = \mu_*(E(x)), \quad x \in R.$$

The immersion ι will be called associated with $\mathcal{R}$. Under condition (C.1), let us further consider the following condition on $\mathcal{R}$:

(C.2) ι is an imbedding.

Assume these two conditions. Then the image $\iota(R)$ of R by ι gives an imbedded submanifold of $J(M, r)$, and the image $\iota_*(D)$ of D by ι_* the contact system of $\iota(R)$. Moreover the image $\iota_*(E)$ of E by ι_* gives a differential system on $\iota(R)$, and the pair $(\iota(R), \iota_*(E))$ defines a generalized pseudo-projective structure of bidegree (n, r) on M, which is denoted by $\mathcal{X}$, and will be called associated with $\mathcal{R}$. Clearly $\mathcal{R}$ is associated with $\mathcal{X}$ or more precisely it is naturally isomorphic to the generalized pseudo-projective system associated with $\mathcal{X}$.

Let us now consider a generalized pseudo-projective structure $\mathcal{X}$ of bidegree (n, r) on a manifold M. Let $\mathcal{R}$ be the associated generalized pseudo-projective system of bidegree (n, r), and (R, E, F) its expression. (Hence (R, E) gives the expression of $\mathcal{X}$.) Then it is clear that R is regular with respect to F, M may be naturally identified with R/F, and the inclusion map of R to $J(M, r)$ may be naturally identified with the immersion ι associated with $\mathcal{R}$. Hence we find that $\mathcal{X}$ is just associated with $\mathcal{R}$.

Let $(\mathcal{X}, z_0)$ and $(\mathcal{X}', z_0')$ be two generalized pseudo-projective structures of bidegree (n, r) with reference points $z_0 \in R$ and $z_0' \in R'$, where R and R' are the underlying fibred manifolds of $\mathcal{X}$ and $\mathcal{X}'$ respectively. Then $(\mathcal{X}, z_0)$ and $(\mathcal{X}', z_0')$ are said to be locally isomorphic, if there is a local isomorphism of $\mathcal{X}$ to $\mathcal{X}'$ at (z_0, z_0'); An equivalence class with respect to this equivalence relation is called a *local isomorphism class* of generalized pseudo-projective structures of bidegree (n, r). Similarly we may speak of a *local isomorphism class* of generalized pseudo-projective systems of bidegree (n, r).

Now, we remark that every local isomorphism class $\mathcal{R}$ of generalized pseudo-projective systems of bidegree (n, r) has a representative $(\mathcal{R}, x_0)$ satisfying conditions (C.1) and (C.2) (cf. Lemma 1.1). By the discussions above we have therefore seen that there is a natural one-to-one correspondence between the local isomorphism classes $\mathcal{X}$ of generalized pseudo-projective structures of bidegree (n, r) and the local isomorphism classes $\mathcal{R}$ of generalized pseudo-projective systems of bidegree (n, r).

3.3. Pseudo-projective structures, and pseudo-projective systems. Let $\mathcal{R}$ be a pseudo-product manifold, and (R, E, F) its expression. Then $\mathcal{R}$ is called a *pseudo-projective system* (of order 2) of bidegree (n, r), if it is a generalized pseudo-projective system of bidegree (n, r), and $\dim R = n + r(n - r)$, $\operatorname{rank} E = r$, and $\operatorname{rank} F = r(n - r)$. We remark that $\mathcal{R}$ is a pseudo-projective system of bidegree (n, r), if and only if it satisfies the following conditions:

((P.1)) The system $D(= E + F)$ is non-degenerate,
((P.2)) $\dim R = n + r(n - r)$, $\operatorname{rank} E = r$, and $\operatorname{rank} F = r(n - r)$.

Indeed these conditions imply that the first derived system of D coincides with $T(R)$, which is a special case of Lemma 2.2 in Chapter II. This fact can be also derived from Lemmas 1.5 and 1.9.

Let $\mathcal{X}$ be a pseudo-projective structure of bidegree (n, r) on a manifold M. Then it is clear that the associated generalized pseudo-projective system $\mathcal{R}$ of bidegree (n, r) is a pseudo-projective system of bidegree (n, r). Conversely let $\mathcal{R}$ be a pseudo-projective system of bidegree (n, r). Assuming that $\mathcal{R}$ satisfies conditions (C.1) and (C.2), let us consider the associated generalized pseudo-projective structure $\mathcal{X}$ of bidegree (n, r) on the manifold $M(= R/F)$. Then it is also clear that $\mathcal{X}$ is a pseudo-projective structure of bidegree (n, r).

We have thus seen that there is a natural one-to-one correspondence between the local isomorphism classes $\mathcal{X}$ of pseudo-projective structures of bidegree (n, r) and the local isomorphism classes $\mathcal{R}$ of pseudo-projective systems of bidegree (n, r).

3.4. Non-degenerate relations in product manifolds $M \times N$. Let M and N be manifolds, and let R be a submanifold of the product manifold $M \times N$, which will be considered particularly as a relation in the product manifold. We denote by μ and ν the projections of R to M and N respectively, and set $n = \dim M$, $n' = \dim N$, $d = \dim R$, $r = d - n'$, and $s = d - n$.

Now, suppose that both μ and ν are submersions, implying that $\operatorname{Max}(n, n') \leqq d \leqq n + n'$. Furthermore suppose that $\operatorname{Max}(n, n') < d < n + n'$ or equivalently $1 \leqq r \leqq n - 1$, and $1 \leqq s \leqq n' - 1$. If $(p, \alpha) \in R$, $\nu^{-1}(\alpha)$ and $\mu(\nu^{-1}(\alpha))$ are r-dimensional submanifolds of R and M respectively, p is in $\mu(\nu^{-1}(\alpha))$, and the tangent space $T_p(\mu(\nu^{-1}(\alpha)))$ gives an r-dimensional contact element to M. This being said, we define a map ι of R to $J(M, r)$ by

$$\iota(x) = T_p(\mu(\nu^{-1}(\alpha))), \quad x = (p, \alpha) \in R.$$

Similarly we define a map κ of R to $J(N, s)$ by

$$\kappa(x) = T_\alpha(\nu(\mu^{-1}(p))), \quad x = (p, \alpha) \in R.$$

Clearly we have

$$\hat{\mu} \circ \iota = \mu, \quad \hat{\nu} \circ \kappa = \nu,$$

where $\hat{\mu}$ and $\hat{\nu}$ are the projections of $J(M, r)$ and $J(N, s)$ onto M and N respectively. Thus we have the following commutative diagram:

$$\begin{tikzcd} J(M,r) \arrow[dr, "\hat{\mu}"'] & & R \arrow[ll, "\iota"'] \arrow[rr, "\kappa"] \arrow[dl, "\mu"] \arrow[dr, "\nu"'] & & J(N,s) \arrow[dl, "\hat{\nu}"] \\ & M & & N & \end{tikzcd}$$

The maps ι and κ will be called associated with the relation R.

Let us consider the following two conditions on the relation R:

(i) $\operatorname{Max}(n, n') < d < n + n'$, and the projections μ and ν are both submersions,
(ii) Under this condition the maps ι and κ are both immersions.

We denote by F and E the kernels of the differentials μ_* and ν_* respectively: $F = \mu_*^{-1}(0)$, and $E = \nu_*^{-1}(0)$. Clearly condition (i) is equivalent to the following condition:

(i′) $\operatorname{Max}(n, n') < d < n + n'$, and $\dim E(x) = r$ and $\dim F(x) = s$ for each $x \in R$.

Assume condition (i). Then we see that both E and F become subbundles of $T(R)$, and the pair (E, F) gives a pseudo-product structure on R, which is nothing but the one naturally induced from the product structure of $M \times N$. The pseudo-product manifold (R, E, F) is denoted by $\mathcal{R}$, and will be called associated with the relation R.

We assert that condition (ii) is equivalent to the following condition:

(ii′) Under condition (i′) the system $D(= E + F)$ is non-degenerate.

Indeed let $x = (p, \alpha)$ be any point of R. Then we have

$$\iota(x) = T_p(\mu(\nu^{-1}(\alpha))) = \mu_*(T_x(\nu^{-1}(\alpha))) = \mu_*(E(x)) = \mu_*(D(x)).$$

Therefore it follows from Lemma 1.9 that $\iota_*^{-1}(0) = \operatorname{ch}(D) \cap F$. Similarly we have $\kappa_*^{-1}(0) = \operatorname{ch}(D) \cap E$. Furthermore we see from Lemma 3.3 that D is non-degenerate, if and only if $\operatorname{ch}(D) \cap E = \operatorname{ch}(D) \cap F = 0$. We have thus proved our assertion.

These having been observed, we say that R is a *non-degenerate relation* in the product manifold $M \times N$, if it satisfies conditions (i) and (ii) or equivalently conditions (i′) and (ii′).

Remark. Let (n, n', d) be a triplet of non-negative integers. Then it can be shown that the following three statements are mutually equivalent:

(1) $\operatorname{Max}(n, n') < d < n + n'$,
$\operatorname{Max}\{(d - n')/(d - n), (d - n)/(d - n')\} \leqq n + n' - d$.
(2) There is a non-degenerate relation in a product manifold $M \times N$ such that $n = \dim M$, $n' = \dim N$, and $d = \dim R$.
(3) There is a pseudo-product manifold (R, E, F) which satisfies condition ((GP.1)) in §3.2 and the conditions: $d = \dim R$, $d - n' = \operatorname{rank} E$, and $d - n = \operatorname{rank} F$.

3.5. Generalized pseudo-projective triplets. Let M and N be manifolds, and let R be an imbedded submanifold of $M \times N$. We preserve the notations as in the previous paragraph. Then the triplet (M, N, R) is called a *generalized pseudo-projective triplet*, if it satisfies the following conditions:

(a) R is a non-degenerate relation in the product manifold $M \times N$,
(b) The full derived system of D coincides with $T(R)$,
(c) The projections μ and ν are both surjective, and their fibres are all connected,
(d) The immersions ι and κ are both imbeddings.

Note that conditions (a) and (b) imply that $\mathcal{R}$ is a generalized pseudo-projective system.

A generalized pseudo-projective triplet is said to be *connected*, if R or equivalently M or equivalently N is connected. It is also said to be *compact*, if R and hence M and N are compact.

Let (M, N, R) and (M', N', R') be two generalized pseudo-projective triplets. Then an *isomorphism* of (M, N, R) to (M', N', R') is a pair (φ, ψ) of diffeomorphisms φ of M onto M' and ψ of N onto N' such that the product diffeomorphism $\varphi \times \psi$ of $M \times N$ onto $M' \times N'$ maps R onto R'. Consider the generalized pseudo-projective systems $\mathcal{R}$ and $\mathcal{R}'$ associated respectively with the relations R and R'. Then it is clear that an isomorphism (φ, ψ) of (M, N, R) to (M', N', R') naturally induces an isomorphism of $\mathcal{R}$ to $\mathcal{R}'$. Conversely it is clear that any isomorphism of $\mathcal{R}$ to $\mathcal{R}'$ can be obtained in this manner.

Let (M, N, R) be as above. We define a relation R^* in $N \times M$ by

$$R^* = \{ (\alpha, p) \in N \times M \mid (p, \alpha) \in R \}.$$

Then the triplet (N, M, R^*) clearly gives a generalized pseudo-projective triplet, which is called the *dual* of (M, N, R). Let us identify R^* with R by the bijection $(\alpha, p) \mapsto (p, \alpha)$. Then we remark that the generalized pseudo-projective system associated with the relation R^* may be identified with the dual $\mathcal{R}^*$ of $\mathcal{R}$.

In the following let (M, N, R) be a fixed generalized pseudo-projective triplet. Consider the generalized pseudo-projective system $\mathcal{R}$ of bidegree (n, r) associated with the relation R. We first remark that R becomes fibred manifolds at once over M and N, their vertical tangent bundles are F and E respectively, and $M = R/F$ and $N = R/E$. Consequently we know that the triplet (M, N, R) is completely determined by $\mathcal{R}$. We then remark that the immersions ι of R to $J(M, r)$ and κ of R to $J(N, s)$ associated with the relation R are nothing but the immersions associated respectively with $\mathcal{R}$ and its dual $\mathcal{R}^*$. Note that these are immediate consequences of conditions (a) and (c).

In view of condition (d) we therefore see that both $\mathcal{R}$ and $\mathcal{R}^*$ satisfy conditions (C.1) and (C.2) in §3.2. Hence we may consider the generalized pseudo-projective structures $(\iota(R), \iota_*(E))$ of bidegree (n, r) on M and $(\kappa(R), \kappa_*(F))$ of bidegree (n', s) on N which are associated respectively with $\mathcal{R}$ and $\mathcal{R}^*$. These structures are denoted respectively by $\mathcal{X}$ and $\mathcal{X}^*$, and will be called associated with the relation R. The structure $\mathcal{X}^*$ will be soon called the *dual* of $\mathcal{X}$, and vice versa.

For simplicity we denote by $\mathcal{P}$ the triplet (M, N, R), and by $\mathcal{P}^*$ its dual. Let us consider the automorphism groups $\mathrm{Aut}(\mathcal{X})$, $\mathrm{Aut}(\mathcal{R})$ and $\mathrm{Aut}(\mathcal{P})$ of $\mathcal{X}$, $\mathcal{R}$ and $\mathcal{P}$ respectively, and similarly the automorphism groups $\mathrm{Aut}(\mathcal{X}^*)$, $\mathrm{Aut}(\mathcal{R}^*)$ and $\mathrm{Aut}(\mathcal{P}^*)$. Then we have the natural isomorphisms as follows: $\mathrm{Aut}(\mathcal{X}) \cong \mathrm{Aut}(\mathcal{R}) \cong \mathrm{Aut}(\mathcal{P}) \cong \mathrm{Aut}(\mathcal{P}^*) \cong \mathrm{Aut}(\mathcal{R}^*) \cong \mathrm{Aut}(\mathcal{X}^*)$.

Now, let $\mathrm{Sol}(\mathcal{X})$ be the set of maximal solutions of the equation $\mathcal{X}$, and let $\mathrm{Sol}(\mathcal{X}, p)$ be the set of maximal solutions of $\mathcal{X}$ through any point $p \in M$. Similarly we consider the sets $\mathrm{Sol}(\mathcal{X}^*)$ and $\mathrm{Sol}(\mathcal{X}^*, \alpha)$, where $\alpha \in N$. Let α be any point of N. By the very definitions of the map ι it follows that the image $\iota(\nu^{-1}(\alpha))$ of $\nu^{-1}(\alpha)$ by ι coincides with the prolongation $p(\mu(\nu^{-1}(\alpha)))$ of $\mu(\nu^{-1}(\alpha))$. Therefore we see that $\mu(\nu^{-1}(\alpha))$ is a maximal solution of $\mathcal{X}$, and the assignment $\alpha \mapsto \mu(\nu^{-1}(\alpha))$ establishes a one-to-one correspondence, say Φ, between N and $\mathrm{Sol}(\mathcal{X})$. Similarly we see that $\nu(\mu^{-1}(p))$ is a maximal solution of $\mathcal{X}^*$ for any point p of M, and the assignment $p \mapsto \nu(\mu^{-1}(p))$ establishes a one-to-one correspondence, say Ψ, between M and $\mathrm{Sol}(\mathcal{X}^*)$. Clearly we have

$$\Psi(p) = \Phi^{-1}(\mathrm{Sol}(\mathcal{X}, p)), \qquad \Phi(\alpha) = \Psi^{-1}(\mathrm{Sol}(\mathcal{X}^*, \alpha)).$$

From these discussions we have seen the following:

(1) $\mathcal{X}$ may be characterized as the differential equation defining the submanifolds $\mu(\nu^{-1}(\alpha))$ of M, and similarly $\mathcal{X}^*$ as the differential equation defining the submanifolds $\nu(\mu^{-1}(p))$ of N.
(2) N may be naturally identified with $\mathrm{Sol}(\mathcal{X})$, so that the submanifolds $\mathrm{Sol}(\mathcal{X},\, p)$ of $\mathrm{Sol}(\mathcal{X})$ form the maximal solutions of $\mathcal{X}^*$. Hence $\mathcal{X}^*$ may be characterized as the differential equation defining the submanifolds $\mathrm{Sol}(\mathcal{X}, p)$ of $\mathrm{Sol}(\mathcal{X})$. Similarly $\mathcal{X}$ may be characterized as the differential equation defining the submanifolds $\mathrm{Sol}(\mathcal{X}^*, \alpha)$ of $\mathrm{Sol}(\mathcal{X}^*)$.

We shall construct local expressions of $\mathcal{X}$ and $\mathcal{X}^*$ in terms of a local expression of the relation R. In the following the indices i, j will range over the integers $1, \dots, r$, the indices δ, ε over the integers $1, \dots, s$, and the indices a, b over the integers $1, \dots, d-r-s$. Note that $d-r-s = n-r = n'-s$.

Let x_0 be any point of R. We take any coordinate system (x, y) of M at $\mu(x_0)$ adapted to the contact element $\iota(x_0)$ to M, where $x = (x^i)$, and $y = (y^a)$. Similarly we take any coordinate system (z, w) of N at $\nu(x_0)$ adapted to the contact element $\kappa(x_0)$ to N, where $z = (z^\delta)$, and $w = (w^a)$. Then (x, y, z, w) naturally gives a coordinate system of $M \times N$ at x_0. For simplicity we assume that x, y, z and w vanish at x_0.

We have $\dim(M \times N) - \dim R = d - r - s$. This being said, let us consider any regular system, $f^a = 0$, of local equations for the submanifold R of $M \times N$ at x_0 (cf. the same system in §2.4). In terms of the coordinate system (x, y, z, w) the relation R is thus defined by the system of relations:

$$f^a(x, y, z, w) = 0.$$

Since $\mu_*(E(x_0)) = \iota(x_0)$, we see that the $n + n'$ differentials df^a, dx^i, dz^δ, and dw^b are linearly independent at x_0, whence $\det((\partial f^a/\partial y^b)(0)) \neq 0$. It follows that the system of relations above can be solved for y, so that y^a are expressed as functions, say ϕ^a, of the variables x, z and w. Hence the relation R is also defined by the system of relations:

$$y^a = \phi^a(x, z, w).$$

Similarly the given system of relations can be solved for w, so that w^a are expressed as functions, say ψ^a, of x, y and z. Hence the relation R is further defined by the system of relations:

$$w^a = \psi^a(x, y, z).$$

Now, we recall that $\mathcal{X}$ may be characterized as the differential equation defining the submanifolds $\mu(\nu^{-1}(\alpha))$ of M, where $\alpha \in N$. Let U and V be sufficiently small neighborhoods of $\mu(x_0)$ and $\nu(x_0)$ respectively. Take any point $\alpha \in V$ with the coordinates (z, w). Then we see that the solution $\mu(\nu^{-1}(\alpha))$ of $\mathcal{X}$ is defined, on U, by the system of relations: $y^a = \phi^a(x, z, w)$. Moreover we have the obvious identities: $f^a(x, \phi(x, z, w), z, w) = 0$. From these facts we deduce the following

Proposition 3.5. *A local expression of $\mathcal{X}$ at $\iota(x_0)$ corresponding to the coordinate system (x, y) can be obtained by eliminating the variables z and w in the following system of relations:*

$$f^a = \frac{df^a}{dx^j} = \frac{d^2 f^a}{dx^i dx^j} = 0.$$

Here, d/dx^j are the formal derivations corresponding to the coordinate system (x, y) (see §1.5). Hence,

$$\begin{aligned}
\frac{df^a}{dx^j} &= \frac{\partial f^a}{\partial x^j} + \sum_b \frac{\partial f^a}{\partial y^b}\frac{\partial y^b}{\partial x^j},\\
\frac{d^2 f^a}{dx^i dx^j} &= \frac{d}{dx^i}\left(\frac{df^a}{dx^j}\right)\\
&= \frac{d}{dx^i}\left(\frac{\partial f^a}{\partial x^j}\right) + \sum_b \frac{d}{dx^i}\left(\frac{\partial f^a}{\partial y^b}\right)\frac{\partial y^b}{\partial x^j} + \sum_b \frac{\partial f^a}{\partial y^b}\frac{\partial^2 y^b}{\partial x^i \partial x^j}.
\end{aligned}$$

We shall now explain how to eliminate the variables z and w. In view of this explanation we can easily give a complete proof of the proposition, which is therefore left to the readers as an exercise.

First of all we remark that the variable w can be easily eliminated by using the relations $w^a = \psi^a(x, y, z)$, so that the system of relations in the proposition is reduced to the following system of relations:

$$\frac{d\psi^a}{dx^j} = \frac{d^2\psi^a}{dx^i dx^j} = 0.$$

We have

$$\frac{d\psi^a}{dx^j} = \frac{\partial \psi^a}{\partial x^j} + \sum_b \frac{\partial \psi^a}{\partial y^b}\frac{\partial y^b}{\partial x^j},$$

and $\det((\partial\psi^a/\partial y^b)(0)) \neq 0$. Hence we may define functions χ_j^b by

$$\frac{\partial \psi^a}{\partial x^j} = \sum_b \frac{\partial \psi^a}{\partial y^b}\chi_j^b.$$

Then the system of relations $d\psi^a/dx^j = 0$ can be replaced by

$$\chi_i^a(x, y, z) + \frac{\partial y^a}{\partial x^i} = 0.$$

By Lemma 3.6 below we can find s pairs $(a_\varepsilon, i_\varepsilon)$ of integers such that $1 \leqq a_\varepsilon \leqq n - r$, $1 \leqq i_\varepsilon \leqq r$ and $\det((\partial\chi_\varepsilon/\partial z^\delta)(0)) \neq 0$, where $\chi_\varepsilon = \chi_{i_\varepsilon}^{a_\varepsilon}$. Therefore the system of relations

$$\chi_\varepsilon(x, y, z) + \frac{\partial y^{a_\varepsilon}}{\partial x^{i_\varepsilon}} = 0$$

can be solved for z, so that z^δ are expressed as functions of x, y and $(\partial y^{a_\varepsilon}/\partial x^{i_\varepsilon})$. Substituting these functions for z^δ in the reduced system of relations, we have thus accomplished the elimination.

Lemma 3.6. *If $(\xi^\delta) \in \mathbb{R}^s$, and $\sum_\delta (\partial\chi_i^a/\partial z^\delta)(0) \cdot \xi^\delta = 0$, then $\xi^\delta = 0$.*

Proof. We first remark that (x, y, z), restricted to R, gives a coordinate system of R at x_0. Hence w^a, restricted to R, can be expressed as functions of x, y and z, which are nothing but the functions $\psi^a(x, y, z)$. Hereafter we work on a sufficiently small neighborhood of x_0 in R. Then we see that E is defined by the system of Pfaffian equations: $dz^\delta = dw^a = 0$, and F by the system of Pfaffian equations: $dx^i = dy^a = 0$. Since $w^a = \psi^a(x, y, z)$, we have

$$dw^a = \sum_i \frac{\partial \psi^a}{\partial x^i}dx^i + \sum_b \frac{\partial \psi^a}{\partial y^b}dy^b + \sum_\delta \frac{\partial \psi^a}{\partial z^\delta}dz^\delta.$$

If we set $\theta^a = \sum_i \chi_i^a dx^i + dy^a$, it therefore follows that $D(= E + F)$ is defined by the system of Pfaffian equations: $\theta^a = 0$. An easy calculation proves that

$$d\theta^a \equiv -\sum_\delta \frac{\partial \chi_i^a}{\partial z^\delta} dx^i \wedge dz^\delta \qquad (\mathrm{mod}\ (\theta^b)).$$

Since D is non-degenerate at x_0, the lemma follows immediately from this fact. □

Similarly to Proposition 3.5 we have the following

Proposition. *A local expression of $\mathcal{X}^*$ at $\kappa(x_0)$ corresponding to the coordinate system (z, w) can be obtained by eliminating the variables x and y in the following system of relations:*

$$f^a = \frac{df^a}{dz^\varepsilon} = \frac{d^2 f^a}{dz^\varepsilon dz^\delta} = 0.$$

Here, d/dz^ε are the formal derivations corresponding to the coordinate system (z, w). Furthermore we remark that the system of relations above can be reduced to the following system of relations

$$\frac{d\phi^a}{dz^\varepsilon} = \frac{d^2\phi^a}{dz^\varepsilon dz^\delta} = 0.$$

We shall refer to the proposition above as the dual form of Proposition 3.5.

3.6. The duality in the generalized pseudo-projective structures. Let $\mathcal{X}$ be a generalized pseudo-projective structure of bidegree (n, r) on a manifold M, and (R, E) its expression. Let $\mathcal{R}$ be the associated generalized pseudo-projective system of bidegree (n, r), which is thus expressed by (R, E, F), F being the vertical tangent bundle of R. We remark that the dual $\mathcal{R}^*$ of $\mathcal{R}$ is a generalized pseudo-projective system of bidegree (n', s), where $n' = \dim R - \operatorname{rank} E$, and $s = \operatorname{rank} F$.

Let us consider the following two conditions on the structure $\mathcal{X}$:

(D.1) The dual $\mathcal{R}^*$ of $\mathcal{R}$ satisfies conditions (C.1) and (C.2) in §3.2. Namely, R is regular with respect to E, and the immersion, say κ, of R to $J(N, s)$ associated with $\mathcal{R}^*$ is an imbedding, where $N = R/E$. (Note that N may be naturally identified with the set of maximal solutions of $\mathcal{X}$ in the wide sense.)

(D.2) Under condition (D.1) the immersion $\mu * \nu$ of R to $M \times N$ is an imbedding, where μ and ν are the projections of R onto M and N respectively.

Assume these two conditions from now on. Then we may consider the generalized pseudo-projective structure $(\kappa(R), \kappa_*(F))$ of bidegree (n', s) on N associated with the dual $\mathcal{R}^*$ of $\mathcal{R}$. We denote this structure by $\mathcal{X}^*$, and call it the *dual* of $\mathcal{X}$. Clearly $\mathcal{X}^*$ satisfies conditions (D.1) and (D.2), and the dual of $\mathcal{X}^*$ is $\mathcal{X}$ or more precisely it is naturally isomorphic to $\mathcal{X}$.

Remark . Of course $\mathcal{X}^*$ makes sense under condition (D.1). Accordingly $\mathcal{X}^*$ will be sometimes called the dual of $\mathcal{X}$ under this weaker condition.

For simplicity let us identify R with a submanifold of $M \times N$ by the imbedding $\mu * \nu$. Then it is clear that the triplet (M, N, R) gives a generalized pseudo-projective triplet, and that $\mathcal{X}$ and $\mathcal{X}^*$ are associated with the relation R. Conversely it is likewise clear that if (M, N, R) is a generalized pseudo-projective triplet, and if $\mathcal{X}$ and $\mathcal{X}^*$ are the generalized pseudo-projective structures associated with the relation R, then $\mathcal{X}^*$ is the dual of $\mathcal{X}$.

Let $\mathcal{X}$ be as in the outset, and let us consider the special case where $\mathcal{X}$ is a pseudo-projective structure of bidegree (n, r). If $\mathcal{X}$ satisfies conditions (D.1) and (D.2), the dual $\mathcal{X}^*$ of $\mathcal{X}$ is a generalized pseudo-projective structure of bidegree $((r+1)(n-r), r(n-r))$, which is not necessarily a pseudo-projective structure. Clearly $\mathcal{X}^*$ is a pseudo-projective structure, if and only if $r = n-1$. Furthermore if $\mathcal{X}$ is a pseudo-projective structure of bidegree $(n, n-1)$, so is $\mathcal{X}^*$, which corresponds to the fact that if $\mathcal{R}$ is a pseudo-projective system of bidegree $(n, n-1)$, so is $\mathcal{R}^*$. It should be noted that the duality in the pseudo-projective structures of bidegree $(n, n-1)$ directly generalizes the duality existing between the points and the hyperplanes in the n-dimensional projective geometry.

Here, we remark that we may always speak of the dual $\mathcal{X}^*$ of a local isomorphism class $\mathcal{X}$ of generalized pseudo-projective structures of bidegree (n, r), and that the dual of $\mathcal{X}^*$ is $\mathcal{X}$. Indeed, in view of Lemma 1.1 it follows that $\mathcal{X}$ has a representative $(\mathcal{X}, z_0)$ satisfying conditions (D.1) and (D.2). Let $\mathcal{X}^*$ denote the local isomorphism class of generalized pseudo-projective structures of bidegree (n', s) which is represented by $(\mathcal{X}^*, z_0^*)$, where $\mathcal{X}^*$ is the dual of $\mathcal{X}$, and $z_0^* = \kappa(z_0)$, κ being as in the definition of condition (D.1). Then it is easy to see that $\mathcal{X}^*$ does not depend on the choice of $(\mathcal{X}, z_0)$, and that the dual of $\mathcal{X}^*$ is $\mathcal{X}$, confirming our claim.

Now, let $\mathcal{X}$ be a pseudo-projective structure of bidegree $(n, 1)$ on a manifold M. We use the notations as in the outset. Assuming conditions (D.1) and (D.2), let us consider the dual $\mathcal{X}^*$ of $\mathcal{X}$, which is a generalized pseudo-projective structure of bidegree $(2(n-1), n-1)$ on N. As before we identify R with a submanifold of $M \times N$.

We shall construct a local expression of $\mathcal{X}^*$ on the basis of a local expression of $\mathcal{X}$. In the following the indices i, j, k will range over the integers $1, \ldots, n-1$.

Let z_0 be any point of R. We take any canonical coordinate system (x, y, p) of R at z_0, where $y = (y^i)$, and $p = (p^i)$, and consider the corresponding local expression, say $(\mathcal{X})$, of $\mathcal{X}$:

$$\frac{d^2 y^i}{dx^2} = f^i\left(x,\, y,\, \frac{dy}{dx}\right).$$

For simplicity we assume that x, y and p vanish at z_0.

By the existence theorem for solutions of system of ordinary differential equations, there are unique functions $\phi^i(x, z, w)$ of the variables x, $z = (z^i)$ and $w = (w^i)$, defined on a neighborhood of the origin, which satisfy the following conditions:

(i) For any fixed z and w the function $x \mapsto (\phi^i(x, z, w))$ gives a solution of $(\mathcal{X})$.

(ii) $\phi^i(0, z, w) = w^i$, and $\dfrac{\partial \phi^i}{\partial x}(0, z, w) = z^i$.

The family of functions ϕ^i will be called the *general solution* of $(\mathcal{X})$.

Now, (x, y) gives a coordinate system of M at $\mu(z_0)$. We also remark that (z, w) may be regarded as a coordinate system of N at $\nu(z_0)$ or rather of $\mathrm{Sol}(\mathcal{X})$ at $\Phi(\nu(z_0))$, so that (z, w) gives the coordinates of the maximal solution of $\mathcal{X}$, being a point of $\mathrm{Sol}(\mathcal{X})$, defined by $y^i = \phi^i(x, z, w)$. Then (x, y, z, w) gives a coordinate system of $M \times N$ at $z_0 \in R$. It is clear that with respect to this coordinate system the relation R in $M \times N$ is defined by the system of relations

$$y^i = \phi^i(x, z, w).$$

Therefore we see from the dual form of Proposition 3.5 that a local expression of $\mathcal{X}^*$ at $\kappa(z_0)$ corresponding to the coordinate system (z, w) can be obtained by eliminating the variables x and y in the following system of relations:

$$\frac{d\phi^i}{dz^j} = \frac{d^2\phi^i}{dz^j dz^k} = 0.$$

For example consider the special case where $f^i = 0$, i.e., system $(\mathcal{X})$ takes the following form:

$$\frac{d^2y^i}{dx^2} = 0.$$

Then we have $\phi^i(x,z,w) = z^i x + w^i$. Therefore the elimination can be easily done, and the eliminated system of relations turns out to be the following system, say $(\mathcal{X}^*)$, of partial differential equations

$$\begin{cases} \dfrac{\partial w^i}{\partial z^j} - \dfrac{1}{n-1}\delta^i_j \displaystyle\sum_k \dfrac{\partial w^k}{\partial z^k} = 0, \\ \dfrac{\partial^2 w^i}{\partial z^j \partial z^k} = 0. \end{cases}$$

In the special case where $n = 2$, system $(\mathcal{X}^*)$ simply means the following single ordinary differential equation:

$$\frac{d^2w}{dz^2} = 0.$$

It is also easy to see that if $n \geqq 3$, system $(\mathcal{X}^*)$ is obtained as the prolongation of the following system of the first order:

$$\frac{\partial w^i}{\partial z^j} - \frac{1}{n-1}\delta^i_j \sum_k \frac{\partial w^k}{\partial z^k} = 0,$$

being a local expression of the equation $\kappa(R)$ at $\kappa(z_0)$. In §2.2 of Chapter II we shall give a further generalization of this fact (see Corollary 2 to Proposition 2.5 in Chapter II).

3.7. Pseudo-projective triplets, and some global facts on compact, connected pseudo-projective triplets. Let (n,r) be a pair of integers with $1 \leqq r \leqq n-1$. A *pseudo-projective triplet* (of order 2) of bidegree (n,r) is a generalized pseudo-projective triplet (M,N,R) which satisfies the following condition:

(e) $\dim M = n$, $\dim N = (r+1)(n-r)$, and $\dim R = n + r(n-r)$.

Let (M,N,R) be a pseudo-projective triplet of bidegree (n,r). Let us consider the generalized pseudo-projective system $\mathcal{R}$ associated with the relation R, and (R,E,F) its expression. Then we have $\operatorname{rank} E = \dim R - \dim N = r$, and $\operatorname{rank} F = \dim R - \dim M = r(n-r)$, from which follows that $\mathcal{R}$ becomes a pseudo-projective system of bidegree (n,r). Note that this is a consequence of conditions (a) and (e).

For example consider the fundamental relation R in the product manifold $\mathbb{P}^n \times \Sigma(\mathbb{P}^n, r)$ (see §2.1). From the observations there we know that the triplet $(\mathbb{P}^n, \Sigma(\mathbb{P}^n,r),\ R)$ satisfies conditions (a), (b), (c), and (e). We also know that the immersion ι associated with the relation R gives a diffeomorphism of R onto $J(\mathbb{P}^n, r)$. Moreover it can be shown that the immersion κ of R to $J(\Sigma(\mathbb{P}^n,r), r(n-r))$ associated with the relation R is an imbedding. Hence the triplet turns out to satisfy condition (d). Since R is compact and connected, we have thus seen that the triplet $(\mathbb{P}^n, \Sigma(\mathbb{P}^n,r), R)$ becomes a compact and connected pseudo-projective triplet of bidegree (n,r), which is called the *standard pseudo-projective triplet* of bidegree (n,r). Incidentally the pseudo-projective system of bidegree (n,r) associated with the relation R is denoted by $\mathcal{R}_0$, and is called the *standard pseudo-projective system* of bidegree (n,r). Clearly the standard pseudo-projective structure $\mathcal{X}_0$ of bidegree (n,r) is associated with $\mathcal{R}_0$, and vice versa. Moreover we remark that if $r = 1$, $\mathcal{X}_0$ is nothing but the pseudo-projective structure $\mathcal{X}(\mathfrak{D}_0)$ associated with the standard projective structure $\mathfrak{D}_0$ on $\mathbb{P}^n$.

These being defined and observed, there naturally arises the following

Problem 1. Study the topological structure of a compact and connected pseudo-projective triplet of bidegree (n, r). In particular, from the view-point of topology are there any compact and connected pseudo-projective triplets of bidegree (n, r) other than the standard pseudo-projective triplet of bidegree (n, r)?

As a first step we treat with this problem in the special case where $r = n-1$. (In Remark 2 below we shall make some comments on the problem in the case where $r = 1$.)

In general let us consider a triplet (M, N, R) formed by manifolds M, N and a submanifold R of $M \times N$ which satisfy conditions (a), (c), and condition (e) with $r = n-1$, and which is compact and connected, i.e., M, N and R are all compact and connected. Thus the triplet is reduced to a compact and connected pseudo-projective triplet of bidegree $(n, n-1)$, if it satisfies condition (d) in addition. (Note that condition (b) follows from condition (a).) For the triplet we preserve the notations as in §3.4. First of all we remark the following: $\dim M = \dim N = n$, and $\dim R = 2n-1$; $\nu(\mu^{-1}(p))$ and $\mu(\nu^{-1}(\alpha))$ are $(n-1)$-dimensional submanifolds of N and M respectively, where $p \in M$, and $\alpha \in N$; ι and κ are immersions of R to $J(M, n-1)$ and $J(N, n-1)$ respectively.

Proposition 3.7. (1) *If ι is injective, $\nu(\mu^{-1}(p))$ is diffeomorphic with the $(n-1)$-dimensional projective space $\mathbb{P}^{n-1}$ for each $p \in M$.*

(2) *If ι is not injective, $\nu(\mu^{-1}(p))$ is diffeomorphic with the $(n-1)$-dimensional sphere S^{n-1} for each $p \in M$.*

We also have the dual form of this proposition, which is concerned with the immersion κ and the submanifolds $\mu(\nu^{-1}(\alpha))$.

Proof of Proposition 3.7. We first see that the pair (R, ι) gives a covering space of $J(M, n-1)$, because both R and $J(M, n-1)$ are compact and connected, and $\dim R = \dim J(M, n-1)$. Let p be any point of M. Since $\hat{\mu} \circ \iota = \mu$, it follows that ι maps $\mu^{-1}(p)$ onto the fibre $\mathrm{Gr}(T_p(M), n-1)$ of $J(M, n-1)$ at p. Furthermore $\mathrm{Gr}(T_p(M), n-1)$ is diffeomorphic with $\mathbb{P}^{n-1}$, and $\mu^{-1}(p)$ is an $(n-1)$-dimensional compact, connected submanifold of R. Hence we see that $\mu^{-1}(p)$ becomes naturally a covering space over $\mathrm{Gr}(T_p(M), n-1)$ ($\cong \mathbb{P}^{n-1}$), from which follows immediately the proposition. □

Example. Let M be an n-dimensional compact, connected Riemannian manifold, and let d be the distance function on it. For any positive number ε we define a relation R in $M \times M$ by

$$R = \{\, (p, q) \in M \times M \mid d(p, q) = \varepsilon \,\}.$$

As is well known, R becomes a submanifold for a sufficiently small ε. Then K. Sugahara showed that if ε is further sufficiently small, the triplet (M, M, R) satisfies conditions (a), (c) and condition (e) with $r = n-1$. By Proposition 3.7 we see that both ι and κ are not injective, which can be directly verified as follows: Take any points $p, q \in M$ with $d(p, q) = 2\varepsilon$, and let r be the middle point of the geodesic segment joining p and q. Then (r, p) and (r, q) are distinct points of R, and we easily see that $\iota((r, p)) = \iota((r, q))$, proving that ι is not injective.

Especially this example indicates that in our present situation conditions (a), (c) and (e) impose no restrictions on the topologies of M and N. At the same time condition (d) together with these conditions may possibly impose any restrictions on the topologies, thus playing an important role in our problem.

Let (M, N, R) be a compact and connected pseudo-projective triplet of bidegree $(n, n-1)$. In his paper [16] K. Sugahara indeed obtained some results on the homotopy groups of M and N, which may be stated as follows:

Theorem S. *If $n = 3$ or $n \geqq 6$, the universal covering spaces of M are homotopy spheres. Furthermore if $n = 3$, the fundamental groups of M and N are $\mathbb{Z}_2$ or 0, and if $n \geqq 6$, they are $\mathbb{Z}_2$.*

Sugahara's proof of the theorem, to a considerable extent, contains geometric studies on the submanifolds $\nu(\mu^{-1}(p))$ and $\mu(\nu^{-1}(\alpha))$ which are diffeomorphic with $\mathbb{P}^{n-1}$ by Proposition 3.7 and its dual form. Here we notice that he actually treats with a class of triplets (M, N, R) more or less wider than ours, and obtains much the same results as the theorem.

Remark 1. In connection with the discussions above we recall a problem proposed in [21] (see page 303 in H.M.J., Vol. XIV, No.3). Let (M, N, R) be a compact and connected pseudo-projective triplet of bidegree $(n, n-1)$. Set $\Omega = (M \times N) - R$. Then, that problem is concerned with the geometry of the product manifold Ω, which is known to be closely related to the geometry of the pseudo-projective system $\mathcal{R}$ of bidegree $(n, n-1)$ associated with the relation R (see Propositions 2.5 and 2.6, loc. cit.).

Remark 2 (cf. [2])**.** Let M be one of the following projective spaces with the standard Riemannian metrics: The m-dimensional real projective spaces $\mathbb{P}^m(\mathbb{R})$ $(m \geqq 2)$, the m-dimensional complex projective spaces $\mathbb{P}^m(\mathbb{C})$ $(m \geqq 2)$, the m-dimensional quaternion projective spaces $\mathbb{P}^m(\mathbb{Q})$ $(m \geqq 2)$, and the Cayley projective plane $\mathbb{P}^2(\mathbb{C}\mathrm{a})$. Then M is a compact symmetric space of rank one, and is what is called a C_π manifold. Now, we denote by N the set of all maximal geodesics of M, where each maximal geodesic should be confounded with its image in M. We also define a relation R in $M \times N$ by

$$R = \{ (p, \alpha) \in M \times N \mid p \subset \alpha \}.$$

If we put $n = \dim M$, it can be shown that N becomes naturally a $2(n-1)$-dimensional manifold, R becomes a $(2n-1)$-dimensional submanifold of $M \times N$, and the triplet (M, N, R) gives a compact and connected pseudo-projective triplet of bidegree $(n, 1)$. Furthermore we know that if $M \neq \mathbb{P}^n(\mathbb{R})$, M is not homotopically equivalent to $\mathbb{P}^n(\mathbb{R})$. Accordingly these observations make a considerable contribution to Problem 1 in the case where $r = 1$.

For the rest of this paragraph everything will be considered in the complex analytic category.

Corresponding to Problem 1 there naturally arises the following

Problem 2. Classify the compact and connected pseudo-projective triplets (M, N, R) of bidegree (n, r) up to (holomorphic) isomorphism. In particular are there any compact and connected pseudo-projective triplets of bidegree (n, r) other than the standard pseudo-projective triplet of bidegree (n, r)?

Let (M, N, R) be a compact and connected pseudo-projective triplet of bidegree $(n, n-1)$. On the basis of Proposition 3.7 and its dual form, both being considered in the complex analytic category, A. Fujiki proved the following theorem, which gives a partial answer to Problem 2.

Theorem F. *The complex manifolds M and N are both biholomorphic to the n-dimensional complex projective space $\mathbb{P}^n(\mathbb{C})$.*

In his proof of the theorem Fujiki above all utilizes Theorem 4.1 in J. Morrow and H. Rossi [11] concerning the characterization of $\mathbb{P}^n(\mathbb{C})$.

CHAPTER II

Geometry of almost pseudo-product manifolds — General theory —

§1. Preliminaries — Some fundamental facts on the geometry of differential systems —

1.1. General remarks on notations and terminologies.

(A) Graded Lie algebras. Let K be the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, and let $\mathbb{Z}$ be the additive group of integers. Then a *graded Lie algebra* or simply GLA over K is a pair formed by a Lie algebra $\mathfrak{g}$ over K and a family $(\mathfrak{g}_p)_{p\in\mathbb{Z}}$ of subspaces of $\mathfrak{g}$ which satisfy the following conditions:

(i) $\dim \mathfrak{g}_p < \infty$,

(ii) $\mathfrak{g} = \sum_p \mathfrak{g}_p$ (direct sum) ,

(iii) $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$.

A graded Lie algebra will be denoted by a capital Germanic letter, e.g., $\mathfrak{G}$, and then the pair $(\mathfrak{g}, (\mathfrak{g}_p))$ will be called the *expression* of $\mathfrak{G}$; The family $(\mathfrak{g}_p)$ itself will be called the *gradation* of $\mathfrak{G}$. Furthermore a graded Lie algebra will be sometimes confounded with the underlying Lie algebra $\mathfrak{g}$.

Concerning graded Lie algebras we have the various notions as follows: isomorphism, homomorphism, graded subalgebra, graded ideal, direct sum, and so on.

If $\mathfrak{G}$ is a graded Lie algebra, we remark that $\mathfrak{g}_0$ becomes a subalgebra, and the Lie algebra $\mathfrak{g}_0$ is naturally represented on each subspace $\mathfrak{g}_p$ of $\mathfrak{g}$, because $[\mathfrak{g}_0, \mathfrak{g}_p] \subset \mathfrak{g}_p$.

(B) Principal fibre bundles. At the outset we make some remarks on Lie groups. In this book Lie groups will be always assumed to satisfy the second axiom of countability. Let G be a Lie group. By the Lie algebra $\mathfrak{g}$ of G we mean the Lie algebra of all left invariant vector fields on G. As usual Ad denotes the adjoint representation of the Lie group G on the vector space $\mathfrak{g}$, while ad the adjoint representation of the Lie algebra $\mathfrak{g}$ on the vector space $\mathfrak{g}$. Furthermore exp denotes the exponential map of $\mathfrak{g}$ to G.

Now, let G be a Lie group. A (differentiable) principal fibre bundle over a manifold M with structure group G is a pair formed by a fibred manifold P over M and an action of G on P which satisfy the following conditions:

(i) G acts freely on P (on the right) by the rule $(z, a) \mapsto z \cdot a$, where $z \in P$, and $a \in G$.

(ii) Two points z and z' of the fibred manifold P are in the same fibre, if and only if $z' = z \cdot a$ for some $a \in G$.

A principal fibre bundle will be denoted simply by P.

Let P and P' be principal fibre bundles over manifolds M and M' with the same structure group G respectively. Then a (bundle) isomorphism of P to P' is a diffeomorphism ψ of P onto P' such that

$$\psi(z \cdot a) = \psi(z) \cdot a, \quad z \in P,\ a \in G.$$

An isomorphism ψ of P to P' naturally induces a diffeomorphism, say φ, of M onto M': $\pi' \circ \psi = \varphi \circ \pi$, π and π' being the projections of P and P' onto M and M' respectively. In particular ψ is called an equivalence of P to P', if ψ is base-preserving, i.e., $M = M'$, and φ is the identity map of M.

Let P be a principal fibre bundle over a manifold M with structure group G. We first remark that P is locally trivial. Indeed let x be any point of M, and g any cross section of the fibred manifold P defined on a neighborhood U of x. Let us consider the inverse image $\pi^{-1}(U)$ of U by the projection π of P onto M, and define a map ψ of $U \times G$ to $\pi^{-1}(U)$ by $\psi(x, a) = g(x) \cdot a$ for all $x \in U$ and $a \in G$. Then it can be easily shown that ψ gives an equivalence of $U \times G$ to $\pi^{-1}(U)$, where both $U \times G$ and $\pi^{-1}(U)$ should be naturally considered as principal fibre bundles.

Example. Let G be a Lie group, and H its closed subgroup. Let us consider the homogeneous space G/H of G over H, and π the projection of G onto G/H. Then the group H naturally acts on G on the right, and G becomes a principal fibre bundle over G/H with structure group H with projection π.

We then explain some notations for our later uses. For each $a \in G$ denote by R_a the diffeomorphism $z \mapsto z \cdot a$ of P, which we call the right translation of P corresponding to a. Let $\mathfrak{g}$ be the Lie algebra of G. For $A \in \mathfrak{g}$ we set $a_t = \exp(tA)$. Then we denote by A^* the vector field on P induced by the one-parameter group (R_{a_t}) of right translations of P, which we call the vertical vector field on P corresponding to A. Let V be the vertical tangent bundle of P. Then we easily see that for each $z \in P$ the map $A \mapsto A^*_z$ gives an isomorphism of $\mathfrak{g}$ onto $V(z)$ as vector spaces, and further that $(R_a)_* B^* = (\mathrm{Ad}(a^{-1})B)^*$ for all $a \in G$ and $B \in \mathfrak{g}$, whence $[A^*, B^*] = [A, B]^*$ for all $A, B \in \mathfrak{g}$.

Now, let f be a homomorphism of a Lie group G to another Lie group G'. Let P and P' be principal fibre bundles over manifolds M and M' with structure groups G and G' respectively. Then a (bundle) homomorphism of P to P' corresponding to f is a map ψ of P to P' such that

$$\psi(z \cdot a) = \psi(z) f(a), \quad z \in P,\, a \in G.$$

A homomorphism ψ of P to P' naturally induces a map of M to M'.

Let G' and P' be as above, and let G be a Lie subgroup of G'. Then a reduction of P' to the group G is a principal fibre bundle P over M' with structure group G such that P is a fibred submanifold of P', and the inclusion map of P to P' is a homomorphism of P to P' corresponding to the inclusion map of G to G'. Clearly this last condition means that the action of G on P is induced by that of G on P'.

For a detailed account of principal fibre bundles we refer to Kobayashi-Nomizu [10].

(C) The frame bundles and linear group structures. In the following we fix an n-dimensional vector space V once for all.

Let M be an n-dimensional manifold. For each point x of M we denote by $F(M, V)_x$ the set of all isomorphisms of V onto $T_x(M)$, and set $F(M, V) = \bigcup_x F(M, V)_x$. We then define the projection π of $F(M, V)$ onto M by $\pi(z) = x$, if $z \in F(M, V)_x$. Let us now consider the general linear group $GL(V)$ of V. For any $z \in F(M, V)$ and $a \in GL(V)$ we define an element za of $F(M, V)$ by

$$(za)v = z(av), \qquad v \in V.$$

Clearly the group $GL(V)$ acts freely on the set $F(M, V)$ by the rule $(z, a) \mapsto za$, and two elements z and z' of $F(M, V)$ are in the same fibre, i.e., $\pi(z) = \pi(z')$, if and only if $z' = za$ for some $a \in GL(V)$. Furthermore we know that the set $F(M, V)$ is endowed with a differentiable structure so that it becomes a differentiable principal fibre bundle over M with structure group $GL(V)$ with projection π. The principal fibre bundle $F(M, V)$, thus obtained, is called the *frame bundle* (of model vector space V) over M.

Let us now consider a diffeomorphism φ of an n-dimensional manifold M onto another n-dimensional manifold M'. Then we define a map $F(\varphi)$ of $F(M,V)$ of $F(M',V)$ by

$$F(\varphi)(z)\cdot v = \varphi_*(z\cdot v), \quad z \in F(M,V),\ v \in V.$$

Clearly $F(\varphi)$ gives a bundle isomorphism of $F(M,V)$ to $F(M',V)$, which is called the *prolongation* of φ.

Now, let G be a Lie subgroup of $GL(V)$. Then a G-structure on an n-dimensional manifold M is a reduction Q of $F(M,V)$ to the group G. Let Q and Q' be G-structures respectively on manifolds M and M'. By an *isomorphism* of Q to Q' we mean a diffeomorphism φ of M onto M' such that the prolongation $F(\varphi)$ of φ sends Q onto Q'. We notice that the restriction of $F(\varphi)$ to Q gives a diffeomorphism of Q onto Q', which is clearly a bundle isomorphism.

Let Q be a G-structure on a manifold M, and π the projection of Q onto M. Then we define a V-valued 1-form ξ on Q by

$$z\cdot\xi(X) = \pi_*(X), \quad X \in T_z(Q),\ z \in Q.$$

The 1-form ξ, thus defined, is called the *basic form* of Q.

Clearly we have the following

Lemma 1.1. (1) *The vertical tangent bundle of Q is defined by the Pfaffian equation: $\xi = 0$.*
(2) $R_a^*\xi = a^{-1}\xi, \quad a \in G.$

We also remark that the basic form ξ of Q is the pull-back of the basic form of the $GL(V)$-structure $F(M,V)$.

Finally we make some remarks on the notion of absolute parallelism. An *absolute parallelism* on an n-dimensional manifold M is usually defined to be a (global) cross section g of $F(M,V)$, which essentially means an $\{e\}$-structure Q on M, where $\{e\}$ stands for the subgroup of $GL(V)$ consisting of the identity element e only. As is easily seen, the geometry of manifolds equipped with absolute parallelism or with $\{e\}$-structures may be represented by the geometry of manifolds M equipped with V-valued 1-forms ω satisfying the following condition:

($*$) For each $x \in M$ the map $X \mapsto \omega(X)$ of $T_x(M)$ to V gives an isomorphism of $T_x(M)$ onto V.

Here, the notion of isomorphism in the latter geometry is of course defined as follows: Let ω and ω' be V-valued 1-forms respectively on manifolds M and M' satisfying condition ($*$). Then an isomorphism of (M,ω) and (M',ω') is a diffeomorphism φ of M onto M' such that $\omega' = \varphi^*\omega$. These being observed, a V-valued 1-form ω on a manifold M satisfying condition ($*$) will be also called an *absolute parallelism* on M.

(D) The prolongations of a space of linear maps. Let V and W be two vector spaces of finite dimension. As usual $\mathrm{Hom}(V,W)$ denotes the vector space of all linear maps of V to W, which may be naturally identified with the tensor product $W\otimes V^*$. Now, let $\mathfrak{g}$ be a subspace of $\mathrm{Hom}(V,W)$. Then the (first) *prolongation* $\mathfrak{g}^{(1)}$ of $\mathfrak{g}$ is the subspace of $\mathrm{Hom}(V,\mathfrak{g})$ consisting of all $A \in \mathrm{Hom}(V,\mathfrak{g})$ such that $A(v)v' = A(v')v$ for all $v, v' \in V$. $\mathfrak{g}^{(1)}$ being a subspace of $\mathrm{Hom}(V,\mathfrak{g})$, we may speak of the prolongation of $\mathfrak{g}^{(1)}$, which we denote by $\mathfrak{g}^{(2)}$, and call the second prolongation of $\mathfrak{g}$. In this way we have the k-th prolongation of $\mathfrak{g}$ for any $k \geqq 1$. We promise that the 0-th prolongation $\mathfrak{g}^{(0)}$ of $\mathfrak{g}$ is the given subspace $\mathfrak{g}$. As is well known, the k-th prolongation $\mathfrak{g}^{(k)}$ of $\mathfrak{g}$ may be simply defined by

$$\mathfrak{g}^{(k)} = \mathfrak{g}\otimes S^k(V^*) \cap W\otimes S^{k+1}(V^*),$$

where $S^\ell(V^*)$ denotes the ℓ-th symmetric product of V^*, and $\mathfrak{g} \otimes S^k(V^*)$ should be regarded as a subspace of $W \otimes V^* \otimes S^k(V^*)$.

1.2. The symbol algebras of differential systems and FGLA's. Let D be a differential system on a manifold R. Let us consider the sheaf $\mathcal{T}$ of Lie algebra of germs of local vector fields on R and the sheaf $\mathcal{D}$ of germs of local cross sections of D. Let us also consider the derived sheaves $\mathcal{D}^p$ $(p < 0)$ of $\mathcal{D}$ as well as the derived systems D^p $(p < 0)$ of D (see (F) of §1.2 in Chapter I). We recall that $\mathcal{T} \supset \cdots \supset \mathcal{D}^p \supset \cdots \supset \mathcal{D}^{-1} = \mathcal{D}$, and $T \supset \cdots \supset D^p \supset \cdots \supset D^{-1} = D$, where $T = T(R)$.

We say that the system D is *regular*, if D^p is a subbundle of T for each $p < 0$.

From now on we assume that D is not reduced to the zero system, and is regular. Clearly $\mathcal{D}^p$ coincides with the sheaf of germs of local cross sections of D^p.

Lemma 1.2. *Let x be a point of R. If $X \in \mathcal{D}^p(x)$, $Y \in \mathcal{D}^q(x)$, and $X_x \in D^{p+1}(x)$, then $[X, Y]_x \in D^{p+q+1}(x)$.*

Proof. Since $X_x \in D^{p+1}(x)$, we can find $Z \in \mathcal{D}^{p+1}(x)$ such that $X_x = Z_x$. If we put $W = X - Z$, we have $W \in \mathcal{D}^p(x)$, $W_x = 0$, and $[X, Y]_x = [Z, Y]_x + [W, Y]_x$. By Lemma 1.4 in Chapter I we have $[Z, Y] \in \mathcal{D}^{p+q+1}(x)$, and hence $[Z, Y]_x \in D^{p+q+1}(x)$. Furthermore let (X_i) be a moving frame of D^p defined on a neighborhood of x. Then W may be expressed as follows: $W = \sum_i f_i \cdot X_i$, and the coefficients f_i satisfy $f_i(x) = 0$. Therefore it follows that $[W, Y]_x = -\sum_i Y_x f_i (X_i)_x \in D^p(x) \subset D^{p+q+1}(x)$. We have thus shown that $[X, Y]_x \in D^{p+q+1}(x)$, proving the lemma. □

Now, we set:

$$\tilde{\mathfrak{g}}_p(x) = \mathcal{D}^p(x)/\mathcal{D}^{p+1}(x), \quad \tilde{\mathfrak{t}}(x) = \sum_p \tilde{\mathfrak{g}}_p(x);$$

$$\mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x), \quad \mathfrak{t}(x) = \sum_p \mathfrak{g}_p(x).$$

Here, we promise that $\mathcal{D}^p(x) = \{0\}$ and $D^p(x) = \{0\}$ for $p \geqq 0$. By Lemma 1.4 in Chapter I we see that $\tilde{\mathfrak{t}}(x)$ becomes naturally a Lie algebra, and $[\tilde{\mathfrak{g}}_p(x), \tilde{\mathfrak{g}}_q(x)] \subset \tilde{\mathfrak{g}}_{p+q}(x)$. (Note that $\tilde{\mathfrak{t}}(x)$ is not a graded Lie algebra, because $\tilde{\mathfrak{g}}_p(x)$ are of infinite dimension.) The natural maps of $\mathcal{D}^p(x)$ onto $D^p(x)$ naturally induce linear maps of $\tilde{\mathfrak{g}}_p(x)$ onto $\mathfrak{g}_p(x)$, and in turn these naturally induce a linear map, say Φ, of $\tilde{\mathfrak{t}}(x)$ onto $\mathfrak{t}(x)$. Then we see from Lemma 1.2 that there is a unique Lie algebra structure in $\mathfrak{t}(x)$ such that Φ gives a Lie algebra homomorphism of $\tilde{\mathfrak{t}}(x)$ onto $\mathfrak{t}(x)$. For any p let us denote by ϖ^p the projection of $D^p(x)$ onto $\mathfrak{g}_p(x)$. Then the bracket operation in the Lie algebra $\mathfrak{t}(x)$ is concretely given by

$$[\varpi^p(X_x), \varpi^q(Y_x)] = \varpi^{p+q}([X, Y]_x),$$

where $X \in \mathcal{D}^p(x)$, and $Y \in \mathcal{D}^q(x)$.

Clearly we have $[\mathfrak{g}_p(x), \mathfrak{g}_q(x)] \subset \mathfrak{g}_{p+q}(x)$, and hence the pair $(\mathfrak{t}(x), (\mathfrak{g}_p(x)))$ defines a graded Lie algebra, which we denote by $\mathfrak{T}(x)$. Moreover, $\mathfrak{g}_{-1}(x) = D(x) \neq \{0\}$, and the Lie algebra $\mathfrak{t}(x)$ is generated by $\mathfrak{g}_{-1}(x)$, i.e., $\mathfrak{g}_{p-1}(x) = [\mathfrak{g}_p(x), \mathfrak{g}_{-1}(x)]$ for $p < 0$, because $\tilde{\mathfrak{g}}_{p-1}(x) = [\tilde{\mathfrak{g}}_p(x), \tilde{\mathfrak{g}}_{-1}(x)]$ for $p < 0$. Incidentally we remark that $\mathfrak{g}_{-2}(x)$ is a subspace of $T(x)/D(x)$, and

$$\gamma_x(X, Y) = [X, Y], \quad X, Y \in D(x),$$

where γ_x stands for the torsion of D at x.

In view of the observation above we now give the following definition. Let $\mathfrak{T}$ be a graded Lie algebra, and $(\mathfrak{t}, (\mathfrak{g}_p))$ its expression. Then $\mathfrak{T}$ is called a *fundamental graded Lie algebra* or briefly a FGLA, if the following conditions are satisfied:

(i) $\dim \mathfrak{t} < \infty$,
(ii) $\mathfrak{g}_{-1} \neq \{0\}$, and the Lie algebra $\mathfrak{t}$ is generated by $\mathfrak{g}_{-1}$, i.e., $\mathfrak{g}_{p-1} = [\mathfrak{g}_p, \mathfrak{g}_{-1}]$ for all $p < 0$.

From this last condition it follows that $\mathfrak{g}_p = \{0\}$ for all $p \geqq 0$. Accordingly the gradation of the FGLA will be denoted by $(\mathfrak{g}_p)_{p<0}$.

Let $\mathfrak{T}$ be a FGLA. Given a positive integer μ, $\mathfrak{T}$ is said to be of the μ-th kind, if $\mathfrak{g}_{-\mu-1} = \{0\}$ and $\mathfrak{g}_{-\mu} \neq \{0\}$ or equivalently $\mathfrak{g}_p = \{0\}$ for all $p < -\mu$, and $\mathfrak{g}_p \neq \{0\}$ for all $-\mu \leqq p \leqq -1$. Furthermore $\mathfrak{T}$ is said to be *non-degenerate*, if the condition "$X \in \mathfrak{g}_{-1}$, $[X, \mathfrak{g}_{-1}] = \{0\}$" implies $X = 0$.

Let D be a differential system on a manifold R. The assumptions and notations being as above, let us consider the graded Lie algebra $\mathfrak{T}(x)$ or $(\mathfrak{t}(x), (\mathfrak{g}_p(x)))$ for each $x \in R$. Then $\mathfrak{T}(x)$ is a FGLA, which is called the *symbol algebra* of D at x.

Let $\mathfrak{T}$ be a FGLA, and $\mathfrak{t}$ its underlying Lie algebra. Then the system D is said to be of type $\mathfrak{T}$, if the following conditions are satisfied:

(a) D is not reduced to the zero system, and is regular,
(b) $\mathfrak{T}(x) \cong \mathfrak{T}$ at each $x \in R$,
(c) $\dim R = \dim \mathfrak{t}$.

1.3. Differential systems of type $\mathfrak{T}$. In this and the subsequent paragraphs $\mathfrak{T}$ will be a fixed FGLA, and $(\mathfrak{t}, (\mathfrak{g}_p)_{p<0})$ its expression.

For each $p < 0$ we set

$$\mathfrak{d}^p = \sum_{j=p}^{-1} \mathfrak{g}_j,$$

and define subgroups H and H_0 of $GL(\mathfrak{t})$ as follows:

$$H = \{\, a \in GL(\mathfrak{t}) \mid a\mathfrak{d}^p = \mathfrak{d}^p,\ p < 0 \,\},$$
$$H_0 = \{\, a \in GL(\mathfrak{t}) \mid a\mathfrak{g}_p = \mathfrak{g}_p,\ p < 0 \,\}.$$

Now, the factor space $\mathfrak{d}^p/\mathfrak{d}^{p+1}$ may be naturally identified with $\mathfrak{g}_p$, where we set $\mathfrak{d}^0 = \{0\}$. Hence every element a of H naturally induces an element, say $\tilde{a}$, of H_0: $\tilde{a}X \equiv aX \pmod{\mathfrak{d}^{p+1}}$ for all $X \in \mathfrak{g}_p$. Clearly the map $a \mapsto \tilde{a}$ gives a homomorphism, say ζ, of H onto H_0. We denote by N^0 the kernel of ζ. Then we have

$$H = H_0 \cdot N^0 \quad \text{(semi-direct)}.$$

Let G_0 be a Lie subgroup of the automorphism group $\mathrm{Aut}(\mathfrak{T})$ of $\mathfrak{T}$. Then we denote by $G_0^\sharp$ the inverse image of G_0 by ζ. Clearly we have

$$G_0^\sharp = G_0 \cdot N^0.$$

Now, let D be a differential system of type $\mathfrak{T}$ on a manifold R, and (D^p) the family of derived systems of D. Let us consider the symbol algebra $\mathfrak{T}(x)$ of D at any $x \in R$. Let us also consider the frame bundle $F(R, \mathfrak{t})$, and let π be the projection of $F(R, \mathfrak{t})$ onto R. We denote by $\tilde{F}(R, D, \mathfrak{T})$ the subset of $F(R, \mathfrak{t})$ consisting of all $z \in F(R, \mathfrak{t})$ such that $z \cdot \mathfrak{d}^p = D^p(x)$, where $x = \pi(z)$, which clearly gives a differentiable reduction of $F(R, \mathfrak{t})$ to the group H. Since $\mathfrak{g}_p = \mathfrak{d}^p/\mathfrak{d}^{p+1}$ and $\mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x)$, every $z \in \tilde{F}(R, D, \mathfrak{T})$ naturally gives rise to an isomorphism, say $\tilde{z}$, of $\mathfrak{t}$ onto $\mathfrak{t}(x)$ as vector spaces. This being said, we denote by $F(R, D, \mathfrak{T})$ the subset of $\tilde{F}(R, D, \mathfrak{T})$ consisting of all $z \in \tilde{F}(R, D, \mathfrak{T})$ such that $\tilde{z}$ give isomorphisms of $\mathfrak{T}$ to $\mathfrak{T}(x)$.

We shall show that $F(R, D, \mathfrak{T})$ gives a differentiable reduction of $\tilde{F}(R, D, \mathfrak{T})$ to the group $\mathrm{Aut}(\mathfrak{T})^\sharp$.

Let S denote the vector space of all anti-symmetric bilinear maps of $\mathfrak{t} \times \mathfrak{t}$ to $\mathfrak{t}$. The group H acts on S by the following rule:

$$(s^a)(X, Y) = \zeta(a)^{-1} s(\zeta(a)X, \zeta(a)Y),$$

where $s \in S$, $a \in H$, and $X, Y \in \mathfrak{t}$. Using the bracket operation in the Lie algebras $\mathfrak{t}(x)$, we now define a map Φ of $\tilde{F}(R, D, \mathfrak{T})$ to S by

$$\Phi(z)(X, Y) = \tilde{z}^{-1}([\tilde{z} \cdot X, \tilde{z} \cdot Y]), \quad z \in \tilde{F}(R, D, \mathfrak{T}),\ X, Y \in \mathfrak{t}.$$

Since $\widetilde{za} = \tilde{z} \cdot \zeta(a)$ for all $z \in \tilde{F}(R, D, \mathfrak{T})$ and $a \in H$, we see that Φ is H-equivariant: $\Phi(za) = \Phi(z)^a$. Now, let s_0 denote the bracket operation in the Lie algebra $\mathfrak{t}$. Clearly $\mathrm{Aut}(\mathfrak{T})^\sharp$ is the isotropy group of H at the point $s_0 \in S$. Since D is of type $\mathfrak{T}$, it follows that the image of $\tilde{F}(R, D, \mathfrak{T})$ by Φ is in the H-orbit through s_0, and further that $F(R, D, \mathfrak{T})$ is the inverse image of s_0 by Φ. By Lemma A.3 in Appendix A we have therefore shown that $F(R, D, \mathfrak{T})$ really gives a differentiable reduction of $\tilde{F}(R, D, \mathfrak{T})$ to the group $\mathrm{Aut}(\mathfrak{T})^\sharp$.

We remark that the construction above of $F(R, D, \mathfrak{T})$ uses the inverse function theorem.

1.4. $G_0^\sharp$-structures of type $\mathfrak{T}$. At the outset we prepare some notations for our later arguments.

Let P be a manifold, and let θ be a $\mathfrak{t}$-valued 1-form on it. Assume that θ is independent in the sense that $\dim P = \dim \mathfrak{t} + \dim \theta_z^{-1}(0)$ at each $z \in P$, where $\theta_z^{-1}(0) = \{\, X \in T_z(P) \mid \theta(X) = 0\,\}$. For each $j < 0$ denote by θ_j the $\mathfrak{g}_j$-component of θ with respect to the decomposition $\mathfrak{t} = \sum_{j<0} \mathfrak{g}_j$. Furthermore let α and β be differential forms on P which take values in a vector space V of finite dimension. Now, let p be any integer < 0. Define a subset $I(p)$ of $\mathbb{Z} \times \mathbb{Z}$ by

$$I(p) = \{\, (r, s) \in \mathbb{Z} \times \mathbb{Z} \mid p < r,\ s < 0, \text{ and } r + s < p\,\}.$$

Then, by the notation "$\alpha \underset{p}{\equiv} \beta$" we mean that

$$\alpha \underset{p}{\equiv} \beta \quad (\text{mod } \theta_r\ (r \leqq p);\ \theta_r \wedge \theta_s\ ((r, s) \in I(p))).$$

Namely, take a basis $(e_{j\lambda})_{1 \leqq \lambda \leqq n_j}$ of $\mathfrak{g}_j$ for each $j < 0$ and a basis $(u_\nu)_{1 \leqq \nu \leqq k}$ of V, and set $\theta_j = \sum_\lambda \theta_{j\lambda} e_{j\lambda}$, $\alpha = \sum_\nu \alpha_\nu u_\nu$, and $\beta = \sum_\nu \beta_\nu u_\nu$. (Note that the independence of θ means the 1-forms $\theta_{j\lambda}$ $(1 \leqq \lambda \leqq n_j,\ j < 0)$ are linearly independent at each point of P.) Furthermore let $\Omega(P)$ denote the sheaf of germs of local differential forms on P. Then the notation "$\alpha \underset{p}{\equiv} \beta$" concretely means that the forms $\alpha_\nu - \beta_\nu$ give cross sections of the subsheaf of ideals of $\Omega(P)$ generated by the following forms:

$$\begin{aligned} &\theta_{r\lambda} \quad (1 \leqq \lambda \leqq n_r,\ r \leqq p), \\ &\theta_{r\lambda} \wedge \theta_{s\kappa} \quad (1 \leqq \lambda \leqq n_r,\ 1 \leqq \kappa \leqq n_s,\ (r, s) \in I(p)). \end{aligned}$$

Let G_0 be a Lie subgroup of $\mathrm{Aut}(\mathfrak{T})$, and let Q be a $G_0^\sharp$-structure on a manifold R. Then Q is said to be *of type* $\mathfrak{T}$, if the basic form ξ of Q or the family $(\xi_p)_{p<0}$ satisfies the following

$$d\xi_p + \frac{1}{2} \sum_{r+s=p} [\xi_r, \xi_s] \underset{p}{\equiv} 0, \quad p \leqq -2,$$

where the notation "$\underset{p}{\equiv}$" should be considered with respect to ξ. Note that the equation $d\xi_{-1} \underset{-1}{\equiv} 0$ is automatically satisfied.

Let D be a differential system of type $\mathfrak{T}$ on a manifold R, and let us consider the $\mathrm{Aut}(\mathfrak{T})^\sharp$-structure $F(R, D, \mathfrak{T})$ on R. Let π be the projection of $F(R, D, \mathfrak{T})$ onto R, and ξ the basic form of the structure. Let us consider the family (D^p) of derived systems of D.

Proposition 1.3. (1) *The pull-back $\pi_*^{-1}(D^p)$ of D^p by π is defined by the system of Pfaffian equations: $\xi_j = 0$ $(j < p)$.*
(2) *$F(R, D, \mathfrak{T})$ is of type $\mathfrak{T}$.*

Proof. For simplicity we set $Q = F(R, D, \mathfrak{T})$.

(1) For any $z \in Q$ and $X \in T_z(Q)$ we have $z \cdot \xi(X) = \pi_*(X)$. Clearly $\xi(X) \in \mathfrak{d}^p$, if and only if $\pi_*(X) \in D^p(x)$, where $x = \pi(z)$. This clearly means that $\pi_*^{-1}(D^p)$ is defined by the system of Pfaffian equations $\xi_j = 0$ $(j < p)$.

(2) The matter being of local character, we may assume that Q is trivial. We take a cross section g of Q, and set $\alpha = g^*\xi$ and $\alpha_p = g^*\xi_p$. Note that α gives an absolute parallelism on R. Now, there is a unique map a of Q to $G_0^\sharp$ such that $z = g(\pi(z)) \cdot a(z)$ for all $z \in Q$. Moreover there are unique maps b of Q to G_0 and c of Q to N^0 such that $a(z) = c(z)b(z)$ for all $z \in Q$. By Lemma 1.1 we have $\xi = a^{-1}\pi^*\alpha$, from which follows that

$$\begin{aligned} \xi_p &\equiv b^{-1}\pi^*\alpha_p \qquad (\mathrm{mod}\ \pi^*\alpha_j\ (j < p)), \\ \pi^*\alpha_p &\equiv b\xi_p \qquad (\mathrm{mod}\ \xi_j\ (j < p)). \end{aligned}$$

Then we can easily show that

$$\Xi_p \equiv b^{-1}\pi^* A_p \qquad (\mathrm{mod}\ \pi^*\alpha_r\ (r \leqq p);\ \pi^*\alpha_r \wedge \pi^*\alpha_s\ ((r,s) \in I(p))),$$

where $\Xi_p = d\xi_p + \dfrac{1}{2}\sum_{r+s=p}[\xi_r, \xi_s]$, and $A_p = d\alpha_p + \dfrac{1}{2}\sum_{r+s=p}[\alpha_r, \alpha_s]$. Therefore (2) follows immediately from the last statement of the next lemma. $\square$

Lemma 1.4. (1) *D^p is defined by the system of Pfaffian equations $\alpha_j = 0$ $(j < p)$.*
(2) *$\alpha_p([X, Y]_x) = [\alpha_r(X_x), \alpha_s(Y_x)]$, where $X \in \mathcal{D}^r(x)$, $Y \in \mathcal{D}^s(x)$, and $p = r + s$.*
(3) *$A_p \equiv 0 \quad (\mathrm{mod}\ \alpha_r\ (r \leqq p);\ \alpha_r \wedge \alpha_s\ ((r,s) \in I(p)))$.*

Proof. (1) follows immediately from (1) of the proposition.

(2) Take any $x \in R$, and set $z = g(x)$. Then, for all $X \in T_x(R)$ we have $z \cdot \alpha(X) = X$ and hence $\tilde{z} \cdot \alpha_p(X) = \varpi^p(X)$. Now, take any $X \in \mathcal{D}^r(x)$, $Y \in \mathcal{D}^s(x)$, and set $p = r + s$. Then it follows that

$$\begin{aligned} \tilde{z} \cdot [\alpha_r(X_x), \alpha_s(Y_x)] &= [\tilde{z} \cdot \alpha_r(X_x), \tilde{z} \cdot \alpha_s(Y_x)] \\ &= [\varpi^r(X_x), \varpi^s(Y_x)] = \varpi^p([X, Y]_x) = \tilde{z} \cdot \alpha_p([X, Y]_x), \end{aligned}$$

whence $[\alpha_r(X_x), \alpha_s(Y_x)] = \alpha_p([X, Y]_x)$.

(3) Since α gives an absolute parallelism on R, we have $A_p \equiv 0 \quad (\mathrm{mod}\ \alpha_r \wedge \alpha_s\ (r, s < 0))$. Moreover, in view of (1) we obtain $A_p(X_x, Y_x) = -\alpha_p([X, Y]_x) + [\alpha_r(X_x), \alpha_s(Y_x)]$, where $X \in \mathcal{D}^r(x)$, $Y \in \mathcal{D}^s(x)$, and $p = r + s$. These being observed, (3) can be easily derived from (1) and (2). $\square$

Now, let G_0 be a Lie subgroup of $\mathrm{Aut}(\mathfrak{T})$. Then it is clear from Proposition 1.3 that any reduction Q of $F(R, D, \mathfrak{T})$ to the group $G_0^\sharp$ is of type $\mathfrak{T}$.

Conversely let Q be any $G_0^\sharp$-structure of type $\mathfrak{T}$ on a manifold R. Let π be the projection of Q onto R, and ξ the basic form of Q. Then we have the following

Proposition 1.5. (1) *For each $p < 0$ there is a unique differential system D^p on R such that $\pi_*^{-1}(D^p)$ is defined by the system of Pfaffian equations $\xi_j = 0$ $(j < p)$.*
(2) *Set $D = D^{-1}$. Then D is of type $\mathfrak{T}$, and (D^p) is the family of derived systems of D.*

(3) *Q gives a reduction of $F(R, D, \mathfrak{T})$ to the group $G_0^\sharp$. Furthermore D is a unique differential system of type $\mathfrak{T}$ on R having this property.*
The system D will be called associated with Q.

Proof. For each $p < 0$ let $\bar{D}^p$ be the differential system on Q defined by the system of Pfaffian equations $\xi_j = 0$ $(j < p)$. Then it is easy to see that $\bar{D}^p$ is invariant under the right translations of Q, and contains the vertical tangent bundle of Q. Hence there is a unique differential system D^p on R such that $\pi_*^{-1}(D^p) = \bar{D}^p$, proving (1). (2) and (3) can be easily verified by a reciprocal argument to the proof of Proposition 1.3. (The uniqueness assertion in (3) is clear from the first assertions of Propositions 1.3 and 1.5). □

1.5. Transitive graded Lie algebras and algebraic prolongations. Let $\mathfrak{G}$ be a graded Lie algebra, and $(\mathfrak{g}, (\mathfrak{g}_p))$ its expression. If we set $\mathfrak{g}_- = \sum_{p<0} \mathfrak{g}_p$, we see that the pair $(\mathfrak{g}_-, (\mathfrak{g}_p)_{p<0})$ or precisely $(\mathfrak{g}_-, (\mathfrak{g}_- \cap \mathfrak{g}_p))$ gives a (truncated) graded subalgebra of $\mathfrak{G}$, which we denote by $\mathfrak{G}_-$. Then $\mathfrak{G}$ is said to be *transitive*, if the following conditions are satisfied:

(i) $\mathfrak{G}_-$ is a FGLA,
(ii) If $p \geqq 0$, $X \in \mathfrak{g}_p$, and $[X, \mathfrak{g}_-] = \{0\}$, then $X = 0$.

If $\mathfrak{G}$ is transitive, $\mathfrak{G}_-$ will be called the *truncated* FGLA of $\mathfrak{G}$.

Now, let $\mathfrak{T}$ be a FGLA. We shall show that there is naturally associated to $\mathfrak{T}$ a graded Lie algebra $\hat{\mathfrak{G}}$ which satisfies the following conditions:

(a) $\hat{\mathfrak{G}}$ is transitive,
(b) $\hat{\mathfrak{G}}_- = \mathfrak{T}$,
(c) $\hat{\mathfrak{G}}$ is maximum among the graded Lie algebras satisfying conditions (a) and (b). Namely, let $\mathfrak{G}$ be any graded Lie algebra satisfying these conditions. Then $\mathfrak{G}$ is naturally imbedded in $\hat{\mathfrak{G}}$.

Let $(\mathfrak{t}, (\mathfrak{g}_p)_{p<0})$ be the expression of $\mathfrak{T}$. For all $p < 0$ we set $\hat{\mathfrak{g}}_p = \mathfrak{g}_p$. For all $p \geqq 0$ let us now define vector space $\hat{\mathfrak{g}}_p$ inductively as follows: First of all we defined $\hat{\mathfrak{g}}_0$ to be the derivation algebra $\mathrm{Der}(\mathfrak{T})$ of $\mathfrak{T}$. k being a positive integer, suppose that we have defined $\hat{\mathfrak{g}}_p$ for all $0 \leqq p < k$ so that $\hat{\mathfrak{g}}_p$ are subspaces of $\mathfrak{q}_p = \sum_{r<0} \mathrm{Hom}(\mathfrak{g}_r, \hat{\mathfrak{g}}_{r+p}) \subset \mathrm{Hom}(\mathfrak{t}, \sum_{r<0} \hat{\mathfrak{g}}_{r+p})$. Then we define $\hat{\mathfrak{g}}_k$ to be the subspace of $\mathfrak{q}_k = \sum_{r<0} \mathrm{Hom}(\mathfrak{g}_r, \hat{\mathfrak{g}}_{r+k})$ which consists of all $X_k \in \mathfrak{q}_k$ satisfying the following equalities:

$$X_k(Y_r)(Z_s) - X_k(Z_s)(Y_r) = X_k([Y_r, Z_s]),$$

where $Y_r \in \mathfrak{g}_r$, $Z_s \in \mathfrak{g}_s$, $r, s < 0$, and we promise that

$$\begin{aligned} X_k(Y_r)(Z_s) &= [X_k(Y_r), Z_s], \quad \text{if } r + k < 0; \\ X_k(Z_s)(Y_r) &= [X_k(Z_s), Y_r], \quad \text{if } s + k < 0. \end{aligned}$$

Thus we have completed our inductive definition. Set $\hat{\mathfrak{g}} = \sum_{p \in \mathbb{Z}} \hat{\mathfrak{g}}_p$. Then we easily see that there is a unique bracket operation $[\ ,\ ]$ in $\hat{\mathfrak{g}}$ having the following properties: (i) The pair $(\hat{\mathfrak{g}}, (\hat{\mathfrak{g}}_p))$ defines a graded Lie algebra, say $\hat{\mathfrak{G}}$, with respect to this bracket operation, (ii) $\hat{\mathfrak{G}}$ satisfies condition (b), and (iii) $[X_k, Y_r] = X_k(Y_r)$ for all $X_k \in \hat{\mathfrak{g}}_k$, $Y_r \in \mathfrak{g}_r$ and all $k \geqq 0$, $r < 0$. Moreover it is clear that $\hat{\mathfrak{G}}$ satisfies conditions (a) and (c).

The graded Lie algebra $\hat{\mathfrak{G}}$, thus defined, is called the *prolongation* of $\mathfrak{T}$. We remark that there is a unique element E in the center of $\hat{\mathfrak{g}}_0$ such that $[E, X] = pX$ for all $X \in \hat{\mathfrak{g}}_p$ and all p.

Now, let $\mathfrak{g}_0$ be a subalgebra of $\hat{\mathfrak{g}}_0$. For all $p \geqq 1$ we define subspaces $\mathfrak{g}_p$ of $\hat{\mathfrak{g}}_p$ inductively as follows: k being a positive integer, suppose that we have defined $\mathfrak{g}_p$ for all $0 < p < k$. Then $\mathfrak{g}_k$ is defined to be the subspace of $\hat{\mathfrak{g}}_k$ which consists of all $X_k \in \hat{\mathfrak{g}}_k$ such that $[X_k, \mathfrak{g}_r] \subset \mathfrak{g}_{k+r}$ for all $r < 0$. If we put $\mathfrak{g} = \sum_{p \in \mathbb{Z}} \mathfrak{g}_p$, we call easily verify that the pair $(\mathfrak{g}, (\mathfrak{g}_p))$ defines a graded subalgebra, say $\mathfrak{G}$, of $\hat{\mathfrak{G}}$.

The graded Lie algebra $\mathfrak{G}$, thus defined, is called the *prolongation* of the pair $(\mathfrak{T}, \mathfrak{g}_0)$.

We shall now recall a proposition concerning the finite dimensionality of a transitive graded Lie algebra.

Let $\mathfrak{G}$ be a transitive graded Lie algebra, and $(\mathfrak{g}, (\mathfrak{g}_p))$ its expression. At the outset we remark that the condition "$p \geqq 0$, $X \in \mathfrak{g}_p$, and $[X, \mathfrak{g}_{-1}] = \{0\}$" implies $X = 0$. In particular we know that the natural representation of the Lie algebra $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is faithful, and hence $\mathfrak{g}_0$ may be naturally identified with a subalgebra of $\mathfrak{gl}(\mathfrak{g}_{-1})$, the Lie algebra of all endomorphisms of $\mathfrak{g}_{-1}$.

For each $p \geqq 0$ we now define a subspace $\mathfrak{h}_p(\mathfrak{G})$ or $\mathfrak{h}_p$ of $\mathfrak{g}_p$ by

$$\mathfrak{h}_p = \{\, X \in \mathfrak{g}_p \mid [X, \mathfrak{g}_j] = \{0\} \quad \text{for all } j \leqq -2 \,\}.$$

Clearly $\mathfrak{h}_0$ is an ideal of $\mathfrak{g}_0$, and hence it may be identified with a subalgebra of $\mathfrak{gl}(\mathfrak{g}_{-1})$. Moreover it is clear that the subspaces $\mathfrak{h}_p$ satisfy the following conditions: (i) $[\mathfrak{h}_p, \mathfrak{g}_{-1}] \subset \mathfrak{h}_{p-1}$, where we promise that $\mathfrak{h}_{-1} = \mathfrak{g}_{-1}$, (ii) $[[X, Y], Z] = [[X, Z], Y]$ for all $X \in \mathfrak{h}_p$ and $Y, Z \in \mathfrak{g}_{-1}$, and (iii) the condition "$p \geqq 0$, $X \in \mathfrak{h}_p$, and $[X, \mathfrak{g}_{-1}] = \{0\}$" implies $X = 0$. Hence it follows that for each $p > 0$, $\mathfrak{h}_p$ may be naturally identified with a subspace of the p-th prolongation $\mathfrak{h}_0^{(p)}$ of the subalgebra $\mathfrak{h}_0$ of $\mathfrak{gl}(\mathfrak{g}_{-1})$ and also with a subspace of the first prolongation of $\mathfrak{h}_{p-1}$. These being observed, we state the following

Proposition 1.6 (Corollary 1 to Theorem 11.1 in [18]). *The Lie algebra $\mathfrak{g}$ is of finite dimension, if and only if $\mathfrak{h}_k$ vanishes for some $k \geqq 0$, and hence $\mathfrak{h}_p$ vanish for all $p \geqq k$.*

We now make some comments in connection with this proposition. Assume that $\mathfrak{G}$ is the prolongation of the pair $(\mathfrak{G}_-, \mathfrak{g}_0)$, $\mathfrak{G}_-$ being the truncated FGLA of $\mathfrak{G}$. From the very definition of the prolongation it follows that $\mathfrak{h}_p$ just coincides with the p-th prolongation $\mathfrak{h}_0^{(p)}$ of $\mathfrak{h}_0$ for each $p \geqq 0$.

By the proposition we therefore see that $\mathfrak{g}$ is of finite dimension, if and only if the subalgebra $\mathfrak{h}_0$ of $\mathfrak{gl}(\mathfrak{g}_{-1})$ is of finite type, i.e., $\mathfrak{h}_0^{(k)}$ vanishes for some $k \geqq 0$ (Corollary 2 to Theorem 11.1 loc. cit.). Furthermore assume that $\mathfrak{g}_0 = \mathrm{Der}(\mathfrak{G}_-)$ or equivalently $\mathfrak{G}$ is the prolongation of $\mathfrak{G}_-$. Then it can be shown that if $\mathfrak{G}_-$ is degenerate or not non-degenerate, then $\mathfrak{g}$ is of infinite dimension. (cf. the proof of (2) of Proposition 2.8).

Remark . Let $\mathfrak{T}$ be a FGLA, and $(\mathfrak{t}, (\mathfrak{g}_p)_{p<0})$ its expression. Let E be a graded (left) $\mathfrak{T}$-module. Namely E is a pair formed by a $\mathfrak{t}$-module E and a family $(E_p)_{p \in \mathbb{Z}}$ of subspaces of E which satisfy the following conditions: (i) $\dim E_p < \infty$, (ii) $E = \sum_p E_p$ (direct sum), and (iii) $\mathfrak{g}_r \cdot E_p \subset E_{r+p}$. We define a subspace H of E by

$$H = \{\, a \in E \mid X \cdot a = 0 \quad \text{for all } X \in \mathfrak{g}_p \text{ and all } p \leqq -2 \,\}.$$

Clearly we have $H = \sum_p H_p$, where $H_p = H \cap E_p$. Then Theorem 11.1 cited above may be restated as follows:

Theorem. *Assume that E_p vanishes for any sufficiently small p. Then E is of finite dimension, if and only if H is of finite dimension.*

We observe that Proposition 1.6 really follows from this theorem, because $\mathfrak{G}$ becomes naturally a graded $\mathfrak{G}_-$-module.

Our original proof of Theorem 11.1 made use of the characterization due to Serre of an involutive module in terms of the Spencer cohomology groups (cf. Guillemin and Sternberg [8]). In his unpublished paper [13] Naruki gave an elementary proof of the theorem just stated which is based on the structure theorem for modules over PID only.

1.6. A general prolongation theorem for $G_0^\sharp$-structures of type $\mathfrak{T}$. Let $\mathfrak{T}$ be a FGLA. Let G_0 be a Lie subgroup of $\mathrm{Aut}(\mathfrak{T})$, and $\mathfrak{g}_0$ its Lie algebra, being a subalgebra of $\mathrm{Der}(\mathfrak{T})$. Let $\mathfrak{G}$ be the prolongation of the pair $(\mathfrak{T}, \mathfrak{g}_0)$, and $\mathfrak{g}$ its underlying Lie algebra.

Theorem 1.7. *Assume that G_0 is connected, and that $\mathfrak{g}$ is of finite dimension.*

(1) *Let Q be any $G_0^\sharp$-structure of type $\mathfrak{T}$ on a manifold R. Then there is canonically associated to Q a pair (P, ω) as follows: P is a fibred manifold over R, and ω is a $\mathfrak{g}$-valued 1-form on P and gives an absolute parallelism on it. The pair (P, ω) is called the prolongation of Q.*

(2) *Let Q and Q' be $G_0^\sharp$-structures of type $\mathfrak{T}$ on manifolds R and R', and let (P, ω) and (P', ω') be their prolongations, respectively. If φ is an isomorphism of Q to Q', then there is a unique isomorphism $\tilde{\varphi}$ of (P, ω) to (P', ω') as manifolds with absolute parallelism such that $\pi' \circ \tilde{\varphi} = \varphi \circ \pi$, where π and π' are the projections of P and P' onto R and R' respectively. Conversely if ψ is an isomorphism of (P, ω) to (P', ω') as manifolds with absolute parallelism, there is a unique isomorphism φ of Q to Q' such that $\psi = \tilde{\varphi}$.*

This theorem follows immediately from Theorem 8.3 in [18] (see also the proof of Theorem 8.4 there).

The notations being as in the outset, assume that $\mathfrak{g}$ is of finite dimension. (We do not assume that G_0 is connected!) Let Q be a $G_0^\sharp$-structure of type $\mathfrak{T}$ on a manifold R, and let us consider the automorphism group $\mathrm{Aut}(Q)$ of Q as well as the sheaf $\mathfrak{a}(Q)$ of germs of local infinitesimal automorphisms of Q. Then, in view of Theorem 1.7 combined with Palais [14] we have the following

Corollary. (1) *Each stalk $\mathfrak{a}(Q)_x$ of $\mathfrak{a}(Q)$ is of finite dimension, and* $\dim \mathfrak{a}(Q)_x \leqq \dim \mathfrak{g}$.

(2) $\mathrm{Aut}(Q)$ *becomes naturally a Lie group, and* $\dim \mathrm{Aut}(Q) \leqq \dim \mathfrak{g}$.

§2. Equivalence problem for almost pseudo-product manifolds

2.1. Almost pseudo-product manifolds and pseudo-product FGLA's. An *almost pseudo-product structure* on a manifold R is a pair formed by differential systems E and F on R which satisfy the following conditions:

(1) $E \cap F = 0$, and hence the sum $E + F$ gives a differential system, say D, on R,

(2) For any local cross sections X and Y of E or of F the bracket $[X, Y]$ gives a local cross section of D.

Then an *almost pseudo-product manifold* is a manifold R equipped with an almost pseudo-product structure (E, F) or the triplet (R, E, F).

Let $\mathcal{R}$ be an almost pseudo-product manifold, and (R, E, F) its expression. Then $\mathcal{R}$ is said to be *integrable*, if both E and F are completely integrable. Clearly an integrable almost pseudo-product manifold means a pseudo-product manifold.

Let us consider the torsion γ_x of D at any $x \in R$. Then we can easily prove the following lemma (cf. the proof of Lemma 3.3 in Chapter I.)

Lemma 2.1. *$\gamma_x(X, Y) = 0$ for any vectors X and Y in $E(x)$ or in $F(x)$. In other words both $E(x)$ and $F(x)$ are integral elements of D at x.*

Assuming that $D \neq 0$, and D is regular, let us now consider the symbol algebra $\mathfrak{T}(x)$ of D at x, and let $(\mathfrak{t}(x), (\mathfrak{g}_p(x))_{p<0})$ be its expression. Clearly we have

$$\mathfrak{g}_{-1}(x) = E(x) + F(x) \quad \text{(direct sum)},$$

and Lemma 2.1 may be restated as follows:

$$[E(x), E(x)] = [F(x), F(x)] = \{0\}.$$

This being said, let $\mathfrak{T}$ be a FGLA, and $(\mathfrak{t}, (\mathfrak{g}_p)_{p<0})$ its expression. Let $\mathfrak{e}$ and $\mathfrak{f}$ be subspaces of $\mathfrak{g}_{-1}$. Then the triplet $(\mathfrak{T}, \mathfrak{e}, \mathfrak{f})$ is called a *pseudo-product* FGLA, if it satisfies the following conditions:

(i) $\mathfrak{g}_{-1} = \mathfrak{e} + \mathfrak{f}$ (direct sum),
(ii) $[\mathfrak{e}, \mathfrak{e}] = [\mathfrak{f}, \mathfrak{f}] = \{0\}$.

A pseudo-product FGLA will be denoted in principle by a capital Germanic letter, e.g., $\mathfrak{L}$, and then the triplet will be called the *expression* of $\mathfrak{L}$.

Let $\mathfrak{L}$ and $\mathfrak{L}'$ be two pseudo-product FGLA's, and $(\mathfrak{T}, \mathfrak{e}, \mathfrak{f})$ and $(\mathfrak{T}', \mathfrak{e}', \mathfrak{f}')$ their expressions. By an *isomorphism* of $\mathfrak{L}$ to $\mathfrak{L}'$ we mean an isomorphism of $\mathfrak{T}$ to $\mathfrak{T}'$ which sends $\mathfrak{e}$ onto $\mathfrak{e}'$ and $\mathfrak{f}$ onto $\mathfrak{f}'$.

Let $\mathfrak{L}$ be as above. Then $\mathfrak{L}$ is said to be of the *μ-th kind*, if $\mathfrak{T}$ is of the μ-th kind. It is also said to be *non-degenerate*, if $\mathfrak{T}$ is non-degenerate.

Now, consider an almost pseudo-product manifold $\mathcal{R}$, and assume that $D \neq 0$, and D is regular. Let x be any point of R. Then we see from the remark above that the triplet $(\mathfrak{T}(x), E(x), F(x))$ defines a pseudo-product FGLA, which we denote by $\mathfrak{L}(x)$, and call the *symbol algebra* of $\mathcal{R}$ at x.

Finally let $\mathfrak{L}$ be a pseudo-product FGLA. Then $\mathcal{R}$ is said to be *of type* $\mathfrak{L}$, if the following conditions are satisfied:

(a) $\mathfrak{L}(x) \cong \mathfrak{L}$ at each $x \in R$,
(b) $\dim R = \dim \mathfrak{t}$, $\mathfrak{t}$ being the underlying Lie algebra of $\mathfrak{L}$.

We remark that if $\mathcal{R}$ is a pseudo-product manifold of type $\mathfrak{L}$, and $\mathfrak{L}$ is non-degenerate, then $\mathcal{R}$ is a generalized pseudo-projective system.

2.2. Almost pseudo-projective systems, and pseudo-projective FGLA's. Let (n, r) be a pair of integers with $1 \leqq r \leqq n-1$. Let $\mathcal{R}$ be an almost pseudo-product manifold, and (R, E, F) its expression. Then $\mathcal{R}$ is called an *almost pseudo-projective system* (of order 2) of bidegree (n, r), if the following conditions are satisfied:

(1) $D(= E + F)$ is non-degenerate,
(2) $\dim R = n + r(n-r)$, $\operatorname{rank} E = r$, and $\operatorname{rank} F = r(n-r)$.

Clearly an integrable almost pseudo-projective system means a pseudo-projective system.

Lemma 2.2. *If $\mathcal{R}$ is an almost pseudo-projective system of bidegree (n, r), then the first derived system of D coincides with $T(R)$, and hence D is regular.*

Proof. Let x be any point of R, and let us consider the torsion γ_x of D at x. Then we have the following: (i) $D(x) = E(x) + F(x)$ (direct sum), (ii) $\gamma_x(E(x), E(x)) = \gamma_x(F(x), F(x)) = \{0\}$ (Lemma 2.1), (iii) γ_x is non-degenerate, and (iv) $\dim E(x) = r$, $\dim F(x) = r(n-r)$, and

$\dim(T(x)/D(x)) = n - r$, where $T = T(R)$. We define a linear map $Y \mapsto \tilde{Y}$ of $F(x)$ to $\mathrm{Hom}(E(x), T(x)/D(x))$ by

$$\tilde{Y}(X) = \gamma_x(Y, X), \quad X \in E(x).$$

Then we easily see from the remark above that the linear map $Y \mapsto \tilde{Y}$ maps $F(x)$ isomorphically onto $\mathrm{Hom}(E(x), T(x)/D(x))$. Hence we obtain

$$T(x)/D(x) = \gamma_x(E(x), F(x)) = \gamma_x(D(x), D(x)),$$

which clearly implies the lemma. □

Let $\mathfrak{L}$ be a pseudo-product FGLA, and $(\mathfrak{T}, \mathfrak{e}, \mathfrak{f})$ its expression. Let $(\mathfrak{t}, (\mathfrak{g}_p)_{p<0})$ be the expression of $\mathfrak{T}$. Then $\mathfrak{L}$ is called a *pseudo-projective* FGLA of bidegree (n, r), if the following conditions are satisfied:

(i) $\mathfrak{L}$ is of the second kind, and hence $\mathfrak{t} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}$,
(ii) $\mathfrak{L}$ is non-degenerate, which means that the condition "$X \in \mathfrak{e}$, $[X, \mathfrak{f}] = \{0\}$" or "$X \in \mathfrak{f}$, $[X, \mathfrak{e}] = \{0\}$" implies $X = 0$,
(iii) $\dim \mathfrak{g}_{-2} = n - r$, $\dim \mathfrak{e} = r$, and $\dim \mathfrak{f} = r(n - r)$.

If $\mathcal{R}$ is an almost pseudo-projective system of bidegree (n, r), we see from Lemma 2.2 together with its proof that the symbol algebra $\mathfrak{L}(x)$ of $\mathcal{R}$ at any $x \in R$ is a pseudo-projective FGLA of bidegree (n, r).

Proposition 2.3. *There is a unique pseudo-projective FGLA of bidegree (n, r) except for isomorphisms.*

Proof. Let $\mathfrak{L}$ be a pseudo-projective FGLA of bidegree (n, r). We define a linear map $Y \mapsto \tilde{Y}$ of $\mathfrak{f}$ to $\mathrm{Hom}(\mathfrak{e}, \mathfrak{g}_{-2})$ by

$$\tilde{Y}(X) = [Y, X], \quad X \in \mathfrak{e}.$$

Then we see that the linear map $Y \mapsto \tilde{Y}$ maps $\mathfrak{f}$ isomorphically onto $\mathrm{Hom}(\mathfrak{e}, \mathfrak{g}_{-2})$ (cf. the proof of Lemma 2.2). In view of this fact we can easily obtain the proposition. □

By Proposition 2.3 we have the following

Proposition 2.4. *Let $\mathfrak{L}$ be a pseudo-projective FGLA of bidegree (n, r), and let $\mathcal{R}$ be an almost pseudo-product manifold. Then $\mathcal{R}$ is an almost pseudo-projective system of bidegree (n, r), if and only if it is of type $\mathfrak{L}$.*

As an application of the symbol algebra of an almost pseudo-projective system we shall prove the following

Proposition 2.5. *Let $\mathcal{R}$ be an almost pseudo-projective system of bidegree (n, r), and (R, E, F) its expression. Let x be any point of R. If $n - r \geqq 2$, then $F(x)$ is a unique $r(n - r)$-dimensional integral element of the system $D (= E + F)$ at x.*

Before proceeding to the proof we shall explain some consequences of the proposition.

First of all we know that if $n - r \geqq 2$, the system F is naturally associated with the system D or it is a covariant system of D in Cartan's terminology. More in detail it can be shown that F can be naturally constructed from D by rational operations and differentiations. (The exact meanings of these operations will be clarified in Part II.) Clearly this fact implies the following

Corollary 1 (cf. Bäcklund [1] and Yamaguchi [23]). *Consider the Grassmann bundle $J(M,r)$ over an n-dimensional manifold M. If $n-r \geqq 2$, then the vertical tangent bundle V of $J(M,r)$ is naturally associated with the contact system C of $J(M,r)$.*

Next, let $\mathcal{X}$ be a pseudo-projective structure of bidegree (n,r) on a manifold M. Let $\mathcal{R}$ be the associated pseudo-projective system of bidegree (n,r), and (R,E,F) its expression. Assuming condition (D.1) in §3.6 of Chapter I, let us consider the dual $\mathcal{X}^*$ of $\mathcal{X}$. We recall that $\mathcal{X}^*$ is a generalized pseudo-projective structure of bidegree $((r+1)(n-r), r(n-r))$, on the manifold $N(=R/E)$, and is expressed by $(\kappa_*(E), \kappa_*(F))$, where κ is the immersion of R to $J(N, r(n-r))$ associated with the dual $\mathcal{R}^*$ of $\mathcal{R}$. Now, the proposition implies the following

Corollary 2. *If $n-r \geqq 2$, then the system $\mathcal{X}^*$ of differential equations of the second order is the prolongation of the system $\kappa(R)$ of differential equations of the first order.*

For the definition of the prolongation of a system of differential equations see [7] and [22].

Proof of Proposition 2.5. Let x be any point of R, and let us consider the symbol algebra $\mathfrak{L}(x)$ or $(\mathfrak{T}(x), E(x), F(x))$ of $\mathcal{R}$ at x. For simplicity set $\mathfrak{g}_p = \mathfrak{g}_p(x)$, $\mathfrak{e} = E(x)$, and $\mathfrak{f} = F(x)$. Now, assume that $n-r \geqq 2$, and let z be any $r(n-r)$-dimensional integral element of D at x, meaning that z is an $r(n-r)$-dimensional subspace of $\mathfrak{g}_{-1}$ such that $[z,z] = \{0\}$. We take a complementary subspace $\mathfrak{e}'$ of $\mathfrak{e} \cap z$ in $\mathfrak{e}$, and define a linear map $X \mapsto \hat{X}$ of z to $\mathrm{Hom}(\mathfrak{e}', \mathfrak{g}_{-2})$ by

$$\hat{X}(Y) = [X,Y], \quad Y \in \mathfrak{e}'.$$

Then we easily see that the kernel of the linear map $X \mapsto \hat{X}$ is $\mathfrak{e} \cap z$. Hence we obtain

$$\dim z \leqq \dim(\mathfrak{e} \cap z) + \dim \mathfrak{e}' \cdot \dim \mathfrak{g}_{-2},$$

whence $(n-r-1)\dim(\mathfrak{e} \cap z) \leqq 0$. Since $n-r \geqq 2$, it follows that $\mathfrak{e} \cap z = \{0\}$, and hence $\mathfrak{g}_{-1} = \mathfrak{e} + z$ (direct sum). Therefore we can find a unique linear map ϕ of $\mathfrak{f}$ to $\mathfrak{e}$ such that

$$z = \{\, \phi(X) + X \mid X \in \mathfrak{f}\,\}.$$

Now, we take bases (e_i) of $\mathfrak{e}$ and (d_a) of $\mathfrak{g}_{-2}$. Here and in the following the indices i,j,k range over the integers $1,\dots,r$, and the indices a,b,c over the integers $1,\dots,n-r$. Then we see from the proof of Proposition 2.3 that there is a unique basis (e^i_a) of $\mathfrak{f}$ such that $[e^i_a, e_j] = \delta^i_j d_a$. Hence ϕ may be expressed as follows:

$$\phi(e^j_b) = \sum_k \phi^{jk}_b e_k.$$

Since $[z,z] = \{0\}$, we have $[e^i_a, \phi(e^j_b)] + [\phi(e^i_a), e^j_b] = 0$, whence $\phi^{ji}_b d_a - \phi^{ij}_a d_b = 0$. Since $n-r \geqq 2$, we may take different indices a and b, and therefore we obtain $\phi^{ij}_a = 0$, i.e., $\phi = 0$.[1] We have thus shown that $z = \mathfrak{f} = F(x)$, proving the proposition. □

2.3. Almost pseudo-product manifolds of type $\mathfrak{L}$ and $\mathrm{Aut}(\mathfrak{L})^\sharp$-structures of type $\mathfrak{T}$. In this paragraph $\mathfrak{L}$ will be a fixed pseudo-product FGLA, $(\mathfrak{T}, \mathfrak{e}, \mathfrak{f})$ its expression, and $(\mathfrak{t}, (\mathfrak{g}_p)_{p<0})$ be the expression of $\mathfrak{T}$.

Let $\mathcal{R}$ be an almost pseudo-product manifold of type $\mathfrak{L}$, and (R,E,F) its expression. Clearly the system $D(=E+F)$ is of type $\mathfrak{T}$, and we apply the argument in §1.3 to this system. Let us consider the symbol algebra $\mathfrak{L}(x)$ or $(\mathfrak{T}(x), E(x), F(x))$ of $\mathcal{R}$ at any $x \in R$. Then we denote by $F(\mathcal{R}, \mathfrak{L})$ the subset of $F(R, D, \mathfrak{T})$ consisting of all $z \in F(R, D, \mathfrak{T})$ such that $\tilde{z}$ give isomorphisms of $\mathfrak{L}$ to $\mathfrak{L}(x)$, where $x = \pi(z)$.

[1] Editor's note: We here slightly modify the proof.

We shall show that $F(\mathcal{R}, \mathfrak{L})$ gives a differentiable reduction of $F(R, D, \mathfrak{T})$ to the group $\mathrm{Aut}(\mathfrak{L})^\sharp$, where $\mathrm{Aut}(\mathfrak{L})$ is the automorphism group of $\mathfrak{L}$.

Let S newly denote the set of all the pairs $\mathfrak{e}'$ and $\mathfrak{f}'$ of $\mathfrak{g}_{-1}$ such that $\mathfrak{g}_{-1} = \mathfrak{e}' + \mathfrak{f}'$ (direct sum), which becomes naturally a real analytic manifold. The group $\mathrm{Aut}(\mathfrak{T})^\sharp$ acts on S by the following rule:

$$(\mathfrak{e}', \mathfrak{f}')^a = (\zeta(a)^{-1}\mathfrak{e}', \zeta(a)^{-1}\mathfrak{f}'), \quad (\mathfrak{e}', \mathfrak{f}') \in S, \ a \in \mathrm{Aut}(\mathfrak{T})^\sharp.$$

Then we define a map Φ of $F(R, D, \mathfrak{T})$ to S by

$$\Phi(z) = (\tilde{z}^{-1} \cdot E(x), \tilde{z}^{-1} \cdot F(x)), \quad z \in F(R, D, \mathfrak{T}),$$

where $x = \pi(z)$. Now, the pair of the subspaces $\mathfrak{e}$ and $\mathfrak{f}$ of $\mathfrak{g}_{-1}$ gives a point of S. Clearly $\mathrm{Aut}(\mathfrak{L})^\sharp$ is the isotropy group of $\mathrm{Aut}(\mathfrak{T})^\sharp$ at the point $(\mathfrak{e}, \mathfrak{f}) \in S$. Since $\mathcal{R}$ is of type $\mathfrak{L}$, it follows that the image of $F(R, D, \mathfrak{T})$ by Φ is in the $\mathrm{Aut}(\mathfrak{T})^\sharp$-orbit through $(\mathfrak{e}, \mathfrak{f})$, and further that $F(\mathcal{R}, \mathfrak{L})$ is the inverse image of $(\mathfrak{e}, \mathfrak{f})$ by Φ. By Lemma A.3 we have therefore shown that $F(\mathcal{R}, \mathfrak{L})$ really gives a differentiable reduction of $F(R, D, \mathfrak{T})$ to the group $\mathrm{Aut}(\mathfrak{L})^\sharp$.

We remark that the construction above of $F(\mathcal{R}, \mathfrak{L})$ uses the inverse function theorem similarly to the construction of $F(R, D, \mathfrak{T})$ in §1.3. However, consider the special case where $\mathfrak{L}$ is a pseudo-projective FGLA. Then it is clear that $F(\mathcal{R}, \mathfrak{L})$ as well as $F(R, D, \mathfrak{T})$ can be constructed by rational operations and differentiations.

We denote by π the projection of $F(\mathcal{R}, \mathfrak{L})$ onto R, and by ξ the basic form of $F(\mathcal{R}, \mathfrak{L})$. We also denote by ξ^+_{-1} (resp. by ξ^-_{-1}) the $\mathfrak{e}$- (resp. $\mathfrak{f}$-) component of ξ_{-1} with respect to the decomposition $\mathfrak{g}_{-1} = \mathfrak{e} + \mathfrak{f}$.

Proposition 2.6. *$F(\mathcal{R}, \mathfrak{L})$ is of type $\mathfrak{T}$. Furthermore the system $\pi_*^{-1}(E)$ is defined by the system of Pfaffian equations $\xi_j = \xi^-_{-1} = 0$ $(j \leqq -2)$, and the system $\pi_*^{-1}(F)$ by the system of Pfaffian equations $\xi_j = \xi^+_{-1} = 0$ $(j \leqq -2)$.*

Proof. The first assertion is clear. The proof of the second assertion is quite similar to that of (1) of Proposition 1.3. □

Now, let Q be any $\mathrm{Aut}(\mathfrak{L})^\sharp$-structure of type $\mathfrak{T}$ on a manifold R. Let π be the projection of Q onto R, and ξ the basic form of Q. As before the $\mathfrak{g}_{-1}$-valued 1-form ξ_{-1} is decomposed as follows: $\xi_{-1} = \xi^+_{-1} + \xi^-_{-1}$.

Proposition 2.7. (1) *There are unique differential systems E and F on R such that $\pi_*^{-1}(E)$ is defined by the system of Pfaffian equations $\xi_j = \xi^-_{-1} = 0$ $(j \leqq -2)$, and $\pi_*^{-1}(F)$ by the system of Pfaffian equations $\xi_j = \xi^+_{-1} = 0$ $(j \leqq -2)$.*

(2) *The triplet (R, E, F) defines an almost pseudo-product manifold, say $\mathcal{R}$, of type $\mathfrak{L}$.*

(3) *$Q = F(\mathcal{R}, \mathfrak{L})$. Furthermore $\mathcal{R}$ is a unique almost pseudo-product manifold of type $\mathfrak{L}$ having this property.*

The manifold $\mathcal{R}$ will be called associated with Q.

Proof. The proof of (1) is quite similar to that of (1) of Proposition 1.5. Let us prove (2) and (3). We first remark that Q gives a reduction of $F(R, D, \mathfrak{T})$ to the group $\mathrm{Aut}(\mathfrak{L})^\sharp$ (by (3) of Proposition 1.5), where D is the differential system of type $\mathfrak{T}$ associated with Q. Now, for our purpose we may assume that Q is trivial. Take a cross section g of Q, and set $\alpha = g^*\xi$, $\alpha_p = g^*\xi_p$, and $\alpha^\pm_{-1} = g^*\xi^\pm_{-1}$. Let (D^p) be the family of derived systems of D. Then it is clear that D^p is defined by the system of Pfaffian equations $\alpha_j = 0$ $(j < p)$, E by the system of Pfaffian equations $\alpha_j = \alpha^-_{-1} = 0$ $(j \leqq -2)$, and F by the system of Pfaffian equations

$\alpha_j = \alpha^+_{-1} = 0$ $(j \leqq -2)$. Clearly we have $D = E + F$ (direct sum). Since Q is of type $\mathfrak{T}$, we obtain

$$d\alpha_p + \frac{1}{2} \sum_{r+s=p} [\alpha_r, \alpha_s] \underset{p}{\equiv} 0,$$

where $\underset{p}{\equiv}$ is considered with respect to α. Let x be any point of R, and let X and Y be any local cross sections of E or of F defined on a neighborhood of x. Then it follows that

$$\begin{aligned} \alpha_p([X, Y]) &= 0 \quad (p < -2), \\ \alpha_{-2}([X, Y]) &= [\alpha_{-1}(X), \alpha_{-1}(Y)] \\ &= [\alpha^+_{-1}(X), \alpha^-_{-1}(Y)] + [\alpha^-_{-1}(X), \alpha^+_{-1}(Y)] = 0, \end{aligned}$$

showing that $[X, Y]$ gives a cross section of D. Consequently we have shown that the pair (E, F) gives an almost pseudo-product structure on R. Let x be any point of R, and set $z = g(x)$. Since $z \cdot \alpha(X) = X$ for all $X \in T_x(R)$, it follows that $z \cdot \mathfrak{e} = E(x)$ and $z \cdot \mathfrak{f} = F(x)$. Therefore $\tilde{z}$ gives an isomorphism of $\mathfrak{L}$ to the symbol algebra $\mathfrak{L}(x)$ of $\mathcal{R}$ at x, and hence $\mathcal{R}$ is of type $\mathfrak{L}$, proving (2). This proof simultaneously indicates that g gives a cross section of $F(\mathcal{R}, \mathfrak{L})$, and hence $Q = F(\mathcal{R}, \mathfrak{L})$. Furthermore the uniqueness assertion in (3) follows immediately from (1) and Proposition 2.6, proving (3). □

2.4. Finiteness for the geometry of almost pseudo-product manifolds of type $\mathfrak{L}$. Let $\mathfrak{G}$ be a transitive graded Lie algebra, and $(\mathfrak{g}, (\mathfrak{g}_p))$ its expression. We denote by $\mathfrak{T}$ the truncated FGLA, $\mathfrak{G}_-$, of $\mathfrak{G}$, which is thus expressed by $(\mathfrak{t}, (\mathfrak{g}_p)_{p<0})$, where $\mathfrak{t} = \mathfrak{g}_- = \sum_{p<0} \mathfrak{g}_p$. Let $\mathfrak{e}$ and $\mathfrak{f}$ be subspaces of $\mathfrak{g}_{-1}$. Then the triplet $(\mathfrak{G}, \mathfrak{e}, \mathfrak{f})$ is called a *pseudo-product graded Lie algebra*, if the following conditions are satisfied:

(i) The triplet $(\mathfrak{T}, \mathfrak{e}, \mathfrak{f})$ defines a pseudo-product FGLA, say $\mathfrak{L}$,
(ii) $\mathfrak{g}_0$ leaves each of $\mathfrak{e}$ and $\mathfrak{f}$ invariant, i.e., $[\mathfrak{g}_0, \mathfrak{e}] \subset \mathfrak{e}$, and $[\mathfrak{g}_0, \mathfrak{f}] \subset \mathfrak{f}$.

The pseudo-product FGLA, $\mathfrak{L}$, will be called the *truncated pseudo-product* FGLA of the pseudo-product graded Lie algebra $(\mathfrak{G}, \mathfrak{e}, \mathfrak{f})$. Furthermore the algebra $(\mathfrak{G}, \mathfrak{e}, \mathfrak{f})$ will be sometimes confounded with the underlying graded Lie algebra $\mathfrak{G}$, and then the pair $(\mathfrak{e}, \mathfrak{f})$ will be called the *pseudo-product structure* of $\mathfrak{G}$.

Let $\mathfrak{L}$ be a pseudo-product FGLA, and $(\mathfrak{T}, \mathfrak{e}, \mathfrak{f})$ its expression. Let us consider the derivation algebra $\mathrm{Der}(\mathfrak{L})$ of $\mathfrak{L}$, i.e., the subalgebra of $\mathrm{Der}(\mathfrak{T})$ consisting of all $X \in \mathrm{Der}(\mathfrak{T})$ which leave each of $\mathfrak{e}$ and $\mathfrak{f}$ invariant. Then the prolongation, say $\mathfrak{G}$, of the pair $(\mathfrak{T}, \mathrm{Der}(\mathfrak{L}))$ is called the *prolongation* of $\mathfrak{L}$. Clearly the triplet $(\mathfrak{G}, \mathfrak{e}, \mathfrak{f})$ gives a pseudo-product graded Lie algebra. Conversely if $(\mathfrak{G}, \mathfrak{e}, \mathfrak{f})$ is a pseudo-product graded Lie algebra, $\mathfrak{G}$ becomes naturally a graded subalgebra of the prolongation of the truncated pseudo-product FGLA, $\mathfrak{L}$, of $(\mathfrak{G}, \mathfrak{e}, \mathfrak{f})$.

Let $(\mathfrak{G}, \mathfrak{e}, \mathfrak{f})$ be a pseudo-product graded Lie algebra. We preserve the notations as in the definition above.

Proposition 2.8 (cf. Corollary 3 to Theorem 11.1 in [18]). (1) *If $\mathfrak{L}$ is non-degenerate, the Lie algebra $\mathfrak{g}$ is of finite dimension.*
(2) *Assume that $\mathfrak{G}$ is the prolongation of $\mathfrak{L}$. Then $\mathfrak{g}$ is of finite dimension, if and only if $\mathfrak{L}$ is non-degenerate.*

Proof. (1) Let us consider the subalgebra $\mathfrak{h}_0 = \mathfrak{h}_0(\mathfrak{G})$ of $\mathfrak{gl}(\mathfrak{g}_{-1})$ (see §1.5). By Proposition 1.6 it suffices to show that the first prolongation $\mathfrak{h}_0^{(1)}$ of $\mathfrak{h}_0$ vanishes. Recall that $\mathfrak{h}_0^{(1)}$ is the vector space of all linear maps A of $\mathfrak{g}_{-1}$ to $\mathfrak{h}_0$ such that $A(X)Y = A(Y)X$ for all $X, Y \in \mathfrak{g}_{-1}$. Now, we define an endomorphism J of $\mathfrak{g}_{-1}$ by

$$JX = X \quad \text{for } X \in \mathfrak{e}, \text{ and } JX = -X \quad \text{for } X \in \mathfrak{f}.$$

Clearly $J^2X = X$, and $[JX, JY] = -[X, Y]$ for all $X, Y \in \mathfrak{g}_{-1}$. We also define a bilinear map $(X, Y) \mapsto \langle X, Y\rangle$ of $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1}$ to $\mathfrak{g}_{-2}$ by

$$\langle X, Y\rangle = [JX, Y], \quad X, Y \in \mathfrak{g}_{-1}.$$

Then, taking account of the assumption that $\mathfrak{L}$ or $\mathfrak{T}$ is non-degenerate, we easily have the following: (i) $\langle \cdot\,, \cdot\rangle$ is symmetric, i.e., $\langle X, Y\rangle = \langle Y, X\rangle$, (ii) $\langle \cdot\,, \cdot\rangle$ is non-degenerate, i.e., the condition "$X \in \mathfrak{g}_{-1}$ and $\langle X, \mathfrak{g}_{-1}\rangle = \{0\}$" implies $X = 0$, and (iii) every $A \in \mathfrak{h}_0$ is skew-symmetric with respect to $\langle \cdot\,, \cdot\rangle$, i.e., $\langle AX, Y\rangle + \langle X, AY\rangle = 0$ for all $X, Y \in \mathfrak{g}_{-1}$. Therefore it follows that $\mathfrak{h}_0^{(1)}$ vanishes, which compares with the well known fact that the first prolongation $\mathfrak{o}(n)^{(1)}$ of the orthogonal Lie algebra $\mathfrak{o}(n)$ vanishes. We have thus proved (1).

(2) Suppose that $\mathfrak{G}$ is the prolongation of $\mathfrak{L}$, and that $\mathfrak{L}$ or $\mathfrak{T}$ is degenerate. We define a subspace $\mathfrak{n}$ of $\mathfrak{g}_{-1}$ by

$$\mathfrak{n} = \{\, X \in \mathfrak{g}_{-1} \mid [X, \mathfrak{g}_{-1}] = \{0\}\,\}.$$

Clearly $\mathfrak{n} = \mathfrak{n} \cap \mathfrak{e} + \mathfrak{n} \cap \mathfrak{f} \neq \{0\}$ (cf. Lemma 3.3 in Chapter I). We also define a subspace $\mathfrak{k}_0$ of $\mathfrak{gl}(\mathfrak{g}_{-1})$ by

$$\mathfrak{k}_0 = \{\, A \in \mathfrak{gl}(\mathfrak{g}_{-1}) \mid A\mathfrak{e} \subset \mathfrak{n} \cap \mathfrak{e},\ A\mathfrak{f} \subset \mathfrak{n} \cap \mathfrak{f}\,\}.$$

Clearly $\mathfrak{k}_0$ is an ideal of $\mathfrak{h}_0 \subset \mathfrak{gl}(\mathfrak{g}_{-1})$, and is of infinite type, i.e., the k-th prolongation $\mathfrak{k}_0^{(k)}$ of $\mathfrak{k}_0$ does not vanish for each $k \geqq 0$. Since $\mathfrak{k}_0^{(k)} \subset \mathfrak{h}_0^{(k)} = \mathfrak{h}_k(\mathfrak{G})$, it follows that $\mathfrak{g}$ is of infinite dimension. Now, (2) follows immediately from this fact and (1). We have thereby completed the proof of the proposition. $\square$

Finally let $\mathfrak{L}$ be a pseudo-product FGLA, and assume that it is non-degenerate. Let $\mathfrak{G}$ be the prolongation of $\mathfrak{L}$, and let G_0 be the connected component of $\mathrm{Aut}(\mathfrak{L})$ containing the identity element. By Proposition 2.8 the underlying Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$ is of finite dimension. Therefore we may apply Theorem 1.7 to the $G_0^\sharp$-structures of type $\mathfrak{T}$. In view of Proposition 2.6 this application indicates the finiteness for the geometry of almost pseudo-product manifold of type $\mathfrak{L}$.

Let $\mathcal{R}$ be an almost pseudo-product manifold of type $\mathfrak{L}$. Let us consider the automorphism group $\mathrm{Aut}(\mathcal{R})$ of $\mathcal{R}$ as well as the sheaf $\mathfrak{a}(\mathcal{R})$ of germs of local infinitesimal automorphisms of $\mathcal{R}$. As a concrete proof of the finiteness stated above we then have the following proposition, which is nothing but a consequence of Corollary to Theorem 1.7.

Proposition 2.9. (1) *Each stalk* $\mathfrak{a}(\mathcal{R})_x$ *of* $\mathfrak{a}(\mathcal{R})$ *is of finite dimension, and* $\dim \mathfrak{a}(\mathcal{R})_x \leqq \dim \mathfrak{g}$.

(2) $\mathrm{Aut}(\mathcal{R})$ *becomes naturally a Lie group, and* $\dim \mathrm{Aut}(\mathcal{R}) \leqq \dim \mathfrak{g}$.

Appendix A. Completely integrable differential systems with singularities and some differentiability lemmas

A.1. Completely integrable differential systems with singularities. In this paragraph everything will be discussed in the real analytic category. Let M be a manifold. Let us consider the sheaf $\mathcal{O}$ of rings of germs of local functions on M, and the sheaf $\mathcal{T}$ of Lie algebras of germs of local vector fields on M. $\mathcal{T}$ is a module over $\mathcal{O}$ or an $\mathcal{O}$-module. In this appendix a differential system on M means an $\mathcal{O}$-submodule $\mathcal{E}$ of $\mathcal{T}$ which satisfies the following conditions : For each $p \in M$ there is a finite or infinite system (X_α) of local cross sections of $\mathcal{E}$ defined on a common neighborhood O of p such that the restriction of $\mathcal{E}$ to O is generated by the cross sections X_α. Such a system (X_α) will be called a *system of local generators of* $\mathcal{E}$. The notion of a differential system just introduced is a formulation of the ambiguous notion of a differential system possibly with singularities. A differential system $\mathcal{E}$ on M is said to be *completely integrable*, if each stalk $\mathcal{E}(p)$ is a subalgebra of the Lie algebra $\mathcal{T}(p)$.

Let $\mathcal{E}$ be a completely integrable system on M. For each $p \in M$ we define a subspace $E(p)$ of $T_p(M)$ by

$$E(p) = \{\, X_p \mid X \in \mathcal{E}(p) \,\}.$$

Then a connected submanifold N of M is called an *integral manifold* of $\mathcal{E}$ or of E, if $T_p(N) = E(p)$ for each $p \in N$. By Theorem 1 of Nagano [12] we know that through every point p of M there passes a unique maximal integral manifold N of $\mathcal{E}$, and hence M is expressed as the disjoint union of maximal integral manifolds of $\mathcal{E}$.

Let $\mathcal{E}$ be as above, and let N be a submanifold of M such that each connected component of the manifold N is a maximal integral manifold of $\mathcal{E}$. We denote by $\mathcal{I}$ the ideal of the closure of N, which is the subsheaf of ideals of $\mathcal{O}$ defined as follows: $\mathcal{I} = \bigcup_p \mathcal{I}(p)$, and $\mathcal{I}(p)$ is the ideal of $\mathcal{O}(p)$ consisting of all the germs $\in \mathcal{O}(p)$ which are represented by local functions f, defined on neighborhoods O of p in M, that vanish on the intersections $O \cap N$. It is clear that if f is a local cross section of $\mathcal{I}$, and if X is a local cross section of $\mathcal{E}$, then Xf is a local cross section of $\mathcal{I}$.

For each $p \in N$ we now define a subspace $\Lambda(p)$ of $T^*_p(M)$ by

$$\Lambda(p) = \{\, (df)_p \mid f \in \mathcal{I}(p) \,\}.$$

Let Γ_0 be the pseudo-group of local transformations of M generated by $\mathcal{E}$ (cf. the proof of Lemma 1.3 in Chapter I). Then we easily see that Γ_0 leaves each connected component of N invariant, and acts transitively on it. It follows that $\dim \Lambda(p)$ is constant on each connected component of N. Furthermore let Γ be a pseudo-group of local transformations of M. Then it follows similarly that if Γ leaves N invariant, and acts transitively on it, then $\dim \Lambda(p)$ is constant on the whole of N. These being observed, let us consider the following condition on N:

$(*)$ $\dim \Lambda(p)$ is constant on the whole of N.

Lemma A.1. *Let $\mathcal{E}$ be a completely integrable system on M, and let N be a submanifold of M such that each connected component of the manifold N is a maximal integral manifold of $\mathcal{E}$. Assume that N satisfies condition $(*)$. Then, for each point p_0 of N there are a neighborhood U of p_0 and an imbedded submanifold V of U which satisfies the following conditions:*

(i) $U \cap N \subset V$,
(ii) $E(p) \subset T_p(V)$ *for each* $p \in V$,
(iii) $\dim E(p) = \dim N$ *for each* $p \in V$.

Proof. We set $k = \dim \Lambda(p_0)$, and take k local functions $x^i \in \mathcal{I}(p_0)$ such that the differentials $(dx^i)_{p_0}$ form a basis of $\Lambda(p_0)$. Then we assert that every local function $f \in \mathcal{I}(p_0)$ may be expressed as a function of (x^i) on some neighborhood of p_0 in M. Indeed we take $n-k$ local functions $y^j \in \mathcal{O}(p_0)$ such that the n local functions x^i and y^j form a coordinate system of M at p_0. Now, let O be a sufficiently small neighborhood of p_0 in M. Then, for each $p \in O \cap N$ we have

$$(df)_p = \sum_i \frac{\partial f}{\partial x^i}(p)(dx^i)_p + \sum_j \frac{\partial f}{\partial y^j}(p)(dy^j)_p.$$

Since $(df)_p \in \Lambda(p)$, and since $(dx^i)_p$ form a basis of $\Lambda(p)$, it follows that $(\partial f/\partial y^j)(p) = 0$, showing that the derivatives $\partial f/\partial y^j$ are local cross sections of $\mathcal{I}$. Therefore we see that all the derivatives $\partial^\alpha f/\partial y^{j_1}\cdots\partial y^{j_\alpha}$ give local cross sections of $\mathcal{I}$, whence $(\partial^\alpha f/\partial y^{j_1}\cdots\partial y^{j_\alpha})(p_0) = 0$. We have thus shown that f may be expressed as a function of (x^i) on some neighborhood of p_0 in M, proving our assertion.

We set $\dim N = r$, and define a subset A of M by

$$A = \{\, p \in M \mid \dim E(p) \leqq r \,\},$$

which clearly contains N. We easily see that there is a system (f_λ) of local functions defined on a neighborhood O_1 of p_0 in M such that the intersection $O_1 \cap A$ is defined by the system of equations $f_\lambda = 0$. Let us also consider a system (X_α) of local generators of the $\mathcal{O}$-module $\mathcal{E}$ defined on a neighborhood O_2 of p_0 in M. Let U be a cubic neighborhood of p_0 with respect to the coordinate system (x^i, y^j) such that $U \subset O_1 \cap O_2$, and such that $\dim E(p) = \dim E(p_0) = r$ for each $p \in U \cap A$. We denote by V the slice of U defined by $V = \{\, p \in U \mid x^i(p) = 0 \,\}$. Clearly we have $U \cap N \subset V$. Since $N \subset A$, f_λ give local cross sections of $\mathcal{I}$, and hence they may be expressed as functions of (x^i) on some neighborhood of p_0 in M. Therefore it follows that f_λ vanish on some neighborhood of p_0 in V and hence on the whole of V, showing that $V \subset A$. Similarly $X_\alpha x^i$ vanish on the whole of V, because they give local cross sections of $\mathcal{I}$. This means that X_α are tangent to V, whence $E(p) \subset T_p(V)$ for each $p \in V$. Since $V \subset U$, we have $\dim E(p) = r$ for each $p \in V$.

We have thus completed the proof of the lemma. □

A.2. Some differentiability lemmas.

Lemma A.2. *Let $\mathcal{E}$ and N be as in Lemma* A.1*, and assume that the submanifold N of M satisfies condition $(*)$ as well as the second axiom of countability. Let φ be a differentiable map of a differentiable manifold B to M such that the image of B by φ is in N. Then the map φ, regarded as a map of B to N, is differentiable.*

Proof. Take any point b_0 of B, and set $p_0 = \varphi(b_0)$, being a point of N. We apply Lemma A.1 to the collection $\{\mathcal{E}, N, p_0\}$. If we set $E' = \bigcup_{p \in V} E(p)$, it follows from that lemma that E' is a subbundle of $T(V)$ and is completely integrable, and that each connected component of the open submanifold $U \cap N$ of the manifold N is a maximal integral manifold of E'. Now, let O be a connected neighborhood of b_0 such that $\varphi(O) \subset U$. Then $\varphi(O) \subset U \cap N \subset V$, and the restriction of φ to O, regarded as a map of O to V, is differentiable. Since the open submanifold $U \cap N$ satisfies the second axiom of countability, we have therefore seen that the restriction of φ to O, regarded as a map of O to $U \cap N$, is differentiable (cf. Chevalley [5]), proving the lemma. □

Let P be a differentiable principal fibre bundle over a manifold M with structure group G. Let S be a real analytic manifold, and assume that the Lie group G acts real analytically on S (on the right) by the rule $(s, a) \mapsto s^a$, where $s \in S$, and $a \in G$. Furthermore let Φ be a differentiable map of P to S which is G-equivariant: $\Phi(za) = \Phi(z)^a$ for all $z \in P$ and $a \in G$. Then we have the following

Lemma A.3. *Let s_0 be a point of S, and assume that the image of P by Φ is in the G-orbit through s_0. Denote by Q the inverse image of s_0 by Φ, and by H the isotropy group of G at s_0. Then Q gives a differentiable reduction of the principal fibre bundle P to the group H.*

Proof. Let π be the projection of P onto M. Then the group H acts freely on the set Q in a natural fashion, π maps Q onto M, and two points z and z' of Q are in the same fibre, i.e, $\pi(z) = \pi(z')$, if and only if $z' = za$ for some $a \in G$. Therefore to prove the lemma, it suffices to show that for each point $p_0 \in M$ there is a differentiable local cross section g of P defined on a neighborhood O of p_0 such that the image of O by g is in Q.

Let us consider the Lie algebra, say $\tilde{\mathfrak{g}}$, of real analytic vector fields on S which is naturally induced by the action of G on S (cf. the Lie algebra consisting of all the vertical vector fields A^* on a principal fibre bundle P). We denote by $\mathcal{E}$ the real analytic differential system on S generated by $\tilde{\mathfrak{g}}$, which is completely integrable. Now, let S_0 denote the G-orbit through s_0. Then the map $a \mapsto s_0^{a^{-1}}$ of G to S_0 naturally induces a bijection, say ι, of G/H onto S_0, and S_0 becomes a real analytic submanifold of S so that ι gives a real analytic homeomorphism of G/H onto S_0. Furthermore we know that each connected component of the manifold S_0 is a maximal integral manifold of $\mathcal{E}$, and that the submanifold S_0 of S satisfies condition $(*)$ as well as the second axiom of countability. (Note that a Lie group is assumed to satisfy condition $(*)$.) Therefore it follows from Lemma A.2 that the map Φ, regarded as a map of P to S_0, is differentiable.

Let us now recall that G may be considered as a principal fibre bundle over G/H with structure group H, and take a local cross section σ of this bundle defined on a neighborhood V of the origin of G/H. Let p_0 be any point of M, and take a local cross section g' of P defined on a neighborhood O of p_0 such that the image of O by $\Phi \circ g'$ is in the image $\iota(V)$ of V by ι. We then define a map g of O to P by

$$g(p) = g'(p) \cdot \sigma(u), \qquad p \in O,$$

where $u \in V$, and $\iota(u) = \Phi(g'(p))$. Then we see that g gives a differentiable cross section of P. Since $s_0 = \iota(u)^{\sigma(u)}$, we also see that $\Phi(g(p)) = s_0$, showing that the image of O by g is in Q.

We have thus proved the lemma. We remark that the construction above of Q uses the inverse function theorem. $\square$

CHAPTER III

Projective graded Lie algebras

§1. Preliminaries: Simple graded Lie algebras

1.1. Simple graded Lie algebras.

$$\mathfrak{G} = \{\mathfrak{g}, (\mathfrak{g}_p)\}, \quad \text{(SGLA1), (SGLA2)}$$

$$\mathfrak{M} = \{\mathfrak{m}, (\mathfrak{g}_p)_{p<0}\}$$

$$\langle \quad , \quad \rangle \qquad \text{Killing form of } \mathfrak{g}$$

Proposition 1.1. (1) $\langle \mathfrak{g}_p, \mathfrak{g}_q \rangle = 0 \quad$ *if* $p + q \neq 0$.
(2)

Proposition 1.2. E

$\mathfrak{g}_0 \subset \mathrm{Der}(\mathfrak{M})$, $\quad \mathfrak{G}$ は $(\mathfrak{M}, \mathfrak{g}_0)$ の prolongation の graded subalgebra

Proposition 1.3. σ

Let $\mathfrak{G}$ be a graded Lie algebra, and $(\mathfrak{g}, (\mathfrak{g}_p))$ its expression. Let us denote by $\mathfrak{T}$ the truncated graded subalgebra $\mathfrak{G}_-$ of $\mathfrak{G}$, which is thus expressed by $(\mathfrak{t}, (\mathfrak{g}_p)_{p<0})$, where $\mathfrak{t} = \mathfrak{g}_- = \sum_{p<0} \mathfrak{g}_p$. Then $\mathfrak{G}$ is called a simple graded Lie algebra or briefly a SGLA, if the following conditions are satisfied:

1) $\dim \mathfrak{g} < \infty$ and the Lie algebra $\mathfrak{g}$ is simple;
2) $\mathfrak{T}$ is a FGLA.

A simple graded Lie algebra $\mathfrak{G}$ is called of the μ-th kind, if the FGLA, $\mathfrak{T}$, is of the μ-th kind.

In the following we shall consider a fixed simple graded Lie algebra $\mathfrak{G}$. We preserve the notation as above.

1.2. The homogeneous spaces $G/G^{(0)}$.

$$\mathrm{Aut}(\mathfrak{g}),$$
$$G_0, \quad G^{(0)},$$
$$\kappa : G^{(0)} \to G_0$$

Proposition 1.4. $G^{(0)} = G_0 \cdot \exp \mathfrak{g}_1 \cdots \exp \mathfrak{g}_\mu$

$$p \geqq 1, \qquad G^{(p)}$$
$$\rho : G^{(0)} \to GL(\mathfrak{m})$$
$$\rho : \mathfrak{g}^{(0)} \to \mathfrak{gl}(\mathfrak{m})$$

Proposition 1.5. G_0 *faithful on* $\mathfrak{g}_{-1}$

Proposition 1.6. $\mathrm{Ker}\,\rho = G^{(\mu)} = \exp \mathfrak{g}_\mu$

Proposition 1.7. G *effective on* $G/G^{(0)}$

Let $\mathfrak{G} = \{\mathfrak{g}, (\mathfrak{g}_p)\}$ be a simple graded Lie algebra, and let us consider the associated filtered Lie algebra $F(\mathfrak{G}) = \{\mathfrak{g}, (\mathfrak{g}^{(p)}\}$.

For any subspace $\mathfrak{h}$ of $\mathfrak{g}$ we define subspaces $P\mathfrak{h}$ and $Q\mathfrak{h}$ of $\mathfrak{g}$ as follows

$$\begin{aligned} P\mathfrak{h} &= \{\, X \in \mathfrak{g} \mid \langle X, \mathfrak{h} \rangle = 0\}, \\ Q\mathfrak{h} &= \{\, X \in \mathfrak{g}^{(0)} \mid [X, \mathfrak{h}] \subset \mathfrak{g}^{(0)}\}. \end{aligned}$$

Lemma 1.8. *Let $\mathfrak{G}$ be of the μ-th kind.*

(1) $\mathfrak{g}^{(-p+1)} = P\mathfrak{g}^{(p)}$, $\quad 1 \leqq p \leqq \mu$.

(2) $\mathfrak{g}^{(p)} = Q\mathfrak{g}^{(-p)}$, $\quad 1 \leqq p \leqq \mu$.

(3) $\mathfrak{g}^{(-\mu+p)} = PQ \cdots PQ\mathfrak{g}$ *(PQ p times)*, $\ 0 \leqq p \leqq \mu$. $\mathfrak{g}^{(\mu-p)} = QP \cdots QPQ\mathfrak{g}$ *(QP p times)*, $\ 1 \leqq p \leqq \mu$.

From (3) of Lemma 1.8 we see that the filtration $(\mathfrak{g}^{(p)})$ depends only on the pair $(\mathfrak{g}, \mathfrak{g}^{(0)})$. More precisely we have the following

Proposition 1.9. *Let $\mathfrak{G}' = \{\mathfrak{g}', (\mathfrak{g}'_p)\}$ be another simple graded Lie algebra, and let $F(\mathfrak{G}') = \{\mathfrak{g}', (\mathfrak{g}'^{(p)}\}$. If $(\mathfrak{g}, \mathfrak{g}^{(0)}) = (\mathfrak{g}', \mathfrak{g}'^{(0)})$, then $\mathfrak{g}^{(p)} = \mathfrak{g}'^{(p)}$ for all p.*

We denote by $\mathrm{Aut}(\mathfrak{g}, \mathfrak{g}^{(0)})$ the automorphism group of the pair $(\mathfrak{g}, \mathfrak{g}^{(0)})$, i.e,

$$\mathrm{Aut}(\mathfrak{g}, \mathfrak{g}^{(0)}) = \{\, a \in \mathrm{Aut}(\mathfrak{g}) \mid a\mathfrak{g}^{(0)} = \mathfrak{g}^{(0)} \,\}.$$

In other words $\mathrm{Aut}(\mathfrak{g}, \mathfrak{g}^{(0)})$ is the normalizer of $\mathfrak{g}^{(0)}$ in $\mathrm{Aut}(\mathfrak{g})$.

Corollary 1. $\mathrm{Aut}(F(\mathfrak{G})) = \mathrm{Aut}(\mathfrak{g}, \mathfrak{g}^{(0)})$.

Corollary 2. *The notations being as in Proposition 1.9, assume that $(\mathfrak{g}, \mathfrak{g}^{(0)}) = (\mathfrak{g}', \mathfrak{g}'^{(0)})$. Then there is $a \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{g}^{(0)})$ such that $a\mathfrak{g}_p = \mathfrak{g}'_p$ for all p.*

1.3. The operators ∂ and ∂^*.

$$C^q(\mathfrak{G}), \qquad \partial$$

$$(\partial c)(X_1 \wedge \cdots \wedge X_{q+1}) =$$

$$(\ ,\ ) \quad \text{inner product}$$

$$\partial^*$$

$$(\partial^* c)(X_1 \wedge \cdots \wedge X_q) =$$

$$\Delta = \partial^* \partial + \partial \partial^*$$

$$H^q(\mathfrak{G})$$

$$C^q(\mathfrak{G}) = H^q(\mathfrak{G}) + \Delta C^q(\mathfrak{G})$$

G_0 on $C^q(\mathfrak{G})$

Proposition 1.10. $(\partial c)^a = \partial(c^a)$, $\quad (\partial^* c)^a = \partial^*(c^a)$.

$G^{(0)}$ on $C^q(\mathfrak{G})$

Proposition 1.11. $(\partial^* c)^a = \partial^*(c^a)$.

$\mathfrak{g}^{(0)}$ on $C^q(\mathfrak{G})$

1.4. The spaces $C^{p,q}(\mathfrak{G})$.

$$\Lambda^q(\mathfrak{m}^*), \qquad \Lambda^q_i$$

$$\mathfrak{g}\otimes\Lambda^q(\mathfrak{m}^*)=\sum_{i,j}\mathfrak{g}_j\otimes\Lambda^q_i$$

$$C^{p,q}(\mathfrak{G})=\sum_j C^{p,q}_j, \text{where } C^{p,q}_j=\mathfrak{g}_j\otimes\Lambda^q_{j-p-q+1}.$$

$$\partial C^{p,q}(\mathfrak{G})\subset C^{p-1,q+1}(\mathfrak{G}),$$
$$\partial^* C^{p,q}(\mathfrak{G})\subset C^{p+1,q-1}(\mathfrak{G}),$$
$$C^{p,q}(\mathfrak{G})=H^{p,q}(\mathfrak{G})+\Delta C^{p,q}(\mathfrak{G})$$
$$H^q(\mathfrak{G})=\sum_p H^{p,q}(\mathfrak{G})$$

Proposition 1.12. $\mathfrak{g}$ *is the prolongation of* $(\mathfrak{M},\mathfrak{g}_0)$ $\iff$
$C^{p,q}(\mathfrak{G})$ は，G_0-invariant

Proposition 1.13. $X\in\mathfrak{g}_r$ $(r\geqq 0)$, $c\in C^{p,q}(\mathfrak{G})$ $\Rightarrow c^X\in C^{p+r,q}(\mathfrak{G})$

§2. Projective graded Lie algebras and the homogeneous spaces $G/G^{(0)}$, $G/A^{(0)}$ and $G/B^{(0)}$[1]

2.1. Projective graded Lie algebras. Let $\mathfrak{L}=\{\mathfrak{M};\mathfrak{e},\mathfrak{f}\}$ be a projective FGLA of type (n,r), and let $\mathfrak{M}=\{\mathfrak{m};(\mathfrak{g}_p)_{p<0}\}$. We first remark that the automorphism group $\mathrm{Aut}(\mathfrak{L})$ of $\mathfrak{L}$ is isomorphic with the product group $GL(\mathfrak{g}_{-2})\times GL(\mathfrak{e})$ through the natural representation of $\mathrm{Aut}(\mathfrak{L})$ on the space $\mathfrak{g}_{-2}+\mathfrak{e}$ (cf. the proof of Proposition $**$), and hence the derivation algebra $\mathrm{Der}(\mathfrak{L})$ of $\mathfrak{L}$ is isomorphic with the product Lie algebra $\mathfrak{gl}(\mathfrak{g}_{-2})\times\mathfrak{gl}(\mathfrak{e})$ through the corresponding representation of $\mathrm{Der}(\mathfrak{L})$ on $\mathfrak{g}_{-2}+\mathfrak{e}$. Especially in the case where $r=1$, $\mathrm{Aut}(\mathfrak{L})$ is also isomorphic with the product group $GL(\mathfrak{e})\times GL(\mathfrak{f})$ through the natural representation of $\mathrm{Aut}(\mathfrak{L})$ on the space $\mathfrak{g}_{-1}=\mathfrak{e}+\mathfrak{f}$, and hence $\mathrm{Der}(\mathfrak{L})$ is isomorphic with the product Lie algebra $\mathfrak{gl}(\mathfrak{e})\times\mathfrak{gl}(\mathfrak{f})$ through the corresponding representation of $\mathrm{Der}(\mathfrak{L})$ on $\mathfrak{g}_{-1}$.

Let $\mathfrak{G}=\{\mathfrak{g},(\mathfrak{g}_p)\}$ be the prolongation of $\mathfrak{L}$ or of $(\mathfrak{M},\mathrm{Der}(\mathfrak{L}))$, which will be called a projective graded Lie algebra of type (n,r). By virtue of Proposition I.2.5, projective graded Lie algebras of type (n,r) are unique up to isomorphism and hence depend only upon the pair (n,r).

A direct calculation of the prolongation yields the following

Proposition 2.1. $\mathfrak{G}$ *is a simple graded Lie algebra of the second kind, and the underlying Lie algebra* $\mathfrak{g}$ *is isomorphic with the simple Lie algebra* $\mathfrak{sl}(n+1,\mathbb{R})$.

In the next paragraph we shall give a matricial representation of $\mathfrak{G}$ together with another proof of the proposition.

We shall now show that to the graded Lie algebra $\mathfrak{G}$ (or precisely to $\mathfrak{L}$) there are associated two kinds of simple graded Lie algebras of the first kind, $\mathfrak{A}$ and $\mathfrak{B}$.

We have $\mathfrak{m}=\mathfrak{g}_{-2}+\mathfrak{e}+\mathfrak{f}$ (direct sum), and then we define a linear endomorphism J of $\mathfrak{m}$ by

$$J\mathfrak{g}_{-2}=0;\quad JX=X,\quad X\in\mathfrak{e};\quad JX=-X,\quad X\in\mathfrak{f}.$$

Clearly J is in the centre of $\mathfrak{g}_0=\mathrm{Der}(\mathfrak{L})$, and

$$[J,\mathfrak{g}_{-2}]=0;\quad [J,[J,X]]=X,\quad X\in\mathfrak{g}_{-1}.$$

[1] III.3.4 のタイトルに対する脚注参照。

Lemma 2.2. $[J,[J,X]] = X, \quad X \in \mathfrak{g}_1$ *and* $[J,\mathfrak{g}_2] = 0$.

Proof. Let $p = 1$ or 2. Then it follows from Proposition 2.1 that the condition "$X \in \mathfrak{g}_p$, $[X,\mathfrak{g}_{-2}] = 0$" implies $X = 0$ (cf. [31, Lemma 3.2]). Lemma 2.2 follows easily from this fact. (Another proof also given by the use of Proposition $**$). □

Let $\varepsilon = 1$ or -1. We define subspaces $\mathfrak{g}_\varepsilon^+$ and $\mathfrak{g}_\varepsilon^-$ by

$$\mathfrak{g}_\varepsilon^{\pm} = \{\, X \in \mathfrak{g}_\varepsilon \mid [J,X] = \pm X \,\}.$$

Clearly we have $\mathfrak{g}_{-1}^+ = \mathfrak{e}$ and $\mathfrak{g}_{-1}^- = \mathfrak{f}$, and it follows from Lemma 2.2 that

$$\mathfrak{g}_1 = \mathfrak{g}_1^+ + \mathfrak{g}_1^- \quad \text{(direct sum)}.$$

Thus the Lie algebra $\mathfrak{g}$ is decomposed as follows:

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}^+ + \mathfrak{g}_{-1}^- + \mathfrak{g}_0 + \mathfrak{g}_1^+ + \mathfrak{g}_1^- + \mathfrak{g}_2.$$

We define subspaces $\mathfrak{a}_p$, $p \in \mathbb{Z}$, of $\mathfrak{g}$ as follows:

$$\mathfrak{a}_p = 0 \quad \text{if } p < -1 \quad \text{or } p > 1; \quad \mathfrak{a}_{-1} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}^+;$$
$$\mathfrak{a}_0 = \mathfrak{g}_{-1}^- + \mathfrak{g}_0 + \mathfrak{g}_1^+; \quad \mathfrak{a}_1 = \mathfrak{g}_1^- + \mathfrak{g}_2,$$

and define an element $E_{\mathfrak{A}}$ in the centre of $\mathfrak{g}_0$ by

$$E_{\mathfrak{A}} = \frac{1}{2}(E - J),$$

where E is the element in the centre of $\mathfrak{g}_0$ such that $[E,X] = pX$, $X \in \mathfrak{g}_p$. Then using Lemma 2.2, we easily have

$$[E_{\mathfrak{A}}, X] = pX, \quad X \in \mathfrak{a}_p.$$

Therefore it follows that $\mathfrak{A} = \{\mathfrak{g}, (\mathfrak{a}_p)\}$ is a simple graded Lie algebra of the first kind.

Similarly we define subspaces $\mathfrak{b}_p$, $p \in \mathbb{Z}$, of $\mathfrak{g}$ as follows:

$$\mathfrak{b}_p = 0 \quad \text{if } p < -1 \quad \text{or } p > 1; \quad \mathfrak{b}_{-1} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}^-;$$
$$\mathfrak{b}_0 = \mathfrak{g}_{-1}^+ + \mathfrak{g}_0 + \mathfrak{g}_1^-; \quad \mathfrak{b}_1 = \mathfrak{g}_1^+ + \mathfrak{g}_2,$$

and define an element $E_{\mathfrak{B}}$ in the centre of $\mathfrak{g}_0$ by

$$E_{\mathfrak{B}} = \frac{1}{2}(E + J).$$

Then we have

$$[E_{\mathfrak{B}}, X] = pX, \quad X \in \mathfrak{b}_p,$$

and hence $\mathfrak{B} = \{\mathfrak{g}, (\mathfrak{b}_p)\}$ becomes a simple graded Lie algebra of the first kind.

Remark. Our discussions above concerning the graded Lie algebras $\mathfrak{A}$ and $\mathfrak{B}$ do not essentially use Proposition 2.1, especially the simplicity of the underlying Lie algebra $\mathfrak{g}$, indicating that the results can be generalized to the cases of any GP FGLA's of the second kind. Indeed let $\mathfrak{L} = \{\mathfrak{M}; \mathfrak{e}, \mathfrak{f}\}$ be a GP FGLA of the second kind, and let $\mathfrak{G} = \{\mathfrak{g}, (\mathfrak{g}_p)\}$ be its prolongation. Then it is shown that there is an element J in the centre of $\mathfrak{g}_0$ such that $[J,X] = X$, $X \in \mathfrak{e}$, and $[J,X] = -X$, $X \in \mathfrak{f}$, and such that $[J,\mathfrak{g}_p] = 0$ if p is even, and $[J,[J,X]] = X$, $X \in \mathfrak{g}_p$, if p is odd. It is also shown that there is an element E in the centre of $\mathfrak{g}_0$ such that $[E,X] = pX$, $X \in \mathfrak{g}_p$. We now know that the elements $E_{\mathfrak{A}} = \frac{1}{2}(E - J)$ and $E_{\mathfrak{B}} = \frac{1}{2}(E + J)$ give rise to graded Lie algebras $\mathfrak{A} = \{\mathfrak{g}, (\mathfrak{a}_p)\}$ and $\mathfrak{B} = \{\mathfrak{g}, (\mathfrak{b}_p)\}$ respectively in the same manner as above. (Analogous results can be also obtained for suitable GP FGLA's of the third order. Such a GP FGLA really appears in the study of single ordinary differential equations of the third order.)

2.2. Matricial representations of projective graded Lie algebras. Let $m, r_1, \dots, r_k$ be positive integers such that $r_1 + \cdots + r_k = m$. Then every matrix X in the Lie algebra $\mathfrak{g} = \mathfrak{sl}(m, \mathbb{R})$ may be written in the form:

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1k} \\ \vdots & & \vdots \\ X_{k1} & \cdots & X_{kk} \end{pmatrix},$$

where X_{ij} are $r_i \times r_j$ matrices. We define subspaces of $\mathfrak{g}_p$, $p \in \mathbb{Z}$, of $\mathfrak{g}$ by

$$\mathfrak{g}_p = \{ X \mid X_{ij} = 0 \quad \text{if } j - i \neq p \}.$$

Then we see that $\{\mathfrak{g}, (\mathfrak{g}_p)\}$ is a simple graded Lie algebra of the $(k-1)$-th kind, which we denote by $\mathfrak{G}(r_1, \dots, r_k)$.

Let n and r be integers such that $1 \leqq r \leqq n-1$. Putting $r_1 = 1$, $r_2 = r$ and $r_3 = n - r$, we have the graded Lie algebra of the second kind, $\mathfrak{G}(1, r, n-r)$, and analogously we have the graded Lie algebras of the first kind, $\mathfrak{G}(1, n)$ and $\mathfrak{G}(r+1, n-r)$. We shall show that $\mathfrak{G} = \mathfrak{G}(1, r, n-r)$ is a projective graded Lie algebra of type (n, r) and the associated graded Lie algebras $\mathfrak{A}$ and $\mathfrak{B}$ are given by $\mathfrak{A} = \mathfrak{G}(1, n)$ and $\mathfrak{B} = \mathfrak{G}(r+1, n-r)$.

We first remark that, the notations being as above, the subspaces $\mathfrak{g}_{-2}, \dots, \mathfrak{g}_2$ of $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R})$ are explicitly given as follows:

$$\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X_{31} & 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ X_{21} & 0 & 0 \\ 0 & X_{32} & 0 \end{pmatrix} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & 0 \\ 0 & 0 & X_{33} \end{pmatrix} \right\}, \quad \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & X_{12} & 0 \\ 0 & 0 & X_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} 0 & 0 & X_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

We also remark that the element E in the centre of $\mathfrak{g}_0$ is given by

$$E = (\lambda_i \delta_{ij})_{1 \leqq i, j \leqq n+1},$$

where $\lambda_1 = \dfrac{2n - r}{n+1}$, $\lambda_2 = \cdots = \lambda_{r+1} = \dfrac{n - r - 1}{n+1}$, $\lambda_{r+2} = \cdots = \lambda_{n+1} = \dfrac{-r-2}{n+1}$.

Let us consider the truncated graded subalgebra $\mathfrak{M} = \{\mathfrak{m}, (\mathfrak{g}_p)_{p<0}\}$ of $\mathfrak{G}$, where $\mathfrak{m} = \sum_{p<0} \mathfrak{g}_p = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}$. Then using Proposition in [**] we easily see that $\mathfrak{G}$ is the prolongation of $(\mathfrak{M}, \mathfrak{g}_0)$, where $\mathfrak{g}_0$ should be naturally identified with a subalgebra of the derivation algebra $\mathrm{Der}(\mathfrak{M})$ of $\mathfrak{M}$. We now define two subspaces $\mathfrak{e}$ and $\mathfrak{f}$ of $\mathfrak{g}$ by

$$\mathfrak{e} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ X_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \qquad \mathfrak{f} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & X_{32} & 0 \end{pmatrix} \right\}.$$

Then it is easy to see that $\mathfrak{L} = \{\mathfrak{M}; \mathfrak{e}, \mathfrak{f}\}$ is a projective FGLA of type (n, r) and that $\mathfrak{g}_0$ coincides with the derivation algebra $\mathrm{Der}(\mathfrak{L})$ of $\mathfrak{L}$. We have thus shown that $\mathfrak{G}$ is the prolongation of $\mathfrak{L}$, and hence a projective graded Lie algebra of type (n, r).

We define a matrix J in $\mathfrak{g}$ by

$$J = (\mu_i \delta_{ij})_{1 \leqq i, j \leqq n+1},$$

where $\mu_1 = \dfrac{-r}{n+1}$, $\mu_2 = \cdots = \mu_{r+1} = \dfrac{n+1-r}{n+1}$, $\mu_{r+2} = \cdots = \mu_{n+1} = \dfrac{-r}{n+1}$. Then we see that J is in the centre of $\mathfrak{g}_0$ and that $[J, \mathfrak{g}_{2\varepsilon}] = 0$ and $[J,[J,X]] = X$, $X \in \mathfrak{g}_\varepsilon$, where $\varepsilon = 1$ or -1. Furthermore we see that $\mathfrak{e} = \mathfrak{g}^+_{-1}$, $\mathfrak{f} = \mathfrak{g}^-_{-1}$ and

$$\mathfrak{g}_1^+ = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & X_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\}, \qquad \mathfrak{g}_1^- = \left\{ \begin{pmatrix} 0 & X_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Therefore the subspaces $\mathfrak{a}_{-1}$, $\mathfrak{a}_0$ and $\mathfrak{a}_1$ of $\mathfrak{g}$ are given by

$$\mathfrak{a}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ X_{21} & 0 & 0 \\ X_{31} & 0 & 0 \end{pmatrix} \right\}, \qquad \mathfrak{a}_0 = \left\{ \begin{pmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & X_{23} \\ 0 & X_{32} & X_{33} \end{pmatrix} \right\},$$

$$\mathfrak{a}_1 = \left\{ \begin{pmatrix} 0 & X_{12} & X_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

and the subspaces $\mathfrak{b}_{-1}$, $\mathfrak{b}_0$ and $\mathfrak{b}_1$ of $\mathfrak{g}$ are given by

$$\mathfrak{b}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X_{31} & X_{32} & 0 \end{pmatrix} \right\}, \qquad \mathfrak{b}_0 = \left\{ \begin{pmatrix} X_{11} & X_{12} & 0 \\ X_{21} & X_{22} & 0 \\ 0 & 0 & X_{33} \end{pmatrix} \right\},$$

$$\mathfrak{b}_1 = \left\{ \begin{pmatrix} 0 & 0 & X_{13} \\ 0 & 0 & X_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

It is now clear that $\mathfrak{A} = \mathfrak{G}(1,n)$ and $\mathfrak{B} = \mathfrak{G}(r+1, n-r)$. We have thus proved our assertions.

2.3. The homogeneous spaces $G/G^{(0)}$, $G/A^{(0)}$ and $G/B^{(0)}$. In this and the next paragraphs, we preserve the notations as in 2.1. First of all we remark that every automorphism of $\mathfrak{L}$ is naturally extensible to an automorphism of $\mathfrak{G}$. Thus the automorphism group $\mathrm{Aut}(\mathfrak{L})$ of $\mathfrak{L}$ may be identified with a subgroup of the automorphism group $\mathrm{Aut}(\mathfrak{G})$ of $\mathfrak{G}$. Since the derivation algebras $\mathrm{Der}(\mathfrak{L})$ of $\mathfrak{L}$ and $\mathrm{Der}(\mathfrak{G})$ of $\mathfrak{G}$ coincide, we know that $\mathrm{Aut}(\mathfrak{L})$ is an open subgroup of $\mathrm{Aut}(\mathfrak{G})$.

Lemma 2.3. (1) *If* $1 \leqq r \leqq n-2$, *then* $\mathrm{Aut}(\mathfrak{L}) = \mathrm{Aut}(\mathfrak{G})$.

(2) *If* $r = n-1$, *then* $\mathrm{Aut}(\mathfrak{g})^0 \cap \mathrm{Aut}(\mathfrak{G}) \subset \mathrm{Aut}(\mathfrak{L})$.

This lemma together with Lemma 2.4 below can be easily proved by using the matricial representation of $\mathfrak{G}$ given in the previous paragraph.

We define subgroups G_0, $G^{(0)}$ and G of the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ as follows:

$$G_0 = \mathrm{Aut}(\mathfrak{L}), \quad G^{(0)} = G_0 \cdot G^{(1)}, \quad G = \mathrm{Aut}(\mathfrak{g})^0 \cdot G_0,$$

where as before $G^{(1)}$ stands for the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by the subalgebra $\mathfrak{g}^{(1)} = \mathfrak{g}_1 + \mathfrak{g}_2$ of $\mathfrak{g}$: $G^{(1)} = \exp \mathfrak{g}_1 \cdot \exp \mathfrak{g}_2$. Hence $G^{(0)}$ and G are the groups associated with the pair $(\mathfrak{G}, G_0)$. (See Remark below Theorem **.) We know from (1) of Lemma 2.3 that if $1 \leqq r \leqq n-2$, then G_0, $G^{(1)}$ and G are the groups naturally associated with $\mathfrak{G}$ itself (See **).

Let us consider the homogeneous space $G/G^{(0)}$. Then we easily see that the subspaces $\mathfrak{e} = \mathfrak{g}^+_{-1}$ and $\mathfrak{f} = \mathfrak{g}^-_{-1}$ naturally give rise to G-invariant differential systems E_0 and F_0 on $G/G^{(0)}$, and that $\mathfrak{P}_0 = \{G/G^{(0)}; E_0, F_0\}$ is a projective system of type (n,r), which will be called the standard projective system of type (n,r).

By using the simple graded Lie algebras of the first kind, $\mathfrak{A}$ and $\mathfrak{B}$, we now define subgroups A_0, $A^{(0)}$, B_0, $B^{(0)}$ of $\mathrm{Aut}(\mathfrak{g})$ as follows:

$$A_0 = \mathrm{Aut}(\mathfrak{A}), \qquad A^{(0)} = A_0 \cdot A^{(1)},$$
$$B_0 = \mathrm{Aut}(\mathfrak{B}),^{2} \qquad B^{(0)} = B_0 \cdot B^{(1)},$$

where $A^{(1)}$ (resp. $B^{(1)}$) stands for the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by the subalgebra $\mathfrak{a}_1$ (resp. $\mathfrak{b}_1$) of $\mathfrak{g}$: $A^{(1)} = \exp \mathfrak{a}_1$, $B^{(1)} = \exp \mathfrak{b}_1$.

Lemma 2.4. (1) *$G = \mathrm{Aut}(\mathfrak{g})^0 \cdot A_0 = \mathrm{Aut}(\mathfrak{g})^0 \cdot B_0$.*

(2) *$A_0 \cap B_0 = G_0$, and $A^{(0)} \cap B^{(0)} = G^{(0)}$.*

Therefore we know that A_0, $A^{(0)}$ and G (resp. B_0, $B^{(0)}$, G) are the groups naturally associated with $\mathfrak{A}$ (resp. $\mathfrak{B}$). Furthermore we know that the diagonal map $a \mapsto (a, a)$ of G to $G \times G$ induces an imbedding

$$\iota : G/G^{(0)} \to G/A^{(0)} \times G/B^{(0)}.$$

We notice that the pseudo-product structure (E_0, F_0) on $G/G^{(0)}$ is induced by this imbedding.

Let us now consider the projective space $\mathbb{P}^n = G^1(\mathbb{R}^{n+1})$, the Grassmann manifold $G^r(\mathbb{P}^n) = G^{r+1}(\mathbb{R}^{n+1})$ and the Grassmann bundle $G^r(T(\mathbb{P}^n))$ (See Chapter I, §1). Let $(e_0, \ldots, e_n)$ be the canonical basis of $\mathbb{R}^{n+1}$. Then the 1-dimensional subspace of $\mathbb{R}^{n+1}$ spanned by e_0 defines a point x_0 of $\mathbb{P}^n$, and similarly the $(r+1)$-dimensional subspace of $\mathbb{R}^{n+1}$ spanned by $e_0, \ldots, e_r$ defines a point y_0 of $G^r(\mathbb{P}^n)$. Moreover, y_0 being an r-dimensional projective subspace of $\mathbb{P}^n$, the tangent space of y_0 at x_0 defines a point z_0 of $G^r(T(\mathbb{P}^n))$.

Now let $\dot{G}$ be the projective transformation group of $\mathbb{P}^n$. Then $\mathbb{P}^n$ may be represented by the homogeneous space $\dot{G}/\dot{A}^{(0)}$, where $\dot{A}^{(0)}$ denotes the isotropy subgroup of $\dot{G}$ at the point x_0. Furthermore $\dot{G}$ naturally acts both on $G^r(\mathbb{P}^n)$ and $G^r(T(\mathbb{P}^n))$, and these spaces may be represented by the homogeneous spaces $\dot{G}/\dot{B}^{(0)}$ and $\dot{G}/\dot{G}^{(0)}$ respectively, where $\dot{B}^{(0)}$ (resp. $\dot{G}^{(0)}$) denotes the isotropy subgroup of $\dot{G}$ at y_0 (resp. at z_0).

These being prepared, we remark that there exist natural identifications as follows: $G = \dot{G}$, $A^{(0)} = \dot{A}^{(0)}$, $B^{(0)} = \dot{B}^{(0)}$ and $G^{(0)} = \dot{G}^{(0)}$, whence $G/A^{(0)} = \mathbb{P}^n$, $G/B^{(0)} = G^r(\mathbb{P}^n)$ and $G/G^{(0)} = G^r(T(\mathbb{P}^n))$.

2.4. Some studies in connection with the imbedding $G/G^{(0)} \to G/A^{(0)} \times G/B^{(0)}$.

For the sake of simplicity we set as follows: $R = G/G^{(0)} = G^r(T(\mathbb{P}^n))$, and $R' = G/A^{(0)} \times G/B^{(0)} = \mathbb{P}^n \times G^r(\mathbb{P}^n)$. The product group $G \times G$ naturally acts on the product manifold R', and hence the group G itself acts on R' through the diagonal map $G \to G \times G$. This being said, it turns out that the imbedding $\iota : R \to R'$ is G-equivariant. If we identify R with a submanifold of R' through the imbedding ι, then it follows that R is a G-orbit, more precisely the G-orbit through the point (x_0, y_0).

Let us now consider the open submanifold $\Omega = R' - R$ of R'. We denote by $\mathrm{Aut}(\Omega)$ the group of all biproduct maps $\Omega \to \Omega$, and similarly by $\mathrm{Aut}(R)$ the group of all biproduct maps $R \to R$, which is nothing but the automorphism group $\mathrm{Aut}(\mathfrak{P}_0)$ of the standard projective system $\mathfrak{P}_0$. These being prepared, we state the following two propositions which can be easily shown.

Proposition 2.5. (1) *Ω is a G-orbit.*

(2) *If $r = n-1$, then Ω may be represented by the homogeneous space G/A_0 or G/B_0 with respect to suitable reference points, and these spaces are affine symmetric spaces, or more precisely $\frac{1}{2}$-kählerian symmetric spaces in the sense of Berger [24].*

[2] $r + 1 = n - r$ のときは不成立!!

Proposition 2.6. (1) *Every biproduct map $\psi \in \mathrm{Aut}(\Omega)$ is extendible to a unique biproduct map $R' \to R'$ and this in turn induces a biproduct map $\varphi \in \mathrm{Aut}(R)$.*

(2) *Conversely every biproduct map $\varphi \in \mathrm{Aut}(R)$ is extendible to a unique biproduct map $R' \to R'$, and this in turn induces a biproduct map $\psi \in \mathrm{Aut}(\Omega)$.*

(3) *Therefore the two groups $\mathrm{Aut}(\Omega)$ and $\mathrm{Aut}(R)$ are naturally isomorphic.*

Concerning these propositions, we now make some remarks.

Remark 1. In §** we shall see that the group $\mathrm{Aut}(R) = \mathrm{Aut}(\mathfrak{P}_0)$ is naturally isomorphic with the group G, i.e., it is obtained from the action of G on R. By Proposition 2.6 we know that the same holds for the group $\mathrm{Aut}(\Omega)$. In particular $\mathrm{Aut}(\Omega)$ is a finite dimensional Lie group. Here we notice that the pseudo-group of all local biproduct maps of a product manifold Ω is necessarily infinite dimensional.

(This fact also follows from the fundamental theorem on projective transformations (cf. Proposition $**$).)

Remark 2. Proposition 2.6 indicates that the geometry of the product manifold Ω is closely related to the geometry of the pseudo-product manifold R, or the global geometry of differential equations $\dfrac{\partial^2 y^\alpha}{\partial x^i \partial x^j} = 0$, $1 \leqq i, j \leqq r$, $1 \leqq \alpha \leqq n - r$, which is very much like the fact that the geometry of a homogeneous Siegel domain is closely related to the geometry of its (completed) Silov boundary (See Tanaka [30]). This analogy is essentially based on the relationship existing between product structures and complex structures.

Remark 3. Our studies in this section depend on the projective FGLA, $\mathfrak{L}$, chosen. It is plausible that most results including Propositions 2.5 and 2.6 can be generalized to the cases of any GP FGLA's of the second kind and of further general GP FGLA's (cf. Remark at the end of 2.1).

2.5. $(\mathfrak{g}, \mathfrak{e}, \mathfrak{f})$ を pseudo-product simple graded Lie algebra とする.

$\mathfrak{g}_0$ の centre の元 J で, 次の条件を満足するものが唯一つある.

$$[J, X] = X, \quad X \in \mathfrak{e}; \quad [J, X] = -X, \quad X \in \mathfrak{f}.$$

そこで, J は次を満足するものと仮定する：

$$[J, \mathfrak{g}_{2p}] = \{0\}; \quad [J, [J, X]] = X, \quad X \in \mathfrak{g}_{2p-1}.$$

こゝで, p は任意の integer である. ($p = 0, 1, -1$ に対してこれらが成り立つことは容易にわかる.)

$\mathfrak{g}_{2p-1}$ の subspaces $\mathfrak{g}^+_{2p-1}$, $\mathfrak{g}^-_{2p-1}$ を

$$\mathfrak{g}^\pm_{2p-1} = \{\, X \in \mathfrak{g}_{2p-1} \mid [J, X] = \pm X \,\}$$

によって定義する. 明らかに $\mathfrak{e} = \mathfrak{g}^+_{-1}$, $\mathfrak{f} = \mathfrak{g}^-_{-1}$ であり, 任意の p に対して

$$\mathfrak{g}_{2p-1} = \mathfrak{g}^+_{2p-1} + \mathfrak{g}^-_{2p-1} \quad \text{(direct sum)}$$

Lemma 2.7. (1) $\mathfrak{g}_{-2} = [\mathfrak{g}^+_{-1}, \mathfrak{g}^-_{-1}]$,

(2) $\mathfrak{g}^\pm_{2p-1} = [\mathfrak{g}_{2p}, \mathfrak{g}^\pm_{-1}]$ *for any* $p < 0$,

(3) $[\mathfrak{g}^\pm_{2p-1}, \mathfrak{g}^\pm_{2q-1}] = \{0\}$ *for any* p, q.

E を simple graded Lie algebra $\mathfrak{G}$ の characteristic element とする. $\mathfrak{g}_0$ の centre の元 $E_\mathfrak{A}$ を

$$E_\mathfrak{A} = \frac{1}{2}(E - J)$$

によって定義し, $\mathfrak{g}$ の subspaces の family $(\mathfrak{a}_p)$ を

$$\mathfrak{a}_p = \{\, X \in \mathfrak{g} \mid [E_\mathfrak{A}, X] = pX \,\}$$

によって定義する．容易に，

$$\mathfrak{a}_p = \mathfrak{g}^-_{2p-1} + \mathfrak{g}_{2p} + \mathfrak{g}^+_{2p+1}$$

がわかる．これより特に $\mathfrak{g} = \sum_p \mathfrak{a}_p$ がわかる．よって $(\mathfrak{g}, (\mathfrak{a}_p))$ が graded Lie algebra になることがわかる．これを $\mathfrak{A}$ で表わす．Lemma 2.7 を使って $\underline{\mathfrak{a}_{p-1} = [\mathfrak{a}_p, \mathfrak{a}_{-1}] \text{ for } p < 0}$ が示される．よって，$\mathfrak{A}$ は simple graded Lie algebra になる．

下線部について，

$$\begin{aligned}
[\mathfrak{a}_p, \mathfrak{a}_{-1}] &= [\mathfrak{g}^-_{2p-1} + \mathfrak{g}_{2p} + \mathfrak{g}^+_{2p+1}, \mathfrak{g}^-_{-3} + \mathfrak{g}_{-2} + \mathfrak{g}^+_{-1}] \\
&= [\mathfrak{g}_{2p}, \mathfrak{g}^-_{-3}] + [\mathfrak{g}^+_{2p+1}, \mathfrak{g}^-_{-3}] + [\mathfrak{g}^-_{2p-1}, \mathfrak{g}_{-2}] + [\mathfrak{g}_{2p}, \mathfrak{g}_{-2}] \\
&\quad + [\mathfrak{g}^+_{2p+1}, \mathfrak{g}_{-2}] + [\mathfrak{g}^-_{2p-1}, \mathfrak{g}^+_{-1}] + [\mathfrak{g}_{2p}, \mathfrak{g}^+_{-1}] \\
\mathfrak{a}_{p-1} &= \mathfrak{g}^-_{2p-3} + \mathfrak{g}_{2p-2} + \mathfrak{g}^+_{2p-1} \\
\mathfrak{g}_{2p-2} &= [\mathfrak{g}_{2p-1}, \mathfrak{g}_{-1}] = [\mathfrak{g}^+_{2p-1} + \mathfrak{g}^-_{2p-1}, \mathfrak{g}^+_{-1} + \mathfrak{g}^-_{-1}] \\
&= [\mathfrak{g}^+_{2p-1}, \mathfrak{g}^-_{-1}] + [\mathfrak{g}^-_{2p-1}, \mathfrak{g}^+_{-1}] \\
&\subset [[\mathfrak{g}_{2p}, \mathfrak{g}^+_{-1}], \mathfrak{g}^-_{-1}] + [\mathfrak{g}^-_{2p-1}, \mathfrak{g}^+_{-1}] \subset [\mathfrak{g}_{2p}, \mathfrak{g}_{-2}] + [\mathfrak{g}^-_{2p-1}, \mathfrak{g}^+_{-1}] \\
\mathfrak{g}^-_{2p-3} &= [\mathfrak{g}_{2p-2}, \mathfrak{g}^-_{-1}] = [[\mathfrak{g}_{2p}, \mathfrak{g}_{-2}] + [\mathfrak{g}^-_{2p-1}, \mathfrak{g}^+_{-1}], \mathfrak{g}^-_{-1}] \\
&\subset [\mathfrak{g}^-_{2p-1}, \mathfrak{g}_{-2}] + [\mathfrak{g}_{2p}, \mathfrak{g}^-_{-3}] \\
\mathfrak{g}^+_{2p-1} &= [\mathfrak{g}_{2p}, \mathfrak{g}^+_{-1}].
\end{aligned}$$

よって，$\mathfrak{a}_{p-1} \subset [\mathfrak{a}_p, \mathfrak{a}_{-1}]$．よって $\mathfrak{a}_{p-1} = [\mathfrak{a}_p, \mathfrak{a}_{-1}]$．

同様に，$\mathfrak{g}_0$ の centre の元 $E_{\mathfrak{B}}$ を

$$E_{\mathfrak{B}} = \frac{1}{2}(E + J)$$

によって定義し，$\mathfrak{g}$ の subspaces の family $(\mathfrak{b}_p)$ を

$$\mathfrak{b}_p = \{ X \in \mathfrak{g} \mid [E_{\mathfrak{B}}, X] = pX \}$$

によって定義する．このとき，

$$\mathfrak{b}_p = \mathfrak{g}^+_{2p-1} + \mathfrak{g}_{2p} + \mathfrak{g}^-_{2p-1}$$

である．そして，$(\mathfrak{g}, (\mathfrak{b}_p))$ が simple graded Lie algebra になる．これを $\mathfrak{B}$ で表わす．

以下の議論において，$(\mathfrak{g}^{(p)})$, $(\mathfrak{a}^{(p)})$, $(\mathfrak{b}^{(p)})$ はそれぞれ，$\mathfrak{G}$, $\mathfrak{A}$, $\mathfrak{B}$ に付随する filtrations とする．

$\mathfrak{G}$ に対して，pseudo-product FGLA, $\mathfrak{L}$, を考える．$\mathrm{Aut}(\mathfrak{L})$ は $\mathrm{Aut}(\mathfrak{G})$ の open subgroup とみなせる．$\mathrm{Aut}(\mathfrak{L}) = \{ a \in \mathrm{Aut}(\mathfrak{G}) \mid \mathrm{Ad}(a)\mathfrak{g}^{\pm}_{-1} = \mathfrak{g}^{\pm}_{-1} \}$．

Lemma 2.8. $\mathrm{Aut}(\mathfrak{L}) = \mathrm{Aut}(\mathfrak{A}) \cap \mathrm{Aut}(\mathfrak{B})$.

$G_0 = \mathrm{Aut}(\mathfrak{L})$ とし，$G^{(0)}$, G を (G, G_0) に付随する groups とする：

$$G^{(0)} = G_0 \cdot G^{(1)}, \quad G = \mathrm{Aut}(\mathfrak{g})^0 \cdot G_0$$

$\mathrm{Aut}(\mathfrak{A})$ の open subgroup A_0 を

$$A_0 = \mathrm{Aut}(\mathfrak{A})^0 \cdot G_0$$

によって定義する．明らかに，$G = \mathrm{Aut}(\mathfrak{g})^0 \cdot A_0$ である．また，G の subgroup $A^{(0)}$ を

$$A^{(0)} = A_0 \cdot A^{(1)}$$

によって定義する．明らかに $A^{(0)}$, G は $(\mathfrak{A}, A_0)$ に付随する groups である．

同様に，$\mathrm{Aut}(\mathfrak{B})$ の open subgroup B_0 を

$$B_0 = \mathrm{Aut}(\mathfrak{B})^0 \cdot G_0$$

によって定義する．このとき, $G = \mathrm{Aut}(\mathfrak{g})^0 \cdot B_0$ である．また G の subgroup $B^{(0)}$ を

$$B^{(0)} = B_0 \cdot B^{(1)}$$

によって定義する．このとき, $B^{(0)}$, G は $(\mathfrak{B}, B_0)$ に付随する groups である．

Lemma 2.9. $G^{(0)} = A^{(0)} \cap B^{(0)}$.

Proof. Lemma 2.8 により, $G_0 = A_0 \cap B_0$. $\mathfrak{g}^{(0)} = \mathfrak{a}^{(0)} \cap \mathfrak{b}^{(0)}$ だから, $G^{(0)} \subset A^{(0)} \cap B^{(0)}$ をえる．$a \in A^{(0)} \cap B^{(0)}$ とする．$\mathrm{Ad}(a)\mathfrak{a}^{(0)} = \mathfrak{a}^{(0)}$, $\mathrm{Ad}(a)\mathfrak{b}^{(0)} = \mathfrak{b}^{(0)}$ だから, $\mathrm{Ad}(a)\mathfrak{g}^{(0)} = \mathfrak{g}^{(0)}$ をえる．よって, Lemma ** により, $a \in \mathrm{Aut}(\mathfrak{G}) \cdot G^{(1)}$. $b \in \mathrm{Aut}(\mathfrak{G})$ を $b^{-1}a \in G^{(1)}$ なるものとする．このとき, $\mathrm{Ad}(a)E \equiv E \pmod{\mathfrak{g}^{(1)}}$, $\mathrm{Ad}(a)J \equiv \mathrm{Ad}(b)J \pmod{\mathfrak{g}^{(1)}}$ をえる．また, $a \in A^{(0)} = A_0 \cdot A^{(1)}$ だから, $\mathrm{Ad}(a)E_{\mathfrak{A}} \equiv E_{\mathfrak{A}} \pmod{\mathfrak{a}^{(1)}}$. これらの事実から, $\mathrm{Ad}(b)J \equiv J \pmod{\mathfrak{g}^{(1)}}$. 従って, $\mathrm{Ad}(b)J = J$ をえる．よって, $b \in G_0$. 故に, $a \in G^{(0)} = G_0 \cdot G^{(1)}$ が示された． □

さて, homogeneous spaces $G/G^{(0)}$, $G/A^{(0)}$, $G/B^{(0)}$ を考えよう．これらは compact, connected である．

Lemma 2.9 により, $G/G^{(0)}$ から $G/A^{(0)}$, $G/B^{(0)}$ の上への自然な maps がある．これらをそれぞれ μ, ν で表わす．$G/G^{(0)}$ は $G/A^{(0)}$, $G/B^{(0)}$ 上の fibred manifolds with projections である．$A^{(0)}/G^{(0)}$, $B^{(0)}/G^{(0)}$ は明らかに connected であるから, これらの fibred manifolds のすべての fibre は connected である．再び, Lemma 2.9 によって, immersion $\mu * \nu : G/G^{(0)} \to G/A^{(0)} \times G/B^{(0)}$ は imbedding である．o_A, o_B を $G/A^{(0)}$, $G/B^{(0)}$ の origins とする．G は diagonal map $G \to G \times G$ を通じて, $G/A^{(0)} \times G/B^{(0)}$ 上に作用する．このとき, $(\mu * \nu)(G/G^{(0)})$ は (o_A, o_B) を通るこの作用による orbit であることを注意する．今後 $\mu * \nu$ によって, $G/G^{(0)}$ を $G/A^{(0)} \times G/B^{(0)}$ の submanifold と同一視することにする．

$(G/A^{(0)}, G/B^{(0)}, G/G^{(0)})$ は generalized projective triplet であることを示そう．$\mathfrak{g}$ の subspaces $\mathfrak{a}^{(0)}$ と $\mathfrak{b}^{(0)}$ は $\mathfrak{g}^{(0)}$ を含み $\mathrm{Ad}(G^{(0)})$-invariant である．よって, $\mathfrak{a}^{(0)}$, $\mathfrak{b}^{(0)}$ は $G/G^{(0)}$ 上の G-invariant differential systems を自然に導く．これらをそれぞれ E, F で表わす．$\mathfrak{a}^{(0)}$, $\mathfrak{b}^{(0)}$ は $\mathfrak{g}$ の subalgebras であり, $\mathfrak{a}^{(0)} \cap \mathfrak{b}^{(0)} = \mathfrak{g}^{(0)}$ であるから, pair (E, F) は $G/G^{(0)}$ 上の pseudo-product structure を与える．Pseudo-product manifold $(G/G^{(0)}, E, F)$ を $\mathfrak{P}_0$ で表わす．明らかに, $E = \nu_*^{-1}(0)$, $F = \mu_*^{-1}(0)$ であるから, $\mathfrak{P}_0$ は relation $G/G^{(0)} \subset G/A^{(0)} \times G/B^{(0)}$ に付随する pseudo-product manifold である．さて, $\mathfrak{g}$ の subspace $\mathfrak{g}^{(-1)}$ は勿論 $\mathrm{Ad}(G^{(0)})$-invariant であり, $\mathfrak{g}^{(-1)} = \mathfrak{a}^{(-1)} + \mathfrak{b}^{(-1)}$. よって system $D(= E + F)$ は $\mathfrak{g}^{(-1)}$ から自然に導かれる $G/G^{(0)}$ 上の G-invariant differential system である．$\mathfrak{L}$ or $\mathfrak{T}$ は non-degenerate であるから, D が non-degenerate であることがわかる．よって relation $G/G^{(0)}$ は non-degenerate relation になる．

さて, relation $G/G^{(0)}$ に付随する immersions $\iota : G/G^{(0)} \to J(G/A^{(0)}, r)$ と $\kappa : G/G^{(0)} \to J(G/B^{(0)}, s)$ を考えよう．こゝで, $r = \mathrm{rank}\, E$, $s = \mathrm{rank}\, F$ である．ι が injective であることを示そう．まず, $\iota(z) = \mu_*(D(z)) = \mu_*(E(z))$, $z \in G/G^{(0)}$ であることを思い出そう．故に, ι が injective ⇔ "$a \in A^{(0)}$, $\mathrm{Ad}(a)\mathfrak{b}^{(0)} = \mathfrak{b}^{(0)}$" ⇒ $a \in G^{(0)}$ がわかる．ところが, Lemma 2.9 の証明から, この条件が満たされることがわかる．よって ι が injective であることが示された．同様に, κ が injective であることも示される．

以上により, $(G/A^{(0)}, G/B^{(0)}, G/G^{(0)})$ が compact, connected generalized projective triplet であることがわかった．これは $(\mathfrak{g}, \mathfrak{e}, \mathfrak{f})$ に付随するとよばれる．

- projective Lie algebra triplet PLAT
- projective Lie group triplet PLGT
- generalized PLAT
- generalized PLGT

(M, N, R) compact, connected generalized pseudo-projective triplet

D について $\mathscr{D}^{-\mu} = \mathscr{T}$ とする

$$\begin{tikzcd} & R \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ M & & N \end{tikzcd}$$

$p_0 \in M$, $\quad S_0$, $S_1, \ldots$, subsets of M inductively defined as follows:

$$\begin{aligned} &S_0 = \{p_0\}, \\ &S_{k+1} = \mu(\nu^{-1}(\nu(\mu^{-1}(S_k)))) \\ &S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_l = M \\ &l = \operatorname*{Max}_q(\rho(p_0, q)) \\ &S_k = \{\, q \in M \mid \rho(p_0, q) = k \,\} \end{aligned}$$

$\dim R < \dim J(M, r)$ の場合には $\{S_k\}$ は M の stratification を与える.
$\mathrm{Aut}(\mathfrak{X})_{p_0}$ the isotropy group of $\mathrm{Aut}(\mathfrak{X})$ at p_0 とするとき,
各 S_k は $\mathrm{Aut}(\mathfrak{X})_{p_0}$ 不変

$\mathrm{Gr}(V, k)$, $\quad \dim V = n$.
$\rho(X_1, X_2) = \dim(X_1/X_1 \cap X_2)$
$X_0 \in \mathrm{Gr}(V, k)$

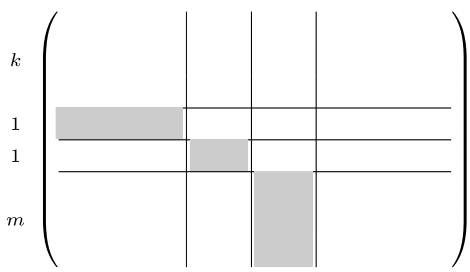
$$\rho(X_0, X) = 1 \quad \Longleftrightarrow \quad \dim(X_0 \cap X) = k - 1$$

$\rho(X_0, X) = 1$ のとき,

$$X_0 \cap X \subset X \subset X_0 + X, \quad \dim(X_0 \cap X) = k - 1, \quad \dim X = k, \quad \dim(X_0 + X) = k + 1$$

$\mathrm{Flag}(V, k-1, k+1) = \{\, (Y, Z) \mid Y \in \mathrm{Gr}(V, k-1), Z \in \mathrm{Gr}(V, k+1), Y \subset Z \,\}$
$M = \mathrm{Gr}(V, k)$, $\quad N = \mathrm{Flag}(V, k-1, k+1)$
$R = \{\, (X, (Y, Z)) \in M \times N \mid Y \subset X \subset Z \,\}$
(M, N, R) generalized pseudo-projective triplet
$\mathfrak{X}$, $\mathfrak{X}^*$, $\mathscr{R}$, ...
$\mathfrak{X}$ の solution は 1-dimensional
$\mathfrak{X}$ の solution が Helgason sphere

$G = GL(n, \mathbb{R})/\text{centre}$
$X \in \mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, $\quad m = n - k - 2$

$X =$ 

これより, pseudo-product SGLA, $\mathfrak{G}$, がえられる.

$$\mathfrak{g}_{-1} = \mathfrak{e} + \mathfrak{f}, \quad \dim \mathfrak{e} = 1, \quad \dim \mathfrak{f} = 2k$$

(M, N, R) $\mathfrak{G}$ に付随する $(G/A^{(0)}, G/B^{(0)}, G/G^{(0)})$ として表現される.

2.6. The spaces of harmonic forms, $H^{p,q}(\mathfrak{G})$. $\mathfrak{G} = \{\mathfrak{g}, (\mathfrak{g}_p)\}$ を simple graded Lie algebra とする.

次の proposition を証明しよう.

Proposition 2.10. 任意の p, q に対して,

$$H^{p,q}(\mathfrak{G}) = \sum_j H^{p,q}(\mathfrak{G}) \cap C_j^{p,q}.$$

任意の positive integer i に対して, ζ_i を $\mathfrak{m} = \sum_{p<0} \mathfrak{g}_p$ から $\mathfrak{g}_{-i}$ の上への projection とする. operator $\partial_i : C^q(\mathfrak{G}) \to C^{q+1}(\mathfrak{G})$ を

$$(\partial_i c)(x_1 \wedge \cdots \wedge x_{q+1}) = \sum_k (-1)^{k+1} [\zeta_i(x_k), c(x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_{q+1})]$$

によって定義する. こゝで $c \in C^q(\mathfrak{G})$, $x_k \in \mathfrak{m}$ である.

さらに operator $\partial_0 : C^q(\mathfrak{G}) \to C^{q+1}(\mathfrak{G})$ を

$$(\partial_0 c)(x_1 \wedge \cdots \wedge x_{q+1}) = \sum_{k<l} (-1)^{k+l} c([x_k, x_l], x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge \hat{x}_l \wedge \cdots \wedge x_{q+1})$$

によって定義する. こゝで $c \in C^q(\mathfrak{G})$, $x_k \in \mathfrak{m}$ である.

明らかに operator $\partial : C^q(\mathfrak{G}) \to C^{q+1}(\mathfrak{G})$ は

$$\partial = \sum_{i \geqq 0} \partial_i$$

と分解する. また容易に

$$\partial_i C_j^{p,q} \subset C_{j-i}^{p-1,q+1} \quad \text{for any } i \geqq 0 \quad \text{and any } p, q, j$$

が確かめられる.

$\forall i \geqq 0$ に対して, $\partial_i^* : C^{q+1}(\mathfrak{G}) \to C^q(\mathfrak{G})$ を operator $\partial_i : C^q(\mathfrak{G}) \to C^{q+1}(\mathfrak{G})$ の adjoint operator とする. 明らかに operator $\partial : C^q(\mathfrak{G}) \to C^{q+1}(\mathfrak{G})$ の adjoint operator $\partial^* : C^{q+1}(\mathfrak{G}) \to C^q(\mathfrak{G})$ は

$$\partial^* = \sum_{i \geqq 0} \partial_i^*$$

と分解し,

$$\partial_i^* C_{j-i}^{p-1,q+1} \subset C_j^{p,q} \quad \text{for any } i \geqq 0 \quad \text{and any } p, q, j$$

が成立する.

$\forall i > 0$ に対して, $n_i = \dim \mathfrak{g}_{-i}$ とおき, $(e_{i\lambda})_{1 \leqq \lambda \leqq n_i}$ を $\mathfrak{g}_{-i}$ の basis とし, $(e^{i\lambda})$ を $\mathfrak{g}$ の Killing form $\langle\ ,\ \rangle$ に関して $(e_{i\lambda})$ に dual な $\mathfrak{g}_i$ の basis とする. $i > 0$ のとき ∂_i^* は

$$(\partial_i^* c)(x_1 \wedge \cdots \wedge x_q) = \sum_{\lambda=1}^{n_i} [e^{i\lambda}, c(e_{i\lambda} \wedge x_1 \wedge \cdots \wedge x_q)]$$

と表現され operator ∂_0^* は

$$(\partial_0^* c)(x_1 \wedge \cdots \wedge x_q) = \frac{1}{2} \sum_k \sum_{i>0} \sum_{\lambda=1}^{n_i} (-1)^{k+1} c([e^{i\lambda}, x_k]_- \wedge e_{i\lambda} \wedge x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_q)$$

と表現される. こゝで $c \in C^{q+1}(\mathfrak{G})$, $x_k \in \mathfrak{m}$ である. これらの式は [20], Proposition の証明から容易に推測される.

次の Lemma を証明しよう.

Lemma 2.11. $\forall i > 0$ に対して,

$$\sum_{r\geqq 0}(\partial_r^*\partial_{r+i} + \partial_{r+i}\partial_r^*) = 0$$

$c \in C^q$, $x_1, \ldots, x_q \in \mathfrak{m}$ とする.

$$\begin{aligned}
&(\partial_0^*\partial_i c)(x_1 \wedge \cdots \wedge x_q)\\
&= \frac{1}{2}\sum_k \sum_{r>0}\sum_{\sigma=1}^{n_r}(-1)^{k+1}(\partial_i c)([e^{r\sigma}, x_k]_- \wedge e_{r\sigma} \wedge x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_q)\\
&= \frac{1}{2}A + \frac{1}{2}B + C
\end{aligned}$$

こゝで A, B, C は次で与えられる.

$$A = \sum_k \sum_{r>0}\sum_{\sigma=1}^{n_r}(-1)^{k+1}[\zeta_i([e^{r\sigma}, x_k]_-), c(e_{r\sigma} \wedge x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_q)],$$

$$B = \sum_k \sum_{r>0}\sum_{\sigma=1}^{n_r}(-1)^{k}[\zeta_i(e_{r\sigma}), c([e^{r\sigma}, x_k]_- \wedge x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_q)],$$

$$\begin{aligned}
C &= \frac{1}{2}\sum_{l<k} \sum_{r>0}\sum_{\sigma=1}^{n_r}(-1)^{k+l}[\zeta_i(x_l), c([e^{r\sigma}, x_k]_- \wedge e_{r\sigma} \wedge x_1 \wedge \cdots \wedge \hat{x}_l \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_q)]\\
&\quad + \frac{1}{2}\sum_{l>k} \sum_{r>0}\sum_{\sigma=1}^{n_r}(-1)^{k+l+1}[\zeta_i(x_l), c([e_{r\sigma}, x_k]_- \wedge e_{r\sigma} \wedge x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge \hat{x}_l \wedge \cdots \wedge x_q)]
\end{aligned}$$

Lemma 2.11 は, 次の Lemma 2.12 から直ちに得られる.

Lemma 2.12. (1) $B = A$.

(2) $A = -\sum_{r>0}(\partial_{r+i}\partial_r^* c)(x_1 \wedge \cdots \wedge x_q) - \sum_{r>0}(\partial_r^*\partial_{r+i} c)(x_1 \wedge \cdots \wedge x_q)$.

(3) $C = -(\partial_i\partial_0^* c)(x_1 \wedge \cdots \wedge x_q)$.

Proof. (1) $r > 0$ を fix するとき, $x_k = \sum_{r+s>0}\zeta_{r+s}(x_k)$ であり, $[e^{r\sigma}, \zeta_{r+s}(x_k)] \in \mathfrak{g}_{-s}$ であるから,

$$[e^{r\sigma}, x_k]_- = \sum_{s>0}[e^{r\sigma}, \zeta_{r+s}(x_k)]$$

である. そして

$$[e^{r\sigma}, \zeta_{r+s}(x_k)] = \sum_{\tau=1}^{n_s}\langle[e^{r\sigma}, \zeta_{r+s}(x_k)], e^{s\tau}\rangle e_{s\tau} = \sum_{\tau=1}^{n_s}\langle e^{r\sigma}, [\zeta_{r+s}(x_k), e^{s\tau}]\rangle e_{s\tau}$$

故に

$$\begin{aligned}
&\sum_{r>0}\sum_{\sigma=1}^{n_r}[\zeta_i(e_{r\sigma}), c([e^{r\sigma}, x_k]_- \wedge x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_q)]\\
&= \sum_{r>0}\sum_{s>0}\sum_{\tau=1}^{n_s}[\zeta_i([\zeta_{r+s}(x_k), e^{s\tau}]), c(e_{s\tau} \wedge x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_q)]\\
&= -\sum_{s>0}\sum_{\tau=1}^{n_s}[\zeta_i([e^{s\tau}, x_k]_-), c(e_{s\tau} \wedge x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_q)]
\end{aligned}$$

故に $B = A$ をえる．

(2) $\zeta_i([e^{r\sigma}, x_k]_-) = [e^{r\sigma}, \zeta_{r+i}(x_k)]$ であるから，

$$\begin{aligned}
A &= \sum_k \sum_{r>0} \sum_{\sigma=1}^{n_r} (-1)^{k+1}[[e^{r\sigma}, \zeta_{r+i}(x_k)], c(e_{r\sigma} \wedge x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_q)] \\
&= \sum_k \sum_{r>0} \sum_{\sigma=1}^{n_r} (-1)^{k+1}[[e^{r\sigma}, c([e_{r\sigma} \wedge x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_q)], \zeta_{r+i}(x_k)] \\
&+ \sum_k \sum_{r>0} \sum_{\sigma=1}^{n_r} (-1)^{k+1}[e^{r\sigma}, [\zeta_{r+i}(x_k), c(e_{r\sigma} \wedge x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_q)]] \\
&= \sum_k \sum_{r>0} (-1)^{k+1}[(\partial_r^* c)(x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_q), \zeta_{r+i}(x_k)] \\
&- \sum_{r>0} \sum_{\sigma=1}^{n_r} [e^{r\sigma}, (\partial_{r+i} c)(e_{r\sigma} \wedge x_1 \wedge \cdots \wedge x_q)] \\
&= -\sum_{r>0} (\partial_{r+i}\partial_r^* c)(x_1 \wedge \cdots \wedge x_q) - \sum_{r>0} (\partial_r^* \partial_{r+i} c)(x_1 \wedge \cdots \wedge x_q)
\end{aligned}$$

(3) は容易に確かめられる． □

さて space $C^{p,q}$ は $C^{p,q} = \sum_j C_j^{p,q}$ と分解し，この分解に対応して $C^{p,q}$ の各元 c は

$$c = \sum_j c_j$$

と分解する．そして Lemma 2.11 を使って容易に次の Lemma 2.13 をえる．

Lemma 2.13. $c \in C^{p,q}$ のとき，

$$(\Delta c)_k = \sum_{r \geqq 0} (\partial_r^* \partial_r + \partial_r \partial_r^*) c_k + \sum_{i>0} \sum_{r \geqq 0} (\partial_{r+i}^* \partial_r + \partial_r \partial_{r+i}^*) c_{k-i}$$

Proposition 2.10 は，この Lemma 2.13 から容易に得られる．$c \in H^{p,q}(\mathfrak{G})$ とし，$\forall k$ に対して $c_k \in H^{p,q}(\mathfrak{G})$ を示そう．これを k に関する induction で証明する．まず十分小さい k に対して $c_k = 0$ である．そこである k について $c_l \in H^{p,q}(\mathfrak{G})$, $\forall l < k$ と仮定する．この仮定は

$$\partial_i c_l = \partial_i^* c_l = 0, \quad \forall\, i \geqq 0, \quad \forall\, l < k$$

と同等である．よって Lemma 2.13 から

$$\sum_{r \geqq 0} (\partial_r^* \partial_r + \partial_r \partial_r^*) c_k = 0$$

をえる．これより直ちに

$$\partial_r c_k = \partial_r^* c_k = 0, \qquad r \geqq 0$$

をえる．これは $c_k \in H^{p,q}(\mathfrak{G})$ を意味する．よって我々の主張が示され，Proposition 2.10 の証明が完了した．

§3. The spaces of harmonic forms, $H^{p,2}(\mathfrak{G})$, associated with projective graded Lie algebras

3.1. The operators ∂_i and ∂_i^*, $i = 0, 1, 2$, associated with simple graded Lie algebras of the second kind. In this and the subsequent two paragraphs let $\mathfrak{G} = \{\mathfrak{g}, (\mathfrak{g}_p)\}$ be a simple graded Lie algebra of the second kind.

Put $\dim \mathfrak{g}_{-2} = n_2$ and $\dim \mathfrak{g}_{-1} = n_1$. Let $(d_1, \dots, d_{n_2}) = (d_\alpha)$ be a basis of $\mathfrak{g}_{-2}$, and (d^α) the basis of $\mathfrak{g}_2$ dual to (d_α). Let $(e_1, \dots, e_{n_1}) = (e_i)$ be a basis of $\mathfrak{g}_{-1}$, and (e^i) the basis of $\mathfrak{g}_1$ dual to (e_i). Let ζ_2 and ζ_1 be the projections of $\mathfrak{m} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}$ onto $\mathfrak{g}_{-2}$ and $\mathfrak{g}_{-1}$ respectively.

Let us recall the operators $\partial_i : C^q \to C^{q+1}$, $i \geqq 0$, and their adjoint operators $\partial_i^* : C^{q+1} \to C^q$, which are defined in the proof of Proposition 2.10. Clearly we have $\partial_i = \partial_i^* = 0$ for $i > 2$.

The operator ∂_2 is defined by

$$(\partial_2 c)(x_1 \wedge \cdots \wedge x_{q+1}) = \sum_k (-1)^{k+1} [\zeta_2(x_k), c(x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_{q+1})],$$

the operator ∂_1 by

$$(\partial_1 c)(x_1 \wedge \cdots \wedge x_{q+1}) = \sum_k (-1)^{k+1} [\zeta_1(x_k), c(x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_{q+1})],$$

and the operator ∂_0 by

$$(\partial_0 c)(x_1 \wedge \cdots \wedge x_{q+1}) = \sum_{k<l} (-1)^{k+l} c([\zeta_1(x_k), \zeta_1(x_l)] \wedge x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge \hat{x}_l \wedge \cdots \wedge x_{q+1}),$$

where $c \in C^q$ and $x_k \in \mathfrak{m}$. In the last formula, note that $[\zeta_1(x_k), \zeta_1(x_l)] = [x_k, x_l]$.

The operator ∂_2^* is defined by

$$(\partial_2^* c)(x_1 \wedge \cdots \wedge x_q) = \sum_\alpha [d^\alpha, c(d_\alpha, \wedge x_1 \wedge \cdots \wedge x_q)],$$

the operator ∂_1^* by

$$(\partial_1^* c)(x_1 \wedge \cdots \wedge x_q) = \sum_i [e^i, c(e_i \wedge x_1 \wedge \cdots \wedge x_q)],$$

and the operator ∂_0^* by

$$(\partial_0^* c)(x_1 \wedge \cdots \wedge x_q) = \frac{1}{2} \sum_{k,i} (-1)^{k+1} c([e^i, \zeta_2(x_k)] \wedge e_i \wedge x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_q),$$

where $c \in C^{q+1}$ and $x_k \in \mathfrak{m}$. In the last formula, note that $[d^\alpha, x_k]_- = 0$, $[e^i, x_k]_- = [e^i, \zeta_2(x_k)]$.

Clearly the operator $\partial : C^q \to C^{q+1}$ may be expressed as follows:

$$\partial = \partial_2 + \partial_1 + \partial_0,$$

and the adjoint operator $\partial^* : C^{q+1} \to C^q$ of ∂ as follows:

$$\partial^* = \partial_2^* + \partial_1^* + \partial_0^*.$$

Now the space $\Lambda^q \mathfrak{m}^*$ is decomposed as follows:

$$\Lambda^q \mathfrak{m}^* = \sum_{r+s=q} \Lambda^{r,s},$$

where

$$\Lambda^{r,s} = \overbrace{\mathfrak{g}_{-2}^* \wedge \cdots \wedge \mathfrak{g}_{-2}^*}^{r} \wedge \overbrace{\mathfrak{g}_{-1}^* \wedge \cdots \wedge \mathfrak{g}_{-1}^*}^{s}.$$

Therefore putting

$$D_j^{r,s} = \mathfrak{g}_j \otimes \Lambda^{r,s},$$

we see that the space C^q is decomposed as follows:

$$C^q = \sum_j \sum_{r+s=q} D_j^{r,s},$$

Clearly we have

$$\begin{aligned} \partial_2 D_j^{r,s} &\subset D_{j-2}^{r+1,s}, & \partial_2^* D_{j-2}^{r+1,s} &\subset D_j^{r,s}, \\ \partial_1 D_j^{r,s} &\subset D_{j-1}^{r,s+1}, & \partial_1^* D_{j-1}^{r,s+1} &\subset D_j^{r,s}, \\ \partial_0 D_j^{r,s} &\subset D_j^{r-1,s+2}, & \partial_0^* D_j^{r-1,s+2} &\subset D_j^{r,s}. \end{aligned}$$

The space C^q is also decomposed as follows: $C^q = \sum_p C^{p,q}$, and each $C^{p,q}$ as follows: $C^{p,q} = \sum_j C_j^{p,q}$. For any pair (p,q) let (r,s) be the pair such that $r+s=q$, $-2r-s = j-p-q+1$. Then we have

$$C_j^{p,q} = D_j^{r,s},$$

from which follows that

$$\partial_i C_j^{p,q} \subset C_{j-i}^{p-1,q+1}, \qquad \partial_i^* C_{j-i}^{p-1,q+1} \subset C_j^{p,q},$$

where $i = 0,1,2$. Hereafter we shall frequently deal with the spaces $D_j^{r,s}$ instead of $C_j^{p,q}$.

3.2. The spaces of harmonic forms, $H^{p,2}(\mathfrak{G})$, associated with simple graded Lie algebras of the second kind. We first explain some preliminary facts on the graded Lie algebra $\mathfrak{G}$.

Lemma 3.1 (cf. Lemmas 3.1 and 3.2 in [31])**.** *Let ε be 1 or -1.*

(1) $[\mathfrak{g}_\varepsilon, \mathfrak{g}_\varepsilon] = \mathfrak{g}_{2\varepsilon}$, $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$, *and* $[\mathfrak{g}_{-2\varepsilon}, \mathfrak{g}_\varepsilon] = \mathfrak{g}_{-\varepsilon}$.

(2) *If $X \in \mathfrak{g}_\varepsilon$ and $[X, \mathfrak{g}_\varepsilon] = \{0\}$, then $X = 0$. If $X \in \mathfrak{g}_{-2\varepsilon}$ and $[X, \mathfrak{g}_\varepsilon] = \{0\}$, then $X = 0$. If $X \in \mathfrak{g}_\varepsilon$ and $[X, \mathfrak{g}_{-2\varepsilon}] = \{0\}$, then $X = 0$. If $X \in \mathfrak{g}_{2\varepsilon}$ and $[X, \mathfrak{g}_{-2\varepsilon}] = \{0\}$, then $X = 0$.*

We define elements E_2 and E_1 of $\mathfrak{g}_0$ respectively by

$$E_2 = \sum_\alpha [d^\alpha, d_\alpha], \quad \text{and } E_1 = \sum_i [e^i, e_i].$$

As is easily observed, both E_2 and E_1 do not depend on the choices of the bases, and are in the centre of $\mathfrak{g}_0$.

Let $x_j \in \mathfrak{g}_j = D_j^{0,0}$. Then we have

$$\partial_2^* \partial_2 x_p = \sum_\alpha [d^\alpha, [d_\alpha, x_p]], \quad \partial_1^* \partial_1 x_p = \sum_i [e^i, [e_i, x_p]],$$

and we can easily prove the following

Lemma 3.2. $\partial_2^*\partial_2 x_1 = [E_2, x_1]$, $\partial_2^*\partial_2 x_2 = [E_2, x_2]$, $\partial_1^*\partial_1 x_{-1} = -[E_2, x_{-1}]$, $\partial_1^*\partial_1 x_1 = [E_2 + E_1, x_1]$, *and* $\partial_1^*\partial_1 x_2 = [E_1, x_2]$.

Lemma 3.3. *The operator $u \mapsto -[E_1, u]$ of $\mathfrak{g}_{-2}$ to itself is symmetric and positive definite with respect to the inner product $(\cdot, \cdot)$.*

Proof. We may assume that the basis (e_i) of $\mathfrak{g}_{-1}$ is orthonormal, i.e., $(e_i, e_j) = \delta_{ij}$. Then we have $\sigma(e_i) = -e^i$, whence $\sigma(E_1) = -E_1$. It follows that

$$([E_1, u], v) = (u, [E_1, v]), \qquad u, v \in \mathfrak{g}_{-2}.$$

Furthermore we have

$$-([E_1, u], u) = \sum_i ([e^i, u], [e^i, u]) \geqq 0,$$

and if the equality holds, we obtain $[e^i, u] = 0$, whence $u = 0$ (Lemma 3.1). □

We shall now study the spaces $H^{p,2}(\mathfrak{G})$. The space $C^{p,2}$ is decomposed as follows:

$$C^{p,2} = C^{p,2}_{p-3} + C^{p,2}_{p-2} + C^{p,2}_{p-1}.$$

In particular we see that $C^{p,2} = 0$ if $p < -1$ or $p > 5$. We also note that $C^{p,2}_{p-3} = D^{2,0}_{p-3}$, $C^{p,2}_{p-2} = D^{1,1}_{p-2}$, $C^{p,2}_{p-1} = D^{0,2}_{p-1}$. Now by Proposition 2.10 we have

$$H^{p,2}(\mathfrak{G}) = H^{p,2}(\mathfrak{G}) \cap D^{2,0}_{p-3} + H^{p,2}(\mathfrak{G}) \cap D^{1,1}_{p-2} + H^{p,2}(\mathfrak{G}) \cap D^{0,2}_{p-1}.$$

Lemma 3.4. $H^{p,2}(\mathfrak{G}) \cap D^{2,0}_{p-3} = \{0\}$.

Proof. Let $c \in D^{2,0}_{p-3}$. For any $u, v \in \mathfrak{g}_{-2}$ we have

$$\begin{aligned}
(\partial_0^*\partial_0 c)(u \wedge v) &= \frac{1}{2}\sum_i (\partial_0 c)([e^i, u] \wedge e_i \wedge v) - \frac{1}{2}\sum (\partial_0 c)([e^i, v] \wedge e_i \wedge u) \\
&= -\frac{1}{2}\sum_i c([[e^i, u], e_i] \wedge v) + \frac{1}{2}\sum_i c([[e^i, v], e_i] \wedge u) \\
&= -\frac{1}{2} c([E_1, u] \wedge v) + \frac{1}{2} c([E_1, v] \wedge u).
\end{aligned}$$

Thus we see from Lemma 3.3 that $\partial_0^*\partial_0 : D^{2,0}_{p-3} \to D^{2,0}_{p-3}$ is a positive definite operator. Since $\partial_0 c = 0$ for $c \in H^{p,2}(\mathfrak{G}) \cap D^{2,0}_{p-3}$, it follows that $H^{p,2}(\mathfrak{G}) \cap D^{2,0}_{p-3} = \{0\}$. □

Therefore we have the following

Proposition 3.5. (1) $H^{p,2}(\mathfrak{G}) = H^{p,2}(\mathfrak{G}) \cap D^{1,1}_{p-2} + H^{p,2}(\mathfrak{G}) \cap D^{0,2}_{p-1}$.

(2) $H^{p,2}(\mathfrak{G}) \cap D^{1,1}_{p-2}$ *consists of all* $c \in D^{1,1}_{p-2}$ *such that*

$$\partial_2 c = \partial_1 c = \partial_0 c = \partial_1^* c = \partial_2^* c = 0.$$

(3) $H^{p,2}(\mathfrak{G}) \cap D^{0,2}_{p-1}$ *consists of all* $c \in D^{0,2}_{p-1}$ *such that*

$$\partial_2 c = \partial_1 c = \partial_0^* c = \partial_1^* c = 0.$$

Using this proposition, we shall now prove the following proposition. Define an ideal $\mathfrak{h}_0$ of $\mathfrak{g}_0$ by

$$\mathfrak{h}_0 = \{\, X \in \mathfrak{g}_0 \mid [X, \mathfrak{g}_{-2}] = 0 \,\}.$$

Proposition 3.6. (1) $H^{5,2}(\mathfrak{G}) = \{0\}$.

(2) *If* $\mathfrak{G}$ *is the prolongation of* $(\mathfrak{M}, \mathfrak{g}_0)$, *then* $H^{4,2}(\mathfrak{G}) = \{0\}$.

(3) $H^{3,2}(\mathfrak{G}) \subset D^{1,1}_1$, $H^{2,2}(\mathfrak{G}) \subset D^{1,1}_0$, *and* $H^{1,2}(\mathfrak{G}) \cap D^{0,2}_0 \subset \mathfrak{h}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$.

Proof. First of all (1) is clear, because $H^{5,2}(\mathfrak{G}) \subset A^{2,0}_2$.

(2) We have $H^{4,2}(\mathfrak{G}) \subset D^{1,1}_2$. Let $c \in H^{4,2}(\mathfrak{G})$. Then we have $\partial_2 c = 0$, and hence

$$(\partial_2 c)(u \wedge v \wedge x) = [u, c(v \wedge x)] - [v, c(u \wedge x)] = 0,$$

where $u, v \in \mathfrak{g}_{-2}$ and $x \in \mathfrak{g}_{-1}$. Fix an $x \in \mathfrak{g}_{-1}$ and define a linear map $f : \mathfrak{g}_{-2} \to \mathfrak{g}_2$ by $f(u) = c(u \wedge x)$, $u \in \mathfrak{g}_{-2}$. Then we have $f \in C^{4,1} = \mathfrak{g}_2 \otimes \mathfrak{g}^*_{-2} = D^{1,0}_2$ and $\partial f = \partial_2 f = 0$. Since $\mathfrak{G}$ is the prolongation of $(\mathfrak{M}, \mathfrak{g}_0)$, and since $\mathfrak{g}_4 = \{0\}$, we see from Proposition 1.12 that the map $\partial : C^{4,1} \to C^{3,2}$ is injective. Therefore we get $f = 0$ and hence $c = 0$, proving (2).

(3) Let $c \in H^{p,2}(\mathfrak{G}) \cap D^{0,2}_{p-1}$. Then we have $\partial_2 c = 0$ and hence

$$(\partial_2 c)(u \wedge x \wedge y) = [u, c(x \wedge y)] = 0,$$

where $u \in \mathfrak{g}_{-2}$, and $x, y \in \mathfrak{g}_{-1}$. Since $c(x \wedge y) \in \mathfrak{g}_{p-1}$, it follows that if $p = 2$ or 3, then $c(x \wedge y) = 0$ (Lemma 3.1) and if $p = 1$, then $c(x \wedge y) \in \mathfrak{h}_0$, proving (3). □

3.3. The case where $\dim \mathfrak{g}_{-2} = 1$. この paragraph では $\dim \mathfrak{g}_{-2} = 1$ と仮定し, $\dim \mathfrak{g}_{-1} = 2(n-1)$ とおく. ([31] で, このような $\mathfrak{G}$ を simple graded Lie algebra of contact type of degree n とよんだ.) 明らかに

$$\mathfrak{g}_0 = \mathbb{R} \cdot E + \mathfrak{h}_0 \quad \text{(direct sum)}$$

が成り立つ. 以下 $\mathfrak{G}$ は $(\mathfrak{M}, \mathfrak{g}_0)$ の prolongation と仮定する. 次の propositions を証明しよう.

Proposition 3.7. $n \geqq 3$ と仮定する.

(1) $H^{p,2}(\mathfrak{G}) = \{0\}$ *if* $p \neq 0$ *or* 1.

(2) $H^{0,2}(\mathfrak{G}) \subset D^{0,2}_{-1}$.

(3) $H^{1,2}(\mathfrak{G}) \subset \mathfrak{h}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$.

Proposition 3.8. $n = 2$ と仮定する.

(1) $\dim \mathfrak{h}_0 = 1$ であり, $\mathfrak{h}_0$ の *basis* A があって, $[A, [A, x]] = x$, $\forall x \in \mathfrak{g}_{-1}$ かまたは $[A, [A, x]] = -x$, $\forall x \in \mathfrak{g}_{-1}$ を満足する.

(2) $H^{p,2}(\mathfrak{G}) = \{0\}$ *if* $p \neq 3$.

(3) $H^{3,2}(\mathfrak{G})$ *consists of all* $c \in A^{1,1}_1$ *satisfying the equation*

$$[A, c(u \wedge v)] + c(u \wedge [A, x]) = 0, \quad u \in \mathfrak{g}_{-2}, \ x \in \mathfrak{g}_{-1}.$$

$\mathfrak{g}_0$ の centre の元 $E_2 = [d^1, d_1]$, $E_1 = \sum_i [e^i, e_i]$ を考える.

Lemma 3.9 ([31, Lemma 5.2])**.** (1) $E_2 = \lambda_2 E$, *where* $\lambda_2 = \dfrac{1}{2(n+1)}$.

(2) $E_1 = \lambda_1 E$, *where* $\lambda_1 = \dfrac{n-1}{2(n+1)}$.

まず $n \geqq 3$ と仮定する.

Lemma 3.10. $H^{p,2}(\mathfrak{G}) \subset D^{0,2}_{p-1}$.

Proof. $c \in H^{p,2}(\mathfrak{G}) \cap D^{1,1}_{p-2}$ とする. このとき $\partial_0 c = 0$ である. $\partial_0^* \partial_0 c$ を計算しよう. $x \in \mathfrak{g}_{-1}$ とする.

$$\begin{aligned}(\partial_0^* \partial_0 c)(d_1 \wedge x) &= \frac{1}{2} \sum_i (\partial_0 c)([e^i, d_1] \wedge e_i \wedge x) \\ &= -\frac{1}{2} \sum_i c([[e^i, d_1], e_i] \wedge x) + \frac{1}{2} \sum_i c([[e^i, d_1], x] \wedge e_i) - \frac{1}{2} \sum_i c([e_i, x] \wedge [e^i, d_1]).\end{aligned}$$

$\sum_i [[e^i, d_1], e_i] = [E_1, d_1]$ であり, $[e^i, d_1] = \sum_j \langle e^i, [d_1, e^j] \rangle e_j$, $[e^i, x] = \langle e^i, [x, d^1] \rangle d_1$ だから

$$\begin{aligned}(\partial_0^* \partial_0 c)(d_1 \wedge x) &= -\frac{1}{2} c([E_1, d_1] \wedge x) - \sum_i c([e_i, x] \wedge [e^i, d_1]) \\ &= -\frac{1}{2} c([E_1, d_1] \wedge x) - c(d_1 \wedge [[x, d^1], d_1]) \\ &= -\frac{1}{2} c([E_1, d_1] \wedge x) - c(d_1 \wedge [x, E_2])\end{aligned}$$

をえる．よって, Lemma 3.9 を使って

$$\partial_0^*\partial_0 c = (\lambda_1 - \lambda_2)c = 0$$

をえる．$n \geqq 3$ だから, $\lambda_1 - \lambda_2 > 0$. よって $c = 0$ をえる．よって Lemma 3.10 が示された． □

Lemma 3.11. $H^{-1,2}(\mathfrak{G}) = \{0\}$.

Proof. $H^{-1,2}(\mathfrak{G}) \subset D^{0,2}_{-2}$ である．$c \in H^{-1,2}(\mathfrak{G})$ とすれば, $\partial_1^* c = 0$. よって, $\forall x \in \mathfrak{g}_{-1}$ に対して,

$$(\partial_1^* c)(x) = \sum_i [e^i, c(e_i \wedge x)] = \sum_i \langle c(e_i \wedge x), d^1\rangle [e^i, d_1] = 0$$

をえる．$([e^i, d_1])$ は linearly independent であるから (Lemma 3.1), $\langle c(e_i \wedge x), d^1\rangle = 0$, i.e., $c = 0$ をえる． □

Lemmas 3.10 and 3.11 と Proposition 3.6 から直ちに Proposition 3.7 をえる．

次に Proposition 3.8 を証明しよう．$\mathfrak{g}_0 \subset \mathfrak{gl}(\mathfrak{g}_{-1})$ をみなせるから, $\dim \mathfrak{g}_0 \leqq 4$. $\dim \mathfrak{g}_0 = 4$ とすれば $(\mathfrak{M}, \mathfrak{g}_0)$ の prolongation は contact graded algebra of degree 2 であり, 無限次元である．($\dim \mathfrak{g}_0 = 4$ のとき, $\mathfrak{G}$ は projective contact graded Lie algebra of degree 2 である.) よって $\dim \mathfrak{g}_0 \leqq 3$. $E \in \mathfrak{g}_0$ だから, $1 \leqq \dim \mathfrak{g}_0 \leqq 3$. よって $7 \leqq \dim \mathfrak{g} \leqq 9$. $\mathfrak{g}$ は simple だから, $\dim \mathfrak{g} = 8$. よって $\dim \mathfrak{g}_0 = 2$. よって $\dim \mathfrak{h}_0 = 1$ である．A を $\mathfrak{h}_0$ の basis とする．$\sigma\mathfrak{h}_0 = \mathfrak{h}_0$ だから, $\sigma A = \pm A$ である．($\langle \sigma A, \sigma A\rangle = \langle A, A\rangle \neq 0$). $\sigma A = -A$ とする．

$$\begin{aligned}&([A, x], y) = (x, [A, y])\\ &[[A, x], y] + [x, [A, y]] = 0, \quad x, y \in \mathfrak{g}_{-1}\end{aligned}$$

だから $\mathfrak{h}_0$ の basis A が存在して, $\mathfrak{g}_{-1}$ の endomorphism $x \mapsto [A, x]$ の eigenvalue は 1 or -1 となる．よって $[A, [A, x]] = x$, $x \in \mathfrak{g}_{-1}$. 次に $\sigma A = A$ とする．このとき

$$([A, x], y) = -(x, [A, y]), \qquad x, y \in \mathfrak{g}_{-1}$$

よって, $\mathfrak{h}_0$ の basis A が存在して, $[A, [A, x]] = -x$, $x \in \mathfrak{g}_{-1}$ となる．

Lemma 3.12. $H^{p,2}(\mathfrak{G}) \subset D^{1,1}_{p-2}$.

Proof. $c \in H^{p,2}(\mathfrak{G}) \cap D^{0,2}_{p-2}$ とする．$\partial_0^* c = 0$ であるから

$$(\partial_0^* c)(d_1) = \frac{1}{2}\sum_i c([e^i, d_1] \wedge e_i) = \frac{1}{2}c([e^1, d_1] \wedge e_1) + \frac{1}{2}c([e^2, d_1] \wedge e_2) = 0.$$

$\lambda = \langle [e^1, d_1], e^2\rangle = \langle [e^2, e^1], d_1\rangle$ とおけば, $\lambda \neq 0$ かつ $[e^1, d_1] = \lambda e_2$, $[e^2, d_1] = -\lambda e_1$ である．よって $-\lambda c(e_1 \wedge e_2) = 0$. よって $c = 0$ をえる． □

Lemma 3.13. $H^{p,2}(\mathfrak{G}) = \{0\}, \quad p = 0, 1, 2.$

Proof. $c \in H^{p,2}(\mathfrak{G})$ とする．$\partial_2^* c = 0$ だから

$$(\partial_2^* c)(x) = [d^1, c(d_1 \wedge x)] = 0, \qquad x \in \mathfrak{g}_{-1}$$

をえる．よって $p = 0, 1$ のとき, $c = 0$ をえる．$p = 2$ と仮定する．このとき $\partial_1 c = 0$ であるから

$$(\partial_1 c)(d_1 \wedge x \wedge y) = -[x, c(d_1 \wedge y)] + [y, c(d_1 \wedge x)] = 0$$

をえる．こゝで $x, y \in \mathfrak{g}_{-1}$. $c(d_1 \wedge y) \in \mathfrak{h}_0$ であるから容易に $c = 0$ をえる．(cf. [20], Lemma 1.4) □

Lemmas 3.11 and 3.13 と Propositions 3.7 and 3.8 から, $H^{p,2}(\mathfrak{G}) = \{0\}$ if $p \neq 3$ 及び $H^{3,2}(\mathfrak{G}) = \{\, c \in D_1^{1,1} \mid \partial_1 c = \partial_1^* c = 0 \,\}$ が分る.

$c \in D_1^{1,1}$ とする. $\partial_1 c = 0$ なるための必要十分条件は

$$\begin{cases} \langle(\partial_1 c)(d_1 \wedge x \wedge y), E\rangle = \langle x, c(d_1 \wedge y)\rangle - \langle y, c(d_1 \wedge x)\rangle = 0, \\ \langle(\partial_1 c)(d_1 \wedge x \wedge y), A\rangle = -\langle [A, x], c(d_1 \wedge y)\rangle + \langle [A, y], c(d_1 \wedge x)\rangle = 0, \end{cases}$$

である. こゝで, $x, y \in \mathfrak{g}_{-1}$. 容易にこの条件は

$$[A, c(d_1 \wedge x)] + c(d_1 \wedge [A, x]) = 0, \qquad x \in \mathfrak{g}_{-1}$$

に equivalent であることがわかる. 次に $\partial_1 c = 0$ から $\partial_1^* c = 0$ が導かれることを示そう. 実際

$$\begin{aligned} \langle d_1, (\partial_1^* c)(d_1)\rangle &= \langle [d_1, e^1], c(e_1 \wedge d_1)\rangle + \langle [d_1, e^2], c(e_2 \wedge d_1)\rangle \\ &= -\lambda\langle e_2, c(e_1 \wedge d_1)\rangle + \lambda\langle e_1, c(e_2 \wedge d_1)\rangle \\ &= \lambda\langle(\partial_1 c)(d_1 \wedge e_1 \wedge e_2), E\rangle = 0. \end{aligned}$$

よって主張をえる. (λ は Lemma 3.12 のもの)

3.4. The spaces of harmonic forms, $H^{p,2}(\mathfrak{G})$, associated with projective graded Lie algebras of type $(n, 1)$, $n \geqq 3$[3]. Let $\mathfrak{L}$ be a projective FGLA of type (n, r), and $\mathfrak{G}$ its prolongation. We first explain some preliminary facts on the graded Lie algebra $\mathfrak{G}$.

Lemma 3.14. *Let ε be 1 or -1. Let δ be $+$ or $-$, and let $-\delta$ be $-$ or $+$ according as δ is $+$ or $-$.*

(1) $[\mathfrak{g}_0, \mathfrak{g}_\varepsilon^\delta] = \mathfrak{g}_\varepsilon^\delta$, $\quad [\mathfrak{g}_\varepsilon^\delta, \mathfrak{g}_{-2\varepsilon}] = \mathfrak{g}_{-\varepsilon}^\delta$, $\quad [\mathfrak{g}_\varepsilon^\delta, \mathfrak{g}_\varepsilon^\delta] = [\mathfrak{g}_1^\delta, \mathfrak{g}_{-1}^\delta] = \{0\}$.

(2) *If $X \in \mathfrak{g}_\varepsilon^\delta$ and $[X, \mathfrak{g}_{-2\varepsilon}] = \{0\}$, then $X = 0$. If $X \in \mathfrak{g}_{-2\varepsilon}$ and $[X, \mathfrak{g}_\varepsilon^\delta] = \{0\}$, then $X = 0$. If $X \in \mathfrak{g}_\varepsilon^\delta$ and $[X, \mathfrak{g}_\varepsilon^{-\delta}] = \{0\}$, then $X = 0$. If $X \in \mathfrak{g}_\varepsilon^\delta$ and $[X, \mathfrak{g}_{-\varepsilon}^{-\delta}] = \{0\}$, then $X = 0$.*

(3) *The spaces $\mathfrak{g}_1^\delta$ and $\mathfrak{g}_{-1}^\delta$ are mutually orthogonal with respect to the Killing form $\langle\ ,\ \rangle$, and hence the bilinear function $\mathfrak{g}_1^\delta \times \mathfrak{g}_{-1}^{-\delta} \ni (X, Y) \mapsto \langle X, Y\rangle \in \mathbb{R}$ is non-degenerate.*

Hereafter we assume that $r = 1$ and $n \geqq 3$.

We know that the Lie algebra $\mathfrak{g}_0$ is naturally isomorphic with the Lie algebra $\mathfrak{gl}(\mathfrak{g}_{-1}^+) + \mathfrak{gl}(\mathfrak{g}_{-1}^-)$. Hence the centre $\mathfrak{c}_0$ of $\mathfrak{g}_0$ is 2-dimensional, and is spanned by E and J. We denote by $\mathfrak{s}_0$ the ideal of $\mathfrak{g}_0$ corresponding to the simple Lie algebra $\mathfrak{sl}(\mathfrak{g}_{-1}^-)$. Then we have

$$\mathfrak{g}_0 = \mathfrak{c}_0 + \mathfrak{s}_0 \quad \text{(direct sum)}, \qquad [\mathfrak{s}_0, \mathfrak{g}_{-1}^+] = [\mathfrak{s}_0, \mathfrak{g}_1^-] = \{0\}.$$

We also note that $\mathfrak{s}_0$ coincides with the ideal of $\mathfrak{g}_0$ orthogonal to $\mathfrak{c}_0$ with respect to the Killing form $\langle\ ,\ \rangle$. Hence

$$\mathfrak{s}_0 = \{\, X \in \mathfrak{g}_0 \mid \langle X, E\rangle = \langle X, J\rangle = 0 \,\}.$$

Let e_0 be a basis of $\mathfrak{g}_{-1}^+$, and e^0 the basis of $\mathfrak{g}_1^-$ dual to e_0 (with respect to the Killing form $\langle\ ,\ \rangle$). Let $(e_1, \ldots, e_{n-1}) = (e_i)$ be a basis of $\mathfrak{g}_{-1}^-$, and (e^i) the basis of $\mathfrak{g}_1^+$ dual to (e_i). Putting

$$d_i = [e_i, e_0],$$

we see that (d_i) is a basis of $\mathfrak{g}_{-2}$. Let (d^i) be the basis of $\mathfrak{g}_2$ dual to (d_i).

In the following the indices i, j, k will range over the integers $1, \ldots, n-1$.

[3] このタイトルは手稿では projective に pseudo- が付加されていた。しかし Chapter III を通して "projective" が Chapter I, II における "pseudo-projective" と同じ意味で使われており、このままとした。

Let us consider the elements E_2 and E_1 in the centre of $\mathfrak{g}_0$ (see §**). We newly define elements E_1^+ and E_1^- of $\mathfrak{g}_0$ respectively by

$$E_1^+ = [e^0, e_0], \quad \text{and} \quad E_1^- = \sum_i [e^i, e_i].$$

Clearly we have

$$E_1 = E_1^+ + E_1^-.$$

Lemma 3.15. (1) *$E_2 = \lambda_2 E + \mu_2 J$, where $\lambda_2 = \dfrac{n}{4(n+1)}$ and $\mu_2 = \dfrac{-(n-2)}{4(n+1)}$.*

(2) *$E_1^+ = \lambda_1^+ E + \mu_1^+ J$, where $\lambda_1^+ = \dfrac{1}{4(n+1)}$ and $\mu_1^+ = \dfrac{-3}{4(n+1)}$.*

(3) *$E_1^- = \lambda_1^- E + \mu_1^- J$, where $\lambda_1^- = \dfrac{1}{4(n+1)}$ and $\mu_1^- = \dfrac{2n-1}{4(n+1)}$.*

(4) *$E_1 = \lambda_1 E + \mu_1 J$, where $\lambda_1 = \dfrac{1}{2(n+1)}$ and $\mu_1 = \dfrac{n-2}{2(n+1)}$.*

Lemma 3.16. *$[e_0, d^i] = e^i$, $[d^i, e_j] = \delta^i_j e^0$, $[e^0, d_i] = (\lambda_1^+ + \mu_1^+)e_i$, $[e^i, e^0] = (\lambda_1^+ + \mu_1^+)d^i$, and $[d_i, e^j] = (\lambda_1^+ + \mu_1^+)\delta_i^j e_0$.*

Finally we mention that all the results in this paragraph can be verified directly or by the use of the matricial representation (or briefly the M.R.) of $\mathfrak{G}$. We shall also use some facts which can be easily obtained from the M.R. of $\mathfrak{G}$.

We shall now study the spaces $H^{p,2}(\mathfrak{G})$. (We are assuming that $r=1$, $n \geqq 3$).

$\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^+ + \mathfrak{g}_{-1}^-$ であるから $D_p^{1,1}$, $D_p^{0,2}$ はそれぞれ

$$D_p^{1,1} = \mathfrak{g}_p \otimes \mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^+)^* + \mathfrak{g}_p \otimes \mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^-)^*,$$
$$D_p^{0,2} = \mathfrak{g}_p \otimes (\mathfrak{g}_{-1}^+)^* \wedge (\mathfrak{g}_{-1}^-)^* + \mathfrak{g}_p \otimes \Lambda^2(\mathfrak{g}_{-1}^-)^*$$

と分解する. 特に $p=1$ or -1 のときには, $\mathfrak{g}_p = \mathfrak{g}_p^+ + \mathfrak{g}_p^-$ であるから, 右辺の 4 つの spaces はそれぞれさらに 2 つの spaces に分解する.

さて Proposition 3.5 によって, $H^{p,2}(\mathfrak{G})$ は

$$H^{p,2}(\mathfrak{G}) = H^{p,2}(\mathfrak{G}) \cap D_{p-2}^{1,1} + H^{p,2}(\mathfrak{G}) \cap D_{p-1}^{0,2}$$

と分解される. 任意の p に対して, $D_{p-2}^{1,1}$ または $D_{p-1}^{0,2}$ の subspace $V^{p,2}(n,1)$ を次のように定義する.

(i) $V^{p,2}(n,1) = 0$ if $p \leqq -2$ or $p \geqq 3$

(ii) $V^{-1,2}(n,1)$ consists of all $c \in \mathfrak{g}_{-2} \otimes \Lambda^2(\mathfrak{g}_{-1}^-)^*$ such that

$$\sum_i \langle d^i, c(e_i \wedge x)\rangle = 0, \quad x \in \mathfrak{g}_{-1}^-,$$

or in other words, the trace of the endomorphism $u \mapsto c([u, e^0] \wedge x)$ of $\mathfrak{g}_{-2}$ vanishes for any $x \in \mathfrak{g}_{-1}^-$. (Note that $[e^0, d_i] = (\lambda_1^+ + \mu_1^+)e_i$ and $\lambda_1^+ + \mu_1^+ \neq 0$ (Lemma 3.15).

(iii)

$$V^{0,2}(n,1) = \begin{cases} 0 & \text{if} \quad n \geqq 4 \\ \mathfrak{g}_{-1}^+ \otimes \Lambda^2(\mathfrak{g}_{-1}^-)^* & \text{if} \quad n = 3 \end{cases}$$

(iv) $V^{1,2}(n,1)$ consists of all $c \in \mathfrak{g}_{-1}^- \otimes \mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^+)^*$ such that

$$\sum_i \langle e^i, c(d_i \wedge e_0)\rangle = 0,$$

or in other words, the trace of the endomorphism $x \mapsto c([x, e_0] \wedge e_0)$ of $\mathfrak{g}_{-1}^-$ vanishes.

(v) $V^{2,2}(n,1)$ consists of all $c \in \mathfrak{s}_0 \otimes \mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^-)^*$ such that $[c([x,e_0]\wedge y), z]$ is symmetric with respect to the variables $x, y, z \in \mathfrak{g}_{-1}^-$.

The being prepared, we shall prove the following

Proposition 3.17. $H^{p,2}(\mathfrak{G}) = V^{p,2}(n,1)$ *for all* p.

Remark. $\mathfrak{G}$ を projective graded Lie algebra of type $(n, n-1)$ とする．$n=2$ のときには，Proposition 3.8 によって spaces $H^{p,2}(\mathfrak{G})$ は既に計算されている．$n \geqq 3$ のときも，Proposition 3.17 を使って容易に計算できる．詳細は読者に任せる．

Proposition 3.17 の証明に入る前に記号についての注意を行う．$p = 1$ or -1 とする．このとき spaces $D_p^{r,s}$ は，

$$D_p^{r,s} = \mathfrak{g}_p^+ \otimes \Lambda^{r,s} + \mathfrak{g}_p^- \otimes \Lambda^{r,s}$$

と分解し，この分解に応じて，$D_p^{r,s}$ の各元 c は

$$c = c^+ + c^-$$

と分解する．

以下我々は $H^{p,2}(\mathfrak{G}) \subset V^{p,2}(n,1)$ を示す．逆向きの inclusion は容易に示される．

3.4.1. The space $H^{-1,2}(\mathfrak{G})$.

Proposition 3.5 によって $H^{-1,2}(\mathfrak{G})$ は $D_{-2}^{0,2}$ の元 c で，

$$\partial_0^* c = \partial_1^* c = 0$$

を満足するもの全体からなる．$c \in H^{-1,2}(\mathfrak{G})$ とする．このとき $\forall x \in \mathfrak{g}_{-1}^-$ に対して，

$$(\partial_1^* c)(x) = [e^0, c(e_0 \wedge x)] + \sum_i [e^i, c(e_i \wedge x)] = 0$$

$[e^0, c(e_0 \wedge x)] \in \mathfrak{g}_{-1}^-$, $\sum_i [e^i, c(e_i \wedge x)] \in \mathfrak{g}_{-1}^+$ だから，

$$[e^0, c(e_0 \wedge x)] = \sum_i [e^i, c(e_i \wedge x)] = 0$$

をえる．これよりまず $c(e_0 \wedge x) = 0$ をえる (Lemma 3.14)．よって $c \in \mathfrak{g}_{-2} \otimes \Lambda^2(\mathfrak{g}_{-1}^-)^*$ である．さらに Lemma 3.16 を使って

$$\sum_i [e^i, c(e_i \wedge x)] = \sum_{i,j} \langle c(e_i \wedge x), d^j \rangle [e^i, d_j] = -(\lambda_1^+ + \mu_1^+) \sum_i \langle c(e_i \wedge x), d^i \rangle e_0 = 0$$

をえる．$\lambda_1^+ + \mu_1^+ > 0$ だから，$\sum_i \langle c(e_i \wedge x), d^i \rangle = 0$ をえる．故に $c \in V^{-1,2}(n,1)$．以上によって $H^{-1,2}(\mathfrak{G}) \subset V^{-1,2}(n,1)$ が示された．

3.4.2. The space $H^{0,2}(\mathfrak{G})$.

Proposition 3.5 によって $H^{0,2}(\mathfrak{G}) \cap D_{-2}^{1,1}$ は $D_{-2}^{1,1}$ の元 c で

$$\partial_0 c = \partial_1^* c = \partial_2^* c = 0$$

を満足するもの全体からなり，$H^{0,2}(\mathfrak{G}) \cap D_{-1}^{0,2}$ は $D_{-1}^{0,2}$ の元 c で

$$\partial_1 c = \partial_0^* c = \partial_1^* c = 0$$

を満足するもの全体からなる．

まず $H^{0,2}(\mathfrak{G}) \subset D_{-1}^{0,2}$ を示そう．$c \in H^{0,2}(\mathfrak{G}) \cap D_{-2}^{1,1}$ をとる．このとき $\forall x \in \mathfrak{g}_{-1}$ に対して

$$(\partial_2^* c)(x) = \sum_i [d^i, c(d_i \wedge x)] = 0$$

をえる．よって $c(d_i \wedge x) = 0$, i.e., $c = 0$ をえる (Consider the M.R. of $\mathfrak{G}$).

$c \in H^{0,2}(\mathfrak{G})$ とする. $\forall x \in \mathfrak{g}_{-1}$ に対して,

$$(\partial_1^* c)(x) = [e^0, c^+(e_0 \wedge x)] + \sum_i [e^i, c^-(e_i \wedge x)] = 0$$

これより $c^+(e_0 \wedge x) = c^-(e_i \wedge x) = 0$ をえる. (Consider the M.R. of $\mathfrak{G}$.) これは, $c \in \mathfrak{g}_{-1}^+ \otimes \Lambda^2(\mathfrak{g}_{-1}^-)^*$ を意味する. $\forall x, y \in \mathfrak{g}_{-1}^-$ に対して, $\gamma(x \wedge y) = \langle c(x \wedge y), e^0 \rangle$ とおく. $\partial_1 c = 0$ であるから

$$(\partial_1 c)(x \wedge y \wedge z) = \gamma(y \wedge z)[x, e_0] - \gamma(x \wedge z)[y, e_0] + \gamma(x \wedge y)[z, e_0] = 0$$

をえる. こゝで $x, y, z \in \mathfrak{g}_{-1}^-$. よって

$$\gamma(y \wedge z)x - \gamma(x \wedge z)y + \gamma(x \wedge y)z = 0$$

これより容易に $(n-3)\gamma = 0$ をえる. よって $n \geqq 4$ のとき, $\gamma = 0$, i.e., $c = 0$ をえる. 以上で $H^{0,2}(\mathfrak{G}) \subset V^{0,2}(n, 1)$ が示された.

3.4.3. The space $H^{1,2}(\mathfrak{G})$. Proposition 3.5 によって $H^{1,2}(\mathfrak{G}) \cap D_{-1}^{1,1}$ は $D_{-1}^{1,1}$ の元 c で

$$\partial_1 c = \partial_0 c = \partial_1^* c = \partial_2^* c = 0$$

を満足するもの全体からなり, $H^{1,2}(\mathfrak{G}) \cap D_0^{0,2}$ は $\mathfrak{h}_0 \otimes \Lambda^2(\mathfrak{g}_{-1})^*$ の元 c で

$$\partial_1 c = \partial_0^* c = \partial_1^* c = 0$$

を満足するもの全体からなる.

まず $H^{1,2}(\mathfrak{G}) \subset D_{-1}^{1,1}$ を示す. $c \in H^{1,2}(\mathfrak{G}) \cap D_0^{0,2}$ とする. 容易にわかるように $\mathfrak{h}_0$ は J により張られる. よって $\forall x, y \in \mathfrak{g}_{-1}$ に対して, $c(x \wedge y)$ は, $c(x \wedge y) = \gamma(x \wedge y)J$ with $\gamma(x \wedge y) \in \mathbb{R}$ と表される. よって $\forall x \in \mathfrak{g}_{-1}$ に対して

$$(\partial_1^* c)(x) = \gamma(e_0 \wedge x)e^0 - \sum_i \gamma(e_i \wedge x)e^i$$

をえる. よって $\gamma(e_0 \wedge x) = \gamma(e_i \wedge x) = 0$, i.e., $\gamma = 0$ をえる. 故に $c = 0$. よって $H^{1,2}(\mathfrak{G}) \subset D_{-1}^{1,1}$.

Lemma 3.18. *$c \in H^{1,2}(\mathfrak{G})$ とする.*

(1) *$c^+(\mathfrak{g}_{-2} \wedge \mathfrak{g}_{-1}^+) = 0$, and $c^-(\mathfrak{g}_{-2} \wedge \mathfrak{g}_{-1}^-) = 0$.*
(2) *$c \in \mathfrak{g}_{-1}^- \otimes \mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^+)^*$.*
(3) *$c \in V^{1,2}(n, 1)$.*

Proof. (1) $\partial_1^* c = 0$ だから, $\forall u \in \mathfrak{g}_{-2}$ に対して,

$$(\partial_1^* c)(u) = [e^0, c^+(e_0 \wedge u)] + \sum_i [e^i, c^-(e_i \wedge u)] = 0$$

をえる. これより直ちに $c^+(e_0 \wedge u) = c^-(e_i \wedge u) = 0$ をえる (Consider the M.R. of $\mathfrak{G}$). よって (1) が従う.

(2) $\forall u \in \mathfrak{g}_{-2}$ と $\forall x \in \mathfrak{g}_{-1}^-$ に対して, $(\partial_1^* \partial_1 c)(u \wedge x)$ は

$$(\partial_1^* \partial_1 c)(u \wedge x) = \partial_1^* \partial_1(c(u \wedge x)) + \sum_{\lambda=0}^{n} [e^\lambda, [x, c(e_\lambda \wedge u)]]$$

と表現される. $\partial_1^* \partial_1 c = 0$ であるから, この式と (1) から

$$\partial_1^* \partial_1(c^+(u \wedge x)) + \sum_i [e^i, [x, c^+(e_i \wedge u)]] = 0$$

がわかる. Lemma 3.2 から

$$\partial_1^* \partial_1(c^+(u \wedge x)) = -[E_2, c^+(u \wedge x)] = (\lambda_2 - \mu_2)c^+(u \wedge x)$$

がわかり, Lemma 3.16 を使って

$$\sum_i [e^i, [x, c^+(e_i \wedge u)]] = \sum_{i,j} \langle x, e^j \rangle \langle c^+(e_i \wedge u), e^0 \rangle [e^i, [e_j, e_0]] = (\lambda_1^+ + \mu_1^+) c^+(u \wedge x)$$

がわかる．よって $(\lambda_2 - \mu_2 + \lambda_1^+ + \mu_1^+) c^+(u \wedge x) = 0$. $\lambda_2 - \mu_2 + \lambda_1^+ + \mu_1^+ > 0$ だから $c^+(u \wedge x) = 0$ をえる．この事実と (1) とから $c \in \mathfrak{g}_{-1}^- \otimes \mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^+)^*$ がわかる．

(3) $\partial_2^* c = 0$ だから, Lemma 3.16 を使って

$$(\partial_2^* c)(e_0) = \sum_i [d^i, c(d_i \wedge e_0)] = \sum_{i,j} \langle c(d_i \wedge e_0), e^j \rangle [d^i, e_j] = \sum_i \langle c(d_i \wedge e_0), e^i \rangle e^0 = 0$$

がわかる．よって $\sum_i \langle c(d_i \wedge e_0), e^i \rangle = 0$. よって, $c \in V^{1,2}(n, 1)$ が示された． □

3.4.4. The space $H^{2,2}(\mathfrak{G})$. Propositions 3.5 and 3.6 によって $H^{2,2}(\mathfrak{G})$ は $D_0^{1,1}$ の元 c で

$$\partial_2 c = \partial_1 c = \partial_0 c = \partial_1^* c = \partial_2^* c = 0$$

を満足するもの全体からなる．

Lemma 3.19. $c \in H^{2,2}(\mathfrak{G})$ とする．

(1) $c \in \mathfrak{s}_0 \otimes (\mathfrak{g}_{-2})^* \wedge (\mathfrak{g}_{-1})^*$,

(2) $c \in V^{2,2}(n, 1)$.

Proof. (1) $\mathfrak{s}_0 = \{ X \in \mathfrak{g}_0 \mid \langle X, E \rangle = \langle X, J \rangle = 0 \}$ だから, $\forall u \in \mathfrak{g}_{-2}$, $\forall x \in \mathfrak{g}_{-1}$ に対して, $\langle c(u \wedge x), E \rangle = \langle c(u \wedge x), J \rangle = 0$ を示せばよい．$\partial_1 c = \partial_1^* c = 0$ だから,

$$\begin{aligned}(\partial_1^* \partial_1 c)(u \wedge x) &= \partial_1^* \partial_1 (c(u \wedge x)) + \sum_{\lambda=0}^n [[e^\lambda, x], c(e_\lambda \wedge u)] + [x, (\partial_1^* c)(u)] \\ &= \partial_1^* \partial_1 (c(u \wedge x)) + \sum_{\lambda=0}^n [[e^\lambda, x], c(e_\lambda \wedge u)] = 0\end{aligned}$$

をえる．Lemma 3.15 を使って

$$\begin{aligned}\langle \partial_1^* \partial_1 (c(u \wedge x)), E \rangle &= \sum_{\lambda=0}^n \langle [e^\lambda, [e_\lambda, c(u \wedge x)]], E \rangle \\ &= \langle c(u \wedge x), E_1 \rangle = \lambda_1 \langle c(u \wedge x), E \rangle + \mu_1 \langle c(u \wedge x), J \rangle\end{aligned}$$

をえる．また J は $\mathfrak{g}_0$ の centre の元だから

$$\langle \sum_{\lambda=0}^n [[e^\lambda, x], c(e_\lambda \wedge u)], E \rangle = 0.$$

よって

$$\langle (\partial_1^* \partial_1 c)(u \wedge x), E \rangle = \lambda_1 \langle c(u \wedge x), E \rangle + \mu_1 \langle c(u \wedge x), J \rangle = 0$$

をえる．同様にして

$$\begin{aligned}&\langle (\partial_1^* \partial_1 c)(u \wedge x), J \rangle = \langle c(u \wedge x), -E_1^+ + E_1^- \rangle \\ &= (-\lambda_1^+ + \lambda_1^-) \langle c(u \wedge x), E \rangle + (-\mu_1^+ + \mu_1^-) \langle c(u \wedge x), J \rangle = \frac{1}{2} \langle c(u \wedge x), J \rangle = 0\end{aligned}$$

をえる．以上から $\langle c(u \wedge x), E \rangle = \langle c(u \wedge x), J \rangle = 0$ をえる．よって (1) が示された．

(2) $\forall u \in \mathfrak{g}_{-2}$, $\forall x \in \mathfrak{g}_{-1}^-$ に対して

$$(\partial c)(u \wedge x \wedge e_0) = -[x, c(u \wedge e_0)] + [e_0, c(u \wedge x)] = 0$$

$c(u \wedge x) \in \mathfrak{s}_0$ だから, $[e_0, c(u \wedge x)] = 0$. よって

$$[x, c(u \wedge e_0)] = 0.$$

よって $c(u \wedge e_0) = 0$ をえる. 故に $c \in \mathfrak{s}_0 \otimes \mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^-)^*$ が示された. $\forall u \in \mathfrak{g}_{-2}$, $\forall x, y \in \mathfrak{g}_{-1}^-$ に対して,

$$(\partial_1 c)(u \wedge x \wedge y) = -[x, c(u \wedge y)] + [y, c(u \wedge x)] = 0,$$
$$(\partial_0 c)(x \wedge y \wedge e_0) = c([x, e_0] \wedge y)] - c([y, e_0] \wedge x)] = 0$$

をえる. これより $[c([x, e_0] \wedge y), z]$ が, $x, y, z \in \mathfrak{g}_{-1}^-$ について symmetric であることがわかる. よって $c \in V^{2,2}(n, 1)$ が示された. □

3.4.5. The space $H^{3,2}(\mathfrak{G})$. Propositions 3.5 and 3.6 によって $H^{3,2}(\mathfrak{G})$ は $D_1^{1,1}$ の元 c で

$$\partial_2 c = \partial_1 c = \partial_0 c = \partial_1^* c = 0$$

を満足するもの全体からなる.

Lemma 3.20. $c \in H^{3,2}(\mathfrak{G})$ とする.

(1) $c \in \mathfrak{g}_1^+ \otimes (\mathfrak{g}_{-2})^* \wedge (\mathfrak{g}_{-1}^-)^*$.

(2) $c = 0$.

Proof. (1) $\forall u, v \in \mathfrak{g}_{-2}$, $x \in \mathfrak{g}_{-1}$ に対して,

$$(\partial_2 c)(u \wedge v \wedge x) = [u, c(v \wedge x)] - [v, c(u \wedge x)] = 0$$

をえる. これより

$$[u, c^-(v \wedge x)] - [v, c^-(u \wedge x)] = 0$$

をえる. $a_j = \langle c^-(d_j \wedge x), e_0 \rangle$ とおけば, $c^-(d_j \wedge x) = a_j e^0$. よって Lemma 3.16 を使って,

$$[d_i, c^-(d_j \wedge x)] - [d_j, c^-(d_i \wedge x)] = a_j [d_i, e^0] - a_i [d_j, e^0]$$
$$= -(\lambda_1^+ + \mu_1^+) \sum_k (a_j \delta_i^k - a_i \delta_j^k) e_k = 0$$

をえる. $\lambda_1^+ + \mu_1^+ < 0$ だから, $a_j \delta_i^k - a_i \delta_j^k = 0$ をえる. これより $(n-2)a_j = 0$ をえる. よって $a_j = 0$, i.e., $c^- = 0$ をえる. $\partial_1^* c = 0$, $c^- = 0$ だから, $\forall u \in \mathfrak{g}_{-2}$ に対して,

$$(\partial_1^* c)(u) = [e^0, c^+(e_0 \wedge u)] + \sum_i [e^i, c^-(e_i \wedge u)] = [e^0, c^+(e_0 \wedge u)] = 0.$$

をえる. これより, $c^+(e_0 \wedge u) = 0$ をえる. 以上により $c \in \mathfrak{g}_1^+ \otimes \mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^-)^*$ が示された.

(2) $u \in \mathfrak{g}_{-2}$, $x, y \in \mathfrak{g}_{-1}^-$ とする. $\partial_1 c = \partial_0 c = 0$ だから

$$\langle (\partial_1 c)(u \wedge x \wedge y), E \rangle = \langle -[x, c(u \wedge y)] + [y, c(u \wedge x)], E \rangle$$
$$= \langle x, c(u \wedge y) \rangle - \langle y, c(u \wedge x) \rangle = 0,$$
$$(\partial_0 c)(x \wedge y \wedge e_0) = c([x, e_0] \wedge y) - c([y, e_0] \wedge x) = 0$$

をえる. よって $\langle c([x, e_0] \wedge y), z \rangle$ は $x, y, z \in \mathfrak{g}_{-1}^-$ について symmetric であることがわかる. $u \in \mathfrak{g}_{-2}$, $x \in \mathfrak{g}_{-1}^-$ とする. $\partial_1 c = 0$ だから

$$(\partial_1^* \partial_1 c)(u \wedge x) = \partial_1^* \partial_1 (c(u \wedge x)) + \sum_i [e^i, [x, c(e_i \wedge u)]] = 0$$

である. ($c(e_0 \wedge u) = 0$ に注意). Lemmas 3.2 and 3.15 を使い

$$\partial_1^* \partial_1 (c(u \wedge x)) = [E_2 + E_1, c(u \wedge x)] = (\lambda_2 + \mu_2 + \lambda_1 + \mu_1) c(u \wedge x)$$

をえる. ($c(u\wedge x)\in\mathfrak{g}_1^+$ に注意) さらに

$$\begin{aligned}\sum_i[e^i,[x,c(e_i\wedge u)]]&=\lambda_1\left(\sum_i\langle x,c(e_i\wedge u)\rangle e^i+\sum_i\langle e^i,x\rangle c(e_i\wedge u)\right)\\&=\lambda_1c(x\wedge u)+\lambda_1\sum_i\langle x,c(e_i\wedge u)\rangle e^i\end{aligned}$$

こゝで, $[z,[x,y]]=\lambda_1(\langle x,y\rangle z+\langle x,z\rangle y)$, $x\in\mathfrak{g}_{-1}^-$, $y,z\in\mathfrak{g}_{-1}^+$ を使った. よって

$$(\lambda_2+\mu_2+\mu_1)c(u\wedge x)+\lambda_1\sum_i\langle x,c(e_i\wedge u)\rangle e^i=0$$

をえる. よって, $x,y,z\in\mathfrak{g}_{-1}^-$ に対して

$$\begin{aligned}&(\lambda_2+\mu_2+\mu_1)\langle c([x,e_0]\wedge y),z\rangle-\lambda_1\langle c([x,e_0]\wedge z),y\rangle\\&=(\lambda_2+\mu_2+\mu_1-\lambda_1)\langle c([x,e_0]\wedge y),z\rangle=0\end{aligned}$$

をえる. $\lambda_2+\mu_2+\mu_1-\lambda_1>0$ だから $\langle c([x,e_0]\wedge y),z\rangle=0$. よって $c=0$ をえる. □

3.5. The spaces of harmonic forms, $H^{p,2}(\mathfrak{G})$, associated with projective graded Lie algebras of type (n,r). この paragraph では $\mathfrak{L}$ は任意の projective FGLA of type (n,r) であり, $\mathfrak{G}$ はその prolongation である.

まず space $H^{0,2}(\mathfrak{G})$ を調べる. vector spaces $V_{(1)}^{0,2}(n,r)$, $V_{(2)}^{0,2}(n,r)$, $V_{(3)}^{0,2}(n,r)$ を次のように定義する.

(i) $V_{(1)}^{0,2}(n,r)$ は $\mathfrak{g}_{-1}^+\otimes\Lambda^2(\mathfrak{g}_{-1}^-)^*$ の元 c で $\partial_1c=0$ を満足するもの全体が作る vector space である.

(ii) $V_{(2)}^{0,2}(n,r)$ は $\mathfrak{g}_{-1}^-\otimes\Lambda^2(\mathfrak{g}_{-1}^+)^*$ の元 c で $\partial_1c=0$ を満足するもの全体が作る vector space である.

(iii) $V_{(3)}^{0,2}(n,r)$ は $\mathfrak{g}_{-1}^-\otimes(\mathfrak{g}_{-1}^+)^*\wedge(\mathfrak{g}_{-1}^-)^*$ の元 c で $\partial_1c=\partial_0^*c=\partial_1^*c=0$ を満足するもの全体が作る vector space である.

次の Proposition を証明しよう.

Proposition 3.21. (1) $H^{0,2}(\mathfrak{G})=V_{(1)}^{0,2}(n,r)+V_{(2)}^{0,2}(n,r)+V_{(3)}^{0,2}(n,r)$.

(2) $n-r\geqq 3$ のとき, $V_{(1)}^{0,2}(n,r)=\{0\}$.

(3) $n-r=1$ *or* $r=1$ のとき, $V_{(3)}^{0,2}(n,r)=\{0\}$

$(e_1,\dots,e_{n-r})=(e_\alpha)$ を $\mathfrak{g}_{-2}$ の basis, $(e_{n-r+1},\dots,e_n)=(e_i)$ を $\mathfrak{g}_{-1}^+$ の basis とする. このとき $\mathfrak{g}_{-1}^-$ の basis (e_α^i) で

$$[e_\alpha^i,e_j]=\delta_j^ie_\alpha$$

を満足するものが唯 1 つ存在する. $(e^1,\dots,e^{n-r})=(e^\alpha)$ を (e_α) に dual な $\mathfrak{g}_2$ の basis, $(e^{n-r+1},\dots,e^n)=(e^i)$ を (e_i) に dual な $\mathfrak{g}_1^-$ の basis, (e_i^α) を (e_α^i) に dual な $\mathfrak{g}_1^+$ の basis とする.

$\mathfrak{g}_0$ の元 E_1^+, E_1^- をそれぞれ次式で定義する:

$$E_1^+=\sum_i[e^i,e_i],\quad\text{and}\quad E_1^-=\sum_{\alpha,i}[e_i^\alpha,e_\alpha^i]$$

Lemma 3.22. (1) $E_1^+=\lambda_1^+E+\mu_1^+J$, *where* $\lambda_1^+=\dfrac{r}{4(n+1)}$ *and* $\mu_1^+=\dfrac{-(r+2)}{4(n+1)}$.

(2) $E_1^-=\lambda_1^-E+\mu_1^-J$, *where* $\lambda_1^-=\dfrac{r}{4(n+1)}$ *and* $\mu_1^-=\dfrac{2n-r}{4(n+1)}$.

さて Proposition 3.21 の証明に入ろう．$r=1$ の場合と全く同様にして $H^{0,2}(\mathfrak{G}) \subset D^{0,2}_{-1}$ が示される．

δ を $+$ or $-$ とする．operator $\partial_1^\delta : \mathfrak{g}_0 \otimes (\mathfrak{g}^\delta_{-1})^* \to \mathfrak{g}_{-1} \otimes \Lambda^2(\mathfrak{g}^\delta_{-1})^*$ を

$$(\partial_1^\delta a)(x \wedge y) = [x, a(y)] - [y, a(x)]$$

によって定義する．こゝで $a \in \mathfrak{g}_0 \otimes (\mathfrak{g}^\delta_{-1})^*$, $x, y \in \mathfrak{g}^\delta_{-1}$ である．inner product (,) に関する ∂_1^δ の adjoint operator $\partial_1^{\delta *} : \mathfrak{g}_{-1} \otimes \Lambda^2(\mathfrak{g}^\delta_{-1})^* \to \mathfrak{g}_0 \otimes (\mathfrak{g}^\delta_{-1})^*$ は

$$(\partial_1^{\delta *} a)(x) = \begin{cases} \sum_i [e^i, a(e_i \wedge x)] & \text{if} \quad \delta \text{ is } + \\ \sum_i [e_i^\alpha, a(e^i_\alpha \wedge x)] & \text{if} \quad \delta \text{ is } - \end{cases}$$

で与えられる．こゝで $a \in \mathfrak{g}^\delta_{-1} \otimes \Lambda^2(\mathfrak{g}^\delta_{-1})^*$, $x \in \mathfrak{g}^\delta_{-1}$ である．$\sigma\mathfrak{g}^+_{-1} = \mathfrak{g}^-_1$, $\sigma\mathfrak{g}^-_{-1} = \mathfrak{g}^+_1$ を注意せよ．

$c \in H^{0,2}(\mathfrak{G}) \subset A^{0,2}_{-1}$ とする．$H^{0,2}(\mathfrak{G})$ は，$A^{0,2}_{-1}$ の元 c で

$$\partial_1 c = \partial_0^* c = \partial_1^* c = 0$$

を満足するもの全体からなる．c は $c = c^+ + c^-$ と分解する．$\delta = +$ or $-$ に対して，$a^\delta \in \mathfrak{g}^\delta_{-1} \otimes \Lambda^2(\mathfrak{g}^\delta_{-1})^*$ を

$$a^\delta(x \wedge y) = c^\delta(x \wedge y), \quad x, y \in \mathfrak{g}^\delta_{-1},$$

によって定義する．

Lemma 3.23. (1) $(-\lambda_1^- + \mu_1^-)a^+ + \partial_1^+ \partial_1^{+*} a^+ = 0.$

(2) $-(\lambda_1^+ + \mu_1^+)a^- + \partial_1^- \partial_1^{-*} a^- = 0$

Proof. $\partial_1^* c = 0$ だから $\forall y \in \mathfrak{g}_{-1}$ に対して，

$$(\partial_1^* c)(y) = \sum_i [e^i, c^+(e_i \wedge y)] + \sum_{\alpha,i} [e_i^\alpha, c^-(e^i_\alpha \wedge y)]$$

をえる．$\partial_1 c = 0$ だから，$\forall x \in \mathfrak{g}^-_{-1}$, $y, z \in \mathfrak{g}^+_{-1}$ に対して，

$$(\partial_1 c)(x \wedge y \wedge z) = [x, c^+(y \wedge z)] - [y, c^-(x \wedge z)] + [z, c^-(x \wedge y)] = 0$$

をえる．よって

$$\sum_{\alpha,i} [e_i^\alpha, [e^i_\alpha, c^+(y \wedge z)]] - \sum_{\alpha,i} [e_i^\alpha, [y, c^-(e^i_\alpha \wedge z)]] + \sum_{\alpha,i} [e_i^\alpha, [z, c^-(e^i_\alpha \wedge y)]] = 0, \quad y, z \in \mathfrak{g}^+_{-1}.$$

よって

$$[E_1^-, c^+(y \wedge z)] - [y, \sum_{\alpha,i} [e_i^\alpha, c^-(e^i_\alpha \wedge z)]] + [z, \sum_{\alpha,i} [e_i^\alpha, c^-(e^i_\alpha \wedge y)]] = 0.$$

よって Lemma 3.22 と $(\partial_1^* c) = 0$ に関する上式から

$$(-\lambda_1^- + \mu_1^-)c^+(y \wedge z) + [y, \sum_i [e^i, c^+(e_i \wedge z)]] - [z, \sum_i [e^i, c^+(e_i \wedge y)]] = 0$$

をえる．これは (1) を意味する．

全く同様にして (2) が示される． □

$(-\lambda_1^- + \mu_1^-) > 0$, $-(\lambda_1^+ + \mu_1^+) > 0$ だから，Lemma 3.23 から $a^+ = a^- = 0$ をえる．よって

$$c^+(\mathfrak{g}^+_{-1} \wedge \mathfrak{g}^+_{-1}) = 0, \qquad c^-(\mathfrak{g}^-_{-1} \wedge \mathfrak{g}^-_{-1}) = 0$$

が示された．この 2 番目の式と $\partial_1^* c = 0$ から

$$\sum_i [e^i, c^+(e_i \wedge y)] = 0, \quad y \in \mathfrak{g}^-_{-1}$$

をえる．これから直ちに $c^+(e_i \wedge y) = 0$ が従う．よって

$$c^+(\mathfrak{g}_{-1}^+ \wedge \mathfrak{g}_{-1}^-) = 0.$$

以上から

$$H^{0,2}(\mathfrak{G}) \subset \mathfrak{g}_{-1}^+ \otimes \Lambda^2(\mathfrak{g}_{-1}^-)^* + \mathfrak{g}_{-1}^- \otimes \Lambda^2(\mathfrak{g}_{-1}^+)^* + \mathfrak{g}_{-1}^- \otimes (\mathfrak{g}_{-1}^+)^* \wedge (\mathfrak{g}_{-1}^-)^*$$

をえる．今や容易に

$$H^{0,2}(\mathfrak{G}) = V_{(1)}^{0,2}(n,r) + V_{(2)}^{0,2}(n,r) + V_{(3)}^{0,2}(n,r)$$

が示される．

$n - r \geqq 3$ のとき，$V_{(1)}^{0,2}(n,r) = \{0\}$ を示そう．$c \in V_{(1)}^{0,2}(n,r)$ とする．$\partial_1 c = 0$ だから

$$(\partial_1 c)(e_\alpha^i \wedge e_\beta^j \wedge e_\gamma^k) = [e_\alpha^i, c(e_\beta^j \wedge e_\gamma^k)] - [e_\beta^j, c(e_\alpha^i \wedge e_\gamma^k)] + [e_\gamma^k, c(e_\alpha^i \wedge e_\beta^j)] = 0$$

$\langle c(e_\alpha^i \wedge e_\gamma^k), e^l \rangle = a_{\alpha\gamma}^{ikl}$ とおけば，

$$a_{\beta\gamma}^{jki}\delta_\alpha^\varepsilon - a_{\alpha\gamma}^{ikj}\delta_\beta^\varepsilon + a_{\alpha\beta}^{ijk}\delta_\gamma^\varepsilon = 0$$

をえる．$n - r \geqq 3$ の仮定の下でこの式より容易に $a_{\alpha\gamma}^{ikl} = 0$ をえる．よって $c = 0$ である．

$n - r = 1$ or $r = 1$ のとき，$V_{(3)}^{0,2}(n,r) = \{0\}$ を示そう．$c \in V_{(3)}^{0,2}(n,r)$ とする．$\partial_1^* c = 0$ だから

$$(\partial_1^* c)(e_k) = \sum_{\alpha,i} [e_i^\alpha, c(e_\alpha^i \wedge e_k)] = \sum_{\alpha,i,\beta,j} \langle c(e_\alpha^i \wedge e_k), e_j^\beta \rangle [e_i^\alpha, e_\beta^j] = 0$$

をえる．$n - r = 1$ or $r = 1$ のとき，この式から $\langle c(e_\alpha^i \wedge e_k), e_j^\beta \rangle = 0$ が従う．よって $c = 0$ をえる．

以上によって Proposition 3.21 の証明が完了した．

さて $r = n - 1$ の場合を考えよう．この仮定の下で，$\dim \mathfrak{g}_{-2} = 1$, $\dim \mathfrak{g}_{-1} = 2(n-1)$ である．$n = 2$ のときには，$H^{0,2}(\mathfrak{G})$ は Proposition 3.21 によって既に決定されている．($J \in \mathfrak{h}_0$ に注意)．$n \geqq 3$ のときには，下に見るように Propositions 3.24 and 3.25 を使って $H^{p,2}(\mathfrak{G})$ は容易に決定される．結果は次の通りである．

Proposition 3.24. *Assume that $r = n - 1$ and $n \geqq 3$.*

(1) $H^{p,2}(\mathfrak{G}) = \{0\}$ *if $p \neq 0$ or 1.*

(2) $H^{0,2}(\mathfrak{G}) = V_{(1)}^{0,1}(n, n-1) + V_{(2)}^{0,1}(n, n-1)$.

(3) $H^{1,2}(\mathfrak{G})$ *consists of all $c \in \mathfrak{h}_0 \otimes (\mathfrak{g}_{-1}^+)^* \wedge (\mathfrak{g}_{-1}^-)^*$ such that $\partial_1 c = \partial_1^* c = 0$.*

Proposition 3.25. *Assume that $r = 1$ and $n = 2$.*

(1) $H^{p,2}(\mathfrak{G}) = \{0\}$ *if $p \neq 3$.*

(2) $H^{3,2}(\mathfrak{G}) = \mathfrak{g}_1^+ \otimes \mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^-)^* + \mathfrak{g}_1^- \otimes \mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^+)^*$.

Proof of Proposition 3.24. It suffices to prove (3). Let us identify $\mathfrak{h}_0$ with a subalgebra of $\mathfrak{gl}(\mathfrak{g}_{-1})$ through the natural representation of $\mathfrak{h}_0$ on $\mathfrak{g}_{-1}$. Then we have $J \in \mathfrak{h}_0$ and $J^2 = 1$. Define a bilinear form α on $\mathfrak{g}_{-1}$ by

$$[Jx, y] = \alpha(x,y)d_1, \qquad x, y \in \mathfrak{g}_{-1},$$

d_1 being a basis of $\mathfrak{g}_{-2}$. Then we see that α is non-degenerate and that $\alpha(x,y) = \alpha(y,x)$ and $\alpha(Jx, Jy) = -\alpha(x,y)$. Now the space $H^{1,2}(\mathfrak{G})$ consists of all $c \in \mathfrak{h}_0 \otimes \Lambda^2(\mathfrak{g}_{-1})^*$ such that $\partial_1 c = \partial_0^* c = \partial_1^* c = 0$. Let $c \in H^{1,2}(\mathfrak{G})$. Then it follows that

$$\begin{aligned} &Jc(x \wedge y) = c(x \wedge y)J, \\ &c(x \wedge y)z + c(y \wedge z)x + c(z \wedge x)y = 0, \\ &\alpha(c(x \wedge y)z, w) = -\alpha(c(x \wedge y)w, z), \quad x, y, z, w \in \mathfrak{g}_{-1}, \end{aligned}$$

whence $c(Jx \wedge Jy) = -c(x \wedge y)$ (cf. the curvature of a Kählarian manifold). Thus we get $c \in \mathfrak{h}_0 \otimes (\mathfrak{g}_{-1}^+)^* \wedge (\mathfrak{g}_{-1}^-)^*$. It is now easy to verify that the equation $\partial_0^* c = 0$ can be derived from the equations $\partial_1 c = \partial_1^* c = 0$. We have thus proved Proposition 3.24. □

最後に次の Proposition をつけ加える. 証明は省略する.

Proposition 3.26. $2 \leqq r \leqq n-2$ と仮定する.
(1) $H^{p,2}(\mathfrak{G}) = \{0\}$ *if* $p \neq -1$ *or* 0.
(2) $H^{-1,2}(\mathfrak{G})$ *consists of all* $c \in \mathfrak{g}_{-2} \otimes \Lambda^2(\mathfrak{g}_{-1}^-)^*$ *such that* $\partial_1^* c = 0$.
(3) $H^{0,2}(\mathfrak{G}) = V_{(1)}^{0,2}(n,r) + V_{(2)}^{0,2}(n,r) + V_{(3)}^{0,2}(n,r)$.

以上のすべての projective graded Lie algebras に対して, spaces $H^{p,2}(\mathfrak{G})$ を計算した.

CHAPTER IV

Almost pseudo-product manifolds and normal connections of type $\mathfrak{G}$

§1. Normal connections of type $\mathfrak{G}$

1.1. Cartan connections.

The notion of isomorphism. (P, ω) (resp. (P', ω')) を M (resp. M') 上の Cartan connection of type $G/G^{(0)}$ とする. $\varphi : M \to M'$ が (P, ω) から (P', ω') への isomorphism であるとは, bundle isomorphism $\tilde{\varphi} : P \to P'$ が存在して,

$$\tilde{\varphi}^* \omega' = \omega$$

を満足し, かつ $\tilde{\varphi}$ が φ を induce することである.

Proposition 1.1. *$G/G^{(0)}$ について次の仮定を行う: i) G は 2nd countability axiom を満足する; ii) $G/G^{(0)}$ は connected である; iii) G の $G/G^{(0)}$ への action は effective である. このとき $\tilde{\varphi}$ は unique.*

$$(P, \omega), \qquad \mathfrak{g} \ni X \mapsto \check{\omega}(X)$$

$$\delta_X = \mathscr{L}_{\check{\omega}(X)}, \quad \delta_X = \mathscr{L}_{X^*} \quad \text{if } X \in \mathfrak{g}^{(0)}$$

Let G/G' be a homogeneous space of a Lie group G over its closed subgroup G', and denote by $\mathfrak{g}$ and $\mathfrak{g}'$ the Lie algebras of G and G' respectively. Then a (Cartan) connection of the model space G/G' on a manifold M is a pair formed by a principal fibre bundle P over M with structure group G' and a $\mathfrak{g}$-valued 1-form ω on P which satisfy the following conditions:

(i) ω gives an absolute parallelism on P, i.e., for each $z \in P$ the linear map which maps $X \in T_z(P)$ to $\omega(X) \in \mathfrak{g}$ is an isomorphism of $T_z(P)$ onto $\mathfrak{g}$.

(ii) $R_a^* \omega = \mathrm{Ad}(a^{-1})\omega, \quad a \in G'$.

(iii) $\omega(X^*) = X, \quad X \in \mathfrak{g}'$.

The connection will be denoted by a capital Greek letter, e.g., Γ, and then the pair (P, ω) will be called the expression of Γ; ω itself will called the connection form of Γ. The same notation and terminology will be used for transverse (Cartan) connections, which will be defined in Part II.

1.2. Connections of type $\mathfrak{G}$.

$$\mathfrak{G}, \qquad G/G^{(0)}$$

$$(P, \omega) : \quad \text{connection of type } \mathfrak{G}$$

$$\omega_-$$

Proposition 1.2. ω_- の性質

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]$$

Proposition 1.3. $K : P \to C^2(\mathfrak{G})$

Proposition 1.4. $R_a^* K = K^a, \quad a \in G^{(0)}$

$$
\begin{aligned}
&\omega = \sum_j \omega_j, \quad \Omega = \sum_j \Omega_j \\
&K = \sum_j K_j = \sum_p K^p, \quad K_j = \sum_p K_j^p, \quad K^p = \sum_j K_j^p, \\
&\omega_j(X_j^*) = \delta_{jr} X_r, \\
&R_a^* \omega_j = \mathrm{Ad}(a^{-1})\omega_j, \\
&\delta_{X_r}\omega_j = -[X_r, \omega_{j-r}], \quad X_r \in \mathfrak{g}_r, \quad r \geqq 1, \\
&R_a^* K^p = (K^p)^a, \quad a \in G_0 \\
&\delta_{X_r} K^p = (K^{p-r})^{X_r}, \quad X_r \in \mathfrak{g}_r, \quad r \geqq 1 \\
&\Omega_j = \frac{1}{2} K_j(\omega_- \wedge \omega_-) = \frac{1}{2}\sum_p K_j^p(\omega_- \wedge \omega_-)
\end{aligned}
$$

$$(P, \omega) \longrightarrow \tilde{P} \subset Q, \qquad Q : G_0^\sharp\text{-structure}$$

1.3. Normal connections of type $\mathfrak{G}$.

Proposition 1.5. *(P,ω) normal $\Longleftrightarrow$ Q of type $\mathfrak{M}$.*

Theorem 1.6. *normal connections of type $\mathfrak{G}$ $\Longleftrightarrow$ $G_0^\sharp$-structure of type $\mathfrak{M}$.*

(1), (2), (3)

Remark. G_0 を $\mathrm{Aut}(\mathfrak{G})$ の open subgroup

$$G^{(0)} = G_0 \cdot G^{(1)}, \quad G = \mathrm{Aut}(\mathfrak{g})^0 G_0$$

$G/G^{(0)}$ は, effective, compact, etc

normal connection of type $(\mathfrak{G}, G_0)$ (of type $\mathfrak{G}$ という)

(P,ω) を weak Cartan connection of type $(\mathfrak{g}, \mathfrak{g}^{(0)})$ on M.
curvature $K : P \to C^2(\mathfrak{G})$ が全く同様にして定義できる.

$$K = \sum K^p$$

(P,ω) が weak normal connection of type $\mathfrak{G}$
(P,ω) が normal connection of type $\mathfrak{G}$ ($\mathfrak{G}$ は, $(\mathfrak{M}, \mathfrak{g}_0)$ の prolongation とは仮定しない)

$$K = \sum K^p$$

$$\Psi^{p-1}(X_1 \wedge X_2 \wedge X_3) = -\delta_{X_1} K^{p+r_1}(X_2 \wedge X_3) - \cdots$$

Proposition 1.7 (Bianchi identities)**.**

$$\partial K^p = \Psi^{p-1}$$

$$H : C^2(\mathfrak{G}) \to H^2(\mathfrak{G}) \quad \text{harmonic projection}$$

H は $C^{p,2}(\mathfrak{G})$ を $H^{p,2}(\mathfrak{G})$ に project する.

HK or $\{HK^p\}$ を the fundamental system of invariants という.

Proposition 1.8. $K = 0 \quad \Longleftrightarrow \quad HK = 0$

§2. Almost pseudo-product manifolds, and normal connections of type $\mathfrak{G}$

2.1. Almost pseudo-product manifolds of type $\mathfrak{L}$ and normal connections of type $\mathfrak{G}$. Let $\mathfrak{L} = \{\mathfrak{M}; \mathfrak{e}, \mathfrak{f}\}$ be a non-degenerate pseudo-product FGLA, and $\mathfrak{G} = \{\mathfrak{g}, (\mathfrak{g}_p)\}$ its prolongation. By Proposition II.2.8 we know that $\mathfrak{g}$ is finite dimensional. In this paragraph we assume that $\mathfrak{g}$ is a simple Lie algebra and hence $\mathfrak{G}$ is a simple graded Lie algebra.

Similarly to the case of a projective FGLA the automorphism group $\mathrm{Aut}(\mathfrak{L})$ of $\mathfrak{L}$ may be identified with an open subgroup of the automorphism group $\mathrm{Aut}(\mathfrak{G})$ of $\mathfrak{G}$. This being said, we set

$$G_0 = \mathrm{Aut}(\mathfrak{L}),$$

and define subgroups $G^{(0)}$ and G of $\mathrm{Aut}(\mathfrak{g})$ respectively by

$$G^{(0)} = G_0 \cdot G^{(1)}, \quad G = G_0 \cdot \mathrm{Aut}(\mathfrak{g})^0,$$

which are just the groups associated with the pair $(\mathfrak{G}, G_0)$ (see Remark $**$). Here $G^{(1)}$ is the Lie subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by the subalgebra $\mathfrak{g}^{(1)} = \sum_{p \geqq 1} \mathfrak{g}_p$ of $\mathfrak{g}$. Let us consider the homogeneous space $G/G^{(0)}$. Then we see as before that the subspaces $\mathfrak{e}$ and $\mathfrak{f}$ of $\mathfrak{g}_{-1}$ give rise to G-invariant differential systems E_0 and F_0 on $G/G^{(0)}$ respectively, and that $\mathfrak{P}_0 = \{R; E_0, F_0\}$ is a pseudo-product manifold of type $\mathfrak{L}$, which will be called the standard pseudo-product manifold of type $\mathfrak{L}$.

Now we have the notion of a normal connection of type $\mathfrak{G}$ (or exactly of type $(\mathfrak{G}, G_0)$) as well as the notion of a $G_0^\sharp$-structure of type $\mathfrak{M}$ (see §§II.1,II.2 and III.2). Let (P, ω) be a normal connection of type $\mathfrak{G}$ on a manifold R, and Q the corresponding $G_0^\sharp$-structure of type $\mathfrak{M}$ on R. Let ρ be the natural homomorphism $P \to Q$ corresponding to the homomorphism $\rho : G^{(0)} \to \tilde{G} \subset G_0^\sharp$, and let π (resp. α) be the projection $P \to R$ (resp. $Q \to R$). Let ξ be the basic form on Q. Then we have

$$\pi = \alpha \circ \rho, \text{ and } \quad \omega_- = \rho^* \xi,$$

where ω_- is the $\mathfrak{m}$-component of ω with respect to the decomposition. Furthermore the connection form ω is decomposed as follows:

$$\omega = \sum_p \omega_p,$$

and ω_{-1} as follows:

$$\omega_{-1} = \omega_{-1}^+ + \omega_{-1}^-,$$

where ω_{-1}^+ (resp. ω_{-1}^-) is the $\mathfrak{e}$-component (resp. the $\mathfrak{f}$-component) of ω_{-1} with respect to the decomposition: $\mathfrak{g}_{-1} = \mathfrak{e} + \mathfrak{f}$. Similarly the basic form ξ is decomposed as follows:

$$\xi = \sum_p \xi_p, \quad \text{and} \quad \xi_{-1} = \xi_{-1}^+ + \xi_{-1}^-.$$

Clearly we have

$$\omega_p = \rho^* \xi_p, \quad p < 0, \text{ and } \quad \omega_{-1}^+ = \rho^* \xi_{-1}^+, \quad \omega_{-1}^- = \rho^* \xi_{-1}^-$$

Therefore from Theorem $**$ and Proposition $**$ we obtain

Theorem 2.1. (1) *To every normal connection of type $\mathfrak{G}$, (P, ω), on a manifold R there is associated a unique almost pseudo-product manifold of type $\mathfrak{L}$, $\mathfrak{P} = \{R; E, F\}$, satisfying the following conditions:*

i) *For each $p < 0$ $\pi_*^{-1}(D_p)$ is defined by the equations $\omega_q = 0$, $q < p$, where D_p is the $(-p-1)$-th derived system of D.*

ii) *$\pi_*^{-1}(E)$ is defined by the equations $\omega_p = \omega_{-1}^- = 0$, $p \leqq -2$,*

iii) $\pi_*^{-1}(F)$ *is defined by the equations* $\omega_p = \omega_{-1}^+ = 0$, $p \leqq -2$,

(2) *Conversely to every almost pseudo-product manifold of type* $\mathfrak{L}$, $\mathfrak{P} = \{R; E, F\}$, *there is associated, in a natural manner, a normal connection of type* $\mathfrak{G}$, (P, ω), *on* R *satisfying the conditions above.*

(3) *Let* (P, ω) *(resp.* (P', ω')*) be a normal connection of type* $\mathfrak{G}$ *on a manifold* R *(resp. on* R'*), and* $\mathfrak{P} = \{R; E, F\}$ *(resp.* $\mathfrak{P}' = \{R'; E', F'\}$*) be the corresponding almost pseudo-product manifold of type* $\mathfrak{L}$. *Then a diffeomorphism* $\varphi : R \to R'$ *is an isomorphism of* $\mathfrak{P}$ *to* $\mathfrak{P}'$ *if and only if it is an isomorphism of* (P, ω) *to* (P', ω').

Let Γ and Γ' be connections of model space G/G' on manifolds M and M', and (P, ω) and (P', ω') their expressions, respectively. By an isomorphism of Γ to Γ' we mean a diffeomorphism φ of M onto M' such that there is a bundle isomorphism $\tilde{\varphi}$ of P onto P' satisfying the following conditions:

(i) $\tilde{\varphi}$ induces φ,

(ii) $\tilde{\varphi}^*\omega' = \omega$.

We remark that if G/G' is connected, and G acts effectively on G/G', then $\tilde{\varphi}$ is uniquely determined by φ.

Furthermore the connections Γ and Γ' are said to be equivalent, if $M = M'$, and the identity transformation of M gives an isomorphism of Γ to Γ'.

By a normal connection of type $\mathfrak{L}$ we mean a normal connection modeled by G/G'. Let us consider a normal connection Γ of type $\mathfrak{L}$ on a manifold M, and (P, ω) its expression. According to the decomposition $\mathfrak{g} = \sum_p \mathfrak{g}_p$ the connection form ω is decomposed as follows: $\omega = \sum_p \omega_p$, and according to the decomposition $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^+ + \mathfrak{g}_{-1}^-$ the $\mathfrak{g}_{-1}$-valued 1-form ω_{-1} as follows: $\omega_{-1} = \omega_{-1}^+ + \omega_{-1}^-$.

Theorem 2.2. (1) *Let* $\mathcal{R}$ *be any almost pseudo-product manifold, and* (R, E, F) *its expression. Then there is naturally associated to* $\mathcal{R}$ *a normal connection* Γ *on* R *which satisfies the following conditions* (i) *and* (ii)*: Let* (P, ω) *be the expression of* Γ, *and* π *the projection of* P *onto* M. *Then,* (i) *the system* $\pi_*^{-1}(E)$ *on* P *is defined by the system of Pfaffian equations* $\omega_p = \omega_{-1}^- = 0$ $(p \leqq -2)$, *and* (ii) *the system* $\pi_*^{-1}(F)$ *on* P *is defined by the system of Pfaffian equations* $\omega_p = \omega_{-1}^+ = 0$ $(p \leqq -2)$.

(2) *Conversely let* Γ *be a normal connection of type* $\mathfrak{L}$ *on a manifold* R. *Then there is a unique almost pseudo-product structure* (E, F) *on* R *such that the almost pseudo-product manifold* $\mathcal{R}$ *expressed by* (R, E, F) *is of type* $\mathfrak{L}$ *and such that* Γ *is naturally isomorphic to the normal connection of type* $\mathfrak{L}$ *associated with* $\mathcal{R}$.

(3) *Let* $\mathcal{R}$ *and* $\mathcal{R}'$ *be two almost pseudo-product manifolds of type* $\mathfrak{L}$, *and* Γ *and* Γ' *be the associated normal connection of type* $\mathfrak{L}$ *respectively. Then an isomorphism of* $\mathcal{R}$ *to* $\mathcal{R}'$ *is an isomorphism of* Γ *to* Γ', *and vice versa.*

Thus we know that the almost pseudo-product manifolds of type $\mathfrak{L}$, $\mathfrak{P}$, are naturally in a one-to-one correspondence with the normal connections of type $\mathfrak{G}$, (P, ω), up to isomorphisms. It is easy to see that the standard pseudo-product manifold of type $\mathfrak{L}$, $\mathfrak{P}_0$, just corresponds to the standard normal connection of type $\mathfrak{G}$, (G, ω), on $G/G^{(0)}$, and hence we get

Corollary. *Let* $\varphi : U \to V$ *be a local transformation of* $G/G^{(0)}$. *Then* φ *is a local automorphism of* $\mathfrak{P}_0$ *if and only if it is obtained from the action of* G *on* $G/G^{(0)}$, *i.e., there is an* $a \in G$ *such that* $\varphi(x) = ax$, $x \in U$.

Suppose now that $\mathfrak{L}$ is a projective FGLA of type (n, r), and let $\mathfrak{P}$ be an almost pseudo-product manifold. Then by Proposition ∗∗ $\mathfrak{P}$ is an almost projective system of type (n, r) if and only if it is of type $\mathfrak{L}$. Accordingly Theorem ∗∗ can be applied to the geometry of almost projective systems of type (n, r). Let us now consider the geometry of single ordinary differential equations of the third order

$$\frac{d^3y}{dx^3} = F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right),$$

where the equivalence is considered with respect to contact transformations of the space of variables x, y. Then it can be shown that this geometry may be represented by the geometry of pseudo-product manifolds of type $\mathfrak{L}$, where $\mathfrak{L}$ is a non-degenerate pseudo-product FGLA of the third kind such that the prolongation $\mathfrak{G}$ of $\mathfrak{L}$ is a simple graded Lie algebra and the underlying Lie algebra $\mathfrak{g}$ is isomorphic with the simple Lie algebra $\mathfrak{o}(3, 2)$. Thus Theorem $**$ can be applied to the former geometry through the latter (cf. Cartan [26], Chern [27]).

2.2. The normal connections of type $\mathfrak{G}$ associated with almost projective systems of type (n, r). The notations being as above, we assume in this paragraph that $\mathfrak{L}$ is a projective FGLA of type (n, r). Let $\mathfrak{P} = \{R; E, F\}$ be an almost projective system of type (n, r), and (P, ω) the corresponding normal connection of type $\mathfrak{G}$.

Let us consider the curvature function $K : P \to C^2(\mathfrak{G})$. The space $C^2(\mathfrak{G})$ is decomposed as follows:

$$C^2(\mathfrak{G}) = \sum_{p=-1}^{5} C^{p,2}(\mathfrak{G}),$$

and hence each $C^{p,2}(\mathfrak{G})$ as follows:

$$C^{p,2}(\mathfrak{G}) = C^{p,2}_{p-3} + C^{p,2}_{p-2} + C^{p,2}_{p-1}.$$

(The notations being as in §III.1, $C^{p,2}_{p-3} = A^{2,0}_{p-3}$, $C^{p,2}_{p-2} = A^{1,1}_{p-2}$ and $C^{p,2}_{p-1} = A^{0,2}_{p-1}$.) Therefore according to the notations given in §III.2, K is decomposed in two ways as follows:

$$K = \sum_{p=-1}^{5} K^p = \sum_{j=-2}^{2} K_j,$$

each K^p as follows:

$$K^p = K^p_{p-3} + K^p_{p-2} + K^p_{p-1},$$

and each K_j as follows:

$$K_j = K_j^{j+3} + K_j^{j+2} + K_j^{j+1}.$$

Here we remark that

$$K^{-1}(= K^{-1}_{-2}) = 0, \quad \text{and} \quad \partial^* K^p = 0, \quad 0 \leqq p \leqq 5,$$

because the connection (P, ω) is normal.

Thus the structure equation $d\omega + \frac{1}{2}[\omega, \omega] = \frac{1}{2}K(\omega_- \wedge \omega_-)$ is decomposed into the five equations as follows:

$$(E_j) \qquad \begin{aligned} &d\omega_j + \frac{1}{2}\sum_{r+s=j} [\omega_r, \omega_s] \\ &= \frac{1}{2}K_j^{j+3}(\omega_{-2} \wedge \omega_{-2}) + K_j^{j+2}(\omega_{-2} \wedge \omega_{-1}) + \frac{1}{2}K_j^{j+1}(\omega_{-1} \wedge \omega_{-1}), \quad -2 \leqq j \leqq 2. \end{aligned}$$

For example equations (E_{-2}) and (E_{-1}) may be explicitly written as follows:

$$(E_{-2}) \qquad d\omega_{-2} + \frac{1}{2}[\omega_{-1}, \omega_{-1}] + [\omega_0, \omega_{-2}] = \frac{1}{2}K^1_{-2}(\omega_{-2} \wedge \omega_{-2}) + K^0_{-2}(\omega_{-2} \wedge \omega_{-1}),$$

$$
(E_{-1}) \quad \begin{aligned} &d\omega_{-1} + [\omega_0, \omega_{-1}] + [\omega_1, \omega_{-2}] \\ &= \frac{1}{2}K^2_{-1}(\omega_{-2} \wedge \omega_{-2}) + K^1_{-1}(\omega_{-2} \wedge \omega_{-1}) + \frac{1}{2}K^0_{-1}(\omega_{-1} \wedge \omega_{-1}). \end{aligned}
$$

As an application of the structure equation we shall now prove an integrability theorem on almost projective systems. We first recall that the space of harmonic forms, $H^{0,2}(\mathfrak{G})$, is calculated as follows:

$$
H^{0,2}(\mathfrak{G}) = V^{0,2}_{(1)}(n,r) + V^{0,2}_{(2)}(n,r) + V^{0,2}_{(3)}(n,r)
$$

(see Proposition $**$), noting that $V^{0,2}_{(1)}(n,r) \subset \mathfrak{g}^+_{-1} \otimes \Lambda^2(\mathfrak{g}^-_{-1})^*$, $V^{0,2}_{(2)}(n,r) \subset \mathfrak{g}^-_{-1} \otimes \Lambda^2(\mathfrak{g}^+_{-1})^*$ and $V^{0,2}_{(3)}(n,r) \subset \mathfrak{g}^-_{-1} \otimes (\mathfrak{g}^+_{-1})^* \wedge (\mathfrak{g}^-_{-1})^*$.

Theorem 2.3. *Let $\mathfrak{P} = \{R; E, F\}$ be an almost projective system of type (n,r), and (P,ω) the corresponding normal connection of type $\mathfrak{G}$ on R. Consider the invariant K^0.*

(1) *$\mathfrak{P}$ is integrable if and only if K^0 takes values in $V^{0,2}_{(3)}(n,r)$.*

(2) *Assume that $n-r=1$ or $r=1$. Then $\mathfrak{P}$ is integrable if and only if $K^0 = 0$.*

(3) *If $n - r \geqq 3$, then F is automatically completely integrable (cf. Yamaguchi [23]).*

(4) *If $r = 1$ and $n \geqq 4$, then $\mathfrak{P}$ is automatically integrable.*

The differential system $\pi_*^{-1}(E)$ (resp. $\pi_*^{-1}(F)$) on P is defined by the equations:

$$
\omega_{-2} = \omega^-_{-1} = 0 \quad (\text{resp. } \omega_{-2} = \omega^+_{-1} = 0),
$$

and it is clear that the differential system E (resp. F) is completely integrable if and only if so is $\pi_*^{-1}(E)$ (resp. $\pi_*^{-1}(F)$). Furthermore $K^0 = K^0_{-2} + K^0_{-1}$ takes values in $H^{0,2}(\mathfrak{G})$ by Proposition $**$. (This implies $K^0_{-2} = 0$.) Therefore the theorem follows immediately from Proposition $**$ and the next

Lemma 2.4. (1) $d\omega_{-2} \equiv 0 \pmod{\omega_{-2}, \omega^-_{-1}}$, $\quad d\omega^-_{-1} \equiv \frac{1}{2}K^0_{-1}(\omega^+_{-1} \wedge \omega^+_{-1}) \pmod{\omega_{-2}, \omega^-_{-1}}$

(2) $d\omega_{-2} \equiv 0 \pmod{\omega_{-2}, \omega^+_{-1}}$, $\quad d\omega^+_{-1} \equiv \frac{1}{2}K^0_{-1}(\omega^-_{-1} \wedge \omega^-_{-1}) \pmod{\omega_{-2}, \omega^+_{-1}}$.

This fact follows easily from equations (E_{-2}) and (E_{-1}), noting that $\frac{1}{2}[\omega_{-1}, \omega_{-1}] = [\omega^+_{-1}, \omega^-_{-1}]$.

CHAPTER V

The geometry of systems of ordinary differential equations of the second order

Throughout this chapter $\mathfrak{L}$ is a projective FGLA of type $(n,1)$, and $\mathfrak{G}$ its prolongation. As to these objects we use the notations as in §III.3 and III.4. Especially we fix bases (d_α) of $\mathfrak{g}_{-2}$, e_0 of $\mathfrak{g}_{-1}^{+}$, (e_i) of $\mathfrak{g}_{-1}^{-}$, etc. in the manner as in §III.4. We always assume that $n \geqq 3$.

§1. Infinitesimal normal connections of type $\mathfrak{G}$

1.1. P を manifold M 上の fibred manifold, V を the system of vertical vectors of the fibred manifold とする. $\mathfrak{g}^{(0)}$ の各 X に対して, P 上の vector filed X^* が対応し, 次の条件を満足しているものとする.

1) 対応 $X \mapsto X^*$ は, Lie algebra homomorphism である. i.e., $[X^*, Y^*] = [X, Y]^*$, $\quad X, Y \in \mathfrak{g}^{(0)}$

2) P の各 z に対して, 対応 $X \mapsto X_z^*$ は $\mathfrak{g}^{(0)}$ から V_z 上への isomorphism を与える.

ω を P 上の $\mathfrak{g}$-valued form とする. 次の条件の下で (P,ω) を M 上の weak Cartan connection of type $(\mathfrak{g},\mathfrak{g}^{(0)})$ とよぶ.

(IC.1) $X \in T_z(P)$ とする. $\omega(X) = 0 \implies X = 0$.

(IC.2) $\omega(X^*) = X, \quad X \in \mathfrak{g}^{(0)}$.

(IC.3) $\mathscr{L}_{X^*}\omega = -[X,\omega], \quad X \in \mathfrak{g}^{(0)}$.

Cartan connection of type $G/G^{(0)}$ は明らかに weak Cartan connection of type $(\mathfrak{g},\mathfrak{g}^{(0)})$ である.

(P,ω) (resp. (P',ω')) を M (resp. M') 上の weak Cartan connection of type $(\mathfrak{g},\mathfrak{g}^{(0)})$ とする. $(z_0, z_0') \in P \times P'$ とする. z_0 の nbd から z_0' の nbd の上への diffeomorphism ψ があって,

$$\psi^*\omega' = \omega, \qquad \psi(z_0) = z_0'$$

が成り立つとき, ψ を local isomorphism of (P,ω) to (P',ω') at (z_0, z_0') とよぶ. 任意の local isomorphism ψ は

$$\psi_*(X^*) = X^*, \qquad X \in \mathfrak{g}^{(0)}$$

を満足する.

Proposition 1.1. 任意の *weak Cartan connection of type* $(\mathfrak{g},\mathfrak{g}^{(0)})$ は *1* つの *Cartan connection of type* $G/G^{(0)}$ によって *represent* される. 即ち, (P,ω) を M 上の *weak Cartan connection of type* $(\mathfrak{g},\mathfrak{g}^{(0)})$, $z_0 \in P$ とする. このとき, *1* つの *manifold* M' 上で定義された *Cartan connection of type* $G/G^{(0)}$, (P',ω') と z_0 の *nbd* で定義された *local isomorphism* $\psi: (P,\omega) \to (P',\omega')$ が存在する.

この Prop. は the Lie algebra of vector fields, $\{\, X^* \mid X \in \mathfrak{g}^{(0)} \,\}$ on P を z_0 の nbd で積分することによって容易に示される.

Proposition 1.2. (P,ω) *(resp.* (P',ω')*)* を M *(resp.*M'*)*上で定義された *Cartan connection of type* $G/G^{(0)}$ とする. ψ を $z_0 \in P$ の *nbd* で定義された *local isomorphism* $(P,\omega) \to (P',\omega')$ *as weak Cartan connections* とする. このとき, $x_0 = \pi(z_0)$ の *nbd* で定義された *local isomorphism* $\varphi: (P,\omega) \to (P',\omega')$ が存在して, $\psi = \tilde{\varphi}$ が z_0 の *nbd* で成立する.

1.2. The space $V^{p,2}(\mathfrak{C})$. Let us consider a class $\mathfrak{C}$ of normal connection of type $\mathfrak{G}$, (P,ω), satisfying certain invariance conditions, for example, such as $K^0 = 0$ or $K^0 = K^1 = 0$, etc. Every connection (P,ω) in the class $\mathfrak{C}$ satisfies the identities

$$\partial^* K^p = 0, \quad \text{and} \quad \partial K^p = \Psi^{p-1}, \quad p \geqq 0.$$

Now the function Ψ^{p-1} takes values in $C^{p-1,3}(\mathfrak{G})$, and we define a subspace $V^{p-1,3}(\mathfrak{C})$ of $C^{p-1,3}(\mathfrak{G})$ by

$$V^{p-1,3}(\mathfrak{C})^{\perp} = \{\, L \in C^{p-1,3}(\mathfrak{G})^* \mid L\Psi^{p-1} = 0 \quad \text{for any } (P,\omega) \in \mathfrak{C} \,\},$$

where $V^{p-1,3}(\mathfrak{C})^{\perp}$ stands for the subspace of $C^{p-1,3}(\mathfrak{G})^*$ orthogonal to $V^{p-1,3}(\mathfrak{C})$. Clearly every $(P,\omega) \in \mathfrak{C}$ satisfies the equations

$$L\partial K^p = \partial^* K^p = 0, \quad p > 0, \quad L \in V^{p-1,3}(\mathfrak{C})^{\perp}.$$

This being said, we define a subspace $V^{p,2}(\mathfrak{C})$ of $C^{p,2}(\mathfrak{G})$ by

$$V^{p,2}(\mathfrak{C}) = \{\, c \in C^{p,2}(\mathfrak{G}) \mid L\partial c = \partial^* c = 0 \quad \text{for any } L \in V^{p-1,3}(\mathfrak{C})^{\perp} \,\},$$

which is nothing but the smallest subspace of $C^{p,2}(\mathfrak{G})$ in which the function K^p corresponding to any $(P,\omega) \in \mathfrak{C}$ takes values.

Here we notice that a considerable amount of informations for the determination of $V^{p-1,3}(\mathfrak{C})$ are obtainable from the informations on the subspaces $V^{q,2}(\mathfrak{C})$, $q < p$. We also notice that $H^{p,2}(\mathfrak{G}) \subset V^{p,2}(\mathfrak{C})$. As we have already seen in §III.3, the determination of $H^{p,2}(\mathfrak{G})$ (at least for projective graded Lie algebras) can be easily carried out by the use of Proposition $**$, but we shall see in §V.3 the works towards the determination of $V^{p,2}(\mathfrak{C})$ require messy and complicated calculations.

1.3. Some remarks in the case where $\mathfrak{G}$ is a simple graded Lie algebra of the first kind. まず simple graded Lie algebra of the first kind は, [29] における irreducible l-system を意味している．よって [29] における結果はこのような graded Lie algebras に対して適用しうる．

さて $\mathfrak{G}$ を simple graded Lie algebra of the first kind とし, 付随する groups $G_0 = \mathrm{Aut}(\mathfrak{G})$, $G^{(0)} = G_0 \cdot G^{(1)}$, $G = \mathrm{Aut}(\mathfrak{g})^0 \cdot G_0$ を考える．$G^{(0)}$ は G における $\mathfrak{g}^{(0)}$ の normalizer である (cf. [29, Lemma 1.5]).

(P,ω) を connection of type $G/G^{(0)}$ on M．K をその curvature とする．K は $C^2(\mathfrak{G}) = \mathfrak{g} \otimes \Lambda^2(\mathfrak{g}_{-1}^*)$ の値を持っており, 分解 $C^2(\mathfrak{G}) = \sum_{p=0}^{2} C^{p,2}(\mathfrak{G})$ に従って

$$K = \sum_{p=0}^{2} K^p$$

と分解される．$C^{p,2}(\mathfrak{G}) = \mathfrak{g}_{p-1} \otimes \Lambda^2(\mathfrak{g}_{-1}^*)$ であることを注意しよう．よって K^0 は (P,ω) の torsion である．$(e_1,\ldots,e_n)$ を $\mathfrak{g}_{-1}$ の basis とすれば,

$$\partial^* K^0 = \qquad , \partial^* K^1 =$$

であり, $\partial^* K^2 = 0$. よって (P,ω) が normal connection of type $\mathfrak{G}$ であるとは, (P,ω) が [29] の意味で normal connection であることを意味していることがわかる．

$GL(\mathfrak{g}_{-1})$ の subgroup $G_0^{\sharp}$ は G_0 に一致しており, $G_0^{\sharp}$-structure of type $\mathfrak{M}$ は単に G_0-structure を意味している．よって Theorem IV.?? より, 次の Proposition をえる．

Proposition 1.3. $\mathfrak{G}$ が $(\mathfrak{M}, \mathfrak{g}_0)$ の *prolongation* であると仮定する, *i.e.,* $H^{0,1}(\mathfrak{G}) = H^{1,1}(\mathfrak{G}) = \{0\}$. このとき

$$\{\, M \text{ 上の } \textit{Cartan connections of type } \mathfrak{G} \,\} \overset{1:1}{\Longleftrightarrow} \{\, M \text{ 上の } G_0\textit{-structures} \,\}$$

この結果は [29] で既に得られている.

G の closed subgroup H を $H = \exp \mathfrak{g}_{-1} \cdot G_0$ よって定義する. H は $G^{(0)}$ の dual である. H の Lie algebra $\mathfrak{h}$ は $\mathfrak{h} = \mathfrak{g}_{-1} + \mathfrak{g}_0$ で与えられている.

Q を M 上の G_0-structure, ξ をその basic form とする. χ を Q 上の $\mathfrak{h}$-valued 1-form とし, χ_{-1} (resp. χ_0) を χ の $\mathfrak{g}_{-1}$- (resp. $\mathfrak{g}_0$-) component とする. (Q, χ) が connection of type H/G_0 であり, かつ $\chi_{-1} = \xi$ であるとき, χ を Q における affine connection という.

χ を Q における affine connection, S をその curvature とする: 即ち S は

$$d\chi + \frac{1}{2}[\chi, \chi] = \frac{1}{2}S(\chi_{-1} \wedge \chi_{-1})$$

を満足する unique な function $Q \to \mathfrak{h} \otimes \Lambda^2(\mathfrak{g}_{-1})^*$ である. 分解 $\mathfrak{h} \otimes \Lambda^2(\mathfrak{g}_{-1})^* = \mathfrak{g}_{-1} \otimes \Lambda^2(\mathfrak{g}_{-1})^* + \mathfrak{g}_0 \otimes \Lambda^2(\mathfrak{g}_{-1})^*$ に従って, S は $S = S_{-1} + S_0$ と分解される. S_{-1} を χ の torsion とよぶ. (S_0 は普通 linear connection χ_0 の curvature とよばれる).

Proposition 1.4 (cf.[29, Theorem 8.1]). *次の条件を満足する function $T : Q \to \mathfrak{g}_{-1} \otimes \Lambda^2(\mathfrak{g}_{-1})^*$ が unique に存在する:*

1) $\partial^* T = 0$.
2) *T は Q における affine connection の torsion S_{-1} である.*

T を Q の torsion とよぶ. これは Q の structure function (cf.[29]) の精密化である.

(P, ω) を M 上の normal connection of type $\mathfrak{G}$, Q を対応する G_0-structure, $\rho : P \to Q$ を自然な homomorphism とする. T を Q の torsion とし, χ を Q における affine connection で, $T = S_{-1}$ なるものとする.

Proposition 1.5 (cf.[29, Proposition 7.5]). (1) $K^0 = \rho^* T$.

(2) $K^0 = 0$. 従って, $T = S_{-1} = 0$ と仮定する. このとき, *function* $U : Q \to \mathfrak{g}_1$ が存在して,

$$K^1(X \wedge Y) = S_0(X \wedge Y) - [U(X), Y] + [U(Y), X], \quad X, Y \in \mathfrak{g}_{-1},$$

こゝで S_0, U は P 上の *functions* と思うべきである.

§2. Normal connections of the type $\mathfrak{G}$, type $\mathfrak{A}$, and type $\mathfrak{B}$

2.1. The normal connections of type $\mathfrak{G}$ associated with projective systems of type $(n, 1)$. Let $\mathfrak{P} = \{R; E, F\}$ be a projective system of type $(n, 1)$, and (P, ω) the corresponding normal connection of type $\mathfrak{G}$ on R. Let us consider the invariants K^p together with their harmonic part HK^p. First of all by Theorem IV.**??** we have

$$K^0 = 0.$$

It follows from Proposition IV.**??** that

$$HK^1 = K^1.$$

By Proposition III.3.17 the spaces of harmonic forms $H^{p,2}(\mathfrak{G})$ are calculated as follows: $H^{p,2}(\mathfrak{G}) = V^{p,2}(n, 1)$ for all p. Since the space $V^{p,2}(n, 1)$ vanishes for $p = 3, 4, 5$, it follows that

$$HK^3 = HK^4 = HK^5 = 0.$$

Therefore we know from Theorem 2.9 in [20] that $K^1 (= HK^1)$ and HK^2 form a fundamental system of invariants of the connection, and especially we get the following

Theorem 2.1. $K^1 = HK^2 = 0$ *if and only if the given projective system $\mathfrak{P}$ is locally isomorphic to the standard projective system of type $(n, 1)$, $\mathfrak{P}_0$.*

Consider the natural homomorphism κ of $G^{(0)}$ onto G_0. By Proposition IV.?? we know that K^1 has a tensorial property in the following sense:

$$R_a^* K^1 = (K^1)^{\kappa(a)}, \quad a \in G^{(0)}.$$

We shall now study the invariants $K^2 = K^2_{-1} + K^2_0 + K^2_1$ and HK^2. K^2_0 takes values in $A_0^{1,1}$, and this space is decomposed as follows:

$$A_0^{1,1} = \mathfrak{g}_0 \otimes \mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^+_{-1})^* + \mathfrak{g}_0 \otimes \mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^-_{-1})^*.$$

Correspondingly K^2_0 is decomposed as follows:

$$K^2_0 = {}^+K^2_0 + {}^-K^2_0.$$

Proposition 2.2. (1) K^2_{-1} *takes values in* $\mathfrak{g}^-_{-1} \otimes \Lambda^2 \mathfrak{g}^*_{-2}$.

(2) $K^2_1 = 0$.

(3) $\partial({}^-K^2_0) = \partial^*({}^-K^2_0) = 0$, *i.e.,* ${}^-K^2_0$ *takes values in* $V^{2,2}(n,1)$.

This fact together with Proposition 2.5 below will be proved in §4.

We assert that $HK^2 = {}^-K^2_0$. Let us consider the inner product $(\,\cdot\,,\,\cdot\,)$ in $C^{2,2}(\mathfrak{G})$. Putting $L = K^2_{-1} + {}^+K^2_0$ we have $K^2 = L + {}^-K^2_0$ and $(L, {}^-K^2_0) = 0$. Since HK^2 takes values in $V^{2,2}(n,1)$, we also see that $(L, HK^2) = 0$. Since $H({}^-K^2_0) = {}^-K^2_0$ by (3) of Proposition 2.2, we therefore obtain

$$(HK^2 - {}^-K^2_0, HK^2 - {}^-K^2_0) = (HK^2 - {}^-K^2_0, K^2 - {}^-K^2_0) = (HK^2 - {}^-K^2_0, L) = 0,$$

whence $HK^2 = {}^-K^2_0$, proving our assertion. Thus we have proved the following

Proposition 2.3. $HK^2 = {}^-K^2_0$

From the fact we can derive a tensorial property for HK^2. Namely we have

Proposition 2.4. $R_a^*(HK^2) = (HK^2)^{\kappa(a)}, \quad a \in G^{(0)}$.

Proof. By Proposition IV.?? we have $R_a^* K^2 = (K^2)^a$, $a \in G_0$, whence $R_a^*({}^-K^2_0) = ({}^-K^2_0)^a$, $a \in G_0$. Let $X \in \mathfrak{g}_r$, where $r = 1$ or 2. Again by Proposition IV.??, we have $\delta_X K^2 = (K^{2-r})^X$. Hence for any $U \in \mathfrak{g}_{-2}$ and $Y \in \mathfrak{g}^-_{-1}$ we obtain

$$(\delta_X K^2)(U \wedge Y) = -[X, K^{2-r}(U \wedge Y)] + K^{2-r}(\rho(X)U \wedge Y) + K^{2-r}(U \wedge \rho(X)Y).$$

Since $K^0 = 0$ and since $K^1 = HK^1$ takes values in $V^{1,2}(n,1)$, we find that the right hand side of the equality above vanishes. Hence $(\delta_X K^2)(U \wedge Y) = 0$, meaning that $\delta_X({}^-K^2_0) = 0$. Thus the proposition follows. □

Finally concerning the invariant $K^3 = K^3_0 + K^3_1 + K^3_2$, we have the following

Proposition 2.5. $K^3_2 = 0$.

We mention that the curvature satisfies further linear relations, which will be stated in Chapter VI[1]. (cf. Proposition 3.1 under the assumption that $HK^2 = 0$; Propositions 3.4 and 3.7 under assumption that $K^1 = 0$.)

2.2. Normal connections of type $\mathfrak{A}$ and projective structures in the usual sense.

simple graded Lie algebra $\mathfrak{A}$ と groups A_0, $A^{(0)}$, G を考えよう．これらの group は $\mathfrak{A}$ に付随する groups である．$G/A^{(0)} = \mathbb{P}^n$ である．A_0 は $G/A^{(0)}$ の原点における linear isotropy group であり，$GL(\mathfrak{a}_{-1})$ に同一視され，従って $\mathfrak{a}_0$ は $\mathfrak{gl}(\mathfrak{a}_{-1})$ に同一視される．$\mathfrak{A}$ は，[28] における graded Lie algebra $\mathfrak{g} = V + \mathfrak{gl}(V) + V^*$ と同型であることを注意する．また，[29] に従えば，$\mathfrak{A}$ は irreducible l-system of type $P^n(\mathbb{R})$ である．$\mathfrak{A}$ は $(\mathfrak{a}_{-1}, \mathfrak{a}_0)$ の prolongation でないことを注意する．

[1] Chapter V, Section 4 と思われる。

容易に $H^{0,2}(\mathfrak{A})=\{0\}$ が示される．よって任意の normal connection of type $\mathfrak{A}$, (P,ω), は torsion を持たない．これは (P,ω) が normal projective connection であることを示している．故に次の Proposition をえる．

Proposition 2.6 (cf. [28] and [29]). *manifold M 上の normal connections of type $\mathfrak{A}$, (P,ω), と M 上の projective structures in the usual sense, $\mathfrak{d}$, とは自然な仕方で 1:1 に対応する．*

(P,ω) を normal connection of type $\mathfrak{A}$ on M (この connection を $(P,\omega)_{\mathfrak{A}}$ とかく), K をその curvature とする．K は $C^2(\mathfrak{A})=\mathfrak{g}\otimes\Lambda^2\mathfrak{a}_{-1}^*$ の値を持っており, 分解 $C^2(\mathfrak{A})=\sum_{p=0}^{2}C^{p,2}(\mathfrak{A})$ に従って,

$$K=\sum_{p=0}^{2}K_{\mathfrak{A}}^{p}$$

と分解する．$K_{\mathfrak{A}}^0$ は $(P,\omega)_{\mathfrak{A}}$ の torsion であり, zero である．$\mathfrak{m}=\mathfrak{a}_{-1}+\mathfrak{g}_{-1}^{-}$ であるから, $C^2(\mathfrak{A})=\mathfrak{g}\otimes\Lambda^2\mathfrak{a}_{-1}^*$ は $C^2(\mathfrak{G})=\mathfrak{g}\otimes\Lambda^2\mathfrak{m}^*$ の subspace とみなされる．よって K は $C^2(\mathfrak{G})$ の値を持つ function とみなされ, 分解 $C^2(\mathfrak{G})=\sum_{p=-1}^{5}C^{p,2}(\mathfrak{G})$ に従って,

$$K=\sum_{p=-1}^{5}K^{p}$$

と分解される．

さて M の frame bundle $F(M)=F(M,\mathfrak{a}_{-1})$ と Grassmann bundle $G^1(T(M))$ を考える．$y\in F(M)$ とする．y は isomorphism $\mathfrak{a}_{-1}\to T_x(M)$ を導く．こゝで x は y の origin である．$\mathfrak{g}_{-1}^{+}$ は $\mathfrak{a}_{-1}$ の 1-subspace であるから $y\cdot\mathfrak{g}_{-1}^{+}$ は M の 1-dim. contact element である: $y\cdot\mathfrak{g}_{-1}^{+}\in G^1(T(M))$. 対応 $y\mapsto y\cdot\mathfrak{g}_{-1}^{+}$ は onto であり, $F(M)$ は $G^1(T(M))$ 上の fibred manifold になる．

$$\begin{tikzcd} F(M) \arrow[rr] \arrow[dr] & & G^1(T(M)) \arrow[dl] \\ & M & \end{tikzcd}$$

$\rho_{\mathfrak{A}}$ を自然な homomorphism $P\to F(M)$ とする．($A_0=GL(\mathfrak{a}_{-1})$ だから $\rho_{\mathfrak{A}}$ は surjective である) $z\in P$, $a\in A^{(0)}$ とすれば, $\rho_{\mathfrak{A}}(za)\mathfrak{g}_{-1}^{+}=\rho_{\mathfrak{A}}(z)\mathfrak{g}_{-1}^{+}\iff a\in G^{(0)}$ であることが容易に示される (Lemma 2.9). よって, $G^1(T(M))$ は

$$G^1(T(M))=P/G^{(0)}$$

と表現される．よって P は $G^1(T(M))$ 上の $G^{(0)}$-principal bundle である．

$$\begin{tikzcd} P \arrow[rr] \arrow[dr] & & G^1(T(M)) \arrow[dl] \\ & M & \end{tikzcd}$$

Proposition 2.7. (1) *(P,ω) は normal connection of type $\mathfrak{G}$ on $G^1(T(M))$ であり, K はその curvature である (この connection を $(P,\omega)_{\mathfrak{G}}$ とかく)*

(2) $K^0=HK^2=0$

$K^0=0$ であるから normal connection $(P,\omega)_{\mathfrak{G}}$ には projective system of type $(n,1)$, $\mathfrak{P}=\{G^1(T(M));E,F\}$ が対応する．容易にわかるように F は the system of vertical vectors of the fibred manifold $G^1(T(M))$ over M である (cf.Th.∗∗). よって $\mathfrak{X}=(G^1(T(M)),E)$ は M 上の projective system of type $(n,1)$ である．$\mathfrak{d}$ を normal connection of type $\mathfrak{A}$ on

M, (P,ω), に対応する projective structure in the usual sense とする. $\mathfrak{X}(\mathfrak{d})$ を $\mathfrak{d}$ に対応する projective structure of the restricted type, $\mathfrak{P}(\mathfrak{d})$ を $\mathfrak{X}(\mathfrak{d})$ に対応する projective system of type $(n,1)$ とする.

Proposition 2.8. $\mathfrak{P}=\mathfrak{P}(\mathfrak{d})$ *and* $\mathfrak{X}=\mathfrak{X}(\mathfrak{d})$.

この定理の証明は, 次の事実を使って容易に示される. $\nabla\in\mathfrak{d}$ を fix する. このとき, $\gamma(t)$ が ∇ の geodesic ならば, $X\in\mathfrak{a}_{-1}$ と $\check{\omega}(X)$ の integral curve $\tilde{\gamma}(s)$ があって, $\gamma=\tilde{\gamma}$ as subsets. また逆も成立する.

Lemma 2.9. $a\in A^{(0)}$, $\rho_{\mathfrak{A}}(a)\mathfrak{g}^{+}_{-1}=\mathfrak{g}^{+}_{-1}\Rightarrow a\in G^{(0)}$

Proof. $a\in A_0$ としてよい. $\rho_{\mathfrak{A}}(a)\mathfrak{g}^{+}_{-1}=\mathfrak{g}^{+}_{-1}$ は, $\mathrm{Ad}(a)\mathfrak{g}^{+}_{-1}\subset\mathfrak{g}^{+}_{-1}+\mathfrak{a}^{(0)}=\mathfrak{g}^{(-1)}$ を意味する. $\langle\cdot,\cdot\rangle$ に関する $\mathfrak{g}^{(-1)}$ の annihilator は $\mathfrak{g}_2$ である. よって $\mathrm{Ad}(a)\mathfrak{g}_2=\mathfrak{g}_2$. $\mathfrak{g}^{(1)}=[\mathfrak{g}^{(-1)},\mathfrak{g}_2]$ であるから $\mathrm{Ad}(a)\mathfrak{g}^{(1)}=\mathfrak{g}^{(1)}$. $\mathfrak{g}^{(0)}=[\mathfrak{g}^{(-1)},\mathfrak{g}^{(1)}]+\mathfrak{g}^{(1)}$ であるから $\mathrm{Ad}(a)\mathfrak{g}^{(0)}=\mathfrak{g}^{(0)}$. よって $a\in\mathrm{Aut}(\mathfrak{G})$. $a\in\mathrm{Aut}(\mathfrak{G})\cap G=\mathrm{Aut}(\mathfrak{g})^0\cap\mathrm{Aut}(\mathfrak{G})\subset G_0$. □

2.3. Normal connections of type $\mathfrak{B}$ and allowable B_0-structures. simple graded Lie algebra of the first kind, $\mathfrak{B}$, と groups B_0, $B^{(0)}$, G を考える. これらの group は $\mathfrak{B}$ に付随する groups である. $G/B^{(0)}=G^r(\mathbb{P}^n)=G^{r+1}(\mathbb{R}^{n+1})$ である. B_0 は $G/B^{(0)}$ の原点における linear isotropy group である. $(a,b)\in GL(n-1,\mathbb{R})\times GL(2,\mathbb{R})$ に対して, $M_{n-1,2}(\mathbb{R})$ の linear automorphism $\eta(a,b)$ を

$$\eta(a,b)X=aXb^{-1},\quad X\in M_{n-1,2}(\mathbb{R})$$

によって定義すれば, B_0 は $\mathfrak{b}_{-1}$ の basis $(e_1,\ldots,e_{2n-2})$ に関して, B_0 は $\eta(a,b)$ なる形の $M_{n-1,2}(\mathbb{R})$ の linear automorphisms 全体の作る linear group として表現される.

容易に $\mathfrak{B}$ は, $(\mathfrak{b}_{-1},\mathfrak{b}_0)$ の prolongation であること, i.e., $H^{1,1}(\mathfrak{B})=H^{2,1}(\mathfrak{B})=\{0\}$ なることが確かめられる. よって次の Prop. をえる.

Proposition 2.10. *manifold N 上の normal connections of type $\mathfrak{B}$, (P,ω), と N 上の B_0-structures, Q, とは自然な仕方で 1:1 に対応する.*

Remark 2.1. $\mathfrak{B}\cong\mathfrak{M}_2(4)$, Möbius algebra. B_0-structure は, conformal structure.

(P,ω) を normal connection of type $\mathfrak{B}$ on N, K をその curvature とする. (この connection を $(P,\omega)_{\mathfrak{B}}$ とかく). K は $C^2(\mathfrak{B})=\mathfrak{g}\otimes\Lambda^2\mathfrak{b}^*_{-1}$ の値を持っており, 分解 $C^2(\mathfrak{B})=\sum\limits_{p=0}^{2}C^{p,2}(\mathfrak{B})$ に従って

$$K=\sum_{p=0}^{2}K^p_{\mathfrak{B}}$$

と分解される. $\mathfrak{m}=\mathfrak{b}_{-1}+\mathfrak{g}^{+}_{-1}$ であるから, $C^2(\mathfrak{B})=\mathfrak{g}\otimes\Lambda^2\mathfrak{b}^*_{-1}$ は, $C^2(\mathfrak{G})=\mathfrak{g}\otimes\Lambda^2\mathfrak{m}^*$ の subspace とみなされる. よって K は $C^2(\mathfrak{G})$ の値を持つ function とみなされ, 分解 $C^2(\mathfrak{G})=\sum\limits_{p=-1}^{5}C^{p,2}(\mathfrak{G})$ に従って

$$K=\sum_{p=-1}^{5}K^p$$

と分解される.

Q を normal connection $(P,\omega)_{\mathfrak{B}}$ に対応する N 上の B_0-structure とする. $y\in Q$ とする. y は isomorphism $\mathfrak{b}_{-1}\to T_x(N)$ を導く. こゝで x は y の origin である. $\mathfrak{g}^{-}_{-1}$ は $\mathfrak{b}_{-1}$ の $(n-1)$-dim. subspace であるから, $y\cdot\mathfrak{g}^{-}_{-1}$ は N の $(n-1)$-dim. contact element である: $y\cdot\mathfrak{g}^{-}_{-1}\in G^{n-1}(T(N))$. 対応 $y\mapsto y\cdot\mathfrak{g}^{-}_{-1}$ による Q の像を $R(Q)$ で表す. $R(Q)$ は $G^{n-1}(T(N))$

の submanifold であり, $\dim R(Q) = 2n-1$ である．また $R(Q)$ は N 上の, Q は $R(Q)$ 上の fibred manifold である．

$$\begin{tikzcd} Q \arrow[r] \arrow[dr] & R(Q) \arrow[r, hook] \arrow[d] & G^{n-1}(T(N)) \arrow[dl] \\ & N & \end{tikzcd}$$

$\rho_{\mathfrak{P}}$ を自然な homomorphism $P \to Q$ とする．$z \in P$, $a \in B^{(0)}$ とすれば, $\rho_{\mathfrak{P}}(za)\mathfrak{g}^-_{-1} = \rho_{\mathfrak{P}}(z)\mathfrak{g}^-_{-1} \Leftrightarrow a \in G^{(0)}$ であることが容易に示される．($a \in B^{(0)}$, $\rho_{\mathfrak{P}}(a)\mathfrak{g}^+_{-1} = \mathfrak{g}^+_{-1} \Rightarrow a \in G^{(0)}$). よって $R(Q)$ は

$$R(Q) = P/G^{(0)}$$

と表現される．よって P は $R(Q)$ 上の $G^{(0)}$-principal bundle である．

$$\begin{tikzcd} P \arrow[r] \arrow[dr] & R(Q) \arrow[r, hook] \arrow[d] & G^{n-1}(T(N)) \arrow[dl] \\ & N & \end{tikzcd}$$

以下 normal connection $(P,\omega)_{\mathfrak{P}}$ は torsion を持たない, i.e., $K^0_{\mathfrak{P}} = 0$ と仮定する．この仮定は Q が torsion を持たないことゝ equivalent である．

Proposition 2.11. (1) (P,ω) は *normal connection of type* $\mathfrak{G}$ *on* $R(Q)$ であり, K はその *curvature* である．この *connection* を $(P,\omega)_{\mathfrak{G}}$ とかく．

(2) $K^0 = 0$ ならば, $K^1 = 0$ である．

χ を Q における torsion zero の affine connection, $S : Q \to \mathfrak{b}_0 \otimes \Lambda^2\mathfrak{b}^*_{-1}$ をその curvature とする．

Lemma 2.12. $X, Y \in \mathfrak{g}^-_{-1}$ とする．このとき, $K^1_{\mathfrak{P}}(X \wedge Y)$ と $(\rho^*_{\mathfrak{P}}S)(X \wedge Y)$ の $\mathfrak{g}^+_{-1}$-*component* は等しい．

そこで function $T' : Q \to \mathfrak{g}^+_{-1} \otimes \Lambda^2(\mathfrak{g}^-_{-1})^*$ を

$$T'(X \wedge Y) = \text{the } \mathfrak{g}^+_{-1}\text{-component of } S(X \wedge Y)$$

によって定義すれば, Lemma 2.12 から T' は, B_0-structure Q だけから決まることがわかる．

$n \geqq 4$ のとき, Theorem IV.**??** から $K^0 = 0$ である．よって Propositions III.3.17 and IV.**??** から $K^1 = 0$ をえる．また, Lemma 2.12 から $T' = 0$ が従うことを注意する．次に, $n = 3$ の場合を考える．このとき, $K^0 = HK^0$ であり, K^0 は $V^{0,2}(\mathfrak{G})$-valued である．$X, Y \in \mathfrak{g}^-_{-1}$ のとき, $K^0(X \wedge Y) =$ the $\mathfrak{g}^+_{-1}$-component of $K^1_{\mathfrak{P}}(X \wedge Y)$ である．よって Lemma 2.12 から, $K^0 = 0 \Leftrightarrow T' = 0$ が分かる．また, $K^0 = 0$ のとき, Propositions III.3.17 and IV.**??** から $K^1 = 0$ が従う．故に次の Prop. をえる．

Proposition 2.13. (1) $n \geqq 4$ のとき, $K^0 = K^1 = 0$

(2) $n = 3$ のとき, $K^0 = 0 \Longleftrightarrow T' = 0$. またこの場合, $T' = 0$ ならば, $K^1 = 0$.

$T' = 0$ と仮定する．このとき $K^0 = 0$ である．よって, normal connection $(P,\omega)_{\mathfrak{G}}$ には projective system of type $(n,1)$, $\mathfrak{P} = \{R(Q); E, F\}$ が対応する．容易に分るように E は the system of vertical vectors of the fibred manifold $R(Q)$ on N である (cf. Theorem $**$). よって $\mathfrak{Y} = (R(Q), F)$ は N 上の GP system of type $(2n-2, n-1)$ である．さらに Proposition 2.10 から F は $R(Q)$ だけから決まることがわかる．そして Proposition 2.13 から Q は $T' = 0$ を満足する任意の B_0-structure without torsion になりうることを注意する．

Q を B_0-structure とする. Q の torsion T が zero であり $T' = 0$ であるとき, Q を allowable B_0-structure とよぶ. 上の議論から任意の allowable B_0-structure Q on N に対して自然な仕方で projective system of type $(n, 1)$, $\mathfrak{P}$, と GP system of type $(2n-2, n-1)$, $\mathfrak{Y}$, が対応することがわかった.

§3. Reduction theorems

3.1. Reduction theorems, (1). $\mathfrak{P} = \{R; E, F\}$ を projective system of type $(n, 1)$ とする. R は E と F の両方について regular であると仮定する. $M = R/F$, $N = R/E$ とし, $\varpi_M : R \to M$, $\varpi_N : R \to N$ を projection とする. $\iota_M : R \to G^1(T(M))$, $\iota_N : R \to G^{n-1}(T(N))$ を canonical immersions とする. $R \to M \times N$, ι_M, ι_N は imbeddings と仮定する. このとき $\mathfrak{X} = (R, E)$ は M 上の projective structure of type $(n, 1)$ であり, $\mathfrak{Y} = (R, F)$ は N 上の GP-system of type $(2n-2, n-1)$ であり, $\mathfrak{X}$ の dual である. (P, ω) を $\mathfrak{P}$ に対応する R 上の normal connection of type $\mathfrak{G}$, K をその curvature とする (この connection を $(P, \omega)_{\mathfrak{G}}$ で表す.)

次の Proposition は §4.4 で証明される.

Proposition 3.1. *$HK^2 = 0$ を仮定する. このとき K は $\mathfrak{a}^{(0)} \otimes \Lambda^2(\mathfrak{a}_{-1})^*$ の値を持つ.*

この Proposition を使って次の Theorem を証明しよう.

Theorem 3.2. *$HK^2 = 0$ と仮定する. このとき $\mathfrak{X}$ は projective structure of the restricted type である.*

$\mathfrak{d}$ を $(P, \omega)_{\mathfrak{G}}$ に対応する projective structure in the usual sense, $\mathfrak{X}(\mathfrak{d}) = \{G^1(T(M)), E(\mathfrak{d})\}$ を $\mathfrak{d}$ に対応する projective structure of the restricted type とする. さらに $(P(\mathfrak{d}), \omega(\mathfrak{d}))$ を $\mathfrak{X}(\mathfrak{d})$ に対応する $G^1(T(M))$ 上の normal connection of type $\mathfrak{G}$ とする. このとき $\mathfrak{X}$ は $\mathfrak{X}(\mathfrak{d})$ の制限として得られる, i.e., $E = E(\mathfrak{d})$ on R (§I.∗∗). また下の証明から $(P, \omega)_{\mathfrak{G}}$ は, $(P(\mathfrak{d}), \omega(\mathfrak{d}))_{\mathfrak{G}}$ の R 上への制限であることが分る.

§5.3 で Theorem 3.2 の別証明を与える.

Theorem 3.2 と §III.8.1[2] の議論から大雑把に言って,

$$\{\text{projective systems of type } (n, 1), \mathfrak{P} \text{ (or projective structures of type } (n, 1), \mathfrak{X}) \text{ with } HK^2 = 0\}$$
$$\overset{1:1}{\Longleftrightarrow} \{\text{projective structures in the usual sense}, \mathfrak{d}\}$$

がわかる.

Theorem 3.2 の証明に入ろう.

Lemma 3.3. *(P, ω) は M 上の infinitesimal normal connection of type $\mathfrak{A}$ である. (この connection を $(P, \omega)_{\mathfrak{A}}$ とかく).*

以下 §III.2 の i.c. connection に関する結果を使う.

$$\begin{tikzcd}[column sep=small]
z_0 \in P & (P, \omega) \arrow[d] \arrow[dd, bend left, "\mathfrak{P}"] & & (P', \omega') \arrow[d] \arrow[dd, bend left, "\mathfrak{P}'"] \\
& R \arrow[d] & & G^1(T(M')) \arrow[d] \\
& M & & M'
\end{tikzcd}$$

$$\begin{aligned}
&(P, \omega)_{\mathfrak{A}} \xrightarrow{\psi} (P', \omega')_{\mathfrak{A}} \quad \text{as i.c.} \\
&(P, \omega)_{\mathfrak{G}} \xrightarrow{\psi} (P', \omega')_{\mathfrak{G}} \quad \text{as i.c.} \\
&(P, \omega)_{\mathfrak{G}} \xrightarrow{\psi} (P', \omega')_{\mathfrak{G}} \quad \text{as c.} \\
&R \xrightarrow[\varphi]{} G^1(T(M))
\end{aligned}$$

3.2. Reduction theorems, (2). $\mathfrak{P} = \{R; E, F\}, \cdots, (P, \omega)$ などは §III.9.1[3] と同様とする.

次の Proposition は §4.5 で証明される.

[2]§V.2.2 と思われる。
[3]§V.3.1 と思われる。

Proposition 3.4. $K^1 = 0$ と仮定する．このとき K は $\mathfrak{b}^{(0)} \otimes \Lambda^2(\mathfrak{b}_{-1})^*$ の値を持つ．

この Proposition を使って次の Theorem を証明しよう．まず任意の allowable B_0-structure on N, Q, には N 上の GP-system $\mathfrak{Y}(Q) = \{R(Q); F(Q)\}$ が対応することを思い出しておこう．

Theorem 3.5. $K^1 = 0$ を仮定する．このとき，*unique* な *allowable* B_0*-structure on* N, Q, が存在して，$\mathfrak{X}$ の *dual* $\mathfrak{Y}$ は，$\mathfrak{Y}(Q)$ の R への制限として得られる．即ち，i) $R \subset R(Q)$，従って，R は $R(Q)$ の *open submanifold* であり，ii) E は，$E(Q)$ の $R(Q)$ への制限である．

$(P(Q), \omega(Q))$ を Q に対応する normal connection of type $\mathfrak{B}$ on $R(Q)$ とする．このとき下の証明から $(P, \omega)_{\mathfrak{G}}$ は，$(P(Q), \omega(Q))_{\mathfrak{G}}$ の $R(Q)$ への制限であることが分る．

Theorem 3.5 と §III.8.2[4] の議論から大雑ぱにいって

$$\{\text{projective systems of type } (n,1), \mathfrak{P}, \ (\text{or projective structures of type } (n,1), \mathfrak{X}) \text{ with } K^1 = 0\}$$
$$\underset{1:1}{\Longleftrightarrow} \{\text{allowable } B_0\text{-structures}, Q\}$$

がわかる．

Theorem 3.5 の証明に入ろう．

Lemma 3.6. (P, ω) は M 上の *infinitesimal normal connection of type* $\mathfrak{B}$ である．*(この connection を* $(P, \omega)_{\mathfrak{B}}$ *とかく)*.

以下 §III.2 の i.C. connections に関する結果を使う．

$$\begin{tikzcd}
z_0 \in P \arrow[d] & (P,\omega) \arrow[d] \arrow[dd, bend left, "\mathfrak{P}"] & (P',\omega') \arrow[d] \arrow[dd, bend left, "\mathfrak{P}(Q')"] & (P'',\omega'') \arrow[d] \arrow[dd, bend left, "\mathfrak{P}(Q'')"] \\
y_0 \in R \arrow[d] & R \arrow[d] & R(Q') \arrow[d] & R(Q'') \arrow[d] \\
x_0 \in N & N & N' & N''
\end{tikzcd}$$

$$(P,\omega)_{\mathfrak{A}} \xrightarrow{\psi} (P',\omega')_{\mathfrak{A}} \quad \text{as i.C.c.}$$

は

$$(P,\omega)_{\mathfrak{G}} \xrightarrow{\psi} (P',\omega')_{\mathfrak{G}} \quad \text{as i.C.c.}$$

を導く．これは

$$\psi = \tilde{\varphi}$$

$$\begin{tikzcd}
(P,\omega)_{\mathfrak{G}} \arrow[r, "\varphi"] \arrow[d] & (P',\omega')_{\mathfrak{G}} \arrow[d] & \text{as C.c} \\
R \arrow[r] \arrow[d] & R(Q') \arrow[d] & \varphi = \hat{\theta} \\
N \arrow[r, "\theta"'] & N' &
\end{tikzcd}$$

$$\begin{array}{ccc}
F(N) & \xrightarrow[\theta_*]{} & F(N') \\
\cup & & \cup \\
Q = \theta_*^{-1}(Q') & & Q'
\end{array}$$

[4]§V.2.3 と思われる。

$$\tau = \theta' \circ \theta^{-1}$$

$$\begin{tikzcd}
(P',\omega')_{\mathfrak{B}} \arrow[rr, "\hat{\tau}"] \arrow[d] \arrow[ddd, bend right] & & (P'',\omega'')_{\mathfrak{B}} \arrow[d] \arrow[ddd, bend left] \\
R(Q') \arrow[d] & R \arrow[l, "\hat{\theta}"'] \arrow[r, "\hat{\theta}'"] \arrow[d] & R(Q'') \arrow[d] \\
N' & N \arrow[l, "\theta"'] \arrow[r, "\theta'"] & N'' \\
F(N') \arrow[u] \arrow[rr, bend right, "\tau_*"'] & F(N) \arrow[l, "\theta_*"'] \arrow[r, "\theta'_*"] \arrow[u] & F(N'') \arrow[u]
\end{tikzcd}$$

$R \to M$ の fibres は connected. よって Q は globally defined

Remark. $\mathfrak{L}$ を projective FGLA of type (n,r), $\mathfrak{G}$ をその prolongation とする. $\mathfrak{G}$ には simple graded Lie algebras $\mathfrak{A}$, $\mathfrak{B}$ が付随する. $\mathfrak{P} = \{R; E, F\}$ を projective system of type (n,r), (P,ω) を対応する normal connection of type $\mathfrak{G}$ とする. $r = n-1$, $n \geqq 3$ ならば,

$$K^0 = HK^2 = HK^3 = HK^4 = HK^5 = 0$$

である. そして $HK^1 = K^1$ が基本的不変量であり, $H^{1,2}(\mathfrak{G}) \subset \mathfrak{h}_0 \otimes (\mathfrak{g}_{-1}^+)^* \wedge (\mathfrak{g}_{-1}^-)^*$ の値を持つ. また $2 \leqq r \leqq n-2$ ならば,

$$HK^1 = HK^2 = HK^3 = HK^4 = HK^5 = 0$$

である. そして HK^0 が基本的不変量であり, $V_{(3)}^{0,2}(n, n-r) \subset \mathfrak{g}_{-1}^- \otimes (\mathfrak{g}_{-1}^+)^* \wedge (\mathfrak{g}_{-1}^-)^*$ の値を持つ.

故に $r = n-1$, $n \geqq 3$ または $2 \leqq r \leqq n-2$ の場合には, (P,ω) が IC connection of type $\mathfrak{A}$ or of type $\mathfrak{B}$ に reduce するとすれば, 即ち curvature K が $\mathfrak{g} \otimes \Lambda^2(\mathfrak{a}_{-1})^*$ または $\mathfrak{g} \otimes \Lambda^2(\mathfrak{b}_{-1})^*$ の値を取れば, 必然的に $K = 0$ となり従って $\mathfrak{P}$ は standard system $\mathfrak{P}_0$ に locally isomorphic になる. 従って Theorems 3.2, 3.5 の形の Reduction theorems は質的な意味を持たない.

$n = 2$, $r = 1$ の場合には, Cartan によって Reduction theorems が得られている. (see Appendix)

Reduction theorems は独立変数が 1 個であるところの常微分方程式系に特有な現象であり, $k \geqq 3$ 階以上のすべての常微分方程式系に対して成立するものと思われる.

3.3. Further properties of the curvatures of projective systems of type $(n,1)$ with $K^1 = 0$. (P,ω) は上と同様とする.

$$K^3 = K_0^3 + K_1^3 + K_2^3, \quad K^4 = K_1^4 + K_2^4, \quad K^5 = K_2^5$$

であることを思い出しておこう. §4.5 で Proposition 3.4 より詳しい次の Proposition を証明する.

Proposition 3.7. *$K^1 = 0$ と仮定する.*

(1) $K^{-1} = K^0 = K^1 = 0$

(2) K^2 *は* $V^{2,2}(n,1)$ *の値を持つ.*

(3) i) $K_0^3 = K_2^3 = 0$, ii) K_1^3 *は,* $\mathfrak{g}_1^+ \otimes (\mathfrak{g}_{-2})^* \wedge (\mathfrak{g}_{-1}^-)^*$ *の値を持つ,* iii) $\langle K_1^3([e_n, x] \wedge y), z\rangle$ *は variables* $x, y, z \in \mathfrak{g}_{-1}^-$ *について symmetric である.*

(4) i) $K_1^4 = 0$, ii) K_2^4 *は,* $\mathfrak{g}_2 \otimes (\mathfrak{g}_{-2})^* \wedge (\mathfrak{g}_{-1}^-)^*$ *の値を持つ,* iii) $\langle K_2^4([e_n, x] \wedge y), [e_n, z]\rangle$ *は variables* $x, y, z \in \mathfrak{g}_{-1}^-$ *について symmetric である.*

(5) $K_2^5 = 0$

(6) $\delta_{e_n} K^2_0 = 0, \quad \delta_{e_n} K^3_1 = -[e_n, K^4_2], \quad \delta_{e_n} K^4_2 = 0.$

3.4. Examples.

§4. Proof of Propositions 2.2, 2.5, 3.1, 3.4, and 3.7

$\mathfrak{C}_1$ (resp. $\mathfrak{C}_2$; resp. $\mathfrak{C}_3$) を normal connection of type $\mathfrak{G}$, (P,ω) で $K^0 = 0$ (resp.$K^0 = HK^2 = 0$; resp. $K^0 = K^1 = 0$) を満足するものゝ作る class とする. $\mathfrak{C}_i$ に対して $C^{p,2}(\mathfrak{G})$ の subspace $V^{p,2}(\mathfrak{C}_i)$ が対応することを思いだそう (see §IV.1). このとき Propositions 2.2 and 2.5 (resp. Proposition 3.1), resp. Proposition 3.4) は $V^{p,2}(\mathfrak{C}_1)$ (resp. $V^{p,2}(\mathfrak{C}_2)$; resp. $V^{p,2}(\mathfrak{C}_3)$) の決定への研究である. また Proposition 3.7 は $V^{p,2}(\mathfrak{C}_3)$ を多分決定しているであろう.

4.1. The Bianchi identity. (P,ω) は $**$ と同様とする. (P,ω) に関する Bianchi identity は次の identities に分解する.

$$\partial K^p = \Psi^{p-1}$$

(§IV.1.4 をみよ). Ψ^{p-1} は $K^0, \dots, K^{p-1}$ だけに depend する量である. $K^0 = 0$ であり, $K^1 = HK^1$ が $V^{1,2}(n,1)$ の値を持つことを考慮して, Ψ^{p-1} を計算して次の Lemma をえる.

Lemma 4.1. $u, v, w \in \mathfrak{g}_{-2}$, $x, y, z \in \mathfrak{g}_{-1}$ とする.

(0) $(\partial K^p)(u \wedge v \wedge w) = -\delta_u K^{p-2}(v \wedge w) - \delta_v K^{p-2}(w \wedge u) - \delta_w K^{p-2}(u \wedge v) - K^{p-3}(K^2(u \wedge v) \wedge w) - K^{p-3}(K^2(v \wedge w) \wedge u) - K^{p-3}(K^2(w \wedge u) \wedge v)$

(1) $(\partial K^p)(u \wedge v \wedge x) = -\delta_u K^{p-2}(v \wedge x) - \delta_v K^{p-2}(x \wedge u) - \delta_x K^{p-1}(u \wedge v) - K^{p-3}(K^2(u \wedge v) \wedge x) - K^{p-2}(K^1(v \wedge x) \wedge u) - K^{p-2}(K^1(x \wedge u) \wedge v)$

(2) $(\partial K^p)(u \wedge x \wedge y) = -\delta_u K^{p-2}(x \wedge y) - \delta_x K^{p-1}(y \wedge u) - \delta_y K^{p-1}(u \wedge x) - K^{p-2}(K^1(u \wedge x) \wedge y) - K^{p-2}(K^1(y \wedge u) \wedge x)$

(3) $(\partial K^p)(x \wedge y \wedge z) = -\delta_x K^{p-1}(y \wedge z) - \delta_y K^{p-1}(z \wedge x) - \delta_z K^{p-1}(x \wedge y)$

K^p は $K^p = K^p_{p-3} + K^p_{p-2} + K^p_{p-1}$ と分解する. At the proofs of the propositions from now on, the invariants K^p, K^p_{p-3}, K^p_{p-2} and K^p_{p-1} will be simply denoted by c, c_{p-3}, c_{p-2} and c_{p-1} respectively. また例えば $p = 2$ のとき, $K^2_{-1} = c_{-1}$ は, $c_{-1} = c^+_{-1} + c^-_{-1}$ と分解する (cf. §III.5).

4.2. Proof of Proposition 2.2. この paragraph では invariant $K^2 = c = c_{-1} + c_0 + c_1$ を考える.

Lemma 4.2. (1) $(\partial c)(u \wedge v \wedge x) = 0$.

(2) $(\partial c)(u \wedge x \wedge y) = 0$ かつ $(\partial c)(u \wedge x \wedge e_n)$ は $\mathfrak{g}^-_{-1}$ の値を持つ.

(3) $(\partial c)(x \wedge y \wedge e_n) = 0$.

(4) $(\partial^*_2 \partial c)(x \wedge e_n) = 0$.

Proof. Lemma 4.1 と $K^0 = 0$ 及び $K^1 = HK^1$ が $V^{1,2}(n,1)$ の値を持つことから, 次の等式をえる.

$$\begin{aligned}
(\partial K^2)(u \wedge v \wedge x) &= 0,\\
(\partial K^2)(u \wedge x \wedge y) &= 0,\\
(\partial K^2)(u \wedge x \wedge e_n) &= -\delta_x K^1(e_n \wedge u),\\
(\partial K^2)(x \wedge y \wedge e_n) &= 0,
\end{aligned}$$

さらに $(\partial^*_2 K^1_{-1})(e_n) = 0$. よって Lemma をえる. □

Lemma 4.3. $c^+_1(x \wedge y) = 0$.

Proof. $(\partial c)(u \wedge x \wedge y) = 0$ より

$$(\partial_0 c_{-1} + \partial_1 c_0 + \partial_2 c_1)(u \wedge x \wedge y) = -[x, c_0(x \wedge y)] + [y, c_0(u \wedge x)] + [u, c_1(x \wedge y)] = 0$$

をえる．これより

$$-[x, c_0(x \wedge y)] + [y, c_0(u \wedge x)] + [u, c_1^-(x \wedge y)] = 0$$

及び

$$[u, c_1^+(x \wedge y)] = 0$$

をえる．よって Lemma III.3.1 より $c_1^+(x \wedge y) = 0$ が従う． □

Lemma 4.4. (1) $2\lambda_1^+ c_{-1}^+(u \wedge v) - [u, a(v)] - \sum_i [e^{n+i}, [u, c_0(v \wedge e_{n+i})]] + [v, c_1^+([e^n, u] \wedge e_n)] = 0$. こゝで $a(v) = \sum_i [e^{n+i}, c_0(e_{n+i} \wedge v)]$.

(2) $(\partial_2^* c_{-1}^+)(u) + a(u) + c_1^+([e^n, u] \wedge e_n) = 0$.

(3) $\partial_1^* \partial_1 c_{-1}^+ + \partial_2 \partial_2^* c_{-1}^+ + 4\lambda_1^+ c_{-1}^+ = 0$.

(4) $c_{-1}^+ = 0$.

Proof. (1)

$$\begin{aligned}(\partial c)(v \wedge x \wedge e_n) &= (\partial_0 c_{-1} + \partial_1 c_0 + \partial_2 c_1)(v \wedge x \wedge e_n) \\ &= -c_{-1}([x, e_n] \wedge v) - [x, c_0(v \wedge e_n)] + [e_n, c_0(v \wedge x)] + [v, c_1(x \wedge e_n)]\end{aligned}$$

$(\partial c)(v \wedge x \wedge e_n)$ は $\mathfrak{g}_{-1}^-$ の値を持つから，

$$-c_{-1}^+([x, e_n] \wedge v) + [e_n, c_0(v \wedge x)] + [v, c_1^+(x \wedge e_n)] = 0$$

よって

$$-c_{-1}^+([[e^n, u], e_n] \wedge v) + [e_n, c_0(v \wedge [e^n, u])] + [v, c_1^+([e^n, u] \wedge e_n)] = 0$$

をえる．$[[e^n, u], e_n] = [E_1^+, u] = -2\lambda_1^+ u$ であり，$[e^n, u] = \sum_i \langle e^n, [u, e^{n+i}] \rangle e_{n+i}$ であるから，

$$\begin{aligned}[e_n, c_0(v \wedge [e^n, u])] &= \sum_i [[u, e^{n+i}], c_0(v \wedge e_{n+i})] \\ &= -[u, a(v)] - \sum_i [e^{n+i}, [u, c_0(v \wedge e_{n+i})]]\end{aligned}$$

故に (1) をえる．

(2) $\partial^* c = 0$ より，

$$\begin{aligned}&(\partial_2^* c_{-1} + \partial_1^* c_0 + \partial_0^* c_1)(u) \\ &= \sum_i [e^i, c_{-1}(e_i \wedge u)] + [e^n, c_0(e^n \wedge u)] + \sum_i [e^{n+i}, c_0(e_{n+i} \wedge u)] + c_1([e^n, u] \wedge e_n)\end{aligned}$$

をえる．こゝで $(\partial_0^* c_1)(u) = \sum_i c_1([e^{n+i}, u] \wedge e_{n+i}) = c_1([e^n, u] \wedge e_n)$ という事実を使った．故に

$$\sum_i [e^i, c_{-1}^+(e_i \wedge u)] + \sum_i [e^{n+i}, c_0(e_{n+i} \wedge u)] + c_1^+([e^n, u] \wedge e_n) = 0$$

をえる．よって (2) が示された．

(3) (1) と (2) から

$$\sum_i [e^{n+i}, [u, c_0(v \wedge e_{n+i})]] = 2\lambda_1^+ c_{-1}^+(u \wedge v) - [u, a(v)] - [v, a(u)] - [v, (\partial_2^* c_{-1}^+)(u)]$$

をえる．さて $(\partial c)(u \wedge v \wedge x) = 0$ だから，

$$(\partial_1 c_{-1} + \partial_2 c_0)(u \wedge v \wedge x) = [x, c_{-1}^+(u \wedge v)] + [u, c_0(v \wedge x)] - [v, c_0(u \wedge x)] = 0$$

これより

$$(\partial_1^* \partial_1 c_{-1}^+)(u \wedge v) + \sum_i [e^{n+i}, [u, c_0(v \wedge e_{n+i})]] - \sum_i [e^{n+i}, [v, c_0(u \wedge e_{n+i})]] = 0$$

をえる．以上により (3) をえる．

(4) $\lambda_1^+ > 0$ だから，(3) から $c_{-1}^+ = 0$ が従う． □

Lemma 4.5. (1) $-\sum_i [e^{n+i}, c_0(e_i \wedge x)] - \sum_i [e_n, [e^{n+i}, c_1^-(e_{n+i} \wedge x)]] + (\lambda_1^+ + \mu_1^+ - \lambda_2 - \mu_2) c_1^+(e^n \wedge x) = 0.$

(2) $\sum_i [e^{n+i}, c_0(e_i \wedge x)] + (\lambda_1^+ + \mu_1^+) c_1^+(e^n \wedge x) - (\partial_1^* \partial_1 (e_n \lrcorner c_1^+))(x) + \sum_i [e_n, [e^{n+i}, c_1^-(e_{n+i} \wedge x)]] = 0.$

(3) $\partial_1^* \partial_1 (e_n \lrcorner c_1^+) + (\lambda_2 + \mu_2 - 2\lambda_1^+ - 2\mu_1^+) e_n \lrcorner c_1^+ = 0$

(4) $c_1^+(e_n \wedge x) = 0$

Proof. (1) $c_{-1}^+ = 0$ であるから

$$[e_n, c_0(u \wedge x)] + [u, c_1^+(x \wedge e_n)] = 0$$

である (Proof of Lemma 4.4, (1))．よって

$$\sum_i [[e^i, e_n], c_0(e_i \wedge x)] + \sum_i [e_n, [e^i, c_0(e_i \wedge x)]] + \partial_2^* \partial_2 (c_1^+(x \wedge e_n)) = 0$$

である．Lemma III.3.2 より

$$\partial_2^* \partial_2 (c_1^+(x \wedge e_n)) = [E_2, c_1^+(x \wedge e_n)] = (\lambda_2 + \mu_2) c_1^+(x \wedge e_n)$$

である．また $\partial^* c = 0$ であるから

$$(\partial_2^* c_0 + \partial_1^* c_1)(x) = \sum_i [e^i, c_0(e_i \wedge x)] + [e^n, c_1^+(e_n \wedge x)] + \sum_i [e^{n+i}, c_1^-(e_{n+i} \wedge x)] = 0$$

よって

$$\begin{aligned}&\sum_i [[e^i, e_n], c_0(e_i \wedge x)] - [e_n, [e^n, c_1^+(e_n \wedge x)]] \\ &- \sum_i [e_n, [e^{n+i}, c_1^-(e_{n+i} \wedge x)]] + (\lambda_2 + \mu_2) c_1^+(x \wedge e_n) = 0.\end{aligned}$$

をえる．Lemma III.3.16 より $[e^i, e_n] = -e^{n+i}$ であり，

$$[e_n, [e^n, c_1^+(e_n \wedge x)]] = -[E_1^+, c_1^+(e_n \wedge x)] = -(\lambda_1^+ + \mu_1^+) c_1^+(e_n \wedge x)$$

であるから，(1) をえる．

(2) $(\partial c)(x \wedge y \wedge e_n) = 0$ から

$$\begin{aligned}&(\partial_0 c_0 + \partial_1 c_1)(x \wedge y \wedge e_n) \\ &= c_0([x, e_n] \wedge y) - c_0([y, e_n] \wedge x) + [x, c_1^+(y \wedge e_n)] - [y, c_1^+(x \wedge e_n)] + [e_n, c_1^-(x \wedge y)] = 0\end{aligned}$$

をえる．よって

$$\begin{aligned}&\sum_i [e^{n+i}, c_0([e_{n+i}, e_n] \wedge y)] - \sum_i [e^{n+i}, c_0([y, e_n] \wedge e_{n+i})] \\ &- (\partial_1^* \partial_1 (e_n \lrcorner c_1^+))(y) + \sum_i [e^{n+i}, [e_n, c_1^-(e_{n+i} \wedge y)]] = 0.\end{aligned}$$

をえる．Lemma III.3.1 より $[e^n,[y,e_n]]=[y,E_1^+]=(\lambda_1^++\mu_1^+)y$ であるから Lemma 4.4 (2) より

$$\sum_i [e^{n+i}, c_0(e_{n+i}\wedge[y,e_n])]+(\lambda_1^++\mu_1^+)c_1^+(y\wedge e_n)=0$$

をえる．以上から (2) が従う．

(3) (1) と (2) から

$$(\partial_1^*\partial_1(e_n\lrcorner c_1^+))(x)+(\lambda_2+\mu_2-2\lambda_1^+-2\mu_1^+)(e_n\lrcorner c_1^+)(x)=0$$

が従う．この式は $x=e_n$ のときにも成立する．よって (3) をえる．

(4) $\lambda_2+\mu_2-2\lambda_1^+-2\mu_1^+>0$ であるから，(3) から $e_n\lrcorner c_1^+=0$, i.e., $c_1^+(e_n\wedge x)=0$ をえる． □

$a\in A_1^{0,2}$ を

$$a(x\wedge y)=c_1^-(x\wedge y),\quad a(x\wedge e_n)=0.$$

によって定義する．

Lemma 4.6. (1) $\partial_1\partial_1^*a+(\lambda_2-\mu_2)a=0$

(2) $c_1^-(x\wedge y)=0$.

Proof. (1)

$$[e_n,c_0(u\wedge x)]+[u,c_1^+(x\wedge e_n)]=0$$

である (Proof of Lemma 4.4, (1))．$c_1^+(x\wedge e_n)=0$ であるから $[e_n,c_0(u\wedge x)]=0$ である．よって $[e^n,c_0(u\wedge x)]=0$.

さて

$$-[x,c_0(u\wedge y)]+[y,c_0(u\wedge x)]+[u,c_1^-(x\wedge y)]=0$$

である (Proof of Lemma 4.3)．よって

$$-\sum_i[x,[e^i,c_0(e_i\wedge y)]]+\sum_i[y,[e^i,c_0(e_i\wedge x)]]+\partial_2^*\partial_2(c_1^-(x\wedge y))=0$$

をえる．($[e^i,x]\in\mathfrak{g}_1^-$ だから，上の注意によって $[[e^i,x],c_0(e_i\wedge y)]=0$ である)．Lemma III.3.2 から

$$\partial_2^*\partial_2(c_1^-(x\wedge y))=[E_2,c_1^-(x\wedge y)]=(\lambda_2-\mu_2)c_1^-(x\wedge y)$$

であり，$c_1^+(e_n\wedge x)=0$ だから，

$$\sum_i[e^i,c_0(e_i\wedge x)]+\sum_i[e^{n+i},c_1^-(e_{n+i}\wedge x)]=0$$

(Proof of Lemma 4.5, (1))．よって

$$\sum_i[x,[e^{n+i},c_1^-(e_{n+i}\wedge y)]]-\sum_i[y,[e^{n+i},c_1^-(e_{n+i}\wedge x)]]+(\lambda_2-\mu_2)c_1^-(x\wedge y)=0$$

をえる．これは明らかに (1) を意味する．

(2) $\lambda_2-\mu_2>0$ だから，(1) から $a=0$, i.e., $c_1^-(x\wedge y)=0$ をえる． □

$a={}^-K_0^2$ とおけば，Lemmas 4.3, 4.4 and 4.5 と Proposition ∗∗ (1), (2) から容易に次をえる．

Lemma 4.7. $\partial_2a=\partial_1a=\partial_0a=\partial_1^*a=\partial_2^*a=0$

これは $\partial a=\partial^*a=0$ を意味する．

Lemma 4.8. (1) $(\partial_2^*c_{-1})([x,e_n])-[x,(\partial_2^*c_0)(e_n)]-[e^n,c_0([x,e_n]\wedge e_n)]+(\lambda_2-\mu_2)c_1(x\wedge e_n)=0$

(2) $(\lambda_2-\mu_2-\lambda_1^+-\mu_1^+)c_1(x\wedge e_n)+[x,(\partial_1^*c_1)(e_n)]=0$

(3) $c_1(x\wedge e_n)=0$

Proof. (1)

$$(\partial c)(u \wedge x \wedge e_n) = -c_{-1}([x, e_n] \wedge u) - [x, c_0(u \wedge e_n)] + [u, c_1(x \wedge e_n)]$$

である．こゝで $[e_n, c_0(u \wedge x)] = 0$ を注意する ($c_0(u \wedge x)$ は，$\mathfrak{s}_0$ の値を持つ)．よって Lemma 4.2, (4) より

$$(\partial_2^* \partial c)(x \wedge e_n) = (\partial_2^* c_{-1})([x, e_n]) - \sum_i [e^i, [x, c_0(e_i \wedge e_n)]] + \partial_2^* \partial_2(c_1(x \wedge e_n)) = 0$$

をえる．まず Lemma III.3.2 より

$$\partial_2^* \partial_2(c(x \wedge e_n)) = [E_2, c_1(x \wedge e_n)] = (\lambda_2 - \mu_2) c_1(x \wedge e_n)$$

をえる．(Lemma 4.5, (4) より，$c_1(x \wedge e_n)$ は $\mathfrak{g}_1^-$ の値を持つ)．次に $[e^i, x] = \langle e^i, [x, e_n] \rangle e^n$ を使って

$$\begin{aligned}\sum_i [e^i, [x, c_0(e_i \wedge e_n)]] &= [x, (\partial_2^* c_0)(e_n)] + \sum_i [[e^i, x], c_0(e_i \wedge e_n)] \\ &= [x, (\partial_2^* c_0)(e_n)] + [e^n, c_0([x, e_n] \wedge e_n)]\end{aligned}$$

をえる．以上により (1) が示された．

(2) $\partial^* c = 0$ より

$$(\partial_2^* c_{-1} + \partial_1^* c_0 + \partial_0^* c_1)(u) = 0, \qquad (\partial_2^* c_0 + \partial_1^* c_1)(e_n) = 0$$

をえる．そして $\sum_i [e^{n+i}, c_0(e_{n+i} \wedge u)] = 0$ (Lemma 4.4 (2)) だから，

$$(\partial_1^* c_0)(u) = [e^n, c_0(e_n \wedge u)]$$

をえる．よって (1) から，

$$-(\partial_0^* c_1)([x, e_n]) + [x, (\partial_1^* c_1)(e_n)] + (\lambda_2 - \mu_2) c_1(x \wedge e_n) = 0$$

をえる．$[e^n, [x, e_n]] = [x, E_1^+] = (\lambda_1^+ + \mu_1^+) x$ だから，

$$(\partial_0^* c_1)([x, e_n]) = c_1([e^n, [x, e_n]] \wedge e_n) = (\lambda_1^+ + \mu_1^+) c_1(x \wedge e_n)$$

である．以上によって (2) が示された．

(3) (2) から

$$(\lambda_2 - \mu_2 - \lambda_1^+ - \mu_1^+)(\partial_1^* c_1)(e_n) + \sum_i [e^{n+i}, [e_{n+i}, (\partial_1^* c_1)(e_n)]] = 0$$

をえる．$\lambda_2 - \mu_2 - \lambda_1^+ - \mu_1^+ > 0$ だから，これより直ちに $(\partial_1^* c_1)(e_n) = 0$ が従う．よって $c_1(x \wedge e_n) = 0$ である． □

4.3. Proof of Proposition 2.5. この paragraph では invariant $K^3 = c = c_0 + c_1 + c_2$ を考える．

Lemma 4.9. (1) $(\partial c)(x \wedge y \wedge e_n) = 0$

(2) $(\partial_2^* \partial c)(x \wedge y) = 0$

(3) $(\partial_2^* \partial c)(x \wedge e_n) = 0$

Proof. Lemma 4.1 と今迄の結果から次の等式をえる．

$$\begin{aligned}&(\partial K^3)(u \wedge x \wedge y) = -\delta_x K^2(y \wedge u) - \delta_y K^2(u \wedge x) \\ &(\partial K^3)(u \wedge x \wedge e_n) = -\delta_x K^2(e_n \wedge u) - \delta_{e_n} K^2(u \wedge x) \\ &(\partial K^3)(x \wedge y \wedge e_n) = 0\end{aligned}$$

をえる．さらに $(\partial_2^* K_0^2)(x) = 0$ であり，$K_1^2 = 0$ だから，$(\partial_2^* K_0^2)(e_n) = 0$ をえる．よって Lemma をえる． □

Lemma 4.10. (1) $-[e^n, c_1^+([x,e_n]\wedge y)] + [e^n, c_1^+([y,e_n]\wedge x)] + 2\lambda_2 c_2(x\wedge y) = 0$,
(2) $c_1^+([x,e_n]\wedge y) - c_1^+([y,e_n]\wedge x) + [e_n, c_2(x\wedge y)] = 0$
(3) $c_2(x\wedge y) = 0$.

Proof. (1)

$$(\partial c)(u\wedge x\wedge y) = (\partial_0 c_0 + \partial_1 c_1 + \partial_2 c_2)(u\wedge x\wedge y)$$
$$= -[x, c_1^+(u\wedge y)] + [y, c_1^+(u\wedge x)] + [u, c_2(x\wedge y)]$$

である. $(\partial_2^*\partial c)(x\wedge y) = 0$ だから,

$$-\sum_i [[e^i, x], c_1^+(e_i\wedge y)] + \sum_i [[e^i, y], c_1^+(e_i\wedge x)] + \partial_2^*\partial_2(c_2(x\wedge y)) = 0$$

をえる. Lemma III.3.2 より

$$\partial_2^*\partial_2(c_2(x\wedge y)) = [E_2, c_2(x\wedge y)] = 2\lambda_2 c_2(x\wedge y)$$

であり, $[e^i, x] = \langle e^i, [x, e_n]\rangle e^n$ だから (1) をえる.

(2) $(\partial c)(x\wedge y\wedge e_n) = 0$ だから,

$$(\partial_0 c_1 + \partial_1 c_2)(x\wedge y\wedge e_n)$$
$$= c_1([x,e_n]\wedge y) - c_1([y,e_n]\wedge x) + [x, c_2(y\wedge e_n)] - [y, c_2(x\wedge e_n)] + [e_n, c_2(x\wedge y)] = 0$$

をえる. よって

$$c_1^+([x,e_n]\wedge y) - c_1^+([y,e_n]\wedge x) + [e_n, c_2(x\wedge y)] = 0,$$

i.e., (2) をえる.

(3) (1) と (2) から

$$[e^n, [e_n, c_2(x\wedge y)]] + 2\lambda_2 c_2(x\wedge y) = 0$$

をえる.

$$[e^n, [e_n, c_2(x\wedge y)]] = [E_1^+, c_2(x\wedge y)] = 2\lambda_1^+ c_2(x\wedge y)$$

だから

$$(\lambda_2 + \lambda_1^+)c_2(x\wedge y) = 0.$$

$\lambda_2 + \lambda_1^+ > 0$ だから, $c_2(x\wedge y) = 0$ をえる. □

Lemma 4.11. (1) $-\sum_i [e^{n+i}, c_1(e_{n+i}\wedge[x,e_n])] - \sum_i [e^{n+i}, c_1([e_{n+i}, e_n]\wedge x)] + (2\lambda_2 - \lambda_1^+ - \mu_1^+)c_2(x\wedge e_n) = 0$
(2) $\sum_i [e^{n+i}, c_1([e_{n+i}, e_n]\wedge x)] - \sum_i [e^{n+i}, c_1([x,e_n]\wedge e_{n+i})] + (\partial_1^*\partial_1 c_2)(x\wedge e_n) = 0$
(3) $(2\lambda_2 - \lambda_1^+ - \mu_1^+)(e_n \lrcorner c_2) + \partial_1^*\partial_1(e_n\lrcorner c_2) = 0.$ [5]
(4) $c_2(x\wedge e_n) = 0$.

Proof. (1)

$$(\partial c)(u\wedge x\wedge e_n) = (\partial_0 c_0 + \partial_1 c_1 + \partial_2 c_2)(u\wedge x\wedge e_n)$$
$$= -c_0([x,e_n]\wedge u) - [x, c_1(u\wedge e_n)] + [e_n, c_1(u\wedge x)] + [u, c_2(x\wedge e_n)]$$

である. $(\partial_2^*\partial c)(x\wedge e_n) = 0$ だから

$$(\partial_2^* c_0)([x,e_n]) - \sum_i [[e^i, x], c_1(e_i\wedge e_n)] + \sum_i [[e^i, e_n], c_1(e_i\wedge x)] + \partial_2^*\partial_2(c_2(x\wedge e_n)) = 0$$

Lemma III.3.2 によって

$$\partial_2^*\partial_2(c_2(x\wedge e_n)) = [E_2, c_2(x\wedge e_n)] = 2\lambda_2 c_2(x\wedge e_n).$$

[5][編注]: (3) の式を $(2\lambda_2 - 3\lambda_1^+ - \mu_1^+)(e_n\lrcorner c_2) + (\partial_1^*\partial_1)(e_n\lrcorner c_2) = 0$ に修正する必要があると思われる.

また $[e^i, x] = \langle e^i, [x, e_n]\rangle e^n$, $[e^i, e_n] = -e^{n+i}$ だから

$$\sum_i [[e^i, x], c_1(e_i \wedge e_n)] = [e^n, c_1([x, e_n] \wedge e_n)],$$
$$\sum_i [[e^i, e_n], c_1(e_i \wedge x)] = -\sum_i [e^{n+i}, c_1([e_{n+i}, e_n] \wedge x)]$$

だから

$$(\partial_2^* c_0)([x, e_n]) - [e^n, c_1([x, e_n] \wedge e_n)] - \sum_i [e^{n+i}, c_1([e^{n+i}, e_n] \wedge x)] + 2\lambda_2 c_2(x \wedge e_n) = 0$$

をえる．さらに $\partial^* c = 0$ より

$$(\partial_2^* c_0)([x, e_n]) + (\partial_1^* c_1)([x, e_n]) + (\partial_0^* c_2)([x, e_n]) = 0$$

であり，

$$(\partial_1^* c_1)([x, e_n]) = [e^n, c_1(e_n \wedge [x, e_n])] + \sum_i [e^{n+i}, c_1(e_{n+i} \wedge [x, e_n])],$$
$$(\partial_0^* c_2)([x, e_n]) = c_2([e^n, [x, e_n]] \wedge e_n) = c_2([x, E_1^+] \wedge e_n) = (\lambda_1^+ + \mu_1^+) c_2(x \wedge e_n)$$

これらの事実から (1) をえる．

(2) $(\partial c)(x \wedge y \wedge e_n) = 0$ より

$$(\partial_0 c_1 + \partial_1 c_2)(x \wedge y \wedge e_n) = c_1([x, e_n] \wedge y) - c_1([y, e_n] \wedge x) + (\partial_1 c_2)(x \wedge y \wedge e_n) = 0$$

をえる．よって

$$\sum_i [e^{n+i}, c_1([e_{n+i}, e_n] \wedge y)] - \sum_i [e^{n+i}, c_1([y, e_n] \wedge e_{n+i})] + (\partial_1^* \partial_1 c_2)(y \wedge e_n) = 0$$

をえる．よって (2) が示された．

(3) (1) と (2) から

$$(2\lambda_2 - \lambda_1^+ - \mu_1^+) c_2(x \wedge e_n) + (\partial_1^* \partial_1 c_2)(x \wedge e_n) = 0$$

をえる．$c_2(x \wedge y) = 0$ だから，これは

$$(2\lambda_2 - \lambda_1^+ - \mu_1^+)(e_n \lrcorner c_2) + (\partial_1^* \partial_1)(e_n \lrcorner c_2) = 0$$

を意味する．$2\lambda_2 - \lambda_1^+ - \mu_1^+ > 0$ だから $e_n \lrcorner c_2 = 0$, i.e., $c_2(x \wedge e_n) = 0$ をえる． □

4.4. Proof of Proposition 3.1. この paragraph では $HK^2 = {}^-K_0^2 = 0$ と仮定する．$K^{-1} = K^0 = 0$ であり，$K^1 = HK^1$ は $V^{1,2}(n,1) \subset \mathfrak{g}_{-1}^- \otimes (\mathfrak{g}_{-2})^* \wedge (\mathfrak{g}_{-1}^+)^*$ の値を持つ．Proposition 2.2 によって $K_{-1}^2(u \wedge v)$ は $\mathfrak{g}_{-1}^-$ の値を持ち，$K_0^2(u \wedge x) = 0$ (仮定)，かつ $K_1^2 = 0$. さらに Proposition 2.5 より $K_2^3 = 0$. よって Proposition 3.1 を証明するには

$$K_1^3(u \wedge x) = K_2^4(u \wedge x) = 0$$

を証明すればよい．

次に invariant $K^3 = c = c_0 + c_1$ を考える．

Lemma 4.12. (1) $(\partial c)(u \wedge x \wedge y) = 0$.

(2) $(\partial_2^* \partial c)(u \wedge x) + [e^n, (\partial c)(e_n \wedge u \wedge x)] = 0$.

Proof. Lemma 4.1 と今迄の結果から次式をえる．

$$(\partial K^3)(u \wedge v \wedge x) = -\delta_x K^2(u \wedge v),$$
$$(\partial K^3)(u \wedge x \wedge y) = 0,$$
$$(\partial K^3)(u \wedge x \wedge e_n) = -\delta_x K^2(e_n \wedge u).$$

さらに $\partial^* K^2 = 0$ より

$$(\partial^* K^2)(u) = (\partial_2^* K_{-1}^2)(u) + (\partial_1^* K_0^2)(u) = (\partial_2^* K_{-1}^2)(u) + [e^n, K^2(e_n \wedge u)]$$

をえる．よって Lemma が従う． □

Lemma 4.13. $c_1^+(u \wedge x) = 0$.

Proof. $(\partial c)(u \wedge x \wedge y) = 0$ より

$$-[x, c_1^+(u \wedge y)] + [y, c_1^+(u \wedge x)] = 0$$

をえる．Lemma III.3.16 により，これは

$$\langle x, c_1^+(u \wedge y)\rangle z + \langle z, c_1^+(u \wedge y)\rangle x = \langle y, c_1^+(u \wedge x)\rangle z + \langle z, c_1^+(u \wedge x)\rangle y$$

を意味する．これより容易に $\langle z, c_1^+(u \wedge y)\rangle = 0$, i.e, $c_1^+(u \wedge y) = 0$ をえる． □

$a \in \mathfrak{g}_1 \otimes (\mathfrak{g}_{-2})^* \wedge (\mathfrak{g}_{-1}^-)^*$ を

$$a(u \wedge x) = c_1(u \wedge x), \qquad a(u \wedge e_n) = 0$$

によって定義する．

Lemma 4.14. (1) $(\partial_1 \partial_1^* a)(u \wedge x) + (\partial_2^* \partial_2 a)(u \wedge x) + (\lambda_1^+ - \mu_1^+) a(u \wedge x) = 0$.
(2) $c_1(u \wedge x) = 0$.

Proof. (1)

$$\begin{aligned}
&(\partial c)(e_i \wedge u \wedge x) = (\partial_1 c_0 + \partial_2 c_1)(e_i \wedge u \wedge x) = [x, c_0(e_i \wedge u)] + (\partial_2 c_1)(e_i \wedge u \wedge x)\\
&(\partial c)(e_n \wedge u \wedge x) = (\partial_0 c_0 + \partial_1 c_1)(e_n \wedge u \wedge x)\\
&= c_0([e_n, x] \wedge u) + [e_n, c_1(u \wedge x)] + [x, c_1(e_n \wedge u)]\\
&(\partial_2^* \partial c)(u \wedge x) + [e^n, (\partial c)(e_n \wedge u \wedge x)] = 0
\end{aligned}$$

だから

$$\begin{aligned}
&\sum_i [e^i, [x, c_0(e_i \wedge u)]] + (\partial_2^* \partial_2 c_1)(u \wedge x)\\
&+ [e^n, c_0([e_n, x] \wedge u)] + [e^n, [e_n, c_1(u \wedge x)]] + [e^n, [x, c_1(e_n \wedge u)]] = 0
\end{aligned}$$

をえる．

$$\begin{aligned}
\sum_i [e^i, [x, c_0(e_i \wedge u)]] &= \sum_i [[e^i, x], c_0(e_i \wedge u)] + [x, (\partial_2^* c_0)(u)]\\
&= [e^n, c_0([x, e_n] \wedge u)] + [x, (\partial_2^* c_0)(u)]
\end{aligned}$$

$\partial^* c = 0$ より

$$\begin{aligned}
&(\partial_2^* c_0)(u) + (\partial_1^* c_1)(u) = 0,\\
&(\partial_1^* c_1)(u) = [e^n, c_1(e_n \wedge u)] + \sum_i [e^{n+i}, c_1(e_{n+i} \wedge u)] = [e^n, c_1(e_n \wedge u)] + (\partial_1^* a)(u)
\end{aligned}$$

をえる．よって

$$\sum_i [e^i, [x, c_0(e_i \wedge u)]] = [e^n, c_0([x, e_n] \wedge u)] - [x, [e^n, c_1(e_n \wedge u)]] - [x, (\partial_1^* a)(u)]$$

さらに

$$[e^n, [e_n, c_1(u \wedge x)]] = [E_1^+, c_1(u \wedge x)] = (\lambda_1^+ - \mu_1^+) c_1(u \wedge x)$$

こゝで $c_1(u \wedge x)$ が $\mathfrak{g}_1^-$ の値を持つことを使った．以上から

$$-[x, (\partial_1^* a)(u)] + (\partial_2^* \partial_2 c_1)(u \wedge x) + (\lambda_1^+ - \mu_1^+) c_1(u \wedge x) = 0$$

をえる．これは

$$(\partial_1\partial_1^* a)(u\wedge x)+(\partial_2^*\partial_2 a)(u\wedge x)+(\lambda_1^+-\mu_1^+)a(u\wedge x)=0$$

を意味する．よって (1) が示された．

(2) $\lambda_1^+-\mu_1^+>0$ だから，この式から $a(u\wedge x)=0$, i.e., $c_1(u\wedge x)=0$ をえる． □

最後に invariant $K^4=c=c_1+c_2$ を考えよう．

Lemma 4.15. $(\partial_2^*\partial c)(u\wedge x)+[e^n,(\partial c)(e_n\wedge u\wedge x)]=0$.

Proof. Lemma 4.1 と今迄の結果から

$$(\partial K^4)(u\wedge v\wedge x)=-\delta_x K^3(u\wedge v),$$
$$(\partial K^4)(u\wedge x\wedge e_n)=-\delta_x K^3(e_n\wedge u)$$

をえる．$\partial^* K^3=0$ から

$$(\partial^* K^3)(u)=(\partial_2^* K^3)(u)+[e^n,K^3(e_n\wedge u)]=0$$

だから Lemma をえる． □

Lemma 4.16. $c_2(u\wedge x)=0$.

Proof.

$$(\partial c)(e_i\wedge u\wedge x)=(\partial_1 c_1+\partial_2 c_2)(e_i\wedge u\wedge x)=[x,c_1(e_i\wedge u)]+(\partial_2 c_2)(e_i\wedge u\wedge x)$$
$$(\partial c)(e_n\wedge u\wedge x)=(\partial_0 c_1+\partial_1 c_2)(e_n\wedge u\wedge x)$$
$$=c_1([e_n,x]\wedge u)]+[e_n,c_2(u\wedge x)]+[x,c_2(e_n\wedge u)]$$

よって，$(\partial_2^*\partial c)(u\wedge x)+[e^n,(\partial c)(e_n\wedge u\wedge x)]=0$ だから

$$\sum_i[[e^i,x],c_1(e_i\wedge u)]+(\partial_2^*\partial_2 c_2)(u\wedge x)+[e^n,c_1([e_n,x]\wedge u)]+[E_1^+,c_2(u\wedge x)]=0,$$

$$[E_1^+,c_2(u\wedge x)]=2\lambda_1^+c_2(u\wedge x),\qquad \sum_i[[e^i,x],c_1(e_i\wedge u)]=[e^n,c_1([x,e_n]\wedge u)]$$

だから

$$(\partial_2^*\partial_2 c_2)(u\wedge x)+2\lambda_1^+c_2(u\wedge x)=0$$

をえる．これより $\lambda_1^+>0$ だから $c_2(u\wedge x)=0$ をえる． □

4.5. Proof of Proposition 3.4. この paragraph では $K^1=0$ と仮定する．この仮定と $K^0=0$ から $K^2=HK^2$ である．従って，K^2 は $V^{2,2}(n,1)\subset\mathfrak{s}_0\otimes(\mathfrak{g}_{-2})^*\wedge(\mathfrak{g}_{-1}^-)^*$ の値を持つ．また Proposition 2.5 より $K_2^3=0$. よって，Proposition 3.4 を証明するには，

$$K_1^3(u\wedge e_n)=K_2^4(u\wedge e_n)=0$$

を証明すればよい．

まず invariant $K^3=c=c_0+c_1$ を考える．

Lemma 4.17. (1) $(\partial c)(u\wedge v\wedge e_n)=0$.

(2) $(\partial_1^*\partial c)(u\wedge e_n)=0$.

Proof. Lemma 4.1 と仮定より次の等式をえる．

$$(\partial K^3)(u\wedge v\wedge e_n)=0$$
$$(\partial K^3)(u\wedge x\wedge e_n)=-\delta_{e_n}K^2(u\wedge x)$$

K^2 は harmonic だから $(\partial_2^* K^2)(x)=(\partial_1^* K_0^2)(u)=0$. よって Lemma をえる． □

Lemma 4.18. $c_1^-(u\wedge e_n)=0$.

Proof. $(\partial c)(u \wedge v \wedge e_n) = 0$ より

$$(\partial_1 c_0 + \partial_2 c_1)(u \wedge v \wedge e_n) = [e_n, c_0(u \wedge v)] + [u, c_1(v \wedge e_n)] - [v, c_1(u \wedge e_n)] = 0$$

よって

$$[u, c_1^-(v \wedge e_n)] - [v, c_1^-(u \wedge e_n)] = 0$$

よって $a_j = \langle c_1^-(e_j \wedge e_n), e_n \rangle$ とおけば

$$a_j[e_i, e^n] - a_i[e_j, e^n] = 0.$$

$[e_i, e^n] = -(\lambda_1^+ + \mu_1^+)e_{n+i}$ だから

$$a_j \delta_i^k - a_i \delta_j^k = 0$$

よって $a_j = 0$, i.e., $c_1^-(e_j \wedge e_n) = 0$ をえる. □

Lemma 4.19. (1) $-\sum_i [e^{n+i}, c_0(e_i \wedge u)] - (\lambda_1^+ + \mu_1^+)c_1(u \wedge e_n) + \sum_i [e^{n+i}, [e_n, c_1(u \wedge e_{n+i})]] + (\partial_2^* \partial_2 c_1)(u \wedge e_n) = 0.$

(2) $-\sum_i [e^{n+i}, c_0(e_i \wedge u)] - (\lambda_1^- + \mu_1^-)c_1(u \wedge e_n) + \sum_i [e^{n+i}, [e_n, c_1(u \wedge e_{n+i})]] = 0.$

(3) $(\lambda_1^- + \mu_1^- - \lambda_1^+ - \mu_1^+)c_1(u \wedge e_n) + (\partial_2^* \partial_2 c_1)(u \wedge e_n) = 0$

(4) $c_1(u \wedge e_n) = 0$

Proof. (1) $(\partial c)(e_i \wedge u \wedge e_n) = 0$ より

$$(\partial_1 c_0 + \partial_2 c_1)(e_i \wedge u \wedge e_n) = [e_n, c_0(e_i \wedge u)] + (\partial_2 c_1)(e_i \wedge u \wedge e_n)$$

よって

$$\sum_i [e^i, [e_n, c_0(e_i \wedge u)]] + (\partial_2^* \partial_2 c_1)(u \wedge e_n) = 0$$

をえる. $\partial^* c = 0$ より

$$(\partial_2^* c_0)(u) + (\partial_1^* c_1)(u) = \sum_i [e^i, c_0(e_i \wedge u)] + [e^n, c_1(e_n \wedge u)] + \sum_i [e^{n+i}, c_1(e_{n+i} \wedge u)] = 0.$$

$[e_n, e^i] = e^{n+i}$ だから,

$$\begin{aligned} &\sum_i [e^{n+i}, c_0(e_i \wedge u)] + \sum_i [e^i, [e_n, c_0(e_i \wedge u)]] \\ &+ [E_1^+, c_1(u \wedge e_n)] + \sum_i [e^{n+i}, [e_n, c_1(e_{n+i} \wedge u)]] = 0 \end{aligned}$$

をえる. こゝで $c_1(u \wedge e_n)$ が $\mathfrak{g}_1^+$ の値を持つことを使った.

$$[E_1^+, c_1(u \wedge e_n)] = (\lambda_1^+ + \mu_1^+)c_1(u \wedge e_n)$$

であるから, (1) をえる.

(2)

$$(\partial c)(u \wedge x \wedge e_n) = -c_0([x, e_n] \wedge u) - [x, c_1(u \wedge e_n)] + [e_n, c_1(u \wedge x)]$$

をえる. $(\partial_1^* \partial c)(u \wedge e_n) = \sum_i [e^{n+i}, (\partial c)(e_{n+i} \wedge u \wedge e_n)] = 0$ だから

$$-\sum_i [e^{n+i}, c_0([e_{n+i}, e_n] \wedge u)] - [E_1^-, c_1(u \wedge e_n)] + \sum_i [e^{n+i}, [e_n, c_1(u \wedge e_{n+i})]] = 0$$

をえる.

$$[E_1^-, c_1(u \wedge e_n)] = (\lambda_1^- + \mu_1^-)c_1(u \wedge e_n)$$

であるから (2) をえる.

(3) (1), (2) より 直ちに (3) をえる.

(4) $\lambda_1^- + \mu_1^- - \lambda_1^+ - \mu_1^+ > 0$ だから, (3) より $c_1(u \wedge e_n) = 0$ をえる. □

invariant $K^4 = c = c_1 + c_2$ を考えよう.

Lemma 4.20. $(\partial_2^*\partial c)(u \wedge e_n) + (\partial_1^*\partial c)(u \wedge e_n) = 0$.

Proof. Lemma 4.1 と今迄の結果から次の等式をえる.

$$(\partial K^4)(u \wedge v \wedge e_n) = -\delta_{e_n} K^3(u \wedge v),$$
$$(\partial K^4)(u \wedge x \wedge e_n) = -\delta_{e_n} K^3(u \wedge x)$$

$\partial^* K^3 = 0$ より

$$(\partial_2^* K^3)(u) + (\partial_1^* K^3)(u) = 0$$

をえる. よって Lemma をえる. □

Lemma 4.21. (1) $(\partial_2^*\partial_2 c_2)(u \wedge e_n) + 2\lambda_1^- c_2(u \wedge e_n) = 0$.
(2) $c_2(u \wedge e_n) = 0$.

Proof. (1)

$$(\partial c)(e_i \wedge u \wedge e_n) = (\partial_1 c_1 + \partial_2 c_2)(e_i \wedge u \wedge e_n) = [e_n, c_1(e_i \wedge u)] + (\partial_2 c_2)(e_i \wedge u \wedge e_n)$$
$$(\partial c)(e_{n+i} \wedge u \wedge e_n) = (\partial_0 c_1 + \partial_1 c_2)(e_{n+i} \wedge u \wedge e_n)$$
$$= c_1([e_{n+i}, e_n] \wedge u) + [e_{n+i}, c_2(u \wedge e_n)] + [e_n, c_2(e_{n+i} \wedge u)]$$

$(\partial_2^*\partial c)(u \wedge e_n) + (\partial_1^*\partial c)(u \wedge e_n) = 0$ だから,

$$\sum_i [[e^i, e_n], c_1(e_i \wedge u)] + (\partial_2^*\partial_2 c_2)(u \wedge e_n) + \sum_i [e^{n+i}, c_1(e_i \wedge u)] + [E_1^-, c_2(u \wedge e_n)] = 0,$$
$$[E_1^-, c_2(u \wedge e_n)] = 2\lambda_1^- c_2(u \wedge e_n)$$

であり, $[e^i, e_n] = -e^{n+i}$ であるから, (1) をえる.
(2) $\lambda_1^- > 0$ だから, (1) から $c_2(u \wedge e_n) = 0$ をえる. □

4.6. Proof of Proposition 3.7. この paragraph では $K^1 = 0$ と仮定する.
まず invariant $K^3 = K_0^3 + K_1^3$ を考える. Lemma 4.1 と今迄の結果から次の Lemma をえる.

Lemma 4.22. (1) $(\partial K^3)(u \wedge v \wedge x) = (\partial K^3)(u \wedge v \wedge e_n) = 0$.
(2) $(\partial K^3)(u \wedge x \wedge y) = -\delta_x K^2(y \wedge u) - \delta_y K^2(u \wedge x)$, $(\partial K^3)(u \wedge x \wedge e_n) = -\delta_{e_n} K^2(u \wedge x)$.
(3) $(\partial K^3)(x \wedge y \wedge z) = (\partial K^3)(x \wedge y \wedge e_n) = 0$

Lemma 4.23. (1) $\delta_{e_n} K_0^2 = 0$.
(2) $K_0^3 = 0$ かつ K_1^3 は $\mathfrak{g}_1^+ \otimes (\mathfrak{g}_{-2})^* \wedge (\mathfrak{g}_{-1}^-)^*$ の値を持つ.
(3) $\langle K_1^3([e_n, x] \wedge y), z\rangle$ は *variables* $x, y, z \in \mathfrak{g}_{-1}^-$ について *symmetric*.

Proof. (1)

$$(\partial K^3)(u \wedge x \wedge e_n) = (\partial_0 K_0^3 + \partial_1 K_1^3)(u \wedge x \wedge e_n) = -K_0^3([x, e_n] \wedge u) + [e_n, K_1^3(u \wedge x)]$$

である. $(\partial K^3)(u \wedge x \wedge e_n) = -\delta_{e_n} K_0^2(u \wedge x)$ であるから,

$$\delta_{e_n} K_0^2(u \wedge x) = -[e_n, K_1^3(u \wedge x)] + K_0^3(u \wedge [e_n, x])$$

をえる. $K_1^3(u \wedge x)$ は

$$K_1^3(u \wedge x) \equiv \alpha(u \wedge x)e^n \qquad (\mathrm{mod}\ \mathfrak{g}_1^+)$$

と表わされる. よって

$$\delta_{e_n} K_0^2(u \wedge x) = \alpha(u \wedge x)E_1^+ + K_0^3(u \wedge [e_n, x])$$

をえる．こゝで α は, function $P \to \mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^-)^*$ である．$u = [e_n, x]$ とおき

$$\delta_{e_n} K_0^2([e_n, x] \wedge x) = \alpha([e_n, x] \wedge x) E_1^+$$

をえる．$K_0^2([e_n, x] \wedge x)$ は, $\mathfrak{s}_0$ の値を持ち, $\langle \mathfrak{s}_0, E_1^+ \rangle = 0$ だから $\alpha([e_n, x] \wedge x) = 0$. 従って

$$\delta_{e_n} K_0^2([e_n, x] \wedge x) = 0$$

をえる．$K_0^2([e_n, x] \wedge y)$ は x, y について symmetric だから,

$$\delta_{e_n} K_0^2([e_n, x] \wedge y) = 0.$$

よって $\delta_{e_n} K_0^2 = 0$ をえる．

(2) (1) から

$$\alpha(u \wedge x) E_1^+ + K_0^3(u \wedge [e_n, x]) = 0$$

をえる．$(\partial K^3)(u \wedge v \wedge e_n) = 0$ より

$$(\partial_1 K_0^3 + \partial_2 K_1^3)(u \wedge v \wedge e_n) = [e_n, K_0^3(u \wedge v)] = 0$$

をえる．故に

$$\alpha(u \wedge x)[E_1^+, e_n] = 0.$$

$[E_1^+, e_n] = (-\lambda_1^+ + \mu_1^+) e_n$ で $-\lambda_1^+ + \mu_1^+ \neq 0$ だから, $\alpha(u \wedge x) = 0$ をえる．よって, $K_1^3(u \wedge x)$ が $\mathfrak{g}_1^+$ の値を持ち $K_0^3 = 0$ がわかった．

(3) $(\partial K^3)(x \wedge y \wedge e_n) = 0$ から

$$(\partial_0 K_1^3)(x \wedge y \wedge e_n) = K_1^3([x, e_n] \wedge y) - K_1^3([y, e_n] \wedge x) = 0$$

をえる．さらに

$$(\partial K^3)(u \wedge x \wedge y) = (\partial_1 K_1^3)(u \wedge x \wedge y) = -[x, K_1^3(u \wedge y)] + [y, K_1^3(u \wedge x)]$$

であり, $(\partial K^3)(u \wedge x \wedge y) = -\delta_x K_0^2(y \wedge u) - \delta_y K_0^2(u \wedge x)$ だから

$$-[x, K_1^3(u \wedge y)] + [y, K_1^3(u \wedge x)] = -\delta_x K_0^2(y \wedge u) - \delta_y K_0^2(u \wedge x)$$

をえる．右辺は $\mathfrak{s}_0$ の値を持ち, $\langle \mathfrak{s}_0, E \rangle = 0$ だから,

$$\langle x, K_1^3(u \wedge y) \rangle = \langle y, K_1^3(u \wedge x) \rangle$$

をえる．以上により $\langle K_1^3([e_n, x] \wedge y), z \rangle$ が x, y, z について symmetric であることがわかった． □

次に invariant $K^4 = K_1^4 + K_2^4$ を考える．Lemma 4.1 と今迄の結果から次の Lemma をえる．

Lemma 4.24. (1) $(\partial K^4)(u \wedge v \wedge x) = -\delta_u K^2(v \wedge x) - \delta_v K^2(x \wedge u)$, $(\partial K^4)(u \wedge v \wedge e_n) = 0$.
(2) $(\partial K^4)(u \wedge x \wedge y) = -\delta_x K^3(y \wedge u) - \delta_y K^3(u \wedge x)$, $(\partial K^4)(u \wedge x \wedge e_n) = -\delta_{e_n} K^3(u \wedge x)$
(3) $(\partial K^4)(x \wedge y \wedge z) = (\partial K^4)(x \wedge y \wedge e_n) = 0$

Lemma 4.25. (1) $K_1^4 = 0$ かつ $\delta_{e_n} K_1^3 = -[e_n, K_2^4]$.
(2) $\langle K_2^4([e_n, x] \wedge y), [e_n, z] \rangle$ は x, y, z について *symmetric* である．

Proof. (1) $(\partial K^4)(u \wedge x \wedge e_n) = -\delta_{e_n} K^3(u \wedge x)$ だから

$$(\partial_0 K_1^4 + \partial_1 K_2^4)(u \wedge x \wedge e_n) = -K_1^4([x, e_n] \wedge u) + [e_n, K_2^4(u \wedge x)] = -\delta_{e_n} K_1^3(u \wedge x)$$

をえる．よって $K_1^4(u \wedge v)$ が $\mathfrak{g}_1^+$ の値を持つことがわかる．$(\partial K^4)(u \wedge v \wedge x) = -\delta_u K_0^2(v \wedge x) - \delta_v K_0^2(x \wedge u)$ だから

$$\begin{aligned}(\partial_1 K_1^4 + \partial_2 K_2^4)(u \wedge v \wedge x) &= [x, K_1^4(u \wedge v)] + [u, K_2^4(v \wedge x)] - [v, K_2^4(u \wedge x)] \\ &= -\delta_u K_0^2(v \wedge x) - \delta_v K_0^2(x \wedge u)\end{aligned}$$

をえる．右辺は $\mathfrak{s}_0$ の値を持ち，$\langle J, \mathfrak{s}_0\rangle = 0$, $[J, \mathfrak{g}_{-2}] = 0$ だから

$$\langle x, K_1^4(u \wedge v)\rangle = 0$$

をえる．$K_1^4(u \wedge v)$ は，$\mathfrak{g}_1^+$ の値を持つから $K_1^4 = 0$ をえる．よって $\delta_{e_n} K_1^3 = -[e_n, K_2^4]$ をえる．

(2)

$$[u, K_2^4(v \wedge x)] - [v, K_2^4(u \wedge x)] = -\delta_u K_0^2((v \wedge x) - \delta_v K_0^2((x \wedge u)$$

である．右辺は $\mathfrak{s}_0$ の値を持ち，$\langle \mathfrak{s}_0, E\rangle = 0$ だから

$$\langle u, K_2^4(v \wedge x)\rangle = \langle v, K_2^4(u \wedge x)\rangle$$

をえる．さらに $(\partial K^4)(x \wedge y \wedge e_n) = 0$ より

$$(\partial_0 K_2^4)(x \wedge y \wedge e_n) = K_2^4([x, e_n] \wedge y) - K_2^4([y, e_n] \wedge x) = 0$$

をえる．以上から，$\langle K_2^4([e_n, x] \wedge y), [e_n, z]\rangle$ が x, y, z について symmetric であることがわかった． □

最後に invariant $K^5 = K_2^5$ を考える．Lemma 4.1 と今迄の結果から次をえる．

Lemma 4.26. (1) $(\partial K^5)(u \wedge v \wedge x) = -\delta_u K^3(v \wedge x) - \delta_v K^3(x \wedge u)$,

(2) $(\partial K^5)(u \wedge x \wedge e_n) = -\delta_{e_n} K^4(u \wedge x)$

Lemma 4.27. (1) $K_2^5 = 0$,

(2) $\delta_{e_n} K_2^4 = 0$

Proof. (1) $(\partial K^5)(u \wedge v \wedge x) = -\delta_u K^3(v \wedge x) - \delta_v K^3(x \wedge u)$ から

$$(\partial_1 K_2^5)(u \wedge v \wedge x) = [x, K_2^5(u \wedge v)] = -\delta_u K^3(v \wedge x) - \delta_v K^3(x \wedge u)$$

をえる．右辺は $\mathfrak{g}_1^+$ の値を持ち，左辺は $\mathfrak{g}_1^-$ の値を持つから，

$$[x, K_2^5(u \wedge v)] = 0$$

をえる．$[e^i, e_{n+j}] = \delta^i_j e^n$ であるから，$K_2^5 = 0$ をえる．

(2) $(\partial K^5)(u \wedge x \wedge e_n) = -\delta_{e_n} K_2^4(u \wedge x)$ かつ $K^5 = K_2^5 = 0$ だから $\delta_{e_n} K_2^4 = 0$ をえる． □

以上で Proposition 3.7 が完全に証明された．

§5. The expressions of the invariants HK^2 and K^1 in terms of canonical coordinate systems

この § では，$\mathfrak{P} = \{R; E, F\}$ を projective system of type $(n, 1)$, (P, ω) を対応する normal connection of type $\mathfrak{G}$, K をその curvature とする．$\pi : P \to R$ を projection とする．

5.1. Fundamental formulas for the covariant derivatives. f を R 上の function とする．P 上の function $f \circ \pi$ を同じ文字 f でかくことにする．

さて任意の $X \in \mathfrak{g}$ に対して，covariant differentiation $\delta_X = \mathscr{L}_{\check{\omega}(X)}$ が対応する．R 上の任意の function f と $X_1, \ldots, X_k \in \mathfrak{g}$ に対して，P 上の function $\tilde{f}(X_1, \ldots, X_k)$ を

$$\tilde{f}(X_1, \ldots, X_k) = \delta_{X_1} \cdots \delta_{X_k} f = \check{\omega}(X_1) \cdots \check{\omega}(X_k) f$$

と定義する．

Lemma 5.1. $a \in G^{(0)}$, $X \in \mathfrak{g}^{(0)}$ とする．

(1) $R_a^* \tilde{f}(X_1, \ldots, X_k) = \tilde{f}(\mathrm{Ad}(a)X_1, \ldots, \mathrm{Ad}(a)X_k)$

(2) $\delta_X \tilde{f}(X_1, \ldots, X_k) = \sum_i \tilde{f}(X_1, \ldots, [X, X_i], \ldots, X_k)$

Proof. $a \in G^{(0)}$ とする．このとき

$$(R_a)_* \check{\omega}(Y) = \check{\omega}(\mathrm{Ad}(a^{-1})Y), \quad Y \in \mathfrak{g}$$

が成立する．さらに

$$R_a^* f = f$$

である．よって

$$R_a^* \tilde{f}(X_1, \ldots, X_k) = \tilde{f}(\mathrm{Ad}(a)X_1, \ldots, \mathrm{Ad}(a)X_k)$$

をえる．よって (1) が従う．$X \in \mathfrak{g}^{(0)}$ のとき, $\delta_X = \mathscr{L}_{X^*}$ であるから, (1) から直ちに (2) が従う． □

space $C^2(\mathfrak{G}) = \mathfrak{g} \otimes \Lambda^2(\mathfrak{m}^*)$ は

$$C^2(\mathfrak{G}) = \mathfrak{m} \otimes \Lambda^2(\mathfrak{m}^*) + \mathfrak{g}^{(0)} \otimes \Lambda^2(\mathfrak{m}^*)$$

と分解する．この分解に応じて, curvature K は

$$K = K_- + K_+$$

と分解する．

$$K_- = \sum_{j<0} K_j, \qquad K_+ = \sum_{j \geqq 0} K_j$$

であることを注意する．$X \in \mathfrak{g}$ に対して, X_- で分解 $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}^{(0)}$ に関する $\mathfrak{m}$-component を表わす．

Lemma 5.2. *$z \in P$, $X_1, X_2 \in \mathfrak{g}$ とする．このとき次式が成立する．*

$$[\check{\omega}(X_1), \check{\omega}(X_2)]_z = \check{\omega}([X_1, X_2])_z - \check{\omega}(A)_z - B_z^*$$

こゝで, A (resp. B) は, $K_-((X_1)_- \wedge (X_2)_-)$ (resp. $K_+((X_1)_- \wedge (X_2)_-)$) の z での値である．

Proof. structure equation より容易に

$$\omega([\check{\omega}(X_1), \check{\omega}(X_2)]) = [X_1, X_2] - K((X_1)_- \wedge (X_2)_-)$$

をえる．この式より直ちに Lemma が従う． □

Lemma 5.3.

$$\begin{aligned}
&\tilde{f}(X_1, X_2, X_3, \ldots, X_k) \\
&= \tilde{f}(X_2, X_1, X_3, \ldots, X_k) + \tilde{f}([X_1, X_2], X_3, \ldots, X_k) \\
&- \tilde{f}(K_-((X_1)_- \wedge (X_2)_-), X_3, \ldots, X_k) - \sum_{i=3}^{k} \tilde{f}(X_3, \ldots, [K_+((X_1)_- \wedge (X_2)_-), X_i], \ldots, X_k)
\end{aligned}$$

Proof. $z \in P$ とする．

$$\begin{aligned}
\tilde{f}(X_1, X_2, X_3, \ldots, X_k)(z) &= \check{\omega}(X_1)_z \check{\omega}(X_2) \tilde{f}(X_3, \ldots, X_k) \\
&= \check{\omega}(X_2)_z \check{\omega}(X_1) \tilde{f}(X_3, \ldots, X_k) + [\check{\omega}(X_1), \check{\omega}(X_2)]_z \tilde{f}(X_3, \ldots, X_k)
\end{aligned}$$

Lemmas 5.1, 5.2 を使って,

$$\begin{aligned}
&[\check{\omega}(X_1), \check{\omega}(X_2)]_z \tilde{f}(X_3, \ldots, X_k) \\
&= \check{\omega}([X_1, X_2])_z \tilde{f}(X_3, \ldots, X_k) - \check{\omega}(A)_z \tilde{f}(X_3, \ldots, X_k) - B_z^* \tilde{f}(X_3, \ldots, X_k) \\
&= \tilde{f}([X_1, X_2], X_3, \ldots, X_k)(z) - \tilde{f}(A, X_3, \ldots, X_k)(z) - \sum_{i=3}^{k} \tilde{f}(X_3, \ldots, [B, X_i], \ldots, X_k)(z)
\end{aligned}$$

故に Lemma をえる. □

f は R 上の function であり, F の first integral であるとする. $\pi_*^{-1}(F)$ は equation $\omega_{-2} = \omega_{-1}^{+} = 0$ で定義されるから, $Z \in \mathfrak{g}_{-1}^{-}$ のとき $\check{\omega}(Z)$ は $\pi_*^{-1}(F)$ の cross section である. よって

$$\tilde{f}(Z) = 0, \qquad Z \in \mathfrak{g}_{-1}^{-}$$

である. 次の Lemma は Lemma 5.3 と Proposition III.3.17, Proposition 2.2 から得られる.

Lemma 5.4. $X \in \mathfrak{g}_{-1}^{+}$, $Z \in \mathfrak{g}_{-1}^{-}$ とする.

(1) $\tilde{f}(Z, X) = \tilde{f}([Z, X])$
(2) $\tilde{f}(Z, Z, X) = \tilde{f}(Z, [Z, X]) = 0$
(3) $\tilde{f}(Z, X, X) = 2\tilde{f}([Z, X], X)$
(4) $\tilde{f}(Z, Z, X, X) = 2\tilde{f}([Z, X], [Z, X])$
(5) $\tilde{f}(Z, Z, Z, X, X) = -2\tilde{f}([[K_0(Z \wedge [Z, X]), Z], X]]) - 2\tilde{f}([K_1(Z \wedge [Z, X]), [Z, X]])$
(6) $\tilde{f}(Z, X, X, X) = 3\tilde{f}([Z, X], X, X) - 2\tilde{f}([K_{-1}(X \wedge [Z, X]), X]) - 2\tilde{f}([K_0(X \wedge [Z, X]), X])$

Proof. この Lemma と下の 2 つの Lemma は Lemma 5.3 と Proposition III.3.17, Proposition 2.2 から得られる.

(1) Lemma 5.3 より

$$\tilde{f}(Z, X) = \tilde{f}(X, Z) + \tilde{f}([Z, X]) - \tilde{f}(K_{-}(Z \wedge X))$$

をえる. $\tilde{f}(Z) = 0$, $K(Z \wedge X) = 0$ (Proposition III.3.17) だから, (1) をえる.

(2) (1) から

$$\tilde{f}(Z, Z, X) = \tilde{f}(Z, [Z, X])$$

をえる. Lemma 5.3 により

$$\tilde{f}(Z, [Z, X]) = \tilde{f}([Z, X], Z) + \tilde{f}([Z, [Z, X]]) - \tilde{f}(K_{-}(Z \wedge [Z, X]))$$

をえる. $\tilde{f}(Z) = 0$, $[Z, [Z, X]] = 0$, $K_{-}(Z \wedge [Z, X]) = 0$ (Proposition III.3.17) だから (2) をえる.

(3) $K(Z \wedge X) = 0$ だから, (1) と同様にして

$$\tilde{f}(Z, X, X) = \tilde{f}(X, Z, X) + \tilde{f}([Z, X], X)$$

をえる. (1) と Lemma 5.3 から

$$\tilde{f}(X, Z, X) = \tilde{f}(X, [Z, X]) = \tilde{f}([Z, X], X) + \tilde{f}([X, [Z, X]]) - \tilde{f}(K_{-}(X \wedge [Z, X]))$$

をえる. $[X, [Z, X]] = 0$ である. また Proposition III.3.17 により $K_{-}(X \wedge [Z, X]) = K_{-1}^{1}(X \wedge [Z, X])$ は $\mathfrak{g}_{-1}^{-}$ の値を持つ. よって (3) が従う.

(4) (3) から

$$\tilde{f}(Z, Z, X, X) = 2\tilde{f}(Z, [Z, X], X)$$

をえる. Lemma 5.3 より

$$\begin{aligned}\tilde{f}(Z, [Z, X], X) = &\tilde{f}([Z, X], Z, X) + \tilde{f}([Z, [Z, X]], X) - \tilde{f}(K_{-}(Z \wedge [Z, X]), X)\\ &- \tilde{f}([K_{+}(Z \wedge [Z, X]), X])\end{aligned}$$

をえる. まず (1) から

$$\tilde{f}([Z, X], Z, X) = \tilde{f}([Z, X], [Z, X])$$

をえる. 次に $[Z, [Z, X]] = 0$, $K_{-}(Z \wedge [Z, X]) = 0$ である. さらに

$$K_{+}(Z \wedge [Z, X]) = K_0(Z \wedge [Z, X]) + K_1(Z \wedge [Z, X]) + K_2(Z \wedge [Z, X])$$

であり, $K_0(Z \wedge [Z,X]) = K_0^2(Z \wedge [Z,X])$ は $\mathfrak{s}_0$ の値を持つ (Proposition 2.2) から, $[K_0(Z \wedge [Z,X]), X] = 0$. よって $[K_+(Z \wedge [Z,X]), X]$ は $\mathfrak{g}_0 + \mathfrak{g}_1 \subset \mathfrak{g}^{(0)}$ の値を持つ. 以上から

$$\tilde{f}(Z,[Z,X],X) = \tilde{f}([Z,X],[Z,X])$$

をえる. よって (4) が従う.

(5) (4) から

$$\tilde{f}(Z,Z,Z,X,X) = 2\tilde{f}(Z,[Z,X],[Z,X])$$

をえる. (3) の証明と同様にして

$$\tilde{f}(Z,[Z,X],[Z,X]) = \tilde{f}([Z,X],Z,[Z,X]) - \tilde{f}([K_+(Z \wedge [Z,X]),[Z,X]])$$

をえる. まず (2) によって $\tilde{f}(Z,[Z,X]) = 0$ である. 次に

$$K_+(Z \wedge [Z,X]) = K_0(Z \wedge [Z,X]) + K_1(Z \wedge [Z,X]) + K_2(Z \wedge [Z,X])$$

であり, $[K_2(Z \wedge [Z,X]),[Z,X]]$ は $\mathfrak{g}_0 \subset \mathfrak{g}^{(0)}$ の値を持ち, $[K_0(Z \wedge [Z,X]),[Z,X]] = [[K_0(Z \wedge [Z,X]),Z],X]$ である. よって,

$$\tilde{f}(Z,[Z,X],[Z,X]) = -\tilde{f}([K_0(Z \wedge [Z,X]),[Z,X]]) - \tilde{f}([K_1(Z \wedge [Z,X]),[Z,X]])$$

よって (5) が従う.

(6) (1) の証明と同様にして,

$$\tilde{f}(Z,X,X,X) = \tilde{f}(X,Z,X,X) + \tilde{f}([Z,X],X,X)$$

をえる. (3) から

$$\tilde{f}(X,Z,X,X) = 2\tilde{f}(X,[Z,X],X)$$

をえる. Lemma 5.3 から

$$\tilde{f}(X,[Z,X],X) = \tilde{f}([Z,X],X,X) - \tilde{f}(K_{-1}(X \wedge [Z,X]),X) - \tilde{f}([K_0(X \wedge [Z,X]),X])$$

をえる. $K_{-1}(X \wedge [Z,X])$ は $\mathfrak{g}_{-1}^-$ の値を持つから (1) を使って

$$\tilde{f}(K_{-1}(X \wedge [Z,X]),X) = \tilde{f}([K_{-1}(X \wedge [Z,X]),X])$$

をえる. よって

$$\tilde{f}(X,[Z,X],X) = \tilde{f}([Z,X],X,X) - \tilde{f}([K_{-1}(X \wedge [Z,X]),X]) - \tilde{f}([K_0(X \wedge [Z,X]),X])$$

をえる. よって (6) が従う. □

5.2. Canonical coordinate systems and the cross sections $\boldsymbol{\sigma}$. R の任意の点 a_0 に対して, a_0 の nbd U で定義された coordinate system $\{x, y^1, \dots, y^{n-1}, p^1, \dots, p^{n-1}\}$ が存在して, F は equation

$$dx = dy^i = 0, \quad 1 \leqq i \leqq n-1$$

で定義され, E は

$$\begin{cases} dy^i - p^i dx = 0, \\ dp^j - F^j dx = 0, \quad 1 \leqq i,j \leqq n-1 \end{cases}$$

なる形の equation で定義される. こゝで F^j は a_0 の nbd で定義された function である.

以下 $U = R$ と仮定する. また indices i,j,k,l は, $\{1,\dots,n-1\}$ を動くものとする. $\mathfrak{g}_{-2}$, $\mathfrak{g}_{-1}^+$, $\mathfrak{g}_{-1}^-$, etc などにはそれぞれ basis (e_i), e_n, (e_{n+j}), etc を §** のように取っておく.

次の Proposition を証明しよう.

Proposition 5.5. 任意の $a \in R$ に対して, P の点 $y = \sigma(a)$ であって, $\pi(y) = a$ かつ次を満足するものが唯一つ存在する:

1) $\tilde{p}^i(e_{n+j})(y) = \delta^i_j, \quad \tilde{x}(e_n)(y) = 1,$
2) $\tilde{x}(e_n, e_n)(y) = \tilde{x}([Z, e_n])(y) = 0, \quad Z \in \mathfrak{g}_{-1}^-.$

3) $\tilde{x}([Z, e_n], e_n)(y) = 0, \quad Z \in \mathfrak{g}_{-1}^{-}$.
さらに 対応 $a \mapsto \sigma(a)$ は, *principal fibre bundle* P の *differentiable cross section* σ を与える.

x は F の first integral だから, Lemma 5.4 から

$$\tilde{x}(Z, e_n)(y) = \tilde{x}([Z, e_n])(y) = 0, \qquad \tilde{x}(Z, e_n, e_n)(y) = 2\tilde{x}([Z, e_n], e_n)(y) = 0.$$

をえる.

まず次の Lemma を証明する.

Lemma 5.6. (1) $\det(\tilde{p}^i(e_{n+j})) \neq 0$ *everywhere.*
(2) $\tilde{x}(e_n) \neq 0$ *everywhere.*

Proof. (1) x, y^i は F の first integrals であるから

$$\tilde{x}(Z) = \tilde{y}^i(Z) = 0, \quad Z \in \mathfrak{g}_{-1}^{-}$$

である. よって, $\tilde{p}^i(Z)(y) = 0$ となる $Z \in \mathfrak{g}_{-1}^{-}$ と $y \in P$ があったとすれば, $\pi_*(\check{\omega}(Z)_y) = 0$ をえる. よって $Z = 0$. この事実は (1) を意味する.

(2) $\pi_*^{-1}(E)$ は equation $\omega_{-2} = \omega_{-1}^{-} = 0$ で定義されるから, $\check{\omega}(e_n)$ は, $\pi_*^{-1}(E)$ の cross section である. よって

$$\tilde{y}^i(e_n) - p^i\tilde{x}(e_n) = 0, \qquad \tilde{p}^j(e_n) - F^j\tilde{x}(e_n) = 0$$

をえる. よって $\tilde{x}(e_n)(y) = 0$ なる $y \in P$ があったとすれば $\pi_*(\check{\omega}(e_n)) = 0$ となる. これは明らかに矛盾である. よって (2) が従う. □

$a \in G^{(0)}$ とする. Proposition III.1.4 により a は

$$a = b \exp X_1 \exp X_2$$

と表わされる. こゝで $b \in G_0$, $X_1 \in \mathfrak{g}_1$, $X_2 \in \mathfrak{g}_2$. さらに X_1 は

$$X_1 = X_1^{+} + X_1^{-}$$

と表される. こゝで $X_1^{+} \in \mathfrak{g}_1^{+}$, $X_1^{-} \in \mathfrak{g}_1^{-}$ である. 以上の記号の下で次の Lemma をえる.

Lemma 5.7. $Z \in \mathfrak{g}_{-1}^{-}$ とする.
(1) $R_a^*\tilde{p}^i(Z) = \tilde{p}^i(\mathrm{Ad}(b)Z), \quad R_a^*\tilde{x}(e_n) = \tilde{x}(\mathrm{Ad}(b)e_n)$
(2) $b = e$ のとき,

$$R_a^*\tilde{x}(e_n, e_n) = \tilde{x}(e_n, e_n) + \tilde{x}([[X_1^{-}, e_n], e_n]),$$
$$R_a^*\tilde{x}([Z, e_n]) = \tilde{x}([Z, e_n]) + \tilde{x}([X_1, [Z, e_n]]).$$

(3) $b = e$, $X_1 = 0$ のとき,

$$R_a^*\tilde{x}([Z, e_n], e_n) = \tilde{x}([Z, e_n], e_n) + \tilde{x}([[X_2, [Z, e_n]], e_n])$$

Proof. (1)

$$\mathrm{Ad}(a)Z \equiv \mathrm{Ad}(b)Z \pmod{\mathfrak{g}^{(0)}}, \qquad \mathrm{Ad}(a)e_n \equiv \mathrm{Ad}(b)e_n \pmod{\mathfrak{g}^{(0)}}$$

である. よって, Lemma 5.1 より

$$R_a^*\tilde{p}^i(Z) = \tilde{p}^i(\mathrm{Ad}(a)Z) = \tilde{p}^i(\mathrm{Ad}(b)Z), \quad R_a^*\tilde{x}(e_n) = \tilde{x}(\mathrm{Ad}(a)e_n) = \tilde{x}(\mathrm{Ad}(b)e_n)$$

をえる. よって (1) が従う.

(2) $b = e$ と仮定する. このとき

$$\mathrm{Ad}(a)e_n \equiv e_n + [X_1^{-}, e_n] \pmod{\mathfrak{g}_1 + \mathfrak{g}_2},$$
$$\mathrm{Ad}(a)[Z, e_n] \equiv [Z, e_n] + [X_1^{+}, [Z, e_n]] \pmod{\mathfrak{g}^{(0)}}.$$

よって Lemma 5.1, 5.3 を使って

$$\begin{aligned}
&R_a^*\tilde{x}(e_n, e_n) = \tilde{x}(\mathrm{Ad}(a)e_n, \mathrm{Ad}(a)e_n) = \tilde{x}(\mathrm{Ad}(a)e_n, e_n)\\
&= \tilde{x}(e_n, \mathrm{Ad}(a)e_n) + \tilde{x}([\mathrm{Ad}(a)e_n, e_n]) - \tilde{x}(K_-((\mathrm{Ad}(a)e_n)_- \wedge e_n))\\
&= \tilde{x}(e_n, e_n) + \tilde{x}([[X_1^-, e_n], e_n])\\
&R_a^*\tilde{x}([Z, e_n]) = \tilde{x}(\mathrm{Ad}(a)[Z, e_n]) = \tilde{x}([Z, e_n]) + \tilde{x}([X_1, [Z, e_n]])
\end{aligned}$$

をえる. $\tilde{x}([X_1, [Z, e_n]]) = \tilde{x}([X_1^+, [Z, e_n]])$ だから, (2) が従う.

(3) $b = e$, $X_1 = 0$ と仮定する. このとき

$$\mathrm{Ad}(a)[Z, e_n] \equiv [Z, e_n] + [X_2, [Z, e_n]] \qquad (\mathrm{mod}\ \mathfrak{g}_1 + \mathfrak{g}_2)$$

である. よって Lemmas 5.1, 5.3 を使って

$$\begin{aligned}
&R_a^*\tilde{x}([Z, e_n], e_n) = \tilde{x}(\mathrm{Ad}(a)[Z, e_n], \mathrm{Ad}(a)e_n) = \tilde{x}(\mathrm{Ad}(a)[Z, e_n], e_n)\\
&= \tilde{x}(e_n, \mathrm{Ad}(a)[Z, e_n]) + \tilde{x}([\mathrm{Ad}(a)[Z, e_n], e_n]) - \tilde{x}(K_-((\mathrm{Ad}(a)[Z, e_n])_- \wedge e_n))\\
&= \tilde{x}(e_n, [Z, e_n]) + \tilde{x}([[X_2, [Z, e_n]], e_n]) - \tilde{x}(K_-([Z, e_n] \wedge e_n))\\
&= \tilde{x}([Z, e_n], e_n) + \tilde{x}([X_2, [Z, e_n]], e_n)
\end{aligned}$$

をえる. よって (3) が従う. □

以上の準備の下で, Proposition 5.5 を証明しよう. $a \in R$ とし, $\pi(y_0) = a$ なる $y_0 \in P$ を 1 つとる.

I. $b \in G_0$ をとり, $y_1 = y_0 \cdot b$ とおく. Lemma 5.7 (1) により

$$\tilde{p}^i(e_{n+j})(y_1) = \tilde{p}^i(\mathrm{Ad}(b)e_{n+j})(y_0), \qquad \tilde{x}(e_n)(y_1) = \tilde{x}(\mathrm{Ad}(b)e_n)(y_0)$$

である. $G_0 = \mathrm{Aut}(\mathfrak{L})$ は, $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^+ + \mathfrak{g}_{-1}^-$ 上への自然な表現を通じて, product group $\mathrm{GL}(\mathfrak{g}_{-1}^+) \times \mathrm{GL}(\mathfrak{g}_{-1}^-)$ と同一視される. また Lemma 5.6 によって, $\tilde{x}(e_n)(y_0) \neq 0$, $\det(\tilde{p}^i(e_{n+j})(y_0)) \neq 0$. よって

$$\tilde{p}^i(e_{n+j})(y_1) = \delta_j^i, \quad \tilde{x}(e_n)(y_1) = 1$$

を満足する b が唯一つ存在する.

II. $X_1 \in \mathfrak{g}_1$ をとり, $a_1 = \exp X_1$, $y_2 = y_1 \cdot a_1$ とおく. このとき Lemma 5.7 (2) から

$$\begin{aligned}
\tilde{x}(e_n, e_n)(y_2) &= \tilde{x}(e_n, e_n)(y_1) + \tilde{x}([[X_1^-, e_n], e_n])(y_1)\\
\tilde{x}([Z, e_n])(y_2) &= \tilde{x}([Z, e_n])(y_1) + \tilde{x}([X_1, [Z, e_n]])(y_1)
\end{aligned}$$

をえる. $X_1^- = \alpha e^n$ とおけば,

$$[[X_1^-, e_n], e_n] = \alpha(-\lambda_1^+ + \mu_1^+)e_n, \qquad [X_1^+, [Z, e_n]] = -(\lambda_1^+ + \mu_1^+)\langle X_1^+, Z\rangle e_n.$$

である. よって

$$\begin{aligned}
\tilde{x}(e_n, e_n)(y_2) &= \tilde{x}(e_n, e_n)(y_1) + \alpha(-\lambda_1^+ + \mu_1^+)\\
\tilde{x}([Z, e_n])(y_2) &= \tilde{x}([Z, e_n])(y_1) - (\lambda_1^+ + \mu_1^+)\langle X_1^+, Z\rangle
\end{aligned}$$

$-\lambda_1^+ + \mu_1^+ \neq 0$, $\lambda_1^+ + \mu_1^+ \neq 0$ だから,

$$\tilde{x}(e_n, e_n)(y_2) = 0, \qquad \tilde{x}([Z, e_n])(y_2) = 0$$

を満足する X_1 が唯一つ存在することがわかる. Lemma 5.7 によって

$$\tilde{p}^i(e_{n+j})(y_2) = \delta_j^i, \qquad \tilde{x}(e_n)(y_2) = 1$$

であることを注意する.

III. $X_2 \in \mathfrak{g}_2$ をとり, $a_2 = \exp X_2$, $y_3 = y_2 \cdot a_2$ とおく. このとき Lemma 5.7 (3) により

$$\tilde{x}([Z, e_n], e_n)(y_3) = \tilde{x}([Z, e_n], e_n)(y_2) + \tilde{x}([[X_2, [Z, e_n]], e_n])(y_2)$$

をえる．

$$[[X_2,[Z,e_n]],e_n] = -2\lambda_1^+\langle X_2,[Z,e_n]\rangle e_n$$

であるから

$$\tilde{x}([Z,e_n],e_n)(y_3) = \tilde{x}([Z,e_n],e_n)(y_2) - 2\lambda_1^+\langle X_2,[Z,e_n]\rangle$$

をえる．$\lambda_1^+ \neq 0$ だから，

$$\tilde{x}([Z,e_n],e_n)(y_3) = 0$$

を満足する X_2 が唯一つ存在することがわかる．Lemma 5.7 により，この y_3 に対して

$$\tilde{p}^i(e_{n+j})(y_3) = \delta^i_j, \qquad \tilde{x}(e_n)(y_3) = 1, \qquad \tilde{x}(e_n,e_n)(y_3) = \tilde{x}([Z,e_n])(y_3) = 0$$

であることを注意する．

以上によって y の存在が示された．一意性は上の議論と Lemma 5.7 から明らかである．また対応 $a \mapsto y$ が P の differentiable cross section を与えることも上の議論から明らかである．

5.3. The invariant HK^2. 今迄の記号を保存する．Proposition 2.3 によって，$HK^2 = {}^-K_0^2$ である．P 上の functions K^i_{jkl} を

$$[{}^-K_0^2([e_{n+j},e_n]\wedge e_{n+k}),e_{n+l}] = [K_0([e_{n+j},e_n]\wedge e_{n+k}),e_{n+l}] = \sum_i K^i_{jkl}e_{n+i}$$

によって定義する．${}^-K_0^2$ は，$V^{2,2}(n,1)$ の値を持つから，K^i_{jkl} は j,k,l について symmetric であり，

$$\sum_i K^i_{jki} = 0$$

を満足する．

R 上の function F^i を使って，R 上の functions F^i_{jkl}, F_{jk} を

$$F^i_{jkl} = \frac{\partial^3 F^i}{\partial p^j \partial p^k \partial p^l}, \qquad F_{jk} = \sum_i F^i_{jki}.$$

によって定義する．そして，R 上の functions Φ^i_{jkl} を

$$\Phi^i_{jkl} = F^i_{jkl} - \frac{1}{n+1}(F_{jk}\delta^i_l + F_{kl}\delta^i_j + F_{lj}\delta^i_k)$$

によって定義する．

$$P^i = \sum F^i_{jkl}z^j z^k z^l$$

とおくと，

$$\sum \Phi^i_{jkl}z^j z^k z^l = P^i - \frac{1}{n+1}z^i \sum_j \frac{\partial P^j}{\partial z^j}$$

以上の準備の下で次の Theorem をえる．

Theorem 5.8. *σ を Proposition 5.5 における cross section $R \to P$ とする．このとき*

$$\sigma^* K^i_{jkl} = \frac{1}{2}\Phi^i_{jkl}$$

が成立する．

Corollary. *$HK^2 = 0$ なるための必要かつ十分な条件は F^i が次の形であることである：*

$$F^i = p^i A + B^i$$

こゝで A, B^i は，$p^1,\dots,p^{n-1}$ について高々2次の polynomials である．

Proof. $HK^2 = 0$ と仮定する．このとき Theorem 5.8 より

$$(*) \qquad F^i_{jkl} - \frac{1}{n+1}(F_{jk}\delta^i_l + F_{kl}\delta^i_j + F_{lj}\delta^i_k) = 0$$

である．この式に $\dfrac{\partial}{\partial p^t}$ を apply して

$$(*') \qquad F^i_{tjkl} - \frac{1}{n+1}(F_{tjk}\delta^i_l + F_{tkl}\delta^i_j + F_{tlj}\delta^i_k) = 0$$

をえる．こゝで

$$F^i_{tjkl} = \frac{\partial}{\partial p^t}F^i_{jkl}, \quad F_{tjk} = \frac{\partial}{\partial p^t}F_{jk}$$

である．明らかに

$$\sum_i F^i_{ijkl} = F_{jkl}$$

であり，F_{jkl} は $j,k.l$ について symmetric であるから，$(*')$ を i と t について縮約することによって

$$F_{jkl} - \frac{3}{n+1}F_{jkl} = \frac{n-2}{n+1}F_{jkl} = 0$$

をえる．よって $F_{jkl} = 0$. よって $(*)$ より $F^i_{tjkl} = 0$ をえる．これは F^i が $p^1, \ldots, p^{n-1}$ について高々3次の polynomials であることを意味する．そして F^i が $(*)$ を満足することから，F^i が Corollary にいう形であることがわかる．逆は容易に確かめられる． □

さて上の Corollary を使って Theorem 3.2 の別証明を与えよう．R は F に関して regular であると仮定し，$M = R/F$ とおく．そして natural immersion $\iota_M : R \to G^1(T(M))$ は imbedding であると仮定する．まず R の coordinate system $\{x, y^i, p^j\}$ は $G^1(T(M))$ の canonical coordinate system から得られていることを注意しよう．この注意と Corollary 及び Proposition $**$ から，「$HK^2 = 0$ なるための必要かつ十分な条件は $\mathfrak{X} = (R, E)$ が projective structure of the restricted type なることである」が分る．よって Theorem 3.2 の別証が得られた．

さて Theorem 5.8 を証明しよう．

Lemma 5.9. $\tilde{y}^i(e_n, e_n) - p^i\tilde{x}(e_n, e_n) = \tilde{x}(e_n)^2 F^i$.

Proof. $\tilde{y}^i(e_n) = p^i\tilde{x}(e_n)$ であるから，

$$\tilde{y}^i(e_n, e_n) = \tilde{p}^i(e_n)\tilde{x}(e_n) + p^i\tilde{x}(e_n, e_n)$$

をえる．$\tilde{p}^i(e_n) = F^i\tilde{x}(e_n)$ であるから，Lemma をえる． □

以下 R の点 a を fix し，$y = \sigma(a)$ とおく．また $\mathfrak{g}^-_{-1}$ の元 Z を fix する．

$$u = \tilde{x}(e_n), \quad u' = \tilde{x}([Z, e_n])$$

とおけば，Lemma 5.4 から

$$\delta_Z u = u', \quad \delta_Z u' = 0$$

をえる．そして，Proposition 5.5 から $u = 1$, $u' = 0$ at y であることを注意する．

Lemma 5.10. y において次が成立する．

(1) $\tilde{p}^i(Z) = \tilde{y}^i([Z, e_n])$.
(2) $\tilde{p}^i(Z, Z) = 0$

Proof. (1) $\tilde{y}^i(e_n) = p^i\tilde{x}(e_n) = p^i u$ である．Lemma 5.4 から $\tilde{y}^i(Z, e_n) = \tilde{y}^i([Z, e_n])$ である．故に

$$\tilde{y}^i([Z, e_n]) = \tilde{p}^i(Z)u + p^i u'$$

をえる．よって (1) が従う．

(2) Lemma 5.4 から $\tilde{y}^i(Z, [Z, e_n]) = 0$ である．よって上式より

$$0 = \tilde{p}^i(Z, Z)u + 2\tilde{p}^i(Z)u'$$

が従う．よって (2) が従う． □

Lemma 5.11. y において次式が成立する．

$$\begin{aligned}&\delta_Z^3(\tilde{y}^i(e_n, e_n) - p^i\tilde{x}(e_n, e_n))\\ &= \tilde{y}^i(Z, Z, Z, e_n, e_n) - p^i\tilde{x}(Z, Z, Z, e_n, e_n) + 3\tilde{p}^i(Z)\tilde{x}(Z, Z, e_n, e_n)\end{aligned}$$

Proof. まず次の 2 つの式をえる．

$$\begin{aligned}\delta_Z(p^i\tilde{x}(e_n, e_n)) &= \tilde{p}^i(Z)\tilde{x}(e_n, e_n) + p^i\tilde{x}(Z, e_n, e_n)\\ \delta_Z^2(p^i\tilde{x}(e_n, e_n)) &= \tilde{p}^i(Z, Z)\tilde{x}(e_n, e_n) + 2\tilde{p}^i(Z)\tilde{x}(Z, e_n, e_n) + p^i\tilde{x}(Z, Z, e_n, e_n)\end{aligned}$$

Proposition 5.5, Lemmas 5.4, 5.10 によって，$\tilde{x}(e_n, e_n) = \tilde{x}(Z, e_n, e_n) = \tilde{p}^i(Z, Z) = 0$ at y である．(Lemma 5.4 より $\tilde{x}(Z, e_n, e_n) = 2\tilde{x}([Z, e_n], e_n)$ である)．よって y において

$$\delta_Z^3(p^i\tilde{x}(e_n, e_n)) = 3\tilde{p}^i(Z)\tilde{x}(Z, Z, e_n, e_n) + p^i\tilde{x}(Z, Z, Z, e_n, e_n)$$

をえる．$\delta_Z^3\tilde{y}^i(e_n, e_n) = \tilde{y}^i(Z, Z, Z, e_n, e_n)$ であるから，Lemma が示された． □

Lemma 5.12. y において次式が成立する．

$$\delta_Z^3(u^2F^i) = \sum_{j,k,l} F^i_{jkl}\tilde{p}^j(Z)\tilde{p}^k(Z)\tilde{p}^l(Z)$$

Proof. $\tilde{x}(Z) = \tilde{y}^i(Z) = 0$ だから，$\tilde{F}^i(Z) = \sum_l \dfrac{\partial F^i}{\partial p^l}\tilde{p}^l(Z)$, etc. をえる．よって次の 2 式をえる．

$$\begin{aligned}\delta_Z(u^2F^i) &= 2uu'F^i + u^2\sum_l \frac{\partial F^i}{\partial p^l}\tilde{p}^l(Z),\\ \delta_Z^2(u^2F^i) &= 2(u')^2F^i + 4uu'\sum_l \frac{\partial F^i}{\partial p^l}\tilde{p}^l(Z)\\ &\quad + u^2\sum_{k,l}\frac{\partial^2 F^i}{\partial p^k\partial p^l}\tilde{p}^k(Z)\tilde{p}^l(Z) + u^2\sum_l \frac{\partial F^i}{\partial p^l}\tilde{p}^l(Z, Z),\end{aligned}$$

故に y において

$$\delta_Z^3(u^2F^i) = \sum_{j,k,l}\frac{\partial^3 F^i}{\partial p^j\partial p^k\partial p^l}\tilde{p}^j(Z)\tilde{p}^k(Z)\tilde{p}^l(Z)$$

をえる．よって Lemma が示された． □

Lemma 5.13. y において次式が成立する．

$$-2\tilde{p}^i([K_0(Z\wedge[Z, e_n]), Z]) + 3\tilde{p}^i(Z)\tilde{x}(Z, Z, e_n, e_n) = \sum_{j,k,l} F^i_{jkl}\tilde{p}^j(Z)\tilde{p}^k(Z)\tilde{p}^l(Z)$$

Proof. 以下の各式は y において評価するものとする．まず Lemmas 5.9, 5.11 and 5.12 から次式をえる．

$$\begin{aligned}&\tilde{y}^i(Z,Z,Z,e_n,e_n) - p^i\tilde{x}(Z,Z,Z,e_n,e_n) + 3\tilde{p}^i(Z)\tilde{x}(Z,Z,e_n,e_n)\\&= \sum_{j,k,l} F^i_{jkl}\tilde{p}^j(Z)\tilde{p}^k(Z)\tilde{p}^l(Z)\end{aligned}$$

そして Lemma 5.4 (5) から

$$\begin{aligned}&\tilde{y}^i(Z,Z,Z,e_n,e_n) - p^i\tilde{x}(Z,Z,Z,e_n,e_n)\\&= -2\tilde{y}^i(T^2(Z)) - 2\tilde{y}^i(T^3(Z)) + 2p^i\tilde{x}(T^2(Z)) + 2p^i\tilde{x}(T^3(Z))\end{aligned}$$

をえる．こゝで

$$T^2(Z) = [[K_0(Z\wedge[Z,e_n]),Z],e_n], \quad T^3(Z) = [K_1(Z\wedge[Z,e_n]),[Z,e_n]]$$

である．Lemma 5.10 から

$$\tilde{y}^i(T^2(Z)) = \tilde{p}^i([K_0(Z\wedge[Z,e_n]),Z])$$

であり，

$$\tilde{x}(T^2(Z)) = 0$$

である．さらに $T^3(Z)$ は $\mathfrak{g}_{-1}$ の値を持ち，R 上の differential system $D = E+F$ は $dy^i - p^i dx = 0$ で定義されるから，

$$\tilde{y}^i(T^3(Z)) - p^i\tilde{x}(T^3(Z)) = 0$$

をえる．以上の事実から Lemma が従う． □

$Z = \sum_i z^i e_{n+i}$ とおけば，$z^i = \tilde{p}^i(Z)$ である (Proposition 5.5)．そして

$$\tilde{x}(Z,Z,e_n,e_n) = \sum_{j,k} A_{jk}z^jz^k, \quad A_{jk} = A_{kj}$$

とおけば，Lemma 5.13 から

$$2\sum_{j,k,l} K^i_{jkl}z^jz^kz^l + 3z^i\sum_{j,k} A_{jk}z^jz^k = \sum_{j,k,l} F^i_{jkl}z^jz^kz^l$$

をえる．$z^1,\ldots,z^{n-1}$ は任意であり，K^i_{jkl} は j,k,l について symmetric だから

$$2K^i_{jkl} + (A_{jk}\delta^i_l + A_{kl}\delta^i_j + A_{lj}\delta^i_k) = F^i_{jkl}$$

をえる．$\sum_i K^i_{jki} = 0$ だから，

$$(n+1)A_{jk} = F_{jk}$$

をえる．よって

$$2K^i_{jkl} = \Phi^i_{jkl}$$

をえる．この式は任意の $a \in R$ に対して $\sigma(a)$ で成立する．よって Theorem 5.8 の証明が完了した．

5.4. The invariant K^1. この paragraph においても今迄の記号を保存する．K^1 は harmonic であり，$V^{1,2}(n,1)$ の値を持つ．K^1 は $K_{-1}([Z,e_n]\wedge e_n)$, $Z\in\mathfrak{g}_{-1}^{-}$, によって決定される．そこで，$P$ 上の functions K^i_k を

$$K_{-1}([e_{n+k},e_n]\wedge e_n)=\sum_i K^i_k e_{n+i}$$

によって定義する．

$$\sum_i K^i_i=0$$

が成立する．

functions F^i を使って，R 上の functions F^i_k を

$$F^i_k=\frac{\partial^2F^i}{\partial p^k\partial x}+\sum_j\frac{\partial^2F^i}{\partial p^k\partial y^j}p^j-2\frac{\partial F^i}{\partial y^k}+\sum_j\frac{\partial^2F^i}{\partial p^k\partial p^j}F^j-\frac{1}{2}\sum_j\frac{\partial F^i}{\partial p^j}\frac{\partial F^j}{\partial p^k}$$

によって定義し，R 上の functions Φ^i_k を

$$\Phi^i_k=F^i_k-\frac{\delta^i_k}{n-1}\sum_j F^j_j$$

によって定義する．

$$\frac{d}{dx}=\frac{\partial}{\partial x}+\sum_j p^j\frac{\partial}{\partial y^j}+\sum_j F^j\frac{\partial}{\partial p^j}$$

を使えば，

$$F^i_k=\frac{d}{dx}\frac{\partial F^i}{\partial p^k}-2\frac{\partial F^i}{\partial y^k}-\frac{1}{2}\sum_j\frac{\partial F^i}{\partial p^j}\frac{\partial F^j}{\partial p^k}$$

以上の記号の下で次の Theorem をえる．

Theorem 5.14. *σ を Proposition 5.5 における cross section $R\to P$ とする．このとき*

$$\sigma^*K^i_k=\frac{1}{2}\Phi^i_k$$

が成立する．

この Theorem の証明に進む前に $K^1=0$, $HK^2\neq 0$ のなる projective system of type $(n,1)$ の例を挙げよう．

R を variables x,y^i,p^j の空間 $\mathbb{R}^{2n-1}$ とする．R 上の functions $F^i,\dots,F^{n-1}$ を次のように定める：$F^i=0$, $2\leqq i\leqq n-1$; F^1 は，$p^2,\dots,p^{n-1}$ だけに depend する任意の functions. $\mathfrak{P}=\{R;E,F\}$ を $F^1,\dots,F^{n-1}$ によって決まる R 上の projective system of type $(n,1)$ とする．即ち F が equation

$$dx=dy^i=0$$

で定義され，E が equation

$$\begin{cases} dy^i-p^idx=0,\\ dp^j-F^jdx=0\end{cases}$$

で定義されるものである．

明らかに，$F^i_k=0$, 従って，$\Phi^i_k=0$ である．よって，Theorem 5.14 よって $K^1=0$ である．そして，Theorem 5.8 から容易に次の命題をえる：

Proposition 5.15. *$HK^2=0$ なるための必要かつ十分な条件は，F^1 が $p^2,\dots,p^{n-1}$ についての高々 2 次の polynomial なることである．*

故にたとえば F^1 が $p^2,\dots,p^{n-1}$ についての 3 次の polynomial ならば, $HK^2 \neq 0$ である.

Theorem 2.1 より $K^1 = HK^2 = 0$ なるための必要かつ十分な条件は, $\mathfrak{P}$ が standard projective system of type $(n,1)$, $\mathfrak{P}_0$, に locally equivalent なることである. よって, 上の Proposition 5.15 は次の様に言換えられる: equation

$$\begin{cases} \dfrac{d^2y^1}{dx^2} = F^1\left(\dfrac{dy^2}{dx},\dots,\dfrac{dy^{n-1}}{dx}\right), \\ \dfrac{d^2y^k}{dx^2} = 0, \quad 2 \leqq k \leqq n-1 \end{cases}$$

が standard equation

$$\frac{d^2y^i}{dx^2} = 0, \quad 1 \leqq i \leqq n-1$$

に locally equivalent になるための必要かつ十分な条件は, F^1 が $p^2,\dots,p^{n-1}$ についての高々 2 次の polynomial なることである.

さて Theorem 5.14 の証明をしよう. 以下 R の点 a を fix し, $y = \sigma(a)$ とおく.

Lemma 5.16. y において次式が成立する.

(1) $\tilde{y}^i(e_n) = p^i$

(2) $\tilde{p}^i(e_n) = F^i$

(3) $\tilde{y}^i(e_{n+k}, e_n) = \tilde{y}^i([e_{n+k}, e_n]) = \delta^i_k$

(4) $\tilde{p}^i(e_{n+k}, e_n) = 2\tilde{p}^i([e_{n+k}, e_n]) = \dfrac{\partial F^i}{\partial p^k}$

Proof. (1) and (2)

$$\tilde{y}^i(e_n) = p^i\tilde{x}(e_n), \quad \tilde{p}^i(e_n) = F^i\tilde{x}(e_n)$$

であり, $\tilde{x}(e_n) = 1$ at y だから, (1), (2) が従う.

(3) $\tilde{y}^i(e_n) = p^i\tilde{x}(e_n)$ より

$$\tilde{y}^i(e_{n+k}, e_n) = \tilde{p}^i(e_{n+k})\tilde{x}(e_n) + p^i\tilde{x}(e_{n+k}, e_n)$$

をえる. $\tilde{y}^i(e_{n+k}, e_n) = \tilde{y}^i([e_{n+k}, e_n])$ (Lemma 5.4 (1)) であり, $\tilde{x}(e_n) = 1$, $\tilde{p}^i(e_{n+k}) = \delta^i_k$, $\tilde{x}(e_{n+k}, e_n) = 0$ at y だから, (3) が従う. (cf. the proof of Lemma 5.10).

(4) $\tilde{p}^i(e_n) = F^i\tilde{x}(e_n)$ より

$$\tilde{p}^i(e_{n+k}, e_n) = \left(\sum_j \frac{\partial F^i}{\partial p^j}\tilde{p}^j(e_{n+k})\right)\tilde{x}(e_n) + F^i\tilde{x}(e_{n+k}, e_n)$$

よって

$$\tilde{p}^i(e_{n+k}, e_n) = \frac{\partial F^i}{\partial p^k} \quad \text{at } y$$

をえる. さらに $\tilde{y}^i(e_n) = p^i\tilde{x}(e_n)$ より

$$\begin{aligned} &\tilde{y}^i([e_{n+k}, e_n], e_n) = \tilde{p}^i([e_{n+k}, e_n])\tilde{x}(e_n) + p^i\tilde{x}([e_{n+k}, e_n], e_n) \\ &\tilde{y}^i(e_n, e_n) = \tilde{p}^i(e_n)\tilde{x}(e_n) + p^i\tilde{x}(e_n, e_n), \\ &\tilde{y}^i(e_{n+k}, e_n, e_n) = \tilde{p}^i(e_{n+k}, e_n)\tilde{x}(e_n) + \tilde{p}^i(e_n)\tilde{x}(e_{n+k}, e_n) \\ &\qquad\qquad + \tilde{p}^i(e_{n+k})\tilde{x}(e_n, e_n) + p^i\tilde{x}(e_{n+k}, e_n, e_n), \end{aligned}$$

をえる. $\tilde{y}^i(e_{n+k}, e_n, e_n) = 2\tilde{y}^i([e_{n+k}, e_n], e_n)$ であり, $\tilde{x}(e_{n+k}, e_n, e_n) = 2\tilde{x}([e_{n+k}, e_n], e_n) = 0$ at y, etc. よって 上式から

$$\tilde{p}^i(e_{n+k}, e_n) = 2\tilde{p}^i([e_{n+k}, e_n]) \quad \text{at } y$$

が従う. 以上で (4) が示された. □

Lemma 5.17. y において次式が成立する.

(1) $\delta_{e_{n+k}}\delta_{e_n}(\tilde{y}^i(e_n,e_n)-p^i\tilde{x}(e_n,e_n)) = \tilde{y}^i(e_{n+k},e_n,e_n,e_n)-p^i\tilde{x}(e_{n+k},e_n,e_n,e_n)-\delta^i_k\tilde{x}(e_n,e_n,e_n)$

(2) $\delta_{[e_{n+k},e_n]}(\tilde{y}^i(e_n,e_n)-p^i\tilde{x}(e_n,e_n)) = \tilde{y}^i([e_{n+k},e_n],e_n,e_n)-p^i\tilde{x}([e_{n+k},e_n],e_n,e_n)$

Proof. (1) まず

$$\delta_{e_n}(p^i\tilde{x}(e_n,e_n)) = \tilde{p}^i(e_n)\tilde{x}(e_n,e_n)+p^i\tilde{x}(e_n,e_n,e_n)$$

をえる. そして, $\tilde{x}(e_n,e_n)=\tilde{x}(e_{n+k},e_n,e_n)=0$, $\tilde{p}^i(e_{n+k})=\delta^i_k$ at y である. よって y において次式をえる.

$$\delta_{e_{n+k}}\delta_{e_n}(p^i\tilde{x}(e_n,e_n)) = \delta^i_k\tilde{x}(e_n,e_n,e_n)+p^i\tilde{x}(e_{n+k},e_n,e_n,e_n)$$

をえる. よって (1) が従う.

(2) 上と同様に, y において次式をえる.

$$\delta_{[e_{n+k},e_n]}(p^i\tilde{x}(e_n,e_n)) = p^i\tilde{x}([e_{n+k},e_n],e_n,e_n)$$

をえる. よって (2) が従う. □

Lemma 5.18. y において次式をえる.

(1) $\delta_{e_{n+k}}\delta_{e_n}(\tilde{x}(e_n)^2F^i) = \dfrac{\partial^2F^i}{\partial p^k\partial x}+\sum_j\dfrac{\partial^2F^i}{\partial p^k\partial y^j}p^j+\dfrac{\partial F^i}{\partial y^k}+\sum_j\dfrac{\partial^2F^i}{\partial p^k\partial p^j}F^j+\sum_j\dfrac{\partial F^i}{\partial p^j}\dfrac{\partial F^j}{\partial p^k}$

(2) $\delta_{[e_{n+k},e_n]}(\tilde{x}(e_n)^2F^i) = \dfrac{\partial F^i}{\partial y^k}+\dfrac{1}{2}\sum_j\dfrac{\partial F^i}{\partial p^j}\dfrac{\partial F^j}{\partial p^k}$

Proof. (1) まず

$$\begin{aligned}\delta_{e_n}(\tilde{x}(e_n)^2F^i) &= 2\tilde{x}(e_n)\tilde{x}(e_n,e_n)F^i\\ &\quad+\tilde{x}(e_n)^2\left(\frac{\partial F^i}{\partial x}\tilde{x}(e_n)+\sum_j\frac{\partial F^i}{\partial y^j}\tilde{y}^j(e_n)+\sum_j\frac{\partial F^i}{\partial p^j}\tilde{p}^j(e_n)\right)\end{aligned}$$

をえる. Proposition 5.5 と Lemma 5.16 を考慮して, $\delta_{e_{n+k}}\delta_{e_n}(\tilde{x}(e_n)^2F^i)$ を計算し, (1) をえる.

(2) (1) と同様にして, (2) をえる. □

Lemma 5.19. y において次式が成立する.

$$2K^i_k-\delta^i_k\tilde{x}(e_n,e_n,e_n) = F^i_k$$

Proof. 以下の各式は y において評価する. Lemma 5.4 (6) と Lemmas 5.18 and 5.9 から直ちに次式をえる:

$$-2\tilde{y}^i(S^1_k)-2\tilde{y}^i(S^2_k)+2p^i\tilde{x}(S^1_k)+2p^i\tilde{x}(S^2_k)-\delta^i_k\tilde{x}(e_n,e_n,e_n) = F^i_k$$

こゝで

$$S^1_k = [K_{-1}(e_n\wedge[e_{n+k},e_n]),e_n], \qquad S^2_k = [K_0(e_n\wedge[e_{n+k},e_n]),e_n]$$

である. Lemma 5.4, etc から

$$\tilde{y}^i(S^1_k) = \tilde{p}^i(K_{-1}(e_n\wedge[e_{n+k},e_n])) = -K^i_k$$

と

$$\tilde{x}(S^1_k) = 0$$

をえる. さらに S^2_k は $\mathfrak{g}^+_{-1}$ の値を持つから,

$$\tilde{y}^i(S^2_k)-p^i\tilde{x}(S^2_k) = 0$$

である．以上から Lemma をえる． $\square$

$\sum_i K_i^i = 0$ だから Lemma 5.19 から

$$-(n-1)\tilde{x}(e_n, e_n, e_n) = \sum_i F_i^i$$

をえる．よって

$$2K_k^i = \Phi_k^i$$

をえる．この式はすべての $a \in R$ に対して, $y = \sigma(a)$ で成立する．よって Theorem 5.14 の証明が完了した．

CHAPTER VI

§1. Contact manifolds of order k

1.1. The Grassmann bundles and their prolongations. Let (n, r) be a pair of integers with $1 \leqq r \leqq n-1$. Let M be an n-dimensional manifold. For every $k \geqq 1$ we shall construct a manifold $J^k(M, r)$ with a differential system C^k inductively in the following manner: First $J^1(M, r)$ is defined to be the Grassmann bundle $J(M, r)$ over M, and C^1 to be the contact system C on it. Suppose that for some $k \geqq 2$ we have constructed $J^\ell(M, r)$ together with C^ℓ for all $1 \leqq \ell \leqq k-1$, so that $J^\ell(M, r)$ are fibred manifolds over $J^{\ell-1}(M, r)$, where we set $J^0(M, r) = M$. Let V^{k-1} be the vertical tangent bundle of the fibred manifold $J^{k-1}(M, r)$ over $J^{k-2}(M, r)$. Then $J^k(M, r)$ is defined to be the set of all r-dimensional integral elements z^k of C^{k-1} such that $z^k \cap V^{k-1}(z^{k-1}) = \{0\}$, where z^{k-1} is the origin of z^k. Note that $J^k(M, r)$ can be inductively shown to become a fibred submanifold of the Grassmann bundle $J(J^{k-1}(M, r), r)$ over $J^{k-1}(M, r)$. Furthermore C^k is defined to be the contact system on the fibred submanifold $J^k(M, r)$, i.e., the pull-back of the contact system on $J(J^{k-1}(M, r), r)$ to $J^k(M, r)$ (see), completing our inductive definition.

We have thus obtained a sequence

$$\cdots \to J^k(M, r) \xrightarrow{\rho^k_{k-1}} J^{k-1}(M, r) \to \cdots \to J^1(M, r) \to J^0(M, r) = M$$

of fibred manifolds and a differential system C^k on each $J^k(M, r)$, where ρ^k_{k-1} denotes the projection. The pair $(J^k(M, r), C^k)$ will be called the $(k-1)$-th prolongation of $(J(M, r), C)$, which will be also called the k-th prolongation of the pair (M, r). As above we denote by V^k the vertical tangent bundle $J^k(M, r)$ over $J^{k-1}(M, r)$, which is called the symbol system of $J^k(M, r)$.

If $0 \leqq \ell \leqq k$, $J^k(M, r)$ becomes naturally a fibred manifold over $J^\ell(M, r)$, and we denote by ρ^k_ℓ its projection: $\rho^k_\ell = \rho^k_{k-1} \circ \cdots \circ \rho^{\ell+1}_\ell$ if $\ell < k$, and ρ^k_k is the identity. This being said, we define differential systems C^k_ℓ and V^k_ℓ $(1 \leqq \ell \leqq k)$ on $J^k(M, r)$ as follows:

$$C^k_\ell = (\rho^k_\ell)^{-1}_*(C^\ell), \quad V^k_\ell = (\rho^k_\ell)^{-1}_*(V^\ell).$$

Clearly we have $C^k_k = C^k$, $V^k_k = V^k$, and

$$C^k_\ell \supset C^k_{\ell+1}, \quad V^k_\ell \supset V^k_{\ell+1}, \quad C^k_\ell \supset V^k_\ell.$$

Let z^k_0 be a point of $J^k(M, r)$. Let $(x^1, \ldots, x^r, y^1, \ldots, y^{n-r})$ be a coordinate system of M at $\rho^k_0(z^k_0)$ such that the 1-forms $dx^1, \ldots, dx^r$ are linearly independent, restricted to the r-dimensional contact element $\rho^k_1(z^k_0)$ to M. We denote by U the domain of the coordinate system, and by $\widetilde{U}$ the set of all $z^k \in J^k(M, r)$ such that $\rho^k_0(z^k) \in U$, and $dx^1, \ldots, dx^r$ are linearly independent, restricted to $\rho^k_1(z^k)$. It is clear that $\widetilde{U}$ is an open set of $J^k(M, r)$, $z^k_0 \in \widetilde{U}$, and $\rho^k_0(\widetilde{U}) = U$. In the following the indices i, i_1, i_2, ... will range over the integers $1, \ldots, r$, and the index α over the integers $1, \ldots, n-r$.

Lemma 1.1. *There are unique functions $p^{\alpha}_{i_1,\dots,i_\ell}$ $(1 \leqq \ell \leqq k)$ on $\widetilde{U}$ satisfying the following conditions:*

1) $p^{\alpha}_{i_1,\dots,i_\ell}$ are symmetric with respect to the indices $i_1, \dots, i_\ell$.

2) The functions x^i $(= x^i \circ \rho^k_0)$, y^α $(= y^\alpha \circ \rho^k_0)$ and $p^{\alpha}_{i_1,\dots,i_\ell}$ $(i_1 \leqq \cdots \leqq i_\ell,\ 1 \leqq \ell \leqq k)$ together give a coordinate system of $J^k(M,r)$ on $\widetilde{U}$.

3) V^k_ℓ is defined by the Pfaffian equations:

$$dx^i = dy^\alpha = dp^{\alpha}_{i_1,\dots,i_m} = 0 \quad (1 \leqq m \leqq \ell - 1),$$

and C^k_ℓ by the Pfaffian equations:

$$\begin{cases} dy^\alpha - \displaystyle\sum_i p^\alpha_i \, dx^i = 0, \\ dp^{\alpha}_{i_1,\dots,i_m} - \displaystyle\sum_i p^{\alpha}_{i_1,\dots,i_m,i} \, dx^i = 0 \quad (1 \leqq m \leqq \ell - 1). \end{cases}$$

The coordinate system $(x^i, y^\alpha, p^{\alpha}_{i_1,\dots,i_\ell})$ $(1 \leqq \ell \leqq k)$ will be called a canonical coordinate system of $J^k(M,r)$ at z^k_0.

Let S be an r-dimensional submanifold of M. Then S is prolonged to the r-dimensional submanifold $p(S)$ of $J^1(M,r)$, and in turn this is prolonged to the r-dimensional submanifold $p^2(S) = p(p(S))$ of $J(J^1(M,r))$, which is clearly in $J^2(M,r)$. In this way S is prolonged to the r-dimensional submanifold $p^k(S)$ of $J^k(M,r)$ for every k. Note that $p^k(S)$ gives an integral manifold of the contact system C^k.

Lemma 1.2. *Let $\widetilde{S}$ be an r-dimensional integral manifold of C^k. Assume that the projection ρ^k_0 maps $\widetilde{S}$ diffeomorphically onto an r-dimensional submanifold S of M. Then $\widetilde{S} = p^k(S)$.*

Let us now consider a canonical coordinate system $(x^i, y^\alpha, p^{\alpha}_{i_1,\dots,i_\ell})$ $(1 \leqq \ell \leqq k)$ of $J^k(M,r)$ at a point z^k_0. Let S be an r-dimensional submanifold of M. Suppose that $p^k(S)$ is in the domain $\widetilde{U}$ of the coordinate system, and S is defined by the equations:

$$y^\alpha = u^\alpha(x^1, \dots, x^r).$$

Then we remark that $p^k(S)$ is defined by the equations:

$$\begin{cases} y^\alpha = u^\alpha(x^1, \dots, x^r), \\ p^{\alpha}_{i_1,\dots,i_\ell} = \dfrac{\partial^\ell u^\alpha}{\partial x^{i_1} \cdots \partial x^{i_\ell}}(x^1, \dots, x^r) \quad (1 \leqq \ell \leqq k). \end{cases}$$

For simplicity we set as follows:

$$R = J^k(M,r), \quad D = C^k,$$
$$D_\ell = C^k_\ell, \quad F_\ell = V^k_\ell \quad (1 \leqq \ell \leqq k).$$

Lemma 1.3. (1) *$D = D_k$ is regular, and D_ℓ is the $(k-\ell)$-th derived system of D_k for $0 \leqq \ell \leqq k-1$, where D_0 is given by $D_0 = T(R)$. Furthermore D_ℓ is the first derived system of $D_{\ell+1}$ for $0 \leqq \ell \leqq k-1$.*

(2)
$$D_\ell = \mathrm{ch}(D_{\ell-1}), \quad 2 \leqq \ell \leqq k,$$
$$F_\ell \subset D_\ell, \quad 1 \leqq \ell \leqq k,$$
$$F_\ell = D_\ell \cap F_{\ell-1}, \quad 2 \leqq \ell \leqq k.$$

(3) *D_k is non-degenerate, i.e.,* $\mathrm{ch}(D_k) = 0$.

$$\text{(4)}\qquad \begin{aligned} \operatorname{rank}(D_\ell/D_{\ell+1}) &= (n-r)\,{}_rH_\ell, \quad 0 \leqq \ell \leqq k-1,\\ \operatorname{rank}(D_\ell/F_\ell) &= r, \quad 1 \leqq \ell \leqq k,\\ \operatorname{rank}(F_\ell/F_{\ell+1}) &= (n-r)\,{}_rH_\ell, \quad 0 \leqq \ell \leqq k-1,\\ \operatorname{rank}(F_k) &= (n-r)\,{}_rH_k. \end{aligned}$$

Recall that ${}_rH_\ell = \binom{r+\ell-1}{\ell}$.

By virtue of Lemma we may speak of the symbol algebra $\mathfrak{F} = (\mathfrak{f}, (\mathfrak{g}_p)_{p<0})$ of (R, D) at any point $z \in R$:

$$\begin{aligned} \mathfrak{g}_{-1} &= D_k(z),\\ \mathfrak{g}_p &= D_{k+p+1}(z)/D_{k+p+2}(z), \quad p \leqq -2, \end{aligned}$$

where we set $D_\ell = T(R)$ if $\ell \leqq 0$. Let f denote the subspace $F_k(z)$ of $\mathfrak{g}_{-1}$.

Lemma 1.4. (1) *The FGLA, $\mathfrak{F}$, is of the $(k+1)$-th kind and non-degenerate.*

(2) $[\mathfrak{g}_p, \mathfrak{g}_q] = \{0\}$ *for $p, q \leqq -2$, $[\mathfrak{g}_p, f] = \{0\}$ for $p \leqq -2$, and $[f, f] = \{0\}$.*

(3) *If $p \leqq -2$, $X \in \mathfrak{g}_p$, and $[X, Y] = 0$ for all $Y \in \mathfrak{g}_{-1}$, then $X = 0$.*

(4) $\dim \mathfrak{g}_{-(k+1)} = n - r$, $\dim \mathfrak{g}_p = (n-r)\,{}_rH_{k+p+1}$ *for $-k \leqq p \leqq -2$,* $\dim(\mathfrak{g}_{-1}/f) = r$, *and* $\dim f = (n-r)\,{}_rH_k$.

(5) *There is a subspace e of $\mathfrak{g}_{-1}$ such that $\mathfrak{g}_{-1} = e + f$ (direct sum) and $[e, e] = \{0\}$.*

This can be easily derived form Lemma together with Lemma .

Here we remark that if $k \geqq 2$, f consists of all $X \in \mathfrak{g}_{-1}$ such that $[X, \mathfrak{g}_{-2}] = \{0\}$, showing that f is naturally associated with $\mathfrak{F}$ (or rather naturally constructed from $\mathfrak{F}$). In the case where $k = 1$ we have the following

Lemma 1.5. *If $k = 1$ and $r \leqq n-2$, then f is naturally associated with $\mathfrak{F}$.*

Indeed, it is known that f is spanned by all the elements $X \in \mathfrak{g}_{-1}$ such that $\operatorname{rank}(ad(X)) = 1$.

It should be noted that if $k = 1$ and $r = n-1$, $\mathfrak{F}$ is a contact FGLA, i.e., $\dim \mathfrak{g}_{-2} = 1$, and hence f is not naturally associated with $\mathfrak{F}$.

By Lemma we know that the systems D_ℓ $(1 \leqq \ell \leqq k)$ and F_ℓ $(2 \leqq \ell \leqq k)$ are all naturally associated with the system D or covariant systems of C^k in E. Cartan's terminology. As for the system F_1 we have the following

Lemma 1.6. *If $r \leqq n-2$, then F_1 is naturally associated with D.*

Indeed this lemma can be easily reduced to the case where $k = 1$, and in this case the lemma follows immediately from Lemma . (Another more direct proof can be given by considering the symbol algebra of F_1 at any $z \in R$ and by using Lemma .)

It should be noted that $(J^1(M, n-1), C^1)$ is a contact manifold of degree n, and hence V^1 is not naturally associated with C^1. It follows that if $r = n-1$, F_1 is not naturally associated with D.

Now let us consider another n-dimensional manifold M' and the prolongations $J^k(M', r)$ of the pair (M', r). Assuming that M is diffeomorphic with M', let φ be a diffeomorphism of M onto M'. Then φ is prolonged to the diffeomorphism $p(\varphi)$ of $J^1(M, r)$ onto $J^1(M'mr)$, and in turn this is prolonged to the diffeomorphism $p^2(\varphi) = p(p(\varphi))$ of $J(J^1(M, r), r)$ onto $J(J^1(M', r), r)$, which clearly sends $J^2(M, r)$ onto $J^2(M', r)$. In this way φ is prolonged to the diffeomorphism $p^k(\varphi)$ of $J^k(M, r)$ onto $J^k(M', r)$ for every k. We note that $p^k(\varphi)$ gives an isomorphism of $(J^k(M, r), C^k)$ onto $(J^k(M', r), C'^k)$.

By Lemma we have

Lemma 1.7. *Let $\widetilde{\varphi}$ be any local isomorphism of $(J^k(M,r), C^k)$ onto $(J^k(M',r), C'^k)$ at any $(z,z') \in J^k(M,r) \times J^k(M',r)$. If $r \leqq n-2$, there is a unique local diffeomorphism φ of M to M' at $(\rho_0^k(z), \rho_0'^k(z'))$ such that $\widetilde{\varphi} = p^k(\varphi)$.*

Proof of Lemma . Let us consider the symbole algebra $\bar{\mathfrak{f}} = \sum_{p<0} \bar{\mathfrak{g}}_p$ of C_1^k at any point $z \in J^k(M,r)$: $\bar{\mathfrak{g}}_p = \{0\}$ for $p \leqq -2$, $\bar{\mathfrak{g}}_{-2} = T_z(J^k(M,r))/C_1^k(z)$, and $\bar{\mathfrak{g}}_{-1} = C_1^k(z)$. Let $\overline{f}$ and $\overline{f}_0$ denote the subspaces $V_1^k(z)$ and $V_2^k(z)$ of $\bar{\mathfrak{g}}_{-1}$ respectively. Clearly $\overline{f}_0$ consists of all $X \in \bar{\mathfrak{g}}_{-1}$ such that $[X, \bar{\mathfrak{g}}_{-1}] = \{0\}$, and hence it is an graded ideal of $\bar{\mathfrak{f}}$. Therefore $\bar{\mathfrak{f}}/\overline{f}_0$ $(\cong \bar{\mathfrak{g}}_{-2} + (\bar{\mathfrak{g}}_{-1}/\overline{f}_0))$ becomes naturally a graded Lie algebra. Let us now consider the symbol algebra $\mathfrak{f}' = \sum_{p<0} \mathfrak{g}'_p$ of C^1 at $x = \rho_1^k(z)$: $\mathfrak{g}'_p = \{0\}$ for $p \leqq -2$, $\mathfrak{g}'_{-2} = T_x(J^1(M,r))/C^1(x)$, and $\mathfrak{g}'_{-1} = C^1(x)$. Let f' denote the subspace $V^1(x)$ of $\mathfrak{g}'_{-1}$. Then we see that there is a natural isomorphism of $\bar{\mathfrak{f}}/\overline{f}_0$ onto $\mathfrak{f}'$ (as graded Lie algebras) which sends the subspace $\overline{f}/\overline{f}_0$ of $\bar{\mathfrak{g}}_1/\overline{f}_0$ onto the subspace f' of $\mathfrak{g}'_{-1}$. Consequently it follows from Lemma that $\overline{f}$ is naturally associated with the graded Lie algebra $\bar{\mathfrak{f}}$, which clearly implies that V_1^k is naturally associated with C_1^k and hence C^k.

1.2. Contact manifolds and their prolongations. Let (M,C) be a contact manifold of degree n. We denote by $L(M,C)$ the set of all $(n-1)$-dimensional integral elements of C, which can be shown to become naturally a fibred submanifold of the Grassmann bundle $J(M,n-1)$ over M. $L(M,C)$ will be called the Lagrange-Grassmann bundle over (M,C). We denote by $\hat{C}$ the contact system on $L(M,C)$.

For every $k \geqq 1$ we shall now construct a manifold $\Gamma^k(M,C)(= L^{k-1}(M,C)?)$ with a differential system C^k inductively in the following manner: First $\Gamma^1(M,c)$ is defined to be the manifold M, and C^1 to be the contact structure C. Second $\Gamma^2(M,c)$ is defined to be the Lagrange-Grassmann bundle $L(M,C)$, and C^2 to be the contact system $\hat{C}$ on it. Suppose that for some $k \geqq 3$ we have constructed $\Gamma^\ell(M,C)$ together with C^ℓ for all $1 \leqq \ell \leqq k-1$, so that $\Gamma^\ell(M,C)$ are fibred submanifolds over $\Gamma^{\ell-1}(M,C)$ for all $2 \leqq \ell \leqq k-1$. Let V^{k-1} be the vertical tangent bundle of the fibred manifold $\Gamma^{k-1}(M,C)$ over $\Gamma^{k-2}(M,C)$. Then $\Gamma^k(M,C)$ is defined to be the set of all $(n-1)$-dimensional integral elements z^k of C^{k-1} such that $z^k \cap V^{k-1}(z^{k-1}) = 0$, where z^{k-1} is the origin of z^k. Note that $\Gamma^k(M.C)$ can be inductively shown to become a fibred submanifold of the Grassmann bundle $J(\Gamma^{k-1}(M,C), n-1)$ over $\Gamma^{k-1}(M,C)$. Furthermore C^k is defined to be the contact system on the fibred submanifold $\Gamma^k(M,C)$.

We have thus obtained a sequence

$$\cdots \Gamma^k(M,C) \overset{\rho^k_{k-1}}{\to} \Gamma^{k-1}(M,C) \to \cdots \to \Gamma^1(M,C) = M$$

of fibred manifolds and a differential system C^k on each $\Gamma^k(M,C)$, where ρ^k_{k-1} denotes the projection. The pair $(\Gamma^k(M,C), C^k)$ will be called the $(k-1)$-th prolongation of the contact manifold (M,C).

As above we denote by V^k the vertical tangent bundle of the fibred manifold $\Gamma^k(M,C)$ over $\Gamma^{k-1}(M,C)$, which is called the symbol system of $\Gamma^k(M,C)$.

Remark: Let us consider the prolongations $(J^k(\bar{M}, n-1), \bar{C}^k)$ of the pair $(\bar{M}, n-1)$, where $\bar{M}$ is an n-dimensional manifold. Then $(M,C) = (J^1(\bar{M}, n-1), \bar{C}^1)$ is a contact manifold of degree n, and for every $k \geqq 1$ $(J^k(\bar{M}, n-1), \bar{C}^k)$ may be naturally identified with an open submanifold of $(\Gamma^k(M,C), C^k)$ (as manifolds with differential system).

If $1 \leqq \ell \leqq k$, $\Gamma^k(M,C)$ becomes naturally a fibred manifold over $\Gamma^\ell(M,C)$, and we denote by ρ^k_ℓ the projection of $\Gamma^k(M,C)$ onto $\Gamma^\ell(M,C)$. As above let V^ℓ ($\ell \geqq 2$) denote the vertical tangent bundle of the fibred manifold $\Gamma^\ell(M,C)$ over $\Gamma^{\ell-1}(M,C)$. Then we define differential systems C^k_ℓ ($1 \leqq \ell \leqq k$) and V^k_ℓ ($2 \leqq \ell \leqq k$) on $\Gamma^k(M,C)$ as follows:

$$C^k_\ell = (\rho^k_\ell)^{-1}_*(C^\ell), \ V^k_\ell = (\rho^k_\ell)^{-1}_*(V^\ell).$$

Let z^1 be any point of M. By Darboux's theorem there is a coordinate system $(x^1, \dots, x^{n-1}, y, p_1, \dots, p_{n-1})$ of M at z^1 such that C is defined by the Pfaffian equation

$$dy - \sum_i p_i dx^i = 0.$$

Such a coordinate system will be called a canonical coordinate system of (M,C).

Now, let z^k_0 be a point of $\Gamma^k(M,C)$. Let $(x^1, \dots, x^{n-1}, y, p_1, \dots, p_{n-1})$ be a canonical coordinate system of (M,C) at $\rho^k_1(z^k_0)$ such that the 1-form $dx^1, \dots, dx^{n-1}$ are linearly independent, restricted to the $(n-1)$-dimensional contact element $\rho^k_2(z^k_0)$ to M. (Since $\rho^k_2(z^k_0) \in L(M,C)$, we easily see that such a canonical coordinate system exists for the given z^k_0.) We denote by U the domain of the coordinate system, and by $\tilde{U}$ the set of all $z^k \in \Gamma^k(M,C)$ such that $\rho^k_1(z^k) \in U$, and $dx^1, \dots, dx^{n-1}$ are linearly independent, restricted to $\rho^k_2(z^k)$. It is clear that $\tilde{U}$ is an open set of $\Gamma^k(M,C)$, $z^k_0 \in \tilde{U}$, and $\rho^k_1(\tilde{U}) = U$. In the following the indices $i, i_1, i_2, \dots$ will range over the integers $1, \dots, n-1$.

Lemma 1.8. *There are unique functions $p_{i_1 \dots i_\ell}$ $(2 \leqq \ell \leqq k)$ on $\tilde{U}$ satisfying the following conditions:*

1) $p_{i_1 \dots i_\ell}$ are symmetric with respect to the indices $i_1, \dots, i_\ell$.

2) The functions $x^i (= x^i \circ \rho^k_1)$, $y (= y \circ \rho^k_1)$, $p_i (= p_i \circ \rho^k_1)$ and $p_{i_1 \dots i_\ell}$ $(i_1 \leqq \dots \leqq i_\ell,\ 2 \leqq \ell \leqq k)$ together give a coordinate system of $\Gamma^k(M,C)$ on $\tilde{U}$.

3) V^k_ℓ is defined by the Pfaffian equations:

$$dx^i = dy = dp_{i_1 \dots i_m} = 0 \quad (1 \leqq m \leqq \ell - 1),$$

and C^k_ℓ by the Pfaffian equations

$$\begin{cases} dy - \sum_i p_i dx^i = 0, \\ dp_{i_1 \dots i_m} - \sum_i p_{i_1 \dots i_m i} dx^i = 0 \quad (1 \leqq m \leqq \ell - 1). \end{cases}$$

The coordinate system $(x^i, y, p_{i_1, \dots, i_\ell})$ $(1 \leqq \ell \leqq k)$ will be called a canonical coordinate system of $\Gamma^k(M,C)$ at z^k_0.

By a Lagrangean submanifold of the contact manifold (M,C) we mean an $(n-1)$-dimensional integral manifold S of C. Let S be a Lagrangean submanifold of (M,C). Then S is prolonged to the $(n-1)$-dimensional submanifold $p(S)$ of $J(M, n-1))$, which is clearly in $\Gamma^2(M,C)$. In this way S is prolonged to the $(n-1)$-dimensional submanifold $p^{k-1}(S)$ of $\Gamma^k(M,C)$ for every k. Note that $p^{k-1}(S)$ gives an integral manifold of the contact system C^k.

Lemma 1.9. *Let $\tilde{S}$ be an $(n-1)$-dimensional integral manifold of C^k. Assume that the projection ρ^k_1 maps $\tilde{S}$ diffeomorphically onto an $(n-1)$-dimensional submanifold S of M. Then S is a Lagrangean submanifold of (M,C), and $\tilde{S} = p^{k-1}(S)$.*

Let us now consider a canonical coordinate system $(x^i, y, p_{i_1, \dots, i_\ell})$ $(1 \leqq \ell \leqq k)$ of $\Gamma^k(M,C)$ at a point z^k_0. Let S be a Lagrangean submanifold of (M,C). Suppose that $p^{k-1}(S)$ is in the domain $\tilde{U}$ of the coordinate system, and S is defined by the equations:

$$\begin{cases} y = u(x^1, \dots, x^{n-1}), \\ p_i = \frac{\partial u}{\partial x^i}(x^1, \dots, x^{n-1}). \end{cases}$$

Then we remark that $p^{k-1}(S)$ is defined by the equations

$$\begin{cases} y = u(x^1, \ldots, x^{n-1}), \\ p_{i_1,\ldots,i_\ell} = \frac{\partial^\ell u}{\partial x^{i_1}\cdots\partial x^{i_\ell}}(x^1, \ldots, x^{n-1}) \quad (1 \leqq \ell \leqq k). \end{cases}$$

For simplicity we set as follows:

$$R = \Gamma^k(M, C), \quad D = C^k_k = C^k, \quad D_\ell = C^k_\ell \ (1 \leqq \ell \leqq k), \quad F_\ell = V^k_\ell \ (2 \leqq \ell \leqq k).$$

Then we notice that Lemma holds true in our present situation, where, of course, the assertions concerning the system F_1 should be neglected, and r should be replaced by $n-1$. Therefore we may speak of the symbol algebra $\mathcal{F} = (\mathfrak{t}, (\mathfrak{g}_p)_{p<0})$ of (R, D) at any point $z \in R$. We then notice that Lemma holds true in addition, where r should be replaced by $n-1$.

Now, let us consider another contact manifold (M', C') of degree n and their prolongations $\Gamma^k(M', C')$. Assuming that (M, C) is isomorphic with (M', C'), let φ be a (contact) isomorphism of (M, C) onto (M', C'). Then φ is prolonged to the diffeomorphism $p(\varphi)$ of $J(M, n-1)$ onto $J(M', n-1)$ which clearly sends $\Gamma^2(M, C)$ onto $\Gamma^2(M', C')$. In this way φ is prolonged to the diffeomorphism $p^{k-1}(\varphi)$ of $\Gamma^k(M, C)$ onto $\Gamma^k(M', C')$ for every k. Note that $p^{k-1}(\varphi)$ gives an isomorphism of $(\Gamma^k(M, C), C^k)$ onto $(\Gamma^k(M', C'), C'^k)$.

Lemma 1.10. *Let $\tilde{\varphi}$ be any local isomorphism of $(\Gamma^k(M, C), C^k)$ to $(\Gamma^k(M', C'), C'^k)$ at any $(z, z') \in \Gamma^k(M, C) \times \Gamma^k(M', C')$. Then there is a unique local (contact) isomorphism φ of (M, C) to (M', C') at $(\rho^k_1(z), \rho'^k_1(z'))$ such that $\varphi = p^{k-1}(\varphi)$.*

1.3. Contact manifolds of order k. Following Yamaguchi, we shall introduce the notion of a contact manifold of order k of bidegree (n, r), where $k \geqq 1$, and $1 \leqq r \leqq n-1$. Hereafter " of bidegree $(n, n-1)$" will be also called "of degree n".

(A) The case where $1 \leqq r \leqq n-2$. Let R be a manifold, and D a differential system on it. Then the pair (R, D) is called a contact manifold of order k of bidegree (n, r), if there are differential systems D_ℓ and F_ℓ $(1 \leqq \ell \leqq k)$ on R satisfying the following conditions:

1) $D_k = D$, and D_ℓ is the first derived system of $D_{\ell+1}$ for $1 \leqq \ell \leqq k-1$.
2) $\operatorname{rank}(T(R)/D_1) = n - r$.
3) $F_\ell = ch(D_{\ell-1}) \subset D_\ell$, and $\operatorname{rank}(D_\ell/F_\ell) = r$ for $2 \leqq \ell \leqq k$.
4) F_1 is completely integrable, $F_2 \subset F_1 \subset D_1$, $\operatorname{rank}(D_1/F_1) = r$, and $F_k = D_k \cap F_1$.
5) D_k is non-degenerate.
6) $\dim R = n + (n-r)\sum_{\ell=1}^k rH_\ell$.

(D_ℓ) family of contact systems
(F_ℓ) family of auxiliary systems

Now, let M be an n-dimensional manifold. Then we see from Lemma that the k-th prolongation $(J^k(M, r), C^k)$ of the pair (M, r) becomes a contact manifold of order k of bidegree (n, r). The next lemma implies that every contact manifold (R, D) of order k of bidegree (n, r) is locally isomorphic with $(J^k(M, r), C^k)$. Therefore we know that the system F_1 together with the systems D_ℓ $(1 \leqq \ell \leqq k)$ and F_ℓ $(2 \leqq \ell \leqq k)$ is naturally associated with D (Lemma). The differential systems D_ℓ and F_ℓ $(1 \leqq \ell \leqq k)$ will be called associated with D. (*)

Let (R, D) be a contact manifold of order k of bidegree (n, r). Assume that R is regular with respect to the completely integrable system F_1. Set $M = R/F_1$, and let ρ_0 be the projection of R onto M. Clearly we have $\dim M = n$.

Lemma 1.11. *There is a unique open immersion ι of R to $J^k(M, r)$ such that $\rho_0 = \rho^k_0 \circ \iota$, and $D = \iota_*^{-1}(C^k)$.*

Note that $D_\ell = \iota_*^{-1}(C^k_\ell)$, and $F_\ell = \iota_*^{-1}(V^k_\ell)$ for $1 \leqq \ell \leqq k$.

(*) In particular the system F_k will be called the symbol system of (R, D), which corresponds to the symbol system V^k of $J^k(M, r)$. We also remark that we may speak of a canonical coordinate system of (R, D), which corresponds to a canonical coordinate system of $J^k(M, r)$.

(B) The case where $r = n - 1$. Let R be a manifold, and D a differential system on it. Then the pair (R, D) is called a contact manifold of order k of degree n, if there are differential systems D_ℓ $(1 \leqq \ell \leqq k)$ and F_ℓ $(2 \leqq \ell \leqq k)$ on R satisfying the following conditions:

1) $D_k = D$, and D_ℓ is the first derived system of $D_{\ell+1}$ for $1 \leqq \ell \leqq k - 1$.
2) $\mathrm{rank}(T(R)/D_1) = 1$, and $\mathrm{rank}(D_1/D_2) = n - 1$.
3) $F_\ell = ch(D_{\ell-1}) \subset D_\ell$, and $\mathrm{rank}(D_\ell/F_\ell) = n - 1$ for $2 \leqq \ell \leqq k$.
4) $F_k = D_k \cap F_2$.
5) D_k is non-degenerate.
6) $\dim R = n + \sum_{\ell=1}^{k} {}_{n-1}H_\ell$.

Let (M, C) be a contact manifold of degree n. Then we remark that the $(k-1)$-th prolongation $(\Gamma^k(M, C), C^k)$ of (M, C) becomes a contact manifold of order k of degree n(cf. Lemma). The next lemma implies that every contact manifold (R, D) of order k of degree n is locally isomorphic with

(D_ℓ) family of contact systems
(F_ℓ) family of auxiliary systems

$(\Gamma^k(M, C), C^k)$. As before the differential systems D_ℓ $(1 \leqq \ell \leqq k)$ and F_ℓ $(2 \leqq \ell \leqq k)$ will be called associated with D. In particular the system F_k will be called the symbol system of (R, D). We further remark that we may speak of a canonical coordinate system of (R, D).

Let (R, D) be a contact manifold of order k of degree n. Assume that R is regular with respect to the completely integrable system $F_2 = ch(D_1)$. Set $M = R/F_2$, and let ρ_1 be the projection of R onto M. Then there is a unique differential system C on M such that $F_2 = (\rho)_*^{-1}(C)$. Clearly the pair (M, C) gives a contact manifold of degree n.

Lemma 1.12. *There is a unique open immersion ι of R to $\Gamma^k(M, C)$ such that $\rho_1 = \rho_1^k \circ \iota$, and $D = \iota_*^{-1}(C^k)$.*

Note that $D_\ell = \iota_*^{-1}(C_\ell^k)$ $(1 \leqq \ell \leqq k)$, and $F_\ell = \iota_*^{-1}(V_\ell^k)$ $(2 \leqq \ell \leqq k)$.

We have thus introduced the notion of a contact manifold of order k.

Incidentally we shall define the prolongation of a contact manifold (M, C) of order $k-1$ of bidegree (n, r), where $k \geqq 2$, and $1 \leqq r \leqq n - 1$. Let V be the symbol system of (M, C), which exists unless $k = 2$, $r = n - 1$. We denote by $L(M, C)$ the set of all r-dimensional integral elements z of C such that $z \cap V(x) = \{0\}$, x being the origin of z. Then $L(M, C)$ can be shown to become a fibred submanifold of the Grassmann bundle $J(M, r)$ over M. Let $\hat{C}$ denote the contact system on the fibred submanifold $L(M, C)$. It is easy to see that the pair $(L(M, C), \hat{C})$ becomes a contact manifold of order k of bidegree (n, r), which will be called the prolongation of (M, C).

Let S be an r-dimensional integral manifold of C which is transversal to V: $T_x(S) \cap V(x) = \{0\}$ at each $x \in M$. Then S is prolonged to the r-dimensional submanifold $p(S)$ of $J(M, C)$, which is clearly in $L(M, C)$. Note that $p(S)$ gives an integral manifold of $\hat{C}$.

Let us now consider another contact manifold (M', C') of order $k-1$ of bidegree (n, r) and its prolongation $(L(M', C'), C')$. Assuming that (M, C) is isomorphic with (M', C'), let φ be an isomorphism of (M, C) onto (M', C'). Then φ is prolonged to the diffeomorphism $p(\varphi)$ of $J(M, r)$ onto $J(M', r)$, which clearly sends $L(M, C)$ onto $L(M', C')$. Note that $p(\varphi)$ gives an isomorphism of $(L(M, C), \hat{C})$ onto $(L(M', C'), \hat{C}')$.

Now, let M be an n-dimensional manifold, and let us consider the prolongations $(J^k(M,r), C^k)$ of the pair (M,r). Then we remark that $J^k(M,r)$ are inductively defined as follows:

$$J^1(M,r) = J(M,r),\ J^k(M,r) = L(J^{k-1}(M,r), C^{k-1}),\ k \geqq 2.$$

Furthermore ler (M,C) be a contact manifold of degree n, and let us consider its prolongations $(\Gamma^k(M,C), C^k)$. Then we remark that $\Gamma^k(M,C)$ are inductively defined as follows

$$\Gamma^1(M,C) = M,\ \Gamma^k(M,C) = L(\Gamma^{k-1}(M,C), C^{k-1}),\ k \geqq 2.$$

§2. Pseudo-projective systems of order k

2.1. OD structures of order k. [1] In this paragraph we shall introduce the notion of an OD structure of order k of bidegree (n,r), where $k \geqq 2$, and $1 \leqq r \leqq n-1$. "OD", of course, means "ordinary differential".

(A) The case where $k=2$ or $k \geqq 3$, $r \leqq n-2$. Let M be an n-dimensional manifold, and let us consider the $(k-1)$-th prolongation $(J^{k-1}(M,r), C^{k-1})$ of the pair (M,r). Let R be an open fibred submanifold of the fibred manifold $J^{k-1}(M,r)$ over M. We denote by D and F the restrictions to R of the contact system C^{k-1} and the symbol system V^{k-1} respectively. Let E be a differential system on R. Then the pair $\mathcal{X} = (R,E)$ is called an OD structure of order k of bidegree (n,r) on M, if it satisfies the following conditions:

1) E is completely integrable,
2) $D = E + F$ (direct sum).

Let $\mathcal{X} = (R,E)$ (resp. $\mathcal{X}' = (R',E')$) be an OD structure of order k of bidegree (n,r) on a manifold M (resp. on M'). By an isomorphism of $\mathcal{X}$ to $\mathcal{X}'$ we mean a diffeomorphism φ of M onto M' such that the prolongation $p^{k-1}(\varphi)$ of φ gives an isomorphism of (R,E) onto (R',E') (as manifolds with differential system). Let (z,z') be a point of $R \times R'$. By a local isomorphism of $\mathcal{X}$ to $\mathcal{X}'$ at (z,z') we mean a local diffeomorphism φ of M to M' at $(\rho_0^{k-1}(z), {\rho'}_0^{k-1}(z'))$ such that $p^{k-1}(\varphi)$ gives a local isomorphism of (R,E) to (R',E') at (z,z').

Clearly an OD structure of order 2 of bidegree (n,r) means a pseudo-projective structure of bidegree (n,r) in the sense of Chapter I.

Let $\mathcal{X} = (R,E)$ be an OD structure of order k of bidegree (n,r) on a manifold M. Then we say that an r-dimensional submanifold S of M is a solution of $\mathcal{X}$, if the prolongation $p^{k-1}(S)$ of S is in R, and is an integral manifold of E. (Note that rank$(E) = r$.) In this way the structure $\mathcal{X}$ may be regarded as a differential equation.

Let z_0 be any point of R, and let $(x^i, y^\alpha, p^\alpha_{i_1\cdots i_l})$ $(1 \leqq l \leqq k-1)$ be any canonical coordinate system of $J^{k-1}(M,r)$ at z_0. Let $\widetilde{U}$ denote the domain of the coordinate system. By Lemma $*$ we easily see that there are unique functions $f^\alpha_{i_1\cdots i_k}$ on the intersection $R \cap \widetilde{U}$ such that E is defined by the Pfaffian equations:

[1] OD $\to$ FD (differential equation of finite type)

$$
(*) \quad \begin{cases} dy^\alpha - \sum_i p_i^\alpha dx^i = 0, \\ dp^\alpha_{i_1\cdots i_l} - \sum_i p^\alpha_{i_1\cdots i_l i}\, dx^i = 0 \quad (1 \leqq l \leqq k-2), \\ dp^\alpha_{i_1\cdots i_{k-1}} - \sum_i f^\alpha_{i_1\cdots i_{k-1} i}\, dx^i = 0. \end{cases}
$$

Now, let S be an r-dimensional submanifold of M. Suppose that $p^{k-1}(S) \subset \widetilde{U}$, and S is defined by the equations $y^\alpha = u^\alpha(x^1, \cdots, x^r)$. Then it is easy to see that S is a solution of $\mathcal{X}$, if and only if (u^α) gives a solution of the system of partial differential equations:

$$
(*') \quad \frac{\partial^k y^\alpha}{\partial x^{i_1} \cdots \partial x^{i_k}} = f^\alpha_{i_1\cdots i_k}\left((x^i), (y^\alpha), \left(\frac{\partial^l y^\alpha}{\partial x^{i_1} \cdots \partial x^{i_l}}\right)\right) \quad (1 \leqq l \leqq k-1).
$$

We note that this system is involutive or satisfies the integrability conditions. We also note that if $r = 1$, $(*')$ simply means the system of ordinary differential equations:

$$
\frac{d^k y^\alpha}{dx^k} = f^\alpha\left(x, (y^\alpha), \left(\frac{dy^\alpha}{dx}\right), \cdots, \left(\frac{d^{k-1}y^\alpha}{dx^{k-1}}\right)\right).
$$

The system $(*')$ will be called a local expression of $\mathcal{X}$ at z_0. Conversely it is clear that any involutive system of the form $(*')$ can be obtained as a local expression of an OD structure $\mathcal{X}$ of order k of bidegree (n, r).

(B) The case where $k \geqq 3$ and $r = n-1$. Let (M, C) be a contact manifold of degree n, and let us consider the $(k-2)$-th prolongation $(\Gamma^{k-1}(M, C), C^{k-1})$ of (M, C). Let R be an open fibred submanifold of the fibred manifold $\Gamma^{k-1}(M, C)$ over M. We denote by D and F the restrictions to R of the contact system C^{k-1} and the symbol system V^{k-1} respectively. Let E be a differential system on R. Then the pair $\mathcal{X} = (R, E)$ is called an OD structure of order k of degree n on (M, C), if it satisfies the conditions 1) and 2) in (A).

Let $\mathcal{X} = (R, E)$ (resp. $\mathcal{X}' = (R', E')$) be an OD structure of order k of degree n on a contact manifold (M, C) (resp. (M', C')). By an isomorphism of $\mathcal{X}$ to $\mathcal{X}'$ we mean a (contact) isomorphism φ of (M, C) onto (M', C') such that the prolongation $p^{k-2}(\varphi)$ of φ gives an isomorphism of (R, E) onto (R', E') (as manifolds with differential system). Let (z, z') be a point of $R \times R'$. By a local isomorphism of $\mathcal{X}$ to $\mathcal{X}'$ at (z, z') we mean a (contact) isomorphism φ of (M, C) to (M', C') at $(\rho_1^{k-1}(z), {\rho'}_1^{k-1}(z))$ such that $p^{k-2}(\varphi)$ gives a local isomorphism of (R, E) to (R', E') at (z, z').

Let $\mathcal{X} = (R, E)$ be an OD structure of order k of degree n on a contact manifold (M, C). Then we say that a Lagrangean submanifold S of (M, C) is a solution of $\mathcal{X}$, if the prolongation $p^{k-2}(S)$ of S is in R, and is an integral manifold of E. In this way the structure $\mathcal{X}$ may be regarded as a differential equation.

Let z_0 be any point of R, and let $(x^i, y, p_{i_1\cdots i_l})$ $(1 \leqq l \leqq k-1)$ be any canonical coordinate system of $\Gamma^{k-1}(M, C)$ at z_0. Let $\widetilde{U}$ denote the domain of the coordinate system. By Lemma $*$ we see that there are unique functions $f_{i_1\cdots i_k}$ on the intersection $R \cap \widetilde{U}$ such that E is defined by the Pfaffian equations of the form $(*)$, where the index α should be taken off everywhere. Now, let S be a Lagrangean submanifold of (M, C). Suppose that $p^{k-2}(S) \subset \widetilde{U}$, and S is defined by the equations

$$\begin{cases} y = u(x^1, \cdots, x^{n-1}), \\ p_i = \dfrac{\partial u}{\partial x^i}(x^1, \cdots, x^{n-1}). \end{cases}$$

Then we see that S is a solution of the equation $\mathcal{X}$, if and only if u gives a solution of the system of partial differential equations:

$$(*'') \qquad \frac{\partial^k y}{\partial x^{i_1} \cdots \partial x^{i_k}} = f_{i_1 \cdots i_k}\left((x^i), y, \left(\frac{\partial^l y}{\partial x^{i_1} \cdots \partial x^{i_l}}\right)\right) \quad (1 \leqq l \leqq k-1).$$

We note that this system is involutive. We also note that if $r = 1$, this system simply means the single ordinary differential equation:

$$\frac{d^k y}{dx^k} = f\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^{k-1}y}{dx^{k-1}}\right).$$

The system $(*'')$ will be called a local expression of $\mathcal{X}$ at z_0. Conversely any involutive system of the form $(*'')$ can be obtained as a local expression of an OD structure $\mathcal{X}$ of order k of degree n.

2.2. Pseudo-projective systems of order *k*. Let (k, n, r) be a triplet of integers with $k \geqq 2$ and $1 \leqq r \leqq n-1$. Let $\mathcal{R} = (R; E, F)$ be a pseudo-product manifold, and let us consider the differential system $D = E + F$ on R. Then $\mathcal{R}$ is called a pseudo-projective system of order k of bidegree (n, r), if it satisfies the following conditions:

1) (R, D) is a contact manifold of order $k-1$ of bidegree (n, r),
2) F coincides with the symbol system F_{k-1} of (R, D), except that $k = 2$ and $r = n-1$.

Clearly a pseudo-projective system of order 2 of bidegree (n, r) means a pseudo-projective system of bidegree (n, r) in the sense of Chapter I.

Let $\mathcal{X} = (R, E)$ be an OD structure of order k of bidegree (n, r) on a manifold M or a contact manifold (M, C). Then it is clear that the triplet $\mathcal{R} = (R; E, F)$ gives a pseudo-projective system of order k of bidegree (n, r), which is called associated with $\mathcal{X}$. Here F is the restriction of the symbol system of $J^{k-1}(M, r)$ or $\Gamma^{k-1}(M, C)$ to R. By Lemmas[2] we have the following

Lemma *Let $\mathcal{X}$ (resp. $\mathcal{X}'$) be an OD structure of order k of bidegree (n, r) on M or (M, C) (resp. on M' or (M', C')), and let $\mathcal{R}$ (resp. $\mathcal{R}'$) be the associated pseudo-projective system of order k of bidegree (n, r). If φ is a local isomorphism of $\mathcal{X}$ to $\mathcal{X}'$ at (z, z'), then the prolongation $p^{k-1}(\varphi)$ or $p^{k-2}(\varphi)$ of φ, gives a local isomorphism of $\mathcal{R}$ to $\mathcal{R}'$ at (z, z'). Conversely if $\tilde{\varphi}$ is a local isomorphism of $\mathcal{R}$ to $\mathcal{R}'$ at (z, z'), then there is a unique local isomorphism φ of $\mathcal{X}$ to $\mathcal{X}'$ at (z, z') such that $\tilde{\varphi} = p^{k-1}(\varphi)$ or $\tilde{\varphi} = p^{k-2}(\varphi)$.*

Furthermore we know from Lemmas that any pseudo-projective system $\mathcal{R}$ of order k of bidegree (n, r) is locally associated with an OD structure $\mathcal{X}$ of order k of bidegree (n, r). We have thereby seen that in the local point of view there is a natural one-to-one correspondence between the OD structures of order k of bidegree (n, r) and the pseudo-projective systems of order k of bidegree (n, r).

[2] By Lemmas we see that the assignment $\mathcal{X} \mapsto \mathcal{R}$ is compatible with the respective isomorphisms.

In connection with OD structures and pseudo-projective systems of order k we shall now introduce the notion of a pseudo-projective structure of order k of bidegree (n, r). In the case where $k = 2$, this is defined to be an OD structure of order 2 of bidegree (n, r) or a pseudo-projective structure of bidegree (n, r) in the sense of Chapter I. Let us consider the case where $k \geqq 3$. Let (M, C) be a contact manifold of order $k-2$ of bidegree (n, r), and $(L(M, C), \hat{C})$ its prolongation. Let R be an open fibred submanifold of $L(M, C)$ over M. We denote by D the restriction of the contact system $\hat{C}$ to R, and by F the vertical tangent bundle of the fibred manifold R over M. Let E be a differential system on R. Then the pair $\mathcal{X} = (R, E)$ is called a pseudo-projective structure of order k of bidegree (n, r) on (M, C), if it satisfies the conditions 1) and 2) in (A), the preceding paragraph.

Concerning pseudo-product structures of order $k \geqq 3$, we remark the following.

(1) A pseudo-projective structure $\mathcal{X} = (R, E)$ of order k of bidegree (n, r) on (M, C) is a generalized pseudo-projective structure of bidegree (n, r) on M in the sense of Chapter I. Therefore, $\mathcal{X}$ may be regarded as a differential equation, which is locally expressed by an involutive system of partial differential equations of the second order. Moreover the dual $\mathcal{X}^*$ of $\mathcal{X}$ can be considered under suitable regularity conditions on $\mathcal{X}$.

(2) A pseudo-projective structure of order 3 of degree n means an OD structure of order 3 of degree n.

(3) Let $\mathcal{X} = (R, E)$ be a pseudo-projective structure of order k of bidegree (n, r) on (M, C). Then the triplet $\mathcal{R} = (R; E, F)$ gives a pseudo-projective system of order k of bidegree (n, r).

(4) Let $\mathcal{X}_0 = (R, E)$ be an OD structure of order k of bidegree (n, r) on a manifold $\overline{M}$ or a contact manifold $(\overline{M}, \overline{C})$. We denote by M the image of R by the projection of R to $J^{k-2}(\overline{M}, r)$ or $\Gamma^{k-2}(\overline{M}, \overline{C})$, and by C the restriction of the contact system $\overline{C}^{k-2}$ on $J^{k-2}(\overline{M}, r)$ or $\Gamma^{k-2}(\overline{M}, \overline{C})$ to R. Then the pair (M, C) gives a contact manifold of order $k-2$ of bidegree (n, r), and the pair (R, E) gives a pseudo-projective structure of order k of bidegree (n, r) on (M, C), which we denote by $\mathcal{X}$. The structure $\mathcal{X}$ will be called associated with $\mathcal{X}_0$.

(5) In the local point of view, the equations $\mathcal{X}_0$ and $\mathcal{X}$ describe essentially the same differential equation.

(6) In the local point of view, there is a natural one-to-one correspondence between the OD structures of order k of bidegree (n, r) and the pseudo-projective structures of order k of bidegree (n, r).

2.3. Pseudo-projective FGLA's of order k. Let $\mathfrak{P} = (R; E, F)$ be a pseudo-projective system of order k of bidegree (n, r). Then D is regular, and therefore we may speak of the symbol algebra of $\mathfrak{P}$ at any point $z \in R$. Let $\mathfrak{F} = (\mathfrak{f}, (\mathfrak{g}_p)_{p<0})$ be the symbol algebra of the contact manifold (R, D) of order $k-1$ at z:

$$\begin{aligned} \mathfrak{g}_{-1} &= D(z), \\ \mathfrak{g}_p &= D_{k+p}(z)/D_{k+p+1}(z), \quad p \leq -2. \end{aligned}$$

Here, $D_l\,(1 \leq l \leq k-1)$ denote the differential systems associated with D, and we set $D_l = T(R)$ for $l \leq 0$. Let e and f denote the subspaces $E(z)$ and $F(z)$ of $\mathfrak{g}_{-1}$ respectively. Then the triplet $\mathcal{L} = (\mathfrak{F}; e, f)$ gives the symbol algebra of $\mathfrak{P}$ at z, which is a pseudo-product FGLA.

By Lemma we know that $\mathcal{L}$ satisfies the following conditions:

1): $\mathcal{L}$ or $\mathfrak{F}$ is of the k-th kind and non-degenerate.

2): $[\mathfrak{g}_p, \mathfrak{g}_q] = \{0\}$ for $p, q \leq -2$, and $[\mathfrak{g}_p, f] = \{0\}$ for $p \leq -2$.

3): If $p \leq -2$, $X \in \mathfrak{g}_p$, and $[X, Y] = 0$ for all $Y \in e$, then $X = 0$.

4): $\dim \mathfrak{g}_{-k} = n - r$, $\dim \mathfrak{g}_p = (n-r)\,{}_rH_{k+p}$ for $-k+1 \leq p \leq -2$, $\dim e = r$, and $\dim f = (n-r)\,{}_rH_{k-1}$.

In general, consider a pseudo-product FGLA, $\mathcal{L}$. Then $\mathcal{L}$ is called a pseudo-projective FGLA of order k of bidegree (n,r), if it satisfies the four conditions above.

Now, let $\mathcal{L}$ be a pseudo-projective FGLA of order k of bidegree (n,r). Let us consider the tensor products $\mathfrak{g}_{-k} \otimes S^l(e^*)$, where $S^l(e^*)$ denotes the l-th symmetric product of the dual space e^* of e. Note that $\mathfrak{g}_{-k} \otimes S^l(e^*)$ may be naturally regarded as the space of all $\mathfrak{g}_{-k}$-valued symmetric l-linear forms on e. For any $X \in \mathfrak{g}_{-k} \otimes S^l(e^*)$ and $Y \in e$, we define an element $Y \lrcorner X \in \mathfrak{g}_{-k} \otimes S^{l-1}(e^*)$ by

$$(Y \lrcorner X)(Y_1, \cdots, Y_{l-1}) = X(Y, Y_1, \cdots, Y_{l-1}).$$

Here and in the following $Y_1, Y_2, \cdots$ denote any elements of e. Now, let p be any integer with $-k+1 \leq p \leq -2$. For any $X \in \mathfrak{g}_p$ we define a $\mathfrak{g}_{-k}$-valued $(k+p)$-linear form $\widetilde{X}$ on e by

$$\widetilde{X}(Y_1, \cdots, Y_{k+p}) = [[\cdots[[X, Y_1], Y_2], \cdots], Y_{k+p}].$$

Since e is abelian, we see that $\widetilde{X}$ is symmetric, and hence $\widetilde{X} \in \mathfrak{g}_{-k} \otimes S^{k+p}(e^*)$. From the conditions 3) and 4) we also see that the assignment $X \to \widetilde{X}$ gives an isomorphism of $\mathfrak{g}_p$ onto $\mathfrak{g}_{-k} \otimes S^{k+p}(e^*)$. For any $X \in f$ we now define an element $\widetilde{X} \in \mathfrak{g}_{-k} \otimes S^{k-1}(e^*)$ in the same manner as above. Then, we see from the conditions 1), 3) and 4) that the assignment $X \to \widetilde{X}$ gives an isomorphism of f onto $\mathfrak{g}_{-k} \otimes S^{k-1}(e^*)$.

Hereafter we shall identify $\mathfrak{g}_p\,(-k+1 \leq p \leq -2)$ with $\mathfrak{g}_{-k} \otimes S^{k+p}(e^*)$, and f with $\mathfrak{g}_{-k} \otimes S^{k-1}(e^*)$, by the isomorphism $X \to \widetilde{X}$. Then, we clearly have

$$[X, Y] = Y \lrcorner X,$$

where $X \in \mathfrak{g}_p\,(-k+1 \leq p \leq -2)$ or $X \in f$, and $Y \in e$.

Form the discussion above we know that there is a unique pseudo-projective FGLA of order k of bidegree (n,r) up to isomorphism.

Accordingly let $\mathcal{L}$ be a pseudo-projective FGLA of order k of bidegree (n,r). Then it follows that every pseudo-projective system $\mathfrak{P}$ of order k of bidegree (n,r) is of type $\mathcal{L}$.

The prolongation of $\mathcal{L}$ will be called a pseudo-projective GLA of order k of bidegree (n,r).

CHAPTER VII

Geometry of systems of ordinary differential equations of higher order, (1)

§1. Pseudo-projective GLA's of order 3 of degree n

1.1. Contact projective geometry. n を integer ≥ 2 とし, $m = 2n-1$ とおく. P^m を m-dim. projective space over $\mathbb{R}$ とする. $\mathbb{R}^{2n}_* = \mathbb{R}^{2n} - \{0\}$ とおき, $p\colon \mathbb{R}^{2n}_* \longrightarrow P^m$ を projection とする. $(z_0, \cdots, z_m)$ を P^m の canonical homogeneous coordinate system or canonical coordinate system of $\mathbb{R}^{2n}_*$ とする. $\mathbb{R}^{2n}_*$ 上の 1-form ω を

$$\omega = z_0\, dz_m - z_m\, dz_0 + \sum_{i=1}^{n-1} (z_i\, dz_{n-1+i} - z_{n-1+i}\, dz_i)$$

によって定義する. C' を Pfaffian equation $\omega = 0$ によって定義される $\mathbb{R}^{2n}_*$ 上の differential system とする. 容易に C' の Cauchy characteristic system $ch(C') =$ the vertical tangent bundle of the fibred manifold $\mathbb{R}^{2n}_*$ over P^m が示される. [1] 故に C' は自然に P^m 上の differential system C を導き, それは contact structure になる.

α を $(n-1)$-dim. projective subspace of P^m とする. α が contact manifold (P^m, C) の Lagrangean submanifold であるとき, α を Lagrangean projective subspace とよぶ. $\widetilde{\alpha} = p^{-1}(\alpha) \cup \{0\}$ とおけば, $\widetilde{\alpha}$ は $\mathbb{R}^{2n}$ の n-dim. linear subspace である. 容易に

$$\begin{aligned} &\alpha : \text{Lagrangean projective subspace} \\ \Longleftrightarrow\quad &\widetilde{\alpha}\text{ が symplectic form} \\ &\qquad d\omega = 2\left(d\, z_0 \wedge d\, z_m + \sum_{i=1}^{n-1} d\, z_i \wedge d\, z_{n-1+i}\right) \\ &\quad\text{に関する Lagrangean subspace} \end{aligned}$$

が示される.

$\Lambda(P^m, C)$ を the set of Lagrangean projective subspaces of P^m とする. $\Lambda(P^m, C)$ は compact connected manifold であり, $\dim \Lambda(P^m, C) = \frac{1}{2} n(n+1)$ が示される.

product manifold $P^m \times \Lambda(P^m, C)$ における relation $R = R(n)$ を

$$R = \{(p, \alpha) \in P^m \times \Lambda(P^m, C) \mid p \subset \alpha\}$$

によって定義する. R は $P^m \times \Lambda(P^m, C)$ の compact connected submanifold であり, $\dim R = 2n - 1 + \frac{n}{2}(n-1)$ であることが示される.

$\pi\colon R \longrightarrow P^m$, $\rho\colon R \longrightarrow \Lambda(P^m, C)$ を projection とする. R は fibred manifold over P^m with projection π であり, それは同時に fibred manifold over $\Lambda(P^m, C)$ with projection ρ である.

さて contact manifold (P^m, C) に付随する Lagrange-Grassmann bundle $L(P^m, C)$ を考えよう. $(p, \alpha) \in R$ ならば, $T_p(\alpha) \in L(P^m, C)$ である. そして対応 $(p, \alpha) \longrightarrow T_p(\alpha)$ は R から $L(P^m, C)$ の上への diffeomorphism ι を与える. $\varpi\colon L(P^m, C) \longrightarrow P^m$ を projection とすれば, $\varpi \circ \iota = \pi$ である.

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[1] $\lambda \in \mathbb{R}_*$ $\quad R_\lambda(x) = \lambda\, x, x \in \mathbb{R}^{2n}_*$, $\underline{R^*_\lambda \omega = \lambda^2\, \omega}$ も必要.

F (resp. E) を fibred manifold R over P^m (resp. over $\Lambda(P^m, C)$) の vertical tangent bundle とする. $D = E + F$ とおく. ι による D の image $\iota_*(D)$ は $\Lambda(P^m, C)$ 上の contact system である. よって (ι によって R と $L(P^m, C)$ を同一視すれば), $\mathfrak{X}_0 = (R, E)$ は P^m 上の pseudo-projective structure of order 3 of degree n を与える. そして triplet $\mathfrak{P} = (R; E, F)$ は $\mathfrak{X}_0$ に付随する pseudo-projective system of order 3 of degree n を与える.

S を P^m の $(n-1)$-dim. submanifold とする. このとき

$$\begin{aligned} & S \text{ is a solution of } \mathfrak{X}_0 \\ \Longleftrightarrow \quad & \alpha \text{ is a part of a Lagrangean projective subspace } \alpha \end{aligned}$$

が成り立つ. よって $\mathfrak{X}_0$ は Lagrangean projective subspaces を定義する differential equation であることがわかった.

U を $z_0 \neq 0$ で定義される P^m の open set とする. U は $\mathbb{R}^m$ に自然に同一視され, $x_\lambda = z_\lambda / z_0\,(1 \leq \lambda \leq m)$ とおけば, (x_λ) は $U = \mathbb{R}^m$ の canonical coordinate system を与える. 以下, i, j は $1, \cdots, n-1$ を動くものとする.

$$\omega' = dx_m + \sum_i (x_i\, dx_{n-1+i} - x_{n-1+i}\, dx_i)$$

とおけば, $\omega = x_0^2\, \omega'$ をえる. さらに,

$$y = x_m + \sum_i x_i\, s_{n-1+i}, \quad p_i = 2\, x_{n-1+i}$$

とおけば, $\omega' = dy - \sum p_i\, dx_i$ をえる. 故に (x_i, y, p_j) は contact manifold (P^m, C) に関する canonical coordinate system を与える.

さて, α を equation

$$\begin{cases} z_{n-1+i} = a_i\, z_0 + \displaystyle\sum_j a_{ij}\, z_j, \\ z_m = a_0\, z_0 + \displaystyle\sum_i b_i\, z_i \end{cases}$$

で定義される P^m の $(n-1)$-dim. projective subspace とする. coordinate system (x_i, y, p_j) に関して $\alpha \cap U$ は

$$\begin{cases} y = a_0 + \displaystyle\sum_i (a_i + b_i)\, x_i + \sum a_{ij}\, x_i x_j, \\ p_i = 2\, a_i + 2 \displaystyle\sum_j a_{ij}\, x_j \end{cases}$$

で定義され, $\alpha \cap U$ 上で,

$$dy - \sum_i p_i\, dx_i = \sum_i (b_i - a_i)\, dx_i + \sum_j (a_{ji} - a_{ij}) x_i\, dx_j$$

が成り立つ. 故に,

$$\begin{aligned} & \alpha \text{ が Lagrangean projective subspace} \\ \Longleftrightarrow \quad & b_i = a_i,\ a_{ij} = a_{ji} \end{aligned}$$

がわかる. よって, α が Lagrangean projective subspace ならば, $\alpha \cap U$ は equation

$$\begin{cases} y = a_0 + 2 \displaystyle\sum_i a_i\, x_i + \sum_j a_{ij} x_i x_j, \\ p_i = 2\, a_i + 2 \displaystyle\sum_j a_{ij}\, x_j \qquad (a_{ij} = a_{ji}) \end{cases}$$

で定義される. 以上によって equation $\mathfrak{X}_0 = (R, E)$ が local に

$$\frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = 0$$

なる形の equation によって表現されることがわかった. (Note that in the discussion above $(z_0, \cdots, z_m)$ may be replaced by any homogeneous coordinate system $(z'_0, \cdots, z'_m)$ such that

$$z'_0\, dz'_m - z'_m\, dz'_0 + \sum_i (z'_i\, dz'_{n-1+i} - z'_{n-1+i}\, dz'_i) = c \cdot \omega,$$

where c is a constant $\neq 0$.)

1.2. Pseudo-projective graded Lie algebras of order 3 of degree *n*. まず pseudo-projective graded Lie algebra of order 3 of degree n の matricial representation を与えよう。$2n$ 次 skew-symmetric matrix I_n を

$$I_n = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1_{n-1} & 0 \\ 0 & -1_{n-1} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

によって定義する。ここで 1_{n-1} は $n-1$ 次 unit matrix である。$\mathfrak{gl}(2n, \mathbb{R})$ の subalgebra $\mathfrak{g}$ を

$$\mathfrak{g} = \{X \in \mathfrak{gl}(2n, \mathbb{R}) \,|\, {}^tXI_n + I_nX = 0\}$$

によって定義する。$\mathfrak{g}$ は simple graded Lie algebra $\mathfrak{sp}(n, \mathbb{R})$ に isomorphic である。

$n_1 = n_2 = 1, n_2 = n_3 = n-1$ とおく。このとき、matrix $X \in \mathfrak{gl}(2n, \mathbb{R})$ は $X = (X_{ij})_{1 \le i,j \le 4}$ と表わせられる。ここで、$X_{i,j}$ は $n_i \times n_j$-matrix である。$X \in \mathfrak{g}$ なるための条件は

$$X_{42} = {}^tX_{31}, \quad X_{43} = -{}^tX_{21}, \quad {}^tX_{32} = X_{32}, \quad X_{44} = -{}^tX_{11},$$
$$X_{33} = -{}^tX_{22}, \quad {}^tX_{23} = X_{23}, \quad X_{34} = -{}^tX_{12}, \quad X_{24} = {}^tX_{13}$$

である。$\mathfrak{g}$ の subspaces の family $(\mathfrak{g}_p)$ を

$$\mathfrak{g}_p = \mathfrak{g} \cap \mathfrak{g}_p(1, n-1, n-1, 1; \mathbb{R}) = \{X \in \mathfrak{g} \,|\, X_{ij} = 0 \text{ if } j - i \neq p\}$$

で定義すれば、$\mathfrak{G} = (\mathfrak{g}, (\mathfrak{g}_p))$ は $\mathfrak{G}(1, n-1, n-1, 1; \mathbb{R})$ の graded subalgebra になる。そして、$\mathfrak{t} = \sum_{p<0} \mathfrak{g}_p$ とおけば、$\mathfrak{T} = (\mathfrak{t}, (\mathfrak{g}_p)_{p<0})$ は FGLA of the third kind である。従って、$\mathfrak{G}$ は SGLA of the third kind になる。

$\mathfrak{g}_{-1}$ の subspaces $\mathfrak{e}, \mathfrak{f}$ をそれぞれ

$$\mathfrak{e} = \{X \in \mathfrak{g}_{-1} \,|\, X_{32} = 0\},$$
$$\mathfrak{f} = \{X \in \mathfrak{g}_{-1} \,|\, X_{21} = 0\}$$

によって定義すれば、$\mathfrak{L} = (\mathfrak{T}; \mathfrak{e}, \mathfrak{f})$ は pseudo-projective FGLA of order 3 of degree n になり、$\mathfrak{g}_0$ は $\mathfrak{L}$ の derivation algebra $\mathrm{Der}(\mathfrak{L})$ に同一視される。従って、Proposition により、$\mathfrak{G}$ は $\mathfrak{L}$ の prolongation になる。この $\mathfrak{G}$ を the standard pseudo-projective graded Lie algebra of order 3 of degree n とよぶ。

E を

$$X_{11} = \frac{3}{2}, \qquad X_{22} = \frac{1}{2} 1_{n-1}$$

で定義される $\mathfrak{g}_0$ の centre の元とすれば、E は $\mathfrak{G}$ の characteristic element である。J を

$$X_{11} = -\frac{1}{2}, \qquad X_{22} \frac{1}{2} 1_{n-1}$$

で定義される $\mathfrak{g}_0$ の centre の元とすれば、

$$[J, \mathfrak{g}_p] = \{0\} \quad \text{if } p \text{ is even}$$
$$[J, [J, X]] = X \quad \text{for all } X \in \mathfrak{g}_p \quad \text{if } p \text{ is odd}$$

が確かめられる。故に odd integer p に対して、$\mathfrak{g}_p$ の subspaces $\mathfrak{g}_p^+$, $\mathfrak{g}_p^-$ を

$$\mathfrak{g}_p^{\pm} = \{X \in \mathfrak{g}_p \,|\, [J, X] = \pm X\}$$

によって定義すれば、

$$\mathfrak{g}_p = \mathfrak{g}_p^+ + \mathfrak{g}_p^- \qquad \text{(direct sum)}$$

をえる。$\mathfrak{g}_{-1}^+ = \mathfrak{e}$, $\mathfrak{g}_{-1}^- = \mathfrak{f}$, $\mathfrak{g}_{-3}^+ = \mathfrak{g}_{-3}$, $\mathfrak{g}_3^- = \mathfrak{g}_3$ 及び

$$\mathfrak{g}_1^+ = \{X \in \mathfrak{g}_1 \,|\, X_{12} = 0\},$$
$$\mathfrak{g}_1^- = \{X \in \mathfrak{g}_1 \,|\, X_{23} = 0\}$$

が確かめられる。

$\mathfrak{g}_0$ の centre の元 $E_{\mathfrak{a}}$ を

$$E_{\mathfrak{a}} = \frac{1}{2}(E - J)$$

によって定義し、$\mathfrak{g}$ の subspaces の family $(\mathfrak{a}_p)$ を

$$\mathfrak{a}_p = \{X \in \mathfrak{g} \,|\, [E_{\mathfrak{a}}, X] = pX\}$$

によって定義する。このとき、

$$\mathfrak{a}_p = \mathfrak{g}_{2p-1}^- + \mathfrak{g}_{2p} + \mathfrak{g}_{2p+1}^+$$

をえる。従って、$\mathfrak{a}_p = \{0\}$ if $p < -2$ or $p > 2$ であり、

$$\mathfrak{a}_{-2} = \mathfrak{g}_{-3}, \quad \mathfrak{a}_{-1} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}^+, \quad \mathfrak{a}_0 = \mathfrak{g}_{-1}^- + \mathfrak{g}_0 + \mathfrak{g}_1^+,$$
$$\mathfrak{a}_1 = \mathfrak{g}_1^- + \mathfrak{g}_2, \quad \mathfrak{a}_2 = \mathfrak{g}_3$$

をえる。故に、$\mathfrak{A} = \{\mathfrak{g}, (\mathfrak{a}_p)\}$ は SGLA of the second kind であり、$\mathfrak{G}(1, 2(n-1), 1; \mathbb{R})$ の graded subalgebra である。$\mathfrak{A}$ は projective contact graded Lie algebra of degree n に isomorphic である。

$\mathfrak{g}_0$ の centre の元 $E_{\mathfrak{b}}$ を

$$E_{\mathfrak{b}} = \frac{1}{2}(E + J)$$

によって定義し、$\mathfrak{g}$ の subspaces の family $(\mathfrak{b}_p)$ を

$$\mathfrak{b}_p = \{X \in \mathfrak{g} \,|\, [E_{\mathfrak{a}}, X] = pX\}$$

によって定義する。このとき、

$$\mathfrak{b}_p = \mathfrak{g}_{2p-1}^+ + \mathfrak{g}_{2p} + \mathfrak{g}_{2p+1}^-$$

をえる。従って、$\mathfrak{b}_p = \{0\}$ if $p < -1$ or $p > 1$ であり、

$$\mathfrak{b}_{-1} = \mathfrak{g}_{-3} + \mathfrak{g}_{-2} + \mathfrak{g}_{-1}^-, \quad \mathfrak{b}_0 = \mathfrak{g}_{-1}^+ + \mathfrak{g}_0 + \mathfrak{g}_1^-,$$
$$\mathfrak{b}_1 = \mathfrak{g}_1^+ + \mathfrak{g}_2 + \mathfrak{g}_3$$

をえる。故に、$\mathfrak{B} = \{\mathfrak{g}, (\mathfrak{b}_p)\}$ は SGLA of the first kind であり、$\mathfrak{G}(n, n; \mathbb{R})$ の graded subalgebra である。[ ] の記号の下、$\mathfrak{B}$ は $\mathfrak{Sp}(n, \mathbb{R})$ に isomorphic である。特に、$n = 2$ のとき、$\mathfrak{B}$ は $\mathfrak{M}_1(3)$ に isomorphic である。

1.3. The homogeneous spaces $G/G^{(0)}$, $G/A^{(0)}$, $G/B^{(0)}$. G を the group of projective transformations of P^m leaving the contact structure C invariant とする。G は P^m 上に transitive に作用する。G は自然に $\wedge(P^m, C)$ 上に作用し、その作用は transitive である。さらに G は diagonal map $G \to G \times G$ を通じて product manifold $P^m \times \wedge(P^m, C)$ 上に作用し、その作用による orbits は $R = R(n)$ とその complement によって与えられる。

p_0 を $z_1 = \cdots = z_m = 0$ によって与えられる P^m の点、α_0 を $z_n = \cdots = z_m = 0$ によって与えられる Lagrangean projective subspace とする。明らかに $(p_0, \alpha_0) \in R$ である。$A^{(0)}$ を p_0 における G の isotropy group、$B^{(0)}$ を α_0 における G の isotropy group、$G^{(0)}$ を (p_0, α_0) における G の isotropy group とする。このとき、P^m, $\wedge(P^m, C)$, R はそれぞれ homogeneous space $G/A^{(0)}$, $G/B^{(0)}$,$G/G^{(0)}$ によって表現される。

$GL(2n, \mathbb{R})$ の subgroup G' を

$$G' = \{A \in GL(2n, \mathbb{R}) \,|\, {}^t A I_n A = \lambda I_n \quad \text{with some } \lambda \in \mathbb{R}_*\}$$

によって定義する。明らかに、

$$G' = \{A \in GL(2n-1, \mathbb{R}) \,|\, A^* d\omega = \lambda d\omega \quad \text{with some } \lambda \in \mathbb{R}_*\}$$

である。($A^* d\omega = \lambda d\omega \Leftrightarrow A^*\omega = \lambda\omega$.) 故に group G は factor group $G'/C(G')$ によって表わされる。ここで、$C(G')$ は G' の centre であり、$\lambda \cdot 1_{2n-1}$ ($\lambda \in \mathbb{R}_*$) なる形の diagonal matrices 全体からなる。

G の Lie algebra は $\mathfrak{g}$ に同一視され、G は $\mathrm{Aut}(\mathfrak{g})$ の open subgroup に同一視される。G' の subgroups G'_0, $G^{(1)}$ を

$$G'_0 = G' \cap G_0(1, n-1, n-1, 1; \mathbb{R}),$$
$$G^{(1)} = G' \cap G^{(1)}(1, n-1, n-1, 1; \mathbb{R})$$

によって定義し、$G_0 = G'_0/C(G')$ とおく。明らかに $G^{(0)} = G_0 \cdot G^{(1)}$ である。ここで $G^{(1)}$ は自然に G の subgroup とみなすべきである。さらに $G_0 = \mathrm{Aut}(\mathfrak{L})$ であり、$G = \mathrm{Aut}(\mathfrak{g})^0 \cdot G_0$ であり、$G^{(1)}$ は $\mathfrak{g}$no nilpotent subalgebra $\mathfrak{g}^{(1)} = \sum_{p \geq 1} \mathfrak{g}_p$ によって生成される G の Lie subgroup であることが示される。以上から groups G_0, $G^{(0)}$, G が $\mathfrak{L}$ に付随する groups であることがわかった。(実は、$\mathrm{Aut}(\mathfrak{L}) = \mathrm{Aut}(\mathfrak{G})$ が示される。よって G_0, $G^{(0)}$, G は $\mathfrak{G}$ に付随する groups にもなっている。)

P^m 上の pseudo-product structure (E, F) は $\mathfrak{g}$ の subalgebras $\mathfrak{a}^{(0)} = \mathfrak{g}^+_{-1} + \mathfrak{g}^{(0)}$ と $\mathfrak{b}^{(0)} = \mathfrak{g}^-_{-1} + \mathfrak{g}^{(0)}$ から導かれる $G/G^{(0)}$ 上の invariant pseudo-product structure と一致する。

$(*)$ $\forall a \in \mathrm{Aut}(\mathfrak{G})$ に対して $\mathrm{Ad}(a)\mathfrak{g}^+_{-1} \neq \mathfrak{g}^-_{-1}$ である。実際、$\mathrm{Ad}(a)\mathfrak{g}^+_{-1} = \mathfrak{g}^-_{-1}$ とすれば、$\mathfrak{g}_{-3} = [\mathfrak{g}_{-2}, \mathfrak{g}^+_{-1}] = [\mathfrak{g}_{-2}, \mathfrak{g}^-_{-1}] = 0$ となり、矛盾が生ずる。$\mathfrak{g}_0$ の $\mathfrak{g}^+_{-1}$, $\mathfrak{g}^-_{-1}$ 上への表現は irreducible だから、$a \in \mathrm{Aut}(\mathfrak{G})$ ならば $\mathrm{Ad}(a)\mathfrak{g}^+_{-1} = \mathfrak{g}^+_{-1}$, $\mathrm{Ad}(a)\mathfrak{g}^-_{-1} = \mathfrak{g}^-_{-1}$, i.e., $a \in \mathrm{Aut}(\mathfrak{L})$ が従う。

G' の subgroups A'_0, $A^{(1)}$ をそれぞれ

$$A'_0 = G' \cap G_0(1, 2(n-1), 1; \mathbb{R}),$$
$$A^{(1)} = G' \cap G^{(1)}(1, 2(n-1), 1; \mathbb{R})$$

によって定義し、$A_0 = A'_0/C(G')$ とおく。明らかに $A^{(0)} = A_0 \cdot A^{(1)}$ である。さらに $A_0 = \mathrm{Aut}(\mathfrak{A})$ であり、$G = \mathrm{Aut}(\mathfrak{g})^0 \cdot A_0$ であり、$A^{(1)}$ は $\mathfrak{g}$ の nilpotent subalgebra $\mathfrak{a}^{(1)} = \sum_{p \geq 1} \mathfrak{a}_p$ によって生成される G の Lie subgroup であることが示される。故に groups A_0, $A^{(0)}$, G は $\mathfrak{A}$ に付随する groups であることがわかる。

P^m 上の contact structure C は $\mathfrak{g}$ の subspace $\mathfrak{a}^{(-1)} = \sum_{p \geq -1} \mathfrak{a}_p$ $(= \mathfrak{g}^{(-2)} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \ldots)$ から導かれる $G/A^{(0)}$ 上の invariant differential system と一致する。

(編注：ここでスキップあり)

$\forall -k \leq p \leq -1$ に対して、S_p を次の条件を満足する $X \in S$ 全体の作る S の subspace とする：

$$X(\overbrace{e_0, \ldots, e_0}^{-q-1}, Y_1, \ldots, Y_{k+q}) = 0.$$

ここで、$q \neq p$, $Y_i \in \mathfrak{g}_{-1}^{+}$. 明らかに $S = \sum_p S_p$ (direct sum) である。

$\forall X \in S$ に対して、$X^{(p)} \in \mathfrak{g}_{-k} \otimes S^{k+p}((\mathfrak{g}_{-1}^{+})^*)$ を

$$X^{(p)}(Y_1, \ldots, Y_{k+p}) = \frac{(k-1)!}{(-p-1)!} X(\overbrace{e_0, \ldots, e_0}^{-p-1}, Y_1, \ldots, Y_{k+p})$$

によって定義する。ここで $Y_i \in \mathfrak{g}_{-1}^{+}$. そして、linear map $\Phi : S \to \mathfrak{b}_{-1}$ を

$$\Phi(X) = \sum_{p=-k}^{-1} X^{(p)}$$

によって定義する。$X \in S_p$ ならば $\Phi(X) = X^{(p)}$ であり、Φ は S_p から $\mathfrak{g}_{-k} \otimes S^{k+p}((\mathfrak{g}_{-1}^{+})^*)$ の上への isomorphism を与える。故に Φ は S から $\mathfrak{b}_{-1}$ の上への isomorphism を与える。

Lemma 1.1. $Z \in l$, $X \in S$

$$\Phi(Z \cdot X) = [\Phi(Z), \Phi(X)]$$

$D \in GL(\mathfrak{g}_{-1}^{+})$ に対して、$\tilde{D} \in GL(V)$ を

$$\tilde{D} = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$$

によって定義する。そして、$\tilde{L}$ の subgroup L_0 を

$$L_0 = \{(\tilde{D}, B) \mid D \in GL(\mathfrak{g}_{-1}^{+}), B \in GL(\mathfrak{g}_{-k})\}$$

によって定義する。L_0 は自然に $L = \tilde{L}/U$ の subgroup と見做される。

group $G_0 = GL(\mathfrak{g}_{-1}^{+}) \times GL(\mathfrak{g}_{-k})$ を考える。group isomorphism $\Phi_0 : L_0 \to G_0$ を

$$\Phi_0((\tilde{D}, B)) = (D, B)$$

によって定義する。

Lemma 1.2. $\forall a \in L_0$, $X \in S$ に対して、

$$\Phi(a \cdot X) = \mathrm{Ad}(\Phi_0(a))\Phi(X).$$

Lemma 1.3. *There is a unique Lie group isomorphism* $\Phi : L \to B_0$ *such that* $\Phi(a) = \Phi_0(a), a \in L_0$, *and such that*

$$\Phi(a \cdot X) = \mathrm{Ad}(\Phi(a))\Phi(X),$$

where $a \in L$, *and* $X \in S$.

Proof. $L = L^0 \cdot L_0$, $\mathrm{Aut}(\mathfrak{B}) = \mathrm{Aut}(\mathfrak{B})^0 \cdot G_0$ である。ここで L^0 (resp. $\mathrm{Aut}(\mathfrak{B})^0$) は L (resp. $\mathrm{Aut}(\mathfrak{B})$) の単位元の connected component である。故に、この Lemma は Lemmas から直ちに従う。 □

1.4.

1.5. A lemma on the graded Lie algebra $\mathfrak{G}$. $\mathfrak{g}_{-1}^{+}$ と $\mathfrak{g}_{-k}$ に inner product $(\cdot,\cdot)$ を入れる. V_0 の inner product $(\cdot,\cdot)$ を $(e_0,e_0)=1$ となるように定義する. $\mathfrak{g}_{-1}^{+}$ と V_0 の inner products $(\cdot,\cdot)$ は自然に V の inner product $(\cdot,\cdot)$ を導く.

S の inner product $(\cdot,\cdot)$ を次のように定義する. $(\bar{e}_\lambda)_{0\leqq\lambda\leqq r}$ を V の orthonormal basis とする. このとき, $X,X'\in V$ に対して,

$$(X,X')=\sum_{\lambda_1,\ldots,\lambda_{k-1}}(X(\bar{e}_{\lambda_1},\ldots,\bar{e}_{\lambda_{k-1}}),X'(\bar{e}_{\lambda_1},\ldots,\bar{e}_{\lambda_{k-1}}))$$

明らかに (X,X') は, $(\bar{e}_\lambda)_{0\leqq\lambda\leqq r}$ のとり方によらない. $(e_i)_{1\leqq i\leqq r}$ を $\mathfrak{g}_{-1}^{+}$ の orthonormal basis とする. $(\bar{e}_\lambda)$ として (e_λ) をとることにより, decomposition $S=\sum_{p=-k}^{-1}S_p$ が orthogonal decomposition であることがわかる.

$\mathfrak{l}$ の inner product $(\cdot,\cdot)$ を次のように定義する. $Z=(A,B)$, $Z'=(A',B')\in\mathfrak{l}$ に対して,

$$(Z,Z')=\mathrm{Tr}({}^tAA')+\mathrm{Tr}({}^tBB')$$

によって定義する. ここで, ${}^tA$ (resp. ${}^tB$) は V の (resp. $\mathfrak{g}_{-k}$) の inner product $(\cdot,\cdot)$ に関する A (resp. B) の transpose である.

map $\hat{\sigma}:\tilde{L}\to\tilde{L}$ を

$$\hat{\sigma}((A,B))=({}^tA^{-1},{}^tB^{-1}),\quad (A,B)\in\tilde{L}$$

によって定義する. 明らかに $\hat{\sigma}$ は $\tilde{L}$ の involutive automorphism である. $\hat{\sigma}(U)=U$ であるから, $\hat{\sigma}$ は L の involutive automorphism $\hat{\sigma}$ を導く. さらにこの $\hat{\sigma}$ は Lie algebra $\mathfrak{l}$ の involutive automorphism σ を導く. 明らかに

$$\sigma((A,B))=(-{}^tA,-{}^tB),\quad (A,B)\in\mathfrak{l}$$

である.

明らかに

$$(aX,X')=(X,\hat{\sigma}(a)^{-1}X'),\quad a\in L,\ X,X'\in S$$

および

$$(aZ,Z')=(Z,\hat{\sigma}(a)^{-1}Z'),\quad a\in L,\ Z,Z'\in\mathfrak{l}$$

が成り立つ. また, $\hat{\sigma}(L_0)=L_0$ を注意する.

$\mathfrak{b}_{-1}$ (resp.$\mathfrak{b}_0$) の inner product $(\cdot,\cdot)$ を isomorphism $\Phi:S\to\mathfrak{b}_{-1}$ (resp. $\Phi:\mathfrak{l}\to\mathfrak{b}_0$) が isometry になるように定義する. また L (resp. $\mathfrak{l}$) の involutive automorphism $\hat{\sigma}$ (resp. σ) と isomorphism $\Phi:L\to B_0$ (resp. $\Phi:\mathfrak{l}\to\mathfrak{b}_0$) は B_0 (resp. $\mathfrak{b}_0$) の involutive automorphism $\hat{\sigma}$ (resp. σ) を導く: $\Phi(\hat{\sigma}(a))=\hat{\sigma}(\Phi(a))$, $a\in B_0$ (resp. $\Phi(\sigma(Z))=\sigma(\Phi(Z))$, $Z\in\mathfrak{b}_0$)

さて $\mathfrak{b}_{-1}$ と $\mathfrak{b}_0$ の inner product $(\cdot,\cdot)$ は $\mathfrak{g}=\mathfrak{b}_{-1}+\mathfrak{b}_0$ の inner product $(\cdot,\cdot)$ を導く.

Lemma 1.4. $\mathfrak{g}$ の *inner product* $(\cdot,\cdot)$ と B_0 の *involutive automorphism* $\hat{\sigma}$ と $\mathfrak{b}_0$ の *involutive automorphism* σ は次の条件を満足する*:*

(1) *decomposition* $\mathfrak{g}=\sum_p\mathfrak{g}_p$*,* $\mathfrak{g}_{-1}=\mathfrak{g}_{-1}^{+}+\mathfrak{g}_{-1}^{-}$*,* $\mathfrak{g}_0=\mathfrak{g}_0'+\mathfrak{g}_0''$ はすべて *orthogonal decomposition* である*.*

(2) $\hat{\sigma}(G_0)=G_0$*,* $\hat{\sigma}(G_0'')=G_0''$ である*.* さらに $\sigma(\mathfrak{g}_0')=\mathfrak{g}_0'$*,* $\sigma(\mathfrak{g}_0'')=\mathfrak{g}_0''$*,* $\sigma(\mathfrak{g}_{-1}^{+})=\mathfrak{g}_1$ である*.*

(3) $(\mathrm{Ad}(a)X,X')=(X,\mathrm{Ad}(\hat{\sigma}(a)^{-1})X')$*,* $a\in B_0$*,* $X,X'\in\mathfrak{g}$*.* 従って*,*

$$(\mathrm{ad}(Z)X,X')=-(X,\mathrm{ad}(\sigma(Z))X'),\quad Z\in\mathfrak{b}_0,\quad X,X'\in\mathfrak{g}.$$

Remark 1.1. $\mathfrak{g}$ の inner product $(\cdot,\cdot)$ が $\mathfrak{g}_p$ $(-k\leqq p\leqq 2)$, $\mathfrak{g}_{-1}^{-}$, $\mathfrak{g}_{-1}^{+}$, $\mathfrak{g}_0$, $\mathfrak{g}_1$ に導く inner products は, 次のように与えられる.

(1) $\mathfrak{g}_{-1}^{+}$, $\mathfrak{g}_{-k}$ の induced inner products は与えられたものと一致する.

(2) $X_p, X'_p \in \mathfrak{g}_p$ $(-k \leqq p \leqq 2)$, $X_{-1}, X'_{-1} \in \mathfrak{g}^-_{-1}$ とする. (e_i) を $\mathfrak{g}^+_{-1}$ の orthonormal basis とする. このとき, $\forall -k \leqq p \leqq -1$ に対して,

$$(X_p, X'_p) = \frac{(-p-1)!}{(k-1)!(k+p)!} \sum_{i_0,\ldots,i_{k+p}} (X_p(e_{i_1}, \ldots, e_{i_{k+p}}), X'_p(e_{i_1}, \ldots, e_{i_{k+p}}))$$

(3) $X_0 = (D, \frac{k-1}{r+1} \operatorname{Tr} D + B)$, $X'_0 = (D', \frac{k-1}{r+1} \operatorname{Tr} D' + B') \in \mathfrak{g}_0$ とする. このとき,

$$(X_0, X'_0) = \operatorname{Tr}({}^tDD') - \frac{1}{r+1} \operatorname{Tr} D \cdot \operatorname{Tr} D' + \operatorname{Tr}({}^tBB')$$

(4) $\mathfrak{g}_1 = (\mathfrak{g}^+_{-1})^*$ の induced inner product は, $\mathfrak{g}^+_{-1}$ の与えられた inner product から自然にえられる. 即ち

$$(X_1, X'_1) = \sum_i X_1(e_i) X'_1(e_i), \quad X_1, X'_1 \in \mathfrak{g}_1$$

序に

$$\begin{aligned} \hat{\sigma}((D, B)) &= ({}^tD^{-1}, {}^tB^{-1}), \quad (D, B) \in G_0, \\ \sigma((D, B)) &= (-{}^tD, -{}^tB), \quad (D, B) \in \mathfrak{g}_0, \\ (\sigma X)(Y) &= -(X, Y), \quad X, Y \in \mathfrak{g}^+_{-1} \end{aligned}$$

等を注意しておく.

Part II

On the integration problems derived from two remarkable classes of systems of ordinary differential equations of the second order

CHAPTER I

Preliminary remarks

By a *foliated manifold* we mean a manifold R equipped with a completely integrable system E or the pair (R, E).

Let (R, E) be a foliated manifold.

1°. A function f on R is called a *first integral* of E if it satisfies the differential equation:

$$Xf = 0 \quad \text{for any } X \in E.$$

2°. Let p be any point of R. If we set $n = \dim R - \operatorname{rank} E$, we see from the Frobenius theorem that there are n first integrals $u^1, \ldots, u^n$ of E defined on a neighborhood of p such that the differentials $du^1, \ldots, du^n$ are linearly independent at p. It can be shown that any first integral f of E defined on a neighborhood of p may be described as a function of $u^1, \ldots, u^n$: $f = F(u^1, \ldots, u^n)$.

3°. By the *integration of E at a point p* we mean to construct a fundamental system $(u^1, \ldots, u^n)$ of first integrals of E at p.

Let ρ be a mapping of a manifold R to a manifold M. Then R is called a *fibred manifold over the base space M with projection ρ*, if ρ is surjective, and for each point p the differential ρ_{*p} is surjective. A vector $X \in T_p(R)$ is called a *vertical vector* if $\rho_{*p}(X) = 0$, and the totality of vertical vectors is called the *vertical tangent bundle of the fibred manifold.*

§1. Transverse Cartan connections and Cartan systems

1.1. Transverse Cartan connections. Let G/G' be a coset of a Lie group G over its closed subgroup G'. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be the Lie algebras of G and G' respectively.

Let us consider a foliated manifold (R, E). Suppose that R becomes a fibred manifold over a manifold R' with projection ρ so that the vertical tangent bundle coincides with E. Furthermore suppose that there is given a Cartan connection $\Gamma' : (P', \omega')$ of model coset space G/G' on R'. Then we denote by P the p.f.b. over R induced from the p.f.b. P' by the mapping ρ, and by $\hat{\rho}$ the natural homomorphism of P to P'. We also denote by ω the $\mathfrak{g}$-valued 1-form $\hat{\rho}^*\omega'$ on P. Then it is clear that the pair (P, ω) satisfies the following conditions:

(C.1) $R_a^*\omega = \operatorname{Ad}(a^{-1})\omega, \quad a \in G'$

(C.2) $\omega(A^*) = A, \quad A \in \mathfrak{g}'$.

Let π' (resp. π) be the projection of P' (resp. P) onto R' (resp. R). Since $\pi' \circ \hat{\rho} = \rho \circ \pi$ and $E = \rho_*^{-1}(0)$, we also see that the pair satisfies the third condition:

(C.3) Let X be a tangent vector to P. Then, $\omega(X) \in \mathfrak{g}'$, if and only if $\pi_*(X) \in E$.

The observation above leads to the following

Definition 1.1. A transverse Cartan connection Γ of model coset space G/G' on a foliated manifold (R, E) is the pair (P, ω) formed by a p.f.b. P over R with structure group G' with projection π and a $\mathfrak{g}$-valued 1-form ω on P satisfying conditions (C.1), (C.2) and (C.3).

Let $\Gamma : (P, \omega)$ be as in the definition. We once for all fix a complementary subspace $\mathfrak{m}$ of $\mathfrak{g}'$ in $\mathfrak{g}$, and denote by θ the $\mathfrak{m}$-component of ω with respect to the decomposition: $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}'$. We also denote by $E(\Gamma)$ the differential system (possibly with singularities) by the equation $\omega = 0$, and by $V(\Gamma)$ the vertical tangent bundle of the p.f.b. P, or simply $V(\Gamma) = \pi_*^{-1}(0)$. Then we have the following

Proposition 1.1. *(1)* $\theta^{-1}(0) = \pi_*^{-1}(E) = E(\Gamma) \oplus V(\Gamma)$.

(2) For each $z \in P$ *the differential* π_* *of* π *maps the fibre* $E(\Gamma)_z$ *of* $E(\Gamma)$ *at* z *isomorphically onto the fibre* $E_{\pi(z)}$ *of* E.

(3) $E(\Gamma)$ *is right invariant.*

(4) There is a unique function $K : P \to \mathfrak{g} \otimes \Lambda^2\mathfrak{m}^*$ *such that*

$$d\omega + \frac{1}{2}[\omega, \omega] = \frac{1}{2}K(\theta \wedge \theta).$$

Proof. (1) $\sim$ (3) can be easily verified. If we set $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$, we have $A^* \lrcorner \Omega = 0$ for any $A \in \mathfrak{g}'$, and $\Omega(X \wedge Y) = 0$ for any cross sections X, Y of $E(\Gamma)$. Since $\theta^{-1}(0) = E(\Gamma) + V(\Gamma)$, it follows that $X \lrcorner \Omega = 0$ for any $X \in \theta^{-1}(0)$. Therefore (4) follows. □

The equality in (4) will be called the *structure equation*, and the function K the *curvature*. We know from the existence of the structure equation that the system $E(\Gamma)$ is completely integrable.

1.2. A reduction proposition for the integration of completely integrable systems. Let $\Gamma : (P, \omega)$ be a transverse Cartan connection of model coset space G/G' on a foliated manifold (R, E).

Proposition 1.2. *Let* x_0 *be any point of* P *and set* $p_0 = \pi(x_0)$. *Then, once the system* $E(\Gamma)$ *was integrated at* x_0, *the integration of the system* E *at* p_0 *can be carried out by elementary operations.*

By elementary operations we here mean a collection of operations formed by rational operations, differentiations and the uses of the implicit function theorem.

Proof. We set $r = \operatorname{rank} E = \operatorname{rank} E(\Gamma)$, $m = \dim P - r$ $(= \dim G)$ and $n = \dim R - r$ $(= \dim G/G')$. Now, let $(u^1, \dots, u^m)$ be a fundamental system of first integrals of $E(\Gamma)$ at x_0.

(1) We take r functions $u^{m+1}, \dots, u^{m+r}$ defined on a neighborhood of x_0 so that $u^1, \dots, u^m$, $u^{m+1}, \dots, u^{m+r}$ form a coordinate system of P at x_0. Let P_0 be a small cubic neighborhood of x_0. We then construct a mapping φ of P_0 to $\mathbf{R}^m$ by

$$\varphi(z) = (u^1(z), \dots, u^m(z)), \quad z \in P_0\,,$$

and set $Q_0 = \varphi(P_0)$. Then P_0 becomes a fibred manifold over the base space Q_0 with projection φ so that the vertical tangent bundle coincides with the system $E(\Gamma)$, restricted to P_0. We set $y_0 = \varphi(x_0)$.

(2) Let us now construct a local right action of the group G' on Q_0. For this purpose we first construct a mapping ψ of Q_0 to P_0 such that $\psi(y_0) = x_0$ and $\varphi \circ \psi$ is the identity mapping. Then the local action is constructed by the following rule:

$$y \cdot a = \varphi(\psi(y) \cdot a),$$

where $(y, a) \in Q_0 \times G'$ with $\psi(y) \cdot a \in P_0$. Since the system $E(\Gamma)$ is right invariant, we see that this rule really gives a local action of G' on Q_0 and that φ is compatible with the respective actions of G'.

(3) For any $A \in \mathfrak{g}'$ we have the (vertical) vector field A^* on P_0. Let us denote by the same symbol A^* the vector field on Q_0 induced from the local 1-parameter group $y \to y \cdot \exp tA$

of local right translations on Q_0. Clearly these two vector fields are φ-related: $\varphi_* A_z^* = A_{\varphi(z)}^*$, $z \in P_0$. Therefore the condition $A_{\varphi(z)}^* = 0$ implies $A_z^* \in E(\Gamma)_z$. By Proposition 1.1 it follows that $A_z^* = 0$ or $A = 0$. We have thereby proved the following:

Lemma 1.3. *The local action of G' on Q_0 is infinitesimally free, that is, the condition "$A_y^* = 0$ for some $A \in \mathfrak{g}'$ and for some $y \in Q_0$" implies $A = 0$.*

(4) We now take an n-dimensional small submanifold, say M_0, of Q_0 through y_0 which is transversal to the G'-orbit through y_0 as well as a small neighborhood, say N_0, of the identity element e of G'. Then we define a mapping Φ of $M_0 \times N_0$ to Q_0 by $\Phi(y, a) = y \cdot a$, $(y, a) \in M_0 \times N_0$. By Lemma 1.3 we easily see that the differential of Φ at (y_0, e) gives an isomorphism. Thus Φ gives a diffeomorphism of a neighborhood of (y_0, e) onto a neighborhood of y_0 so that $\Phi(y_0, e) = y_0$, and we construct the inverse, say Ψ, of Φ.

The matter being highly of local character, we may assume that Ψ gives a diffeomorphism of Q_0 onto $M_0 \times N_0$. This being said, we denote by ϖ the projection of $M_0 \times N_0$ onto M_0, and set $\varepsilon = \varpi \circ \Psi$. Then Q_0 becomes a fibred manifold over M_0 with projection ε. Furthermore if we set $R_0 = \pi(P_0)$, R_0 is an open submanifold of R, and P_0 becomes a fibred manifold over R_0 with projection π.

(5) We set $p_0 = \pi(x_0)$, and take a local cross section σ of the fibred manifold P_0 over R_0 defined on a neighborhood V_0 of p_0 and such that $\sigma(p_0) = x_0$. Then we define a mapping ρ of V_0 to M_0 by $\rho = \varepsilon \circ \varphi \circ \sigma$.

Lemma 1.4. *There is a neighborhood U_0 of x_0 such that $\pi(U_0) \subset V_0$ and the two mappings $\rho \circ \pi$ and $\varepsilon \circ \varphi$ coincide on U_0.*

Proof. Let $z \in \pi^{-1}(V_0) \cap P_0$. Then we have $(\rho \circ \pi)(z) = (\varepsilon \circ \varphi)(\sigma(\pi(z)))$. Since $\pi(\sigma(\pi(z))) = \pi(z)$, there is an element $a(z)$ of G' with $\sigma(\pi(z)) = z \cdot a(z)$. It follows $(\rho \circ \pi)(z) = (\varepsilon \circ \varphi)(z \cdot a(z)) = \varepsilon(\varphi(z) \cdot a(z))$. In view of the structure of Q_0, φ may be expressed as follows: $\varphi(z) = \varepsilon(\varphi(z)) \cdot b(z)$, $z \in P_0$, where $b(z) \in N_0$. Since $\varphi(x_0) = y_0$ and $\varepsilon(y_0) = y_0$, we have $b(x_0) = e$. Similarly we have $\sigma(p_0) = x_0 = x_0 \cdot a(x_0)$, whence $a(x_0) = e$. Therefore it follows that there is a neighborhood U_0 of x_0 such that $\pi(U_0) \subset V_0$ and $b(z) \cdot a(z) \in N_0$ for any $z \in U_0$. Consequently for any $z \in U_0$ we obtain

$$\varepsilon(\varphi(z) \cdot a(z)) = \varepsilon(\varepsilon(\varphi(z)) \cdot b(z)a(z)) = \varepsilon(\varphi(z)),$$

whence $(\rho \circ \pi)(z) = \varepsilon(\varphi(z))$, proving the lemma. □

(6) By Proposition 1.1 we have

$$\pi_*^{-1}(E) = E(\Gamma) + V(\Gamma)\,.$$

By Lemma 1.4 we have

$$(\rho \circ \pi)_*^{-1}(0) = (\varepsilon \circ \varphi)_*^{-1}(0) \quad \text{on } U_0.$$

Furthermore we can easily verify that

$$(\varepsilon \circ \varphi)_*^{-1}(0) = E(\Gamma) + V(\Gamma) \quad \text{on } U_0.$$

Consequently it follows that

$$(\rho \circ \pi)_*^{-1}(0) = \pi_*^{-1}(E) \quad \text{on } U_0,$$

which clearly means that

$$\rho_*^{-1}(0) = E \quad \text{on } \rho(U_0).$$

Now, let $(\xi^1, \cdots, \xi^n)$ be a coordinate system of M_0 at $\rho(p_0)$. Then this last fact implies that $(\xi^1 \circ \rho, \cdots, \xi^n \circ \rho)$ gives a fundamental system of first integrals of E at p_0. We have thus completed the proof of Proposition 1.2, because all the constructions above can be carried out by elementary operations. We have thus completed the proof of the proposition. □

§2. Cartan systems

As for a detailed account of the theory of Cartan systems, see the appendix.

2.1. Cartan systems. At the outset we give the following

Definition 2.1. (E.Cartan[**]) A Cartan system Σ (of degree n) is the pair (P, ω) formed by a manifold P and an $\mathbf{R}^n$-valued 1-form ω or a system of 1-forms $\omega^1, \cdots, \omega^n$ on P satisfying the following conditions:

1) $\omega^1, \ldots, \omega^n$ are linearly independent at each point of P.
2) There are functions $c^i_{jk}(1 \leq i, j, k \leq n)$ on P such that

$$d\omega^i + \frac{1}{2}\sum_{j,k} c^i_{j,k}\omega^j \wedge \omega^k = 0, \quad c^i_{jk} + c^i_{kj} = 0.$$

Given a Cartan system Σ, the equality above will be called the *structure equation*, and the system (c^i_{jk}) of functions the *structure functions*. We also denote by $E(\Sigma)$ the differential system on P defined by the equation $\omega = 0$, which is clearly completely integrable.

Now, a local vector field X on P is called a *local infinitesimal automorphism* of Σ, if X leaves ω invariant or $L_X\omega = 0$. In view of the structure equation we know that a local cross section of $E(\Sigma)$ is an infinitesimal automorphism of Σ and that if X is a local infinitesimal automorphism of Σ and Y is a local cross section of $E(\Sigma)$, then the bracket $[X, Y]$ is a local cross section of $E(\Sigma)$. This being said, we denote by $\mathcal{A}$ the sheaf of Lie algebras of local infinitesimal automorphisms of Σ, and by $\mathcal{E}$ the sheaf of local cross sections of $E(\Sigma)$, being a subsheaf of ideals of $\mathcal{A}$. Furthermore for each $x \in P$ the factor Lie algebra $\mathcal{A}_x/\mathcal{E}_x$ is denoted by $\mathrm{aut}(\Sigma)_x$, and is called the *reduced Lie algebra of infinitesimal automorphisms of* Σ *at* x, where $\mathcal{A}_x$ (resp. $\mathcal{E}_x$) denotes the stalk at x of $\mathcal{A}$ (resp. of $\mathcal{E}$). We shall see that under a suitable regularity condition on Σ $\mathrm{aut}(\Sigma)_x$ is naturally isomorphic with the stalk at x of the factor Lie algebra sheaf $\mathcal{A}/\mathcal{E}$.

For example let us consider a transverse Cartan connection $\Gamma : (P, \omega)$ of model coset space $G/G^{(0)}$ on a foliated manifold (R, E). Then the pair (P, ω) may be regarded as a Cartan system, say Σ, which is called *associated*. Proposition 1.2 indicates that the integration problem in the foliated manifold (R, E) can be reduced to the integration problem in the associated Cartan system Σ.

2.2. Non-degenerate Cartan systems. A Cartan system of degree n: $\Sigma : (P, \omega)$ is called *non-degenerate*, if $\dim P = n$ or in other words ω gives an absolute parallelism on P. A non-degenerate Cartan system no more possesses the meaning of a differential equation, but is a geometric object.

Proposition 2.1. *Let* $\Sigma : (P, \omega)$ *be a Cartan system of degree* n*, and let* x_0 *be any point of* P.

(1) There are a neighborhood V *of* x_0 *and a non-degenerate Cartan system* $\Sigma' : (V', \omega')$ *satisfying the following conditions: 1)* V *becomes a fibred manifold over* V' *with projection* κ *so that the vertical tangent bundle coincides with the system* $E(\Sigma)$*, restricted to* V*, and 2)* $\kappa^*\omega'$ *coincides with* ω*, restricted to* V.

(2) The projection κ *naturally induces an isomorphism of* $\mathrm{aut}(\Sigma)_{x_0}$ *onto* $\mathrm{aut}(\Sigma')_{\kappa(x_0)}$.

Corollary 2.2. $\dim \mathrm{aut}(\Sigma)_x \leq n,\ x \in P$.

Proof of Proposition 2.1. (0) We set $r = \operatorname{rank} E(\Sigma)$, whence $\dim P = n + r$. In the following the letters i, j will range over the integers $1, \ldots, n$, and the letters λ, μ over the integers $n + 1, \ldots, n + r$.

(1) Let x_0 be any point of P, and let $(u^1, \dots, u^{n+r})$ be a coordinate system of P at x_0 such that (u^i) gives a fundamental system of first integrals of the system $E(\Sigma)$ at x_0.

Let V be a cubic neighborhood of x_0 with respect to this coordinate system, and let us consider the moving frame $(\partial/\partial u^1, \dots, \partial/\partial u^{n+r})$ of the tangent bundle $T(V)$ of V. For simplicity we set $Y_\lambda = \partial/\partial u^\lambda$. Then we see that (Y_λ) gives a moving frame of the system $E(\Sigma)$.

We define a mapping κ of V to $\mathbf{R}^n$ by $\kappa(x) = (u^i(x))$ for any $x \in V$, and set $V' = \kappa(V)$. Then we see that V becomes a fibred manifold over V' with projection κ so that the vertical tangent bundle coincides with the system $E(\Sigma)$, restricted to V.

Now, the $\mathbf{R}^n$-valued 1-form ω may be described as follows:

$$\omega = \sum_i f_i du^i + \sum_\mu f_\mu du^\mu,$$

where f_i, f_μ are $\mathbf{R}^n$-valued functions on V. Since $\omega(Y_\lambda) = 0$ and $Y_\lambda u^\mu = \delta^\mu_\lambda$, we have $f_\lambda = 0$. Furthermore since $L_{Y_\lambda}\omega = 0$ and $Y_\lambda u^i = 0$, we obtain $Y_\lambda f_i = 0$. It follows that f_i are first integrals of $E(\Sigma)$, and hence may be described as follows: $f_i = \hat{f}_i(u^1, \dots, u^n)$, $\hat{f}_i$ being functions on V'.

Let (v^i) be the canonical coordinate system of V'. Then we have $u^i = v^i \circ \kappa$ and $f_i = \hat{f}_i \circ \kappa$. If we set

$$\omega' = \sum_i f_i dv^i,$$

we find that $\omega = \kappa^*\omega'$, and the pair (V', ω') gives a non-degenerate Cartan system, say Σ', of degree n, proving the first assertion.

(2) Let X be any local infinitesimal automorphism of Σ defined on a neighborhood of x_0, which may be described as follows:

$$X = \sum_i g^i \partial/\partial u^i + \sum_\mu g^\mu Y_\mu.$$

If we set $\bar{X} = \sum g^i \partial/\partial u^i$, we see that $\bar{X}$ is a local infinitesimal automorphism of Σ.

Lemma 2.3. *The vector fields* $[Y_\lambda, \bar{X}]$ *and* $[Y_\lambda, \partial/\partial u^i]$ *are local cross sections of* $E(\Sigma)$.

For the proof of this fact we remark that i) $L_{\bar{X}}\omega = 0$ and $\omega(Y_\lambda) = 0$, and ii) $du^j(\partial/\partial u^i) = \delta^j_i$ and $Y_\lambda u^j = 0$.

Now, we have

$$[Y_\lambda, \bar{X}] = \sum_i Y_\lambda g^i \partial/\partial u^i + \sum_i g^i\, [Y_\lambda, \partial/\partial u^i].$$

By lemma 2.3 it follows that $Y_\lambda g^i = 0$. Therefore g^i are first integrals of $E(\Sigma)$ and hence may be described as follows: $g^i = \hat{g}^i(u^1, \dots, u^n)$, $\hat{g}^i$ being functions defined on a neighborhood of $\kappa(x_0)$. If we set

$$X' = \sum_i \hat{g}^i \partial/\partial v^i,$$

we know that 1) X' is a local infinitesimal automorphism of Σ', 2) X and X' are κ-related, and 3) the assignment $X \to X'$ induces an isomorphism of $\mathrm{aut}(\Sigma)_{x_0}$ with $\mathrm{aut}(\Sigma')_{\kappa(x_0)}$, proving the second assertion. □

2.3. The regularity condition. Let $\Sigma : (P, \omega)$ be a Cartan system of degree n. First of all we remark that a function f on P is a first integral of $E(\Sigma)$, if and only if

$$df \equiv 0 \pmod{\omega^1, \dots, \omega^n}.$$

Therefore if f is a first integral of $E(\Sigma)$, the differential df may be described as follows:

$$df = \sum_i f_i \omega^i.$$

The coefficients f_i will be denoted by $\delta_i f$ or $\partial f/\partial\omega^i$.

By using the structure equation we can easily verify the following

Lemma 2.4. *(1) The structure functions c^i_{jk} are first integrals of $E(\Sigma)$.*

(2) If f is a first integral of $E(\Sigma)$, then the coefficients $\delta_i f$ of df are first integrals of $E(\Sigma)$.

This being said, let us denote by $I'(\Sigma)$ the family of all (global) first integrals of $E(\Sigma)$. Then we denote by $I(\Sigma)$ the smallest subfamily among the subfamilies I of $I'(\Sigma)$ satisfying the following conditions:

1) $c^i_{jk} \in I$,
2) If $f \in I$, then $\delta_i f \in I$.

Namely $I(\Sigma)$ consists of all functions of the following form:

$$\delta_{i_1} \cdots \delta_{i_\ell} c^i_{jk} \quad (1 \leq i, j, k, i_1, \ldots, i_\ell \leq n,\ \ell \geq 0)$$

Now, for each point $x \in P$ we denote by $\Delta(\Sigma)_x$ the subspace of $T_x(P)$ consisting of all tangent vectors $X \in T_x(P)$ such that

$$Xf = 0 \ \text{ for all } f \in I(\Sigma),$$

and set $\Delta(\Sigma) = \bigcup_x \Delta(\Sigma)_x$, being a differential system (possibly with singularities) on P. Clearly we have $E(\Sigma) \subset \Delta(\Sigma)$.

We say that the Cartan system Σ is *regular*, if $\Delta(\Sigma)$ gives a differential system in the usual sense, restricted to each connected component of P, or equivalently $\dim \Delta(\Sigma)_x$ is locally constant.

It can be shown that in the real analytic category any Cartan system is necessarily regular.

2.4. The Cartan's integration theorem for Cartan systems. We are now in a position to state the following

Theorem 2.5. *(E.Cartan[**]) Let $\Sigma : (P, \omega)$ be a Cartan system, and let x_0 be any point of P. If Σ is regular, then the integration of the completely integrable system $E(\Sigma)$ can be carried out by elementary operations, quadratures and the integrations of the Lie systems associated with the simple components of the Lie algebra* $\mathrm{aut}(\Sigma)_{x_0}$.

§3. The integration problems associated with pseudo-projective systems of the restricted types

3.1. The transverse Cartan connections associated with pseudo-projective systems of the restricted types. (1) Let us consider the graded Lie algebra $\mathfrak{B} : \mathfrak{g} = \mathfrak{b}_{-1} + \mathfrak{b}_0 + \mathfrak{b}_1$ as well as the coset space $G/B^{(0)}$ defined in I.2. We denote by ρ_2 the linear isotropy representation of the group $B^{(0)}$ on the space $\mathfrak{b}_{-1}$ identified with the tangent space to $G/B^{(0)}$ at the origin o:

$$\rho_2(a)X \equiv \mathrm{Ad}(a)X \pmod{\mathfrak{b}^{(0)}},$$

where $a \in B^{(0)}$ and $X \in \mathfrak{b}_{-1}$. Furthermore the group $B^{(0)}$ acts on the space $\mathfrak{g} \otimes \wedge^2 \mathfrak{b}^*_{-1}$ by the following rule:

$$L^a(X \wedge Y) = \mathrm{Ad}(a^{-1})L(\rho_2(a)X \wedge \rho_2(a)Y),$$

where $X, Y \in \mathfrak{b}_{-1}$, $L \in \mathfrak{g} \otimes \wedge^2 \mathfrak{b}_{-1}^*$ and $a \in B^{(0)}$. Clearly $\mathfrak{b}^{(0)} \otimes \wedge^2 \mathfrak{b}_{-1}^*$ is an invariant subspace in this action.

Now, let $\mathcal{R} : (R, E, F)$ be a pseudo-projective system of degree n and of the second restricted type, and let $\Gamma : (P, \omega)$ be the associated pseudo-projective connection, being a Cartan connection of model coset space $G/G^{(0)}$ on R. $G^{(0)}$ being a subgroup of $B^{(0)}$, let P_2 denote the extension of the p.f.b. P to the group $B^{(0)}$. It is easy to see that the connection form ω of Γ is naturally extended to a $\mathfrak{g}$-valued 1-form, say $\bar{\omega}$, on P_2 so that the pair $(P_2, \bar{\omega})$ satisfies conditions (C.1) and (C.2) in the definition of a transverse Cartan connection.

By Theorem I.4.1 we know that the curvature K of Γ takes values in $\mathfrak{b}^{(0)} \otimes \wedge^2 \mathfrak{b}_{-1}^*$ and hence the structure equation for Γ takes the following form:

$$d\omega + \frac{1}{2}[\omega, \omega] = \frac{1}{2} K(\beta_{-1} \wedge \beta_{-1}),$$

where β_{-1} denotes the $\mathfrak{b}_{-1}$-component of ω with respect to the decomposition: $\mathfrak{g} = \mathfrak{b}_{-1} + \mathfrak{b}^{(0)}$. It follows that

$$R_a^* K = K^a, \quad a \in G^{(0)}.$$

Therefore K is naturally extended to a function, say $\bar{K}$, on P_2 taking values in $\mathfrak{b}^{(0)} \otimes \wedge^2 \mathfrak{b}_{-1}^*$ so that

$$R_a^* \bar{K} = \bar{K}^a, \quad a \in B^{(0)}.$$

Then we have the following

Proposition 3.1. *The pair $(P_2, \bar{\omega})$ gives a transverse Cartan connection, say Γ_2, of model coset space $G/B^{(0)}$ on the foliated manifold (R, E), and its curvature (with respect to the decomposition $\mathfrak{g} = \mathfrak{b}_{-1} + \mathfrak{b}^{(0)}$) coincides with the natural extension $\bar{K}$ of the curvature K of Γ.*

Proof. To prove the first assertion we must show that the pair $(P_2, \bar{\omega})$ satisfies condition (C.3). Namely, let $x \in P_2$ and $X \in T_x(P_2)$. Then we must show that $\bar{\omega}(X) \in \mathfrak{b}^{(0)}$, if and only if $\pi_{2*}(X) \in E$, where π_2 denotes the projection of P_2 onto R. Clearly it suffices to deal with the case where $x \in P$. Then there are a vector $Y \in T_x(P)$ and an element $A \in \mathfrak{b}^{(0)}$ such that $X = Y + A_x^*$. Since $\omega_{-1}^+(Y) \in \mathfrak{b}^{(0)}$, we have

$$\begin{aligned} \bar{\omega}(X) &= \omega(Y) + A \\ &\equiv \omega_-(Y) \pmod{\mathfrak{b}^{(0)}} \\ &\equiv \omega_{-2}(Y) + \omega_{-1}^-(Y) \pmod{\mathfrak{b}^{(0)}}. \end{aligned}$$

Therefore it follows that $\bar{\omega}(X) \equiv 0 \pmod{\mathfrak{b}^{(0)}}$, if and only if $\omega_{-2}(Y) + \omega_{-1}^-(Y) = 0$. On the other hand we have

$$\begin{aligned} \bar{\pi}_*(X) &= \pi_*(Y) = \rho(x)\omega_-(Y) \\ &= \rho(x)(\omega_{-2}(Y) + \omega_{-1}^-(Y) + \omega_{-1}^+(Y)), \end{aligned}$$

where π denotes the projection of P onto R, and ρ the natural homomorphism of P to the frame bundle $F(R, \mathfrak{g}_-)$. Therefore it follows that $\bar{\pi}_*(X)$ is in E, if and only if $\omega_{-2}(Y) + \omega_{-1}^-(Y) = 0$, proving the first assertion.

Let us now prove the second assertion. Let L be the curvature of Γ_2. Then the structure equation for $\bar{\Gamma}$ takes the following form:

$$d\bar{\omega} + \frac{1}{2}[\bar{\omega}, \bar{\omega}] = \frac{1}{2} L(\bar{\beta}_{-1} \wedge \bar{\beta}_{-1}),$$

where $\bar{\beta}_{-1}$ denotes the $\mathfrak{b}_{-1}$-component of $\bar{\omega}$ with respect to the decomposition: $\mathfrak{g} = \mathfrak{b}_{-1} + \mathfrak{b}^{(0)}$. It follows that

$$R_a^* L = L^a, \quad a \in B^{(0)}.$$

Furthermore it is clear that the restriction of the equation above to P yields the structure equation for Γ, whence the restriction of L to P coincides with K. Therefore we obtain $L = \bar{K}$, proving the second assertion. □

(2) Let us now consider the graded Lie algebra $\mathfrak{A} : \mathfrak{g} = \mathfrak{a}_{-1} + \mathfrak{a}_0 + \mathfrak{a}_1$ as well as the coset space $G/A^{(0)}$ defined in §I.2. We denote by ρ_1 the linear isotropy representation of the group $A^{(0)}$ on the space $\mathfrak{a}_{-1}$:

$$\rho_1(a)X \equiv \mathrm{Ad}(a)X \pmod{\mathfrak{a}^{(0)}},$$

where $a \in A^{(0)}$ and $X \in \mathfrak{a}_{-1}$. Furthermore the group $A^{(0)}$ acts on the space $\mathfrak{g} \otimes \wedge^2 \mathfrak{a}_{-1}^*$ by the following rule:

$$L^a(X \wedge Y) = \mathrm{Ad}(a^{-1}) L(\rho_1(a)X \wedge \rho_1(a)Y),$$

where $X, Y \in \mathfrak{a}_{-1}$, $L \in \mathfrak{g} \otimes \wedge^2 \mathfrak{a}_{-1}^*$ and $a \in A^{(0)}$.

Now let $\mathcal{R} : (R, E, F)$ be a pseudo-projective system of degree n and of the first restricted type, and let $\Gamma : (P, \omega)$ be the associated pseudo-projective connection. $G^{(0)}$ being a subgroup of $A^{(0)}$, let P_1 denote the extension of the p.f.b. P to the group $A^{(0)}$. Then we see that the connection form ω of Γ is naturally extended to a $\mathfrak{g}$-valued 1-form, say $\hat{\omega}$, on P_1 so that the pair $(P_1, \hat{\omega})$ satisfies conditions (C.1) and (C.2).

By Theorem I.4.1 we know that the curvature K of Γ takes values in $\mathfrak{a}^{(0)} \otimes \wedge^2 \mathfrak{a}_{-1}^*$ and hence the structure equation for Γ takes the following form:

$$d\omega + \frac{1}{2}[\omega, \omega] = \frac{1}{2} K(\alpha_{-1} \wedge \alpha_{-1}),$$

where α_{-1} denotes the α_{-1}-component of ω with respect to the decomposition: $\mathfrak{g} = \mathfrak{a}_{-1} + \mathfrak{a}^{(0)}$. Therefore, as before K is naturally extended to a function, say $\hat{K}$, on P_1 taking values in $\mathfrak{a}^{(0)} \otimes \wedge^2 \mathfrak{a}_{-1}^*$ so that

$$R_a^* \hat{K} = \hat{K}^a, \quad a \in A^{(0)}.$$

Reasoning just in the same manner as in the proof of the preceding proposition, we can prove the following

Proposition 3.2. *The pair $(P_1, \hat{\omega})$ gives a transverse Cartan connection, say Γ_1, of model coset space $G/A^{(0)}$ on the foliated manifold (R, F), and its curvature (with respect to the decomposition $\mathfrak{g} = \mathfrak{a}_{-1} + \mathfrak{a}^{(0)}$) coincides with the natural extension $\hat{K}$ of the curvature K of Γ.*

Finally we add that both the transverse Cartan connections Γ_2 and Γ_1 can be constructed from the pseudo-projective system $\mathcal{R}$ by differentiations, because so is the Cartan connection Γ.

3.2. The Lie algebras $\mathrm{aut}(\mathcal{R})_p$, $\mathrm{aut}(\Sigma)_x$, $\mathrm{aut}(\Sigma_2)_x$ and $\mathrm{aut}(\Sigma_1)_x$. (0) Let $\mathcal{R} : (R, E, F)$ be a pseudo-projective system of degree n, and let $\Gamma : (P, \omega)$ be the associated pseudo-projective connection on R. We denote by $\mathrm{aut}(\mathcal{R})$ the sheaf of germs of local infinitesimal automorphisms of $\mathcal{R}$, which coincides with the sheaf of germs of local infinitesimal automorphisms of Γ.

Let $\Sigma : (P, \omega)$ be the Cartan system associated with the connection Γ. We take any point x of P and set $p = \pi(x)$, π being the projection of P onto R. Then we know from Lemma 3.3 just below that the projection π naturally induces a (Lie algebra) isomorphism of $\mathrm{aut}(\Sigma)_x$ with $\mathrm{aut}(\mathcal{R})_p$.

Lemma 3.3. *Let U be a simply connected neighborhood of x. Then any germ of* $\mathrm{aut}(\Sigma)_x$ *can be represented by a right invariant infinitesimal automorphism of Σ defined on the whole of $\pi^{-1}(U)$.*

For the proof of this fact we only remark that if Y is a local infinitesimal automorphism of Σ, then $[Y, A^*] = 0$ for any $A \in \mathfrak{g}^{(0)}$, which can be easily obtained from the structure equation for Γ.

(1) Now, assume that $\mathcal{R}$ is of the second restricted type, and consider the transverse Cartan connection $\Gamma_2 : (P_2, \bar{\omega})$ on the foliated manifold (R, E) as well as the Cartan system $\Sigma_2 : (P_2, \bar{\omega})$. Then we easily see that any right invariant vector field X on $\pi^{-1}(U)$ is extended to a right invariant vector field, say $\bar{X}$, on $\pi_2^{-1}(U)$, π_2 being the projection of P_2 onto R.

Lemma 3.4. *Let X and Y be any right invariant vector fields on $\pi^{-1}(U)$. Then the following hold:*

$$(L_{\bar{X}}\bar{\omega}(\bar{Y}_{ya}) = \mathrm{Ad}(a^{-1})(L_X\omega)(Y_y),$$
$$(L_{\bar{X}}\bar{\omega})(A^*_{ya}) = 0,$$

where $y \in P$, $a \in B^{(0)}$ and $A \in \mathfrak{b}^{(0)}$.

It follows from Lemma 2.4 that if X is a right invariant infinitesimal automorphism of Σ, then $\bar{X}$ is a right invariant infinitesimal automorphism of Σ_2, and hence the assignment $X \to \bar{X}$ yields an (injective) homomorphism, say $\boldsymbol{\iota}_2$, of $\mathrm{aut}(\Sigma)_x$ to $\mathrm{aut}(\Sigma_2)_x$.

Then we have the following

Proposition 3.5. *(1) $\boldsymbol{\iota}_2$ gives an isomorphism of* $\mathrm{aut}(\Sigma)_x$ *with* $\mathrm{aut}(\Sigma_2)_x$.

(2) The injection, say ι_2, of P to P_2 induces an isomorphism of $\Delta(\Sigma)_x$ *with* $\Delta(\Sigma_2)_x/E(\Sigma_2)_x$.

Corollary 3.6. *If the Cartan system Σ is regular, so is the Cartan system Σ_2.*

For the proof of this fact we note that the system $\Delta(\Sigma_2)$ together with the system $E(\Sigma_2)$ is right invariant.

Proof of Proposition 3.5.

(1) By Proposition 2.1 we know that there are a neighborhood V of x and a non-degenerate Cartan system $\Sigma' : (V', \omega')$ satisfying the following conditions: 1) V becomes a fibred manifold over V' with projection κ so that the vertical tangent bundle coincides with the system $E(\Sigma_2)$, restricted to V, and 2) $\kappa^*\omega'$ coincides with ω_2, restricted to V.

Now, both Σ and Σ' are non-degenerate, and $\dim P = \dim V'$. Furthermore, $\omega = \iota_2^*\omega_2$ and $\kappa^*\omega' = \omega_2$. Therefore, if we set $\varphi = \kappa \circ \iota_2$, we see that $\varphi^*\omega' = \omega$, meaning that φ gives a local isomorphism of Σ with Σ' defined on a neighborhood of x. Accordingly φ induces an isomorphism of $\mathrm{aut}(\Sigma)_x$ with $\mathrm{aut}(\Sigma')_{\kappa(x)}$. Furthermore we know from the second assertion of Proposition 2.1 that there is a natural isomorphism of $\mathrm{aut}(\Sigma_2)_x$ with $\mathrm{aut}(\Sigma')_{\kappa(x)}$. Since the homomorphism $\boldsymbol{\iota}_2$ of $\mathrm{aut}(\Sigma)_x$ to $\mathrm{aut}(\Sigma_2)_x$ is injective, we have thereby shown that $\boldsymbol{\iota}_2$ is an isomorphism, completing the proof of the first assertion.

(2) From the proof of Proposition 2.1 we know that κ induces an isomorphism of $\Delta(\Sigma_2)_x/E(\Sigma)_x$ with $\Delta(\Sigma')_x$. Furthermore it is clear that φ induces an isomorphism of $A(\Sigma)_x$ with $A(\Sigma')_x$. Consequently the second assertion follows. □

(3) Next, assume that $\mathcal{R}$ is of the first restricted type, and consider the transverse Cartan connection $\Gamma_1 : (P_1, \hat{\omega})$ on the foliated manifold (R, F) as well as the Cartan system $\Sigma_1 : (P_1, \hat{\omega})$. Then just as above we have a natural (injective) homomorphism, say $\boldsymbol{\iota}_1$, of $\mathrm{aut}(\Sigma)_x$ to $\mathrm{aut}(\Sigma_1)_x$ as well as the following

Proposition 3.7. *(1) $\boldsymbol{\iota}_1$ gives an isomorphism of* $\mathrm{aut}(\Sigma)_x$ *with* $\mathrm{aut}(\Sigma_1)_x$.

(2) The injection, say ι_1, of P to P_1 induces an isomorphism of $\Delta(\Sigma_1)_x$ *with* $\Delta(\Sigma_1)_x/E(\Sigma_1)_x$.

Corollary 3.8. *If the Cartan system Σ is regular, so is the Cartan system Σ_1.*

3.3. An integration theorem for the system E or F associated with a pseudo-projective system $\mathcal{R}$ of the second or the first restricted type. A pseudo-projective system $\mathcal{R}$ will be called *regular*, if the associated Cartan system $\Sigma : (P, \omega)$ is regular.

Theorem 3.9. *Let $\mathcal{R} : (R, E, F)$ be a pseudo-projective system, and let p_0 be any point of R.*

(1) Assume that $\mathcal{R}$ is regular and of the second restricted type. Then the integration of the system E at the point p_0 can be carried out by elementary operations, quadratures and the integrations of the Lie systems associated with the simple components of the Lie algebra $\mathrm{aut}(\mathcal{R})_{p_0}$.

(2) Assume that $\mathcal{R}$ is regular and of the first restricted type. Then the integration of the system F at the point p_0 can be carried out by the same operations as above.

Above all we notice that the first assertion settles the integration problem for system of ordinary differential equations of the second order and of the second restricted type:

$$(\mathfrak{X}) \qquad \frac{d^2y^j}{dx^2} = f^j\left(x, (y^k), (\frac{dy^k}{dx})\right).$$

Proof of Theorem 3.9. To prove the first assertion let us consider the following objects:

$$\Gamma : (P, \omega),\ \Sigma : (P, \omega),\ \Gamma_2 : (P_2, \bar{\omega}),\ \Sigma_2 : (P_2, \bar{\omega}).$$

As we have remarked, the transverse Cartan connection Γ_2 together with the Cartan connection Γ can constructed from the system $\mathcal{R}$ by differentiations.

Now, we take a point x_0 of P with $\pi(x_0) = p_0$. Then we have the following:

1°. Since Γ_2 is a transverse Cartan connection on the foliated manifold (R, E), we know from Proposition 1.2 that the integration problem for the system E at p_0 can be reduced to the integration problem for the system $E(\Sigma_2)$ at x_0.

2°. By Proposition 3.5 we know that

$$\mathrm{aut}(\mathcal{R})_{p_0} \cong \mathrm{aut}(\Sigma)_{x_0} \cong \mathrm{aut}(\Sigma_2)_{x_0}.$$

3°. Since the system $\mathcal{R}$ or the Cartan system Σ is regular by the assumption, we know from Corollary 3.6 that the Cartan system Σ_2 as well is regular.

Therefore the first assertion follows immediately from Theorem 2.5. Here, we mention that the second assertion can be dealt with just in the same manner as above, thus completing the proof of the theorem. □

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