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Title	Geometric Theory of Systems of Ordinary Differential Equations II
Author(s)	Tanaka, Noboru; Kiyohara, K.; Morimoro, T. et al.
Citation	Hokkaido University technical report series in mathematics, 170, 1
Issue Date	2017-03-28
DOI	https://doi.org/10.14943/81532
Doc URL	https://hdl.handle.net/2115/68096
Type	departmental bulletin paper
File Information	tech170.pdf



Geometric Theory of Systems of Ordinary Differential Equations II

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Series #170. March, 2017

HOKKAIDO UNIVERSITY
TECHNICAL REPORT SERIES IN MATHEMATICS
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Foreword

This volume contains the scanned images of the original manuscripts. The first half of them which are written with typewriter is a copy of the report to JSPS for the scientific grant. For the general remarks, please check “Explanatory Notes” in the first volume.

Geometric theory of systems of ordinary differential equations, I

NOBORU TANAKA

Introduction. For around ten years we have worked on geometric theory of systems of ordinary differential equations, and obtained many results on the geometry as well as the geometric integration of these systems, which will be published as a book in the near future.

This report just corresponds to the first two chapters of the book, and is concerned with the following geometries:

(1) The geometry of pseudo-projective structures of order 2 of bidegree (n, r) , which formulates the geometry of involutive systems of partial differential equations of the second order

$$\frac{\partial^2 y^a}{\partial x^i \partial x^j} = f_{ij}^a((x^k), (y^b), (\frac{\partial y^b}{\partial x^k})).$$

Here, (n, r) is a given pair of integers with $1 \leq r \leq n - 1$, and the indices i, j, k range over the integers $1, \dots, r$, and the indices a, b over the integers $1, \dots, n - r$.

(2) The geometry of generalized pseudo-projective structures of bidegree (n, r) , which formulates the geometry of involutive and "generic" systems of partial differential equations of the second order

$$\begin{cases} \frac{\partial^2 y^a}{\partial x^i \partial x^j} = f_{ij}^a((x^k), (y^b), (\frac{\partial y^b}{\partial x^k})), \\ f^\lambda((x^k), (y^b), (\frac{\partial y^b}{\partial x^k})) = 0. \end{cases}$$

For rigorous formulations of these structures or rather differential equations see §2. Here we only remark the following: A generalized pseudo-projective structure $\mathcal{K}$ of bidegree (n, r) on an n -dimensional manifold M is a pair formed by a fibred submanifold R of $J(M, r)$ and a differential system E of rank r on R which satisfy certain conditions, where

$J(M, r)$ is the Grassmann bundle of r -dimensional contact elements to M . Then a pseudo-projective structure of order 2 of bidegree (n, r) on M simply means a generalized pseudo-projective structure $\mathcal{K}$ of bidegree (n, r) on M such that R is an open fibred submanifold of $J(M, r)$. It is shown that there is naturally associated to every projective structure $\mathcal{D}$ on an n -dimensional manifold M a pseudo-projective structure $\mathcal{K}(\mathcal{D})$ of bidegree $(n, 1)$.

Remark. Let (k, n, r) be a triplet of integers with $k \geq 3$ and $1 \leq r \leq n - 1$. Let us consider the geometry of involutive systems of partial differential equations of the k -th order

$$\frac{\partial^k y^a}{\partial x^{i_1} \dots \partial x^{i_k}} = f_{i_1 \dots i_k}^a((x^j), (y^b), \dots, (\frac{\partial^\ell y^b}{\partial x^{j_1} \dots \partial x^{j_\ell}}), \dots),$$

where the indices $i_1, \dots, j, j_1, \dots$ range over the integers $1, \dots, r$, the indices a, b over the integers $1, \dots, n - r$, and the index ℓ over the integers $1, \dots, k - 1$. In the classical fashion the isomorphism in the geometry is considered with respect to point transformations of the space of the variables x, y^b , if $k = 2$ or $k \geq 3, r \leq n - 2$, and with respect to contact transformations of the same space otherwise. Then we remark that the geometry can be formulated as the geometry of pseudo-projective structures of order k of bidegree (n, r) , being a kind of generalized pseudo-projective structure. Thus we have seen that the generalized pseudo-projective structures constitute a considerably large class of differential equations.

In §3 we introduce the notion of a (generalized) pseudo-projective system. First of all a generalized pseudo-projective system is defined to be a triplet formed by a manifold R and two differential systems, E and F , on it which satisfy the following conditions:

- (i) (R, E, F) is a pseudo-product manifold. Namely $E \cap F = 0$, and both E and F are completely integrable.
- (ii) The system $D(= E + F)$ is not reduced to the zero system, and is non-degenerate.
- (iii) The full derived system of D coincides with the tangent bundle $T(R)$.

Next, a pseudo-projective system (by which we here mean that of order 2 of bidegree (n, r)) is defined to be a generalized pseudo-projective system (R, E, F) such that $\text{rank } (T(R)/D) = n - r$, $\text{rank } E = r$, and $\text{rank } F = r(n - r)$.

Let us consider a (generalized) pseudo-projective structure $\mathcal{X}$ on a manifold M which is expressed by the pair (R, E) . Let F be the vertical tangent bundle of R . Then the triplet (R, E, F) turns out to be a (generalized) pseudo-projective system, which is called associated. Furthermore it is shown that (from the local view-point) the geometry of (generalized) pseudo-projective structures may be represented by the geometry of (generalized) pseudo-projective systems.

Now, a pseudo-product manifold is an abstraction of a submanifold R of a product manifold $M \times N$ equipped with the induced structure (E, F) from the product structure. Since the conditions (ii) and (iii) above are genericness conditions on the system D , we therefore see that a generalized pseudo-projective system essentially means a generic submanifold or a generic relation in a product manifold. This reminds us of the axiomatic projective geometry, i.e., the geometry of relations R in product sets $M \times N$ satisfying the so-called axioms of projective geometry. Accordingly we have known that the geometry of generalized pseudo-projective systems well inherits the basic thought in the axiomatic projective geometry.

Let us now consider a generalized pseudo-projective triplet (M, N, R) , which is a rigorous formulation of a generic relation R in a product manifold $M \times N$. As we have seen, there is associated to the relation R the generalized pseudo-projective system (R, E, F) . By the non-degeneracy assumption on the system $D(= E + F)$ it follows that there is naturally associated to the relation R two kinds of generalized pseudo-projective structure, $\mathcal{X}$ on M and $\mathcal{X}^*$ on N , which are called the dual of each other. This fact assures the existence of the dual $\mathcal{X}^*$ of any generalized pseudo-projective structure $\mathcal{X}$.

§§4 and 5 are concerned with the prolongations of generalized pseudo-projective systems. Let (R, E, F) be a generalized pseudo-projective system, and assume that the system $D(= E + F)$ is regular in the sense that the k -th derived system D^{-k-1} of D is a subbundle of $T(R)$ for each $k \geq 1$. Then there is associated to every point $x \in R$ the symbol algebra $\mathfrak{T}(x)$ of D at x , which means the graded Lie algebra $\mathfrak{t}(x) = \sum_{p \leq 0} \mathfrak{g}_p(x)$ with the natural bracket operation, where $\mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x)$. We have $\mathfrak{g}_{-1}(x) = D(x) = E(x) + F(x)$ (direct sum), and both $E(x)$ and $F(x)$ are abelian subalgebras of $\mathfrak{t}(x)$. Then

the triplet $\mathcal{L}(z)$ formed by $\mathfrak{T}(z)$, $E(z)$ and $F(z)$ is called the symbol algebra of (R, E, F) at z .

The symbol algebra $\mathcal{L}(z)$ can be easily abstracted to yield the notion of an abstract symbol algebra of generalized pseudo-projective system or a non-degenerate pseudo-product FGPA. Let $\mathcal{L}$ be a model (abstract) symbol algebra of generalized pseudo-projective system. Then it is prolonged to a transitive graded Lie algebra $\mathfrak{G}$, and this is shown to be necessarily of finite dimension. Now, a generalized pseudo-projective system (R, E, F) is called of type $\mathcal{L}$, if the symbol algebra $\mathcal{L}(z)$ of the system at each $z \in R$ is isomorphic to $\mathcal{L}$. Then it is shown that through a prolongation process the geometry of generalized pseudo-projective systems of type $\mathcal{L}$ may be represented by the geometry of a class of manifolds with absolute parallelism (cf. Theorem 4.7).

Besides these we study some global problems on generalized pseudo-projective structures, etc. In §2 we show that there is naturally associated to every generalized pseudo-projective structure $\mathcal{X}$ on a manifold M the "arithmetic distance" $\rho(p, q)$ in M . In paragraph 3.7 we moreover propose some global problems on compact pseudo-projective triplets of order 2.

For the rest of the introduction we shall explain our main results which have been hitherto obtained and are not included in the text of this report.

(I) Let $\mathcal{L}$ and $\mathfrak{G}$ be as above, and let $\mathfrak{g} = \sum_p \mathfrak{g}_p$ be the expression of $\mathfrak{g}$. Assume that $\mathfrak{g}$ is simple. Then there is naturally associated to $\mathfrak{g}$ a homogeneous space $G/G^{(0)}$ such that the Lie algebra of G (resp. of $G^{(0)}$) is $\mathfrak{g}$ (resp. $\mathfrak{g}^{(0)} = \sum_{p \geq 0} \mathfrak{g}_p$). Then we have the following

Theorem 1. *There is naturally associated to every generalized pseudo-projective system (R, E, F) of type $\mathcal{L}$ a normal Cartan connection Γ of model space $G/G^{(0)}$ on R . Furthermore Γ can be constructed by elementary operations, i.e., rational operations, differentiations and the inverse function theorem regarded as operations.*

Note that the connection Γ is expressed by the pair of a principal fibre bundle P over the base space R with structure group $G^{(0)}$ and a $\mathfrak{g}$ -valued 1-form ω on P called the connection form. For the normality condition see [20].

Theorem 1 can be derived from the main theorem there. It should be noted that the

same existence theorem, a variant of Theorem 1, can be also obtained in the case where $\mathfrak{g}$ is not semi-simple and $\mathfrak{G}$ satisfies certain additional conditions. These conditions include a condition assuring the existence of harmonic theory in the complex $\{C^q(\mathfrak{G}), \partial\}$ associated with $\mathfrak{G}$, the notation being as in [20].

(II) First of all we remark that the geometry of pseudo-projective structures of order k of bidegree (n, r) may be represented by the geometry of pseudo-projective systems of order k of bidegree (n, r) , being a kind of generalized pseudo-projective system. (This has been already remarked in the case where $k = 2$.)

Let us consider a pseudo-projective system (R, E, F) of order k of bidegree (n, r) . Then it is shown that the symbol algebras $\mathfrak{L}(x)$ of the system at any points $x \in R$ are mutually isomorphic, and essentially depend on the triplet (k, n, r) . Thus we may speak of the model symbol algebra $\mathfrak{L}$ of pseudo-projective system of order k of bidegree (n, r) . Then it is shown that the prolongation $\mathfrak{G}$ of $\mathfrak{L}$ is as follows:

(a) If $k = 2$, then the underlying Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$ is isomorphic to the simple Lie algebra $\mathfrak{sp}(n+1, \mathbb{R})$, and $\mathfrak{G}$ takes the following form: $\mathfrak{g} = \sum_{p=-2}^2 \mathfrak{g}_p$.

(b) If $k = 3$ and $r = n - 1$, then $\mathfrak{g}$ is isomorphic to the simple Lie algebra $\mathfrak{sp}(n, \mathbb{R})$, and $\mathfrak{G}$ takes the following form: $\mathfrak{g} = \sum_{p=-3}^3 \mathfrak{g}_p$.

(c) Otherwise $\mathfrak{g}$ is not semi-simple, and $\mathfrak{g}$ takes the following form: $\mathfrak{g} = \sum_{p=-k}^1 \mathfrak{g}_p$. Furthermore $\mathfrak{G}$ satisfies the additional conditions mentioned above.

By Theorem 1 and its variant we therefore obtain the following

Theorem 2. *There is naturally associated to every pseudo-projective system (R, E, F) of order k of bidegree (n, r) a normal connection Γ of model space $G/G^{(0)}$ on R . Furthermore Γ can be constructed by rational operations and differentiations.*

Now, there are naturally associated to $\mathfrak{G}$ two kinds of homogeneous space, $G/A^{(0)}$ and $G/B^{(0)}$, together with $G/G^{(0)}$ such that $G^{(0)} = A^{(0)} \cap B^{(0)}$. Hence $G/G^{(0)}$ may be identified with a submanifold of $G/A^{(0)} \times G/B^{(0)}$. Then it is shown that the triplet $(G/A^{(0)}, G/B^{(0)}, G/G^{(0)})$ gives a generalized pseudo-projective triplet, and that the generalized pseudo-projective system associated with the relation $G/G^{(0)}$ is a pseudo-projective

system of order k of bidegree (n, r) , which is called standard. It follows that the generalized pseudo-projective structure $\mathcal{X}_0$ on $G/A^{(0)}$ associated with the relation $G/G^{(0)}$ is a pseudo-projective structure of order k of bidegree (n, r) , which is also called standard. Then we notice that the standard structure $\mathcal{X}_0$ can be locally expressed by the system $\partial^k y^a / \partial x^{i_1} \dots \partial x^{i_k} = 0$ or precisely by the associated system of the second order, and that the standard structure $\mathcal{X}_0$ together with the triplet $(G/A^{(0)}, G/B^{(0)}, G/G^{(0)})$ may be characterized in terms of the projective geometry. In more detail we have the following.

(a) The case where $k = 2$. Let P^n be the n -dimensional projective space. Then $\mathcal{X}_0$ may be characterized as the differential equation defining the r -dimensional projective subspaces of P^n . Incidentally we denote by $\Sigma(P^n, r)$ the Grassmann manifold of r -dimensional projective subspaces of P^n , and define a relation in the product manifold $P^n \times \Sigma(P^n, r)$ by

$$R = \{(p, \alpha) \in P^n \times \Sigma(P^n, r) \mid p \subset \alpha\}.$$

We also denote by $\hat{G}$ the group of projective transformations of P^n . Then we have the following natural identifications:

$$\begin{aligned} G &= \hat{G}, \quad G/A^{(0)} = P^n, \quad G/B^{(0)} = \Sigma(P^n, r), \\ G/G^{(0)} &= R = J(P^n, r). \end{aligned}$$

(b) The case where $k = 3$ and $r = n - 1$. Let us consider the $(2n - 1)$ -dimensional projective space P^{2n-1} with the standard contact structure. Then $\mathcal{X}_0$ may be characterized as the differential equation defining the Lagrangean projective subspaces of P^{2n-1} .

(c) The case where $k = 3$, $r \leq n - 2$ or $k \geq 4$. Let L be the universal line bundle over the r -dimensional projective space P^r , and set $V = \mathbb{R}^{r+1}$ and $W = \mathbb{R}^{n-r}$. Let us consider the vector bundle $H = W \otimes L^{-(k-1)}$ over P^r , where $L^{-(k-1)}$ denotes the $(k-1)$ -th power of the dual L^{-1} of L . Set $S = W \otimes S^{k-1}(V^*)$, where $S^{k-1}(V^*)$ stands for the $(k-1)$ -th symmetric product of the dual space V^* of V . As is well known, there is a unique natural injective linear map θ of S into the space of cross sections of H . These being prepared, $\mathcal{X}_0$ may be characterized as the differential equation defining the cross sections $\theta(\alpha)$ ($\alpha \in S$) of H . Incidentally the group G is naturally isomorphic to a certain

subgroup of the affine transformation group $AF(S)$ of S containing the translations, and this group may be characterized as the group of fibre-preserving transformations of H which naturally induce transformations on the cross sections $\theta(\alpha)$.

(III) A profound study has been made of systems of ordinary differential equations of the second order.

(III.1) Let $\mathcal{X}$ be a pseudo-projective structure of order 2 of bidegree $(n, 1)$, and let (R, E, F) be the associated pseudo-projective system of order 2 of bidegree $(n, 1)$. Furthermore let Γ be the normal connection on R associated with (R, E, F) . In the following we assume that $n \geq 3$. For the graded Lie algebra $\mathfrak{G}$ and the connection Γ we shall use the notations as in [20].

The curvature K of the connection Γ is decomposed as follows: $K = \sum_{p=0}^5 K^p$, and hence the harmonic part HK of K as follows: $HK = \sum_{p=0}^5 HK^p$. Calculating the spaces $H^{p,2}(\mathfrak{G})$ of harmonic forms, we see that $K^0 = 0$, $K^1 = HK^1$, and $HK = K^1 + HK^2$. Since HK gives a fundamental system of invariants, it follows that both the invariants K^1 and HK^2 vanish, if and only if the equation $\mathcal{X}$ is equivalent to the standard equation $\mathcal{X}_0$.

Now, $\mathcal{X}$ is said to be of the first restricted type, if the invariant HK^2 vanishes, and of the second restricted type, if the invariant K^1 vanishes. Then we have the following

Theorem 3. (1) $\mathcal{X}$ is of the first restricted type, if and only if $\mathcal{X}$ is equivalent to the differential equation $\mathcal{X}(\mathcal{D})$ of geodesics associated with a projective structure $\mathcal{D}$ on an n -dimensional manifold M .

(2) $\mathcal{X}$ is of the second restricted type, if and only if the dual $\mathcal{X}^*$ of $\mathcal{X}$ is equivalent to the differential equation $\mathcal{Y}(\Psi)$ associated with an M_2 -structure Φ on a $2(n-1)$ -dimensional manifold N .

Here, M_2 means the algebra $M_2(\mathbb{R})$ of square matrices of degree 2, and an M_2 -structure is a kind of geometric structure analogous to a quaternion structure. More precisely, let N be a manifold, and consider the algebra $\text{End}(T_\alpha(N))$ of endomorphisms of the tangent vector space $T_\alpha(N)$ to N at each point $\alpha \in N$. Then an almost M_2 -structure on N is an assignment Φ which assigns to every point $\alpha \in N$ a subalgebra Φ_α of $\text{End}(T_\alpha(N))$

isomorphic to $M_2(\mathbb{R})$ and which is differentiable in a suitable sense. It can be shown that if N admits an M_2 -structure, N is necessarily of even dimension. Clearly an M_2 -structure on a 2-dimensional manifold is a trivial structure. Now, an almost M_2 -structure Φ on N is called an M_2 -structure, either if $\dim N \geq 6$, and there is an affine connection ∇ on N which leaves Φ parallel or if $\dim N = 4$, and there is an affine connection ∇ on N which leaves Φ parallel and of which the curvature satisfies certain conditions. It should be noted that the complexification N^c of a real analytic quaternion manifold N is endowed with an M_2 -structure in the complex analytic category, provided $\dim N \geq 8$.

Example. Let $f(t^1, \dots, t^{n-2})$ be any function of the $n-2$ variables $t^1, \dots, t^{n-2}$, where $n \geq 3$. Let us consider the following system $\mathcal{K}$ of ordinary differential equations of the second order:

$$\begin{cases} \frac{d^2 y^a}{dx^2} = 0 & (1 \leq a \leq n-2), \\ \frac{d^2 y^{n-1}}{dx^2} = f\left(\frac{dy^1}{dx}, \dots, \frac{dy^{n-2}}{dx}\right). \end{cases}$$

Then $\mathcal{K}$ is of the second restricted type, and it is equivalent to the standard system $d^2 y^i / dx^2 = 0$ ($1 \leq i \leq n-1$), if and only if $f(t^1, \dots, t^{n-2})$ is a polynomial of the variables $t^1, \dots, t^{n-2}$ of degree at most 2.

Finally let (R, E, F) be a pseudo-projective system of order 2 of bidegree $(n, 1)$, and let P be the principal fibre bundle of the associated connection Γ . Then we remark that the curvature K of Γ may be represented by $n+1$ homogeneous polynomials of $n-1$ variables of degree 3 with functions on P as coefficients. In particular it follows that if $n = 3$, the invariant HK^2 may be represented by a homogeneous polynomial $\mathcal{F}$ of two variables of degree 4, and the curvature K by $\mathcal{F}$ and two homogeneous polynomials, $\mathcal{G}$ and $\mathcal{H}$, of two variables of degree 3. This fact plays an important role in the classification problem for pseudo-projective systems of order 2 of bidegree (3.1) which will be explained in (III.3).

(III.2) Let $\mathcal{K}$, (R, E, F) and Γ be as in (III.1). We preserve the assumption that $n \geq 3$. Now, let α be the sheaf of germs of local infinitesimal automorphisms of (R, E, F) , being a sheaf of Lie algebras. Since (R, E, F) admits the connection Γ , it follows that

each stalk $\alpha(x)$ of α is of finite dimension or more precisely $\dim \alpha(x) \leq \dim \mathfrak{g} = n^2 + 2n$. From now on we assume that $\dim \alpha(x)$ is locally constant. Note that this assumption is automatically satisfied in the real or complex analytic category.

Theorem 4. *Assume that $\mathcal{X}$ is of the second restricted type. Then the integration of $\mathcal{X}$ at any point $x \in R$ can be carried out by quadratures and the integrations of Lie's differential equations associated with the simple components of the Lie algebra $\alpha(x)$.*

For example Lie's differential equation associated with the simple Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is the Riccati equation

$$\frac{dy}{dx} = a_0(x) + a_1(x)y + a_2(x)y^2.$$

As an immediate consequence of the theorem we have the following

Corollary. *If $\alpha(x) = \{0\}$, then the integration of $\mathcal{X}$ at x can be carried out by elementary operations.*

Theorem 4 compares with Elie Cartan's theorems for certain classes of single ordinary differential equations of the second and third order. (See [4].)

Clearly the integration of $\mathcal{X}$ is reduced to the integration of the system E of rank 1. Let us consider the triplet $(G/A^{(0)}, G/B^{(0)}, G/G^{(0)})$. Then the proof of Theorem 4 is based on the following geometric facts:

(i) There is naturally associated to the normal connection Γ of model space $G/G^{(0)}$ a transverse M_2 -structure $\tilde{\Phi}$ on the foliated manifold (R, E) , and in turn there is associated to $\tilde{\Phi}$ a normal transverse connection $\tilde{\Gamma}$ of model space $G/B^{(0)}$ on (R, E) . Furthermore both $\tilde{\Phi}$ and $\tilde{\Gamma}$ can be constructed by rational operations and differentiations, from which follows that $\tilde{\Gamma}$ can be constructed from $\mathcal{X}$ by the same operations.

(ii) The stalk $\alpha(x)$ of α is naturally isomorphic to the stalk $\tilde{\alpha}(x)$ of the reduced sheaf $\tilde{\alpha}$ of germs of local infinitesimal automorphisms of $\tilde{\Gamma}$. Here, $\tilde{\alpha}$ means the quotient sheaf $\tilde{\alpha}/\mathcal{E}$, where $\tilde{\alpha}$ is the sheaf of germs of local infinitesimal automorphisms of $\tilde{\Gamma}$, and $\mathcal{E}$ is the sheaf of germs of local cross sections of E , being a subsheaf of ideals of $\tilde{\alpha}$.

Note that these facts have been inspired by (2) of Theorem 3.

Finally we add the following

Theorem 4'. *Assume that α is transitive at a point $x \in R$, i.e., $T_x(R) = \{X_x | X \in \alpha(x)\}$. Then the integration of $\mathcal{X}$ at x can be carried out by quadratures and the integrations of Lie's differential equations associated with the simple components of $\alpha(x)$.*

This fact is a consequence of a general theorem on the equivalence of linear group structures of finite type. Accordingly the proofs of Theorems 4 and 4' are essentially different.

(III.3) The final explanation in (III) is concerned with the (local) classification problem for systems of ordinary differential equations of the second order for two independent variables. Clearly the problem is reduced (locally) to classify the pseudo-projective systems of order 2 of bidegree (3.1). In the following these systems will be simply called pseudo-projective systems. Moreover everything will be considered in the complex analytic category.

Let $\mathcal{R}$ be a pseudo-projective system, and let us consider the invariants K^1 and HK^2 . From now on the underlying manifold R of $\mathcal{R}$ is assumed to be connected. Then $\mathcal{R}$ is called of class (A), if $K^1 = 0$, of class (B), if $HK^2 = 0$, and of class (C), if $K^1 \neq 0$ and $HK^2 \neq 0$.

Let α be the sheaf of germs of local infinitesimal automorphisms of $\mathcal{R}$. As we have remarked, the dimension of each stalk $\alpha(x)$ of α is constant, whence the stalks $\alpha(x)$ are mutually isomorphic as Lie algebras. This being said, $\alpha(\mathcal{R})$ denotes a model Lie algebra isomorphic to $\alpha(x)$. Now, $\mathcal{R}$ is said to be transitive, if α is transitive at each $x \in R$, and to be intransitive, otherwise.

For each $x \in R$ we denote by $\alpha^0(x)$ the isotropy subalgebra of $\alpha(x)$: $\alpha^0(x) = \{X \in \alpha(x) | X_x = 0\}$. We have

$$\dim(\alpha(x)/\alpha^0(x)) \leq \dim R = 5.$$

Then we consider the following conditions on the system $\mathcal{R}$:

- (i) $\dim \alpha^0(x) \geq 1$ at each $x \in R$,
- (ii) $\dim \alpha^0(x)$ is constant, and $\dim(\alpha(x)/\alpha^0(x)) = 5$ or 4 . In other words, either α is transitive or all the orbits of α are of codimension 1.

(A) We have completely classified the pseudo-projective systems of class (A) satisfying conditions (i) and (ii). In more detail we have obtained the following:

(1) These systems are classified into 7 transitive systems $\mathcal{R}'$ and 8 intransitive systems $\mathcal{R}''$.

(2) The dimensions of the Lie algebras $\alpha(\mathcal{R}')$ for the transitive systems $\mathcal{R}'$ are 15, 9, 8, 7, 6, 6, 6, and the dimensions of the Lie algebras $\alpha(\mathcal{R}'')$ for the intransitive systems $\mathcal{R}''$ are 6, 5, 5, 5, 5, 5, 5, 5.

(3) The transitive system $\mathcal{R}'$ with $\dim \alpha(\mathcal{R}') = 7$ is parameterized by an arbitrary constant, the intransitive system $\mathcal{R}''$ with $\dim \alpha(\mathcal{R}'') = 6$ by an arbitrary function $f(t)$ ($\neq 0$) of one variable t , and three of the intransitive systems $\mathcal{R}''$ with $\dim \alpha(\mathcal{R}'') = 5$ by arbitrary constants. Furthermore one of the intransitive system $\mathcal{R}''$ with $\dim \alpha(\mathcal{R}'') = 5$ is parameterized by the solutions of the single ordinary differential equation

$$4 \cdot f f'' - f'^2 + 20f + 9t^2 + 36 = 0,$$

and another of the same systems by the solutions of the ordinary differential equation

$$f'^3 + 27f(f' + 3t) = 0 \quad \text{with} \quad f \neq 0, -\frac{9}{4}t^2.$$

Now, let $\mathcal{R}$ be a pseudo-projective system of class (A), and assume that $\mathcal{R}$ is not locally equivalent to any one of the 15 ($= 7 + 8$) classified systems. Then it is shown that $\dim \alpha(\mathcal{R}) \leq 5$.

(B) Let $\mathcal{R}$ be a pseudo-projective system of class (B). Then we proved that $\dim \alpha(\mathcal{R}) \leq 8$, and classified the systems $\mathcal{R}$ with $\dim \alpha(\mathcal{R}) = 8$ or 7. The result shows that there is a unique system in either case, and it is transitive. Furthermore the system $\mathcal{R}'$ with $\dim \alpha(\mathcal{R}') = 7$ is parametrized by an arbitrary constant.

(C) Let $\mathcal{R}$ be a pseudo-projective system of class (C). Then we proved that $\dim \alpha(\mathcal{R}) \leq 7$, and that there is a unique system $\mathcal{R}'$ with $\dim \alpha(\mathcal{R}') = 7$, and it is transitive.

Let us now consider a pseudo-projective structure $\mathcal{X}$, and let $\mathcal{R}$ be the associated pseudo-projective system. By Theorems 4 and 4' combined with the results above we have the following:

(1) If $\dim \mathfrak{a}(\mathcal{R}) = 15$, then $\mathcal{X}$ is of class (A), and can be locally expressed by the system

$$\frac{d^2 y_1}{dx^2} = \frac{d^2 y_2}{dx^2} = 0.$$

Furthermore the Lie algebra $\mathfrak{a}(\mathcal{R})$ is isomorphic to the simple Lie algebra $\mathfrak{sl}(4, \mathbb{C})$, and hence the integration of $\mathcal{X}$ can be carried out by the integration of the generalized Riccati equation of degree 4, i.e., Lie's differential equation associated with $\mathfrak{sl}(4, \mathbb{C})$.

(2) If $\dim \mathfrak{a}(\mathcal{R}) < 15$, we have $\dim \mathfrak{a}(\mathcal{R}) \leq 9$. If $\dim \mathfrak{a}(\mathcal{R}) = 9$, then $\mathcal{X}$ is of class (A), and can be locally expressed by the system

$$\frac{d^2 y_1}{dx^2} = 0, \quad \frac{d^2 y_2}{dx^2} = \frac{1}{3} \left(\frac{dy_1}{dx} \right)^3.$$

(Compare the system given in Example, (III.2).) Furthermore the semi-simple part of $\mathfrak{a}(\mathcal{R})$ is isomorphic to the simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, and hence the integration of $\mathcal{X}$ can be carried out by quadratures and the integration of the Riccati equation.

(3) If $\dim \mathfrak{a}(\mathcal{R}) = 8$, there occur the following two cases: (i) $\mathcal{X}$ is of class (A), and can be locally expressed by the system

$$\frac{d^2 y_1}{dx^2} = 0, \quad \frac{d^2 y_2}{dx^2} = 2 \frac{dy_1}{dx} \left(\frac{dy_2}{dx} \right)^2 / \left(y_2 \frac{dy_1}{dx} - 1 \right).$$

Furthermore $\mathfrak{a}(\mathcal{R})$ is isomorphic to the simple Lie algebra $\mathfrak{sl}(3, \mathbb{C})$, and hence the integration of $\mathcal{X}$ can be carried out by the integration of the generalized Riccati equation of degree 3.

(ii) $\mathcal{X}$ is of class (B), and can be locally expressed by the system

$$\frac{d^2 y_1}{dx^2} = 0, \quad \frac{d^2 y_2}{dx^2} = \frac{2}{3} \left(y_1 - x \frac{dy_1}{dx} \right).$$

Furthermore $\mathfrak{a}(\mathcal{R})$ is solvable, and hence the integration of $\mathcal{X}$ can be carried out by quadratures.

And so forth.

(IV) In physics it is known that the differential equation of the orbits of rays in the space-time or generally in an indefinite Riemannian space is a conformal invariant, and hence the equation is naturally associated with the conformal structure on the space. In our framework of the geometry of generalized pseudo-projective systems, we have obtained

a thorough generalization of an indefinite conformal structure as well as the associated equation of the orbits of rays. Indeed we have introduced the notions of a pseudo-conformal structure R of degree n and a co-pseudo-conformal structure $\bar{R}$ of degree n , both defined on n -dimensional manifolds M ($n \geq 3$), which have the following properties (1)~(4):

(1) A pseudo-conformal structure R of degree n on M is a fibred submanifold of codimension 1 of $J(M, 1)$, the Grassmann bundle of 1-dimensional contact elements to M . Hence it may be regarded as a differential equation for 1-dimensional submanifolds of M , and can be locally expressed by a single ordinary differential equation of the first order:

$$f(x, (y^j), (\frac{dy^j}{dx})) = 0.$$

Here and in the following the indices i and j range over the integers $1, \dots, n-1$. Note that a pseudo-conformal structure may be regarded as a generalization of the light cone or the Monge cone in an indefinite conformal structure, while a co-pseudo-conformal structure as a generalization of the co-light cone.

(2) A co-pseudo-conformal structure $\bar{R}$ of degree n on M is a fibred submanifold of codimension 1 of $J(M, n-1)$, the Grassmann bundle of contact elements of codimension 1 to M . Hence it may be regarded as a differential equation for submanifolds of codimension 1 of M , and can be expressed by a single partial differential equation of the first order:

$$g((x^j), y, (\frac{\partial y}{\partial x^j})) = 0.$$

It should be noted that a co-pseudo-conformal structure $\bar{R}$ is "generic" among the fibred submanifolds of codimension 1 of $J(M, n-1)$, while a pseudo-conformal structure is not generic at all.

(3) There is a natural one-to-one correspondence between the pseudo-conformal structures R of degree n on M and the co-pseudo-conformal structures $\bar{R}$ of degree n on M , and the correspondences $R \rightarrow \bar{R}$ and $\bar{R} \rightarrow R$ are kinds of Bäcklund transformation. Furthermore if $\bar{R}$ corresponds to R , then there is a natural fibre-preserving diffeomorphism δ_R of R onto $\bar{R}$ which induces the identity transformation of M .

(4) Let R be a pseudo-conformal structure of degree n on M , and $\bar{R}$ the associated co-pseudo-conformal structure of degree n on M . Let $\bar{D}'$ be the contact system of $\bar{R}$, and

let $\tilde{E}$ be the Cauchy characteristic system of $\tilde{D}'$, which is a subbundle of $\tilde{D}'$ of rank 1. We now denote by E the pull-back of $\tilde{E}$ to R by δ_R^{-1} . Then the pair $\mathcal{K}$ formed by R and E gives a generalized pseudo-projective structure on M , and hence may be regarded as a differential equation for 1-dimensional submanifolds of M , which can be expressed by a system of ordinary differential equations of the following form:

$$\begin{cases} \frac{d^2 y_i}{dx^2} = f^i(x, (y^j), (\frac{dy^j}{dx})), \\ f(x, (y^j), (\frac{dy^j}{dx})) = 0. \end{cases}$$

The equation $\mathcal{K}$ just generalizes the equation of the orbits of rays associated with an indefinite conformal structure.

(5) Let F be the vertical tangent bundle of the fibred manifold R . Then the system (R, E, F) gives a generalized pseudo-projective system, which can be abstracted to yield the notion of a pseudo-conformal system of degree n . Thus the geometry of the equations $\mathcal{K}$ may be represented by the geometry of pseudo-conformal systems of degree n . Then it follows from Theorem 1 that there is naturally associated to every pseudo-conformal system (R, E, F) of degree n a normal Cartan connection Γ on R . We notice that the model symbol algebra $\mathcal{L}$ admits the index r ($2 \leq 2r \leq n$) as an invariant, and that the prolongation $\mathcal{G}$ of $\mathcal{L}$ takes the form $\mathfrak{g} = \sum_{p=-3}^3 \mathfrak{g}_p$, and $\mathfrak{g}$ is isomorphic to the simple Lie algebra $so(r+1, n-r+1)$.

At this point we mention that the geometry of the equations $\mathcal{K}$, to a considerable extent, resembles the geometry of systems of ordinary differential equations of the second order. Therefore most results in the latter geometry are expected to find their counterparts in the former geometry. For example we shall have the notion of a contact M_2 -structure in connection with the dual $\mathcal{K}^*$ of $\mathcal{K}$, which will correspond to the notion of an M_2 -structure. Finally we add that the dual (R, F, E) of a pseudo-conformal system (R, E, F) of degree 3 means a pseudo-projective system of order 3 of bidegree (2.1), which implies that the geometry of pseudo-conformal structures of degree 3 is closely related to the geometry of single ordinary differential equations of the third order.

§1. Preliminaries — Differential systems and the
Grassmann bundles —

1.1. General remarks on terminologies and notations.

(a) As usual R denotes the field of real numbers, and C the field of complex numbers.

(b) Let M be a manifold. As usual $T_x(M)$ denotes the tangent vector space to M at $x \in M$, and $T(M)$ the tangent vector bundle of M : $T(M) = \bigcup_x T_x(M)$. $T^*(M)$ denotes the cotangent bundle of M : $T^*(M) = \bigcup_x T_x^*(M)$, $T_x^*(M)$ being the dual space of $T_x(M)$. Let ϕ be a map of a manifold M to a manifold M' . ϕ_* denotes the differential of ϕ , which is a bundle homomorphism of $T(M)$ to $T(M')$. Given a differential form ω' on M' , $\phi^*\omega'$ denotes the pull-back of ω' by ϕ .

(c) Let E be a vector bundle over a manifold M . For $x \in M$, $E(x)$ or E_x denotes the fibre of E at x . $\text{rank } E$ denotes the fibre dimension of E : $\text{rank } E = \dim E(x)$. Given an open set U of M , $\Gamma(U, E)$ denotes the space of all local cross sections of E defined on U .

(d) Let $\mathcal{E}$ be a sheaf over a manifold M . For $x \in M$, $\mathcal{E}(x)$ or $\mathcal{E}_x$ denotes the stalk of $\mathcal{E}$ at x . Given an open set U of M , $\Gamma(U, \mathcal{E})$ denotes the space of local cross sections of $\mathcal{E}$ defined on U . Let E be a vector bundle over a manifold M , and $\mathcal{E}$ the sheaf of germs of local cross sections of E . Then a germ of a stalk $\mathcal{E}(x)$ will be often confounded with a local cross section s of E defined on a neighborhood of x which represents the germ.

(e) An imbedding of a manifold M to a manifold M' is an injective immersion ϕ of M to M' which gives a homeomorphism of M onto the image $\phi(M)$. As usual a submanifold of a manifold M' is

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a manifold M such that M is a subset of M' , and the inclusion map i of M to M' is an immersion. An imbedded submanifold of M' is a submanifold M of M' such that i is an imbedding, i.e., the topology of M coincides with the relative topology.

(f) Let π be a map of a manifold R to a manifold M . Then R is called a fibred manifold over M with projection π , if the following conditions are satisfied: (i) π is surjective, and (ii) π is a submersion, i.e., the differential π_* of π maps $T_x(R)$ onto $T_{\pi(x)}(M)$ for each $x \in R$. Let R be a fibred manifold over M with projection π . Let R' be a submanifold of R , and π' the restriction of π to R' . Then R' is called a fibred submanifold of R , if R' becomes a fibred manifold over M with projection π' .

(g) A local diffeomorphism of a manifold M to a manifold M' is a diffeomorphism ϕ of an open set of M onto an open set of M' . Given a point (x_0, x'_0) of $M \times M'$, a local diffeomorphism of M to M' at (x_0, x'_0) is a local diffeomorphism ϕ of M to M' such that x_0 is in the domain of ϕ and $\phi(x_0) = x'_0$.

1.2. Differential systems.

(A) Differential systems. A differential system on a manifold R is a subbundle D of the tangent vector bundle $T(R)$ of R .

Let D be a differential system on a manifold R . Set $\sigma = \text{rank}(T(R)/D)$, and let x_0 be any point of R . Then there are 1-forms θ^a ($1 \leq a \leq \sigma$) defined on a neighborhood O of x_0 such that D is defined there by the system of Pfaffian equations:

$$\theta^a = 0,$$

that is, the restriction $D|O$ of D to O consists of all $X \in T(O)$

such that $\theta^a(X) = 0$. Note that θ^a are automatically linearly independent at each point of O .

Let R and R' be two manifolds, and let $(D_i)_{i \in I}$ and $(D'_i)_{i \in I}$ be finite families of differential systems on them respectively. Then an isomorphism of $(R, (D_i))$ to $(R', (D'_i))$ is a diffeomorphism ϕ of R onto R' such that its differential ϕ_* maps D_i onto D'_i for each i . We remark that the notion of isomorphism naturally gives rise to the notions of local isomorphism, of (local) automorphism, and of (local) infinitesimal automorphism:

A local isomorphism of $(R, (D_i))$ to $(R', (D'_i))$ is a diffeomorphism ϕ of an open set O of R onto an open set O' of R' which gives an isomorphism of $(O, (D_i|_O))$ to $(O', (D'_i|_{O'}))$. Let (x_0, x'_0) be a point of $R \times R'$. A local isomorphism of $(R, (D_i))$ to $(R', (D'_i))$ at (x_0, x'_0) is a local diffeomorphism ϕ of R to R' at (x_0, x'_0) which gives a local isomorphism of $(R, (D_i))$ to $(R', (D'_i))$. A (local) automorphism of $(R, (D_i))$ is a (local) isomorphism of $(R, (D_i))$ to itself. A (local) infinitesimal automorphism of $(R, (D_i))$ is a (local) vector field X on R such that for any local 1-parameter group (ϕ_t) of local transformations of R generated by X , each ϕ_t gives a local automorphism of $(R, (D_i))$. It is easy to see that a local vector field X is a (local) infinitesimal automorphism of $(R, (D_i))$, if and only if for any i and any local cross section Y of D_i the bracket $[X, Y]$ is a local cross section of D_i .

In this book we shall be concerned with various kinds of manifolds with family of differential systems, e.g., product manifolds, pseudo-product manifolds and (generalized) pseudo-

projective systems.

Finally let ϕ be a map of a manifold to a manifold R' , and let D' be a differential system on R' . Let us consider the inverse image $\phi_*^{-1}(D')$ of D' by the differential ϕ_* . Set $D = \phi_*^{-1}(D')$, and $D(x) = D \cap T_x(R)$ for each $x \in R$. Then $D(x)$ is a subspace of $T_x(R)$, and $D = \bigcup_x D(x)$. Now let x_0 be any point of R , and suppose that D' is defined, on a neighborhood O' of $\phi(x_0)$, by a system of Pfaffian equations: $\theta'^a = 0$ ($1 \leq a \leq \sigma$), where $\sigma = \text{rank}(T(R')/D')$. Then D is defined, on $\phi^{-1}(O')$, by the system of Pfaffian equations: $\phi^*\theta'^a = 0$.

Thus D is, so to speak, a differential system possibly with singularities, which is called the pull-back of D' by ϕ . Especially the pull-back of the zero differential system, 0 , on R' by ϕ means the kernel of the differential ϕ_* , and we prefer to use this last terminology. Furthermore, if R is a submanifold of R' , and ϕ is the inclusion map of R to R' , D is particularly called the pull-back of D' to R .

(B) The torsion of a differential system. Let R be a manifold, and let D be a differential system on it. Let x be any point of R . As is easily verified, there is a unique skew-symmetric bilinear map γ_x of $D(x) \times D(x)$ to $T(x)/D(x)$ such that

$$\gamma_x(X_x, Y_x) = \pi([X, Y]_x),$$

where $T = T(R)$, π is the projection of T onto T/D , and X and Y are any local cross sections of D defined on a neighborhood of x . The bilinear map γ_x thus obtained is called the torsion of D at x .

Now let x_0 be any point of R , and suppose that D is defined, on a neighborhood of x_0 , by a system of Pfaffian equations:

$\theta^a = 0$ ($1 \leq a \leq \sigma$), where $\sigma = \text{rank}(T(R)/D)$. Let O be a sufficiently small neighborhood of x_0 , and take 1-forms ω^i ($1 \leq i \leq \rho$) on O such that the 1-forms θ^a and ω^i are linearly independent at each point $x \in O$, where $\rho = \text{rank } D$. Then there are unique functions T_{ij}^a ($1 \leq i, j \leq \rho$) on O such that $T_{ij} = -T_{ji}^a$, and

$$d\theta^a \equiv \frac{1}{2} \sum_{i,j} T_{ij}^a \omega^i \wedge \omega^j \pmod{\theta^1, \dots, \theta^\sigma}.$$

It is easy to see that

$$\theta^a(\gamma_x(X, Y)) = -d\theta^a(X, Y) = - \sum_{i,j} T_{ij}^a(x) \omega^i(X) \omega^j(Y),$$

where x is any point of O , and X and Y are any vectors of $D(x)$.

By definition the system D is completely integrable, if for any local cross sections X and Y of D (with the same domain) the bracket $[X, Y]$ is likewise a cross section of D . Clearly D is completely integrable, if and only if the torsion γ_x vanishes at each $x \in R$.

The system D is called non-degenerate, if the torsion γ_x is non-degenerate at each $x \in R$, i.e., the condition " $x \in R$, $X \in D(x)$, and $\gamma_x(X, Y) = 0$ for all $Y \in D(x)$ " implies $X = 0$. In particular D is called a contact structure on R , if D is non-degenerate, $\text{rank}(T(R)/D) = 1$, and $\dim R > 1$. A manifold R equipped with a contact structure D is called a contact manifold. Note that a contact manifold is necessarily odd dimensional.

(C) Integral manifolds and integral elements of a differential system. Let R be a manifold, and let D be a differential system on it. An integral manifold of D is a connected submanifold N of R such that $T_x(N) \subset D(x)$ at each $x \in R$. In particular if D is completely integrable, an integral manifold of D will always

mean that of the maximal dimension, unless otherwise stated.

If N is an integral manifold of D , and x is a point of N , the torsion γ_x of D at x clearly vanishes on the tangent space $T_x(N)$. Now, let z be a contact element to R at a point x . Then z is called an integral element of D , if z is a subspace of $D(x)$, and the torsion γ_x vanishes on z .

(D) Completely integrable systems. Let R be a manifold, and let E be a completely integrable (differential) system on it. A first integral of E is a function f defined on an open set O of R which satisfies the following differential equation:

$$Xf = 0 \text{ for any } X \in E(x) \text{ and any } x \in O.$$

Let x_0 be any point of R . Then the famous Frobenius's theorem states that there are first integrals f^a ($1 \leq a \leq \sigma$) of E defined on a neighborhood of x_0 such that the differentials df^a are linearly independent at x_0 , where $\sigma = \text{rank}(T(R)/E)$. The system (f^a) is called a fundamental system of first integrals of E at x_0 .

Let R be a fibred manifold over a manifold M with projection π . Let V denote the kernel of the differential π_* : $V = \pi_*^{-1}(0)$. Clearly V is a subbundle of $T(R)$, and is completely integrable, which is called the vertical tangent bundle of the fibred manifold R .

Let R and E be as in the outset. We denote by R/E the set of all (maximal) integral manifolds of E , and by π the projection of R onto R/E . Now, R is said to be regular with respect to E , if the set R/E becomes naturally a manifold or more precisely if R/E is equipped with a differentiable structure so that (i) R becomes a fibred manifold over R/E with projection π , and (ii)

the vertical tangent bundle of this fibred manifold coincides with E . Note that there is at most one differentiable structure on R/E satisfying the first condition, and that if R satisfies the second axiom of countability the first condition implies the second.

Finally we add the following

Lemma 1.1. Let R be a manifold, and let (E_i) be a finite family of completely integrable systems on it. Then every point x_0 of R has a neighborhood O which is regular with respect to E_i for each i .

This fact follows easily from Frobenius's theorem and Lemma 1.2 below.

For any positive number ϵ we denote by $S^{n-1}(\epsilon)$ the sphere of the standard n -dimensional Euclidean space R^n of radius ϵ centred at the origin o .

Lemma 1.2. Let ϕ be any local transformation of R^n at (o,o) , i.e., such that o is in the domain of ϕ and $\phi(o) = o$. If ϵ is sufficiently small, then the image $\phi(S^{n-1}(\epsilon))$ of $S^{n-1}(\epsilon)$ by ϕ becomes a convex hypersurface of R^n .

Indeed it can be shown that if ϵ is sufficiently small, the second fundamental form of the hypersurface $\phi(S^{n-1}(\epsilon))$ of R^n becomes definite everywhere, meaning that $\phi(S^{n-1}(\epsilon))$ is a convex hypersurface of R^n .

(E) The Cauchy characteristic system of a differential system. Let R be a manifold, and let D be a differential system on it. For each $x \in R$ we denote by $E(x)$ the null space of the torsion γ_x of D at x :

$$E(x) = \{X \in D(x) \mid \gamma_x(X, Y) = 0 \text{ for all } Y \in D(x)\},$$

and set $E = \bigcup_x E(x)$. It is clear that E is a subbundle of $T(R)$, if and only if $\dim E(x)$ is constant. We now assert that E is completely integrable, provided E is a subbundle of $T(R)$. Indeed let x be any point of R . Let X and Y be any local cross sections of E defined on neighborhoods of x , and Z any local cross section of D defined on a neighborhood of x . Clearly $[X, Y]_x$, $[X, Z]_x$ and $[Y, Z]_x$ are all in $D(x)$. Since

$$[[X, Y], Z] = [[X, Z], Y] + [X, [Y, Z]],$$

it follows that

$$\gamma_x([X, Y]_x, Z_x) = \gamma_x([X, Z]_x, Y_x) + \gamma_x(X_x, [Y, Z]_x) = 0,$$

showing that $[X, Y]_x \in E(x)$. This proves our assertion.

Thus E is, so to speak, a completely integrable system possibly with singularities, which is denoted by $ch(D)$, and is called the Cauchy characteristic system of D .

Now, assume that E is a subbundle of $T(R)$, and that R is regular with respect to E . Set $M = R/E$, and let π be the projection of R onto M .

Lemma 1.3.(E.Cartan). There is a unique differential system $\bar{D}$ on M such that the given system D is the pull-back of $\bar{D}$ by π : $D = \pi_*^{-1}(\bar{D})$. Furthermore the system $\bar{D}$ is non-degenerate.

Proof. We first recall that a local vector field X on R is a local infinitesimal automorphism of (R, D) , if and only if for any local cross section Y of D $[X, Y]$ is a local cross section of D . Then we see that a local vector field X on R is a local cross section of E , if and only if it is at once a local cross section

of D and a local infinitesimal automorphism of (R, D) . Now, let $\mathcal{E}$ be the sheaf of all germs of local cross sections X of E , which becomes naturally a sheaf of Lie algebras. Clearly each fibre N of the fibred manifold R over M is an orbit of $\mathcal{E}$: $T_x(N) = E(x) = \{X_x \mid X \in \mathcal{E}(x)\}$ for each $x \in N$.

Let Γ be the pseudo-group of local transformations of R generated by $\mathcal{E}$ (cf. [15]). For any local cross section X of $\mathcal{E}$ let (ϕ_t) be any local 1-parameter group of local transformations of R generated by X . Then Γ is generated by the local transformations of the form ϕ_t . From the observations above we easily obtain the following: (i) Every $\phi \in \Gamma$ is a local automorphism of (R, D) , (ii) Γ leaves each fibre of R invariant, i.e., if $\phi \in \Gamma$, and x is a point in the domain of ϕ , then the two points x and $\phi(x)$ are in the same fibre of R , and (iii) Γ acts transitively on each fibre of R , i.e., if x and x' are two points in the same fibre of R , then there is $\phi \in \Gamma$ such that $x' = \phi(x)$. It is now easy to see that there is a unique differential system $\bar{D}$ on M such that $D = \pi_*^{-1}(\bar{D})$, proving the first assertion.

Clearly the differential $\pi_*: T(R) \rightarrow T(M)$ naturally induces a bundle homomorphism, say π_* , of $T(R)/D$ onto $T(M)/\bar{D}$, which gives an isomorphism of $T_x(R)/D(x)$ onto $T_{\pi(x)}(M)/\bar{D}(\pi(x))$ for each $x \in R$. Now, let x be any point of R , and let us consider the torsion γ_x of D at x as well as the torsion $\bar{\gamma}_{\pi(x)}$ of $\bar{D}$ at $\pi(x)$. Then we can easily show that

$$\bar{\gamma}_{\pi(x)}(\pi_*(X), \pi_*(Y)) = \pi_*(\gamma_x(X, Y)), \quad X, Y \in D(x),$$

from which follows immediately that $\bar{\gamma}_{\pi(x)}$ is non-degenerate.

Hence the system $\bar{D}$ is non-degenerate, proving the second assertion.

(F) The derived systems of a differential system. Let R be a manifold. For simplicity we denote by T the tangent bundle $T(R)$ of R . We also denote by $\mathcal{O}$ the sheaf of germs of local functions on M , which is a sheaf of rings. Furthermore we denote by $\mathcal{T}$ the sheaf of germs of local vector fields on M or of local cross sections of T , which is a sheaf of Lie algebras, and is a module over $\mathcal{O}$ or an $\mathcal{O}$ -module.

Now, let D be a differential system on R . We denote by $\mathcal{D}$ the sheaf of germs of local cross sections of D , which is an $\mathcal{O}$ -submodule of the $\mathcal{O}$ -module $\mathcal{T}$. Let x be any point of R . For any integer $p < 0$ we define a subspace $\mathcal{D}^p(x)$ of $\mathcal{T}(x)$ inductively as follows:

$$\mathcal{D}^{-1}(x) = \mathcal{D}(x),$$

$$\mathcal{D}^p(x) = [\mathcal{D}^{p+1}(x), \mathcal{D}(x)] + \mathcal{D}^{p+1}(x) \quad \text{for } p \leq -2.$$

We then set $\mathcal{D}^p = \bigcup_x \mathcal{D}^p(x)$. Clearly $\mathcal{D}^p$ are $\mathcal{O}$ -submodules of $\mathcal{T}$, and

$$\mathcal{T} \supset \dots \supset \mathcal{D}^p \supset \dots \supset \mathcal{D}^{-2} \supset \mathcal{D}^{-1} = \mathcal{D}.$$

We can easily prove the following

Lemma 1.4. $[\mathcal{D}^p(x), \mathcal{D}^q(x)] \subset \mathcal{D}^{p+q}(x)$.

For each $x \in R$ let us now consider the natural map of $\mathcal{T}(x)$ onto $T(x)$, i.e., the map which maps every $X \in \mathcal{T}(x)$ to $X_x \in T(x)$. Then we denote by $D^p(x)$ the image of $\mathcal{D}^p(x)$ by the natural map, and set $D^p = \bigcup_x D^p(x)$. Clearly we have

$$T \supset \dots \supset D^p \supset \dots \supset D^{-1} = D.$$

Moreover it is clear that D^p becomes a subbundle of T , if and

only if $\dim D^p(x)$ is constant. Thus D^p is a differential system possibly with singularities. For any positive integer k the system D^{-k-1} will be called the k -th derived system of D , and the sheaf $\mathcal{D}^{-k-1}$ the k -th derived sheaf of $\mathcal{D}$.

Let us now consider the torsion γ_x of D at any point $x \in R$. Then we remark that the factor space $D^{-2}(x)/D(x)$ is the subspace of $T(x)/D(x)$ spanned by the vectors $\gamma_x(X, Y)$, where $X, Y \in D(x)$.

Now, we denote by $D^{-\infty}$ the union $\bigcup_{p \geq 0} D^p$, which is an $\mathcal{O}$ -submodule of $\mathcal{T}$. For each $x \in R$ we also denote by $D^{-\infty}(x)$ the image of $\mathcal{D}^{-\infty}(x)$ by the natural map of $\mathcal{T}(x)$ onto $T(x)$, and set $D^{-\infty} = \bigcup_x D^{-\infty}(x)$. By Lemma 1.4 we see that $D^{-\infty}(x)$ is a subalgebra of the Lie algebra $\mathcal{T}(x)$. Moreover it is clear that $D^{-\infty}$ becomes a subbundle of $T(R)$, if and only if $\dim D^{-\infty}(x)$ is constant. Thus $D^{-\infty}$ is a completely integrable system possibly with singularities. The system $D^{-\infty}$ will be called the full derived system of D , and the sheaf $\mathcal{D}^{-\infty}$ the full derived sheaf of $\mathcal{D}$.

For example consider the case where D is a contact structure. Then it is clear that the first derived system of D coincides with T : $T = D^{-\infty} = D^{-2}$.

1.3. The Grassmann bundles. Let V be an n -dimensional vector space. Then $\text{Gr}(V, r)$ will denote the Grassmann manifold of r -dimensional subspaces of V , which is a compact, connected manifold of dimension $r(n-r)$.

Now, let (n, r) be a pair of integers with $1 \leq r \leq n-1$. Let M be an n -dimensional manifold. We denote by $J(M, r)$ the set of all r -dimensional contact elements to M or in other words,

$$J(M,r) = \bigcup_p \text{Gr}(T_p(M),r).$$

We then denote by π the natural map of $J(M,r)$ onto M : $\pi(z) = p$, if $z \in \text{Gr}(T_p(M))$. As is easily seen, $J(M,r)$ is a fibre bundle over the base space M with projection π (associated with the frame bundle of M), which is called the Grassmann bundle of r -dimensional contact elements to M . Note that $J(M,1)$ and $J(M,n-1)$ may be regarded as the projective bundles associated with the vector bundles $T(M)$ and $T^*(M)$ respectively.

In the following the indices i,j range over the integers $1, \dots, r$, and the indices a,b over the integers $1, \dots, n-r$.

Let z_0 be any point of $J(M,r)$, and let (x^i, y^a) be any coordinate system of M at $\pi(z_0)$ such that the differentials dx^i are linearly independent, restricted to the contact element z_0 . Such a coordinate system will be called adapted to z_0 . Now, let U be the domain of the coordinate system. We denote by $\tilde{U}$ the subset of $J(M,r)$ consisting of all $z \in J(M,r)$ such that $\pi(z) \in U$, and the coordinate system is adapted to z . Clearly $\tilde{U}$ is an open set of $J(M,r)$, $z_0 \in \tilde{U}$, and $\pi(\tilde{U}) = U$. Moreover it is clear that there are unique functions p_i^a on $\tilde{U}$ such that

$$dy^a(X) = \sum_i p_i^a(z) dx^i(X),$$

where z is any point of $\tilde{U}$, and X is any vector in the contact element z . Then we see that the functions $x^i (= x^i \circ \pi)$, $y^a (= y^a \circ \pi)$ and p_j^b together form a coordinate system of $J(M,r)$, which is defined on $\tilde{U}$. The coordinate system, thus obtained, is called the canonical coordinate system of $J(M,r)$ at z_0 corresponding to the coordinate system (x^i, y^a) of M or simply a canonical coordinate

system of $J(M, r)$ at z_0 .

Consider the following three special cases: (A) $r = 1$; (B) $r = n-1$; (C) $n = 2$, and hence $r = 1$. In case (A) the canonical coordinate system will be described as $(x, y^1, \dots, y^{n-1}, p^1, \dots, p^{n-1})$ by setting $x = x^1$ and $p^a = p_1^a$, and in case (B) as $(x^1, \dots, x^{n-1}, y, p_1, \dots, p_{n-1})$ by setting $y = y^1$ and $p_i = p_i^1$, and in case (C) as (x, y, p) by setting $x = x^1$, $y = y^1$ and $p = p_1^1$.

For any $z \in J(M, r)$ we denote by $C(z)$ the subspace of $T_z(J(M, r))$ consisting of all vectors $X \in T_z(J(M, r))$ such that $\pi_*(X)$ is in the contact element z to M :

$$C(z) = \{X \in T_z(J(M, r)) \mid \pi_*(X) \in z\}.$$

We then set $C = \bigcup_z C(z)$, which is easily shown to become a subbundle of $T(J(M, r))$. The system C is called the contact system of $J(M, r)$. Let V be the vertical tangent bundle of $J(M, r)$. Then it is clear that $V \subset C$, and $\text{rank}(C/V) = r$.

Let us consider any canonical coordinate system (x^i, y^a, p_j^b) of $J(M, r)$ at any point z_0 . We define 1-forms θ^a on the domain $\tilde{U}$ of the coordinate system by

$$\theta^a = dy^a - \sum_i p_i^a dx^i.$$

Then it is clear that V is defined, on $\tilde{U}$, by the system of Pfaffian equations: $dx^i = dy^a = 0$, and C by the system of Pfaffian equations: $\theta^a = 0$. We have

$$d\theta^a = \sum_i dx^i \wedge dp_i^a,$$

from which follows immediately that C is non-degenerate, and the first derived system of C coincides with $T(J(M, r))$.

Summarizing the results above on the systems C and V , we have

Lemma 1.5. (1) V is completely integrable, and $V \subset C$.

(2) C is non-degenerate, and the first derived system of C coincides with $T(J(M,r))$.

(3) $\text{rank}(T(J(M,r))/C) = n-r$, $\text{rank}(C/V) = r$, and $\text{rank } V = r(n-r)$.

Now, let α be any r -dimensional submanifold of M . For each point $p \in \alpha$ the tangent space $T_p(\alpha)$ gives an r -dimensional contact element to M and hence a point of $J(M,r)$. This being said, we denote by $p(\alpha)$ the image of the map $p \mapsto T_p(\alpha)$ of α to $J(M,r)$. Clearly $p(\alpha)$ is an r -dimensional submanifold of $J(M,r)$, and the projection π maps $p(\alpha)$ diffeomorphically onto the submanifold α of M . The submanifold $p(\alpha)$ thus obtained is called the prolongation of α . We now assert that $p(\alpha)$ is an integral manifold of the contact system C . Indeed take any point z of $p(\alpha)$. If we put $p = \pi(z)$, we have

$$\pi_*(T_z(p(\alpha))) = T_p(\alpha) = z,$$

whence $T_z(p(\alpha)) \subset C(z)$. This proves our assertion.

Lemma 1.6. Let $\tilde{\alpha}$ be an r -dimensional integral manifold of C . If π maps $\tilde{\alpha}$ diffeomorphically onto a submanifold, say α , of M , then $\tilde{\alpha} = p(\alpha)$.

Proof. Take any point z of $\tilde{\alpha}$. If we put $p = \pi(z)$, we have

$$T_p(\alpha) = \pi_*(T_z(\tilde{\alpha})) \subset \pi_*(C(z)) = z.$$

It follows that $T_p(\alpha) = z$, proving the lemma.

Let us now consider any canonical coordinate system (x^i, y^a, p_j^b)

of $J(M, r)$ at any point z_0 . Let α be an r -dimensional submanifold of M . Suppose that $p(\alpha)$ is in the domain $\tilde{U}$ of the coordinate system, and that α is defined by a system of equations of the following form: $y^a = u^a(x^1, \dots, x^r)$. Then, in virtue of Lemma 1.6 we can easily see that $p(\alpha)$ is defined by the following system of equations:

$$\begin{cases} y^a = u^a(x^1, \dots, x^r), \\ p_j^a = \frac{\partial u^a}{\partial x^j}(x^1, \dots, x^r). \end{cases}$$

Finally let us consider another n -dimensional manifold M' as well as the Grassmann bundle $J(M', r)$. Let π' be the projection of $J(M', r)$ onto M' , and let C' and V' be the contact system and the vertical tangent bundle of $J(M', r)$ respectively. Now, suppose that there is given a diffeomorphism ϕ of M onto M' . If z is an r -dimensional contact element to M , $\phi_*(z)$ is an r -dimensional contact element to M' , and the map $z \mapsto \phi_*(z)$ gives a diffeomorphism of $J(M, r)$ onto $J(M', r)$, which we denote by $p(\phi)$, and call the prolongation of ϕ . Clearly

$$\pi' \circ p(\phi) = \phi \circ \pi,$$

and $p(\phi)$ gives an isomorphism of $(J(M, r), C, V)$ to $(J(M', r), C', V')$.

1.4. Fibred submanifolds of the Grassman bundles, and Realization lemma. Let M be an n -dimensional manifold, and let R be a fibred submanifold of the Grassmann bundle $J(M, r)$. The notations being as in the previous paragraph, we denote by D and F the pull-backs of C and V to R respectively: $D = i_*^{-1}(C)$, and

$F = \iota_*^{-1}(V)$, ι being the inclusion map of R to $J(M,r)$. It is clear that F is nothing but the vertical tangent bundle of the fibred manifold R over M , and that for each $z \in R$

$$D(z) = \{X \in T_z(R) \mid \pi_*(X) \in z\}.$$

In particular it follows that D is a subbundle of $T(R)$. The system D thus obtained is called the contact system of R .

Lemma 1.7. (1) F is completely integrable, and $F \subset D$.

(2) $\text{ch}(D) \cap F = 0$.

(3) $\text{rank}(T(R)/D) = n-r$, and $\text{rank}(D/F) = r$.

Proof. (1) is clear, and (3) follows immediately from the expression of $D(z)$ given above. (2) follows from Lemma 1.9 below, proving the lemma.

Let us now consider another n -dimensional manifold M' , and let R' be a fibred submanifold of $J(M',r)$. Let π' be the projection of $J(M',r)$ onto M' , and let D' and F' be the contact system and the vertical tangent bundle of R' respectively.

Lemma 1.8. Let (z_0, z'_0) be a point of $R \times R'$. If $\tilde{\phi}$ is a local isomorphism of (R, D, F) to (R', D', F') at (z_0, z'_0) , then there is a unique local diffeomorphism ϕ of M to M' at $(\pi(z_0), \pi(z'_0))$ such that $\tilde{\phi} = p(\phi)$ on a neighborhood of z_0 in R .

Proof. Since $\tilde{\phi}$ gives a local isomorphism of (R, F) to (R', F') at (z_0, z'_0) , we easily see that there is a unique local diffeomorphism ϕ of M to M' at $(\pi(z_0), \pi(z'_0))$ such that $\pi' \circ \tilde{\phi} = \phi \circ \pi$ on a neighborhood of z_0 in R . Now, let O be a sufficiently small neighborhood of z_0 in R , and take any point z of O . Since $\tilde{\phi}$ gives a local isomorphism of (R, D) to (R', D') at (z_0, z'_0) , it follows

that

$$\begin{aligned}\tilde{\phi}(z) &= \pi'_*(D'(\tilde{\phi}(z))) = \pi'_*(\tilde{\phi}_*(D(z))) \\ &= \phi_*(\pi_*D(z)) = \phi_*(z) = p(\phi)(z),\end{aligned}$$

whence $\tilde{\phi} = p(\phi)$ on O . This proves the lemma.

Let R be a fibred manifold over a manifold M with projection μ , and let D be a differential system on R . Let F be the vertical tangent bundle of R , and assume that $F \subset D$, and $1 \leq r \leq n-1$, where $n = \dim M$, and $r = \text{rank}(D/F)$. As for the Grassmann bundle $J(M, r)$, we shall use the notations as in the previous paragraph.

Lemma 1.9 (Realization lemma). There is a unique map ϕ of R to $J(M, r)$ satisfying the following conditions:

$$(i) \quad \pi \circ \phi = \mu,$$

$$(ii) \quad D \text{ is the pull-back of } C \text{ by } \phi: D = \phi_*^{-1}(C).$$

Furthermore the kernel $\phi_*^{-1}(0)$ of the differential ϕ_* coincides with the intersection $\text{ch}(D) \cap F$.

In particular it follows that if D is non-degenerate, i.e., $\text{ch}(D) = 0$, there is a unique immersion ϕ of R to $J(M, r)$ satisfying conditions (i) and (ii).

Proof of Lemma 1.9. We first remark that for each $x \in R$ $\mu_*(D(x))$ gives an r -dimensional contact element to M , because $F(x) \subset D(x)$, and $r = \dim(D(x)/F(x))$. This being said, we define a map ϕ of R to $J(M, r)$ by

$$\phi(x) = \mu_*(D(x)), \quad x \in R.$$

Clearly $\pi \circ \phi = \mu$. For $X \in T_x(R)$ we have: $X \in \phi_*^{-1}(C) \iff$

$$\phi_*(X) \in C(\phi(x)) \iff \pi_*(\phi_*(X)) \in \phi(x) \iff \mu_*(X) \in \mu_*(D(x)) \iff$$

$X \in D(x)$. Hence we obtain $\phi_*^{-1}(C) = D$. Now, let ϕ' be any map of R to $J(M, r)$ such that $\pi \circ \phi' = \mu$, and $\phi'^{-1}(C) = D$. For $X \in T_x(R)$ we have: $\mu_*(X) \in \phi'(x) \iff \pi_*(\phi'_*(X)) \in \phi'(x) \iff \phi'_*(X) \in C(\phi'(x)) \iff X \in D(x) \iff \mu_*(X) \in \mu_*(D(x)) = \phi(x)$. Therefore we obtain $\phi'(x) = \phi(x)$, and hence $\phi' = \phi$, proving the first assertion.

Let x be any point of R . We take a coordinate system (x^i, y^a) of M adapted to the contact element $\phi(x)$ and consider the corresponding canonical coordinate system $(x^i \circ \pi, y^a \circ \pi, p_j^b)$ of $J(M, r)$ at $\phi(x)$. Let X be a vector of $T_x(R)$. Clearly $\phi_*(X) = 0$, if and only if $X(x^i \circ \pi \circ \phi) = X(y^a \circ \pi \circ \phi) = X(p_i^a \circ \phi) = 0$. Since $\pi \circ \phi = \mu$, it follows that $\phi_*(X) = 0$, if and only if $X \in F(x)$, and $X(p_i^a \circ \phi) = 0$. Let us now consider the 1-forms $\theta^a = d(y^a \circ \pi) - \sum_i p_i^a d(x^i \circ \pi)$, and recall that C is defined, on a neighborhood of $\phi(x)$ by the system of Pfaffian equations: $\theta^a = 0$. Set $\eta^a = \phi^* \theta^a$. Since $D = \phi_*^{-1}(C)$, we see that D is defined, on a neighborhood of x , by the system of Pfaffian equations $\eta^a = 0$. We have $\eta^a = d(y^a \circ \mu) - \sum_i p_i^a \circ \phi d(x^i \circ \mu)$, and hence $d\eta^a = \sum_i d(x^i \circ \mu) \wedge d(p_i^a \circ \phi)$. Therefore we obtain

$$d\eta^a(X, Y) = - \sum_i X(p_i^a \circ \phi) \cdot Y(x^i \circ \mu),$$

where $X \in F(x)$, and $Y \in D(x)$. We also recall that $\text{ch}(D)(x)$ consists of all $X \in D(x)$ such that $d\eta^a(X, Y) = 0$ for all $Y \in D(x)$. Furthermore we see that the linear map $Y \mapsto (Y(x^i \circ \mu))$ maps $D(x)$ onto $\mathbb{R}^r$, because $F(x) \subset D(x)$. Therefore it follows that the intersection $\text{ch}(D)(x) \cap F(x)$ consists of all $X \in F(x)$ with $X(p_i^a \circ \phi) = 0$. As we have seen, this set consists of all $X \in T_x(R)$ with $\phi_*(X) = 0$. We have thus shown that $\phi_*^{-1}(0) \cap T_x(R) = \text{ch}(D)(x) \cap F(x)$, and hence $\phi_*^{-1}(0) = \text{ch}(D) \cap F$, proving the second assertion.

1.5. The formal derivations d/dx^i . Let (n, r) be a pair of integers with $1 \leq r \leq n-1$. In the following the indices $i, j, i_1, i_2, \dots$ will range over the integers $1, \dots, r$, and the indices a, b over the integers $1, \dots, n-r$.

Let (x^i) and (y^a) be the canonical coordinate systems of R^r and R^{n-r} respectively. Then (x^i, y^a) gives the canonical coordinate system of $R^n = R^r \times R^{n-r}$. Now, let x be any point of R^r , and let u be any map of a neighborhood of x to R^{n-r} . For any integer $k \geq 0$ we may consider the k -jet $j_x^k(u)$ of u at x (cf. []). Then we denote by J^k the set of all the k -jets $j_x^k(u)$. Clearly J^0 may be identified with R^n . For any $0 \leq \ell \leq k$ we also denote by ρ_ℓ^k the projection of J^k onto J^ℓ : $\rho_\ell^k(j_x^k(u)) = j_x^\ell(u)$. We now define functions $p_{i_1 \dots i_\ell}^a$ ($1 \leq \ell \leq k$) on J^k by

$$p_{i_1 \dots i_\ell}^a(j_x^k(u)) = \frac{\partial^\ell u^a}{\partial x^{i_1} \dots \partial x^{i_\ell}}(x),$$

where $u^a = y^a \circ u$. The functions $p_{i_1 \dots i_\ell}^a$ will be sometimes denoted by $\partial^\ell y^a / \partial x^{i_1} \dots \partial x^{i_\ell}$. Now, J^k becomes a manifold so that the functions $x^i (= x^i \circ \rho_0^k)$, $y^a (= y^a \circ \rho_0^k)$ and $p_{i_1 \dots i_\ell}^b$ ($i_1 \leq \dots \leq i_\ell$, $1 \leq \ell \leq k$) together form a coordinate system of J^k globally defined on it. The coordinate system of J^k thus defined is called the canonical coordinate system of J^k .

Let f be any function defined on an open set O of J^k . Then we define functions df/dx^i on $(\rho_k^{k+1})^{-1}(O)$ in the following manner: Take any point $j_k^{k+1}(u)$ of $(\rho_k^{k+1})^{-1}(O)$, and denote by $j^k(u)$ the map $x' \mapsto j_x^k(u)$ of a neighborhood of x to J^k . Then df/dx^i are defined by

$$\frac{df}{dx^i}(j_x^{k+1}(u)) = \frac{\partial f \circ j^k(u)}{\partial x^i}(x).$$

In terms of the canonical coordinate systems of J^k and J^{k+1} the functions df/dx^i may be expressed as follows:

$$\frac{df}{dx^i} = \frac{\partial f}{\partial x^i} + \sum_a \frac{\partial f}{\partial y^a} p_i^a + \sum_{\ell=1} \sum' \frac{\partial f}{\partial p_{i_1 \dots i_\ell}^a} p_{i_1 \dots i_\ell}^a,$$

where $\sum'$ means a summation with respect to the index a and the system $(i_1, \dots, i_\ell)$ of indices with $i_1 \leq \dots \leq i_\ell$.

The differential operators d/dx^i thus obtained are called the formal derivations corresponding to the coordinate system (x^i, y^a) , x^i being considered as independent variables, and y^a as dependent variables. Clearly we have

$$\frac{d}{dx^i} \left(\frac{df}{dx^j} \right) = \frac{d}{dx^j} \left(\frac{df}{dx^i} \right).$$

a manifold M such that M is a subset of M' , and the inclusion map i of M to M' is an immersion. An imbedded submanifold of M' is a submanifold M of M' such that i is an imbedding, i.e., the topology of M coincides with the relative topology.

(f) Let π be a map of a manifold R to a manifold M . Then R is called a fibred manifold over M with projection π , if the following conditions are satisfied: (i) π is surjective, and (ii) π is a submersion, i.e., the differential π_* of π maps $T_x(R)$ onto $T_{\pi(x)}(M)$ for each $x \in R$. Let R be a fibred manifold over M with projection π . Let R' be a submanifold of R , and π' the restriction of π to R' . Then R' is called a fibred submanifold of R , if R' becomes a fibred manifold over M with projection π' .

(g) A local diffeomorphism of a manifold M to a manifold M' is a diffeomorphism ϕ of an open set of M onto an open set of M' . Given a point (x_0, x'_0) of $M \times M'$, a local diffeomorphism of M to M' at (x_0, x'_0) is a local diffeomorphism ϕ of M to M' such that x_0 is in the domain of ϕ and $\phi(x_0) = x'_0$.

1.2. Differential systems.

(A) Differential systems. A differential system on a manifold R is a subbundle D of the tangent vector bundle $T(R)$ of R .

Let D be a differential system on a manifold R . Set $\sigma = \text{rank}(T(R)/D)$, and let x_0 be any point of R . Then there are 1-forms θ^a ($1 \leq a \leq \sigma$) defined on a neighborhood O of x_0 such that D is defined there by the system of Pfaffian equations:

$$\theta^a = 0,$$

that is, the restriction $D|O$ of D to O consists of all $X \in T(O)$

such that $\theta^a(X) = 0$. Note that θ^a are automatically linearly independent at each point of O .

Let R and R' be two manifolds, and let $(D_i)_{i \in I}$ and $(D'_i)_{i \in I}$ be finite families of differential systems on them respectively. Then an isomorphism of $(R, (D_i))$ to $(R', (D'_i))$ is a diffeomorphism ϕ of R onto R' such that its differential ϕ_* maps D_i onto D'_i for each i . We remark that the notion of isomorphism naturally gives rise to the notions of local isomorphism, of (local) automorphism, and of (local) infinitesimal automorphism:

A local isomorphism of $(R, (D_i))$ to $(R', (D'_i))$ is a diffeomorphism ϕ of an open set O of R onto an open set O' of R' which gives an isomorphism of $(O, (D_i|_O))$ to $(O', (D'_i|_{O'}))$. Let (x_0, x'_0) be a point of $R \times R'$. A local isomorphism of $(R, (D_i))$ to $(R', (D'_i))$ at (x_0, x'_0) is a local diffeomorphism ϕ of R to R' at (x_0, x'_0) which gives a local isomorphism of $(R, (D_i))$ to $(R', (D'_i))$. A (local) automorphism of $(R, (D_i))$ is a (local) isomorphism of $(R, (D_i))$ to itself. A (local) infinitesimal automorphism of $(R, (D_i))$ is a (local) vector field X on R such that for any local 1-parameter group (ϕ_t) of local transformations of R generated by X , each ϕ_t gives a local automorphism of $(R, (D_i))$. It is easy to see that a local vector field X is a (local) infinitesimal automorphism of $(R, (D_i))$, if and only if for any i and any local cross section Y of D_i the bracket $[X, Y]$ is a local cross section of D_i .

In this book we shall be concerned with various kinds of manifolds with family of differential systems, e.g., product manifolds, pseudo-product manifolds and (generalized) pseudo-

projective systems.

Finally let ϕ be a map of a manifold to a manifold R' , and let D' be a differential system on R' . Let us consider the inverse image $\phi_*^{-1}(D')$ of D' by the differential ϕ_* . Set $D = \phi_*^{-1}(D')$, and $D(x) = D \cap T_x(R)$ for each $x \in R$. Then $D(x)$ is a subspace of $T_x(R)$, and $D = \bigcup_x D(x)$. Now let x_0 be any point of R , and suppose that D' is defined, on a neighborhood O' of $\phi(x_0)$, by a system of Pfaffian equations: $\theta'^a = 0$ ($1 \leq a \leq \sigma$), where $\sigma = \text{rank}(T(R')/D')$. Then D is defined, on $\phi^{-1}(O')$, by the system of Pfaffian equations: $\phi^* \theta'^a = 0$.

Thus D is, so to speak, a differential system possibly with singularities, which is called the pull-back of D' by ϕ . Especially the pull-back of the zero differential system, 0 , on R' by ϕ means the kernel of the differential ϕ_* , and we prefer to use this last terminology. Furthermore, if R is a submanifold of R' , and ϕ is the inclusion map of R to R' , D is particularly called the pull-back of D' to R .

(B) The torsion of a differential system. Let R be a manifold, and let D be a differential system on it. Let x be any point of R . As is easily verified, there is a unique skew-symmetric bilinear map γ_x of $D(x) \times D(x)$ to $T(x)/D(x)$ such that

$$\gamma_x(X_x, Y_x) = \pi([X, Y]_x),$$

where $T = T(R)$, π is the projection of T onto T/D , and X and Y are any local cross sections of D defined on a neighborhood of x . The bilinear map γ_x thus obtained is called the torsion of D at x .

Now let x_0 be any point of R , and suppose that D is defined, on a neighborhood of x_0 , by a system of Pfaffian equations:

$\theta^a = 0$ ($1 \leq a \leq \sigma$), where $\sigma = \text{rank}(T(R)/D)$. Let O be a sufficiently small neighborhood of x_0 , and take 1-forms ω^i ($1 \leq i \leq \rho$) on O such that the 1-forms θ^a and ω^i are linearly independent at each point $x \in O$, where $\rho = \text{rank } D$. Then there are unique functions T_{ij}^a ($1 \leq i, j \leq \rho$) on O such that $T_{ij} = -T_{ji}^a$, and

$$d\theta^a \equiv \frac{1}{2} \sum_{i,j} T_{ij}^a \omega^i \wedge \omega^j \pmod{\theta^1, \dots, \theta^\sigma}.$$

It is easy to see that

$$\theta^a(\gamma_x(X, Y)) = -d\theta^a(X, Y) = - \sum_{i,j} T_{ij}^a(x) \omega^i(X) \omega^j(Y),$$

where x is any point of O , and X and Y are any vectors of $D(x)$.

By definition the system D is completely integrable, if for any local cross sections X and Y of D (with the same domain) the bracket $[X, Y]$ is likewise a cross section of D . Clearly D is completely integrable, if and only if the torsion γ_x vanishes at each $x \in R$.

The system D is called non-degenerate, if the torsion γ_x is non-degenerate at each $x \in R$, i.e., the condition " $x \in R$, $X \in D(x)$, and $\gamma_x(X, Y) = 0$ for all $Y \in D(x)$ " implies $X = 0$. In particular D is called a contact structure on R , if D is non-degenerate, $\text{rank}(T(R)/D) = 1$, and $\dim R > 1$. A manifold R equipped with a contact structure D is called a contact manifold. Note that a contact manifold is necessarily odd dimensional.

(C) Integral manifolds and integral elements of a differential system. Let R be a manifold, and let D be a differential system on it. An integral manifold of D is a connected submanifold N of R such that $T_x(N) \subset D(x)$ at each $x \in R$. In particular if D is completely integrable, an integral manifold of D will always

mean that of the maximal dimension, unless otherwise stated.

If N is an integral manifold of D , and x is a point of N , the torsion γ_x of D at x clearly vanishes on the tangent space $T_x(N)$. Now, let z be a contact element to R at a point x . Then z is called an integral element of D , if z is a subspace of $D(x)$, and the torsion γ_x vanishes on z .

(D) Completely integrable systems. Let R be a manifold, and let E be a completely integrable (differential) system on it. A first integral of E is a function f defined on an open set O of R which satisfies the following differential equation:

$$Xf = 0 \text{ for any } X \in E(x) \text{ and any } x \in O.$$

Let x_0 be any point of R . Then the famous Frobenius's theorem states that there are first integrals f^a ($1 \leq a \leq \sigma$) of E defined on a neighborhood of x_0 such that the differentials df^a are linearly independent at x_0 , where $\sigma = \text{rank}(T(R)/E)$. The system (f^a) is called a fundamental system of first integrals of E at x_0 .

Let R be a fibred manifold over a manifold M with projection π . Let V denote the kernel of the differential π_* : $V = \pi_*^{-1}(0)$. Clearly V is a subbundle of $T(R)$, and is completely integrable, which is called the vertical tangent bundle of the fibred manifold R .

Let R and E be as in the outset. We denote by R/E the set of all (maximal) integral manifolds of E , and by π the projection of R onto R/E . Now, R is said to be regular with respect to E , if the set R/E becomes naturally a manifold or more precisely if R/E is equipped with a differentiable structure so that (i) R becomes a fibred manifold over R/E with projection π , and (ii)

the vertical tangent bundle of this fibred manifold coincides with E . Note that there is at most one differentiable structure on R/E satisfying the first condition, and that if R satisfies the second axiom of countability the first condition implies the second.

Finally we add the following

Lemma 1.1. Let R be a manifold, and let (E_i) be a finite family of completely integrable systems on it. Then every point x_0 of R has a neighborhood O which is regular with respect to E_i for each i .

This fact follows easily from Frobenius's theorem and Lemma 1.2 below.

For any positive number ϵ we denote by $S^{n-1}(\epsilon)$ the sphere of the standard n -dimensional Euclidean space R^n of radius ϵ centred at the origin o .

Lemma 1.2. Let ϕ be any local transformation of R^n at (o,o) , i.e., such that o is in the domain of ϕ and $\phi(o) = o$. If ϵ is sufficiently small, then the image $\phi(S^{n-1}(\epsilon))$ of $S^{n-1}(\epsilon)$ by ϕ becomes a convex hypersurface of R^n .

Indeed it can be shown that if ϵ is sufficiently small, the second fundamental form of the hypersurface $\phi(S^{n-1}(\epsilon))$ of R^n becomes definite everywhere, meaning that $\phi(S^{n-1}(\epsilon))$ is a convex hypersurface of R^n .

(E) The Cauchy characteristic system of a differential system. Let R be a manifold, and let D be a differential system on it. For each $x \in R$ we denote by $E(x)$ the null space of the torsion γ_x of D at x :

$$E(x) = \{X \in D(x) \mid \gamma_x(X, Y) = 0 \text{ for all } Y \in D(x)\},$$

and set $E = \bigcup_x E(x)$. It is clear that E is a subbundle of $T(R)$, if and only if $\dim E(x)$ is constant. We now assert that E is completely integrable, provided E is a subbundle of $T(R)$. Indeed let x be any point of R . Let X and Y be any local cross sections of E defined on neighborhoods of x , and Z any local cross section of D defined on a neighborhood of x . Clearly $[X, Y]_x$, $[X, Z]_x$ and $[Y, Z]_x$ are all in $D(x)$. Since

$$[[X, Y], Z] = [[X, Z], Y] + [X, [Y, Z]],$$

it follows that

$$\gamma_x([X, Y]_x, Z_x) = \gamma_x([X, Z]_x, Y_x) + \gamma_x(X_x, [Y, Z]_x) = 0,$$

showing that $[X, Y]_x \in E(x)$. This proves our assertion.

Thus E is, so to speak, a completely integrable system possibly with singularities, which is denoted by $ch(D)$, and is called the Cauchy characteristic system of D .

Now, assume that E is a subbundle of $T(R)$, and that R is regular with respect to E . Set $M = R/E$, and let π be the projection of R onto M .

Lemma 1.3.(E.Cartan). There is a unique differential system $\bar{D}$ on M such that the given system D is the pull-back of $\bar{D}$ by π : $D = \pi_*^{-1}(\bar{D})$. Furthermore the system $\bar{D}$ is non-degenerate.

Proof. We first recall that a local vector field X on R is a local infinitesimal automorphism of (R, D) , if and only if for any local cross section Y of D $[X, Y]$ is a local cross section of D . Then we see that a local vector field X on R is a local cross section of E , if and only if it is at once a local cross section

of D and a local infinitesimal automorphism of (R, D) . Now, let $\mathcal{E}$ be the sheaf of all germs of local cross sections X of E , which becomes naturally a sheaf of Lie algebras. Clearly each fibre N of the fibred manifold R over M is an orbit of $\mathcal{E}$: $T_x(N) = E(x) = \{X_x \mid X \in \mathcal{E}(x)\}$ for each $x \in N$.

Let Γ be the pseudo-group of local transformations of R generated by $\mathcal{E}$ (cf. [15]). For any local cross section X of $\mathcal{E}$ let (ϕ_t) be any local 1-parameter group of local transformations of R generated by X . Then Γ is generated by the local transformations of the form ϕ_t . From the observations above we easily obtain the following: (i) Every $\phi \in \Gamma$ is a local automorphism of (R, D) , (ii) Γ leaves each fibre of R invariant, i.e., if $\phi \in \Gamma$, and x is a point in the domain of ϕ , then the two points x and $\phi(x)$ are in the same fibre of R , and (iii) Γ acts transitively on each fibre of R , i.e., if x and x' are two points in the same fibre of R , then there is $\phi \in \Gamma$ such that $x' = \phi(x)$. It is now easy to see that there is a unique differential system $\bar{D}$ on M such that $D = \pi_*^{-1}(\bar{D})$, proving the first assertion.

Clearly the differential $\pi_*: T(R) \rightarrow T(M)$ naturally induces a bundle homomorphism, say π_* , of $T(R)/D$ onto $T(M)/\bar{D}$, which gives an isomorphism of $T_x(R)/D(x)$ onto $T_{\pi(x)}(M)/\bar{D}(\pi(x))$ for each $x \in R$. Now, let x be any point of R , and let us consider the torsion γ_x of D at x as well as the torsion $\bar{\gamma}_{\pi(x)}$ of $\bar{D}$ at $\pi(x)$. Then we can easily show that

$$\bar{\gamma}_{\pi(x)}(\pi_*(X), \pi_*(Y)) = \pi_*(\gamma_x(X, Y)), \quad X, Y \in D(x),$$

from which follows immediately that $\bar{\gamma}_{\pi(x)}$ is non-degenerate.

Hence the system $\bar{D}$ is non-degenerate, proving the second assertion.

(F) The derived systems of a differential system. Let R be a manifold. For simplicity we denote by T the tangent bundle $T(R)$ of R . We also denote by $\mathcal{O}$ the sheaf of germs of local functions on M , which is a sheaf of rings. Furthermore we denote by $\mathcal{T}$ the sheaf of germs of local vector fields on M or of local cross sections of T , which is a sheaf of Lie algebras, and is a module over $\mathcal{O}$ or an $\mathcal{O}$ -module.

Now, let D be a differential system on R . We denote by $\mathcal{D}$ the sheaf of germs of local cross sections of D , which is an $\mathcal{O}$ -submodule of the $\mathcal{O}$ -module $\mathcal{T}$. Let x be any point of R . For any integer $p < 0$ we define a subspace $\mathcal{D}^p(x)$ of $\mathcal{T}(x)$ inductively as follows:

$$\mathcal{D}^{-1}(x) = \mathcal{D}(x),$$

$$\mathcal{D}^p(x) = [\mathcal{D}^{p+1}(x), \mathcal{D}(x)] + \mathcal{D}^{p+1}(x) \quad \text{for } p \leq -2.$$

We then set $\mathcal{D}^p = \bigcup_x \mathcal{D}^p(x)$. Clearly $\mathcal{D}^p$ are $\mathcal{O}$ -submodules of $\mathcal{T}$, and

$$\mathcal{T} \supset \dots \supset \mathcal{D}^p \supset \dots \supset \mathcal{D}^{-2} \supset \mathcal{D}^{-1} = \mathcal{D}.$$

We can easily prove the following

$$\text{Lemma 1.4. } [\mathcal{D}^p(x), \mathcal{D}^q(x)] \subset \mathcal{D}^{p+q}(x).$$

For each $x \in R$ let us now consider the natural map of $\mathcal{T}(x)$ onto $T(x)$, i.e., the map which maps every $X \in \mathcal{T}(x)$ to $X_x \in T(x)$. Then we denote by $D^p(x)$ the image of $\mathcal{D}^p(x)$ by the natural map, and set $D^p = \bigcup_x D^p(x)$. Clearly we have

$$T \supset \dots \supset D^p \supset \dots \supset D^{-1} = D.$$

Moreover it is clear that D^p becomes a subbundle of T , if and

only if $\dim D^p(x)$ is constant. Thus D^p is a differential system possibly with singularities. For any positive integer k the system D^{-k-1} will be called the k -th derived system of D , and the sheaf $\mathcal{D}^{-k-1}$ the k -th derived sheaf of $\mathcal{D}$.

Let us now consider the torsion γ_x of D at any point $x \in R$. Then we remark that the factor space $D^{-2}(x)/D(x)$ is the subspace of $T(x)/D(x)$ spanned by the vectors $\gamma_x(X, Y)$, where $X, Y \in D(x)$.

Now, we denote by $D^{-\infty}$ the union $\bigcup_{p \leq 0} D^p$, which is an $\mathcal{O}$ -submodule of $\mathcal{T}$. For each $x \in R$ we also denote by $D^{-\infty}(x)$ the image of $\mathcal{D}^{-\infty}(x)$ by the natural map of $\mathcal{T}(x)$ onto $T(x)$, and set $D^{-\infty} = \bigcup_x D^{-\infty}(x)$. By Lemma 1.4 we see that $D^{-\infty}(x)$ is a subalgebra of the Lie algebra $\mathcal{T}(x)$. Moreover it is clear that $D^{-\infty}$ becomes a subbundle of $T(R)$, if and only if $\dim D^{-\infty}(x)$ is constant. Thus $D^{-\infty}$ is a completely integrable system possibly with singularities. The system $D^{-\infty}$ will be called the full derived system of D , and the sheaf $\mathcal{D}^{-\infty}$ the full derived sheaf of $\mathcal{D}$.

For example consider the case where D is a contact structure. Then it is clear that the first derived system of D coincides with T : $T = D^{-\infty} = D^{-2}$.

1.3. The Grassmann bundles. Let V be an n -dimensional vector space. Then $\text{Gr}(V, r)$ will denote the Grassmann manifold of r -dimensional subspaces of V , which is a compact, connected manifold of dimension $r(n-r)$.

Now, let (n, r) be a pair of integers with $1 \leq r \leq n-1$. Let M be an n -dimensional manifold. We denote by $J(M, r)$ the set of all r -dimensional contact elements to M or in other words,

$$J(M,r) = \bigcup_p \text{Gr}(T_p(M),r).$$

We then denote by π the natural map of $J(M,r)$ onto M : $\pi(z) = p$, if $z \in \text{Gr}(T_p(M))$. As is easily seen, $J(M,r)$ is a fibre bundle over the base space M with projection π (associated with the frame bundle of M), which is called the Grassmann bundle of r -dimensional contact elements to M . Note that $J(M,1)$ and $J(M,n-1)$ may be regarded as the projective bundles associated with the vector bundles $T(M)$ and $T^*(M)$ respectively.

In the following the indices i,j range over the integers $1, \dots, r$, and the indices a,b over the integers $1, \dots, n-r$.

Let z_0 be any point of $J(M,r)$, and let (x^i, y^a) be any coordinate system of M at $\pi(z_0)$ such that the differentials dx^i are linearly independent, restricted to the contact element z_0 . Such a coordinate system will be called adapted to z_0 . Now, let U be the domain of the coordinate system. We denote by $\tilde{U}$ the subset of $J(M,r)$ consisting of all $z \in J(M,r)$ such that $\pi(z) \in U$, and the coordinate system is adapted to z . Clearly $\tilde{U}$ is an open set of $J(M,r)$, $z_0 \in \tilde{U}$, and $\pi(\tilde{U}) = U$. Moreover it is clear that there are unique functions p_i^a on $\tilde{U}$ such that

$$dy^a(X) = \sum_i p_i^a(z) dx^i(X),$$

where z is any point of $\tilde{U}$, and X is any vector in the contact element z . Then we see that the functions $x^i (= x^i \circ \pi)$, $y^a (= y^a \circ \pi)$ and p_j^b together form a coordinate system of $J(M,r)$, which is defined on $\tilde{U}$. The coordinate system, thus obtained, is called the canonical coordinate system of $J(M,r)$ at z_0 corresponding to the coordinate system (x^i, y^a) of M or simply a canonical coordinate

system of $J(M,r)$ at z_0 .

Consider the following three special cases: (A) $r = 1$; (B) $r = n-1$; (C) $n = 2$, and hence $r = 1$. In case (A) the canonical coordinate system will be described as $(x, y^1, \dots, y^{n-1}, p^1, \dots, p^{n-1})$ by setting $x = x^1$ and $p^a = p_1^a$, and in case (B) as $(x^1, \dots, x^{n-1}, y, p_1, \dots, p_{n-1})$ by setting $y = y^1$ and $p_i = p_i^1$, and in case (C) as (x, y, p) by setting $x = x^1$, $y = y^1$ and $p = p_1^1$.

For any $z \in J(M,r)$ we denote by $C(z)$ the subspace of $T_z(J(M,r))$ consisting of all vectors $X \in T_z(J(M,r))$ such that $\pi_*(X)$ is in the contact element z to M :

$$C(z) = \{X \in T_z(J(M,r)) \mid \pi_*(X) \in z\}.$$

We then set $C = \bigcup_z C(z)$, which is easily shown to become a subbundle of $T(J(M,r))$. The system C is called the contact system of $J(M,r)$. Let V be the vertical tangent bundle of $J(M,r)$. Then it is clear that $V \subset C$, and $\text{rank}(C/V) = r$.

Let us consider any canonical coordinate system (x^i, y^a, p_j^b) of $J(M,r)$ at any point z_0 . We define 1-forms θ^a on the domain $\tilde{U}$ of the coordinate system by

$$\theta^a = dy^a - \sum_i p_i^a dx^i.$$

Then it is clear that V is defined, on $\tilde{U}$, by the system of Pfaffian equations: $dx^i = dy^a = 0$, and C by the system of Pfaffian equations: $\theta^a = 0$. We have

$$d\theta^a = \sum_i dx^i \wedge dp_i^a,$$

from which follows immediately that C is non-degenerate, and the first derived system of C coincides with $T(J(M,r))$.

Summarizing the results above on the systems C and V , we have

Lemma 1.5. (1) V is completely integrable, and $V \subset C$.

(2) C is non-degenerate, and the first derived system of C coincides with $T(J(M,r))$.

(3) $\text{rank}(T(J(M,r))/C) = n-r$, $\text{rank}(C/V) = r$, and $\text{rank } V = r(n-r)$.

Now, let α be any r -dimensional submanifold of M . For each point $p \in \alpha$ the tangent space $T_p(\alpha)$ gives an r -dimensional contact element to M and hence a point of $J(M,r)$. This being said, we denote by $p(\alpha)$ the image of the map $p \mapsto T_p(\alpha)$ of α to $J(M,r)$. Clearly $p(\alpha)$ is an r -dimensional submanifold of $J(M,r)$, and the projection π maps $p(\alpha)$ diffeomorphically onto the submanifold α of M . The submanifold $p(\alpha)$ thus obtained is called the prolongation of α . We now assert that $p(\alpha)$ is an integral manifold of the contact system C . Indeed take any point z of $p(\alpha)$. If we put $p = \pi(z)$, we have

$$\pi_*(T_z(p(\alpha))) = T_p(\alpha) = z,$$

whence $T_z(p(\alpha)) \subset C(z)$. This proves our assertion.

Lemma 1.6. Let $\tilde{\alpha}$ be an r -dimensional integral manifold of C . If π maps $\tilde{\alpha}$ diffeomorphically onto a submanifold, say α , of M , then $\tilde{\alpha} = p(\alpha)$.

Proof. Take any point z of $\tilde{\alpha}$. If we put $p = \pi(z)$, we have

$$T_p(\alpha) = \pi_*(T_z(\tilde{\alpha})) \subset \pi_*(C(z)) = z.$$

It follows that $T_p(\alpha) = z$, proving the lemma.

Let us now consider any canonical coordinate system (x^i, y^a, p_j^b)

of $J(M,r)$ at any point z_0 . Let α be an r -dimensional submanifold of M . Suppose that $p(\alpha)$ is in the domain $\tilde{U}$ of the coordinate system, and that α is defined by a system of equations of the following form: $y^a = u^a(x^1, \dots, x^r)$. Then, in virtue of Lemma 1.6 we can easily see that $p(\alpha)$ is defined by the following system of equations:

$$\begin{cases} y^a = u^a(x^1, \dots, x^r), \\ p_j^a = \frac{\partial u^a}{\partial x^j}(x^1, \dots, x^r). \end{cases}$$

Finally let us consider another n -dimensional manifold M' as well as the Grassmann bundle $J(M',r)$. Let π' be the projection of $J(M',r)$ onto M' , and let C' and V' be the contact system and the vertical tangent bundle of $J(M',r)$ respectively. Now, suppose that there is given a diffeomorphism ϕ of M onto M' . If z is an r -dimensional contact element to M , $\phi_*(z)$ is an r -dimensional contact element to M' , and the map $z \mapsto \phi_*(z)$ gives a diffeomorphism of $J(M,r)$ onto $J(M',r)$, which we denote by $p(\phi)$, and call the prolongation of ϕ . Clearly

$$\pi' \circ p(\phi) = \phi \circ \pi,$$

and $p(\phi)$ gives an isomorphism of $(J(M,r), C, V)$ to $(J(M',r), C', V')$.

1.4. Fibred submanifolds of the Grassman bundles, and Realization lemma. Let M be an n -dimensional manifold, and let R be a fibred submanifold of the Grassmann bundle $J(M,r)$. The notations being as in the previous paragraph, we denote by D and F the pull-backs of C and V to R respectively: $D = i_*^{-1}(C)$, and

$F = \iota_*^{-1}(V)$, ι being the inclusion map of R to $J(M,r)$. It is clear that F is nothing but the vertical tangent bundle of the fibred manifold R over M , and that for each $z \in R$

$$D(z) = \{X \in T_z(R) \mid \pi_*(X) \in z\}.$$

In particular it follows that D is a subbundle of $T(R)$. The system D thus obtained is called the contact system of R .

Lemma 1.7. (1) F is completely integrable, and $F \subset D$.

(2) $\text{ch}(D) \cap F = 0$.

(3) $\text{rank}(T(R)/D) = n-r$, and $\text{rank}(D/F) = r$.

Proof. (1) is clear, and (3) follows immediately from the expression of $D(z)$ given above. (2) follows from Lemma 1.9 below, proving the lemma.

Let us now consider another n -dimensional manifold M' , and let R' be a fibred submanifold of $J(M',r)$. Let π' be the projection of $J(M',r)$ onto M' , and let D' and F' be the contact system and the vertical tangent bundle of R' respectively.

Lemma 1.8. Let (z_0, z'_0) be a point of $R \times R'$. If $\tilde{\phi}$ is a local isomorphism of (R, D, F) to (R', D', F') at (z_0, z'_0) , then there is a unique local diffeomorphism ϕ of M to M' at $(\pi(z_0), \pi(z'_0))$ such that $\tilde{\phi} = p(\phi)$ on a neighborhood of z_0 in R .

Proof. Since $\tilde{\phi}$ gives a local isomorphism of (R, F) to (R', F') at (z_0, z'_0) , we easily see that there is a unique local diffeomorphism ϕ of M to M' at $(\pi(z_0), \pi(z'_0))$ such that $\pi' \circ \tilde{\phi} = \phi \circ \pi$ on a neighborhood of z_0 in R . Now, let O be a sufficiently small neighborhood of z_0 in R , and take any point z of O . Since $\tilde{\phi}$ gives a local isomorphism of (R, D) to (R', D') at (z_0, z'_0) , it follows

that

$$\begin{aligned}\tilde{\phi}(z) &= \pi'_*(D'(\tilde{\phi}(z))) = \pi'_*(\tilde{\phi}_*(D(z))) \\ &= \phi_*(\pi_*D(z)) = \phi_*(z) = p(\phi)(z),\end{aligned}$$

whence $\tilde{\phi} = p(\phi)$ on O . This proves the lemma.

Let R be a fibred manifold over a manifold M with projection μ , and let D be a differential system on R . Let F be the vertical tangent bundle of R , and assume that $F \subset D$, and $1 \leq r \leq n-1$, where $n = \dim M$, and $r = \text{rank}(D/F)$. As for the Grassmann bundle $J(M, r)$, we shall use the notations as in the previous paragraph.

Lemma 1.9 (Realization lemma). There is a unique map ϕ of R to $J(M, r)$ satisfying the following conditions:

$$(i) \quad \pi \circ \phi = \mu,$$

$$(ii) \quad D \text{ is the pull-back of } C \text{ by } \phi: D = \phi_*^{-1}(C).$$

Furthermore the kernel $\phi_*^{-1}(0)$ of the differential ϕ_* coincides with the intersection $\text{ch}(D) \cap F$.

In particular it follows that if D is non-degenerate, i.e., $\text{ch}(D) = 0$, there is a unique immersion ϕ of R to $J(M, r)$ satisfying conditions (i) and (ii).

Proof of Lemma 1.9. We first remark that for each $x \in R$ $\mu_*(D(x))$ gives an r -dimensional contact element to M , because $F(x) \subset D(x)$, and $r = \dim(D(x)/F(x))$. This being said, we define a map ϕ of R to $J(M, r)$ by

$$\phi(x) = \mu_*(D(x)), \quad x \in R.$$

Clearly $\pi \circ \phi = \mu$. For $X \in T_x(R)$ we have: $X \in \phi_*^{-1}(C) \iff$

$$\phi_*(X) \in C(\phi(x)) \iff \pi_*(\phi_*(X)) \in \phi(x) \iff \mu_*(X) \in \mu_*(D(x)) \iff$$

$X \in D(x)$. Hence we obtain $\phi_*^{-1}(C) = D$. Now, let ϕ' be any map of R to $J(M, r)$ such that $\pi \circ \phi' = \mu$, and $\phi'^{-1}(C) = D$. For $X \in T_x(R)$ we have: $\mu_*(X) \in \phi'(x) \iff \pi_*(\phi'_*(X)) \in \phi'(x) \iff \phi'_*(X) \in C(\phi'(x)) \iff X \in D(x) \iff \mu_*(X) \in \mu_*(D(x)) = \phi(x)$. Therefore we obtain $\phi'(x) = \phi(x)$, and hence $\phi' = \phi$, proving the first assertion.

Let x be any point of R . We take a coordinate system (x^i, y^a) of M adapted to the contact element $\phi(x)$ and consider the corresponding canonical coordinate system $(x^i \circ \pi, y^a \circ \pi, p_j^b)$ of $J(M, r)$ at $\phi(x)$. Let X be a vector of $T_x(R)$. Clearly $\phi_*(X) = 0$, if and only if $X(x^i \circ \pi \circ \phi) = X(y^a \circ \pi \circ \phi) = X(p_i^a \circ \phi) = 0$. Since $\pi \circ \phi = \mu$, it follows that $\phi_*(X) = 0$, if and only if $X \in F(x)$, and $X(p_i^a \circ \phi) = 0$. Let us now consider the 1-forms $\theta^a = d(y^a \circ \pi) - \sum_i p_i^a d(x^i \circ \pi)$, and recall that C is defined, on a neighborhood of $\phi(x)$ by the system of Pfaffian equations: $\theta^a = 0$. Set $\eta^a = \phi^* \theta^a$. Since $D = \phi_*^{-1}(C)$, we see that D is defined, on a neighborhood of x , by the system of Pfaffian equations $\eta^a = 0$. We have $\eta^a = d(y^a \circ \mu) - \sum_i p_i^a \circ \phi d(x^i \circ \mu)$, and hence $d\eta^a = \sum_i d(x^i \circ \mu) \wedge d(p_i^a \circ \phi)$. Therefore we obtain

$$d\eta^a(X, Y) = - \sum_i X(p_i^a \circ \phi) \cdot Y(x^i \circ \mu),$$

where $X \in F(x)$, and $Y \in D(x)$. We also recall that $\text{ch}(D)(x)$ consists of all $X \in D(x)$ such that $d\eta^a(X, Y) = 0$ for all $Y \in D(x)$. Furthermore we see that the linear map $Y \mapsto (Y(x^i \circ \mu))$ maps $D(x)$ onto R^r , because $F(x) \subset D(x)$. Therefore it follows that the intersection $\text{ch}(D)(x) \cap F(x)$ consists of all $X \in F(x)$ with $X(p_i^a \circ \phi) = 0$. As we have seen, this set consists of all $X \in T_x(R)$ with $\phi_*(X) = 0$. We have thus shown that $\phi_*^{-1}(0) \cap T_x(R) = \text{ch}(D)(x) \cap F(x)$, and hence $\phi_*^{-1}(0) = \text{ch}(D) \cap F$, proving the second assertion.

1.5. The formal derivations d/dx^i . Let (n,r) be a pair of integers with $1 \leq r \leq n-1$. In the following the indices $i, j, i_1, i_2, \dots$ will range over the integers $1, \dots, r$, and the indices a, b over the integers $1, \dots, n-r$.

Let (x^i) and (y^a) be the canonical coordinate systems of R^r and R^{n-r} respectively. Then (x^i, y^a) gives the canonical coordinate system of $R^n = R^r \times R^{n-r}$. Now, let x be any point of R^r , and let u be any map of a neighborhood of x to R^{n-r} . For any integer $k \geq 0$ we may consider the k -jet $j_x^k(u)$ of u at x (cf. []). Then we denote by J^k the set of all the k -jets $j_x^k(u)$. Clearly J^0 may be identified with R^n . For any $0 \leq \ell \leq k$ we also denote by ρ_ℓ^k the projection of J^k onto J^ℓ : $\rho_\ell^k(j_x^k(u)) = j_x^\ell(u)$. We now define functions $p_{i_1 \dots i_\ell}^a$ ($1 \leq \ell \leq k$) on J^k by

$$p_{i_1 \dots i_\ell}^a(j_x^k(u)) = \frac{\partial^\ell u^a}{\partial x^{i_1} \dots \partial x^{i_\ell}}(x),$$

where $u^a = y^a \circ u$. The functions $p_{i_1 \dots i_\ell}^a$ will be sometimes denoted by $\partial^\ell y^a / \partial x^{i_1} \dots \partial x^{i_\ell}$. Now, J^k becomes a manifold so that the functions $x^i (= x^i \circ \rho_0^k)$, $y^a (= y^a \circ \rho_0^k)$ and $p_{i_1 \dots i_\ell}^b$ ($i_1 \leq \dots \leq i_\ell$, $1 \leq \ell \leq k$) together form a coordinate system of J^k globally defined on it. The coordinate system of J^k thus defined is called the canonical coordinate system of J^k .

Let f be any function defined on an open set O of J^k . Then we define functions df/dx^i on $(\rho_k^{k+1})^{-1}(O)$ in the following manner: Take any point $j_k^{k+1}(u)$ of $(\rho_k^{k+1})^{-1}(O)$, and denote by $j^k(u)$ the map $x' \rightarrow j_x^k(u)$ of a neighborhood of x to J^k . Then df/dx^i are defined by

$$\frac{df}{dx^i}(j_x^{k+1}(u)) = \frac{\partial f \circ j^k(u)}{\partial x^i}(x).$$

In terms of the canonical coordinate systems of J^k and J^{k+1} the functions df/dx^i may be expressed as follows:

$$\frac{df}{dx^i} = \frac{\partial f}{\partial x^i} + \sum_a \frac{\partial f}{\partial y^a} p_i^a + \sum_{\ell=1} \sum' \frac{\partial f}{\partial p_{i_1 \dots i_\ell}^a} p_{i_1 \dots i_\ell}^a,$$

where $\sum'$ means a summation with respect to the index a and the system $(i_1, \dots, i_k)$ of indices with $i_1 \leq \dots \leq i_k$.

The differential operators d/dx^i thus obtained are called the formal derivations corresponding to the coordinate system (x^i, y^a) , x^i being considered as independent variables, and y^a as dependent variables. Clearly we have

$$\frac{d}{dx^i} \left(\frac{df}{dx^j} \right) = \frac{d}{dx^j} \left(\frac{df}{dx^i} \right).$$

§2. Pseudo-projective and generalized pseudo-projective structures

2.1. Projective geometry. Let (n, r) be a pair of integers with $1 \leq r \leq n-1$. Let P^n be the n -dimensional projective space over R . In this paragraph we shall give a global formulation of the differential equation defining the r -dimensional projective subspaces of P^n , and explain some related facts.

Let $\Sigma(P^n, r)$ be the set of all r -dimensional projective subspaces of P^n , which may be naturally identified with the Grassmann manifold $Gr(R^{n+1}, r+1)$. Hence $\Sigma(P^n, r)$ is a compact, connected manifold of dimension $(r+1)(n-r)$. We define a relation R in the product manifold $P^n \times \Sigma(P^n, r)$ by

$$R = \{ (p, \alpha) \in P^n \times \Sigma(P^n, r) \mid p \subset \alpha \},$$

where " $p \subset \alpha$ " means the point p lies on the projective subspace α . This relation will be called the fundamental relation in the product manifold. It is easy to see that R is a compact, connected submanifold of the product manifold, and $\dim R = n+r(n-r)$.

Let μ and ν be the projections of R to P^n and $\Sigma(P^n, r)$ respectively: $\mu(x) = p$ and $\nu(x) = \alpha$, where $x = (p, \alpha) \in R$. We easily see that R is a fibred manifold over P^n with projection μ , and at the same time is a fibred manifold over $\Sigma(P^n, r)$ with projection ν . We also see that the fibres of these fibred manifolds are all connected. We denote by F and E the vertical tangent bundles of the fibred manifolds respectively: $F = \mu_*^{-1}(0)$ and $E = \nu_*^{-1}(0)$.

Let us now consider the Grassmann bundle $J(P^n, r)$. If $(p, \alpha) \in R$, the tangent space $T_p(\alpha)$ to the projective subspace α

at p is an r -dimensional contact element to P^n , and hence gives a point of $J(P^n, r)$. We easily see that the correspondence $(p, \alpha) \rightarrow T_p(\alpha)$ gives a diffeomorphism, say ι , of R onto $J(P^n, r)$. Clearly we have $\hat{\mu} \circ \iota = \mu$, $\hat{\mu}$ being the projection of $J(P^n, r)$ onto P^n .

Now, consider the differential system $\iota_*(E)$ on $J(P^n, r)$, which is completely integrable and of rank r .

Lemma 2.1. The maximal integral manifolds of $\iota_*(E)$ consist of all the prolongations $p(\alpha)$ of r -dimensional projective subspaces α .

Proof. If α is an r -dimensional projective subspace, we clearly have $p(\alpha) = \iota(v^{-1}(\alpha))$, where α in the right hand side should be considered as a point of $\Sigma(P^n, r)$. Moreover it is clear that the maximal integral manifolds of E consist: of all the fibres $v^{-1}(\alpha)$ of the fibred manifold R over $\Sigma(P^n, r)$. Therefore the lemma follows.

Let β be an r -dimensional connected submanifold of P^n . By Lemma 2.1 it follows that the prolongation $p(\beta)$ of β is an integral manifold of $\iota_*(E)$, if and only if β is an open submanifold of some r -dimensional projective subspace α . Consequently the pair $(J(P^n, r), \iota_*(E))$ may be regarded as a differential equation for r -dimensional submanifolds of P^n , which defines the r -dimensional projective subspaces.

Let $(u^0, \dots, u^n)$ be any homogeneous coordinate system of P^n . As usual the ratios $u^1/u^0, \dots, u^n/u^0$ define a coordinate system of P^n , which we denote by $(x^1, \dots, x^r, y^1, \dots, y^{n-r})$. Then we remark the obvious fact: With respect to this coordinate system the

equation $(J(P^n, r), \iota_*(E))$ may be expressed as the system of partial differential equations of the second order:

$$\frac{\partial^2 y^a}{\partial x^i \partial x^j} = 0 \quad (1 \leq i, j \leq r, 1 \leq a \leq n-r).$$

Let C and V be the contact system and the vertical tangent bundle of $J(P^n, r)$ respectively.

Lemma 2.2. $V = \iota_*(F)$, and $C = \iota_*(E) + V$ (direct sum).

Proof. Since $\hat{\mu} \circ \iota = \mu$, we have $V = \iota_*(F)$. Since $E \cap F = 0$, it follows that $\iota_*(E) \cap V = 0$. By Lemma 2.1 we have $\iota_*(E) \subset C$. Since $\text{rank } (C/V) = \text{rank } \iota_*(E) = r$, we have therefore seen that $C = \iota_*(E) + V$ (direct sum), proving the lemma.

Finally we write down the fundamental properties of the triplet (R, E, F) :

- (1) $E \cap F = 0$.
- (2) Both E and F are completely integrable.
- (3) The system $E + F$ is non-degenerate.
- (4) $\dim R = n+r(n-r)$, $\text{rank } E = r$, and $\text{rank } F = r(n-r)$.

Indeed (1), (2) and (4) are clear, and (3) follows immediately from Lemmas 1.4 and 2.2.

It should be noted that the triplet $(P^n, \Sigma(P^n, r), R)$, the triplet (R, E, F) and the equation $(J(P^n, r), \iota_*(E))$ are closely interrelated.

2.2. Pseudo-projective structures. Let (n, r) be a pair of integers with $1 \leq r \leq n-1$. Let M be an n -dimensional manifold, and let R be an open fibred submanifold of $J(M, r)$. Let us consider the contact system D and the vertical tangent bundle F

of R , which are nothing but the restrictions of the contact system C and the vertical tangent bundle V of $J(M, r)$ to R respectively. By Lemma 1.5 we know that (i) F is completely integrable, and $F \subset D$, (ii) D is non-degenerate, and the first derived system of D coincides with $T(R)$, and (iii) $\text{rank}(T(R)/D) = n-r$, $\text{rank}(D/F) = r$, and $\text{rank } F = r(n-r)$. Now, let E be a differential system on R . Then the pair (R, E) is called a pseudo-projective structure (of order 2) of bidegree (n, r) on M , if the following conditions are satisfied:

- (P.1) $D = E + F$ (direct sum),
- (P.2) E is completely integrable,
- (P.3) Each fibre of R is connected.

A pseudo-projective structure will be denoted in principle by a capital script, e.g., $\mathcal{X}$, and then the pair (R, E) will be called the expression of $\mathcal{X}$; R itself will be called the underlying fibred manifold of $\mathcal{X}$. The same notation and terminology will be also used for generalized pseudo-projective structures, which will be defined in 2.4. Incidentally we shall introduce, in that paragraph, the notion of (local) isomorphism for the generalized pseudo-projective structures, which of course holds good for the pseudo-projective structures.

The notations being as in the previous paragraph, let us consider the completely integrable system $\iota_*(E)$ on $J(P^n, r)$. By Lemma 2.2 we see that the pair $(J(M, r), \iota_*(E))$ defines a pseudo-projective structure of bidegree (n, r) on the projective space P^n , which we denote by $\mathcal{X}_0$, and call the standard pseudo-projective structure of bidegree (n, r) . We recall that $\mathcal{X}_0$ may be regarded

as the differential equation defining the r -dimensional projective subspaces of P^n .

In general let X be a pseudo-projective structure of bidegree (n,r) on a manifold M , and (R,E) its expression. Then an r -dimensional submanifold α of M is said to be a solution of X , if the prolongation $p(\alpha)$ of α is in R , and is an integral manifold of E .

In this fashion X may be regarded as a differential equation for r -dimensional submanifolds of M . From the global view-point the notion of a solution of X should be generalized as follows: A solution of X in the wide sense is the image α of some integral manifold $\tilde{\alpha}$ of E by the projection μ of R onto M . (Note that the restriction of μ to $\tilde{\alpha}$ is an immersion, and that $\tilde{\alpha}$ is uniquely determined by α .) However, in our later arguments a solution of X will always mean a solution in the former sense, unless otherwise stated.

We shall now give a local expression of X in terms of a canonical coordinate system. In the following the indices i,j,k,l will range over the integers $1, \dots, r$, and the indices a,b over the integers $1, \dots, n-r$.

Let z_0 be any point of R , and let (x^i, y^a, p_j^k) be any canonical coordinate system of R or of $J(M,r)$ at z_0 . Let $\tilde{U}$ be the domain of the coordinate system. Then F is defined, on the intersection $R \cap \tilde{U}$, by the system of Pfaffian equations: $dx^i = dy^a = 0$, and D by the system of Pfaffian equations: $dy^a - \sum_j p_j^a dx^j = 0$. Since $D = E + F$ (direct sum), it follows that there are unique functions f_{ij}^a on the intersection $R \cap \tilde{U}$ such that E is defined there by the

system of Pfaffian equations:

$$\begin{cases} dy^a - \sum_j p_j^a dx^j = 0, \\ dp_i^a - \sum_j f_{ij}^a dx^j = 0. \end{cases}$$

Now, f_{ij}^a may be expressed as functions of (x^k) , (y^b) and (p_k^b) . Let α be an r -dimensional connected submanifold of M . Suppose that $p(\alpha)$ is in $\tilde{U}$, and that α is defined by a system of equations of the following form: $y^a = u^a(x^1, \dots, x^r)$. Then we easily see that α is a solution of X , if and only if (u^a) gives a solution of the following system, say (X) , of partial differential equations of the second order:

$$\frac{\partial^2 y^a}{\partial x^i \partial x^j} = f_{ij}^a((x^k), (y^b), (\frac{\partial y^b}{\partial x^k})).$$

In the special case where $r = 1$, this system simply means the following system of ordinary differential equations of the second order:

$$\frac{d^2 y^a}{dx^2} = f^a(x, (y^b), (\frac{dy^b}{dx})),$$

where the indices a, b range over the integers $1, \dots, n-1$, and we set $x = x_1$ and $f^a = f_{11}^a$.

Using the functions f_{ij}^a , we now define functions f_{ijk}^a by

$$f_{ijk}^a = \frac{\partial f_{ij}^a}{\partial x^k} + \sum_b \frac{\partial f_{ij}^a}{\partial y^b} \cdot p_k^b + \sum_{b, \ell} \frac{\partial f_{ij}^a}{\partial p_\ell^b} \cdot f_{\ell k}^b.$$

Then it can be shown that the system (X) is involutive or it satisfies the integrability conditions:

$$f_{ij}^a = f_{ji}^a, \quad f_{ijk}^a = f_{ikj}^a,$$

which describe that the system E , restricted to $R \cap \tilde{U}$, is completely integrable. For the definition of an involutive system, see [7].

System (X) , thus obtained, is called the local expression of X at z_0 corresponding to the canonical coordinate system (x^i, y^a, p_j^b) of $J(M, r)$ or corresponding to the coordinate system (x^i, y^a) of M . Conversely it is clear that any involutive system of the form (X) can be obtained as a local expression of some pseudo-projective structure X of bidegree (n, r) .

2.3. Projective structures and the associated pseudo-projective structures of bidegree $(n, 1)$. First of all we recall the definition of an affine connection together with some related definitions. For the details see [9] and [10].

Let M be a manifold. We denote by $\mathcal{T}$ and $\mathcal{T}^{(1,1)}$ the sheaves of germs of local cross sections of T and $T^* \otimes T$ respectively, where $T = T(M)$. Let ∇ be a sheaf homomorphism of $\mathcal{T}$ to $\mathcal{T}^{(1,1)}$, which naturally induces a linear map, denoted by the same letter ∇ , of $\Gamma(U, T)$ to $\Gamma(U, T^* \otimes T)$ for any open set U of M . Then ∇ is called an affine connection on M , if it satisfies the following conditions:

$$\nabla(fY) = f\nabla Y + df \otimes Y,$$

where Y is any vector field defined on any open set U of M , and f any function defined on U .

Let ∇ be an affine connection on M . Then ∇ is said to be without torsion, if $\nabla_X Y - \nabla_Y X = [X, Y]$ for any local vector fields X, Y on M . (Note that $\nabla_X Y$ means the contraction of ∇Y by X .)

Now, let $\sigma(t)$ ($t \in I$) be a curve of M , I being an open interval. Then σ is called a geodesic of (M, ∇) , if the tangent vector field $(d\sigma/dt)(t)$ ($t \in I$) to the curve σ is parallel with respect to the connection ∇ . We remark that for any $p \in M$, $X \in T_p(M)$ and $t_0 \in \mathbb{R}$ there is a unique maximal geodesic $\sigma(t)$ ($t \in I$) of (M, ∇) such that $t_0 \in I$, $\sigma(t_0) = p$, and $(d\sigma/dt)(t_0) = X$.

We shall now recall the definition of a projective structure. In the following, affine connections will be always assumed to be without torsion.

Let ∇ and ∇' be two affine connections on a manifold M . Then ∇' is said to be projectively equivalent to ∇ , if the geodesics of (M, ∇') coincide with the geodesics of (M, ∇) . Here, each geodesic of (M, ∇) or of (M, ∇') should be confounded with its image in M . Clearly this relation becomes an equivalence relation. As is well known, ∇' is projectively equivalent to ∇ , if and only if there is a 1-form ζ on M such that

$$\nabla'_X Y = \nabla_X Y + \zeta(X)Y + \zeta(Y)X,$$

where X and Y are any local vector fields on M .

Let M be a manifold, and let $D(M)$ be the set of all affine connections ∇ defined on open sets, say $U(\nabla)$, of M . Then a projective structure on M is a subset D of $D(M)$ which satisfies the following conditions:

- (i) $M = \bigcup_{\nabla \in D} U(\nabla)$,
- (ii) If $\nabla, \nabla' \in D$, and $U(\nabla) \cap U(\nabla') \neq \emptyset$, then ∇' is projectively equivalent to ∇ on the intersection $U(\nabla) \cap U(\nabla')$,
- (iii) D is maximal in the following sense: Let D' be

another subset $D(M)$ satisfying conditions (i) and (ii). If $D \subset D'$, then D' coincides with D .

This definition is particularly valid in the real or complex analytic category, while in the C^∞ category the projective structure D can be reasonably replaced by its subset

$$D_0 = \{ \nabla \in D \mid U(\nabla) = M \},$$

which is nothing but a class of mutually projectively equivalent affine connections on M .

We shall show that to every projective structure there is naturally associated a pseudo-projective structure of bidegree $(n, 1)$.

Let M be an n -dimensional manifold, and let us consider its tangent bundle $T(M)$. Let $\mathring{T}(M)$ be the complement of the zero cross-section of $T(M)$, which is an open fibred submanifold of $T(M)$. We denote by π the projection of $\mathring{T}(M)$ onto M . For each $\lambda \in \mathbb{R} - \{0\}$ we define a transformation τ_λ of $\mathring{T}(M)$ by $\tau_\lambda(y) = \lambda y$ for all $y \in \mathring{T}(M)$. Now, let ∇ be an affine connection on M . As is well known, there is a unique vector field, say $X(\nabla)$, on $\mathring{T}(M)$ having the following property: Let $\sigma(t)$ ($t \in I$) be any geodesic of (M, ∇) with $\dot{\sigma}(t) \neq 0$, where $\dot{\sigma}(t) = (d\sigma/dt)(t)$. Then $\dot{\sigma}(t)$ ($t \in I$) is an integral curve of $X(\nabla)$. We can easily prove the following

Lemma 2.3. (1) $\pi_*(X(\nabla)_y) = y$, $y \in \mathring{T}(M)$.

(2) $(\tau_\lambda)_*(X(\nabla)) = \frac{1}{\lambda} \cdot X(\nabla)$, $\lambda \in \mathbb{R} - \{0\}$.

The vector field $X(\nabla)$ is usually called the spray associated with (M, ∇) .

Let us now consider the Grassmann bundle $J(M, 1)$, and let C

and V be the contact system and the vertical tangent bundle of $J(M,1)$ respectively. We denote by π the projection of $J(M,1)$ onto M , and by ρ the natural map of $\mathring{T}(M)$ onto $J(M,1)$: For each $y \in \mathring{T}(M)$ $\rho(y)$ is the 1-dimensional contact element to M spanned by the vector y . Clearly we have $\pi \circ \rho = \varpi$. We also note that $\mathring{T}(M)$ is a fibred manifold over $J(M,1)$ with projection ρ . Now, let z be any point of $J(M,1)$. Take any point y of $\mathring{T}(M)$ with $z = \rho(y)$, and set $Y = \rho_*(X(\nabla)_y)$. By (1) of Lemma 2.3 we have $\pi_*(Y) = y(\in z)$, from which follows that $Y \in C(z)$, and $Y \notin V(z)$. This being said, we denote by $E(z)$ the 1-dimensional subspace of $C(z)$ spanned by Y . By (2) of Lemma 2.3 we then find that $E(z)$ does not depend on the choice of y . Now, set $E(\nabla) = \bigcup_z E(z)$. Then we see that $E(\nabla)$ is a subbundle of C , and $C = E(\nabla) + V$ (direct sum). Hence the pair $(J(M,1), E(\nabla))$ defines a pseudo-projective structure of bidegree $(n,1)$, which we denote by $X(\nabla)$.

Let $\sigma(t)$ ($t \in I$) be any geodesic of (M, ∇) . We define a curve $\hat{\sigma}(t)$ ($t \in I$) of $J(M,1)$ by $\hat{\sigma}(t) = \rho(\dot{\sigma}(t))$ for all $t \in I$. $\dot{\sigma}(t)$ being an integral curve of $X(\nabla)$, we can easily prove the following

Lemma 2.4. The image $\hat{\sigma}(I)$ of I by $\hat{\sigma}$ is an integral manifold of $E(\nabla)$, and hence the image $\sigma(I)$ of I by σ is a solution of $X(\nabla)$ in the wide sense. Furthermore the integral manifolds of $E(\nabla)$ are exhausted by the submanifolds $\hat{\sigma}(I)$ of $J(M,1)$ corresponding to geodesics $\sigma(t)$ ($t \in I$) of (M, ∇) .

Now, let D be a projective structure on an n -dimensional manifold M . For any $\nabla \in D$ let us consider the pseudo-projective structure $X(\nabla)$ or $(J(U(\nabla), 1), E(\nabla))$ of bidegree $(n,1)$ on the

domain $U(\nabla)$ of ∇ . Note that $J(U(\nabla), 1)$ is an open submanifold of $J(M, 1)$, and $J(M, 1) = \bigcup_{\nabla \in D} J(U(\nabla), 1)$. By Lemma 2.4 it follows that if $\nabla, \nabla' \in D$, and $U(\nabla) \cap U(\nabla') \neq \emptyset$, the systems $E(\nabla)$ and $E(\nabla')$ coincide on the intersection $J(U(\nabla), 1) \cap J(U(\nabla'), 1)$. Therefore the systems $E(\nabla)$ can be pieced together to yield a differential system, say $E(D)$, defined on the whole of $J(M, 1)$, and the pair $(J(M, 1), E(D))$ defines a pseudo-projective structure of bidegree $(n, 1)$ on M , which we denote by $X(D)$. Clearly a solution (in the wide sense) of $X(D)$ means a geodesic of the projective structure D . Thus $X(D)$ will be sometimes called the differential equation of geodesics of D .

Let X be a pseudo-projective structure of bidegree $(n, 1)$ on a manifold M , and (R, E) its expression. In terms of a local expression of X we shall discuss conditions that X can be obtained as a restriction of the pseudo-projective structure $X(D)$ associated with some projective structure D on M . In the following the indices i, j, k will range over the integers $1, \dots, n-1$, and the indices a, b, c over the integers $0, 1, \dots, n-1$.

Let (x, y, p) be any canonical coordinate system of R at any point z_0 , where $y = (y^i)$ and $p = (p^i)$, and let us consider the corresponding local expression of X at z_0 :

$$\frac{d^2 y^i}{dx^2} = f^i(x, y, \frac{dy}{dx}).$$

Proposition 2.5 (cf. [3]). The following two statements are mutually equivalent:

- (1) There is a projective structure D on M such that E is the restriction of $E(D)$ to R .

(2) The functions $f^i(x, y, p)$ take the following forms:

$$f^i(x, y, p) = p^i g^0(x, y, p) + g^i(x, y, p).$$

Here, $g^a(x, y, p)$ are polynomials of the variable p of order at most 2 with functions of the variables x, y as coefficients.

In particular we consider the case where $n = 2$. Let (x, y, p) be any canonical coordinate system of R at any point z_0 , and let us consider the corresponding local expression of X at z_0 :

$$\frac{d^2 y}{dx^2} = f(x, y, \frac{dy}{dx}).$$

Then we remark that the second statement in the proposition means the following: The function $f(x, y, p)$ is a polynomial of the variable p of order at most 3 with functions of the variables x, y as coefficients, that is, $f(x, y, p)$ takes the following form:

$$f(x, y, p) = A(x, y) + 3B(x, y)p + 3C(x, y)p^2 + D(x, y)p^3.$$

Proof of proposition 2.5. The implication (1) $\Rightarrow$ (2).

Clearly we may assume that there is an affine connection ∇ defined on the whole of M such that $X = X(\nabla)$, i.e., $R = J(M, 1)$ and $E = E(\nabla)$. Let (x, y, p) be any canonical coordinate system of R at any point z_0 , and let $\tilde{U}$ be its domain. Denote by $(x^0, \dots, x^{n-1})$ the coordinate system $(x, y^1, \dots, y^{n-1})$ of M , and consider the coefficients Γ_{bc}^a of ∇ with respect to (x^a) : $\nabla_{X_b} X_c = \sum_a \Gamma_{bc}^a X_a$, where $X_a = \partial/\partial x^a$. Note that $\Gamma_{bc}^a = \Gamma_{cb}^a$, because ∇ is without torsion. Now, let $\sigma(t)$ ($t \in I$) be any geodesic of (M, ∇) with $\hat{\sigma}(I) = p(\sigma(I)) \subset \tilde{U}$. If we put $x^a(t) = x^a(\sigma(t))$, the functions $x^a(t)$ satisfy the following system of ordinary differential equations:

$$\frac{d^2 x^a}{dt^2} + \sum_{b,c} \Gamma_{bc}^a \cdot \frac{dx^b}{dt} \cdot \frac{dx^c}{dt} = 0,$$

where Γ_{bc}^a should be preferably replaced by $\Gamma_{bc}^a(x^0, \dots, x^{n-1})$. By Lemma 2.4 $\sigma(I)$ is a solution of X , and is defined by a system of equations of the following form: $x^i = x^i(x^0)$. Then our task is to seek the system of ordinary differential equations satisfied by the functions $x^i(x^0)$. Since $x^i(t) = x^i(x^0(t))$, we have

$$\frac{dx^i}{dt} = \frac{dx^i}{dx^0} \cdot \frac{dx^0}{dt}, \quad \frac{d^2 x^i}{dt^2} = \frac{d^2 x^i}{(dx^0)^2} \cdot \left(\frac{dx^0}{dt}\right)^2 + \frac{dx^i}{dx^0} \cdot \frac{d^2 x^0}{dt^2}.$$

Hence we easily find that the desired system is

$$\frac{d^2 x^i}{(dx^0)^2} = \frac{dx^i}{dx^0} \cdot p^0 - p^i,$$

where

$$p^a = \Gamma_{00}^a + 2 \sum_k \Gamma_{0k}^a \cdot \frac{dx^k}{dx^0} + \sum_{j,k} \Gamma_{jk}^a \cdot \frac{dx^j}{dx^0} \cdot \frac{dx^k}{dx^0}.$$

We have therefore seen that the functions f^i corresponding to the canonical coordinate system (x, y, p) take the forms as stated in the proposition.

The implication $(2) \Rightarrow (1)$. From the discussion above we deduce that for each $z_0 \in R$ there is an affine connection $\nabla \in D(M)$ such that $z_0 \in J(U(\nabla), 1)$, and $E = E(\nabla)$ on some neighborhood of z_0 . Now, let D be the set of all affine connections $\nabla \in D(M)$ such that $E = E(\nabla)$ on the intersection $R \cap J(U(\nabla), 1)$. Then, in view of the remark above we can easily verify that D is a projective structure on M , and E is the restriction of $E(D)$ to R . (Note that each fibre of R is assumed to be connected.)

we have thus completed the proof of the proposition.

2.4. Generalized pseudo-projective structures. Let (n, r) be a pair of integers with $1 \leq r \leq n-1$. Let M be an n -dimensional manifold, and let R be an imbedded fibred submanifold of $J(M, r)$. Let us consider the contact system D and the vertical tangent bundle F of R . Concerning these systems, recall Lemma 1.7. Now, let E be a differential system on R . Then the pair (R, E) is called a generalized pseudo-projective structure of bidegree (n, r) on M , if the following conditions are satisfied:

- (GP.1) $D = E + F$ (direct sum),
- (GP.2) E is completely integrable,
- (GP.3) Each fibre of R is connected,
- (GP.4) D is non-degenerate,
- (GP.5) The full derived system of D coincides with $T(R)$.

We notice that under conditions (GP.1) and (GP.2), condition (GP.4) is equivalent to the condition that $\text{ch}(D) \cap E = 0$, $\text{ch}(D)$ being the Cauchy characteristic system of D . Indeed, we have $\text{ch}(D) \cap F = 0$ by Lemma 1.7, and $\text{ch}(D) = \text{ch}(D) \cap E + \text{ch}(D) \cap F$ by Lemma 3.3 in the next section.

Let X be a generalized pseudo-projective structure of bidegree (n, r) on M , and (R, E) its expression. In the same fashion as in the case of a pseudo-projective structure X may be regarded as a differential equation: A solution of X is an r -dimensional submanifold α of M of which the prolongation $P(\alpha)$ is in R , and is an integral manifold of E . It should be noted that if $\dim R < \dim J(M, r)$, R itself may be regarded as a differential equation.

Furthermore a solution of X in the wide sense is the image $\mu(\tilde{\alpha})$ of an integral manifold $\tilde{\alpha}$ of E by the projection μ of R onto M .

Let X and X' be two generalized pseudo-projective structures of bidegree (n,r) on manifolds M and M' , and let (R,E) and (R',E') be their expressions, respectively. By an isomorphism of X to X' we mean a diffeomorphism ϕ of M onto M' such that the prolongation $p(\phi)$ of ϕ naturally induces an isomorphism of (R,E) to (R',E') as manifolds with differential system. Clearly a diffeomorphism ϕ of M onto M' is an isomorphism of X to X' , if and only if it naturally induces a one-to-one correspondence between the solutions of X and the solutions of X' . Now, let μ and μ' be the projections of R onto M and M' respectively, and let (z_0, z'_0) be a point of $R \times R'$. By a local isomorphism of X to X' at (z_0, z'_0) we mean a local diffeomorphism ϕ of M to M' at $(\mu(z_0), \mu'(z'_0))$ such that the prolongation $p(\phi)$ of ϕ naturally induces a local isomorphism of (R,E) to (R',E') at (z_0, z'_0) as manifolds with differential system.

We shall now make some remarks on the geometry of generalized pseudo-projective structures X .

(1) In the next paragraph we shall introduce the notion of the arithmetic distance ρ in M associated with X , and in the next section the notion of the dual X^* of X under some additional conditions on X , which is a generalized pseudo-projective structure on the manifold of maximal solutions of X . We shall find that conditions (GP.5) and (GP.4) play crucial roles in the definitions of these notions respectively.

(2) Let us consider the sheaf $\mathcal{a}$ of germs of local

infinitesimal automorphisms of X or precisely of the associated generalized pseudo-projective system R . (For the definition of this notion see the next section.) In §5 we shall show that under certain natural conditions the structure X is of finite type in the sense that each stalk of a is of finite dimension (see Proposition 5.9). We then remark that conditions (GP.4) and (GP.5) likewise play crucial roles in the proof of this fact.

(3) In Chapter VI we shall introduce the notion of pseudo-projective structures of order $k \geq 3$, being special kinds of generalized pseudo-projective structure. Then we shall see that the geometry of systems of ordinary differential equations of order $k \geq 3$ may be represented by the geometry of pseudo-projective structures of order k .

Let X be a generalized pseudo-projective structure of bidegree (n,r) on a manifold M , and (R,E) its expression. We shall give a local expression of X in terms of a canonical coordinate system of $J(M,r)$. In the following C and V will denote the contact system and the vertical tangent bundle of $J(M,r)$ respectively. The indices i,j,k will range over the integers $1, \dots, r$, the indices a,b over the integers $1, \dots, n-r$, and the index λ over the integers $1, \dots, \delta$, where $\delta = \dim J(M,r) - \dim R$.

Let z_0 be any point of R , and let (x^i, y^a, p_j^b) be any canonical coordinate system of $J(M,r)$ at z_0 . Let us consider any regular system of local equations $f^\lambda = 0$ for the submanifold R of $J(M,r)$ at z_0 . Namely f^λ are functions defined on a neighborhood O_1 of z_0 in $J(M,r)$ such that $R \cap O_1 = \{ z \in O_1 \mid f^\lambda(z) = 0 \}$, and the

differentials df^λ are linearly independent at z_0 . Let us also take any differential system $\tilde{E}$ defined on a neighborhood O_2 of z_0 in $J(M, r)$ such that $\tilde{E} \subset C$, and $\tilde{E} = E$ on the intersection $R \cap O_2$. Clearly we have $C = \tilde{E} + V$ (direct sum) on a sufficiently small neighborhood O of z_0 in $J(M, r)$. Therefore it follows that there are unique functions f_{ij}^a on the neighborhood O such that $\tilde{E}$ is defined there by the system of Pfaffian equations:

$$\begin{cases} dy^a - \sum_j p_j^a dx^j = 0, \\ dp_i^a - \sum_j f_{ij}^a dx^j = 0. \end{cases}$$

Now, f^λ and f_{ij}^a may be expressed as functions of $(x^k), (y^b)$ and (p_k^b) . Let α be an r -dimensional connected submanifold of M . Suppose that the prolongation $p(\alpha)$ of α is in O , and that α is defined by a system of equations of the following form: $y^a = u^a(x^1, \dots, x^r)$. We first see that α is a solution of the equation R , if and only if (u^a) gives a solution of the following system, say (R) , of partial differential equations of the first order:

$$f^\lambda((x^k), (y^b), (\frac{\partial y^b}{\partial x^k})) = 0.$$

We next see that α is a solution of the equation X , if and only if (u^a) gives a solution of the following system, say (X) , of partial differential equations of the second order:

$$\begin{cases} f^\lambda((x^k), (y^b), (\frac{\partial y^b}{\partial x^k})) = 0, \\ \frac{\partial^2 y^a}{\partial x^i \partial x^j} = f_{ij}^a((x^k), (y^b), (\frac{\partial y^b}{\partial x^k})). \end{cases}$$

We note that system (X) is involutive, because the system E is

completely integrable.

System (X) (resp. (R)), thus obtained, will be called a local expression of X (resp. of R) at z_0 corresponding to the canonical coordinate system (x^i, y^a, p_j^b) of $J(M, r)$ or corresponding to the coordinate system (x^i, y^a) of M .

2.5. The arithmetic distance associated with a generalized pseudo-projective structure. Let D be a differential system on a manifold R , and let us consider the full derived system $D^{-\infty}$ of D . By an integral curve of D we mean a curve $\sigma(t)$ ($a \leq t \leq b$) of R such that the tangent vector $(d\sigma/dt)(t)$ to the curve is in $D(\sigma(t))$ for each t . It is well known that if R is connected, and $D^{-\infty} = T(R)$, then any two points x and y of R can be joined by a piece-wise integral curve of D (cf. Chow [6] and Tanaka [19]). Now, let (E_i) be a family of differential systems on R such that $D = \sum_i E_i$, i.e., $D(x) = \sum_i E_i(x)$ for each $x \in R$. Then we have the following stronger result.

Lemma 2.6. If R is connected, and $D^{-\infty} = T(R)$, then any two points x and y of R can be joined by a composite of integral curves of E_i .

Before proceeding to the proof of this fact we shall apply it to the study of generalized pseudo-projective structures.

Let X be a generalized pseudo-projective structure of bidegree (n, r) on a manifold M , and (R, E) its expression. BY a chain in X we mean a system Λ formed by a finite number of points p_i ($0 \leq i \leq k$) of M and $k-1$ maximal solutions α_i ($1 \leq i \leq k$) of X in the wide sense such that $p_{i-1}, p_i \in \alpha_i$ for each $1 \leq i \leq k$.

The non-negative integer k will be called the length of the chain Λ and will be denoted by $k(\Lambda)$. Furthermore the points p_0 and p_k will be denoted respectively by $p(\Lambda)$ and $q(\Lambda)$.

From now on we assume that R or equivalently M is connected.

Lemma 2.7. For any two points p and q of M there is a chain Λ in X joining p and q : $p = p(\Lambda)$ and $q = q(\Lambda)$.

Proof. Let us consider the contact system D of R . Then $D = E + F$, F being the vertical tangent bundle of R , and the full derived system $D^{-\infty}$ of D coincides with $T(R)$. Therefore we know from Lemma 2.6 that any two points z and w of R can be joined by a composite of integral curves of E and F . Furthermore an integral curve of E means a curve in some maximal integral manifold of E , and an integral curve of F a curve in some fibre of R . Now, the lemma follows immediately from these facts.

Let p and q be any two points of M . Owing to Lemma 2.7 we may define a non-negative integer $\rho(p, q)$ to be the minimum of the lengths $k(\Lambda)$ of chains Λ in X joining p and q . It is easy to see that the function $(p, q) \rightarrow \rho(p, q)$ gives a distance (function) in M , which we call the arithmetic distance associated with X . Clearly the arithmetic distance or the assignment $X \rightarrow \rho$ is a global invariant in the geometry of generalized pseudo-projective structures. In particular we know that the arithmetic distance ρ is invariant under the automorphism group $\text{Aut}(X)$ of X : $\rho(\phi(p), \phi(q)) = \rho(p, q)$ for all $\phi \in \text{Aut}(X)$ and $p, q \in M$.

Remark. Let M be an irreducible symmetric R -space, which is a compact and connected symmetric space. Then a submanifold α of M is called a Helgason sphere, if the following conditions are

satisfied: (i) α is a totally geodesic sphere with minimum radius, and (ii) α is of maximum dimension among the submanifolds satisfying condition (i). Among others it is known that Helgason spheres are congruent to each other under the largest connected group of isometries of M , and that any two points p and q of M can be joined by a chain of Helgason spheres. For any $p, q \in M$ one may therefore define a non-negative integer $\rho(p, q)$ to be the minimum of the lengths of chains of Helgason spheres joining the two points. Then the function $(p, q) \mapsto \rho(p, q)$ gives a distance in M , which is called the arithmetic distance in the symmetric R-space M . (A detailed account of these can be found in Takeuchi [17].) In §7 we shall make some studies on the arithmetic distances associated with generalized pseudo-projective structures. Especially we shall see that the arithmetic distance in a Grassmann manifold, being an irreducible symmetric R-space, is obtained as the arithmetic distance associated with a suitable generalized pseudo-projective structure on the manifold. It is reasonably expected that the same fact exists concerning an arbitrary irreducible symmetric R-space.

We shall now prove Lemma 2.6.

First of all we prepare some general notations. Given a differential system H on R , $\Gamma(H)$ denotes the space of all cross sections of H , and $\Gamma_0(H)$ the subspace of $\Gamma(H)$ consisting of all cross sections of H with compact support. We set $T = T(R)$. For any $X \in \Gamma_0(T)$ let $(\phi_t)_{t \in R}$ be the one-parameter group of transformations of R generated by X . Then e^X denotes the transformation ϕ_1 , which is usually called the exponential map corresponding to X .

Note that $\phi_t = e^{tX}$.

Now, we denote by G the group of all the transformations of R of the following form: $e^{X_1} \dots e^{X_k}$, where each X_λ is a vector field in $\Gamma_0(E_i)$ for some i . We then denote by $\tilde{G}$ the set of all the one-parameter families $(\phi_t)_{t \in R}$ of transformations of R such that $\phi_0 = 1$, the identity transformation of R , and $\phi_t \in G$ for all t . Furthermore, for each $(\phi_t) \in \tilde{G}$ we denote by $V((\phi_t))$ the vector field on R induced by (ϕ_t) :

$$V((\phi_t))_x = \left. \frac{\partial}{\partial t} \phi_t(x) \right|_{t=0}, \quad x \in R.$$

We then denote by g the set of all the vector fields on R of the form: $V((\phi_t))$.

Lemma 2.8. g is a subspace of $\Gamma(T)$, and if $\phi \in G$ and $X \in g$, then $\phi_*(X) \in g$.

Proof. let (ϕ_t) and (ψ_t) be any two families in $\tilde{G}$. Then we have $(\phi_t \psi_t) \in \tilde{G}$, and $V((\phi_t \psi_t)) = V((\phi_t)) + V((\psi_t))$. If $\lambda \in R$, we have $(\phi_{\lambda t}) \in \tilde{G}$, and $V((\phi_{\lambda t})) = \lambda V((\phi_t))$. Furthermore if $\phi \in G$, we have $(\phi \circ \phi_t \circ \phi^{-1}) \in \tilde{G}$, and $V((\phi \circ \phi_t \circ \phi^{-1})) = \phi_* V((\phi_t))$, proving the lemma.

For each $x \in R$ we define a subspace $g(x)$ of $T(x)$ by $g(x) = \{ X_x \mid X \in g \}$.

Lemma 2.9. If $x_1, \dots, x_\ell \in g$ and $x \in R$, then $(\text{ad}(X_1) \dots \text{ad}(X_{\ell-1})X_\ell)_x \in g(x)$.

Proof. We prove this by induction on the integer ℓ . k being a positive integer, assume that the lemma is true for any $\ell < k$. Let $X_1, \dots, X_k$ be any k vector fields in g , and take $(\phi_t) \in \tilde{G}$ with

$X_k = V((\phi_t))$. By Lemma 2.8 we have $\phi_{t*}(X_{k-1}) \in g$. By the assumption of induction we therefore obtain

$(\text{ad}(X_1) \cdots \text{ad}(X_{k-2}) \phi_{t*}(X_{k-1}))_x \in g(x)$. Since

$$\frac{\partial}{\partial t}(\phi_{t*}(X_{k-1}))_y \big|_{t=0} = [X_{k-1}, X_k]_y, \quad y \in R,$$

it follows that

$$\begin{aligned} & (\text{ad}(X_1) \cdots \text{ad}(X_{k-1}) X_k)_x \\ &= \frac{\partial}{\partial t}(\text{ad}(X_1) \cdots \text{ad}(X_{k-1}) \phi_{t*}(X_{k-1}))_x \big|_{t=0} \in g(x), \end{aligned}$$

proving the lemma.

If $X \in \Gamma_0(E_i)$, we have $e^{tX} \in G$, and hence $X = V((e^{tX})) \in g$. Therefore we obtain $\Gamma_0(E_i) \subset g$. Since $D = \bigcup_i E_i$, and $D^{-\infty} = T$, it follows from Lemma 2.9 that $g(x) = T(x)$ for each $x \in R$.

We show that the group G acts transitively on R . Indeed, let x_0 be any point of R . Set $n = \dim R$. Since $g(x_0) = T(x_0)$, we can find n vector fields $X_1, \dots, X_n$ in g such that the n vectors $(X_1)_{x_0}, \dots, (X_n)_{x_0}$ form a basis of $T(x_0)$. For each $1 \leq a \leq n$ take $(\phi_t^a) \in \tilde{G}$ with $X_a = V((\phi_t^a))$, and set $\phi_t = \phi_{t1}^1 \cdots \phi_{tn}^n$ for any $t = (t^1, \dots, t^n) \in R^n$. Clearly we have $\phi_0 = 1$, and $(\partial/\partial t^a) \phi_t(x_0) \big|_{t=0} = (X_a)_{x_0}$. Therefore the map $t \mapsto \phi_t(x_0)$ gives a diffeomorphism of a neighborhood of 0 onto a neighborhood of x_0 . Since R is connected, we have thus shown that G acts transitively on R .

Now, let x_0 and x be any two points of R . Since G acts transitively on R , we can find $\phi \in G$ such that $x = \phi(x_0)$. By the very definition of the group G ϕ may be written in the form:

$\phi = e^{Y_N} \dots e^{Y_1}$, where for each $1 \leq A \leq N$ Y_A is in $\Gamma_0(E_i)$ for some i . If we set $x_A = e^{Y_A} \dots e^{Y_1}(x_0)$, we have $x = x_N$ and $x_A = e^{Y_A}(x_{A-1})$. We also see that the curve $e^{tY_A}(x_{A-1})$ ($0 \leq t \leq 1$) of R joins the two points p_{A-1} and p_A , and is an integral curve of a suitable system E_i .

We have thus completed the proof of Lemma 2.6.

Remark. We recall that $D^{-\infty}$ is a completely integrable system possibly with singularities. By an integral manifold of $D^{-\infty}$ we mean a connected submanifold N of R such that $T_x(N) = D^{-\infty}(x)$ at each $x \in N$. Now, assume that D together with R is real analytic. Then we know that through every point $x_0 \in R$ there passes a unique (real analytic) maximal integral manifold, say R' , of $D^{-\infty}$ (see Appendix A). Let R'_0 be the set of all the points $x \in R$ which can be joined by composites of (differentiable) integral curves of E_i . Then we assert that $R'_0 = R'$. Indeed we easily have $R'_0 \subset R'$. Let D' and E'_i denote the restrictions of D and E_i to R' respectively, which are differential systems on R' . Clearly $D' = \sum_i E'_i$, and the full derived system of D' coincides with the restriction, say $D'^{-\infty}$, of $D^{-\infty}$ to R' . Since $D'^{-\infty} = T(R')$, we see from Lemma 2.6 that $R'_0 = R'$, proving our assertion.

In particular it follows that if $R'_0 = R$, then R is connected, and $D^{-\infty} = T(R)$. This shows that the converse of Lemma 2.6 is true, provided D together with R is real analytic. Accordingly consider a generalized pseudo-projective structure X . Then we have seen that under the real analyticity assumption on X condition (GP.5) is the very condition assuring the existence of the arithmetic distance associated with X .

§3. Generalized pseudo-projective systems and the duality in the generalized pseudo-projective structures

3.1. Product and pseudo-product manifolds. Let us consider the product $M \times N$ of two manifolds M and N , which we simply denote by R . Then R becomes naturally fibred manifolds at once over M and N , and we denote by F and E their vertical tangent bundles respectively: $F = \mu_*^{-1}(0)$ and $E = \nu_*^{-1}(0)$, μ and ν being the projections of R onto M and N respectively. Clearly the pair (E, F) satisfies the following conditions:

- (1) $T(R) = E + F$ (direct sum),
- (2) Both E and F are completely integrable.

This beins said, a product structure on a manifold R is a pair formed by differential systems E and F on R satisfying conditions (1) and (2). Then a product manifold is a manifold R equipped with a product structure (E, F) .

Now, let R' be a product manifold, and (E', F') its product structure. Let R be a submanifold of R' . Then we denote by E and F the pull-backs of E' and F' to R respectively: $E = \iota_*^{-1}(E')$ and $F = \iota_*^{-1}(F')$, ι being the inclusion map of R to R' . These are differential systems possibly with singularities on R , and let us thus consider the following regularity condition on R :

. Both E and F are subbundles of $T(R)$ or equivalently both $\dim E(x)$ and $\dim F(x)$ ($x \in R$) are constant.

If R satisfies the regularity condition, the pair (E, F) clearly satisfies the following conditions:

- (i) $E \cap F = 0$,
- (ii) Both E and F are completely integrable.

This being said, a pseudo-product structure on a manifold R is a pair formed by differential systems E and F on R satisfying conditions (i) and (ii). Then a pseudo-product manifold is a manifold R equipped with a pseudo-product structure (E, F) or the triplet (R, E, F) . It will be denoted in principle by a capital script, e.g., $\mathcal{R}$, and then the triplet (R, E, F) will be called the expression of $\mathcal{R}$. The same notation and terminology will be also used for almost pseudo-product manifolds which will be defined in 5.1.

Here, we remark that we already have the notion of (local) isomorphism for the pseudo-product manifolds, because they form a special class of manifolds with family of differential systems (see (a) in 1.2.). The same remark holds good for the almost pseudo-product manifolds.

We shall now explain some fundamental facts on pseudo-product manifolds.

As we have seen, a submanifold R of a product manifold R' becomes naturally a pseudo-product manifold, provided R satisfies the regularity condition. Conversely let R be a pseudo-product manifold, and (E, F) its pseudo-product structure. Assume that R is regular with respect to E and F , and set $M = R/F$, $N = R/E$, and $R' = M \times N$. Denote by μ and ν the projections of R onto M and N respectively, and define a map $\mu * \nu$ of R to R' by

$$(\mu * \nu)(x) = (\mu(x), \nu(x)), \quad x \in R.$$

Since $E \cap F = 0$, we see that $\mu * \nu$ is an immersion. Moreover it is clear that the pseudo-product structure (E, F) of R is induced from the product structure (E', F') of R' by $\mu * \nu$, that is, E and F

are the pull-backs of E' and F' by $\mu \ast \nu$ respectively.

In view of Lemma 1.1. we have therefore proved the following.

Proposition 3.1. Any pseudo-product manifold R can be locally realized as a submanifold of a product manifold. More precisely, let x_0 be any point of R . Then there are a product manifold R' and an imbedding ι of a neighborhood of x_0 to R' such that the pseudo-structure (E, F) of R , restricted to the neighborhood, is induced from the product structure (E', F') of R' by ι , and such that

$$2 \dim R = \dim R' + \text{rank } E + \text{rank } F.$$

In connection with this proposition, we also add the following proposition, of which the proof is left to the readers as an exercise.

Proposition 3.2. For $i (=1 \text{ or } 2)$ let R'_i be a product manifold, and R_i its submanifold. Assume that R_i satisfies the regularity condition, and $2 \dim R_i = \dim R'_i + \text{rank } E_i + \text{rank } F_i$, where (E_i, F_i) denotes the pseudo-product structure of R_i . Then a diffeomorphism ϕ of R_1 onto R_2 is an isomorphism of R_1 onto R_2 as pseudo-product manifolds, if and only if it is extended to a local isomorphism, say $\tilde{\phi}$, of R'_1 to R'_2 as product manifolds. Furthermore $\tilde{\phi}$ is uniquely, determined as a germ.

Finally, let (R, E, F) be a pseudo-product manifold. Since $E \cap F = 0$, the sum $E + F$ gives a subbundle of $T(R)$, which we denote by D . Let us consider the Cauchy characteristic system $\text{ch}(D)$ of D (see (e) in 1.2.). Then we have the following

Lemma 3.3. $\text{ch}(D) = \text{ch}(D) \cap E + \text{ch}(D) \cap F.$

Proof. Consider the torsion γ_x of D at $x \in R$. Since both E and F are completely integrable, we see that $\gamma_x(X, Y) = 0$ for any vectors X and Y in $E(x)$ or in $F(x)$. The lemma follows easily from this fact.

3.2. Generalized pseudo-projective structures and generalized pseudo-projective systems. Let R be a pseudo-product manifold, and (R, E, F) its expression. Then R is called a generalized pseudo-projective system, if the following conditions are satisfied:

((GP.1)) The system $D(= E + F)$ is not reduced to the zero system, and is non-degenerate,

((GP.2)) The full derived system of D coincides with $T(R)$.

Our discussions from now on will be mainly concerned with condition ((GP.1)). Incidentally we know from Lemma 3.3 that this condition is equivalent to the following condition: $D \neq 0$, and $\text{ch}(D) \cap E = \text{ch}(D) \cap F = 0$.

We shall now give a few definitions on a generalized pseudo-projective system R .

Clearly the triplet (R, F, E) defines a generalized pseudo-projective system, which we denote by R^* , and call the dual of R .

We easily have $1 \leq \text{rank } E \leq \dim R - \text{rank } F - 1$. This being said, let (n, r) be a pair of integers with $1 \leq r \leq n-1$. Then R is said to be of bidegree (n, r) , if $n = \dim R - \text{rank } F$, and $r = \text{rank } E$.

Set $r = \text{rank } E$. Then an r -dimensional submanifold of R is said to be a solution of R , if it is an integral manifold of E . In this fashion R may be regarded as a differential equation for

r -dimensional submanifolds of R .

The next proposition indicates that the correspondence $X \rightarrow \mathcal{R}$ is compatible with the respective local isomorphisms.

Proposition 3.4. Let X and X' be two generalized pseudo-projective structures of bidegree (n,r) on manifolds M and M' , and let $\mathcal{R}$ and $\mathcal{R}'$ be the generalized pseudo-projective systems of bidegree (n,r) , respectively. Let (z_0, z'_0) be a point of $R \times R'$, where R and R' are the underlying fibred manifolds of X and X' respectively. If ϕ is a local isomorphism of X to X' at (z_0, z'_0) , then the prolongation $p(\phi)$ of ϕ , restricted to R , gives a local isomorphism $\tilde{\phi}$ of $\mathcal{R}$ to $\mathcal{R}'$ at (z_0, z'_0) . Conversely if $\tilde{\phi}$ is a local isomorphism of $\mathcal{R}$ to $\mathcal{R}'$ at (z_0, z'_0) , then there is a unique local isomorphism ϕ of X to X' at (z_0, z'_0) such that $\tilde{\phi}$ coincides with $p(\phi)$, restricted to R .

Indeed, the first assertion is clear, and the second follows immediately from Lemma 1.8.

We also remark that the correspondence $X \rightarrow \mathcal{R}$ is compatible with the respective isomorphisms.

Conversely, let $\mathcal{R}$ be a generalized pseudo-projective system of bidegree (n,r) , and (R,E,F) its expression. First of all let us consider the following condition on $\mathcal{R}$:

(C.1) R is regular with respect to F .

Assume this condition, and set $M = R/F$. Since $\text{ch}(D) \cap F = 0$, we see from Lemma 1.9 that there is a unique immersion, say ι , of R to $J(M,r)$ such that $\mu = \hat{\mu} \circ \iota$, and $D = \iota_*^{-1}(C)$, where μ and $\hat{\mu}$ are the projections of R and $J(M,r)$ onto M respectively, and C is the contact system of $J(M,R)$. Recall that ι is concretely given by

$$\iota(x) = \mu_*(D(x)) = \mu_*(E(x)), x \in R.$$

The immersion ι will be called associated with $\mathcal{R}$. Under condition (C.1), let us further consider the following condition on $\mathcal{R}$:

(C.2) ι is an imbedding.

Assume these two conditions. Then the image $\iota(R)$ of R by ι gives an imbedded submanifold of $J(M, r)$, and the image $\iota_*(D)$ of D by ι_* the contact system of $\iota(R)$. Moreover the image $\iota_*(E)$ of E by ι_* gives a differential system on $\iota(R)$, and the pair $(\iota(R), \iota_*(E))$ defines a generalized pseudo-projective structure of bidegree (n, r) on M , which is denoted by X , and will be called associated with $\mathcal{R}$. Clearly $\mathcal{R}$ is associated with X or more precise it is naturally isomorphic to the generalized pseudo-projective system associated with X .

Let us now consider a generalized pseudo-projective structure X of bidegree (n, r) on a manifold M . Let $\mathcal{R}$ be the associated generalized pseudo-projective system of bidegree (n, r) , and (R, E, F) its expression. (Hence (R, E) gives the expression of X .) Then it is clear that R is regular with respect to F , M may be naturally identified with R/F , and the inclusion map of R to $J(M, r)$ may be naturally identified with the immersion ι associated with $\mathcal{R}$. Hence we find that X is just associated with $\mathcal{R}$.

Let (X, z_0) and (X', z'_0) be two generalized pseudo-projective structures of bidegree (n, r) with reference points $z_0 \in R$ and $z'_0 \in R'$, where R and R' are the underlying fibred manifolds of X and X' respectively. Then (X, z_0) and (X', z'_0) are said to be locally isomorphic; if there is a local isomorphism of X to X' at (z_0, z'_0) ; An equivalence class with respect to this equivalence

relation is called a local isomorphism class of generalized pseudo-projective structures of bidegree (n,r) . Similarly we may speak of a local isomorphism class of generalized pseudo-projective systems of bidegree (n,r) .

Now, we remark that every local isomorphism class $\mathcal{R}$ of generalized pseudo-projective systems of bidegree (n,r) has a representative (R, x_0) satisfying conditions (C.1) and (C.2) (cf. Lemma 1.1). By the discussions above we have therefore seen that there is a natural one-to-one correspondence between the local isomorphism classes $\mathcal{X}$ of generalized pseudo-projective structures of bidegree (n,r) and the local isomorphism classes $\mathcal{R}$ of generalized pseudo-projective systems of bidegree (n,r) .

3.3. Pseudo-projective structures, and pseudo-projective systems. Let R be a pseudo-product manifold, and (R, E, F) its expression. Then R is called a pseudo-projective system (of order 2) of bidegree (n,r) , if it is a generalized pseudo-projective system of bidegree (n,r) , and $\dim R = n+r(n-r)$, $\text{rank } E = r$, and $\text{rank } F = r(n-r)$. We remark that R is a pseudo-projective system of bidegree (n,r) , if and only if it satisfies the following conditions:

((P.1)) The system $D(= E + F)$ is non-degenerate,

((P.2)) $\dim R = n+r(n-r)$, $\text{rank } E = r$, and $\text{rank } F = r(n-r)$.

Indeed these conditions imply that the first derived system of D coincides with $T(R)$, which is a special case of Lemma 5.2 in §5. This fact can be also derived from Lemmas 1.5 and 1.9.

Let $\mathcal{X}$ be a pseudo-projective structure of bidegree (n,r) on

a manifold M . Then it is clear that the associated generalized pseudo-projective system $\mathcal{R}$ of bidegree (n, r) is a pseudo-projective system of bidegree (n, r) . Conversely let $\mathcal{R}$ be a pseudo-projective system of bidegree (n, r) . Assuming that $\mathcal{R}$ satisfies conditions (C.1) and (C.2), let us consider the associated generalized pseudo-projective structure X of bidegree (n, r) on the manifold $M(= R/F)$. Then it is also clear that X is a pseudo-projective structure of bidegree (n, r) .

We have thus seen that there is a natural one-to-one correspondence between the local isomorphism classes X of pseudo-projective structures of bidegree (n, r) and the local isomorphism classes $\mathcal{R}$ of pseudo-projective systems of bidegree (n, r) .

3.4. Non-degenerate relations in product manifolds $M \times N$.

Let M and N be manifolds, and let R be a submanifold of the product manifold $M \times N$, which will be considered particularly as a relation in the product manifold. We denote by μ and ν the projections of R to M and N respectively, and set $n = \dim M$, $n' = \dim N$, $d = \dim R$, $r = d - n'$, and $s = d - n$.

Now, suppose that both μ and ν are submersions, implying that $\max(n, n') \leq d \leq n + n'$. Furthermore suppose that $\max(n, n') < d < n + n'$ or equivalently $1 \leq r \leq n - 1$, and $1 \leq s \leq n' - 1$. If $(p, \alpha) \in R$, $\nu^{-1}(\alpha)$ and $\mu(\nu^{-1}(\alpha))$ are r -dimensional submanifolds of R and M respectively, p is in $\mu(\nu^{-1}(\alpha))$, and the tangent space $T_p(\mu(\nu^{-1}(\alpha)))$ gives an r -dimensional contact element to M . This being said, we define a map ι of R to $J(M, r)$ by

$$\iota(x) = T_p(\mu(\nu^{-1}(\alpha))), \quad x = (p, \alpha) \in R.$$

Similarly we define a map κ of R to $J(N,s)$ by

$$\kappa(x) = T_\alpha(v(\mu^{-1}(p))), \quad x = (p, \alpha) \in R$$

Clearly we have

$$\hat{\mu} \circ \iota = \mu, \quad \hat{\nu} \circ \kappa = \nu,$$

where $\hat{\mu}$ and $\hat{\nu}$ are the projections of $J(M,r)$ and $J(N,s)$ onto M and N respectively. Thus we have the following commutative diagram:

$$\begin{array}{ccccc} J(M,r) & \xleftarrow{\iota} & R & \xrightarrow{\kappa} & J(N,s) \\ & \searrow \hat{\mu} & \swarrow \mu \quad \searrow \nu & \swarrow \hat{\nu} & \\ & M & & N & \end{array}$$

The maps ι and κ will be called associated with the relation R .

Let us consider the following two conditions on the relation R :

- (i) $\text{Max}(n,n') < d < n+n'$, and the projections μ and ν are both submersions,
- (ii) Under this condition the maps ι and κ are both immersions.

We denote by F and E the kernels of the differentials μ_* and ν_* respectively: $F = \mu_*^{-1}(0)$, and $E = \nu_*^{-1}(0)$. Clearly condition (i) is equivalent to the following condition:

- (i') $\text{Max}(n,n') < d < n+n'$, and $\dim E(x) = r$ and $\dim F(x) = s$ for each $x \in R$.

Assume condition (i). Then we see that both E and F become subbundles of $T(R)$, and the pair (E,F) gives a pseudo-product structure on R , which is nothing but the one naturally induced from the product structure of $M \times N$. The pseudo-product manifold (R,E,F) is denoted by $\mathcal{R}$, and will be called associated with the

relation R .

We assert that condition (ii) is equivalent to the following condition:

(ii'). Under condition (i') the system $D(= E + F)$ is non-degenerate.

Indeed let $x = (p, \alpha)$ be any point of R . Then we have

$$\iota(x) = T_p(\mu(v^{-1}(\alpha))) = \mu_*(T_x(v^{-1}(\alpha))) = \mu_*(E(x)) = \mu_*(D(x)).$$

Therefore it follows from Lemma 1.9 that $\iota_*^{-1}(0) = \text{ch}(D) \cap F$.

Similarly we have $\kappa_*^{-1}(0) = \text{ch}(D) \cap E$. Furthermore we see from Lemma 3.3 that D is non-degenerate, if and only if $\text{ch}(D) \cap E = \text{ch}(D) \cap F = 0$. We have thus proved our assertion.

These having been observed, we say that R is a non-degenerate relation in the product manifold $M \times N$, if it satisfies conditions (i) and (ii) or equivalently conditions (i') and (ii').

Remark. Let (n, n', d) be a triplet of non-negative integers. Then it can be shown that the following three statements are mutually equivalent:

$$(1) \quad \text{Max}(n, n') < d < n + n',$$

$$\text{Max}\{(d - n')/(d - n), (d - n)/(d - n')\} \leq n + n' - d.$$

(2) There is a non-degenerate relation in a product manifold $M \times N$ such that $n = \dim M$, $n' = \dim N$, and $d = \dim R$.

(3) There is a pseudo-product manifold (R, E, F) which satisfies condition ((GP.1)) in 3.2 and the conditions: $d = \dim R$, $d - n' = \text{rank } E$, and $d - n = \text{rank } F$.

3.5. Generalized pseudo-projective triplets. Let M and N be manifolds, and let R be an imbedded submanifold of $M \times N$. We

preserve the notations as in the previous paragraph. Then the triplet (M, N, R) is called a generalized pseudo-projective triplet, if it satisfies the following conditions:

- (a) R is a non-degenerate relation in the product manifold $M \times N$,
- (b) The full derived system of D coincides with $T(R)$,
- (c) The projections μ and ν are both surjective, and their fibres are all connected,
- (d) The immersions ι and κ are both imbeddings.

Note that conditions (a) and (b) imply $\mathcal{R}$ is a generalized pseudo-projective system.

A generalized pseudo-projective triplet is said to be connected, if R or equivalently M or equivalently N is connected. It is also said to be compact, if R and hence M and N are compact.

Let (M, N, R) and (M', N', R') be two generalized pseudo-projective triplets. Then an isomorphism of (M, N, R) to (M', N', R') is a pair (ϕ, ψ) of diffeomorphisms ϕ of M onto M' and ψ of N onto N' such that the product diffeomorphism $\phi \times \psi$ of $M \times N$ onto $N' \times M'$ maps R onto R' . Consider the generalized pseudo-projective systems $\mathcal{R}$ and $\mathcal{R}'$ associated respectively with the relations R and R' . Then it is clear that an isomorphism (ϕ, ψ) of (M, N, R) to (M', N', R') naturally induces an isomorphism of $\mathcal{R}$ to $\mathcal{R}'$. Conversely it is clear that any isomorphism of $\mathcal{R}$ to $\mathcal{R}'$ can be obtained in this manner.

Let (M, N, R) be as above. We define a relation R^* in $N \times M$ by

$$R^* = \{(\alpha, p) \in N \times M \mid (p, \alpha) \in R\}.$$

Then the triplet (N, M, R^*) clearly gives a generalized pseudo-

projective triplet, which is called the dual of (M, N, R) . Let us identify R^* with R by the bijection $(\alpha, p) \rightarrow (p, \alpha)$. Then we remark that the generalized pseudo-projective system associated with the relation R^* may be identified with the dual $\bar{R}^*$ of $\bar{R}$.

In the following let (M, N, R) be a fixed generalized pseudo-projective triplet. Consider the generalized pseudo-projective system $\bar{R}$ of bidegree (n, r) associated with the relation R . We first remark that R becomes fibred manifolds at one over M and N , their vertical tangent bundles are F and E respectively, and $M = R/F$ and $N = R/E$. Consequently we know that the triplet (M, N, R) is completely determined by $\bar{R}$. We then remark that the immersions ι of R to $J(M, r)$ and κ of R to $J(N, s)$ associated with the relation R are nothing but the immersions associated respectively with $\bar{R}$ and its dual $\bar{R}^*$. Note that these are immediate consequences of conditions (a) and (c).

In view of condition (d) we therefore see that both $\bar{R}$ and $\bar{R}^*$ satisfy conditions (C.1) and (C.2) in 3.2. Hence we may consider the generalized pseudo-projective structures $(\iota(R), \iota_*(E))$ of bidegree (n, r) on M and $(\kappa(R), \kappa_*(F))$ of bidegree (n', s) on N which are associated respectively with $\bar{R}$ and $\bar{R}^*$. These structures are denoted respectively by X and X^* , and will be called associated with the relation R . The structure X^* will be soon called the dual of X , and vice versa.

For simplicity we denote by $\mathcal{P}$ the triplet (M, N, R) , and by $\mathcal{P}^*$ its dual. Let us consider the automorphism groups $\text{Aut}(X)$, $\text{Aut}(\bar{R})$ and $\text{Aut}(\mathcal{P})$ of X , $\bar{R}$ and $\mathcal{P}$ respectively, and similarly the automorphism groups $\text{Aut}(X^*)$, $\text{Aut}(\bar{R}^*)$ and $\text{Aut}(\mathcal{P}^*)$. Then we have the

natural isomorphisms as follows: $\text{Aut}(X) \approx \text{Aut}(R) \approx \text{Aut}(P) \approx \text{Aut}(P^*) \approx \text{Aut}(R^*) \approx \text{Aut}(X^*)$.

Now, let $\text{Sol}(X)$ be the set of maximal solutions of the equation X , and let $\text{Sol}(X, p)$ be the set of maximal solutions of X through any point $p \in M$. Similarly we consider the sets $\text{Sol}(X^*)$ and $\text{Sol}(X^*, \alpha)$, where $\alpha \in N$. Let α be any point of N . By the very definition of the map ι it follows that the image $\iota(v^{-1}(\alpha))$ of $v^{-1}(\alpha)$ by ι coincides with the prolongation $p(\mu(v^{-1}(\alpha)))$ of $\mu(v^{-1}(\alpha))$. Therefore we see that $\mu(v^{-1}(\alpha))$ is a maximal solution of X , and the assignment $\alpha \rightarrow \mu(v^{-1}(\alpha))$ establishes a one-to-one correspondence, say ϕ , between N and $\text{Sol}(X)$. Similarly we see that $v(\mu^{-1}(p))$ is a maximal solution of X^* for any point p of M , and the assignment $p \rightarrow v(\mu^{-1}(p))$ establishes a one-to-one correspondence, say ψ , between M and $\text{Sol}(X^*)$. Clearly we have

$$\psi(p) = \phi^{-1}(\text{Sol}(X, p)), \quad \phi(\alpha) = \psi^{-1}(\text{Sol}(X^*, \alpha)).$$

From these discussions we have seen the following:

(1) X may be characterized as the differential equation defining the submanifolds $\mu(v^{-1}(\alpha))$ of M , and similarly X^* as the differential equation defining the submanifolds $v(\mu^{-1}(p))$ of N .

(2) N may be naturally identified with $\text{Sol}(X)$, so that the submanifolds $\text{Sol}(X, p)$ of $\text{Sol}(X)$ form the maximal solutions of X^* . Hence X^* may be characterized as the differential equation defining the submanifolds $\text{Sol}(X, p)$ of $\text{Sol}(X)$. Similarly X may be characterized as the differential equation defining the submanifolds $\text{Sol}(X^*, \alpha)$ of $\text{Sol}(X^*)$.

We shall construct local expressions of X and X^* in terms of a local expression of the relation R . In the following the

indices i, j will range over the integers $1, \dots, r$, the indices δ, ε over the integers $1, \dots, s$, and the indices a, b over the integers $1, \dots, d-r-s$. Note that $d-r-s = n-r = n'-s$.

Let x_0 be any point of R . We take any coordinate system (x, y) of M at $\mu(x_0)$ adapted to the contact element $\iota(x_0)$ to M , where $x = (x^i)$, any $y = (y^a)$. Similarly we take any coordinate system (z, w) of N at $\nu(x_0)$ adapted to the contact element $\kappa(x_0)$ to N , where $z = (z^\delta)$, and $w = (w^a)$. Then (x, y, z, w) naturally gives a coordinate system of $M \times N$ at x_0 . For simplicity we assume that x, y, z and w vanish at x_0 .

We have $\dim (M \times N) - \dim R = d-r-s$. This being said, let us consider any regular system, $f^a = 0$, of local equations for the submanifold R of $M \times N$ at x_0 (cf. the same system in 2.4). In terms of the coordinate system (x, y, z, w) the relation R is thus defined by the system of relations:

$$f^a(x, y, z, w) = 0.$$

Since $\mu_*(E(x_0)) = \iota(x_0)$, we see that the $n+n'$ differentials df^a , dx^i , dz^δ , and dw^b are linearly independent at x_0 , whence $\det((\partial f^a / \partial y^b)(0)) \neq 0$. It follows that the system of relations above can be solved for y , so that y^a are expressed as functions, say ϕ^a , of the variables x, z and w . Hence the relation R is also defined by the system of relations:

$$y^a = \phi^a(x, z, w).$$

Similarly the given system of relations can be solved for w , so that w^a are expressed as functions, say ψ^a , of x, y and z . Hence the relation R is further defined by the system of relations:

$$w^a = \psi^a(x, y, z).$$

Now, we recall that X may be characterized as the differential equation defining the submanifolds $\mu(v^{-1}(\alpha))$ of M , where $\alpha \in N$. Let U and V be sufficiently small neighborhoods of $\mu(x_0)$ and $v(x_0)$ respectively. Take any point $\alpha \in V$ with the coordinates (z, w) . Then we see that the solution $\mu(v^{-1}(\alpha))$ of X is defined, on U , by the system of relations: $y^a = \phi^a(x, z, w)$. Moreover we have the obvious identities: $f^a(x, \phi(x, z, w), z, w) = 0$. From these facts we deduce the following

Proposition 3.5. A local expression of X at $\mu(x_0)$ corresponding to the coordinate system (x, y) can be obtained by eliminating the variables z and w in the following system of relations:

$$f^a = \frac{df^a}{dx^j} = \frac{d^2 f^a}{dx^i dx^j} = 0.$$

Here, d/dx^j are the formal derivations corresponding to the coordinate system (x, y) (see 1.5). Hence,

$$\begin{aligned} \frac{df^a}{dx^j} &= \frac{\partial f^a}{\partial x^j} + \sum_b \frac{\partial f^a}{\partial y^b} \frac{\partial y^b}{\partial x^j}, \\ \frac{d^2 f^a}{dx^i dx^j} &= \frac{d}{dx^i} \left(\frac{df^a}{dx^j} \right) \\ &= \frac{d}{dx^i} \left(\frac{\partial f^a}{\partial x^j} \right) + \sum_b \frac{d}{dx^i} \left(\frac{\partial f^a}{\partial y^b} \right) \frac{\partial y^b}{\partial x^j} + \sum_b \frac{\partial f^a}{\partial y^b} \frac{\partial^2 y^b}{\partial x^i \partial x^j}. \end{aligned}$$

We shall now explain how to eliminate the variables z and w . In view of this explanation we can easily give a complete proof of the proposition, which is therefore left to the readers as an exercise.

First of all we remark that the variable w can be easily

eliminated by using the relations $w^a = \psi^a(x, y, z)$, so that the system of relations in the proposition is reduced to the following system of relations:

$$\frac{d\psi^a}{dx^j} = \frac{d^2\psi^a}{dx^i dx^j} = 0.$$

We have

$$\frac{d\psi^a}{dx^j} = \frac{\partial\psi^a}{\partial x^j} + \sum_b \frac{\partial\psi^a}{\partial y^b} \frac{\partial y^b}{\partial x^j},$$

and $\det((\partial\psi^a/\partial y^b)(0)) \neq 0$. Hence we may define functions χ_j^b by

$$\frac{\partial\psi^a}{\partial x^j} = \sum_b \frac{\partial\psi^a}{\partial y^b} \chi_j^b.$$

Then the system of relations $d\psi^a/dx^j = 0$ can be replaced by

$$\chi_i^a(x, y, z) + \frac{\partial y^a}{\partial x^i} = 0.$$

By Lemma 3.6 below we can find s pairs $(a_\varepsilon, i_\varepsilon)$ of integers such that $1 \leq a_\varepsilon \leq n-r$, $1 \leq i_\varepsilon \leq r$, and $\det((\partial\chi_\varepsilon/\partial z^\delta)(0)) \neq 0$, where $\chi_\varepsilon = \chi_{i_\varepsilon}^{a_\varepsilon}$. Therefore the system of relations

$$\chi_\varepsilon(x, y, z) + \frac{\partial y^{a_\varepsilon}}{\partial x^{i_\varepsilon}} = 0$$

can be solved for z , so that z^δ are expressed as functions of x, y and $(\partial y^{a_\varepsilon}/\partial x^{i_\varepsilon})$. Substituting these functions for z^δ in the reduced system of relations, we have thus accomplished the elimination.

Lemma 3.6. If $(\xi^\delta) \in R^s$, and $\sum_\delta (\partial\chi_\varepsilon/\partial z^\delta)(0) \cdot \xi^\delta = 0$, then $\xi^\delta = 0$.

Proof. We first remark that (x, y, z) , restricted to R , gives

a coordinate system of R at x_0 . Hence w^a , restricted to R , can be expressed as functions of x, y and z , which are nothing but the functions $\psi^a(x, y, z)$. Hereafter we work on a sufficiently small neighborhood of x_0 in R . Then we see that E is defined by the system of Pfaffian equations: $dz^\delta = dw^a = 0$, and F by the system of Pfaffian equations: $dx^i = dy^a = 0$. Since $w^a = \psi^a(x, y, z)$, we have

$$dw^a = \sum_i \frac{\partial \psi^a}{\partial x^i} dx^i + \sum_b \frac{\partial \psi^a}{\partial y^b} dy^b + \sum_\delta \frac{\partial \psi^a}{\partial z^\delta} dz^\delta.$$

If we set $\theta^a = \sum_i \chi_i^a dx^i + dy^a$, it therefore follows that $D (= E + F)$ is defined by the system of Pfaffian equations: $\theta^a = 0$. An easy calculation proves that

$$d\theta^a \equiv - \sum_\delta \frac{\partial \chi_i^a}{\partial z^\delta} dx^i \wedge dz^\delta \pmod{(\theta^b)}.$$

Since D is non-degenerate at x_0 , the lemma follows immediately from this fact.

Similarly to Proposition 3.5 we have the following

Proposition. A local expression of χ^* at $\kappa(x_0)$ corresponding to the coordinate system (z, w) can be obtained by eliminating the variables x and y in the following system of relations:

$$f^a = \frac{df^a}{dz^\epsilon} = \frac{d^2 f^a}{dz^\epsilon dz^\delta} = 0.$$

Here, d/dz^ϵ are the formal derivations corresponding to the coordinate system (z, w) . Furthermore we remark that the system of relations above can be reduced to the following system of relations

$$\frac{d\phi^a}{dz^\epsilon} = \frac{d^2 \phi^a}{dz^\epsilon dz^\delta} = 0.$$

We shall refer to the proposition above as the dual form of Proposition 3.5.

3.6. The duality in the generalized pseudo-projective structures. Let X be a generalized pseudo-projective structure of bidegree (n,r) on a manifold M , and (R,E) its expression. Let R be the associated generalized pseudo-projective system of bidegree (n,r) , which is thus expressed by (R,E,F) , F being the vertical tangent bundle of R . We remark that the dual R^* of R is a generalized pseudo-projective system of bidegree (n',s) , where $n' = \dim R - \text{rank } E$, and $s = \text{rank } F$.

Let us consider the following two conditions on the structure X :

(D.1) The dual R^* of R satisfies conditions (C.1) and (C.2) in 3.2. Namely, R is regular with respect to E , and the immersion, say κ , of R to $J(N,s)$ associated with R^* is an imbedding, where $N = R/E$. (Note that N may be naturally identified with the set of maximal solutions of X in the wide sense.)

(D.2) Under condition (D.1) the immersion $\mu \times v$ of R to $M \times N$ is an imbedding, where μ and v are the projections of R onto M and N respectively.

Assume these two conditions from now on. Then we may consider the generalized pseudo-projective structure $(\kappa(R), \kappa_*(F))$ of bidegree (n',s) on N associated with the dual R^* of R . We denote this structure by X^* , and call it the dual of X . Clearly X^* satisfies conditions (D.1) and (D.2), and the dual of X^* is X or more precisely it is naturally isomorphic to X .

Remark. Of course X^* makes sense under condition (D.1). Accordingly X^* will be sometimes called the dual of X under this weaker condition.

For simplicity let us identify R with a submanifold of $M \times N$ by the imbedding $\mu \times \nu$. Then it is clear that the triplet (M, N, R) gives a generalized pseudo-projective triplet, and that X and X^* are associated with the relation R . Conversely it is likewise clear that if (M, N, R) is a generalized pseudo-projective triplet, and if X and X^* are the generalized pseudo-projective structures associated with the relation R , then X^* is the dual of X .

Let X be as in the outset, and let us consider the special case where X is a pseudo-projective structure of bidegree (n, r) . If X satisfies conditions (D.1) and (D.2), the dual X^* of X is a generalized pseudo-projective structure of bidegree $((r+1)(n-r), r(n-r))$, which is not necessarily a pseudo-projective structure. Clearly X^* is a pseudo-projective structure, if and only if $r = n-1$. Furthermore if X is a pseudo-projective structure of bidegree $(n, n-1)$, so is X^* , which corresponds to the fact that if R is a pseudo-projective system of bidegree $(n, n-1)$, so is R^* . It should be noted that the duality in the pseudo-projective structures of bidegree $(n, n-1)$ directly generalizes the duality existing between the points and the hyperplanes in the n -dimensional projective geometry.

Here, we remark that we may always speak of the dual X^* of a local isomorphism class X of generalized pseudo-projective structures of bidegree (n, r) , and that the dual of X^* is X . Indeed, in view of Lemma 1.1 it follows that X has a representative (X, z_0)

satisfying conditions (D.1) and (D.2). Let X^* denote the local isomorphism class of generalized pseudo-projective structures of bidegree (n', s) which is represented by (X^*, z_0^*) , where X^* is the dual of X , and $z_0^* = \kappa(z_0)$, κ being as in the definition of condition (D.1). Then it is easy to see that X^* does not depend on the choice of (X, z_0) , and that the dual of X^* is X , confirming our claim.

Now, let X be a pseudo-projective structure of bidegree $(n, 1)$ on a manifold M . We use the notations as in the outset. Assuming conditions (D.1) and (D.2), let us consider the dual X^* of X , which is a generalized pseudo-projective structure of bidegree $(2(n-1), n-1)$ on N . As before we identify R with a submanifold of $M \times N$.

We shall construct a local expression of X^* on the basis of a local expression of X . In the following the indices i, j, k will range over the integers $1, \dots, n-1$.

Let z_0 be any point of R . We take any canonical coordinate system (x, y, p) of R at z_0 , where $y = (y_i)$, and $p = (p_i)$, and consider the corresponding local expression, say (X) , of X :

$$\frac{d^2 y^i}{dx^2} = f^i(x, y, \frac{dy}{dx}).$$

For simplicity we assume that x, y and p vanish at z_0 .

By the existence theorem for solutions of systems of ordinary differential equations, there are unique functions $\phi^i(x, z, w)$ of the variables $x, z = (z^i)$ and $w = (w^i)$, defined on a neighborhood of the origin, which satisfies the following conditions:

- (i) For any fixed z and w the function $x \rightarrow (\phi^i(x, z, w))$ gives

a solution of (X) ,

$$(ii) \quad \phi^i(0, z, w) = w^i, \text{ and } \frac{\partial \phi^i}{\partial x}(0, z, w) = z^i.$$

The family of functions ϕ^i will be called the general solution of (X) .

Now, (x, y) gives a coordinate system of M at $\mu(z_0)$. We also remark that (z, w) may be regarded as a coordinate system of N at $v(z_0)$ or rather of $\text{Sol}(X)$ at $\phi^{-1}(v(z_0))$, so that (z, w) gives the coordinates of the maximal solution of X , being a point of $\text{Sol}(X)$, defined by $y^i = \phi^i(x, z, w)$. Then (x, y, z, w) gives a coordinate system of $M \times N$ at $z_0 \in R$. It is clear that with respect to this coordinate system the relation R in $M \times N$ is defined by the system of relations

$$y^i = \phi^i(x, z, w).$$

Therefore we see from the dual form of Proposition 3.5 that a local expression of X^* at $\kappa(z_0)$ corresponding to the coordinate system (z, w) can be obtained by eliminating the variables x and y in the following system of relations:

$$\frac{d\phi^i}{dz^j} = \frac{d^2\phi^i}{dz^j dz^k} = 0.$$

For example consider the special case where $f^i = 0$, i.e., system (X) takes the following form:

$$\frac{d^2 y^i}{dx^2} = 0.$$

Then we have $\phi^i(x, z, w) = z^i x + w^i$. Therefore the elimination can be easily done, and the eliminated system of relations turns out to be the following system, say (X^*) , of partial differential

$$\begin{cases} \frac{\partial w^i}{\partial z^j} - \frac{1}{n-1} \delta_j^i \sum_k \frac{\partial w^k}{\partial z^k} = 0, \\ \frac{\partial^2 w^i}{\partial z^j \partial z^k} = 0. \end{cases}$$

In the special case where $n = 2$, system (X^*) simply means the following single ordinary differential equation:

$$\frac{d^2 w}{dz^2} = 0.$$

It is also easy to see that if $n \geq 3$, system (X^*) is obtained as the prolongation of the following system of the first order:

$$\frac{\partial w^i}{\partial z^j} - \frac{1}{n-1} \delta_j^i \sum_k \frac{\partial w^k}{\partial z^k} = 0,$$

being a local expression of the equation $\kappa(R)$ at $\kappa(z_0)$. In 5.2 we shall give a further generalization of this fact (see Corollary to Proposition 5.5).

3.7. Pseudo-projective triplets, and some global facts on compact, connected pseudo-projective triplets. Let (n, r) be a pair of integers with $1 \leq r \leq n-1$. A pseudo-projective triplet (of order 2) of bidegree (n, r) is a generalized pseudo-projective triplet (M, N, R) which satisfies the following condition:

(e) $\dim M = n$, $\dim N = (r+1)(n-r)$, and $\dim R = n+r(n-r)$.

Let (M, N, R) be a pseudo-projective triplet of bidegree (n, r) . Let us consider the generalized pseudo-projective system $\tilde{R}$ associated with the relation R , and (R, E, F) its expression. Then we have $\text{rank } E = \dim R - \dim N = r$, and $\text{rank } F = \dim R - \dim M = r(n-r)$, from which follows that $\tilde{R}$ becomes a pseudo-projective

system of bidegree (n, r) . Note that this is a consequence of conditions (a) and (e).

For example consider the fundamental relation R in the product manifold $P^n \times \Sigma(P^n, r)$ (see 2.1). From the observations there we know that the triplet $(P^n, \Sigma(P^n, r), R)$ satisfies conditions (a), (b), (c), and (e). We also know that the immersion ι associated with the relation R gives a diffeomorphism of R onto $J(P^n, r)$. Moreover it can be shown that the immersion κ of R to $J(\Sigma(P^n, r), r(n-r))$ associated with the relation R is an imbedding. Hence the triplet turns out to satisfy condition (d). Since R is compact and connected, we have thus seen that the triplet $(P^n, \Sigma(P^n, r), R)$ becomes a compact and connected pseudo-projective triplet of bidegree (n, r) , which is called the standard pseudo-projective triplet of bidegree (n, r) . Incidentally the pseudo-projective system of bidegree (n, r) associated with the relation R is denoted by $\mathcal{R}_0$, and is called the standard pseudo-projective system of bidegree (n, r) . Clearly the standard pseudo-projective structure $\mathcal{X}_0$ of bidegree (n, r) is associated with $\mathcal{R}_0$, and vice versa. Moreover we remark that if $r = 1$, $\mathcal{X}_0$ is nothing but the pseudo-projective structure $\mathcal{X}(D_0)$ associated with the standard projective structure D_0 on P^n .

These being defined and observed, there naturally arises the following

Problem 1. Study the topological structure of a compact and connected pseudo-projective triplet of bidegree (n, r) . In particular, from the view-point of topology are there any compact and connected pseudo-projective triplets of bidegree (n, r) other than the

standard pseudo-projective triplet of bidegree (n, r) ?

As a first step we treat with this problem in the special case where $r = n-1$. (In Remark 2 below we shall make some comment on the problem in the case where $r = 1$.)

In general let us consider a triplet (M, N, R) formed by manifolds M, N and a submanifold R of $M \times N$ which satisfy conditions (a), (c) and condition (e) with $r = n-1$, and which is compact and connected, i.e., M, N and R are all compact and connected. Thus the triplet is reduced to a compact and connected pseudo-projective triplet of bidegree $(n, n-1)$, if it satisfies condition (d) in addition. (Note that condition (b) follows from condition (a).) For the triplet we preserve the notations as in 3.4. First of all we remark the following: $\dim M = \dim N = n$, and $\dim R = 2n-1$; $v(\mu^{-1}(p))$ and $\mu(v^{-1}(\alpha))$ are $(n-1)$ -dimensional submanifolds of N and M respectively, where $p \in M$, and $\alpha \in N$; ι and κ are immersions of R to $J(M, n-1)$ and $J(N, n-1)$ respectively.

Proposition 3.7 (1) If ι is injective, $v(\mu^{-1}(p))$ is diffeomorphic with the $(n-1)$ -dimensional projective space P^{n-1} for each $p \in M$.

(2) If ι is not injective, $v(\mu^{-1}(p))$ is diffeomorphic with the $(n-1)$ -dimensional sphere S^{n-1} for each $p \in M$.

We also have the dual form of this proposition, which is concerned with the immersion κ and the submanifolds $\mu(v^{-1}(\alpha))$.

Proof of Proposition 3.7. We first see that the pair (R, ι) gives a covering space of $J(M, n-1)$, because both R and $J(M, n-1)$ are compact and connected, and $\dim R = \dim J(M, n-1)$. Let p be any point of M . Since $\hat{\mu} \circ \iota = \mu$, it follows that ι maps $\mu^{-1}(p)$ onto the

fibre $\text{Gr}(T_p(M), n-1)$ of $J(M, n-1)$ at p . Furthermore $\text{Gr}(T_p(M), n-1)$ is diffeomorphic with P^{n-1} , and $\mu^{-1}(p)$ is an $(n-1)$ -dimensional compact, connected submanifolds of R . Hence we see that $\mu^{-1}(p)$ becomes naturally a covering space over $\text{Gr}(T_p(M), n-1) (\approx P^{n-1})$, from which follows immediately the proposition.

Example. Let M be an n -dimensional compact, connected Riemannian manifold, and let d be the distance function on it. For any positive number ϵ we define a relation R in $M \times M$ by

$$R = \{(p, q) \in M \times M \mid d(p, q) = \epsilon\}.$$

As is well known, R becomes a submanifold for a sufficiently small ϵ . Then K.Sugawara showed that if ϵ is further sufficiently small, the triplet (M, N, R) satisfies conditions (a), (c) and condition (e) with $r = n-1$. By Proposition 3.7 we see that both ι and κ are not injective, which can be directly verified as follows: Take any points $p, q \in M$ with $d(p, q) = 2\epsilon$, and let r be the middle point of the geodesic segment joining p and q . Then (r, p) and (r, q) are distinct points of R , and we easily see that $\iota((r, p)) = \iota((r, q))$, proving that ι is not injective.

Especially this example indicates that in our present situation conditions (a), (c) and (e) impose no restrictions on the topologies of M and N . At the same time condition (d) together with these conditions may possibly impose any restrictions on the topologies, thus playing an important role in our problem.

Let (M, N, R) be a compact and connected pseudo-projective triplet of bidegre $(n, n-1)$. In his paper [16] K.Sugawara indeed obtained some results on the homotopy groups of M and N , which may be stated as follows:

Theorem S. If $n = 3$ or $n \geq 6$, the universal covering spaces of M are homotopy spheres. Furthermore if $n = 3$, the fundamental groups of M and N are z_2 or 0 , and if $n \geq 6$, they are z_2 .

Sugawara's proof of the theorem, to a considerable extent, contains geometric studies on the submanifolds $v(\mu^{-1}(p))$ and $\mu(v^{-1}(\alpha))$ which are diffeomorphic with P^{n-1} by Proposition 3.7 and its dual form. Here, we notice that he actually treats with a class of triplets (M, N, R) more or less wider than ours, and obtains much the same results as the theorem.

Remark 1. In connection with the discussions above we recall a problem proposed in [21] (see page 303 in H.M.J., Vol. XIV, NO.3). Let (M, N, R) be a compact and connected pseudo-projective triplet of bidegree $(n, n-1)$. Set $\Omega = (M \times N) - R$. Then, that problem is concerned with the geometry of the product manifold Ω , which is known to be closely related to the geometry of the pseudo-projective system R of bidegree $(n, n-1)$ associated with the relation R (see Propositions 2.5 and 2.6, loc. cit.).

Remark 2 (cf. [2]). Let M be one of the following projective spaces with the standard Riemannian metrics: The m -dimensional projective spaces $P^m(R)$ ($m \geq 2$), the m -dimensional complex projective spaces $P^m(C)$ ($m \geq 2$), the m -dimensional quaternion projective spaces $P^m(Q)$ ($m \geq 2$), and the Cayley projective plane $P^2(Ca)$. Then M is a compact symmetric space of rank one, and is what is called a C_π manifold. Now, we denote by N the set of all maximal geodesics of M , where each maximal geodesic should be confounded with its image in M . We also define a relation R in $M \times N$ by

$$R = \{(p, \alpha) \in M \times N \mid p \subset \alpha\}.$$

If we put $n = \dim M$, it can be shown that N becomes naturally a $2(n-1)$ -dimensional manifold, R becomes a $(2n-1)$ -dimensional submanifold of $M \times N$, and the triplet (M, N, R) gives a compact and connected pseudo-projective triplet of bidegree $(n, 1)$. Furthermore we know that if $M \neq P^n(R)$, M is not homotopically equivalent to $P^n(R)$. Accordingly these observations make a considerable contribution to Problem 1 in the case where $r = 1$.

For the rest of this paragraph everything will be considered in the complex analytic category.

Corresponding to Problem 1 there naturally arises the following

Problem 2. Classify the compact and connected pseudo-projective triplets (M, N, R) of bidegree (n, r) up to (holomorphic) isomorphism. In particular are there any compact and connected pseudo-projective triplets of bidegree (n, r) other than the standard pseudo-projective triplet of bidegree (n, r) ?

Let (M, N, R) be a compact and connected pseudo-projective triplet of bidegree $(n, n-1)$. On the basis of Proposition 3.7 and its dual form, both being considered in the complex analytic category, A. Fujiki proved the following theorem, which gives a partial answer to Problem 2.

Theorem F. The complex manifold M and N are both biholomorphic to the n -dimensional complex projective space $P^n(C)$.

In his proof of the theorem Fujiki above all utilizes Theorem 4.1 in I. Morrow and H. Rossi [11] concerning the characterization of $P^n(C)$.

§4. Preliminaries — Some fundamental facts on the geometry of differential systems —

4.1. General remarks on notations and terminologies.

(A) Graded Lie algebras. Let K be the field R of real numbers or the field C of complex numbers, and let Z be the additive group of integers. Then a graded Lie algebra or simply a GLA over K is a pair formed by a Lie algebra g over K and a family $(g_p)_{p \in Z}$ of subspaces of g which satisfy the following conditions:

- (i) $\dim g < \infty$,
- (ii) $g = \sum_p g_p$ (direct sum),
- (iii) $[g, g_q] \subset g_{p+q}$.

A graded Lie algebra will be denoted by a capital Germanic letter, e.g., G , and then the pair $(g, (g_p))$ will be called the expression of G ; The family (g_p) itself will be called the gradation of G . Furthermore a graded Lie algebra will be sometimes confounded with the underlying Lie algebra g .

Concerning graded Lie algebras we have the various notions as follows: isomorphism, homomorphism, graded subalgebra, graded ideal, direct sum, and so on.

If G is a graded Lie algebra, we remark that g_0 becomes a subalgebra, and the Lie algebra g_0 is naturally represented on each subspace g_p of g , because $[g_0, g_p] \subset g_p$.

(B) Principal fibre bundles. At the outset we make some remarks on Lie groups. In this book Lie groups will be always assumed to satisfy the second axiom of countability. Let G be a Lie group. By the Lie algebra g of G we mean the Lie algebra of all left invariant vector fields on G . As usual Ad denotes the

adjoint representation of the Lie group G on the vector space $\mathfrak{g}$, while ad the adjoint representation of the Lie algebra $\mathfrak{g}$ on the vector space $\mathfrak{g}$. Furthermore $\exp$ denotes the exponential map of $\mathfrak{g}$ to G .

Now, let G be a Lie group. A (differentiable) principal fibre bundle over a manifold M with structure group G is a pair formed by a fibred manifold P over M and an action of G on P which satisfy the following conditions:

(i) G acts freely on P (on the right) by the rule $(z, a) \mapsto z \cdot a$, where $z \in P$, and $a \in G$.

(ii) Two points z and z' of the fibred manifold P are in the same fibre, if and only if $z' = z \cdot a$ for some $a \in G$.

A principal fibre bundle will be denoted simply by P .

Let P and P' be principal fibre bundles over manifolds M and M' with the same structure group G respectively. Then a (bundle) isomorphism of P to P' is a diffeomorphism ψ of P onto P' such that

$$\psi(z \cdot a) = \psi(z) \cdot a, \quad z \in P, a \in G.$$

An isomorphism ψ of P to P' naturally induces a diffeomorphism, say ϕ , of M onto M' : $\pi' \circ \psi = \phi \circ \pi$, π and π' being the projections of P and P' onto M and M' respectively. In particular ψ is called an equivalence of P to P' , if ψ is base-preserving, i.e., $M = M'$, and ϕ is the identity map of M .

Let P be a principal fibre bundle over a manifold M with structure group G . We first remark that P is locally trivial. Indeed let x be any point of M , and σ any local cross section of the fibred manifold P defined on a neighborhood U of x . Let us

consider the inverse image $\pi^{-1}(U)$ of U by the projection π of P onto M , and define a map ψ of $U \times G$ to $\pi^{-1}(U)$ by $\psi(x,a) = g(x) \cdot a$ for all $x \in U$ and $a \in G$. Then it can be easily shown that ψ gives an equivalence of $U \times G$ to $\pi^{-1}(U)$, where both $U \times G$ and $\pi^{-1}(U)$ should be naturally considered as principal fibre bundles.

Example. Let G be a Lie group, and H its closed subgroup. Let us consider the homogeneous space G/H of G over H , and π the projection of G onto G/H . Then the group H naturally acts on G on the right, and G becomes a principal fibre bundle over G/H with structure group H with projection π .

We then explain some notations for our later uses. For each $a \in G$ denote by R_a the diffeomorphism $z \mapsto z \cdot a$ of P , which we call the right translation of P corresponding to a . Let $\mathfrak{g}$ be the Lie algebra of G . For $A \in \mathfrak{g}$ we set $a_t = \exp(tA)$. Then we denote by A^* the vector field on P induced by the one-parameter group (R_{a_t}) of right translations of P , which we call the vertical vector field on P corresponding to A . Let V be the vertical tangent bundle of P . Then we easily see that for each $z \in P$ the map $A \mapsto A_z^*$ gives an isomorphism of $\mathfrak{g}$ onto $V(z)$ as vector spaces, and further that $(R_a)_* B^* = (\text{Ad}(a^{-1})B)^*$ for all $a \in G$ and $B \in \mathfrak{g}$, whence $[A^*, B^*] = [A, B]^*$ for all $A, B \in \mathfrak{g}$.

Now, let f be a homomorphism of a Lie group G to another Lie group G' . Let P and P' be principal fibre bundles over manifolds M and M' with structure groups G and G' respectively. Then a (bundle) homomorphism of P to P' corresponding to f is a map ψ of P to P' such that

$$\psi(z \cdot a) = \psi(z)F(a), \quad z \in P, \quad a \in G.$$

A homomorphism ψ of P to P' naturally induces a map of M to M' .

Let G' and P' be as above, and let G be a Lie subgroup of G' . Then a reduction of P' to the group G is a principal fibre bundle P over M' with structure group G such that P is a fibred submanifold of P' , and the inclusion map of P to P' is a homomorphism of P to P' corresponding to the inclusion map of G to G' . Clearly this last condition means that the action of G on P is induced by that of G on P' .

For a detailed account of principal fibre bundles we refer to Kobayashi-Nomizu [10].

(C) The frame bundles and linear group structures. In the following we fix an n -dimensional vector space V once for all.

Let M be an n -dimensional manifold. For each point x of M we denote by $F(M,V)_x$ the set of all isomorphisms of V onto $T_x(M)$, and set $F(M,V) = \bigcup_x F(M,V)_x$. We then define the projection π of $F(M,V)$ onto M by $\pi(z) = x$, if $z \in F(M,V)_x$. Let us now consider the general linear group $GL(V)$ of V . For any $z \in F(M,V)$ and $a \in GL(V)$ we define an element za of $F(M,V)$ by

$$(za)v = z(av), \quad v \in V.$$

Clearly the group $GL(V)$ acts freely on the set $F(M,V)$ by the rule $(z,a) \rightarrow za$, and two elements z and z' of $F(M,V)$ are in the same fibre, i.e., $\pi(z) = \pi(z')$, if and only if $z' = za$ for some $a \in GL(V)$. Furthermore we know that the set $F(M,V)$ is endowed with a differentiable structure so that it becomes a differentiable principal fibre bundle over M with structure group $GL(V)$ with projection π . The principal fibre bundle $F(M,V)$, thus obtained, is called the frame bundle (of model vector space V)

over M .

Let us now consider a diffeomorphism ϕ of an n -dimensional manifold M onto another n -dimensional manifold M' . Then we define a map $F(\phi)$ of $F(M, V)$ to $F(M', V)$ by

$$F(\phi)(z) \cdot v = \phi_*(z \cdot v), \quad z \in F(M, V), \quad a \in GL(V).$$

Clearly $F(\phi)$ gives a bundle isomorphism of $F(M, V)$ to $F(M', V)$, which is called the prolongation of ϕ .

Now, let G be a Lie subgroup of $GL(V)$. Then a G -structure on an n -dimensional manifold M is a reduction Q of $F(M, V)$ to the group G . Let Q and Q' be G -structures respectively on manifolds M and M' . By an isomorphism of Q to Q' we mean a diffeomorphism ϕ of M onto M' such that the prolongation $F(\phi)$ of ϕ sends Q onto Q' . We notice that the restriction of $F(\phi)$ to Q gives a diffeomorphism of Q onto Q' , which is clearly a bundle isomorphism.

Let Q be a G -structure on a manifold M , and π the projection of Q onto M . Then we define a V -valued 1-form ξ on Q by

$$z \cdot \xi(X) = \pi_*(X), \quad X \in T_z(Q), \quad z \in Q.$$

The 1-form ξ , thus defined, is called the basic form of Q .

Clearly we have the following

Lemma 4.1. (1) The vertical tangent bundle of Q is defined by the Pfaffian equation: $\xi = 0$.

$$(2) \quad R_a^* \xi = a^{-1} \xi, \quad a \in G.$$

We also remark that the basic form ξ of Q is the pull-back of the basic form of the $GL(V)$ -structure $F(M, V)$.

Finally we make some remarks on the notion of absolute parallelism. An absolute parallelism on an n -dimensional manifold

M is usually defined to be a (global) cross section g of $F(M,V)$, which essentially means an e -structure Q on M , where e stands for the subgroup of $GL(V)$ consisting of the identity element e only. As is easily seen, the geometry of manifolds equipped with absolute parallelisms or with e -structures may be represented by the geometry of manifolds M equipped with V -valued 1-forms satisfying the following condition:

(*) For each $x \in M$ the map $X \mapsto \omega(X)$ of $T_x(M)$ to V gives an isomorphism of $T_x(M)$ onto V .

Here, the notion of isomorphism in the latter geometry is of course defined as follows: Let ω and ω' be V -valued 1-forms respectively on manifolds M and M' satisfying condition (*). Then an isomorphism of (M,ω) to (M',ω') is a diffeomorphism ϕ of M onto M' such that $\omega' = \phi^*\omega$. These being observed, a V -valued 1-form ω on a manifold M satisfying condition (*) will be also called an absolute parallelism on M .

(D) The prolongations of a space of linear maps. Let V and W be two vector spaces of finite dimension. As usual $\text{Hom}(V,W)$ denotes the vector space of all linear maps of V to W , which may be naturally identified with the tensor product $W \otimes V^*$. Now, let g be a subspace of $\text{Hom}(V,W)$. Then the (first) prolongation $g^{(1)}$ of g is the subspace of $\text{Hom}(V,g)$ consisting of all $A \in \text{Hom}(V,g)$ such that $A(v)v' = A(v')v$ for all $v, v' \in V$. $g^{(1)}$ being a subspace of $\text{Hom}(V,g)$, we may speak of the prolongation of $g^{(1)}$, which we denote by $g^{(2)}$, and call the second prolongation of g . In this way we have the k -th prolongation of g for any $k \geq 1$. We promise that the 0-th prolongation $g^{(0)}$ of g is the given subspace g . As

is well known, the k -th prolongation $g^{(k)}$ of g may be simply defined by

$$g^{(k)} = g \otimes S^k(V^*) \cap W \otimes S^{k+1}(V^*),$$

where $S^l(V^*)$ denotes the l -th symmetric product of V^* , and $g \otimes S^k(V^*)$ should be regarded as a subspace of $W \otimes V^* \otimes S^k(V^*)$.

4.2. The symbol algebras of differential systems and FGLA's. Let D be a differential system on a manifold R . Let us consider the sheaf $\mathcal{T}$ of Lie algebras of germs of local vector fields on R and the sheaf $\mathcal{D}$ of germs of local cross sections of D . Let us also consider the derived sheaves $\mathcal{D}^p$ ($p < 0$) of $\mathcal{D}$ as well as the derived systems D^p ($p < 0$) of D (see(F) of 1.2). We recall that $\mathcal{T} \supset \dots \supset \mathcal{D}^p \supset \dots \supset \mathcal{D}^{-1} = \mathcal{D}$, and $T \supset \dots \supset D^p \supset \dots \supset D^{-1} = D$, where $T = T(R)$.

We say that the system D is regular, if D^p is a subbundle of T for each $p < 0$.

From now on we assume that D is not reduced to the zero system, and is regular. Clearly $\mathcal{D}^p$ coincides with the sheaf of germs of local cross sections of D^p .

Lemma 4.2. Let x be a point of R . If $X \in \mathcal{D}^p(x)$, $Y \in \mathcal{D}^q(x)$, and $X_x \in D^{p+1}(x)$, then $[X, Y]_x \in D^{p+q+1}(x)$.

Proof. Since $X_x \in D^{p+1}(x)$, we can find $Z \in \mathcal{D}^{p+1}(x)$ such that $X_x = Z_x$. If we put $W = X - Z$, we have $W \in \mathcal{D}^p(x)$, $W_x = 0$, and $[X, Y]_x = [Z, Y]_x + [W, Y]_x$. By Lemma 1.4 we have $[Z, Y] \in \mathcal{D}^{p+q+1}(x)$, and hence $[Z, Y]_x \in D^{p+q+1}(x)$. Furthermore let (X_i) be a moving frame of D^p defined on a neighborhood of x . Then W may

be expressed as follows: $W = \sum_i f_i \cdot X_i$, and the coefficients f_i satisfy $f_i(x) = 0$. Therefore it follows that $[W, Y]_x = - \sum_i Y_x f_i(X_i)_x \in D^p(x) \subset D^{p+q+1}(x)$. We have thus shown that $[X, Y]_x \in D^{p+q+1}(x)$, proving the lemma.

Now, we set:

$$\begin{aligned}\tilde{g}_p(x) &= D^p(x)/D^{p+1}(x), & \tilde{t}(x) &= \sum_p \tilde{g}_p(x); \\ g_p(x) &= D^p(x)/D^{p+1}(x), & t(x) &= \sum_p g_p(x).\end{aligned}$$

Here, we promise that $D^p(x) = \{0\}$ and $D^p(x) = \{0\}$ for $p \geq 0$. By Lemma 1.4 we see that $\tilde{t}(x)$ becomes naturally a Lie algebra, and $[\tilde{g}_p(x), \tilde{g}_q(x)] \subset \tilde{g}_{p+q}(x)$. (Note that $\tilde{t}(x)$ is not a graded Lie algebra, because $\tilde{g}_p(x)$ are of infinite dimension.) The natural maps of $D^p(x)$ onto $D^p(x)$ naturally induce linear maps of $\tilde{g}_p(x)$ onto $g_p(x)$, and in turn these naturally induce a linear map, say ϕ , of $\tilde{t}(x)$ onto $t(x)$. Then we see from Lemma 4.2 that there is a unique Lie algebra structure in $t(x)$ such that ϕ gives a Lie algebra homomorphism of $\tilde{t}(x)$ onto $t(x)$. For any p let us denote by w^p the projection of $D^p(x)$ onto $g_p(x)$. Then the bracket operation in the Lie algebra $t(x)$ is concretely given by

$$[w^p(X_x), w^q(Y_x)] = w^{p+q}([X, Y]_x),$$

where $X \in D^p(x)$, and $Y \in D^q(x)$.

Clearly we have $[g_p(x), g_q(x)] \subset g_{p+q}(x)$, and hence the pair $(t(x), (g_p(x)))$ defines a graded Lie algebra, which we denote by $T(x)$. Moreover, $g_{-1}(x) = D(x) \neq \{0\}$, and the Lie algebra $t(x)$ is generated by $g_{-1}(x)$, i.e., $g_{p-1}(x) = [g_p(x), g_{-1}(x)]$ for $p < 0$, because $\tilde{g}_{p-1}(x) = [\tilde{g}_p(x), \tilde{g}_{-1}(x)]$ for $p < 0$. Incidentally we remark that $g_{-2}(x)$ is a subspace of $T(x)/D(x)$, and

$$\gamma_x(X,Y) = [X,Y], \quad X,Y \in D(x),$$

where γ_x stands for the torsion of D at x .

In view of the observation above we now give the following definition. Let T be a graded Lie algebra, and $(t, (g_p))$ its expression. Then T is called a fundamental graded Lie algebra or briefly a FGLA, if the following conditions are satisfied:

- (i) $\dim t < \infty$,
- (ii) $g_{-1} \neq \{0\}$, and the Lie algebra t is generated by g_{-1} , i.e., $g_{p-1} = [g_p, g_{-1}]$ for all $p < 0$.

From this last condition it follows that $g_p = \{0\}$ for all $p \geq 0$. Accordingly the gradation of the FGLA will be denoted by $(g_p)_{p < 0}$.

Let T be a FGLA. Given a positive integer μ , T is said to be of the μ -th kind, if $g_{-\mu-1} = \{0\}$ and $g_{-\mu} \neq \{0\}$ or equivalently $g_p = \{0\}$ for all $p < -\mu$, and $g_{-p} \neq \{0\}$ for all $-\mu \leq p \leq -1$. Furthermore T is said to be non-degenerate, if the condition " $X \in g_{-1}$, $[X, g_{-1}] = \{0\}$ " implies $X = 0$.

Let D be a differential system on a manifold R . The assumptions and notations being as above, let us consider the graded Lie algebra $T(x)$ or $(t(x), (g_p(x)))$ for each $x \in R$. Then $T(x)$ is a FGLA, which is called the symbol algebra of D at x .

Let T be a FGLA, and t its underlying Lie algebra. Then the system D is said to be of type T , if the following conditions are satisfied:

- (a) D is not reduced to the zero system, and is regular,
- (b) $T(x) = T$ at each $x \in R$,
- (c) $\dim R = \dim t$.

4.3. Differential systems of type T. In this and the subsequent paragraphs T will be a fixed FGLA, and $(t, (g_p)_{p < 0})$ its expression.

For each $p < 0$ we set

$$d^p = \sum_{j=p}^{-1} g_j,$$

and define subgroups H and H_0 of $GL(t)$ as follows:

$$H = \{ a \in GL(t) \mid ad^p = d^p, p < 0 \},$$

$$H_0 = \{ a \in GL(t) \mid ag_p = g_p, p < 0 \}.$$

Now, the factor space d^p/d^{p+1} may be naturally identified with g_p , where we set $d^0 = \{0\}$. Hence every element a of H naturally induces an element, say $\tilde{a}$, of H_0 : $\tilde{a}X \equiv aX \pmod{d^{p+1}}$ for all $X \in g_p$. Clearly the map $a \mapsto \tilde{a}$ gives a homomorphism, say ζ , of H onto H_0 . We denote by N^0 the kernel of ζ . Then we have

$$H = H_0 \cdot N^0 \text{ (semi-direct)}.$$

Let G_0 be a Lie subgroup of the automorphism group $Aut(T)$ of T. Then we denote by $G_0^\#$ the inverse image of G_0 by ζ . Clearly we have

$$G_0^\# = G_0 \cdot N^0.$$

Now, let D be a differential system of type T on a manifold R, and (D^p) the family of derived systems of D. Let us consider the symbol algebra $T(x)$ of D at any $x \in R$. Let us also consider the frame bundle $F(R, t)$, and let π be the projection of $F(R, t)$ onto R. We denote by $\tilde{F}(R, D, T)$ the subset of $F(R, t)$ consisting of all $z \in F(R, t)$ such that $z \cdot d^p = D^p(x)$, where $x = \pi(z)$, which

clearly gives a differentiable reduction of $F(R,t)$ to the group H . Since $g_p = d^p/d^{p+1}$ and $g_p(x) = D^p(x)/D^{p+1}(x)$, every $z \in \tilde{F}(R,D,T)$ naturally gives rise to an isomorphism, say $\tilde{z}$, of t onto $t(x)$ as vector spaces. This being said, we denote by $F(R,D,T)$ the subset of $\tilde{F}(R,D,T)$ consisting of all $z \in \tilde{F}(R,D,T)$ such that $\tilde{z}$ give isomorphisms of T to $T(x)$.

We shall show that $F(R,D,T)$ gives a differentiable reduction of $\tilde{F}(R,D,T)$ to the group $\text{Aut}(T)^\#$.

Let S denote the vector space of all anti-symmetric bilinear maps s of $t \times t$ to t . The group H acts on S by the following rule:

$$(s^a)(X,Y) = \zeta(a)^{-1} s(\zeta(a)X, \zeta(a)Y),$$

where $s \in S$, $a \in H$, and $X,Y \in t$. Using the bracket operation in the Lie algebras $t(x)$, we now define a map ϕ of $\tilde{F}(R,D,T)$ to S by

$$\phi(z)(X,Y) = \tilde{z}^{-1}[\tilde{z}X, \tilde{z}Y], \quad z \in \tilde{F}(R,D,T), \quad X,Y \in t.$$

Since $(za)\tilde{z} = \tilde{z} \cdot \zeta(a)$ for all $z \in \tilde{F}(R,D,T)$ and $a \in H$, we see that ϕ is H -equivariant: $\phi(za) = \phi(z)^a$. Now, let s_0 denote the bracket operation in the Lie algebra t . Clearly $\text{Aut}(T)^\#$ is the isotropy group of H at the point $s_0 \in S$. Since D is of type T , it follows that the image of $\tilde{F}(R,D,T)$ by ϕ is in the H -orbit through s_0 , and further that $F(R,D,T)$ is the inverse image of s_0 by ϕ . By Lemma A.3 in Appendix A we have therefore shown that $F(R,D,T)$ really gives a differentiable reduction of $\tilde{F}(R,D,T)$ to the group $\text{Aut}(T)^\#$.

We remark that the construction above of $F(R,D,T)$ uses the inverse function theorem.

4.4. $G_0^\#$ -structures of type T. At the outset we prepare some notations for our later arguments.

Let P be a manifold, and let θ be a t -valued 1-form on it. Assume that θ is independent in the sense that $\dim P = \dim t + \dim \theta_z^{-1}(0)$ at each $z \in P$, where $\theta_z^{-1}(0) = \{ X \in T_z(P) \mid \theta(X) = 0 \}$. For each $j < 0$ denote by θ_j the g_j -component of θ with respect to the decomposition $t = \sum_{j < 0} g_j$. Furthermore let α and β differential forms on P which take values in a vector space V of finite dimension. Now, let p be any integer < 0 . Define a subset $I(p)$ of $\mathbb{Z} \times \mathbb{Z}$ by

$$I(p) = \{ (r, s) \in \mathbb{Z} \times \mathbb{Z} \mid p < r, s < 0, \text{ and } r+s < p \}.$$

Then, by the notation " $\alpha \equiv_p \beta$ " we mean that

$$\alpha \equiv_p \beta \pmod{\theta_r \text{ } (r \leq p); \theta_r \wedge \theta_s \text{ } ((r, s) \in I(p))}.$$

Namely, take a basis $(e_{j\lambda})_{1 \leq \lambda \leq n_j}$ of g_j for each $j < 0$ and a basis $(u_\nu)_{1 \leq \nu \leq k}$ of V , and set $\theta_j = \sum_{\lambda} \theta_{j\lambda} e_{j\lambda}$, $\alpha = \sum_{\nu} \alpha_\nu u_\nu$, and $\beta = \sum_{\nu} \beta_\nu u_\nu$. (Note that the independence of θ means the 1-forms $\theta_{j\lambda}$ ($1 \leq \lambda \leq n_j$, $j < 0$) are linearly independent at each point of P .) Furthermore let $\Omega(P)$ denote the sheaf of germs of local differential forms on P . Then the notation " $\alpha \equiv_p \beta$ " concretely means that the forms $\alpha_\nu - \beta_\nu$ give cross sections of the subsheaf of ideals of $\Omega(P)$ generated by the following forms:

$$\theta_{r\lambda} \quad (1 \leq \lambda \leq n_r, r \leq p),$$

$$\theta_{r\lambda} \wedge \theta_{s\kappa} \quad (1 \leq \lambda \leq n_r, 1 \leq \kappa \leq n_s, (r, s) \in I(p)).$$

Let G_0 be a Lie subgroup of $\text{Aut}(T)$, and let Q be a $G_0^\#$ -structure on a manifold R . Then Q is said to be of type T, if the basic form ξ of Q or the family $(\xi_p)_{p < 0}$ satisfies the following

$$d\xi_p + \frac{1}{2} \sum_{r+s=p} [\xi_r, \xi_s] \equiv 0, \quad p \leq -2,$$

where the notation " $\equiv$ " should be considered with respect to ξ . Note the equation $d\xi_{-1} \equiv 0$ is automatically satisfied.

Let D be a differential system of type T on a manifold R , and let us consider the $\text{Aut}(T)^\#$ -structure $F(R, D, T)$ on R . Let π be the projection of $F(R, D, T)$ onto R , and ξ the basic form of the structure. Let us consider the family (D^p) of derived systems of D .

Proposition 4.3. (1) The pull-back $\pi_*^{-1}(D^p)$ of D^p by π is defined by the system of Pfaffian equations: $\xi_j = 0$ ($j < p$).

(2) $F(R, D, T)$ is of type T .

Proof. For simplicity we set $Q = F(R, D, T)$.

(1) For any $z \in R$ and $X \in T_z(R)$ we have $z \cdot \xi(X) = \pi_*(X)$.

Clearly $\xi(X) \in d^p$, if and only if $\pi_*(X) \in D^p(x)$, where $x = \pi(z)$. This clearly means that $\pi_*^{-1}(D^p)$ is defined by the system of Pfaffian equations $\xi_j = 0$ ($j < p$).

(2) The matter being of local character, we may assume that Q is trivial. We take a cross section g of Q , and set $\alpha = g^*\xi$ and $\alpha_p = g^*\xi_p$. Note that α gives an absolute parallelism on R . Now, there is a unique map a of Q to $G_0^\#$ such that $z = g(\pi(z)) \cdot a(z)$ for all $z \in Q$. Moreover there are unique maps b of Q to G_0 and c of Q to N^0 such that $a(z) = c(z)b(z)$ for all $z \in Q$. By Lemma 4.1 we have $\xi = a^{-1}\pi^*\alpha$, from which follows that

$$\xi_p \equiv b^{-1}\pi^*\alpha_p \quad (\text{mod } \pi^*\alpha_j (j < p)),$$

$$\pi^*\alpha_p \equiv b\xi_p \quad (\text{mod } \xi_j (j < p)).$$

Then we can easily show that

$$\Xi_p \equiv b^{-1} \pi^* A_p$$

$$(\text{mod } \pi^* \alpha_r \ (r \leq p); \pi^* \alpha_r \wedge \pi^* \alpha_s \ ((r,s) \in I(p))),$$

where $\Xi_p = d\xi_p + \frac{1}{2} \sum_{r+s=p} [\xi_r, \xi_s]$, and $A_p = d\alpha_p + \frac{1}{2} \sum_{r+s=p} [\alpha_r, \alpha_s]$.

Therefore (2) follows immediately from the last statement of the next lemma.

Lemma 4.4. (1) D^p is defined by the system of Pfaffian equations $\alpha_j = 0$ ($j < p$).

(2) $\alpha_p([X, Y]_x) = [\alpha_r(X_x), \alpha_s(Y_x)]$, where $X \in D^r(x)$, $Y \in D^s(x)$, and $p = r+s$.

$$(3) \ A_p \equiv 0 \ (\text{mod } \alpha_r \ (r \leq p); \alpha_r \wedge \alpha_s \ ((r,s) \in I(p)))$$

Proof. (1) follows immediately from (1) of the proposition.

(2) Take any $x \in R$, and set $z = g(x)$. Then, for all $X \in T_x(R)$ we have $z \cdot \alpha(X) = X$ and hence $\tilde{z} \cdot \alpha_p(X) = \omega^p(X)$. Now, take any $X \in D^r(x)$, $Y \in D^s(x)$, and set $p = r+s$. Then it follows that

$$\begin{aligned} \tilde{z} \cdot [\alpha_r(X_x), \alpha_s(Y_x)] &= [\tilde{z} \cdot \alpha_r(X_x), \tilde{z} \cdot \alpha_s(Y_x)] \\ &= [\omega^r(X_x), \omega^s(Y_x)] = \omega^p([X, Y]_x) = \tilde{z} \cdot \alpha_p([X, Y]_x), \end{aligned}$$

whence $[\alpha_r(X_x), \alpha_s(Y_x)] = \alpha_p([X, Y]_x)$.

(3) Since α gives an absolute parallelism on R , we have $A_p \equiv 0 \ (\text{mod } \alpha_r \wedge \alpha_s \ (r, s < p))$. Moreover, in view of (1) we obtain $A_p(X_x, Y_x) = -\alpha_p([X, Y]_x) + [\alpha_r(X_x), \alpha_s(Y_x)]$, where $X \in D^r(x)$, $Y \in D^s(x)$, and $p = r+s$. These being observed, (3) can be easily derived from (1) and (2).

Now, let G_0 be a Lie subgroup of $\text{Aut}(T)$. Then it is clear

from Proposition 4.3 that any reduction Q of $F(R,D,T)$ to the group $G_0^\#$ is of type T .

Conversely let Q be any $G_0^\#$ -structure of type T on a manifold R . Let π be the projection of Q onto R , and ξ the basic form of Q . Then we have the following

Proposition 4.5. (1) For each $p < 0$ there is a unique differential system D^p on R such that $\pi_*^{-1}(D^p)$ is defined by the system of Pfaffian equations $\xi_j = 0$ ($j < p$).

(2) Set $D = D^{-1}$. Then D is of type T , and (D^p) is the family of derived systems of D .

(3) Q gives a reduction of $F(R,D,T)$ to the group $G_0^\#$. Furthermore D is a unique differential system of type T on R having this property.

The system D will be called associated with Q .

Proof. For each $p < 0$ let $\bar{D}^p$ be the differential system on Q defined by the system of Pfaffian equations $\xi_j = 0$ ($j < p$). Then it is easy to see that $\bar{D}^p$ is invariant under the right translations of Q , and contains the vertical tangent bundle of Q . Hence there is a unique differential system D^p on R such that $\pi_*^{-1}(D^p) = \bar{D}^p$, proving (1). (2) and (3) can be easily verified by a reciprocal argument to the proof of Proposition 4.3. (The uniqueness assertion in (3) is clear from the first assertions of Propositions 4.3 and 4.5)

4.5. Transitive graded Lie algebras and algebraic prolongations. Let G be a graded Lie algebra, and $(g, (g_p))$ its expression. If we set $g_- = \sum_{p < 0} g_p$, we see that the pair $(g_-, (g_p)_{p < 0})$ or

precisely $(g_-, (g_- \cap g_p))$ gives a (truncated) graded subalgebra of G , which we denote by G_- . Then G is said to be transitive, if the following conditions are satisfied:

- (i) G_- is a FGLA,
- (ii) If $p \geq 0$, $X \in g_p$, and $[X, g_-] = \{0\}$, then $X = 0$.

If G is transitive, G_- will be called the truncated FGLA of G .

Now, let T be a FGLA. We shall show that there is naturally associated to T a graded Lie algebra $\hat{G}$ which satisfies the following conditions:

- (a) $\hat{G}$ is transitive,
- (b) $\hat{G}_- = T$,

(c) $\hat{G}$ is maximum among the graded Lie algebras satisfying conditions (a) and (b). Namely, let G be any graded Lie algebra satisfying these conditions. Then G is naturally imbedded in $\hat{G}$.

Let $(t, (g_p)_{p < 0})$ be the expression of T . For all $p < 0$ we set $\hat{g}_p = g_p$. For all $p \geq 0$ let us now define vector spaces $\hat{g}_p$ inductively as follows: First of all we define $\hat{g}_0$ to be the derivation algebra $\text{Der}(T)$ of T . k being a positive integer, suppose that we have defined $\hat{g}_p$ for all $0 \leq p < k$ so that $\hat{g}_p$ are subspaces of $q_p = \sum_{r < 0} \text{Hom}(g_r, \hat{g}_{r+p}) \subset \text{Hom}(t, \sum_{r < 0} \hat{g}_{r+p})$. Then we define $\hat{g}_k$ to be the subspace $q_k = \sum_{r < 0} \text{Hom}(g_r, \hat{g}_{r+k})$ which consists of all $X_k \in q_k$ satisfying the following equalities:

$$X_k(Y_r)(Z_s) - X_k(Z_s)(Y_r) = X_k([Y_r, Z_s]),$$

where $Y_r \in g_r$, $Z_s \in g_s$, $r, s < 0$, and we promise that

$$X_k(Y_r)(Z_s) = [X_k(Y_r), Z_s], \text{ if } r+k < 0;$$

$$X_k(Z_s)(Y_r) = [X_k(Z_s), Y_r], \text{ if } s+k < 0.$$

Thus we have completed our inductive definition. Set $\hat{g} = \sum_{p \in \mathbb{Z}} \hat{g}_p$. Then we easily see that there is a unique bracket operation $[\cdot, \cdot]$ in $\hat{g}$ having the following properties: (i) The pair $(\hat{g}, (\hat{g}_p))$ defines a graded Lie algebra, say $\hat{G}$, with respect to this bracket operation, (ii) $\hat{G}$ satisfies condition (b), and (iii) $[X_k, Y_r] = X_k(Y_r)$ for all $X_k \in \hat{g}_k$, $Y_r \in g_r$ and all $k \geq 0$, $r < 0$. Moreover it is clear that $\hat{G}$ satisfies conditions (a) and (c).

The graded Lie algebra $\hat{G}$, thus defined, is called the prolongation of T . We remark that there is a unique element E in the centre of $\hat{g}_0$ such that $[E, X] = pX$ for all $X \in \hat{g}_p$ and all p .

Now, let g_0 be a subalgebra of $\hat{g}_0$. For all $p \geq 1$ we define subspaces g_p of $\hat{g}_p$ inductively as follows: k being a positive integer, suppose that we have defined g_p for all $0 < p < k$. Then g_k is defined to be the subspace of $\hat{g}_k$ which consists of all $X_k \in \hat{g}_k$ such that $[X_k, g_r] \subset g_{k+r}$ for all $r < 0$. If we put $g = \sum_{p \in \mathbb{Z}} g_p$, we can easily verify that the pair $(g, (g_p))$ defines a graded subalgebra, say G , of $\hat{G}$.

The graded Lie algebra G , thus defined, is called the prolongation of the pair (T, g_0) .

We shall now recall a proposition concerning the finite dimensionality of a transitive graded Lie algebra.

Let G be a transitive graded Lie algebra, and $(g, (g_p))$ its expression. At the outset we remark that the condition " $p \geq 0$, $X \in g_p$, and $[X, g_{-1}] = \{0\}$ " implies $X = 0$. In particular we know that the natural representation of the Lie algebra g_0 on g_{-1} is faithful, and hence g_0 may be naturally identified with a

subalgebra of $gl(g_{-1})$, the Lie algebra of all endomorphisms of g_{-1} .

For each $p \geq 0$ we now define a subspace $h_p(G)$ or h_p of g_0 by

$$h_p = \{ X \in g_p \mid [X, g_j] = \{0\} \text{ for all } j \leq -2 \}.$$

Clearly h_0 is an ideal of g_0 , and hence it may be identified with a subalgebra of $gl(g_{-1})$. Moreover it is clear that the subspaces h_p satisfy the following conditions: (i) $[h_p, g_{-1}] \subset h_{p-1}$, where we promise that $h_{-1} = g_{-1}$, (ii) $[[X, Y], Z] = [[X, Z], Y]$ for all $X \in h_p$ and $Y, Z \in g_{-1}$, and (iii) the condition " $p \geq 0, X \in h_p$, and $[X, g_{-1}] = \{0\}$ " implies $X = 0$. Hence it follows that for each $p > 0$ h_p may be naturally identified with a subspace of the p -th prolongation $h_0^{(p)}$ of the subalgebra h_0 of $gl(g_{-1})$ and also with a subspace of the first prolongation of h_{p-1} . These being observed, we state the following

Proposition 4.6 (Corollary to Theorem 11.1 in [18]). The Lie algebra g is of finite dimension, if and only if h_k vanishes for some $k \geq 0$, and hence h_p vanish for all $p \geq k$.

We now make some comments in connection with this proposition. Assume that G is the prolongation of the pair (G_-, g_0) , G_- being the truncated FGLA of G . From the very definition of the prolongation it follows that h_p just coincides with the p -th prolongation $h_0^{(p)}$ of h_0 for each $p \geq 0$.

By the proposition we therefore see that g is of finite dimension, if and only if the subalgebra h_0 of $gl(g_{-1})$ is of finite type, i.e., $h_0^{(k)}$ vanishes for some $k \geq 0$ (Corollary 2 to Theorem 11.1 loc. cit.). Furthermore assume that $g_0 = \text{Der}(G_-)$ or

equivalently G is the prolongation of G_- . Then it can be shown that if G_- is degenerate or not non-degenerate, then g is of infinite dimension. (cf. the proof of (2) of Proposition 5.8).

Remark. Let T be a FGLA, and $(t, (g_p)_{p < 0})$ its expression. Let E be a graded (left) T -module. Namely E is a pair formed by a t -module E and a family $(E_p)_{p \in \mathbb{Z}}$ of subspaces of E which satisfy the following conditions: (i) $\dim E_p < \infty$, (ii) $E = \sum_p E_p$ (direct sum), and (iii) $g_r \cdot E_p \subset E_{r+p}$. We define a subspace H of E by

$$H = \{ a \in E \mid X \cdot a = 0 \text{ for all } X \in g_p \text{ and all } p \leq -2 \}.$$

Clearly we have $H = \sum_p H_p$, where $H_p = H \cap E_p$. Then Theorem 11.1 cited above may be restated as follows.

Theorem. Assume that E_p vanishes for any sufficiently small p . Then E is of finite dimension, if and only if H is of finite dimension.

We observe that Proposition 4.5 really follows from this theorem, because G becomes naturally a graded G_- -module.

Our original proof of Theorem 11.1 made use of the characterization due to Serre of an involutive module in terms of the Spencer cohomology groups (cf. Guillemin [8]). In his unpublished paper [13] Naruki gave an elementary proof of the theorem just stated which is based on the structure theorem for modules over PID only.

4.6. A general prolongation theorem for $G_0^\#$ -structures of type T . Let T be a FGLA. Let G_0 be a Lie subgroup of $\text{Aut}(T)$, and g_0 its Lie algebra, being a subalgebra of $\text{Der}(T)$. Let G be

the prolongation of the pair (T, g_0) , and g its underlying Lie algebra.

Theorem 4.7. Assume that G_0 is connected, and that g is of finite dimension.

(1) Let Q be any $G_0^\#$ -structure of type T on a manifold R . Then there is canonically associated to Q a pair (P, ω) as follows: P is a fibred manifold over R , and ω is a g -valued 1-form on P and gives an absolute parallelism on it. The pair (P, ω) is called the prolongation of Q .

(2) Let Q and Q' be $G_0^\#$ -structures of type T on manifolds R and R' , and let (P, ω) and (P', ω') be their prolongations, respectively. If ϕ is an isomorphism of Q to Q' , then there is a unique isomorphism $\tilde{\phi}$ of (P, ω) to (P', ω') as manifolds with absolute parallelism such that $\pi' \circ \tilde{\phi} = \phi \circ \pi$, where π and π' are the projections of P and P' onto R and R' respectively. Conversely, if ψ is an isomorphism of (P, ω) to (P', ω') as manifolds with absolute parallelism, there is a unique isomorphism ψ of Q to Q' such that $\psi = \tilde{\phi}$.

This theorem follows immediately from Theorem 8.3 in [18] (see also the proof of Theorem 8.4 there).

The notations being as in the outset, assume that g is of finite dimension. (We do not assume that G_0 is connected !) Let Q be a $G_0^\#$ -structure of type T on a manifold R , and let us consider the automorphism group $\text{Aut}(Q)$ of Q as well as the sheaf $a(Q)$ of germs of local infinitesimal automorphisms of Q . Then, in view of Theorem 4.7 combined with Palais [14] we have the following

Corollary. (1) Each stalk $a(Q)_x$ of $a(Q)$ is of finite dimension, and $\dim a(Q)_x \leq \dim g$.

(2) $\text{Aut}(Q)$ becomes naturally a Lie group, and $\dim \text{Aut}(Q) \leq \dim g$.

§5. Equivalence problem for almost pseudo-product manifolds

5.1. Almost pseudo-product manifolds and pseudo-product FGLA's. An almost pseudo-product structure on a manifold R is a pair formed by differential systems E and F on R which satisfy the following conditions:

(1) $E \cap F = 0$, and hence the sum $E + F$ gives a differential system, say D , on R ,

(2) For any local cross sections X and Y of E or of F the bracket $[X, Y]$ gives a local cross section of D .

Then an almost pseudo-product manifold is a manifold R equipped with an almost pseudo-product structure (E, F) or the triplet (R, E, F) .

Let $\tilde{R}$ be an almost pseudo-product manifold, and (R, E, F) its expression. Then $\tilde{R}$ is said to be integrable, if both E and F are completely integrable. Clearly an integrable almost pseudo-product manifold means a pseudo-product manifold.

Let us consider the torsion γ_x of D at any $x \in R$. Then we can easily prove the following lemma (cf. the proof of Lemma 3.3).

Lemma 5.1. $\gamma_x(X, Y) = 0$ for any vectors X and Y in $E(x)$ or in $F(x)$. In other words both $E(x)$ and $F(x)$ are integral elements of D at x .

Assuming that $D \neq 0$, and D is regular, let us now consider the symbol algebra $T(x)$ of D at x , and let $(t(x), (g_p(x))_{p < 0})$ be its expression. Clearly we have

$$g_{-1}(x) = E(x) + F(x) \quad (\text{direct sum}),$$

and Lemma 5.1 may be restated as follows:

$$[E(x), E(x)] = [F(x), F(x)] = \{0\}.$$

This being said, let T be a FGLA, and $(t, (g_p)_{p < 0})$ its expression. Let e and f be subspaces of g_{-1} . Then the triplet (T, e, f) is called a pseudo-product FGLA, if it satisfies the following conditions:

- (i) $g_{-1} = e + f$ (direct sum),
- (ii) $[e, e] = [f, f] = \{0\}$.

A pseudo-product FGLA will be denoted in principle by a capital Germanic letter, e.g., L , and then the triplet will be called the expression of L .

Let L and L' be two pseudo-product FGLA's, and (T, e, f) and (T', e', f') their expressions. By an isomorphism of L to L' we mean an isomorphism of T to T' which sends e onto e' and f onto f' .

Let L be as above. Then L is said to be of the μ -th kind, if T is of the μ -th kind. It is also said to be non-degenerate, if T is non-degenerate.

Now, consider an almost pseudo-product manifold $\bar{R}$, and assume that $D \neq 0$, and D is regular. Let x be any point of R . Then we see from the remark above that the triplet $(T(x), E(x), F(x))$ defines a pseudo-product FGLA, which we denote by $L(x)$, and call the symbol algebra of $\bar{R}$ at x .

Finally let L be a pseudo-product FGLA. Then $\bar{R}$ is said to be of type L , if the following conditions are satisfied:

- (a) $L(x) \cong L$ at each $x \in R$,
- (b) $\dim R = \dim t$, t being the underlying Lie algebra of L .

We remark that if $\bar{R}$ is a pseudo-product manifold of type L ,

and L is non-degenerate, then $\mathcal{R}$ is a generalized pseudo-projective system.

5.2. Almost pseudo-projective systems, and pseudo-projective FGLA's. Let (n, r) be a pair of integers with $1 \leq r \leq n-1$. Let $\mathcal{R}$ be an almost pseudo-product manifold, and (R, E, F) its expression. Then $\mathcal{R}$ is called an almost pseudo-projective system (of order 2) of bidegree (n, r) , if the following conditions are satisfied:

- (1) $D(= E + F)$ is non-degenerate,
- (2) $\dim R = n+r(n-r)$, $\text{rank } E = r$, and $\text{rank } F = r(n-r)$.

Clearly an integrable almost pseudo-projective system means a pseudo-projective system.

Lemma 5.2. If $\mathcal{R}$ is an almost pseudo-projective system of bidegree (n, r) , then the first derived system of D coincides with $T(R)$, and hence D is regular.

Proof. Let x be any point of R , and let us consider the torsion γ_x of D at x . Then we have the following: (i) $D(x) = E(x) + F(x)$ (direct sum), (ii) $\gamma_x(E(x), E(x)) = \gamma_x(F(x), F(x)) = \{0\}$ (Lemma 5.1), (iii) γ_x is non-degenerate, and (iv) $\dim E(x) = r$, $\dim F(x) = r(n-r)$, and $\dim(T(x)/D(x)) = n-r$, where $T = T(R)$. We define a linear map $Y \mapsto \tilde{Y}$ of $F(x)$ to $\text{Hom}(E(x), T(x)/D(x))$ by

$$\tilde{Y}(x) = \gamma_x(Y, X), \quad X \in E(x).$$

Then we easily see from the remark above that the linear map $Y \mapsto \tilde{Y}$ maps $F(x)$ isomorphically onto $\text{Hom}(E(x), T(x)/D(x))$. Hence we obtain

$$T(x)/D(x) = \gamma_x(E(x), F(x)) = \gamma_x(D(x), D(x)),$$

which clearly implies the lemma.

Let L be a pseudo-product FGLA, and (T, e, f) its expression. Let $(t, (g_p)_{p < 0})$ be the expression of T . Then L is called a pseudo-projective FGLA of bidegree (n, r) , if the following conditions are satisfied:

- (i) L is of the second kind, and hence $t = g_{-2} + g_{-1}$.
- (ii) L is non-degenerate, which means that the condition " $X \in e, [X, f] = \{0\}$ " or " $X \in f, [X, e] = \{0\}$ " implies $X = 0$,
- (iii) $\dim g_{-2} = n - r$, $\dim e = r$, and $\dim f = r(n - r)$.

If $\mathcal{R}$ is an almost pseudo-projective system of bidegree (n, r) , we see from Lemma 5.2 together with its proof that the symbol algebra $L(x)$ of $\mathcal{R}$ at any $x \in R$ is a pseudo-projective FGLA of bidegree (n, r) .

Proposition 5.3. There is a unique pseudo-projective FGLA of bidegree (n, r) except for isomorphisms.

Proof. Let L be a pseudo-projective FGLA of bidegree (n, r) . We define a linear map $Y \mapsto \tilde{Y}$ of f to $\text{Hom}(e, g_{-2})$ by

$$\tilde{Y}(X) = [Y, X], \quad X \in e.$$

Then we see that the linear map $Y \mapsto \tilde{Y}$ maps f isomorphically onto $\text{Hom}(e, g_{-2})$ (cf. the proof of Lemma 5.2). In view of this fact we can easily obtain the proposition.

By Proposition 5.3 we have the following

Proposition 5.4. Let L be a pseudo-projective FGLA of bidegree (n, r) , and let $\mathcal{R}$ be an almost pseudo-product manifold. Then $\mathcal{R}$ is an almost pseudo-projective system of bidegree (n, r) , if and only if it is of type L .

As an application of the symbol algebra of an almost pseudo-projective system we shall prove the following

Proposition 5.5. Let $\mathcal{R}$ be an almost pseudo-projective system of bidegree (n,r) , and (R,E,F) its expression. Let x any point of R . If $n-r \geq 2$, then $F(x)$ is a unique $r(n-r)$ -dimensional integral element of the system $D(= E + F)$ at x .

Before proceeding to the proof we shall explain some consequences of the proposition.

First of all we know that if $n-r \geq 2$, the system F is naturally associated with the system D or it is a convariant system of D in Cartan's terminology. More in detail it can be shown that F can be naturally constructed from D by rational operations and differentiations. (The exact meanings of these operations will be clarified in Part II.) Clearly this fact implies the following

Corollary 1 (cf. Bäcklund [1] and Yamaguchi [23]).

Consider the Grassmann bundle $J(M,r)$ over an n -dimensional manifold M . If $n-r \geq 2$, then the vertical tangent bundle V of $J(M,r)$ is naturally associated with the contact system C of $J(M,r)$.

Next, let $\mathcal{X}$ be a pseudo-projective structure of bidegree (n,r) on a manifold M . Let $\mathcal{R}$ be the associated pseudo-projective system of bidegree (n,r) , and (R,E,F) its expression. Assuming condition (D.1) in 3.6, let us consider the dual $\mathcal{X}^*$ of $\mathcal{X}$. We recall that $\mathcal{X}^*$ is a generalized pseudo-projective structure of bidegree $((r+1)(n-r), r(n-r))$, on the manifold $N(= R/E)$, and is

expressed by $(\kappa_*(R), \kappa_*(F))$, where κ is the immersion of R to $J(N, r(n-r))$ associated with the dual R^* of R . Now, the proposition implies the following

Corollary 2. If $n-r \geq 2$, then the system χ^* of differential equations of the second order is the prolongation of the system $\kappa(R)$ of differential equations of the first order.

For the definition of the prolongation of a system of differential equations see [7] and [22].

Proof of Proposition 5.5. Let x be any point of R , and let us consider the symbol algebra $L(x)$ or $(T(x), E(x), F(x))$ of R at x . For simplicity set $g_p = g_p(x)$, $e = E(x)$, and $f = F(x)$. Now, assume that $n-r \geq 2$, and let z be any $r(n-r)$ -dimensional contact element of D at x , meaning that z is an $r(n-r)$ -dimensional subspace of g_{-1} such that $[z, z] = \{0\}$. We take a complementary subspace e' of $e \cap z$ in e , and define a linear map $X \mapsto \hat{X}$ of z to $\text{Hom}(e', g_{-2})$ by

$$\hat{X}(Y) = [X, Y], \quad Y \in e'.$$

Then we easily see that the kernel of the linear map $X \mapsto \hat{X}$ is $e \cap z$. Hence we obtain

$$\dim z \leq \dim (e \cap z) + \dim e' \cdot \dim g_{-2},$$

whence $(n-r-1)\dim (e \cap z) \leq 0$. Since $n-r \geq 2$, it follows that $e \cap z = \{0\}$, and hence $g_{-1} = e + z$ (direct sum). Therefore we can find a unique linear map ϕ of f to e such that

$$z = \{\phi(X) + X \mid X \in f\}.$$

Now, we take bases (e_i) of e and (d_a) of f .

Here and in the following the indices i, j, k range over the integers $1, \dots, r$, and the indices a, b, c over the integers $1, \dots, n-r$. Then we see from the proof of Proposition 5.3 that there is a unique basis (d_a^i) of f such that $[e_a^i, e_j] = \delta_j^i d_a^i$. Hence ϕ may be expressed as follows:

$$\phi(e_b^j) = \sum_k \phi_b^{jk} e_k.$$

Since $[z, z] = \{0\}$, we have $[e_a^i, \phi(e_b^j)] + [\phi(e_a^i), e_b^j] = 0$, whence

$$\sum_c (\phi_b^{ji} \delta_a^c - \phi_a^{ij} \delta_b^c) = 0.$$

Contracting both sides of the equalities above with respect to a and c , we obtain

$$(n-r)\phi_b^{ji} - \phi_b^{ij} = 0,$$

from which follows that

$$(n-r)^2 \phi_b^{ij} = \phi_b^{ij}.$$

Since $n-r \geq 2$, we therefore obtain $\phi_b^{ij} = 0$, i.e., $\phi = 0$. We have thus shown that $z = f = F(x)$, proving the proposition.

5.3. Almost pseudo-product manifolds of type L and $\text{Aut}(L)^\#$ -structures of type T . In this paragraph L will be a fixed pseudo-product FGLA, (T, e, f) its expression, and $(t, (g_p)_{p < 0})$ the expression of T .

Let $\mathcal{R}$ be an almost pseudo-product manifold of type L , and (R, E, F) its expression. Clearly the system $D(= E + F)$ is of type T , and we apply the argument in 4.3 to this system. Let us consider the symbol algebra $L(x)$ or $(T(x), E(x), F(x))$ of $\mathcal{R}$ at any

$x \in R$. Then we denote by $F(\mathcal{R}, L)$ the subset of $F(R, D, T)$ consisting of all $z \in F(R, D, T)$ such that $\tilde{z}$ give isomorphisms of L to $L(x)$, where $x = \pi(z)$.

We shall show that $F(\mathcal{R}, L)$ gives a differentiable reduction of $F(R, D, T)$ to the group $\text{Aut}(L)^\#$, where $\text{Aut}(L)$ is the automorphism group of L .

Let S newly denote the set of all the pairs e' and f' of g_{-1} such that $g_{-1} = e' + f'$ (direct sum), which becomes naturally a real analytic manifold. The group $\text{Aut}(T)^\#$ acts on S by the following rule:

$$(e', f')^a = (\zeta(a)^{-1}e', \zeta(a)^{-1}f'), \quad (e', f') \in S, \quad a \in \text{Aut}(T)^\#.$$

Then we define a map ϕ of $F(R, D, T)$ to S by

$$\phi(z) = (\tilde{z}^{-1} \cdot E(x), \tilde{z}^{-1} \cdot F(x)), \quad z \in F(R, D, T),$$

where $x = \pi(z)$. Now, the pair of the subspaces e and f of g_{-1} gives a point of S . Clearly $\text{Aut}(L)^\#$ is the isotropy group of $\text{Aut}(T)^\#$ at the point $(e, f) \in S$. Since $\mathcal{R}$ is of type L , it follows that the image of $F(R, D, T)$ by ϕ is in the $\text{Aut}(T)^\#$ -orbit through (e, f) , and further that $F(\mathcal{R}, L)$ is the inverse image of (e, f) by ϕ . By Lemma A.3 we have therefore shown that $F(\mathcal{R}, L)$ really gives a differentiable reduction of $F(R, D, T)$ to the group $\text{Aut}(L)^\#$.

We remark that the construction above of $F(\mathcal{R}, L)$ uses the inverse function theorem similarly to the construction of $F(R, D, T)$ in 4.3. However, consider the special case where L is a pseudo-projective FGLA. Then it is clear that $F(\mathcal{R}, L)$ as well as $F(R, D, T)$ can be constructed by rational operations and differentiations.

We denote by π the projection of $F(\mathcal{R}, L)$ onto R , and by ξ the

basic form of $F(\mathcal{R}, L)$. We also denote by ξ_{-1}^+ (resp. by ξ_{-1}^-) the e- (resp. f-) component of ξ_{-1} with respect to the decomposition $g_{-1} = e + f$.

Proposition 5.6. $F(\mathcal{R}, L)$ is of type T. Furthermore the system $\pi_*^{-1}(E)$ is defined by the system of Pfaffian equations $\xi_j = \xi_{-1}^- = 0$ ($j \leq -2$), and the system $\pi_*^{-1}(F)$ by the system of Pfaffian equations $\xi_j = \xi_{-1}^+ = 0$ ($j \leq -2$).

Proof. The first assertion is clear. The proof of the second assertion is quite similar to that of (1) of Proposition 4.3.

Now, let Q be any $\text{Aut}(L)^\#$ -structure of type T on a manifold R . Let π be the projection of Q onto R , and ξ the basic form of Q . As before the g_{-1} -valued 1-form ξ_{-1} is decomposed as follows: $\xi_{-1} = \xi_{-1}^+ + \xi_{-1}^-$.

Proposition 5.7. (1) There are unique differential systems E and F on R such that $\pi_*^{-1}(E)$ is defined by the system of Pfaffian equations $\xi_j = \xi_{-1}^- = 0$ ($j \leq -2$), and $\pi_*^{-1}(F)$ by the system of Pfaffian equations $\xi_j = \xi_{-1}^+ = 0$ ($j \leq -2$).

(2) The triplet (R, E, F) defines an almost pseudo-product manifold, say $\mathcal{R}$, of type L.

(3) $Q = F(\mathcal{R}, L)$. Furthermore $\mathcal{R}$ is a unique almost pseudo-product manifold of type L having this property.

The manifold $\mathcal{R}$ will be called associated with Q .

Proof. The proof of (1) is quite similar to that of (1) of Proposition 4.5. Let us prove (2) and (3). We first remark that Q gives a reduction of $F(R, D, T)$ to the group $\text{Aut}(L)^\#$ (by (3) of

Proposition 4.5, where D is the differential system of type T associated with Q . Now, for our purpose we may assume that Q is trivial. Take a cross section g of Q , and set $\alpha = g^* \xi$, $\alpha_p = g^* \xi_p$, and $\alpha_{-1}^\pm = g^* \xi_{-1}^\pm$. Let (D^p) be the family of derived systems of D . Then it is clear that D^p is defined by the system of Pfaffian equations $\alpha_j = 0$ ($j < p$), E by the system of Pfaffian equations $\alpha_j = \alpha_{-1}^- = 0$ ($j \leq -2$), and F by the system of Pfaffian equations $\alpha_j = \alpha_{-1}^+ = 0$ ($j \leq -2$). Clearly we have $D = E + F$ (direct sum). Since Q is of type T , we obtain

$$d\alpha_p + \frac{1}{2} \sum_{r+s=p} [\alpha_r, \alpha_s] \equiv_p 0,$$

where $\equiv_p$ is considered with respect to α . Let x be any point of R , and let X and Y be any local cross sections of E or of F defined on a neighborhood of x . Then it follows that

$$\begin{aligned} \alpha_p([X, Y]) &= 0 \quad (p < -2), \\ \alpha_{-2}([X, Y]) &= [\alpha_{-1}(X), \alpha_{-1}(Y)] \\ &= [\alpha_{-1}^+(X), \alpha_{-1}^-(Y)] + [\alpha_{-1}^-(X), \alpha_{-1}^+(Y)] = 0, \end{aligned}$$

showing that $[X, Y]$ gives a cross section of D . Consequently we have shown that the pair (E, F) gives an almost pseudo-product structure on R . Let x be any point of R , and set $z = g(x)$. Since $z \cdot \alpha(X) = X$ for all $X \in T_x(R)$, it follows that $z \cdot e = E(x)$ and $z \cdot f = F(x)$. Therefore $\tilde{z}$ gives an isomorphism of L to the symbol algebra $L(x)$ of $\tilde{R}$ at x , and hence $\tilde{R}$ is of type L , proving (2). This proof simultaneously indicates that g gives a cross section of $F(\tilde{R}, L)$, and hence $Q = F(\tilde{R}, L)$. Furthermore the uniqueness assertion in (3) follows immediately from (1) and Proposition

5.6, proving (3).

5.4. Finiteness for the geometry of almost pseudo-product manifolds of type L. Let G be a transitive graded Lie algebra, and $(g, (g_p))$ its expression. We denote by T the truncated FGLA, G_- , of G , which is thus expressed by $(t, (g_p)_{p < 0})$, where $t = g_- = \sum_{p < 0} g_p$. Let e and f be subspaces of g_{-1} . Then the triplet (G, e, f) is called a pseudo-product graded Lie algebra, if the following conditions are satisfied:

- (i) The triplet (T, e, f) defines a pseudo-product FGLA, say L ,
- (ii) g_0 leaves each of e and f invariant, i.e., $[g_0, e] \subset e$, and $[g_0, f] \subset f$.

The pseudo-product FGLA, L , will be called the truncated pseudo-product FGLA of the pseudo-product graded Lie algebra (G, e, f) . Furthermore the algebra (G, e, f) will be sometimes confounded with the underlying graded Lie algebra G , and then the pair (e, f) will be called the pseudo-product structure of G .

Let L be a pseudo-product FGLA, and (T, e, f) its expression. Let us consider the derivation algebra $\text{Der}(L)$ of L , i.e., the subalgebra of $\text{Der}(T)$ consisting of all $X \in \text{Der}(T)$ which leave each of e and f invariant. Then the prolongation, say G , of the pair $(T, \text{Der}(L))$ is called the prolongation of L . Clearly the triplet (G, e, f) gives a pseudo-product graded Lie algebra. Conversely if (G, e, f) is a pseudo-product graded Lie algebra, G becomes naturally a graded subalgebra of the prolongation of the truncated pseudo-product FGLA, L , of (G, e, f) .

Let (G, e, f) be a pseudo-product graded Lie algebra. We

preserve the notations as in the definition above.

Proposition 5.8. (cf. Corollary 3 to Theorem 11.1 in []).

(1) If L is non-degenerate, the Lie algebra $\mathfrak{g}$ is of finite dimension.

(2) Assume that G is the prolongation of L . Then $\mathfrak{g}$ is of finite dimension, if and only if L is non-degenerate.

Proof. (1) Let us consider the subalgebra $h_0 = h_0(G)$ of $\mathfrak{gl}(\mathfrak{g}_{-1})$ (see 4.5). By Proposition 4.6 it suffices to show that the first prolongation $h_0^{(1)}$ of h_0 vanishes. Recall that $h_0^{(1)}$ is the vector space of all linear maps A of $\mathfrak{g}_{-1}$ to h_0 such that $A(X)Y = A(Y)X$ for all $X, Y \in \mathfrak{g}_{-1}$. Now, we define an endomorphism J of $\mathfrak{g}_{-1}$ by

$$JX = X \text{ for } X \in \mathfrak{e}, \text{ and } JX = -X \text{ for } X \in \mathfrak{f}.$$

Clearly $J^2X = X$, and $[JX, JY] = -[X, Y]$ for all $X, Y \in \mathfrak{g}_{-1}$. We also define a bilinear map $(X, Y) \mapsto \langle X, Y \rangle$ of $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1}$ to $\mathfrak{g}_{-2}$ by

$$\langle X, Y \rangle = [JX, Y], \quad X, Y \in \mathfrak{g}_{-1}.$$

Then, taking account of the assumption that L or T is non-degenerate, we easily have the following : (i) $\langle, \rangle$ is symmetric, i.e., $\langle X, Y \rangle = \langle Y, X \rangle$, (ii) $\langle, \rangle$ is non-degenerate, i.e., the condition " $X \in \mathfrak{g}_{-1}$, and $\langle X, \mathfrak{g}_{-1} \rangle = \{0\}$ " implies $X = 0$, and (iii) every $A \in h_0$ is skew-symmetric with respect to $\langle, \rangle$, i.e., $\langle AX, Y \rangle + \langle X, AY \rangle = 0$ for all $X, Y \in \mathfrak{g}_{-1}$. Therefore it follows that $h_0^{(1)}$ vanishes, which compares with the well known fact that the first prolongation $\mathfrak{o}(n)^{(1)}$ of the orthogonal Lie algebra $\mathfrak{o}(n)$ vanishes. We have thus proved (1).

(2) Suppose that G is the prolongation of L , and that L or

T is degenerate. We define a subspace n of g_{-1} by

$$n = \{X \in g_{-1} \mid [X, g_{-1}] = \{0\}\}.$$

Clearly $n = n \cap e + n \cap f \neq \{0\}$ (cf. Lemma 3.3). We also define a subspace k_0 of $gl(g_{-1})$ by

$$k_0 = \{A \in gl(g_{-1}) \mid Ae \subset n \cap e, Af \subset n \cap f\}.$$

Clearly k_0 is an ideal of $h_0 \subset gl(g_{-1})$, and is of infinite type, i.e., the k -th prolongation $k_0^{(k)}$ of k_0 does not vanish for each $k \geq 0$. Since $k_0^{(k)} \subset h_0^{(k)} = h_k(G)$, it follows that g is of infinite dimension. Now, (2) follows immediately from this fact and (1). We have thereby completed the proof of the proposition.

Finally let L be a pseudo-product FGLA, and assume that it is non-degenerate. Let G be the prolongation of L , and let G_0 be the connected component of $Aut(L)$ containing the identity element. By Proposition 5.8 the underlying Lie algebra g of G is of finite dimension. Therefore we may apply Theorem 4.7 to the $G_0^\#$ -structures of type T . In view of Proposition 5.6 this application indicates the finiteness for the geometry of almost pseudo-product manifold of type L .

Let R be an almost pseudo-product manifold of type L . Let us consider the automorphism group $Aut(R)$ of R as well as the sheaf $a(R)$ of germs of local infinitesimal automorphisms of R . As a concrete proof of the finiteness stated above we then have the following proposition, which is nothing but a consequence of Corollary to Theorem 4.7.

Proposition 5.9. (1) Each stalk $a(R)_x$ of $a(R)$ is of finite dimension, and $\dim a(R)_x \leq \dim g$.

(2) $\text{Aut}(\mathcal{R})$ becomes naturally a Lie group, and $\dim \text{Aut}(\mathcal{R}) \leq \dim \mathfrak{g}$.

Appendix A

Completely integrable differential systems with singularities and some differentiability lemmas

A.1. Completely integrable differential systems with singularities. In this paragraph everything will be discussed in the real analytic category. Let M be a manifold. Let us consider the sheaf $\mathcal{O}$ of rings of germs of local functions on M , and the sheaf $\mathcal{T}$ of Lie algebras of germs of local vector fields on M . $\mathcal{T}$ is a module over $\mathcal{O}$ or an $\mathcal{O}$ -module. In this appendix a differential system on M means an $\mathcal{O}$ -submodule $\mathcal{E}$ of $\mathcal{T}$ which satisfies the following conditions: For each $p \in M$ there is a finite or infinite system (X_α) of local cross sections of $\mathcal{E}$ defined on a common neighborhood U of p such that the restriction of $\mathcal{E}$ to U is generated by the cross sections X_α . Such a system (X_α) will be called a system of local generators of $\mathcal{E}$. The notion of a differential system just introduced is a formulation of the ambiguous notion of a differential system possibly with singularities. A differential system $\mathcal{E}$ on M is said to be completely integrable, if each stalk $\mathcal{E}(p)$ is a subalgebra of the Lie algebra $\mathcal{T}(p)$.

Let $\mathcal{E}$ be a completely integrable system on M . For each $p \in M$ we define a subspace $E(p)$ of $T_p(M)$ by

$$E(p) = \{ X_p \mid X \in \mathcal{E}(p) \}.$$

Then a connected submanifold N of M is called an integral manifold of $\mathcal{E}$ or of E , if $T_p(N) = E(p)$ for each $p \in N$. By Theorem 1 of Nagano [12] we know that through every point p of M there passes a unique maximal integral manifold N of $\mathcal{E}$, and hence M is expressed

as the disjoint union of maximal integral manifolds of $\mathcal{E}$.

Let $\mathcal{E}$ be as above, and let N be a submanifold of M such that each connected component of the manifold N is a maximal integral manifold of $\mathcal{E}$. We denote by I the ideal of the closure of N , which is the subsheaf of ideals of $\mathcal{O}$ defined as follows:

$I = \bigcup_p I(p)$, and $I(p)$ is the ideal of $\mathcal{O}(p)$ consisting of all the germs $\in \mathcal{O}(p)$ which are represented by local functions f , defined on neighborhoods U of p_0 in M , that vanish on the intersections $U \cap N$. It is clear that if f is a local cross section of I , and if X is a local cross section of $\mathcal{E}$, then Xf is a local cross section of I .

For each $p \in N$ we now define a subspace $\Lambda(p)$ of $T_p^*(M)$ by

$$\Lambda(p) = \{ (df)_p \mid f \in I(p) \}.$$

Let Γ_0 be the pseudo-group of local transformations of M generated by $\mathcal{E}$ (cf. the proof of Lemma 1.3). Then we easily see that Γ_0 leaves each connected component of N invariant, and acts transitively on it. It follows that $\dim \Lambda(p)$ is constant on each connected component of N . Furthermore let Γ be a pseudo-group of local transformations of M . Then it follows similarly that if Γ leaves N invariant, and acts transitively on it, then $\dim \Lambda(p)$ is constant on the whole of N . These being observed, let us consider the following condition on N :

(*) $\dim \Lambda(p)$ is constant on the whole of N .

Lemma A.1. Let $\mathcal{E}$ be a completely integrable system on M , and let N be a submanifold of M such that each connected component of the manifold N is a maximal integral manifold of $\mathcal{E}$. Assume

that N satisfies condition (*). Then, for each point p_0 of N there are a neighborhood U of p_0 and an imbedded submanifold V of U which satisfies the following conditions:

- (i) $U \cap N \subset V$,
- (ii) $E(p) \subset T_p(V)$ for each $p \in V$,
- (iii) $\dim E(p) = \dim N$ for each $p \in V$.

Proof. We set $k = \dim \Lambda(p_0)$, and take k local functions $x^i \in I(p_0)$ such that the differentials $(dx^i)_p$ form a basis of $\Lambda(p_0)$. Then we assert that every local function $f \in I(p_0)$ may be expressed as a function of (x^i) on some neighborhood of p_0 in M . Indeed we take $n-k$ local functions $y^j \in O(p_0)$ such that the n local functions x^i and y^j form a coordinate system of M at p_0 . Now, let O be a sufficiently small neighborhood of p_0 in M . Then, for each $p \in O \cap N$ we have

$$(df)_p = \sum_i \frac{\partial f}{\partial x^i}(p) (dx^i)_p + \sum_j \frac{\partial f}{\partial y^j}(p) (dy^j)_p.$$

Since $(df)_p \in \Lambda(p)$, and since $(dx^i)_p$ form a basis of $\Lambda(p)$, it follows that $(\partial f / \partial y^j)(p) = 0$, showing that the derivatives $\partial f / \partial y^j$ are local cross sections of I . Therefore we see that all the derivatives $\partial^\alpha f / \partial y^{j_1} \dots \partial y^{j_\alpha}$ give local cross sections of I , whence $(\partial^\alpha f / \partial y^{j_1} \dots \partial y^{j_\alpha})(p_0) = 0$. We have thus shown that f may be expressed as a function of (x^i) on some neighborhood of p_0 in M , proving our assertion.

We set $\dim N = r$, and define a subset A of M by

$$A = \{ p \in M \mid \dim E(p) \leq r \},$$

which clearly contains N . We easily see that there is a system (f_λ) of local functions defined on a neighborhood O_1 of p_0 in M

such that the intersection $O_1 \cap A$ is defined by the system of equations $f_\lambda = 0$. Let us also consider a system (X_α) of local generators of the $\mathcal{O}$ -module $\mathcal{E}$ defined on a neighborhood O_2 of p_0 in M . Let U be a cubic neighborhood of p_0 with respect to the coordinate system (x^i, y^j) such that $U \subset O_1 \cap O_2$, and such that $\dim E(p) = \dim E(p_0) = r$ for each $p \in U \cap A$. We denote by V the slice of U defined by $V = \{ p \in U \mid x^i(p) = 0 \}$. Clearly we have $U \cap N \subset V$. Since $N \subset A$, f_λ give local cross sections of I , and hence they may be expressed as functions of (x^i) on some neighborhood of p_0 in M . Therefore it follows that f_λ vanish on some neighborhood of p_0 in V and hence on the whole of V , showing that $V \subset A$. Similarly $X_\alpha x^i$ vanish on the whole of V , because they give local cross sections of I . This means that X_α are tangent to V , whence $E(p) \subset T_p(V)$ for each $p \in V$. Since $V \subset U$, we have $\dim E(p) = r$ for each $p \in V$.

We have thus completed the proof of the lemma.

A.2. Some differentiability lemmas.

Lemma A.2. Let $\mathcal{E}$ and N be as in Lemma A.1, and assume that the submanifold N of M satisfies condition (*) as well as the second axiom of countability. Let ϕ be a differentiable map of a differentiable manifold B to M such that the image of B by ϕ is in N . Then the map ϕ , regarded as a map of B to N , is differentiable.

Proof. Take any point b_0 of B , and set $p_0 = \phi(b_0)$, being a point of N . We apply Lemma A.1 to the collection $\{\mathcal{E}, N, p_0\}$. If we set $E' = \bigcup_{p \in V} E(p)$, it follows from that lemma that E' is a

subbundle of $T(V)$ and is completely integrable, and that each connected component of the open submanifold $U \cap N$ of the manifold N is a maximal integral manifold of E' . Now, let O be a connected neighborhood of b_0 such that $\phi(O) \subset U$. Then $\phi(O) \subset U \cap N \subset V$, and the restriction of ϕ to O , regarded as a map of O to V , is differentiable. Since the open submanifold $U \cap N$ satisfies the second axiom of countability, we have therefore seen that the restriction of ϕ to O , regarded as a map O to $U \cap N$, is differentiable (cf. Chevalley []), proving the lemma.

Let P be a differentiable principal fibre bundle over a manifold M with structure group G . Let S be a real analytic manifold, and assume that the Lie group G acts real analytically on S (on the right) by the rule $(s, a) \mapsto s^a$, where $s \in S$, and $a \in G$. Furthermore let ϕ be a differentiable map of P to S which is G -equivariant: $\phi(za) = \phi(z)^a$ for all $z \in P$ and $a \in G$. Then we have the following

Lemma A.3. Let s_0 be a point of S , and assume that the image of P by ϕ is in the G -orbit through s_0 . Denote by Q the inverse image of s_0 by ϕ , and by H the isotropy group of G at s_0 . Then Q gives a differentiable reduction of the principal fibre bundle P to the group H .

Proof. Let π be the projection of P onto M . Then the group H acts freely on the set Q in a natural fashion, π maps Q onto M , and two points z and z' of Q are in the same fibre, i.e., $\pi(z) = \pi(z')$, if and only if $z' = za$ for some $a \in G$. Therefore to prove the lemma, it suffices to show that for each point $p_0 \in M$ there is a differentiable local cross section g of P defined on a

neighborhood O of p_0 such that the image of O by g is in Q .

Let us consider the Lie algebra, say $\tilde{g}$, of real analytic vector fields on S which is naturally induced by the action of G on S (cf. the Lie algebra consisting of all the vertical vector fields A^* on a principal fibre bundle P). We denote by $\mathcal{L}$ the real analytic differential system on S generated by $\tilde{g}$, which is completely integrable. Now, let S_0 denote the G -orbit through s_0 . Then the map $a \mapsto s_0^{a^{-1}}$ of G to S_0 naturally induces a bijection, say ι , of G/H onto S_0 , and S_0 becomes a real analytic submanifold of S so that ι gives a real analytic homeomorphism of G/H onto S_0 . Furthermore we know that each connected component of the manifold S_0 is a maximal integral manifold of $\mathcal{L}$, and that the submanifold S_0 of S satisfies condition (*) as well as the second axiom of countability. (Note that a Lie group is assumed to satisfy condition (*).) Therefore it follows from Lemma A.2 that the map ϕ , regarded as a map of P to S_0 , is differentiable.

Let us now recall that G may be considered as a principal fibre bundle over G/H with structure group H , and take a local cross section σ of this bundle defined on a neighborhood V of the origin of G/H . Let p_0 be any point of M , and take a local cross section g' of P defined on a neighborhood O of p_0 such that the image of O by $\phi \circ g'$ is in the image $\iota(V)$ of V by ι . We then define a map g of O to P by

$$g(p) = g'(p) \cdot \sigma(u), \quad p \in O,$$

where $u \in V$, and $\iota(u) = \phi(g'(p))$. Then we see that g gives a differentiable cross section of P . Since $s_0 = \iota(u)^{\sigma(u)}$, we also

see that $\phi(g(p)) = s_0$, showing that the image of O by g is in Q .

We have thus proved the lemma. We remark that the construction above of Q uses the inverse function theorem.

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Part I Chapter III

Part I Chapter III

Chapter III. Projective graded Lie algebras,

§III.1. Preliminaries: Simple graded Lie algebras

1. Simple 'graded' Lie algebras

$$\mathcal{G} = \{\mathcal{G}, (\mathcal{G}_p)\}.$$

(SGLA.1), (SGLA.2)

$$\mathcal{M} = \{\mathcal{M}, (\mathcal{G}_p)_{p \leq 0}\}$$

$\langle, \rangle$ Killing form of $\mathcal{G}$

Proposition. (1) $\langle \mathcal{G}_p, \mathcal{G}_q \rangle = 0$ if $p+q \neq 0$

(2)

Proposition. E

$\mathcal{G}_0 \subset \text{Der}(\mathcal{M})$ $\mathcal{G}$ is $(\mathcal{M}, \mathcal{G}_0)$ a prolongation
of graded subalgebra

Proposition $\sim$

Let $\mathcal{G}$ be a graded Lie algebra, and $(\mathcal{G}, (\mathcal{G}_p))$ its expression. Let us denote by $\mathcal{F}$ the truncated graded subalgebra $\mathcal{F}$ of $\mathcal{G}$, which is thus expressed by $(\mathcal{F}, (\mathcal{F}_p)_{p \leq 0})$, where $\mathcal{F} = \mathcal{F}_{-1} = \sum_{p \leq 0} \mathcal{G}_p$. Then $\mathcal{G}$ is called a simple graded Lie algebra or briefly a SGLA, if the following conditions are satisfied:

- 1) $\dim \mathcal{G} < \infty$,
- 2) $\mathcal{F}$ is a FGLA.

A simple graded Lie algebra $\mathcal{G}$ is called of the μ -th kind, if the FGLA, $\mathcal{F}$, is of the μ -th kind.

In the following we shall consider a fixed simple graded Lie algebra $\mathcal{G}$. We preserve the notation as above.

The homogeneous spaces $G/G^{(0)}$

$\text{Aut}(\mathcal{G})$

$G_0, G^{(0)}$

$$K: G^{(0)} \rightarrow G_0$$

Proposition $G^{(0)} = G_0 \cdot \exp \mathcal{G}_1 \cdots \exp \mathcal{G}_n$

$p \geq 1, G^{(p)}$

$$\rho: G^{(0)} \rightarrow GL(m)$$

$$\rho: \mathcal{G}^{(0)} \rightarrow \mathfrak{gl}(m)$$

Proposition G_0 faithful on $\mathcal{G}_1$

Proposition $\text{Ker } \rho = G^{(u)} = \exp \mathcal{G}_n$

Proposition G effective on $G/G^{(0)}$

Let $\mathfrak{g} = \{\mathfrak{g}, (\mathfrak{g}^p)\}$ be a simple graded Lie algebra, and let us consider the associated filtered Lie algebra $F(\mathfrak{g}) = \{\mathfrak{g}, (\mathfrak{g}^p)\}$.

For any subspace $\mathfrak{f}$ of $\mathfrak{g}$ we define a subspaces $P\mathfrak{f}$ and $Q\mathfrak{f}$ of $\mathfrak{g}$ as follows

$$P\mathfrak{f} = \{X \in \mathfrak{g} \mid \langle X, \mathfrak{f} \rangle = 0\},$$

$$Q\mathfrak{f} = \{X \in \mathfrak{g}^{(0)} \mid [X, \mathfrak{f}] \subset \mathfrak{g}^{(0)}\}.$$

Lemma Let $\mathfrak{g}$ be of the μ -th kind.

$$(1) \quad \mathfrak{g}^{(-p+1)} = P\mathfrak{g}^{(p)}, \quad 1 \leq p \leq \mu$$

$$(2) \quad \mathfrak{g}^{(p)} = Q\mathfrak{g}^{(-p)}, \quad 1 \leq p \leq \mu$$

$$(3) \quad \mathfrak{g}^{(-\mu+p)} = \overline{PQ} \cdots \overline{PQ} \mathfrak{g} \text{ (PQ } p \text{ times)}, \quad 0 \leq p \leq \mu,$$

$$\mathfrak{g}^{(\mu-p)} = \overline{QP} \cdots \overline{QP} \mathfrak{g} \text{ (QP } p \text{ times)}, \quad 1 \leq p \leq \mu.$$

From (3) of Lemma — we see that the filtration $(\mathfrak{g}^{(p)})$ depends only on the pair $(\mathfrak{g}, \mathfrak{g}^{(0)})$. More precisely we have the following

Proposition. Let $\mathfrak{g}' = \{\mathfrak{g}', (\mathfrak{g}')^p\}$ be another simple graded Lie algebra, and let $F(\mathfrak{g}') = \{\mathfrak{g}', (\mathfrak{g}')^p\}$. If $(\mathfrak{g}, \mathfrak{g}^{(0)}) = (\mathfrak{g}', \mathfrak{g}'^{(0)})$, then $\mathfrak{g}^{(p)} = \mathfrak{g}'^{(p)}$ for all p .

We denote by $\text{Aut}(\mathcal{G}, \mathcal{G}^{(0)})$ the automorphism group of the pair $(\mathcal{G}, \mathcal{G}^{(0)})$, i.e.,

$$\text{Aut}(\mathcal{G}, \mathcal{G}^{(0)}) = \{a \in \text{Aut}(\mathcal{G}) \mid a\mathcal{G}^{(0)} = \mathcal{G}^{(0)}\}.$$

In other words $\text{Aut}(\mathcal{G}, \mathcal{G}^{(0)})$ is the normalizer of $\mathcal{G}^{(0)}$ in $\text{Aut}(\mathcal{G})$.

Corollary 1. $\text{Aut}(F(\mathcal{G})) = \text{Aut}(\mathcal{G}, \mathcal{G}^{(0)})$.

Corollary 2. The notations being as in Proposition ..., assume that $(\mathcal{G}, \mathcal{G}^{(0)}) = (\mathcal{G}', \mathcal{G}'^{(0)})$. Then there is $a \in \text{Aut}(\mathcal{G}, \mathcal{G}^{(0)})$ such that $a\mathcal{G}_p = \mathcal{G}'_p$ for all p .

3. The operators ∂ and ∂^* $C^{\infty}(\mathcal{G})$, ∂

$$(\partial c)(x_1 \wedge \cdots \wedge x_{g+1}) =$$

(,) inner product

 ∂^*

$$(\partial^* c)(x_1 \wedge \cdots \wedge x_g) =$$

$$\Delta = \partial^* \partial + \partial \partial^*$$

 $H^{\infty}(\mathcal{G})$

$$C^{\infty}(\mathcal{G}) = H^{\infty}(\mathcal{G}) + \Delta C^{\infty}(\mathcal{G})$$

 G_0 on $C^{\infty}(\mathcal{G})$

Proposition

$$(\partial c)^{\dagger} = \partial(c^{\dagger}), \quad (\partial^* c)^{\dagger} = \partial^*(c^{\dagger})$$

 $G^{(0)}$ on $C^{\infty}(\mathcal{G})$

Proposition

$$(\partial^* c)^{\dagger} = \partial^*(c^{\dagger})$$

 $G^{(0)}$ on $C^{\infty}(\mathcal{G})$

The spaces $C^{p,q}(\mathcal{G})$

$$\Lambda^q(\mathfrak{m}^*), \quad \Lambda_i^q$$

$$\mathcal{G} \otimes \Lambda^q(\mathfrak{m}^*) = \sum_{i,j} \mathcal{G}_j \otimes \Lambda_i^q$$

$$C^{p,q}(\mathcal{G}) = \sum_j C_j^{p,q}, \text{ where } C_j^{p,q} = \mathcal{G}_j \otimes \Lambda_{j+p-q+1}^q$$

$$\partial C^{p,q}(\mathcal{G}) \subset C^{p-1,q+1}(\mathcal{G}),$$

$$\partial^* C^{p,q}(\mathcal{G}) \subset C^{p+1,q-1}(\mathcal{G})$$

$$C^{p,q}(\mathcal{G}) = H^{p,q}(\mathcal{G}) + \Delta C^{p,q}(\mathcal{G})$$

$$H^q(\mathcal{G}) = \sum_p H^{p,q}(\mathcal{G})$$

Proposition. $\mathcal{G}$ is the prolongation of $(\mathcal{M}, \mathcal{I}_0) \iff$

$C^{p,0}(\mathcal{G})$ is G_0 -invariant

Proposition. $X \in \mathcal{G}_r$ ($r \geq 0$), $c \in C^{p,q}(\mathcal{G})$
 $\Rightarrow c^X \in C^{p+r,q}(\mathcal{G})$

3. The spaces of harmonic forms, $H^{p,q}(\mathcal{G})$.

$\mathcal{G} = \{\mathcal{G}, (\mathcal{G}_p)\}$ is simple graded Lie algebra & $\exists$ $H^{p,q}(\mathcal{G})$ is

次の proposition を証明しよう.

Proposition 任意の p, q に対して,

$$H^{p,q}(\mathcal{G}) = \sum_j H^{p,q}(\mathcal{G}) \cap C_j^{p,q}.$$

任意の positive integer i に対して,

S_i は $m = \sum_{p \leq 0} \mathcal{G}_p$ から $\mathcal{G}_{-i}$ の $\perp$ への projection とする. operator $\partial_i: C^i(\mathcal{G}) \rightarrow C^{i+1}(\mathcal{G})$ を

$$(\partial_i c)(x_1 \wedge \cdots \wedge x_{q+1}) = \sum_k (-1)^{k+i} [S_i(x_k), c(x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_{q+1})]$$

によって定義する. $c \in C^i(\mathcal{G})$, $x_k \in m$ である. ∂_i は operator $\partial_i: C^i \rightarrow C^{i+1}$ を

$$(\partial_0 c)(x_1 \wedge \cdots \wedge x_{q+1}) = \sum_{k \leq 0} (-1)^{k+1} c([x_k, x_1] \wedge x_2 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_{q+1})$$

によって定義する. $c \in C^i(\mathcal{G})$, $x_k \in m$ である. 明らかに operator $\partial: C^i(\mathcal{G}) \rightarrow C^{i+1}(\mathcal{G})$ は

$$\partial = \sum_{i \geq 0} \partial_i$$

$$\begin{aligned} x_1 \in \mathcal{G}_{i-1}, \dots, x_{q+1} \in \mathcal{G}_{q+1-i} \\ x_1 + \cdots + x_{q+1} \neq j-i-(q-i)-(q+1-i) \\ j-i-p-q+1 \end{aligned}$$

と分解する. また容易に

$$\partial_i C_j^{p,q} \subset C_{j-i}^{p+1, q+1} \text{ for any } i \geq 0 \text{ and any } p, q, j$$

が確かめられる。

$\forall i \geq 0$ に $i \neq 1, 2$, $\partial_i^* : C^{s+1}(\mathcal{G}) \rightarrow C^s(\mathcal{G})$ を operator $\partial_i : C^s(\mathcal{G}) \rightarrow C^{s+1}(\mathcal{G})$ の adjoint operator とする。明らか に operator $\partial : C^s(\mathcal{G}) \rightarrow C^{s+1}(\mathcal{G})$ の adjoint operator $\partial^* : C^{s+1}(\mathcal{G}) \rightarrow C^s(\mathcal{G})$ は

$$\partial^* = \sum_{i \geq 0} \partial_i^*$$

と分解し,

$$\partial_i^* C_{j-i}^{p+1, s+1} \subset C_j^{p, s} \text{ for any } i \geq 0 \text{ and any } p, s, j$$

が成立する。

$\forall i > 0$ に $i \neq 1, 2$, $n_i = \dim \mathcal{G}_i$ とおき, $(e_{i\lambda})_{1 \leq \lambda \leq n_i}$ を $\mathcal{G}_i$ の basis とし, $(e^{i\lambda})$ を $\mathcal{G}$ の Killing form $\langle \cdot, \cdot \rangle$ に 関し $(e_{i\lambda})$ に dual な $\mathcal{G}_i$ の basis とする。 $i > 0$ のとき operator ∂_i^* は

$$(\partial_i^* c)(x_1 \wedge \cdots \wedge x_q) = \sum_{\lambda=1}^{n_i} [e^{i\lambda}, c(e_{i\lambda} \wedge x_1 \wedge \cdots \wedge x_q)]$$

と表現され operator ∂_0^* は

$$(\partial_0^* c)(x_1 \wedge \cdots \wedge x_q)$$

$$= \frac{1}{2} \sum_k \sum_{i>0} \sum_{\lambda=1}^{n_i} (-1)^{k+1} c([e^{i\lambda}, x_k] \wedge e_{i\lambda} \wedge x_1 \wedge \cdots \wedge \widehat{x_k} \wedge \cdots \wedge x_q)$$

と表現される。ここで $c \in C^{s+1}(\mathcal{G})$, $x_k \in \mathcal{M}$ とする。これらの式は [1, Proposition —] の証明から容易に推測される。

次の Lemma を 証明しよう.

Lemma $\forall i > 0$ に $\hat{x} \neq 1$ に

$$\sum_{r \geq 0} (\partial_r^* \partial_{r+i} + \partial_{r+i} \partial_r^*) = 0$$

$$C \in C^2, x_1, \dots, x_g \in M \text{ と } \exists$$

$$(\partial_0^* \partial_i C)(x_1 \wedge \dots \wedge x_g)$$

$$= \frac{1}{2} \sum_R \sum_{r \geq 0} \sum_{\sigma=1}^{n_r} (-1)^{R+1} (\partial_i C)([e_{r\sigma}^{\vee}, x_R]_{-1} e_{r\sigma} \wedge x_1 \wedge \dots \wedge \hat{x}_R \wedge \dots \wedge x_g)$$

$$= \frac{1}{2} A + \frac{1}{2} B + C.$$

" $\Rightarrow$ " A, B, C は 次式 で 与えられる.

$$A = \sum_R \sum_{r \geq 0} \sum_{\sigma=1}^{n_r} (-1)^{R+1} [\zeta_i([e_{r\sigma}^{\vee}, x_R]_{-1}), C(e_{r\sigma} \wedge x_1 \wedge \dots \wedge \hat{x}_R \wedge \dots \wedge x_g)]$$

$$B = \sum_R \sum_{r \geq 0} \sum_{\sigma=1}^{n_r} (-1)^R [\zeta_i(e_{r\sigma}), C([e_{r\sigma}^{\vee}, x_R]_{-1} x_1 \wedge \dots \wedge \hat{x}_R \wedge \dots \wedge x_g)]$$

$$C = \frac{1}{2} \sum_{\ell \in R} \sum_{r \geq 0} \sum_{\sigma=1}^{n_r} (-1)^{R+1} [\zeta_i(x_\ell), C([e_{r\sigma}^{\vee}, x_R]_{-1} e_{r\sigma} \wedge x_1 \wedge \dots \wedge \hat{x}_R \wedge \dots \wedge \hat{x}_\ell \wedge \dots \wedge x_g)]$$

$$+ \frac{1}{2} \sum_{\ell \in R} \sum_{r \geq 0} \sum_{\sigma=1}^{n_r} (-1)^{R+1} [\zeta_i(x_\ell), C([e_{r\sigma}^{\vee}, x_R]_{-1} e_{r\sigma} \wedge x_1 \wedge \dots \wedge \hat{x}_R \wedge \dots \wedge \hat{x}_\ell \wedge \dots \wedge x_g)]$$

Lemma 12 の Lemma から 直ちに 得られる

Lemma (1) $B = A$.

$$(2) A = - \sum_{r \geq 0} (\partial_{r+i} \partial_r^* C)(x_1 \wedge \dots \wedge x_g) - \sum_{r \geq 0} (\partial_r^* \partial_{r+i} C)(x_1 \wedge \dots \wedge x_g),$$

$$(3) C = -(\partial_i \partial_0^* C)(x_1 \wedge \dots \wedge x_g)$$

$r > 0$ 是 fix 的数.

No. 4

Proof. (1) $x_R = \sum_{r+s>0} S_{r+s}(x_R) \tau''$ 有,

$[e^{r_0}, S_{r+s}(x_R)] \in \mathcal{I}_{-s}$ τ'' 有 3 个,

$$[e^{r_0}, x_R]_- = \sum_{s>0} [e^{r_0}, S_{r+s}(x_R)]$$

τ'' 有 3 个 $\sum 1 \tau$

$$\begin{aligned} [e^{r_0}, S_{r+s}(x_R)] &= \sum_{T=1}^{n_s} \langle [e^{r_0}, S_{r+s}(x_R)], e^{s_T} \rangle e_{s_T} \\ &= \sum_{T=1}^{n_s} \langle e^{r_0}, [S_{r+s}(x_R), e^{s_T}] \rangle e_{s_T} \end{aligned}$$

故 12

$$\begin{aligned} &\sum_{r>0} \sum_{\sigma=1}^{n_r} [S_i(e_{r_0}), c([e^{r_0}, x_R]_- \wedge x_1 \wedge \dots \wedge \hat{x}_R \wedge \dots \wedge x_g)] \\ &= \sum_{r>0} \sum_{s>0} \sum_{T=1}^{n_s} [S_i([S_{r+s}(x_R), e^{s_T}]), c(e_{s_T} \wedge x_1 \wedge \dots \wedge \hat{x}_R \wedge \dots \wedge x_g)] \\ &= - \sum_{s>0} \sum_{T=1}^{n_s} [S_i([e^{s_T}, x_R]_-), c(e_{s_T} \wedge x_1 \wedge \dots \wedge \hat{x}_R \wedge \dots \wedge x_g)] \end{aligned}$$

故 12 $B = A$ 是 3.

(2) $S_i([e^{r_0}, x_R]_-) = [e^{r_0}, S_{r+i}(x_R)] \tau''$ 有 3 个,

$$\begin{aligned} A &= \sum_R \sum_{r>0} \sum_{\sigma=1}^{n_r} (-1)^{R+1} [[e^{r_0}, S_{r+i}(x_R)], c(e_{r_0} \wedge x_1 \wedge \dots \wedge \hat{x}_R \wedge \dots \wedge x_g)] \\ &= \sum_R \sum_{r>0} \sum_{\sigma=1}^{n_r} (-1)^{R+1} [[e^{r_0}, c(e_{r_0} \wedge x_1 \wedge \dots \wedge \hat{x}_R \wedge \dots \wedge x_g)], S_{r+i}(x_R)] \\ &\quad + \sum_R \sum_{r>0} \sum_{\sigma=1}^{n_r} (-1)^{R+1} [e^{r_0}, [S_{r+i}(x_R), c(e_{r_0} \wedge x_1 \wedge \dots \wedge \hat{x}_R \wedge \dots \wedge x_g)]] \end{aligned}$$

$$\begin{aligned}
&= \sum_R \sum_{r \geq 0} (-1)^{R+1} [(\partial_r^* C)(x_1 \wedge \cdots \wedge \hat{x}_R \wedge \cdots \wedge x_g), \zeta_{r+1}(x_R)] \\
&\quad - \sum_{r \geq 0} \sum_{\sigma=1}^{n_r} [e^{r\sigma}, (\partial_{r+1} C)(\varrho_{r\sigma} \wedge x_1 \wedge \cdots \wedge x_g)] \\
&= - \sum_{r \geq 0} (\partial_{r+1} \partial_r^* C)(x_1 \wedge \cdots \wedge x_g) \\
&\quad - \sum_{r \geq 0} (\partial_r^* \partial_{r+1} C)(x_1 \wedge \cdots \wedge x_g)
\end{aligned}$$

(3) は容易に確かめられる

さて space $C^{p,q}$ は $C^{p,q} = \sum_j C_j^{p,q}$ と分解し、
この分解に対して $C^{p,q}$ の各元 C は

$$C = \sum_j C_j$$

と分解する。ここで Lemma — を使って容易に
次の Lemma をえる。

Lemma $C \in C^{p,q}$ のとき

$$(\Delta C)_R = \sum_{r \geq 0} (\partial_r^* \partial_r + \partial_r \partial_r^*) C_R + \sum_{r \geq 0} \sum_{i \geq 0} (\partial_{r+i}^* \partial_r + \partial_r \partial_{r+i}^*) C_{R-i}$$

Proposition はこの Lemma から容易に得
られる。 $C \in H^{p,q}(\mathcal{O}_Y)$ とし、 $\forall R$ に對して $C_R \in H^{p,q}(\mathcal{O}_Y)$
を示そう。これを R に對する induction で証明
する。まず " n 以下の R に對して $C_R = 0$ である" と
して " n である R に對して $C_R \in H^{p,q}(\mathcal{O}_Y)$, $\forall l < R$ と假
定する。この假定は

$$\partial_i C_l = \partial_i^* C_l = 0, \quad \forall i \geq 0, \forall l < R$$

と同等である。よって Lemma から

$$\sum_{r \geq 0} (\partial_r^* \partial_r + \partial_r \partial_r^*) C_R = 0$$

をえる。これより直ちに

$$\partial_r C_R = \partial_r^* C_R = 0, \quad r \geq 0$$

をえる。これは $C_R \in H^{p,q}(\mathcal{M})$ を意味する。よって
我々の主張が示され、Proposition 1 の証明
が完了した。

§III 2. Projective graded Lie algebras and the homogeneous spaces $G/G^{(0)}$, $G/A^{(0)}$ and $G/B^{(0)}$

1. Projective graded Lie algebras. Let $\mathcal{L} = \{\mathcal{W}; e, f\}$ be a projective FGLA of type (n, r) , and let $\mathcal{W} = \{m; (\mathfrak{g}_p)_{p \leq 0}\}$. We first remark that the automorphism group $\text{Aut}(\mathcal{L})$ of $\mathcal{L}$ is isomorphic with the product group $GL(\mathbb{Z}_2) \times GL(e)$ through the natural representation of $\text{Aut}(\mathcal{L})$ on the space $\mathbb{Z}_2 + e$ (cf. the proof of Proposition —), and hence the derivation algebra $\text{Der}(\mathcal{L})$ of $\mathcal{L}$ is isomorphic with the product Lie algebra $gl(\mathbb{Z}_2) \times gl(e)$ through the corresponding representation of $\text{Der}(\mathcal{L})$ on $\mathbb{Z}_2 + e$. Especially, in the case where $r = 1$, $\text{Aut}(\mathcal{L})$ is also isomorphic with the product group $GL(e) \times GL(f)$ through the natural representation of $\text{Aut}(\mathcal{L})$ on the space $\mathbb{Z}_1 = e + f$, and hence $\text{Der}(\mathcal{L})$ is isomorphic with the product Lie algebra $gl(e) \times gl(f)$ through the corresponding representation of $\text{Der}(\mathcal{L})$ on $\mathbb{Z}_1$.

Let $\mathcal{G} = \{g, (\mathfrak{g}_p)\}$ be the prolongation of $\mathcal{L}$ or of $(\mathcal{W}, \text{Der}(\mathcal{L}))$, which will be called a projective graded Lie algebra of type (n, r) . By virtue of Proposition —, projective graded Lie algebras of type (n, r) are unique up to isomorphism, and hence depend only upon the pair (n, r) .

A direct calculation of the prolongation yields the following

Proposition $\mathcal{G}$ is a simple graded Lie algebra of the second kind, and the underlying Lie algebra $\mathfrak{g}$ is isomorphic with the simple Lie algebra $sl(n+1, \mathbb{R})$.

In the next paragraph we shall give a matricial representation of $\mathcal{G}$ together with another proof of the proposition.

We shall now show that to the graded Lie algebra $\mathcal{G}$ (or precisely to $\mathcal{L}$) there are associated two kinds of simple graded Lie algebras of the first kind, $\mathcal{A}$ and $\mathcal{L}$.

We have $m = \mathcal{G}_{-2} \oplus \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2$ (direct sum), and then we define a linear endomorphism J of m by

$$J\mathcal{G}_{-2} = 0; \quad JX = X, X \in \mathcal{G}_{-1}; \quad JX = -X, X \in \mathcal{G}_1.$$

Clearly J is in the centre of $\mathcal{D}_0 = \text{Der}(\mathcal{L})$, and

$$[J, \mathcal{G}_{-2}] = 0; \quad [J, [J, X]] = X, \quad X \in \mathcal{G}_1.$$

Lemma $[J, [J, X]] = X, X \in \mathcal{G}_1$, and $[J, \mathcal{G}_2] = 0$.

Proof Let $p=1$ or 2 . Then it follows from

Proposition that the condition " $X \in \mathfrak{g}_p, [X, \mathfrak{g}_{-2}] = 0$ " implies $X = 0$ (cf. Lemma —). Lemma follows easily from this fact. (Another proof also given by the use of Proposition —).

Let $\varepsilon = 1$ or -1 . We define subspaces $\mathfrak{g}_\varepsilon^+$ and $\mathfrak{g}_\varepsilon^-$ by

$$\mathfrak{g}_\varepsilon^\pm = \{X \in \mathfrak{g}_\varepsilon \mid [J, X] = \pm X\}.$$

Clearly we have $\mathfrak{g}_1^+ = \mathfrak{e}$ and $\mathfrak{g}_1^- = \mathfrak{f}$, and it follows from Lemma that

$$\mathfrak{g}_1 = \mathfrak{g}_1^+ + \mathfrak{g}_1^- \text{ (direct sum).}$$

Thus the Lie algebra $\mathfrak{g}$ is decomposed as follows:

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_1^+ + \mathfrak{g}_1^- + \mathfrak{g}_0 + \mathfrak{g}_1^+ + \mathfrak{g}_1^- + \mathfrak{g}_2.$$

We define subspaces α_p , $p \in \mathbb{Z}$, of $\mathfrak{g}$ as follows:

$$\alpha_p = 0 \text{ if } p < -1 \text{ or } p > 1; \quad \alpha_{-1} = \mathfrak{g}_{-2} + \mathfrak{g}_1^+;$$

$$\alpha_0 = \mathfrak{g}_1^- + \mathfrak{g}_0 + \mathfrak{g}_1^+; \quad \alpha_1 = \mathfrak{g}_1^- + \mathfrak{g}_2,$$

and define an element E_0 in the centre of $\mathfrak{g}_0$ by

$$E_{\alpha} = \frac{1}{2}(E - J),$$

where E is the element in the centre of $\mathcal{G}_0$ such that $[E, X] = pX$, $X \in \mathcal{G}_p$. Then using Lemma, we easily have

$$[E_{\alpha}, X] = pX, \quad X \in \mathcal{G}_p.$$

Therefore it follows that $\mathcal{A} = \{\mathcal{G}, (\mathcal{G}_p)\}$ is a simple graded Lie algebra of the first kind.

Similarly we define subspaces $\mathcal{G}_p$, $p \in \mathbb{Z}$, of $\mathcal{G}$ as follows:

$$\mathcal{G}_p = 0 \text{ if } p < -1 \text{ or } p > 1; \quad \mathcal{G}_{-1} = \mathcal{G}_2 + \mathcal{G}_1^-;$$

$$\mathcal{G}_0 = \mathcal{G}_1^+ + \mathcal{G}_0 + \mathcal{G}_1^-; \quad \mathcal{G}_1 = \mathcal{G}_1^+ + \mathcal{G}_2,$$

and define an element $E_{\mathcal{L}}$ in the centre of $\mathcal{G}_0$ by

$$E_{\mathcal{L}} = \frac{1}{2}(E + J).$$

Then we have

$$[E_{\mathcal{L}}, X] = pX, \quad X \in \mathcal{G}_p,$$

and hence $\mathcal{L} = \{\mathcal{G}, (\mathcal{G}_p)\}$ becomes a simple graded Lie algebra of the first kind.

Remark. Our discussions above concerning the graded Lie algebras $\mathcal{A}$ and $\mathcal{B}$ do not essentially use Proposition —, especially the simplicity of the underlying Lie algebra $\mathfrak{g}$, indicating that the results can be generalized to the cases of any GP FGLA's of the second kind. Indeed let $\mathcal{A} = \{\mathfrak{g}; e, f\}$ be a GP FGLA of the second kind, and let $\mathcal{G} = \{\mathfrak{g}, (\mathfrak{g}_p)\}$ be its prolongation. Then it is shown that there is an element J in the centre of $\mathcal{G}_0$ such that $[J, X] = X$, $X \in e$, and $[J, X] = -X$, $X \in f$, and such that $[J, \mathfrak{g}_p] = 0$ if p is even, and $[J, [J, X]] = X$, $X \in \mathfrak{g}_p$, if p is odd. It is also shown that there is an element E in the centre of $\mathcal{G}_0$ such that $[E, X] = pX$, $X \in \mathfrak{g}_p$. We now know that the elements $E_{\mathcal{A}} = \frac{1}{2}(E - J)$ and $E_{\mathcal{B}} = \frac{1}{2}(E + J)$ give rise to graded Lie algebras $\mathcal{A} = \{\mathfrak{g}, (\mathfrak{a}_p)\}$ and $\mathcal{B} = \{\mathfrak{g}, (\mathfrak{b}_p)\}$ respectively in the same manner as above. (Analogous results can be also obtained for suitable GP FGLA's of the third order. Such a GP FGLA ^{really} appears in the study of single ordinary differential equations of the third order.)

2. Matricial representations of projective graded Lie algebras. Let $m, r_1, \dots, r_k$ be positive integers such that $r_1 + \dots + r_k = m$. Then every matrix X in the Lie algebra $\mathfrak{g} = \mathfrak{sl}(m, \mathbb{R})$ may be written in the form:

$$X = \begin{pmatrix} X_{11} & \dots & X_{1r_k} \\ \vdots & & \vdots \\ X_{r_1 1} & \dots & X_{r_k r_k} \end{pmatrix},$$

where X_{ij} are $r_i \times r_j$ matrices. We define subspaces $\mathfrak{g}_p, p \in \mathbb{Z}$, of $\mathfrak{g}$ by

$$\mathfrak{g}_p = \{X \mid X_{ij} = 0 \text{ if } j-i \neq p\}.$$

Then we see that $\{\mathfrak{g}, (\mathfrak{g}_p)\}$ is a simple graded Lie algebra of the $(k-1)$ -th kind, which we denote by $\mathfrak{g}(r_1, \dots, r_k)$.

Let n and r be integers such that $1 \leq r \leq n-1$. Putting $r_1 = 1, r_2 = r$ and $r_3 = n-r$, we have a graded Lie algebra of the second kind, $\mathfrak{g}(1, r, n-r)$, and analogously we have the algebras of the first kind, $\mathfrak{g}(1, n)$ and $\mathfrak{g}(r+1, n-r)$. We shall show that $\mathfrak{g} = \mathfrak{g}(1, r, n-r)$ is a projective graded Lie algebra of type (n, r) and the associated graded Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are given by $\mathfrak{g}_1 = \mathfrak{g}(1, n)$ and $\mathfrak{g}_2 = \mathfrak{g}(r+1, n-r)$.

We first remark that, the notations being as above, the subspaces $\mathfrak{g}_2, \dots, \mathfrak{g}_r$ of $\mathfrak{g} = \mathfrak{sl}(n+r, \mathbb{R})$ are explicitly given as follows

$$\mathcal{I}_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X_{31} & 0 & 0 \end{pmatrix} \right\}, \quad \mathcal{I}_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ X_{21} & 0 & 0 \\ 0 & X_{32} & 0 \end{pmatrix} \right\},$$

$$\mathcal{I}_0 = \left\{ \begin{pmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & 0 \\ 0 & 0 & X_{33} \end{pmatrix} \right\}, \quad \mathcal{I}_1 = \left\{ \begin{pmatrix} 0 & X_{12} & 0 \\ 0 & 0 & X_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$\mathcal{I}_2 = \left\{ \begin{pmatrix} 0 & 0 & X_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

We also remark that the element E in the centre of $\mathcal{I}_0$ is given by

$$E = (\lambda_i \delta_{ij})_{1 \leq i, j \leq n+1},$$

where $\lambda_1 = \frac{2n-r}{n+1}$, $\lambda_2 = \dots = \lambda_{r+1} = \frac{n-r-1}{n+1}$, $\lambda_{r+2} = \dots = \lambda_{n+1} = \frac{-r-2}{n+1}$.

Let us consider the truncated graded subalgebra $\mathcal{M} = \{m, (\mathcal{I}_p)_{p \leq 0}\}$ of $\mathcal{G}$, where $m = \sum_{p \leq 0} \mathcal{I}_p = \mathcal{I}_2 + \mathcal{I}_1$. Then using Proposition 1.7 we see that $\mathcal{G}$ is the prolongation of $(\mathcal{M}, \mathcal{I}_0)$, where $\mathcal{I}_0$ should be identified with a subalgebra of the derivation algebra $\text{Der}(\mathcal{M})$ of $\mathcal{M}$. We now define two subspaces e and f of $\mathcal{G}$ by

$$e = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ X_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad f = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & X_{32} & 0 \end{pmatrix} \right\}.$$

Then it is easy to see that $\mathcal{L} = \{\mathcal{M}, e, f\}$ is a projective FGLA of type (m, r) and that

$\mathcal{G}$ coincides with the derivation algebra $D(L)$ of L . We have thus shown that $\mathcal{G}$ is the prolongation of L , and hence a projective gradient Lie algebra of type (n, r) . We define a matrix J in $\mathcal{G}$ by

$$J = (\mu_i \delta_{ij})_{1 \leq i, j \leq n+1},$$

where $\mu_1 = \frac{-r}{n+1}$, $\mu_2 = \dots = \mu_{n+1} = \frac{n+1-r}{n+1}$, $\mu_{n+2} = \dots = \mu_{n+1} = \frac{-r}{n+1}$. Then we see that J is in the centre of $\mathcal{G}_0$ and that $[J, \mathcal{G}_\varepsilon] = 0$ and $[J, [J, X]] = X$, $X \in \mathcal{G}_\varepsilon$, where $\varepsilon = 1$ or -1 . Furthermore we see that $e = \mathcal{G}_1^+$ and $f = \mathcal{G}_1^-$ and

$$\mathcal{G}_1^+ = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad \mathcal{G}_1^- = \left\{ \begin{pmatrix} 0 & x_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Therefore the subspaces σ_{-1} , σ_0 and σ_1 of $\mathcal{G}$ are ~~respectively~~ given by

$$\sigma_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & 0 & 0 \\ x_{31} & 0 & 0 \end{pmatrix} \right\}, \quad \sigma_0 = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ 0 & x_{22} & x_{23} \\ 0 & x_{32} & x_{33} \end{pmatrix} \right\},$$

$$\sigma_1 = \left\{ \begin{pmatrix} 0 & x_{12} & x_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

and the subspaces τ_{-1} , τ_0 and τ_1 of $\mathcal{G}$ are ~~respectively~~ given by

$$\tau_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_{11} & x_{20} & 0 \end{pmatrix} \right\}, \quad \tau_0 = \left\{ \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & x_{33} \end{pmatrix} \right\},$$

$$b_1 = \left\{ \begin{pmatrix} 0 & 0 & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

It is now clear that $\mathcal{O} = \mathcal{O}(1, n)$ and $\mathcal{L} = \mathcal{O}(r+1, n-r)$. We have thus proved our assertions.

3. The homogeneous spaces $G/G^{(0)}$, $G/A^{(0)}$ and $G/B^{(0)}$. In this and the next paragraphs, we preserve the notations as in 1. First of all we remark that every automorphism of L is naturally extensible to an automorphism of $\mathfrak{g}$. Thus the automorphism group $\text{Aut}(L)$ of L may be identified with a subgroup of the automorphism group $\text{Aut}(\mathfrak{g})$ of $\mathfrak{g}$. Since the derivations algebras $\text{Der}(L)$ of L and $\text{Der}(\mathfrak{g})$ of $\mathfrak{g}$ coincide, we know that $\text{Aut}(L)$ is an open subgroup of $\text{Aut}(\mathfrak{g})$.

Lemma (1) If $1 \leq r \leq n-2$, then $\text{Aut}(L) = \text{Aut}(\mathfrak{g})$.

(2) If $r = n-1$, then $\text{Aut}(\mathfrak{g}) \cap \text{Aut}(\mathfrak{g}) \subset \text{Aut}(L)$.

This lemma together with Lemma — below can be easily proved by using the ^{paragraph} matrixial representation of $\mathfrak{g}$ given in the previous

We define subgroups G_0 , $G^{(0)}$ and G of the automorphism group $\text{Aut}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ as follows:

$$G_0 = \text{Aut}(L), \quad G^{(0)} = G_0 G^{(1)}, \quad G = \text{Aut}(\mathfrak{g}) \cap G_0,$$

where as before $G^{(1)}$ stands for the subgroup of $\text{Aut}(\mathfrak{g})$ generated by the subalgebra $\mathfrak{g}^{(1)} = \mathfrak{g}_1 + \mathfrak{g}_2$ of $\mathfrak{g}$: $G^{(1)} = \exp \mathfrak{g}_1 \cdot \exp \mathfrak{g}_2$. Hence $G^{(0)}$ and G are the groups associated with the pair $(\mathfrak{g}, G_0)$. (See Remark below Theorem —). We know from (1) of Lemma that if $1 \leq r \leq n-2$, then G_0 ,

$G^{(0)}$ and G are the groups naturally associated with $\mathcal{G}$ itself (see —)

Let us ~~now~~ consider the homogeneous space $G/G^{(0)}$. Then we easily see that the subspaces $\mathfrak{e} = \mathfrak{g}_+^+$ and $\mathfrak{f} = \mathfrak{g}_-^+$ naturally give rise to G -invariant differential systems E_0 and F_0 on $G/G^{(0)}$, and that $\mathcal{P}_0 = \{G/G^{(0)}; E_0, F_0\}$ is a projective system of type (n, r) , which will be called the standard projective system of type (n, r) .

By using the simple graded Lie algebras of the first kind, $\mathcal{A}$ and $\mathcal{B}$, we now define subgroups $A_0, A^{(0)}, B_0, B^{(0)}$ of $\text{Aut}(\mathcal{G})$ as follows:

$$A_0 = \text{Aut}(\mathcal{A}), \quad A^{(0)} = A_0 \cdot A^{(1)}$$

$$B_0 = \text{Aut}(\mathcal{B}), \quad B^{(0)} = B_0 \cdot B^{(1)}$$

$r+1 = n-r$ のときは不成立!!

where $A^{(1)}$ (resp $B^{(1)}$) stands for the subgroup of $\text{Aut}(\mathcal{G})$ generated by the subalgebra $\mathcal{A}_1$ (resp $\mathcal{B}_1$) of $\mathcal{G}$: $A^{(1)} = \exp \mathcal{A}_1$, $B^{(1)} = \exp \mathcal{B}_1$.

Lemma (1) $G = \text{Aut}(\mathcal{G}) \circ A_0 = \text{Aut}(\mathcal{G}) \cdot B_0$.

(2) $A_0 \cap B_0 = G_0$, and $A^{(0)} \cap B^{(0)} = G^{(0)}$.

Therefore we know that $A_0, A^{(0)}$ and G (resp $B_0, B^{(0)}$ and G) are the groups naturally associated with $\mathcal{A}$ (resp $\mathcal{B}$). Furthermore we know that the diagonal map $a \rightarrow (a, a)$

of G to $G \times G$ induces an imbedding

$$2: G/G^{(0)} \rightarrow G/A^{(0)} \times G/B^{(0)}.$$

We notice that the pseudo-product structure (E_0, F_0) on $G/G^{(0)}$ is induced by this imbedding.

Let us now consider the projective space $P^n = G^r(\mathbb{R}^{n+1})$, the Grassmann manifold $G^r(P^n) = G^{r+1}(\mathbb{R}^{n+1})$ and, the Grassmann bundle $G^r(T(P^n))$ (See § 7). Let $(e_0, \dots, e_n)$ be the canonical basis of $\mathbb{R}^{n+1}$. Then the 1-dimensional subspace of $\mathbb{R}^{n+1}$ spanned by e_0 defines a point x_0 of P^n , and similarly the $(r+1)$ -dimensional subspace of $\mathbb{R}^{n+1}$ spanned by $e_0, \dots, e_r$ defines a point y_0 of $G^r(P^n)$. Moreover γ_0 being an r -dimensional projective subspace of P^n , the tangent space of γ_0 at x_0 defines a point z_0 of $G^r(T(P^n))$.

Now let G be the projective transformation group of P^n . Then P^n may be represented by the homogeneous space $G/A^{(0)}$, where $A^{(0)}$ denotes the isotropy subgroup of G at the point x_0 . Furthermore G naturally acts both on $G^r(P^n)$ and $G^r(T(P^n))$, and these spaces may be represented by the homogeneous spaces $G/B^{(0)}$ and $G/G^{(0)}$ respectively, where $B^{(0)}$ (resp. $G^{(0)}$) denotes the isotropy subgroup of G at y_0 (resp. at z_0).

There being prepared, we remark that there exist natural identifications as follows:

$$G = \dot{G}, A^{(0)} = \dot{A}^{(0)}, B^{(0)} = \dot{B}^{(0)} \text{ and } G^{(0)} = \dot{G}^{(0)}, \text{ whence} \\ G/A^{(0)} = P^n, G/B^{(0)} = G^r(P^n) \text{ and } G/G^{(0)} = G^r(T(P^n)).$$

4. Some studies in connection with the imbedding $G/G^{(0)} \rightarrow G/A^{(0)} \times G/B^{(0)}$. For the sake of simplicity we set as follows: $R = G/G^{(0)} = G^r(TP^n)$, and $R' = G/A^{(0)} \times G/B^{(0)} = P^n \times G^r(P^n)$. The product group $G \times G$ naturally acts on the product manifold R' , and hence the group G itself acts on R' through the diagonal map $G \rightarrow G \times G$. This being said, it turns out that the imbedding $\iota: R \rightarrow R'$ is G -equivariant. If we identify R with a submanifold of R' through the imbedding ι , then it follows that R is a G -orbit, more precisely the G -orbit through the point (x_0, y_0) .

Let us now consider the open submanifold $\Omega = R' - R$ of R' . We denote by $\text{Aut}(\Omega)$ the group of all biproduct maps $\Omega \rightarrow \Omega$, and similarly by $\text{Aut}(R)$ the group of all biproduct maps $R \rightarrow R$, which is nothing but the automorphism group $\text{Aut}(P_0)$ of the standard projective system P_0 . These being prepared, we state the following two propositions which can be easily shown.

Proposition (1) Ω is a G -orbit.

(2) If $r = n-1$, then Ω may be represented by the homogeneous space G/A_0 or G/B_0 with respect to suitable reference points, and these spaces are affine symmetric spaces, or more precisely $\frac{1}{2}$ -Kählerian symmetric spaces in the sense of Berger [1].

Proposition (1) Every biproduct map $\varphi \in \text{Aut}(\Omega)$ is extendible to a unique biproduct map $R' \rightarrow R'$ and this in turn induces a biproduct map $\varphi \in \text{Aut}(R)$. (1)

(2) Conversely every biproduct map $\varphi \in \text{Aut}(R)$ is extendible to a unique biproduct map $R' \rightarrow R'$ and this in turn induces a biproduct map $\varphi \in \text{Aut}(\Omega)$.

(3) Therefore the two groups $\text{Aut}(\Omega)$ and $\text{Aut}(R)$ are naturally isomorphic.

Concerning these propositions, we now make some remarks.

Remark 1. In § — we shall see that the group $\text{Aut}(R) = \text{Aut}(\mathcal{P}_0)$ is naturally isomorphic with the group G , i.e., it is obtained from the action of G on R . By Proposition we know that the same holds for the group $\text{Aut}(\Omega)$. In particular $\text{Aut}(\Omega)$ is a finite dimensional Lie group. Here we notice that the pseudo-groups of all ^{local} biproduct maps of a product manifold Ω is necessarily infinite dimensional. (This fact also follows from the fundamental theorem on projective transformations. (cf. Proposition — 2))

Remark 2. Proposition indicates that the geometry of the product manifold Ω is closely related to the geometry of the pseudo-product manifold R or the global geometry of differential equations $\frac{\partial^2 x^i}{\partial x^j \partial x^k} = 0$, $1 \leq i, j \leq r$, $1 \leq \alpha \leq r-r$, which is very much like the fact that the geometry of a homogeneous Siegel domain is closely related to the geometry

of its (completed) Sylow boundary (see Tanaka [V]). This analogy is essentially based on the relationship existing between product structures and complex structures.

Remark 3. Our studies in this section depend on the projective FGLA, $\mathcal{L}$, chosen. It is plausible that most results including Propositions — and — can be generalized to the cases of any GP FGLA's of the second kind and of further general GP FGLA's (cf. Remark at the end of 1.)

$(\mathcal{G}, e, f)$ を pseudo-product simple graded Lie algebra とす.

$\mathcal{G}_0$ の centre の元 J で, 次の条件を満足するものが唯一つある:

$$[J, X] = X, X \in e; [J, X] = -X, X \in f.$$

そこで, J は 次を満足するものと仮定する:

$$[J, \mathcal{G}_{2p}] = \{0\}, [J, [J, X]] = X, X \in \mathcal{G}_{2p-1}.$$

ここで, p は 任意の integer である. ($p=0, 1, -1$ に対してもこれらが成り立つことは容易にわかる.)

$\mathcal{G}_{2p-1}$ の subspaces $\mathcal{G}_{2p-1}^+, \mathcal{G}_{2p-1}^-$ を

$$\mathcal{G}_{2p-1}^{\pm} = \{X \in \mathcal{G}_{2p-1} \mid [J, X] = \pm X\}$$

によって定義する. 明らかに $e = \mathcal{G}_{-1}^+, f = \mathcal{G}_{-1}^-$ であり, 任意の p に対して

$$\mathcal{G}_{2p-1} = \mathcal{G}_{2p-1}^+ + \mathcal{G}_{2p-1}^- \text{ (direct sum)}$$

Lemma. (1) $\mathcal{G}_{-2} = [\mathcal{G}_{-1}^+, \mathcal{G}_{-1}^-]$,

$$(2) \quad \mathcal{G}_{2p-1}^{\pm} = [\mathcal{G}_{2p}, \mathcal{G}_{-1}^{\pm}] \text{ for any } p < 0,$$

$$(3) \quad [\mathcal{G}_{2p-1}^{\pm}, \mathcal{G}_{2q-1}^{\pm}] = \{0\} \text{ for any } p, q.$$

E を simple graded Lie algebra $\mathcal{G}$ の characteristic element とする. $\mathcal{G}_0$ の centre の元 E_0 を

$$E_0 = \frac{1}{2} (E - J)$$

によって定義し, $\mathcal{G}$ の subspaces の family (α_p) を

$$\alpha_p = \{X \in \mathcal{G} \mid [E_0, X] = pX\}$$

によって定義する. 容易に,

$$\alpha_p = \mathcal{G}_{2p-1}^- + \mathcal{G}_{2p} + \mathcal{G}_{2p+1}^+$$

がわかる. これより特に $\mathcal{G} = \sum_p \alpha_p$ がわかる. よって $(\mathcal{G}, (\alpha_p))$ が graded Lie algebra になることがわかる. これを $\mathcal{A}$ で表わす. Lemma を使って $\alpha_{p-1} = [\alpha_p, \alpha_{-1}]$ for $p < 0$ が示される. よって, $\mathcal{A}$ は simple graded Lie algebra になる.

次に 12 ついて,

$$\begin{aligned} [\alpha_p, \alpha_{-1}] &= [\mathcal{G}_{2p-1}^- + \mathcal{G}_{2p} + \mathcal{G}_{2p+1}^+, \mathcal{G}_3^- + \mathcal{G}_2 + \mathcal{G}_1^+] \\ &= \underbrace{[\mathcal{G}_{2p}, \mathcal{G}_3^-]} + \underbrace{[\mathcal{G}_{2p+1}^+, \mathcal{G}_3^-]} + \underbrace{[\mathcal{G}_{2p-1}^-, \mathcal{G}_2]} + \underbrace{[\mathcal{G}_{2p}, \mathcal{G}_2]} \\ &\quad + \underbrace{[\mathcal{G}_{2p+1}^+, \mathcal{G}_2]} + \underbrace{[\mathcal{G}_{2p-1}^-, \mathcal{G}_1^+]} + [\mathcal{G}_{2p}, \mathcal{G}_1^+] \end{aligned}$$

$$\alpha_{p-1} = \mathcal{I}_{2p-3}^- + \mathcal{I}_{2p-2} + \mathcal{I}_{2p-1}^+$$

$$\mathcal{I}_{2p-2} = [\mathcal{I}_{2p-1}, \mathcal{I}_{-1}] = [\mathcal{I}_{2p-1}^+ + \mathcal{I}_{2p-1}^-, \mathcal{I}_{-1}^+ + \mathcal{I}_{-1}^-]$$

$$= [\mathcal{I}_{2p-1}^+, \mathcal{I}_{-1}^-] + [\mathcal{I}_{2p-1}^-, \mathcal{I}_{-1}^+]$$

$$\subset [[\mathcal{I}_{2p}, \mathcal{I}_{-1}^+], \mathcal{I}_{-1}^-] + [\mathcal{I}_{2p-1}^-, \mathcal{I}_{-1}^+]$$

$$\subset [\mathcal{I}_{2p}, \mathcal{I}_{-2}] + [\mathcal{I}_{2p-1}^-, \mathcal{I}_{-1}^+]$$

$$\mathcal{I}_{2p-3}^- = [\mathcal{I}_{2p-2}, \mathcal{I}_{-1}^-] = [[\mathcal{I}_{2p}, \mathcal{I}_{-2}] + [\mathcal{I}_{2p-1}^-, \mathcal{I}_{-1}^+], \mathcal{I}_{-1}^-]$$

$$\in [\mathcal{I}_{2p-1}, \mathcal{I}_{-2}] + [\mathcal{I}_{2p}, \mathcal{I}_{-3}]$$

$$\mathcal{I}_{2p-1}^+ = [\mathcal{I}_{2p}, \mathcal{I}_{-1}^+]$$

$$j > 2, \quad \alpha_{p-1} \subset [\alpha_p, \alpha_{-1}]. \quad j > 2 \quad \alpha_{p-1} = [\alpha_p, \alpha_{-1}]$$

同様12, $\mathcal{G}_0$ の centre の元 $E_{\mathcal{G}}$ を

$$E_{\mathcal{G}} = \frac{1}{2} (E + J)$$

によって定義し, $\mathcal{G}$ の subspaces の family $(\mathcal{G}_p)$ を

$$\mathcal{G}_p = \{X \in \mathcal{G} \mid [E_{\mathcal{G}}, X] = pX\}$$

によって定義する. このとき,

$$\mathcal{G}_p = \mathcal{G}_{2p-1}^+ + \mathcal{G}_{2p} + \mathcal{G}_{2p-1}^-$$

である. よって, $(\mathcal{G}, (\mathcal{G}_p))$ が "simply graded Lie algebra" になる. これを $\mathfrak{g}$ で表わす.

以下の議論において, $(\mathcal{G}^{(p)})$, $(\mathcal{O}^{(p)})$, $(\mathfrak{g}^{(p)})$ はそれぞれ $\mathcal{G}$, $\mathcal{O}$, $\mathfrak{g}$ に付随する filtrations とする.

$\mathcal{O}$ 12 に対して, pseudo-product FGLA, $\mathcal{L}$, を考
える. $\text{Aut}(\mathcal{L})$ は $\text{Aut}(\mathcal{O})$ の open subgroup とみ
なせる. $\text{Aut}(\mathcal{L}) = \{g \in \text{Aut}(\mathcal{O}) \mid A_0(g)z_i^\pm = z_i^\pm\}$.

Lemma $\text{Aut}(\mathcal{L}) = \text{Aut}(\mathcal{O}) \cap \text{Aut}(\mathcal{L})$.

$G_0 = \text{Aut}(\mathcal{L})$ とし, $G^{(0)}, G \in (\mathcal{O}, G_0)$ 12 付随
する group とする:

$$G^{(0)} = G_0 \cdot G^{(1)}, \quad G = \text{Aut}(\mathcal{O})^\circ \cdot G_0$$

$\text{Aut}(\mathcal{O})$ の open subgroup A_0 を

$$A_0 = \text{Aut}(\mathcal{O})^\circ \cdot G_0$$

12 よって定義する. 明らかに, $G = \text{Aut}(\mathcal{O})^\circ \cdot A_0$ である.
 $\forall t_2, G$ の subgroup $A^{(0)}$ を

$$A^{(0)} = A_0 \cdot A^{(1)}$$

12 よって定義する. 明らかに 12 $A^{(0)}, G$ は $(\mathcal{O}, A_0)$
12 付随する group である.

同様 12, $\text{Aut}(\mathcal{L})$ の open subgroup B_0 を

$$B_0 = \text{Aut}(\mathcal{L})^\circ \cdot G_0$$

12 よって定義する. このとき, $G = \text{Aut}(\mathcal{O})^\circ \cdot B_0$ である.

また G の subgroup $B^{(0)}$ を

$$B^{(0)} = B_0 \cdot B^{(1)}$$

によって定義する。このとき、 $B^{(0)}, G$ は $(\mathcal{G}, B_0)$ に付随する groups である。

Lemma. $G^{(0)} = A^{(0)} \cap B^{(0)}$.

Proof Lemma により、 $G_0 = A_0 \cap B_0$. $\mathcal{G}^{(0)} = \mathcal{O}^{(0)} \cap \mathcal{F}^{(0)}$ であるから、 $G^{(0)} \subset A^{(0)} \cap B^{(0)}$ を示す。 $a \in A^{(0)} \cap B^{(0)}$ とする。
 $\text{Ad}(a)\mathcal{O}^{(0)} = \mathcal{O}^{(0)}$, $\text{Ad}(a)\mathcal{F}^{(0)} = \mathcal{F}^{(0)}$ であるから、 $\text{Ad}(a)\mathcal{G}^{(0)} = \mathcal{G}^{(0)}$ を示す。よって、Lemma により、 $a \in \text{Aut}(\mathcal{G})$. $G^{(1)}$ $\mathcal{F} \in \text{Aut}(\mathcal{G})$ を $\mathcal{F}^{-1}a \in G^{(1)}$ なるものとする。このとき、 $\text{Ad}(a)E \equiv E \pmod{\mathcal{G}^{(1)}}$, $\text{Ad}(a)J \equiv \text{Ad}(\mathcal{F})J \pmod{\mathcal{G}^{(1)}}$ を示す。
 また、 $a \in A^{(0)} = A_0 \cdot A^{(1)}$ であるから、 $\text{Ad}(a)E_0 \equiv E_0 \pmod{\mathcal{O}^{(1)}}$.
 これらの事実から、 $\text{Ad}(\mathcal{F})J \equiv J \pmod{\mathcal{G}^{(1)}}$, 従って、 $\text{Ad}(\mathcal{F})J = J$ を示す。よって、 $\mathcal{F} \in G_0$. 故に、 $a \in G^{(0)} = G_0 \cdot G^{(1)}$ が示された。

さて, homogeneous spaces $G/G^{(0)}$, $G/A^{(0)}$, $G/B^{(0)}$ を考えよう. こからは compact, connected である.

Lemma 12 より, $G/G^{(0)}$ から $G/A^{(0)}$, $G/B^{(0)}$ への自然な maps がある. これらをそれぞれ μ, ν である. $G/G^{(0)}$ は $G/A^{(0)}$, $G/B^{(0)}$ 上の fibered manifolds with projections である. $A^{(0)}/G^{(0)}$, $B^{(0)}/G^{(0)}$ は明らかに connected であるから, こからの fibered manifolds のすべての fibre は connected である. 再び, Lemma 12 より, immersion $\mu * \nu: G/G^{(0)} \rightarrow G/A^{(0)} \times G/B^{(0)}$ は imbedding である. o_A, o_B を $G/A^{(0)}$, $G/B^{(0)}$ の origins とする. G は diagonal map $G \rightarrow G \times G$ を通じて, $G/A^{(0)} \times G/B^{(0)}$ 上に作用する. このとき, $(\mu * \nu)(G/G^{(0)})$ は (o_A, o_B) を通るこの作用による orbit であることを注意する. 今後 $\mu * \nu$ により, $G/G^{(0)}$ を $G/A^{(0)} \times G/B^{(0)}$ の submanifold と同一視することにする.

$(G/A^{(0)}, G/B^{(0)}, G/G^{(0)})$ は generalized projective triplet であることを示そう. $\mathfrak{g}$ の subspaces $\alpha^{(0)}$ と $\beta^{(0)}$ は $\mathfrak{g}^{(0)}$ を含み $\text{Ad}(G^{(0)})$ -invariant である. よって, $\alpha^{(0)}$, $\beta^{(0)}$ は $G/G^{(0)}$ 上の G -invariant differential systems を自然に導く. これらをそれぞれ E, F である. $\alpha^{(0)}$, $\beta^{(0)}$ は $\mathfrak{g}$ の subalgebras であり, $\alpha^{(0)} \cap \beta^{(0)} = \mathfrak{g}^{(0)}$ であるから, pair (E, F) は $G/G^{(0)}$ 上の pseudo-product structure を与える. pseudo-product manifold $(G/G^{(0)}, E, F)$ を P_0 である. 明らかに, $E = \mu_*^{-1}(0)$, $F = \nu_*^{-1}(0)$ であるから, P_0 は relation $G/G^{(0)} \subset G/A^{(0)} \times G/B^{(0)}$ に付随する pseudo-product

manifold T がある。さて, $\mathcal{G}$ の subspace $\mathcal{G}^{(-)}$ は勿
論 $\text{Ad}(\mathcal{G}^{(0)})$ -invariant T あり, $\mathcal{G}^{(-)} = \mathcal{O}^{(-)} + \mathcal{F}^{(-)}$.
よって system $D(=E+F)$ は $\mathcal{G}^{(-)}$ から自然に導
かれる $\mathcal{G}/\mathcal{G}^{(0)}$ 上の $\mathcal{G}$ -invariant differential
system T がある。 L or F は non-degenerate T
があるから, D が non-degenerate T があることが
わかる。よって relation $\mathcal{G}/\mathcal{G}^{(0)}$ は non-degenerate
relation になる。

さて, relation $\mathcal{G}/\mathcal{G}^{(0)}$ に付随する immersions
 $z: \mathcal{G}/\mathcal{G}^{(0)} \rightarrow J(\mathcal{G}/A^{(0)}, r)$ と $k: \mathcal{G}/\mathcal{G}^{(0)} \rightarrow J(\mathcal{G}/B^{(0)}, s)$
を考へよう。ここで, $r = \text{rank } E$, $s = \text{rank } F$ T がある。
 z が injective T があることを示そう。まず, $z(z)$
 $= \mu_*(D(z)) = \mu_*(E(z))$, $z \in \mathcal{G}/\mathcal{G}^{(0)}$ T があることを思い出す。
故に, z が injective $\iff "a \in A^{(0)}, \text{Ad}(a)\mathcal{F}^{(0)} = \mathcal{F}^{(0)}"$
 $\implies a \in \mathcal{G}^{(0)}$ がわかる。ところが, Lemma の証明
から, この条件が満たされることかわかる。よって
 z が injective T があることが示された。同様に, k
が injective T があることも示される。

以上より, $(\mathcal{G}/A^{(0)}, \mathcal{G}/B^{(0)}, \mathcal{G}/\mathcal{G}^{(0)})$ が compact,
connected generalized projective triplet T あり
ることがわかった。これは $(\mathcal{G}, e, f)$ に付随すると
よばれる。

① projective Lie algebra triplet.
PLAT

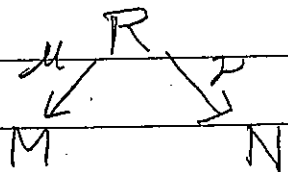
② projective Lie group triplet
PLGT

③ generalized PLAT

④ generalized PLGT

(M, N, R) compact, connected
generalized pseudo-projective
triplet

$$D \text{ is a } \mathbb{R} \text{ } D^{\perp} = J \circ J \circ$$



$p_0 \in M$, $S_0, S_1, \dots$ subsets of M
inductively defined as follows:

$$S_0 = \{p_0\}$$

$$S_{k+1} = \mu(\nu(\mu^{-1}(S_k)))$$

$$S_0 \subsetneq S_1 \subsetneq \dots \subsetneq S_l = M$$

$$l = \max_{(p_0)} (j(p_0, q))$$

$$S_k = \{q \in M \mid j(p_0, q) = k\}$$

$\dim R < \dim J(M, R)$ の場合には

$\{S_k\}$ は M の stratification を与える

よって

$\text{Aut}(X)_{p_0}$ the isotropy group of $\text{Aut}(X)$ at p_0

とすると,

S_k は $\text{Aut}(X)_{p_0}$ 不変

$$\text{Gr}(V, \mathbb{R}) \quad \dim V = n.$$

$$\rho(X_1, X_2) = \dim(X_1 / X_1 \cap X_2)$$

$$X_0 \in \text{Gr}(V, \mathbb{R})$$

$$\rho(X_0, X) = 1 \iff \dim(X_0 \cap X) = \mathbb{R} - 1.$$

$\rho(X_0, X) = 1$ のとき,

$$\underbrace{X_0 \cap X}_{\mathbb{R}-1} \subset \underbrace{X}_{\mathbb{R}} \subset \underbrace{X_0 + X}_{\mathbb{R}+1}$$

$$\text{Flag}(V, \mathbb{R}-1, \mathbb{R}+1) \dots$$

$$= \{(Y, Z) \mid Y \in \text{Gr}(V, \mathbb{R}-1), Z \in \text{Gr}(V, \mathbb{R}+1), Y \subset Z\}$$

$$M = \text{Gr}(V, \mathbb{R}), \quad N = \text{Flag}(V, \mathbb{R}-1, \mathbb{R}+1)$$

$$R = \{(X, (Y, Z)) \in M \times N \mid Y \subset X \subset Z\}$$

(M, N, R) generalized pseudo-projective triplet.

$$X, X^*, R \dots$$

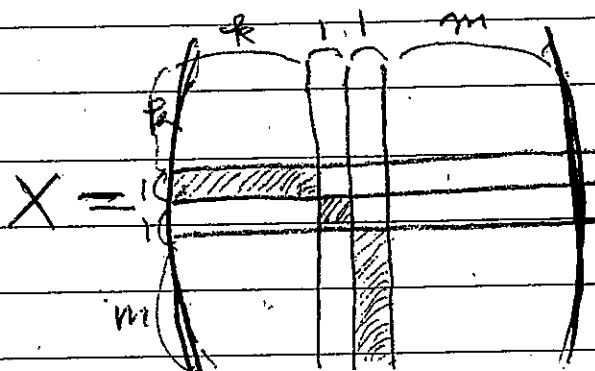
X の solution は 1-dimensional

X の solution x'' Delgasen sphere

$$G = GL(n, \mathbb{R}) / \text{centre}$$

$$X \in \mathfrak{g} = \mathfrak{sl}(m, \mathbb{R})$$

$$m = n - k - 2$$



これより, pseudo-product
SGLA, $\mathfrak{g}$, が $\mathbb{R}^2$ 上の

$$\mathfrak{g} = \underbrace{\mathfrak{e}}_1 + \underbrace{\mathfrak{f}}_{2k}$$

(M, N, R) $\mathfrak{g}$ に付随する $(G/A^{(0)}, G/B^{(0)}, G/G^{(0)})$
として表現される。

(**) (Let p and q be two points on a maximal geodesic α of M . The notations being as in 3.4, we set

$$\Lambda(\alpha, p, q) = K(p, \alpha) \cap K(q, \alpha) = T_\alpha(\mathcal{D}(\mu(p))) \cap T_\alpha(\mathcal{D}(\mu(q))).$$

Then we remark the following : (i) q is a conjugate point of p along α , if and only if $\Lambda(\alpha, p, q) \neq \{0\}$,
 (ii) $\dim \Lambda(\alpha, p, q)$.

この2つの内容について、
 いくつか質問があるか？

§III.3. The spaces of harmonic forms, $H^{p,2}(\mathcal{G})$ associated with projective graded Lie algebras

1. The operators ∂_i and ∂_i^* , $i=0,1,2$, associated with simple graded Lie algebras of the second kind. In this and the subsequent two paragraphs let $\mathcal{G} = \{\mathcal{G}, (\mathcal{G}_p)\}$ be a simple graded Lie algebra of the second kind.

Put $\dim \mathcal{G}_2 = n_2$ and $\dim \mathcal{G}_1 = n_1$. Let $(d_1, \dots, d_{n_2}) = (d_\alpha)$ be a basis of $\mathcal{G}_2$, and (d^α) the basis of $\mathcal{G}_2$ dual to (d_α) . Let $(e_1, \dots, e_{n_1}) = (e_i)$ be a basis of $\mathcal{G}_1$, and (e^i) the basis of $\mathcal{G}_1$ dual to (e_i) . Let S_2 and S_1 be the projections of $m = \mathcal{G}_2 + \mathcal{G}_1$ onto $\mathcal{G}_2$ and $\mathcal{G}_1$, respectively.

Let us recall the operators $\partial_i: C^8 \rightarrow C^{8+1}$, $i \geq 0$, and their adjoint operators $\partial_i^*: C^{8+1} \rightarrow C^8$, which are defined in the proof of Proposition —. Clearly we have $\partial_i = \partial_i^* = 0$ for $i \geq 2$.

The operator ∂_2 is defined by

$$(\partial_2 c)(x_1 \wedge \dots \wedge x_{g+1}) = \sum_{\alpha} (-1)^{g+1} [S_2(x_\alpha), c(x_1 \wedge \dots \wedge \hat{x}_\alpha \wedge \dots \wedge x_{g+1})],$$

the operator ∂_1 by

$$(\partial_1 c)(x_1 \wedge \dots \wedge x_{g+1}) = \sum_{\alpha} (-1)^{g+1} [S_1(x_\alpha), c(x_1 \wedge \dots \wedge \hat{x}_\alpha \wedge \dots \wedge x_{g+1})],$$

and the operator ∂_0 by

$$(\partial_0 c)(x_1 \wedge \dots \wedge x_{g+1}) = \sum_{\alpha < \beta} (-1)^{g+1} c([S_1(x_\alpha), S_1(x_\beta)]) \wedge x_1 \wedge \dots \wedge \hat{x}_\alpha \wedge \dots \wedge \hat{x}_\beta \wedge \dots \wedge x_{g+1}.$$

where $c \in C^8$ and $x_k \in m$. In the last formula, note that $[\zeta_1(x_k), \zeta_1(x_q)] = [\pi_k, \pi_q]$.

The operator ∂_2^* is defined by

$$(\partial_2^* c)(x_1 \wedge \cdots \wedge x_g) = \sum_k [d^k, c(d_k \wedge x_1 \wedge \cdots \wedge x_g)],$$

the operator ∂_1^* by

$$(\partial_1^* c)(x_1 \wedge \cdots \wedge x_g) = \sum_i [e_i^*, c(e_i \wedge x_1 \wedge \cdots \wedge x_g)],$$

and the operator ∂_0^* by

$$(\partial_0^* c)(x_1 \wedge \cdots \wedge x_g) = \frac{1}{2} \sum_{k,l} (-1)^{k+l} c([e_i^*, \zeta_2(x_k)] \wedge e_l \wedge x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge \hat{x}_l \wedge \cdots \wedge x_g),$$

where $c \in C^{g+1}$ and $x_k \in m$. In the last formula note that $[d^k, x_k] = 0$, $[e_i^*, x_k] = [e_i, \zeta_2(x_k)]$.

Clearly the operator $\partial : C^g \rightarrow C^{g+1}$ may be expressed as follows:

$$\partial = \partial_2 + \partial_1 + \partial_0,$$

and the adjoint operator $\partial^* : C^{g+1} \rightarrow C^g$ of ∂ as follows

$$\partial^* = \partial_2^* + \partial_1^* + \partial_0^*.$$

Now the space $\Lambda^g m^*$ is decomposed as follows:

$$\Lambda^g m^* = \sum_{r+s=g} \Lambda^{r,s},$$

where

$$A^{r,s} = \overbrace{Z_2^* \wedge \cdots \wedge Z_2^*}^r \wedge \overbrace{Z_1^* \wedge \cdots \wedge Z_1^*}^s.$$

Therefore putting

$$D_j^{r,s} = Z_j \otimes A^{r,s},$$

we see that the space C^g is decomposed as follows:

$$C^g = \sum_j \sum_{r+s=g} D_j^{r,s}.$$

Clearly we have

$$Z_2 D_j^{r,s} \subset D_{j-2}^{r+1,s}, \quad Z_2^* D_{j-2}^{r+1,s} \subset D_j^{r,s},$$

$$Z_1 D_j^{r,s} \subset D_{j-1}^{r,s+1}, \quad Z_1^* D_{j-1}^{r,s+1} \subset D_j^{r,s},$$

$$Z_0 D_j^{r,s} \subset D_j^{r-1,s+2}, \quad Z_0^* D_j^{r-1,s+2} \subset D_j^{r,s}.$$

The space C^g is also decomposed as follows:
 $C^g = \sum_p C^{p,g}$, and each $C^{p,g}$ as follows $C^{p,g} = \sum_j C_j^{p,g}$.
 For any pair (p,g) let (r,s) be the pair such that $r+s=g$, $-2r-s=j-p-g+1$. Then we have

$$C_j^{p,g} = D_j^{r,s},$$

from which follows that

$$\partial_i C_j^{p,q} \subset C_{j-i}^{p-1, q+1}, \quad \partial_i^* C_{j-i}^{p-1, q+1} \subset C_j^{p,q},$$

where $i = 0, 1, 2$. Hereafter we shall frequently deal with the spaces $D_j^{p,q}$ instead of $C_j^{p,q}$.

2. The spaces of harmonic forms, $H^p(\mathcal{G})$, associated with simple graded Lie algebras of the second kind. We first explain some preliminary facts on the graded Lie algebra $\mathcal{G}$.

Lemma (cf. [1], Lemmas —, —). Let ε be 1 or -1.

$$(1) [\mathcal{G}_\varepsilon, \mathcal{G}_\varepsilon] = \mathcal{G}_{2\varepsilon}, [\mathcal{G}_1, \mathcal{G}_1] = \mathcal{G}_0, \text{ and } [\mathcal{G}_{-2\varepsilon}, \mathcal{G}_\varepsilon] = \mathcal{G}_{-\varepsilon}.$$

(2) If $X \in \mathcal{G}_\varepsilon$ and $[X, \mathcal{G}_\varepsilon] = 0$, then $X = 0$. If $X \in \mathcal{G}_{2\varepsilon}$ and $[X, \mathcal{G}_\varepsilon] = 0$, then $X = 0$. If $X \in \mathcal{G}_\varepsilon$ and $[X, \mathcal{G}_{-2\varepsilon}] = 0$, then $X = 0$. If $X \in \mathcal{G}_{2\varepsilon}$ and $[X, \mathcal{G}_{-2\varepsilon}] = 0$, then $X = 0$.

We define elements E_2 and E_1 of $\mathcal{G}_0$ respectively by

$$E_2 = \sum_{\alpha} [d^\alpha, d_\alpha], \text{ and } E_1 = \sum_i [e^i, e_i].$$

As is easily observed, both E_2 and E_1 do not depend on the choices of the bases, and are in the centre of $\mathcal{G}_0$.

Let $x_j \in \mathcal{G}_j = D_j^{0,0}$. Then we have

$$2_2^* 2_2 x_p = \sum_{\alpha} [d^\alpha, [d_\alpha, x_p]], \text{ and } 2_1^* 2_1 x_p = \sum_i [e^i, [e_i, x_p]],$$

and we can easily prove the following

Lemma $\partial_2^* \partial_2 x_1 = [E_2, x_1]$, $\partial_2^* \partial_2 x_2 = [E_2, x_2]$,

$\partial_1^* \partial_1 x_1 = -[E_2, x_1]$, $\partial_1^* \partial_1 x_1 = [E_2 + E_1, x_1]$, and

$\partial_1^* \partial_1 x_2 = [E_1, x_2]$.

Lemma. The operator $u \rightarrow -[E_1, u]$ of $\mathcal{I}_2$ to itself is symmetric and positive definite with respect to the inner product $(,)$.

Proof. We may assume that the basis (e_i) of $\mathcal{I}_1$ is orthonormal, i.e., $(e_i, e_j) = \delta_{ij}$. Then we have $\sigma e_i = -e^i$, whence $\sigma E_1 = -E_1$. It follows that

$$([E_1, u], v) = (u, [E_1, v]), \quad u, v \in \mathcal{I}_2.$$

Furthermore we have

$$-([E_1, u], u) = \sum_i ([e^i, u], [e^i, u]) \geq 0,$$

and if the equality holds, we obtain $[e^i, u] = 0$, whence $u = 0$ (Lemma —).

We shall now study the spaces $H^{p,2}(\mathcal{G})$. The space $C^{p,2}$ is decomposed as follows:

$$C^{p,2} = C_{p-3}^{p,2} + C_{p-2}^{p,2} + C_{p-1}^{p,2}.$$

In particular we see that $C^{p,2} = 0$ if $p < -1$ or $p > 5$.

We also note that $C_{p-3}^{1,2} = D_{p-3}^{2,0}$, $C_{p-2}^{1,2} = D_{p-2}^{1,1}$ and $C_{p-1}^{1,2} = D_{p-1}^{0,2}$. Now by Proposition — we have

$$H^{p,2}(\mathcal{G}) = H^{p,2}(\mathcal{G}) \cap D_{p-3}^{2,0} + H^{p,2}(\mathcal{G}) \cap D_{p-2}^{1,1} + H^{p,2}(\mathcal{G}) \cap D_{p-1}^{0,2}$$

Lemma $H^{p,2}(\mathcal{G}) \cap D_{p-3}^{2,0} = 0$

Proof. Let $c \in D_{p-3}^{2,0}$. For any $u, v \in \mathcal{G}_2$ we have

$$\begin{aligned} (\partial_0^* \partial_0 c)(u \wedge v) &= \frac{1}{2} \sum_i (\partial_0 c)([e_i^1, u] \wedge e_i \wedge v) - \frac{1}{2} \sum_i (\partial_0 c)([e_i^1, v] \wedge e_i \wedge u) \\ &= -\frac{1}{2} \sum_i c([e_i^1, u], e_i] \wedge v) + \frac{1}{2} \sum_i c([e_i^1, v], e_i] \wedge u) \\ &= -\frac{1}{2} c([E_1, u] \wedge v) + \frac{1}{2} c([E_1, v] \wedge u). \end{aligned}$$

Thus we see from Lemma — that $\partial_0^* \partial_0 : D_{p-3}^{2,0} \rightarrow D_{p-3}^{2,0}$ is a positive definite operator. Since $\partial_0 c = 0$ for $c \in H^{p,2}(\mathcal{G}) \cap D_{p-3}^{2,0}$, it follows that $H^{p,2}(\mathcal{G}) \cap D_{p-3}^{2,0} = 0$.

Therefore we have the following

Proposition — (1) $H^{p,2}(\mathcal{G}) = H^{p,2}(\mathcal{G}) \cap D_{p-2}^{1,1} + H^{p,2}(\mathcal{G}) \cap D_{p-1}^{0,2}$

(2) $H^{p,2}(\mathcal{G}) \cap D_{p-2}^{1,1}$ consists of all $c \in D_{p-2}^{1,1}$ such that

$$\begin{array}{ccccc} D_{p-4}^{3,1} & D_{p-3}^{1,2} & D_{p-2}^{0,3} & D_{p-1}^{1,0} & D_p^{0,1} \\ \partial_2 c = \partial_1 c = \partial_0 c = \partial_1^* c = \partial_2^* c = 0. \end{array}$$

(3) $H^{p,2}(\mathcal{G}) \cap D_{p-1}^{0,2}$ consists of all $c \in D_{p-1}^{0,2}$ such that

$$D_{p-3}^{1,2} \quad D_{p-2}^{0,3} \quad D_{p-1}^{1,0} \quad D_p^{0,1}$$

$$\partial_2 c = \partial_1 c = \partial_0^* c = \partial_1^* c = 0$$

Using this proposition, we shall now prove the following proposition. Define an ideal $\mathcal{I}_0$ of $\mathcal{G}_0$ by

$$\mathcal{I}_0 = \{x \in \mathcal{G}_0 \mid [x, \mathcal{G}_2] = 0\}.$$

Proposition —. (1) $H^{5,2}(\mathcal{G}) = 0$.

(2) If $\mathcal{G}$ is the prolongation of $(\mathcal{M}, \mathcal{G}_0)$, then $H^{4,2}(\mathcal{G}) = 0$.

(3) $H^{3,2}(\mathcal{G}) \subset D_1^{1,1}$, $H^{2,2}(\mathcal{G}) \subset D_0^{1,1}$, and $H^{1,2}(\mathcal{G}) \cap D_0^{0,2} \subset \mathcal{I}_0 \oplus \wedge^2 \mathcal{G}_1^*$.

First of all (1) is clear, because $H^{5,2}(\mathcal{G}) \subset A_2^{2,0}$.

(2) We have $H^{4,2}(\mathcal{G}) \subset D_2^{1,1}$. Let $c \in H^{4,2}(\mathcal{G})$. Then we have $\partial_2 c = 0$, and hence

$$(\partial_2 c)(u \wedge v \wedge x) = [u, c(v \wedge x)] - [v, c(u \wedge x)] = 0,$$

where $u, v \in \mathcal{G}_2$ and $x \in \mathcal{G}_1$. Fix an $x \in \mathcal{G}_1$ and define a linear map $f: \mathcal{G}_2 \rightarrow \mathcal{G}_2$ by $f(u) = c(u \wedge x)$, $u \in \mathcal{G}_2$. Then we have $f \in C^{1,1} = \mathcal{G}_2 \oplus \mathcal{G}_2^* = D_2^{1,0}$ and $\partial f = \partial_2 f = 0$. Since $\mathcal{G}$ is the prolongation of $(\mathcal{M}, \mathcal{G}_0)$,

and since $\mathcal{I}_4 = 0$, we see from Proposition — that the map $\partial: C^{4,1} \rightarrow C^{3,2}$ is injective. Therefore we get $f = 0$ and hence $c = 0$, proving (2).

(3) Let $c \in H^{p,2}(\mathcal{G}) \cap \mathcal{D}_{p-1}^{0,2}$. Then we have $\partial_2 c = 0$ and hence

$$(\partial_2 c)(u \wedge x \wedge y) = [u, c(x \wedge y)] = 0$$

where $u \in \mathcal{I}_2$, and $x, y \in \mathcal{I}_1$. Since $c(x \wedge y) \in \mathcal{I}_{p-1}$, it follows that if $p = 2$ or 3 , then $c(x \wedge y) = 0$ (Lemma —) and if $p = 1$, then $c(x \wedge y) \in \mathcal{I}_0$, proving (3).

3. The case where $\dim \mathcal{G}_2 = 1$. この paragraph 2 は $\dim \mathcal{G}_2 = 1$ と仮定し, $\dim \mathcal{G}_1 = 2(n-1)$ とおく. (EJ 2).
 このような $\mathcal{G}$ は simple graded Lie algebra of contact type of degree n とよんた.) 明 5 頁 12

$$\mathcal{G}_0 = \mathbb{R} \cdot E + \mathcal{F}_0 \quad (\text{direct sum})$$

が成り立つ. 以下 $\mathcal{G}$ は $(\mathcal{M}, \mathcal{G}_0)$ の prolongation と仮定する. 次の propositions を証明しよう

Proposition 1 $n \geq 3$ と仮定する.

$$(1) \quad H^{p,2}(\mathcal{G}) = 0 \quad \text{if } p \neq 0 \text{ or } 1.$$

$$(2) \quad H^{0,2}(\mathcal{G}) \subset \mathcal{D}_{-1}^{0,2}.$$

$$(3), \quad H^{1,2}(\mathcal{G}) \subset \mathcal{F}_0 \oplus \wedge^2 \mathcal{G}_1^*.$$

Proposition. $n=2$ と仮定する

(1) $\dim \mathcal{F}_0 = 1$ であり, $\mathcal{F}_0$ の basis A があって,
 $[A, [A, x]] = x, \forall x \in \mathcal{G}_1$, かつ $\mathcal{F}_2$ は $[A, [A, x]] = -x, \forall x \in \mathcal{G}_{-1}$,
 を満足する.

$$(2) \quad H^{p,2}(\mathcal{G}) = 0 \quad \text{if } p \neq 3.$$

(3) $H^{3,2}(\mathcal{G})$ consists of all $c \in A_1^{1,1}$ satisfying the equation

$$[A, c(u \wedge x)] + c(u \wedge [A, x]) = 0, \quad u \in \mathcal{G}_{-2}, \quad x \in \mathcal{G}_1$$

$$\text{II. 同型写像 } c^+(u \wedge x) - \bar{c}(u \wedge x) + c(u \wedge x^+) - \bar{c}(u \wedge x^+) = 0$$

$\mathcal{I}_0$ の centre の $\bar{\pi}$ $E_2 = [d', d_1]$, $E_1 = \sum_i [e_i', e_i]$ と
考えよう.

Lemma ([1], Lemma). (1) $E_2 = \lambda_2 E$, where
 $\lambda_2 = \frac{1}{2(n+D)}$.

(2) $E_1 = \lambda_1 E$, where $\lambda_1 = \frac{n-1}{2(n+D)}$.

まず " $n \geq 3$ と仮定する

Lemma $H^{p,2}(\mathcal{I}) \subset \mathcal{D}_{p-1}^{0,2}$.

Proof. $c \in H^{p,2}(\mathcal{I}) \cap \mathcal{D}_{p-2}^{1,1}$ とする. このとき $\partial_0 c = 0$
である. $\partial_0^* \partial_0 c$ を計算しよう. $x \in \mathcal{I}_1$ とする.

$$\begin{aligned}
(\partial_0^* \partial_0 c)(d_1 \wedge x) &= \frac{1}{2} \sum_i (\partial_0 c)([e^i, d_1] \wedge e_i \wedge x) \\
&= -\frac{1}{2} \sum_i c([e^i, d_1], e_i \wedge x) + \frac{1}{2} \sum_i c([e^i, d_1] \wedge x, e_i) \\
&= \frac{1}{2} \sum_i c([e_i, x] \wedge [e^i, d_1]).
\end{aligned}$$

$$\begin{aligned}
\sum [e^i, d_1], e_i] &= [E_1, d_1] \quad \text{7" 3}, \quad [e^i, d_1] = \sum_j \langle e^i, [d_1, e^j] \rangle e_j \\
[e_i, x] &= \langle e_i, [x, d_1] \rangle d_1 \quad \text{7" 3}
\end{aligned}$$

$$\begin{aligned}
(\partial_0^* \partial_0 c)(d_1 \wedge x) &= -\frac{1}{2} c([E_1, d_1] \wedge x) - \sum_i c([e_i, x] \wedge [e^i, d_1]) \\
&= -\frac{1}{2} c([E_1, d_1] \wedge x) - c(d_1 \wedge [x, d_1], d_1) \\
&= -\frac{1}{2} c([E_1, d_1] \wedge x) - c(d_1 \wedge [x, E_2])
\end{aligned}$$

7 3. $\lambda > 2$ Lemma 7 3 2.

$$\partial_0^* \partial_0 c = (\lambda_1 - \lambda_2) c = 0$$

7 3. $n \geq 3$ 7 3 2, $\lambda_1 - \lambda_2 > 0$, $\lambda > 2$ $c = 0$

7 3. $\lambda > 2$ Lemma が示した

Lemma. $H^{1,2}(\mathcal{G}) = 0$.

Proof. $H^{1,2}(\mathcal{G}) \subset D_{-2}^{0,2}$ 7 3. $c \in H^{1,2}(\mathcal{G})$ とすれば, $\partial_1^* c = 0$. $\lambda > 2$, $\forall x \in \mathcal{G}_1, 1 \leq i \leq 2$.

$$(\partial_1^* c)(x) = \sum_i [e^i, c(e_i \wedge x)] = \sum_i \langle c(e_i \wedge x), d' \rangle [e^i, d_1] = 0$$

7 3. $([e^i, d_1])$ は linearly independent 7 3 3
 7 3 (Lemma), $\langle c(e_i \wedge x), d' \rangle = 0$, i.e., $c = 0$ 7 3

Lemma と Prop. 7 3 12 Prop. 7 3

次に Prop. を証明しよう. $\mathcal{G}_0 \subset \mathcal{GL}(\mathcal{G}_{-1})$ と
 みたせるから, $\dim \mathcal{G}_0 \leq 4$. $\dim \mathcal{G}_0 = 4$ とすれば
 は" $(\mathcal{M}, \mathcal{G}_0)$ の prolongation は contact graded algebra
 of degree 2 であり, 無次元である. ($\dim \mathcal{G}_0 = 4$
 のとき, $\mathcal{G}$ は projective contact graded Lie
 algebra of degree 2 である.) よって $\dim \mathcal{G}_0 \leq 3$
 である. $\mathcal{G}_0$ は $\mathfrak{sl}(2)$ から, $1 \leq \dim \mathcal{G}_0 \leq 3$. よって
 $7 \leq \dim \mathcal{G} \leq 9$. $\mathcal{G}$ は simple $\mathfrak{sl}(2)$ から,
 $\dim \mathcal{G} = 8$, よって $\dim \mathcal{G}_0 = 2$, よって $\dim \mathcal{F}_0 = 1$
 である. $A \in \mathcal{F}_0$ の basis とする. $\sigma \mathcal{F}_0 = \mathcal{F}_0$
 $\mathfrak{sl}(2)$ から, $\sigma A = \pm A$ である. ($\langle \sigma A, \sigma A \rangle = \langle A, A \rangle \neq 0$)
 $\sigma A = -A$ とする.

$$([A, x], y) = (x, [A, y]),$$

$$[A, x], y + [x, [A, y]] = 0, \quad x, y \in \mathcal{G}_1,$$

$\mathfrak{sl}(2)$ から, $\mathcal{F}_0$ の basis A が存在して, $\mathcal{G}_1$ の
 endomorphism $x \rightarrow [A, x]$ の eigenvalue
 は 1 or -1 となる. よって $[A, [A, x]] = x, x \in \mathcal{G}_{-1}$.
 したがって $\sigma A = A$ とする. このとき

$$([A, x], y) = -(x, [A, y]), \quad x, y \in \mathcal{G}_1,$$

よって, $\mathcal{F}_0$ の basis A が存在して,
 $[A, [A, x]] = -x, x \in \mathcal{G}_1$ となる.

Lemma. $H^{p,2}(\mathcal{G}) \subset \mathcal{D}_{p-2}^{1,1}$.

Proof. $c \in H^{p,2}(\mathcal{G}) \cap \mathcal{D}_{p-1}^{0,2}$ とする. $\partial_0^* c = 0$ である.

から

$$(\partial_0^* c)(d_1) = \frac{1}{2} \sum_i c([e^i, d_1] \wedge e_i)$$

$$= \frac{1}{2} c([e^1, d_1] \wedge e_1) + \frac{1}{2} c([e^2, d_1] \wedge e_2) = 0$$

$\lambda = \langle [e^1, d_1], e^2 \rangle = \langle [e^2, e^1], d_1 \rangle$ とおけば, $\lambda \neq 0$ かつ

$[e^1, d_1] = \lambda e_2$, $[e^2, d_1] = -\lambda e_1$, である. よって

$-\lambda c(e_1 \wedge e_2) = 0$. よって $c = 0$ である.

Lemma. $H^{p,2}(\mathcal{G}) = 0$, $p = 0, 1, 2$.

Proof. $c \in H^{p,2}(\mathcal{G})$ とする. $\partial_2^* c = 0$ である.

$$(\partial_2^* c)(x) = [d^1, c(d_1 \wedge x)] = 0, \quad x \in \mathcal{G}_1$$

である. よって $p = 0, 1$ のときは, $c = 0$ である.

$p = 2$ と仮定する. このとき $\partial_1 c = 0$ である.

$$(\partial_1 c)(d_1 \wedge x \wedge y) = -[x, c(d_1 \wedge y)] + [y, c(d_1 \wedge x)] = 0$$

である. よって $x, y \in \mathcal{G}_1$, $c(d_1 \wedge y) \in \mathcal{L}_0$ である.

容易に $c = 0$ である (cf. [], Proposition —)

Lemmas と Props から, $H^{p,2}(\mathcal{G}) = 0$ if $p \neq 3$

である. $H^{3,2}(\mathcal{G}) = \{c \in \mathcal{D}_1^{1,1} \mid \partial_1 c = \partial_1^* c = 0\}$ である.

$c \in \mathcal{D}_1^{1,1}$ とする. $\partial_1 c = 0$ なるための必要十分

条件は

$$\begin{cases} \langle (\partial_1 c)(d_1 \wedge x \wedge y), E \rangle = \langle x, c(d_1 \wedge y) \rangle - \langle y, c(d_1 \wedge x) \rangle = 0, \\ \langle (\partial_1 c)(d_1 \wedge x \wedge y), A \rangle = -\langle [A, x], c(d_1 \wedge y) \rangle + \langle [A, y], c(d_1 \wedge x) \rangle = 0 \end{cases}$$

τ'' あり. $\tau > \tau''$, $x, y \in \mathcal{F}_1$. 容易にこの条件は

$$[A, c(d_1 \wedge x)] + c(d_1 \wedge [A, x]) = 0, \quad x \in \mathcal{F}_1$$

に equivalent τ'' あり $\Rightarrow$ がわかる. 次に

$\partial_1 c = 0$ から $\partial_1^* c = 0$ が導かれることを示す. 実際

$$\langle d_1, (\partial_1^* c)(d_1) \rangle = \langle [d_1, c], c(e_1 \wedge d_1) \rangle + \langle [d_1, c^2], c(e_2 \wedge d_1) \rangle$$

$$= -\lambda \langle e_2, c(e_1 \wedge d_1) \rangle + \lambda \langle e_1, c(e_2 \wedge d_1) \rangle$$

$$= \lambda \langle (\partial_1 c)(d_1 \wedge e_1 \wedge e_2), E \rangle = 0.$$

よって主張は正しい. (λ は Lemma のもの.)

4. The spaces of harmonic forms, $H^{p,2}(\mathcal{G})$, associated with projective graded Lie algebras of type $(n,1)$, $n \geq 3$. Let $\mathcal{L}$ be a projective EGLA of type $(n,1)$, and $\mathcal{G}$ its prolongation. We first explain some preliminary facts on the graded Lie algebra $\mathcal{G}$.

Lemma Let ε be 1 or -1. Let δ be + or -, and let $-\delta$ be - or + according as δ is + or -.

$$(1) [\mathcal{G}_0, \mathcal{G}_\varepsilon^\delta] = \mathcal{G}_\varepsilon^\delta, [\mathcal{G}_\varepsilon^\delta, \mathcal{G}_{-2\varepsilon}^\delta] = \mathcal{G}_{-\varepsilon}^\delta,$$

$$[\mathcal{G}_\varepsilon^\delta, \mathcal{G}_\varepsilon^\delta] = [\mathcal{G}_1^\delta, \mathcal{G}_{-1}^\delta] = 0.$$

(2) If $X \in \mathcal{G}_\varepsilon^\delta$ and $[X, \mathcal{G}_{-2\varepsilon}^\delta] = 0$, then $X = 0$. If $X \in \mathcal{G}_{-2\varepsilon}^\delta$ and $[X, \mathcal{G}_\varepsilon^\delta] = 0$, then $X = 0$. If $X \in \mathcal{G}_\varepsilon^\delta$ and $[X, \mathcal{G}_\varepsilon^{-\delta}] = 0$, then $X = 0$. If $X \in \mathcal{G}_\varepsilon^\delta$ and $[X, \mathcal{G}_{-\varepsilon}^{-\delta}] = 0$, then $X = 0$.

(3) The spaces $\mathcal{G}_1^\delta$ and $\mathcal{G}_{-1}^\delta$ are mutually orthogonal with respect to the Killing form $\langle \cdot, \cdot \rangle$, and hence the bilinear function $\mathcal{G}_1^\delta \times \mathcal{G}_{-1}^{-\delta} \ni (X, Y) \rightarrow \langle X, Y \rangle \in \mathbb{R}$ is non-degenerate.

Hereafter we assume that $r=1$ and $n \geq 3$.

We know that the Lie algebra $\mathcal{G}_0$ is naturally isomorphic with the Lie algebra $\mathfrak{gl}(\mathcal{G}_1^+) + \mathfrak{gl}(\mathcal{G}_1^-)$.

Hence the centre $\mathfrak{L}_0$ of $\mathfrak{g}$ is 2-dimensional, and is spanned by E and J . We denote by $\mathfrak{l}_0$ the ideal of $\mathfrak{g}$ corresponding to the simple Lie algebra $sl(\mathfrak{g}_1)$. Then we have

$$\mathfrak{g}_0 = \mathfrak{L}_0 + \mathfrak{l}_0 \quad (\text{direct sum}),$$

$$[\mathfrak{l}_0, \mathfrak{g}_1^+] = [\mathfrak{l}_0, \mathfrak{g}_1^-] = 0.$$

We also note that $\mathfrak{l}_0$ coincides with the ideal of $\mathfrak{g}_0$ orthogonal to $\mathfrak{L}_0$ with respect to the Killing form $\langle \cdot, \cdot \rangle$. Hence

$$\mathfrak{l}_0 = \{X \in \mathfrak{g}_0 \mid \langle X, E \rangle = \langle X, J \rangle = 0\}.$$

Let e_0 be a basis of $\mathfrak{g}_1^+$, and e^0 the basis of $\mathfrak{g}_1^-$ dual to e_0 (with respect to the Killing form $\langle \cdot, \cdot \rangle$). Let $(e_1, \dots, e_{n-1}) = (e_i)$ be a basis of $\mathfrak{g}_1^-$, and (e^i) the basis of $\mathfrak{g}_1^+$ dual to (e_i) . Putting

$$d_i = [e_i, e_0],$$

we see that (d_i) is a basis of $\mathfrak{g}_2$. Let (d^i) be the basis of $\mathfrak{g}_2$ dual to (d_i) .

In the following the indices i, j, k will range over the integers $1, \dots, n-1$.

Let us consider the elements E_2 and E_1 in the centre of $\mathfrak{g}$ (see § —). We now

define elements E_1^+ and E_1^- of $\mathcal{J}_0$ respectively by

$$E_1^+ = [e^0, e_0], \text{ and } E_1^- = \sum_i [e^i, e_i].$$

Clearly we have

$$E_1 = E_1^+ + E_1^-.$$

Lemma (1) $E_2 = \lambda_2 E + \mu_2 J$, where $\lambda_2 = \frac{n}{4(n+1)}$ and $\mu_2 = \frac{-(n-2)}{4(n+1)}$.

(2) $E_1^+ = \lambda_1^+ E + \mu_1^+ J$, where $\lambda_1^+ = \frac{1}{4(n+1)}$ and $\mu_1^+ = \frac{-3}{4(n+1)}$.

(3) $E_1^- = \lambda_1^- E + \mu_1^- J$, where $\lambda_1^- = \frac{1}{4(n+1)}$ and $\mu_1^- = \frac{2n-1}{4(n+1)}$.

(4) $E_1 = \lambda_1 E + \mu_1 J$, where $\lambda_1 = \frac{1}{2(n+1)}$ and $\mu_1 = \frac{n-2}{2(n+1)}$.

Lemma $[e_0, d^i] = e^i$, $[d^i, e_j] = \delta_j^i e^0$,

$$[e^0, d_i] = (\lambda_1^+ + \mu_1^+) e_i, [e^i, e^0] = (\lambda_1^+ + \mu_1^+) d^i, \text{ and}$$

$$[d_i, e^j] = (\lambda_1^+ + \mu_1^+) \delta_i^j e_0.$$

Finally we mention that all the results in this paragraph can be verified directly or by the use of the matricial representation (or briefly the M.R.) of $\mathcal{G}$. We shall also use some facts which can be easily obtained from the M.R. of $\mathcal{G}$.

We shall now study the spaces $H^{p,2}(\mathcal{G})$.
(We are assuming that $r=1$, $n \geq 3$).

$\mathcal{G}_1 = \mathcal{G}_1^+ + \mathcal{G}_1^-$ であるから $D_p^{1,1}, D_p^{0,2}$ は 2 次元

$$D_p^{1,1} = \mathcal{G}_p \otimes \mathcal{G}_2^* \wedge (\mathcal{G}_1^+)^* + \mathcal{G}_p \otimes \mathcal{G}_2^* \wedge (\mathcal{G}_1^-)^*,$$

$$D_p^{0,2} = \mathcal{G}_p \otimes (\mathcal{G}_1^+)^* \wedge (\mathcal{G}_1^-)^* + \mathcal{G}_p \otimes \wedge^2 (\mathcal{G}_1^-)^*$$

と分解する。特に $p=1$ 或 -1 のときは,
 $\mathcal{G}_p = \mathcal{G}_p^+ + \mathcal{G}_p^-$ であるから, 2π の 4 つの spaces
は 2 次元 2 つの spaces に分解する
さて Prop. 12 より, $H^{p,2}(\mathcal{G})$ は

$$H^{p,2}(\mathcal{G}) = H^{p,2}(\mathcal{G}) \cap D_{p-2}^{1,1} + H^{p,2}(\mathcal{G}) \cap D_{p-1}^{0,2}$$

と分解される。任意の p に対して, $D_{p-2}^{1,1}$ または
 $D_{p-1}^{0,2}$ の subspace $V^{p,2}(n,1)$ を次のように定義する

$$V^{p,2}(n,1) = 0 \quad \text{if } p \leq -2 \text{ or } p \geq 3.$$

$V^{1,2}(n,1)$ consists of all $c \in \mathcal{G}_2 \otimes \wedge^2 (\mathcal{G}_1^-)^*$
such that

$$\sum_i \langle d_i^*, c(e_i \wedge x) \rangle = 0, \quad x \in \mathcal{G}_1^-.$$

or in other words, the trace of the
endomorphism $u \rightarrow c([u, e^0] \wedge x)$ of $\mathcal{G}_2$ vanishes
for any $x \in \mathcal{G}_1^-$. (Note that $[e^0, d_i^*] = (\lambda_i^* + \mu_i^*)e_i$ and

$\lambda_1^+ + \mu_1^+ \neq 0$ (Lemma —).

$$V^{0,2}(n,1) = 0 \quad \text{if } n \geq 4,$$

$$= \mathcal{G}_1^+ \otimes \wedge^2(\mathcal{G}_1^-)^* \quad \text{if } n=3.$$

$V^{1,2}(n,1)$ consists of all $c \in \mathcal{G}_1^- \otimes \mathcal{G}_2^* \wedge (\mathcal{G}_1^+)^*$ such that

$$\sum_i \langle e^i, c(d_i \wedge e_0) \rangle = 0,$$

or in other words, the trace of the endomorphism $x \rightarrow c([x, e_0] \wedge e_0)$ of $\mathcal{G}_1^-$ vanishes

$V^{2,2}(n,1)$ consists of all $c \in \mathcal{G}_0 \otimes \mathcal{G}_2^* \wedge (\mathcal{G}_1^-)^*$ such that $[c([x, e_0] \wedge y), z]$ is symmetric with respect to the variables $x, y, z \in \mathcal{G}_1^-$.

These being prepared, we shall prove the following

Proposition $H^{p,2}(\mathcal{G}) = V^{p,2}(n,1)$ for all p .

Remark. $\mathcal{G}$ is projective graded Lie algebra of type $(n, n-1)$ とする. $n=2$ のとき 12 は, Proposition 12 より 2 spaces $H^{p,2}(\mathcal{G})$ は 既に計算されてゐる. $n \geq 3$ のときも, Prop. — を使って容易に計算できる. 詳細は読者にまかせる.

Prop. の証明に入る前に記号についての注意を行う. $p=1.02-1$ とする. このとき spaces D_p^{ns} は

$$D_p^{ns} = \mathcal{G}_p^+ \otimes \Lambda^{ns} + \mathcal{G}_p^- \otimes \Lambda^{ns}$$

と分解し, この分解に応じて, D_p^{ns} の各元 c は

$$c = c^+ + c^-$$

と分解する

以下我々は $H^{p,2}(\mathcal{G}) \subset V^{p,2}(m,1)$ を示す. 逆向きの inclusion は容易に示される.

The space $H^{-1,2}(\mathcal{G})$. Proposition — 12.572 $H^{-1,2}(\mathcal{G})$ は $D_{-2}^{0,2}$ の元 $c \in \mathcal{G}$,

$$\partial_0^* c = \partial_1^* c = 0$$

を満足するものの全体からなる. $c \in H^{-1,2}(\mathcal{G})$ とする

$\therefore c \in H^{1,2}(\mathcal{G})$ とする. このとき $\forall x \in \mathcal{G}_1^-$ に対して,

$$(\partial_1^* c)(x) = [e^0, c(e_0 \wedge x)] + \sum_i [e^i, c(e_i \wedge x)] = 0$$

$[e^0, c(e_0 \wedge x)] \in \mathcal{G}_1^-$, $\sum_i [e^i, c(e_i \wedge x)] \in \mathcal{G}_1^+$ であるから,

$$[e^0, c(e_0 \wedge x)] = - \sum_i [e^i, c(e_i \wedge x)] = 0$$

とある. これより $c(e_0 \wedge x) = 0$ とある (Lemma —)

よって $c \in \mathcal{G}_2 \otimes \Lambda^2(\mathcal{G}_1^-)^*$ である. 4.5.5.12 (Lemma —)

従って

$$\sum_i [e^i, c(e_i \wedge x)] = \sum_{i,j} \langle c(e_i \wedge x), d^j \rangle [e^i, d_j]$$

$$= -(\lambda_1^* + \mu_1^*) \sum_i \langle c(e_i \wedge x), d^i \rangle e_0 = 0$$

したがって $\lambda_1^* + \mu_1^* > 0$ ならば $\sum_i \langle c(e_i \wedge x), d^i \rangle = 0$ となる

したがって 故に $c \in V^{-1,2}(n,1)$. 以上より $H^{-1,2}(\mathcal{F}) \subset V^{-1,2}(n,1)$ が示された。

The space $H^{0,2}(\mathcal{F})$. Proposition — 以上より $H^{0,2}(\mathcal{F}) \cap D_{-2}^{1,1}$ は $D_{-2}^{1,1}$ の元である

$$\partial_0 c = \partial_1^* c = \partial_2^* c = 0$$

を満足するものの全体からなり、 $H^{0,2}(\mathcal{F}) \cap D_{-1}^{0,2}$ は $D_{-1}^{0,2}$ の元である

$$\partial_1 c = \partial_0^* c = \partial_2^* c = 0$$

を満足するものの全体からなり

したがって $H^{0,2}(\mathcal{F}) \subset D_{-1}^{0,2}$ を示そう。 $c \in H^{0,2}(\mathcal{F}) \cap D_{-2}^{1,1}$ とする。このとき $\forall x \in \mathcal{F}_1$ に $\bar{x} \neq 1$ なる

$$(\partial_2^* c)(x) = \sum_i [d^i, c(e_i \wedge x)] = 0$$

したがって $\forall x \in \mathcal{F}_1$, i.e., $c = 0$ となる (Consider the M.R. of $\mathcal{F}$).

$c \in H^{0,2}(\mathcal{F})$ とする。 $\forall x \in \mathcal{F}_1$ に $\bar{x} \neq 1$ なる

$$(\partial_1^* c)(x) = [e^0, c^+(e_0 \wedge x)] + \sum_i [e^i, c^-(e_i \wedge x)] = 0,$$

これより $c^+(e_0 \wedge x) = c^-(e_i \wedge x) = 0$ である。 (Consider the M.R of $\mathcal{G}_1$) これは,

$c \in \mathcal{G}_1^+ \otimes \wedge^2(\mathcal{G}_1^-)^*$ 意味する。 $\forall x, y \in \mathcal{G}_1^-$ には
 $12, \gamma(x \wedge y) = \langle c(x \wedge y), e^0 \rangle$ とおく。 $\partial_1 c = 0$ であるから

$$(\partial_1 c)(x \wedge y \wedge z) = \gamma(y \wedge z)[x, e_0] - \gamma(x \wedge z)[y, e_0] + \gamma(x \wedge y)[z, e_0] = 0.$$

である。 $\Rightarrow$ である $x, y, z \in \mathcal{G}_1^-$ によって

$$\gamma(y \wedge z)x - \gamma(x \wedge z)y + \gamma(x \wedge y)z = 0.$$

これより容易に $(n-3)\gamma = 0$ である。 よって
 $n \geq 4$ のとき, $\gamma = 0$, i.e., $c = 0$ である。 以上
 $\Rightarrow H^{0,2}(\mathcal{G}) \subset V^{0,2}(n,1)$ が示された。

The space $H^{1,2}(\mathcal{G})$. Propositions —, — 12522
 $H^{1,2}(\mathcal{G}) \cap D_{-1}^{1,1}$ は $D_{-1}^{1,1}$ の元である。

$$\partial_1 c = \partial_0 c = \partial_1^* c = \partial_2^* c = 0$$

を満足するものの全体からなり, $H^{1,2}(\mathcal{G}) \cap D_0^{0,2}$ は
 $\mathcal{G}_0 \otimes \wedge^2 \mathcal{G}_1^*$ の元である。

$$\partial_1 c = \partial_0^* c = \partial_1^* c = 0$$

を満足するものの全体からなる。

まず $H^1(\mathcal{G}) \subset \mathcal{D}'_1$ を示す. $c \in H^1(\mathcal{G}) \cap \mathcal{D}_0^{0,2} \subset \mathcal{F}_3$.
 容易にわかるように $\mathcal{F}_0$ は $\mathcal{F}_1$ に含まれる. よって
 $\forall x, y \in \mathcal{F}_1$ に対して $c(x \wedge y)$ は $c(x \wedge y) = \gamma(x \wedge y) J$ with
 $\gamma(x \wedge y) \in \mathbb{R}$ と表わされる. よって $\forall x \in \mathcal{F}_1$ に対して,

$$(\partial_1^* c)(x) = \gamma(e_0 \wedge x) e^0 - \sum_i \gamma(e_i \wedge x) e^i$$

である. よって $\gamma(e_0 \wedge x) = \gamma(e_i \wedge x) = 0$, i.e., $\gamma = 0$ である.
 故に $c = 0$. よって $H^1(\mathcal{G}) \subset \mathcal{D}'_1$.

Lemma $c \in H^1(\mathcal{G}) \subset \mathcal{F}_3$.

(1) c^+

$$(1) \quad c^+(\mathcal{F}_2 \wedge \mathcal{F}_1^+) = 0, \text{ and } c^-(\mathcal{F}_2 \wedge \mathcal{F}_1^-) = 0.$$

$$(2) \quad c \in \mathcal{F}_1^- \oplus \mathcal{F}_2^* \wedge (\mathcal{F}_1^+)^*$$

$$(3) \quad c \in V^{1,2}(\mathcal{G}).$$

Proof. (1) $\partial_1^* c = 0$ より, $\forall u \in \mathcal{F}_2$ に対して,

$$(\partial_1^* c)(u) = [e^0, c^+(e_0 \wedge u)] + \sum_i [e^i, c^-(e_i \wedge u)] = 0$$

である. これより直ちに $c^+(e_0 \wedge u) = c^-(e_i \wedge u) = 0$ である.
 (Consider the M.R. of $\mathcal{G}$). よって

(1) が従う.

$$(2) \quad \forall u \in \mathcal{F}_2 \text{ と } \forall x \in \mathcal{F}_1, \text{ に対して } (\partial_1^* \partial_1 c)(u \wedge x) \text{ は}$$

$$(\partial_1^* \partial_1 c)(u \wedge x) = \partial_1^* \partial_1 (c(u \wedge x)) + \sum_{\lambda=0}^m [e^\lambda, [x, c(e_\lambda \wedge u)]]$$

と表現される

$\partial_1^* \partial_1 c = 0$ であるから, この式と (1)

かゝる

$$\partial_1^* \partial_1 (c^+(u \wedge x)) + \sum_i [e_i^*, [x, c^+(e_i \wedge u)]] = 0$$

かわかゝる Lemmas —, — かゝる

$$\partial_1^* \partial_1 (c^+(u \wedge x)) = - [E_2, c^+(u \wedge x)]$$

$$= (\lambda_2 - \mu_2) c^+(u \wedge x)$$

かわかゝる, Lemmas 3.12, 3.2

$$\begin{aligned} \sum_i [e_i^*, [x, c^+(e_i \wedge u)]] &= \sum_{i,j} \langle x, e_j^* \rangle \langle c^+(e_i \wedge u), e_j \rangle [e_i^*, [e_j, e_1]] \\ &= (\lambda_1^+ + \mu_1^+) c^+(u \wedge x) \end{aligned}$$

$$\text{かわかゝる, } \lambda_2 - \mu_2 + \lambda_1^+ + \mu_1^+ > 0 \text{ ならば } c^+(u \wedge x) = 0 \text{ である}$$

$$\lambda_2 - \mu_2 + \lambda_1^+ + \mu_1^+ > 0 \text{ ならば } c^+(u \wedge x) = 0 \text{ である}$$

この事実と (1) とから $C \in \mathcal{F}_1^- \oplus \mathcal{F}_2^* \cap (\mathcal{F}_1^+)^*$ かわかゝる

$$(3) \quad \partial_2^* C = 0 \text{ ならば, Lemma 3.12, 3.2}$$

$$(\partial_2^* C)(e_0) = \sum_i [d_i^*, C(d_i \wedge e_0)]$$

$$= \sum_{i,j} \langle C(d_i \wedge e_0), e_j^* \rangle [d_i^*, e_j] = \sum_i \langle C(d_i \wedge e_0), e_i^* \rangle e^0 = C$$

かわかゝる, $\lambda_2 - \mu_2 + \lambda_1^+ + \mu_1^+ > 0$ ならば $\sum_i \langle C(d_i \wedge e_0), e_i^* \rangle = 0$ ならば $C \in V^{1,2}(n,1)$ ならば $C \in \mathcal{F}_1^-$

The space $H^{2,2}(\mathcal{G})$. Propositions —, — 12

よって $H^{2,2}(\mathcal{G})$ は $\mathcal{D}_0^{1,1}$ の元 $c \in \mathcal{C}$

$$\partial_2 c = \partial_1 c = \partial_0 c = \partial_1^* c = \partial_2^* c = 0$$

を満足するもの全体からなる

Lemma. $c \in H^{2,2}(\mathcal{G})$ とする

$$(1) \quad c \in \mathcal{I}_0 \oplus \mathcal{G}_2^* \wedge \mathcal{G}_1^*$$

$$(2) \quad c \in V^{2,2}(n, 1).$$

Proof. (1) $\mathcal{I}_0 = \{X \in \mathcal{G}_0 \mid \langle X, E \rangle = \langle X, J \rangle = 0\}$ なるから、 $\forall u \in \mathcal{G}_2, \forall x \in \mathcal{G}_1$ には $\nexists \lambda \in \mathbb{C}, \langle C(u \wedge x), E \rangle = \langle C(u \wedge x), J \rangle = 0$ を示せばよい。 $\partial_1 c = \partial_1^* c = 0$ なるから、

$$\begin{aligned} (\partial_1^* \partial_1 c)(u \wedge x) &= \partial_1^* \partial_1 (C(u \wedge x)) + \sum_{\lambda=0}^n [[e^\lambda x], C(e_\lambda u)] \\ &\quad + [x, (\partial_1^* c)(u)] \end{aligned}$$

$$= \partial_1^* \partial_1 (C(u \wedge x)) + \sum_{\lambda=0}^n [[e^\lambda x], C(e_\lambda u)] = 0$$

よって Lemma を使って

$$\langle \partial_1^* \partial_1 (C(u \wedge x)), E \rangle = \sum_{\lambda=0}^n \langle [e^\lambda x], [e_\lambda, C(u \wedge x)], E \rangle$$

$$= \langle C(u \wedge x), E \rangle = \lambda_1 \langle C(u \wedge x), E \rangle + \mu_1 \langle C(u \wedge x), J \rangle$$

§ 2.3. $\exists t_2$ J は $\mathcal{G}_0$ の centre の $\bar{t}_2$ かし

$$\langle \sum_{\lambda=0}^n [C^\lambda x], C(u \wedge x), E \rangle = 0.$$

よって

$$\langle (Q_1^*, C)(u \wedge x), E \rangle = \lambda_1 \langle C(u \wedge x), E \rangle + \mu_1 \langle C(u \wedge x), J \rangle = 0$$

§ 2.3. 同様にして

$$\langle (Q_1^*, C)(u \wedge x), J \rangle = \langle C(u \wedge x), -E_1^+ + E_1^- \rangle$$

$$= (-\lambda_1^+ + \lambda_1^-) \langle C(u \wedge x), E \rangle + (-\mu_1^+ + \mu_1^-) \langle C(u \wedge x), J \rangle$$

$$= \frac{1}{2} \langle C(u \wedge x), J \rangle = 0$$

§ 2.3. 以上より $\langle C(u \wedge x), E \rangle = \langle C(u \wedge x), J \rangle = 0$

§ 2.3. よって (1) が示され t_2 .

(2) $\forall u \in \mathcal{G}_2, \forall x \in \mathcal{G}_1, i \neq 1$

$$(C)(u \wedge x \wedge e_0) = -[x, C(u \wedge e_0)] + [e_0, C(u \wedge x)] = 0.$$

$C(u \wedge x) \in \mathcal{G}_0$ t_2 かし, $[e_0, C(u \wedge x)] = 0$. よって

[

$$[x, C(u \wedge e_0)] = 0.$$

よって $C(u \wedge e_0) \in \mathcal{G}_0$. 故に $C \in \mathcal{G}_0 \otimes \mathcal{G}_2^* \wedge (\mathcal{G}_1^-)^*$

が示され t_2 . $\forall u \in \mathcal{G}_2, x, y \in \mathcal{G}_1, i \neq 1$.

$$(\partial_1 C)(u \wedge v) = -[x, C(u \wedge v)] + [y, C(u \wedge x)] = 0,$$

$$(\partial_0 C)(x \wedge y \wedge e_0) = C([x, e_0] \wedge y) - C([y, e_0] \wedge x) = 0$$

と 2.3. これより $[C([x, e_0] \wedge y), z]$ が, $x, y, z \in \mathcal{G}_1^-$
 12.7 (1) symmetric 7" あり = 0 がわかる
 5.7 2 $C \in V^{2,2}(n, 1)$ が示さね 12.

The space $H^{3,2}(\mathcal{G})$. Propositions —, —
 12.57 2 $H^{3,2}(\mathcal{G})$ は $\mathcal{D}_1^{1,1}$ の元 C 7".

$$\partial_2 C = \partial_1 C = \partial_0 C = \partial_1^* C = 0$$

を満足するものの全体からなる

Lemma. $C \in H^{3,2}(\mathcal{G})$ とする

$$(1) \quad C \in \mathcal{G}_1^+ \otimes \mathcal{G}_2^* \wedge (\mathcal{G}_4^-)^*$$

$$(2) \quad C = 0$$

Proof. (1) $\forall u, v \in \mathcal{G}_2, x \in \mathcal{G}_1, 12.7 \# 12.$

$$(\partial_2 C)(u \wedge v \wedge x) = [u, C(v \wedge x)] - [v, C(u \wedge x)] = 0$$

と 2.3. これより

$$[u, C(v \wedge x)] - [v, C(u \wedge x)] = 0$$

と 2.3. $q_j = \langle C(d_j \wedge x), e_0 \rangle$ とおくと $C(d_j \wedge x) = q_j e^0$

$J > 2$ Lemma — Σ 使 $\gamma > 2$

$$[d_i, c^-(d_j \wedge x)] - [d_j, c^-(d_i \wedge x)] = q_j [d_i, e^0] - q_i [d_j, e^0]$$

$$= -(\lambda_i^+ + \mu_i^+) \sum_k (q_j d_i^{\cdot k} - q_i d_j^{\cdot k}) e_k = 0$$

$\Sigma \ni 3$. $\lambda_i^+ + \mu_i^+ < 0$ P_2'' 不成立, $q_j d_i^{\cdot k} - q_i d_j^{\cdot k} = 0$ $\Sigma \ni 3$.

$\Sigma \ni 4$ より $(n-2)q_j = 0$ $\Sigma \ni 3$. $J > 2$ $q_j = 0$, i.e., $C^- = 0$.

$\Sigma \ni 3$. $\partial_1^* C = 0, C^- = 0$ P_2'' 不成立, $\forall u \in \mathcal{G}_2$ 123812,

$$(\partial_1^* C)(u) = [e^0, C^+(e_0 \wedge u)] + \sum_i [e^i, C^-(e_i \wedge u)]$$

$$= [e^0, C^+(e_0 \wedge u)] = 0.$$

$\Sigma \ni 3$. $\Sigma \ni 4$ より $C^+(e_0 \wedge u) = 0$ $\Sigma \ni 3$. 以上より

$C \in \mathcal{G}_1^+ \oplus \mathcal{G}_2^* \wedge (\mathcal{G}_1^-)^*$ 不成立 $\Sigma \ni 4$ P_2 .

(2). $u \in \mathcal{G}_2$, $x, y \in \mathcal{G}_1^-$ とする. $\partial_1 C = \partial_0 C = 0$
 P_2'' 不成立

$$\langle (\partial_1 C)(u \wedge x \wedge y), E \rangle = \langle [x, C(u \wedge y)] + [y, C(u \wedge x)], E \rangle$$

$$= \langle x, C(u \wedge y) \rangle - \langle y, C(u \wedge x) \rangle = 0.$$

$$(\partial_0 C)(x \wedge y \wedge e_0) = C([x, e_0] \wedge y) - C([y, e_0] \wedge x) = 0$$

$\Sigma \ni 3$. $J > 2$ $\langle C([x, e_0] \wedge y), z \rangle$ は $x, y, z \in \mathcal{G}_1^-$

123417 symmetric 7" あり $\Sigma \ni 4$ 不成立

$u \in \mathcal{G}_2$, $x \in \mathcal{G}_1^-$ とする. $\partial_1 C = 0$, P_2'' 不成立

$$(\mathfrak{a}_1^* \mathfrak{a}_1, c)(u \wedge x) = \mathfrak{a}_1^* \mathfrak{a}_1 (c(u \wedge x)) + \sum_i [e_i^c, [x, c(e_i \wedge u)]] = 0$$

2" 3. ($c(u \wedge u) = 0$ 12 13 14) Lemmas —, —
 3 15 16

$$\mathfrak{a}_1^* \mathfrak{a}_1 (c(u \wedge x)) = [E_2 + E_1, c(u \wedge x)]$$

$$= (\lambda_2 + \mu_2 + \lambda_1 + \mu_1) c(u \wedge x)$$

3 17 3. ($c(u \wedge x) \in \mathfrak{g}_1^+$ 12 13 14) 3 15 16

$$\sum_i [e_i^c, [x, c(e_i \wedge u)]] = \lambda_1 \left(\sum_i \langle x, c(e_i \wedge u) \rangle e_i^c + \sum_i \langle e_i^c, x \rangle c(e_i \wedge u) \right)$$

$$= \lambda_1 c(x \wedge u) + \lambda_1 \sum_i \langle x, c(e_i \wedge u) \rangle e_i^c$$

2 > 2", $[z, [x, y]] = \lambda_1 (\langle x, y \rangle z + \langle x, z \rangle y)$, $x \in \mathfrak{g}_1^-, y, z \in \mathfrak{g}_1^+$ 3 15 16
 3 17 2

$$(\lambda_2 + \mu_2 + \mu_1) c(u \wedge x) + \lambda_1 \sum_i \langle x, c(e_i \wedge u) \rangle e_i^c = 0$$

3 17 3. 3 17 2, $x, y, z \in \mathfrak{g}_1^-$ 12 13 14

$$(\lambda_2 + \mu_2 + \mu_1) \langle c([x, y], z), z \rangle + \lambda_1 \langle c([x, y], z), y \rangle$$

$$= (\lambda_2 + \mu_2 + \mu_1 - \lambda_1) \langle c([x, y], z), z \rangle = 0$$

3 17 3. $\lambda_2 + \mu_2 + \mu_1 - \lambda_1 > 0$ 3 17 2" 3 15 $\langle c([x, y], z), z \rangle = 0$

3 17 2 $c = 0$ 3 17 3

4. The spaces of harmonic forms, $H^{p,2}(\mathcal{G})$, associated with projective graded Lie algebras of type (n, r) . この paragraph では $\mathcal{L}$ は 4.4 の projective FGLA of type (n, r) であり, $\mathcal{G}$ は $\mathcal{L}$ の prolongation である.

vector spaces $V_{(1)}^{0,2}(n, r)$, $V_{(2)}^{0,2}(n, r)$, $V_{(3)}^{0,2}(n, r)$ を次のように定義する:

$V_{(1)}^{0,2}(n, r)$ は $\mathcal{G}_1^+ \oplus \wedge^2(\mathcal{G}_1^-)^*$ の元 c で $\partial_1 c = 0$ を満足するものの全体が作る vector space である.

$V_{(2)}^{0,2}(n, r)$ は $\mathcal{G}_1^- \oplus \wedge^2(\mathcal{G}_1^+)^*$ の元 c で $\partial_1 c = 0$ を満足するものの全体が作る vector space である.

$V_{(3)}^{0,2}(n, r)$ は $\mathcal{G}_1^- \oplus (\mathcal{G}_1^+)^* \wedge (\mathcal{G}_1^-)^*$ の元 c で $\partial_1 c = \partial_0^* c = \partial_1^* c = 0$ を満足するものの全体が作る vector space である.

次の Prop. を証明しよう.

Proposition (1) $H^{0,2}(\mathcal{G}) = V_{(1)}^{0,2}(n, r) + V_{(2)}^{0,2}(n, r) + V_{(3)}^{0,3}(n, r)$

(2) $n-r \geq 3$ のとき, $V_{(1)}^{0,2}(n, r) = 0$.

(3) $n-r=1$ or $r=1$ のとき, $V_{(3)}^{0,2}(n, r) = 0$.

$(e_1, \dots, e_{n-r}) = (e_\alpha)$ は $\mathcal{G}_2$ の basis, $(e_{n-r+1}, \dots, e_n) = (e_i)$ は $\mathcal{G}_1^+$ の basis とする. このとき $\mathcal{G}_1^-$ の basis (e_α^i) である.

$$[e_\alpha^i, e_j] = \delta_j^i e_\alpha$$

を満足するものが唯一つ存在する. $(e_1^*, \dots, e_{n-r}^*) = (e^\alpha)$ は (e_α) に dual な $\mathcal{G}_2$ の basis, $(e_{n-r+1}^*, \dots, e_n^*) = (e^i)$ は (e_i) に dual な $\mathcal{G}_1^+$ の basis, (e_i^α) は

(e_α^i) is dual to $\mathcal{F}_1^+$ a basis of $\mathcal{F}_1^+$.

$\mathcal{F}_0$ of E , E_1^+ , E_1^- is given by the following definition:

$$E_1^+ = \sum_i [e_i^+ e_i^-], \text{ and } E_1^- = \sum_{\alpha, i} [e_i^+, e_\alpha^i].$$

Lemma (1) $E_1^+ = \lambda_1^+ E + \mu_1^+ J$, where $\lambda_1^+ = \frac{r}{4(n+1)}$ and $\mu_1^+ = \frac{-(r+2)}{4(n+1)}$

(2) $E_1^- = \lambda_1^- E + \mu_1^- J$, where $\lambda_1^- = \frac{r}{4(n+1)}$ and $\mu_1^- = \frac{2n-r}{4(n+1)}$

is Proposition of the proof of λ_3 , $r=1$ case is completely the same as $H^{0,2}(\mathcal{F}) \subset D_1^{0,2}$ is proved. δ is $+$ or $-$ and operator $\partial_1^\delta: \mathcal{F}_0 \otimes (\mathcal{F}_1^\delta)^* \rightarrow \mathcal{F}_1^\delta \otimes \wedge^2(\mathcal{F}_1^\delta)^*$ is

$$(\partial_1^\delta a)(x \wedge y) = [x, a(y)] - [y, a(x)]$$

is by definition. $\partial_1^\delta a \in \mathcal{F}_0 \otimes (\mathcal{F}_1^\delta)^*$, $x, y \in \mathcal{F}_1^\delta$ is. inner product $(,)$ is defined by ∂_1^δ of adjoint operator $\partial_1^{\delta*}: \mathcal{F}_1^\delta \otimes \wedge^2(\mathcal{F}_1^\delta)^* \rightarrow \mathcal{F}_0 \otimes (\mathcal{F}_1^\delta)^*$ is

$$\begin{aligned} (\partial_1^{\delta*} a)(x) &= \sum_i [e_i^+, a(e_i^- \wedge x)] \quad \text{if } \delta \text{ is } +, \\ &= \sum_i [e_i^-, a(e_i^+ \wedge x)] \quad \text{if } \delta \text{ is } - \end{aligned}$$

$$H^{0,2}(\mathcal{G}) \text{ は } A_1^{0,2} \cap \bar{z} \subset \tau''$$

$$\partial_1 C = \partial_0^* C = \partial_1^* C = 0$$

$$\exists \text{ 満足する } \delta \text{ の全体が } \mathbb{R}^3$$

τ'' と $\bar{z}$ と τ'' $\subset \tau''$ $a \in \mathcal{G}_1^\delta \oplus \wedge^2(\mathcal{G}_1^\delta)^*$, $x \in \mathcal{G}_1^\delta$
 τ'' あり. $\sigma \mathcal{G}_1^+ = \mathcal{G}_1^-$, $\sigma \mathcal{G}_1^- = \mathcal{G}_1^+$ を注意せよ.

$C \in H^{0,2}(\mathcal{G}) \subset A_1^{0,2}$ とする. C は $C = C^+ + C^-$ と
 分解する. $\delta = +\alpha - 1 \neq 1, 2$, $a^\delta \in \mathcal{G}_1^\delta \oplus \wedge^2(\mathcal{G}_1^\delta)^*$ と

$$a^\delta(x \wedge y) = C^\delta(x \wedge y), \quad x, y \in \mathcal{G}_1^\delta,$$

によって定義する.

$$\text{Lemma (1)} \quad (-\lambda_1^- + \mu_1^-) a^+ + \partial_1^+ \partial_1^{+*} a^+ = 0.$$

$$(2) \quad -(\lambda_1^+ + \mu_1^+) a^- + \partial_1^- \partial_1^{-*} a^- = 0$$

Proof. $\partial_1^* C = 0$ より, $\forall y \in \mathcal{G}_1$ $1 \neq 2 \neq 1, 2$,

$$(\partial_1^* C)(y) = \sum_i [e_i^*, C^+(e_i \wedge y)] + \sum_{\alpha, i} [e_i^\alpha, C^-(e_\alpha^i \wedge y)].$$

とある. $\partial_1 C = 0$ より, $\forall x \in \mathcal{G}_1$, $y, z \in \mathcal{G}_1^+$ $1 \neq 2 \neq 1, 2$

$$(\partial_1 C)(x \wedge y \wedge z) = [x, C^+(y \wedge z)] - [y, C^-(x \wedge z)]$$

$$+ [z, C^-(x \wedge y)] = 0$$

とある. よって

$$\sum_{\alpha, i} [e_i^\alpha, [e_\alpha^i, C^+(y \wedge z)]] - \sum_{\alpha, i} [e_i^\alpha, [y, C^-(e_\alpha^i \wedge z)]]$$

$$+ \sum_{\alpha, i} [e_i^\alpha, [z, C^-(e_\alpha^i \wedge y)]] = 0, \quad y, z \in \mathcal{G}_1^+.$$

よって

$$[E_1^-, C^+(y \wedge z)] - [y, \sum_{\alpha, i} [e_i^\alpha, C^-(e_{\alpha}^i \wedge z)]] \\ + [z, \sum_{\alpha, i} [e_i^\alpha, C^-(e_{\alpha}^i \wedge y)]] = 0.$$

よって Lemma と $(\partial_1^* C) = 0$ 12/27 33 上式から

$$(-\lambda_1^- + \mu_1^-) C^+(y \wedge z) + [y, \sum_i [e_i^-, C^+(e_i^- \wedge z)]] \\ - [z, \sum_i [e_i^-, C^+(e_i^- \wedge y)]] = 0$$

よって これは (1) を意味する

全く同様に (2) が示される。

$$(-\lambda_1^- + \mu_1^-) > 0, \quad -(\lambda_1^+ + \mu_1^+) > 0 \text{ から}$$

Lemma から $a^+ = a^- = 0$ である。よって

$$C^+(z_1^+ \wedge z_1^+) = 0,$$

$$C^-(z_1^- \wedge z_1^-) = 0$$

が示された。この 2 番目の式と $\partial_1^* C = 0$ から

$$\sum_i [e_i^-, C^+(e_i^- \wedge y)] = 0, \quad y \in z_1^-$$

よって。これから 直ちに $C^+(e_i^- \wedge y) = 0$ が従う。
よって

$$C^+(z_1^+ \wedge z_1^-) = 0.$$

以上から

$$H^{0,2}(\mathcal{G}) \subset \mathcal{G}_1^+ \otimes \wedge^2(\mathcal{G}_1^-)^* + \mathcal{G}_1^- \otimes \wedge^2(\mathcal{G}_1^+)^* \\ + \mathcal{G}_1^- \otimes (\mathcal{G}_1^+)^* \wedge (\mathcal{G}_1^-)^*$$

2.2.3. 今や容易に

$$H^{0,2}(\mathcal{G}) = V_{(1)}^{0,2}(n,r) + V_{(2)}^{0,2}(n,r) + V_{(3)}^{0,2}(n,r)$$

が示される。

$n-r \geq 3$ のとき, $V_{(1)}^{0,2}(n,r) = 0$ を示そう。

$c \in V_{(1)}^{0,2}(n,r)$ とする。 $\partial_1 c = 0$

$$(\partial_1 c)(e_\alpha^i \wedge e_\beta^j \wedge e_\gamma^k) = [e_\alpha^i, c(e_\beta^j \wedge e_\gamma^k)]$$

$$- [e_\beta^j, c(e_\alpha^i \wedge e_\gamma^k)] + [e_\gamma^k, c(e_\alpha^i \wedge e_\beta^j)] = 0$$

$\langle c(e_\alpha^i \wedge e_\beta^j), e_\gamma^k \rangle = a_{\alpha\beta}^{ijk}$ とおけば, $a_{\alpha\beta}^{ijk} = 0$

$$a_{\beta\gamma}^{jki} \delta_\alpha^i - a_{\alpha\gamma}^{ikj} \delta_\beta^i + a_{\alpha\beta}^{ikj} \delta_\gamma^i = 0$$

2.2.3. $n-r \geq 3$ の仮定の下でこの式より容易

に $a_{\alpha\beta}^{ikj} = 0$ を示す。よって $c = 0$ である。

$n-r=1$ 或 $r=1$ のとき, $V_{(3)}^{0,2}(n,r) = 0$

を示そう。 $c \in V_{(3)}^{0,2}(n,r)$ とする。 $\partial_1^* c = 0$ から

$$(\partial_1^* c)(e_k) = \sum_{\alpha, i} [e_i^\alpha, c(e_\alpha^i \wedge e_k)]$$

$$= \sum_{\substack{\alpha, i \\ \beta, j}} \langle c(e_\alpha^i \wedge e_\beta^j), e_j^\beta \rangle [e_i^\alpha, e_j^\beta] = 0$$

とえる。 $n-r=1$ のとき、この式から

$$\langle c(e_\alpha^i \wedge e_\beta^j), e_j^\beta \rangle = 0 \text{ が従う。 よって } c=0 \text{ とえる。}$$

以上によって Prop. の証明が完了した。

さて $r=n-1$ の場合を考えよう。この仮定の下で、

$$\dim \mathcal{G}_2 = 1, \dim \mathcal{G}_1 = 2(n-1) \text{ である。 } n=2 \text{ のとき}$$

には、 $H^{p,2}(\mathcal{G})$ は Proposition — によって既に

決定されている。(J ∈ $\mathcal{G}_0$ に注意) $n \geq 3$ のとき

には、 $\wedge$ Propositions —, — を使って $H^{p,2}(\mathcal{G})$

は容易に決定される。結果は次の通りである。

Proposition Assume that $r=n-1$ and $n \geq 3$.

$$(1) \quad H^{p,2}(\mathcal{G}) = 0 \text{ if } p \neq 0 \text{ or } 1.$$

$$(2) \quad H^{0,2}(\mathcal{G}) = V_{(1)}^{0,1}(n, n-1) + V_{(2)}^{0,1}(n, n-1).$$

$$(3) \quad H^{1,2}(\mathcal{G}) \text{ consists of all } c \in \mathcal{G}_0 \oplus (\mathcal{G}_1^+)^* \wedge (\mathcal{G}_1^-)^* \text{ such that } \partial_1 c = \partial_1^* c = 0$$

Proposition Assume that $r=1$ and $n=2$.

$$(1) \quad H^{p,2}(\mathcal{G}) = 0 \text{ if } p \neq 3.$$

$$(2) \quad H^{3,2}(\mathcal{G}) = \mathcal{G}_1^+ \oplus \mathcal{G}_2^* \wedge (\mathcal{G}_1^-)^* + \mathcal{G}_1^- \oplus \mathcal{G}_2^* \wedge (\mathcal{G}_1^+)^*$$

Proof of Proposition—. It suffices to prove (3). Let us identify $\mathcal{L}_0$ with a subalgebra of $\mathfrak{gl}(\mathcal{I}_1)$ through the natural representation of $\mathcal{L}_0$ on $\mathcal{I}_1$. Then we have $J \in \mathcal{L}_0$ and $J^2 = 1$. Define a bilinear form α on $\mathcal{I}_1$ by

$$[Jx, y] = \alpha(x, y) d_1, \quad x, y \in \mathcal{I}_1,$$

d_1 being a basis of $\mathcal{I}_2$. Then we see that α is non-degenerate and that $\alpha(x, y) = \alpha(y, x)$ and $\alpha(Jx, Jy) = -\alpha(x, y)$. Now the space $H^{1,2}(\mathcal{G})$ consists of all $c \in \mathcal{L}_0 \otimes \wedge^2 \mathcal{I}_1^*$ such that $\partial_1 c = \partial_0^* c = \partial_1^* c = 0$. Let $c \in H^{1,2}(\mathcal{G})$. Then it follows that

$$Jc(x \wedge y) = c(x \wedge y)J,$$

$$c(x \wedge y)z + c(y \wedge z)x + c(z \wedge x)y = 0,$$

$$\alpha(c(x \wedge y)z, w) = -\alpha(c(x \wedge y)w, z), \quad x, y, z, w \in \mathcal{I}_1,$$

whence $c(Jx \wedge Jy) = -c(x \wedge y)$ (cf. the curvature of a Kählerian manifold). Thus we get $c \in \mathcal{L}_0 \otimes (\mathcal{I}_1^*)^* \wedge (\mathcal{I}_1)^*$. It is now easy to verify that the equation $\partial_0^* c = 0$ can be derived from the equations $\partial_1 c = \partial_1^* c = 0$. We have thus proved Proposition—.

最後に次の Proposition をつけ加える. 証明は省略する.

Proposition $2 \leq r \leq n-2$ と仮定する.

$$(1) \quad H^{p,2}(\mathcal{G}) = 0 \quad \text{if } p \neq +020.$$

(2) $H^{-1,2}(\mathcal{G})$ consists of all $c \in \mathcal{G}_2 \otimes \wedge^2(\mathcal{G}_1)^*$ such that $\partial_1^* c = 0$

$$(3) \quad H^{0,2}(\mathcal{G}) = V_{(1)}^{0,2}(n,r) + V_{(2)}^{0,2}(n,r) + V_{(3)}^{0,2}(n,r).$$

以上のすべての projective graded Lie algebras に対して spaces $H^{p,2}(\mathcal{G})$ を計算した.

Part I Chapter IV

Part I Chapter IV

Chapter IV Almost pseudo-product manifolds, and normal connections of type G

§IV.1 Normal connections of type G

1. Cartan connections

The notion of isomorphism (P, ω) (resp. (P', ω')) $\in M$ (resp. M') 上の Cartan connection of type $G/G^{(0)}$ と $\exists$ $\varphi: M \rightarrow M'$ が (P, ω) から (P', ω') への isomorphism であるとは,

bundle isomorphism $\tilde{\varphi}: P \rightarrow P'$ が $\exists$ して,
 $\tilde{\varphi}^* \omega' = \omega$

を満足し, かつ $\tilde{\varphi}$ が φ を induce する $\Rightarrow$ $\tilde{\varphi}$ である

Proposition $G/G^{(0)}$ に関する次の仮定を行う:

i) G は 2nd countability axiom を満足する, ii) $G/G^{(0)}$ は connected である, iii) G の $G/G^{(0)}$ への action は effective である. このとき $\tilde{\varphi}$ は unique.

$$(P, \omega) \quad X \in \mathfrak{g} \rightarrow \tilde{\omega}(X)$$

$$\delta_X = \tilde{\omega}(X), \quad \delta_X = \mathcal{L}_X^* \text{ if } X \in \mathfrak{g}^{(0)}$$

Let G/G' be a homogeneous space of a Lie group G over its closed subgroup G' and denote by $\mathfrak{g}$ and $\mathfrak{g}'$ the Lie algebras of G and G' respectively. Then a (Cartan) connection of the model space G/G' on a manifold M is a pair formed by a principal fibre bundle P over M with structure group G' and a $\mathfrak{g}$ -valued 1-form ω on P which satisfy the following conditions:

(i) ω gives an absolute parallelism on P , i.e., for each $z \in P$ the linear map which maps $X \in T_z(P)$ to $\omega(X) \in \mathfrak{g}$ is an isomorphism of $T_z(P)$ onto $\mathfrak{g}$.

$$(ii) \quad R_a^* \omega = \text{Ad}(a^{-1}) \omega, \quad a \in G'.$$

$$(iii) \quad \omega(X^*) = X, \quad X \in \mathfrak{g}'.$$

The connection will be denoted by a capital Greek letter, e.g., Γ , and then the pair (P, ω) will be called the expression of Γ ; ω itself will be called the connection form of Γ . The same notation and terminology will be used for transverse (Cartan) connections, which will be defined in Part II.

2 Connections of type $\mathcal{G}$

$\mathcal{G}$ $G/G^{(0)}$
 (P, ω) connection of type $\mathcal{G}$

ω -

Proposition ω -性质

$$\Omega = d\omega + \frac{1}{2} [\omega, \omega]$$

Proposition $K: P \rightarrow C^2(\mathcal{G})$

Proposition $R_a^* K = K^a, a \in G^{(0)}$

$$\omega = \sum \omega_j, \quad \Omega = \sum \Omega_j$$

$$K = \sum_j K_j = \sum_P K^P, \quad K_j = \sum_P K_j^P, \quad K^P = \sum_j K_j^P$$

$$\omega_j(X_r^*) = \delta_{jr} X_r$$

$$R_a^* \omega_j = \text{Ad}(a^{-1}) \omega_j$$

$$\delta_{X_r} \omega_j = -[X_r, \omega_{j-r}], \quad X_r \in \mathcal{G}_r, r \geq 1$$

$$R_a^* K^P = (K^P)^a, \quad a \in G_0$$

$$\delta_{X_r} K^P = (K^{P-r})^{X_r}, \quad X_r \in \mathcal{G}_r, r \geq 1$$

$$\Omega_j = \frac{1}{2} K_j (\omega_- \wedge \omega_-) = \frac{1}{2} \sum_P K_j^P (\omega_- \wedge \omega_-)$$

$$(P, \omega) \rightarrow \tilde{P} \subset \mathcal{Q}, \quad \mathcal{Q} \text{ } G_0^\# \text{-structure}$$

normal connections of type $\mathcal{G}$

Proposition (P, ω) normal $\Leftrightarrow \mathcal{Q}$ of type $\mathcal{K}$

3. Normal connections of type $\mathcal{G}$

Proposition (P, ω) normal $\iff Q$ of type $\mathcal{H}$

Theorem. normal connections of type $\mathcal{G}$

$\iff G_0^\#$ -structures of type $\mathcal{H}$

(1), (2), (3)

Remark. $G_0 \subseteq \text{Aut}(\mathcal{G})$ の open subgroup

$$G^{(0)} = G_0 \cdot G^{(1)}, \quad G = \text{Aut}(\mathcal{G}) \cdot G_0$$

$G/G^{(0)}$ は effective, compact, etc.

normal connections of type $(\mathcal{G}, G_0)$

of type $\mathcal{G}$ という

$(P, \omega) \in$ weak Cartan connection of type $(\mathcal{G}, G^{(0)})$ on M
curvature $K: P \rightarrow C^2(\mathcal{G})$ が 全く同様

定義 2" を 3

$$K = \sum K^P$$

(P, ω) が weak normal connection of type $\mathcal{G}$.

(P, ω) normal connection of type $\mathcal{G}$

($\mathcal{G}$ は $(\mathcal{H}, G_0)$ の prolongation とは仮定しない)

$$K = \sum K^P$$

$$\Psi^{P-1}(X_1 \wedge X_2 \wedge X_3) = -\partial_{X_1} K^{P+1}(X_2 \wedge X_3) -$$

Proposition (Bianchi identities)

$$\partial K^P = \Psi^{P-1}$$

$H: C^2(\mathcal{G}) \rightarrow H^2(\mathcal{G})$ harmonic projection.

H is $C^1(\mathcal{G}) \subseteq H^1(\mathcal{G})$ is project 3.3.

HK or $\{HKP\}$ is the fundamental system of invariants 2.11.3.

Proposition. $K=0 \iff HK=0$

§ IV. 2 Almost pseudo-product manifolds, and normal connections of type $\mathcal{G}$

1. Almost pseudo-product manifolds of type $\mathcal{L}$ and normal connections of type $\mathcal{G}$.
 Let $\mathcal{L} = \{\mathcal{N}\mathcal{E}; e, f\}$ be a non-degenerate pseudo-product FGLA, and $\mathcal{G} = \{\mathcal{g}, (\mathcal{g}_p)\}$ its prolongation. By Proposition we know that $\mathcal{g}$ is finite dimensional. In this paragraph we assume that $\mathcal{g}$ is a simple Lie algebra and hence $\mathcal{G}$ is a simple graded Lie algebra.

Similarly to the case of a projective FGLA, the automorphism group $\text{Aut}(\mathcal{L})$ of $\mathcal{L}$ may be identified with an open subgroup of the automorphism group $\text{Aut}(\mathcal{G})$ of $\mathcal{G}$. This being said, we set

$$G_0 = \text{Aut}(\mathcal{L}),$$

and define subgroups $G^{(0)}$ and G of $\text{Aut}(\mathcal{G})$ respectively by

$$G^{(0)} = G_0 \cdot G^{(1)}, \text{ and } G = G_0 \cdot \text{Aut}(\mathcal{g}),$$

which are just the groups associated with the pair $(\mathcal{G}, G_0)$ (see Remark). Here $G^{(1)}$ is the Lie subgroup of $\text{Aut}(\mathcal{g})$ generated by the subalgebra $\mathcal{g}^{(1)} = \sum_{p \geq 1} \mathcal{g}_p$ of $\mathcal{G}$. Let us consider the homogeneous space $G/G^{(0)}$. Then we see as before that the subspaces e and f

of $\mathcal{L}$, give rise to G -invariant differential systems E_0 and F_0 on $G/G^{(0)}$ respectively, and that $\mathcal{P}_0 = \{R; E_0, F_0\}$ is a pseudo-product manifold of type $\mathcal{L}$, which will be called the standard pseudo-product manifold of type $\mathcal{L}$.

Now we have the notion of a normal connection of type G (or exactly of type (G, G_0)) as well as the notion of a $G_0^\#$ -structure of type $\mathcal{M}$ (see §§ II.1, II.2 and III.2). Let (P, ω) be a normal connection of type G on a manifold R , and $\mathcal{Q}$ the corresponding $G_0^\#$ -structure of type $\mathcal{M}$ on R . Let p be the natural homomorphism $P \rightarrow Q$ corresponding to the homomorphism $\varphi: G^{(0)} \rightarrow G \subset G_0^\#$, and let π (resp. α) be the projection $P \rightarrow R$ (resp. $Q \rightarrow R$). Let ξ be the basic form of $\mathcal{Q}$. Then we have

$$\pi = \alpha \circ p, \text{ and } \omega_- = p^* \xi,$$

where ω_- is the m -component of ω with respect to the decomposition. Furthermore the connection form ω is decomposed as follows

$$\omega = \sum_p \omega_p,$$

and ω_{-1} as follows:

$$\omega_{-1} = \omega_{-1}^+ + \omega_{-1}^-,$$

where ω_{-1}^+ (resp. ω_{-1}^-) is the e -component (resp. the f -component) of ξ_{-1} with respect to the decomposition $\mathcal{L}_1 = e + f$. Similarly the basic ξ is decomposed as follows:

$$\xi = \sum_p \xi_p, \text{ and } \xi_{-1} = \xi_{-1}^+ + \xi_{-1}^-.$$

Clearly we have

$$\omega_p = p^* \xi_p, \quad p < 0, \text{ and } \omega_{-1}^+ = p^* \xi_{-1}^+, \quad \omega_{-1}^- = p^* \xi_{-1}^-.$$

Therefore from Theorem — and Proposition — we obtain

Theorem —. (1) To every normal connection of type $\mathcal{G}$, (P, ω) , on a manifold R there is associated a unique almost pseudo-product manifold of type $\mathcal{L}$, $\mathcal{P} = \{R; E, F\}$, satisfying the following conditions:

i) For each $p < 0$, $\pi_*^{-1}(D_p)$ is defined by the equations $\omega_q = 0$, $q \leq p$, where D_p is the $(-p-1)$ -th derived system of D .

ii) $\pi_*^{-1}(E)$ is defined by the equations $\omega_p = \omega_{-1}^- = 0$, $p \leq -2$,

iii) $\pi_*^{-1}(F)$ is defined by the equations $\omega_p = \omega_{-1}^+ = 0$, $p \leq -2$.

(2) Conversely to every almost pseudo-product manifold of type $\mathcal{L}$, $\mathcal{P} = \{R; E, F\}$, there is associated, in a natural manner, a normal connection of type $\mathcal{G}$, (P, ω) , on R satisfying the conditions above.

(3) Let (P, ω) (resp. (P', ω')) be a normal connection of type $\mathcal{G}$ on a manifold R (resp. on R'), and $\mathcal{P} = \{R; E, F\}$ (resp. $\mathcal{P}' = \{R'; E', F'\}$) be the corresponding almost pseudo-product manifold of type $\mathcal{L}$. Then a diffeomorphism $g: R \rightarrow R'$ is an isomorphism of $\mathcal{P}$ to $\mathcal{P}'$ if and only if it is an isomorphism of (P, ω) to (P', ω') .

Let Γ and Γ' be connections of model space G/G' on manifolds M and M' , and (P, ω) and (P', ω') their expressions, respectively. By an isomorphism of Γ to Γ' we mean a diffeomorphism φ of M onto M' such that there is a bundle isomorphism $\tilde{\varphi}$ of P onto P' satisfying the following conditions:

(i) $\tilde{\varphi}$ induces φ ,

(ii) $\tilde{\varphi}^* \omega' = \omega$

We remark that if G/G' is connected, and G acts effectively on G/G' , then $\tilde{\varphi}$ is uniquely determined by φ .

Furthermore the connections Γ and Γ' are said to be equivalent, if $M = M'$, and the identity transformation of M gives an isomorphism of Γ to Γ' .

By a normal connection of type $\mathcal{L}$ we mean a normal connection modeled by G/G' . Let us consider a normal connection Γ of type $\mathcal{L}$ on a manifold M , and (P, ω) its expression.

According to the decomposition $\mathcal{G} = \sum_p \mathcal{G}_p$ the connection form ω is decomposed as follows: $\omega = \sum_p \omega_p$, and according to the decomposition $\mathcal{G}_- = \mathcal{G}_-^+ + \mathcal{G}_-^-$ the $\mathcal{G}_-$ -valued 1-form ω_- as follows: $\omega_- = \omega_-^+ + \omega_-^-$.

Theorem (1) Let R be any almost pseudo-product manifold, and (R, E, F) its expression. Then there is naturally associated to R a normal connection Γ on R which satisfies the following conditions (i) and (ii): Let (P, ω) be the expression of Γ , and π the projection of P onto M . Then, (i) the system $\pi_*^{-1}(E)$ on P is defined by the system of Pfaffian equations $\omega_p = \omega_-^- = 0$ ($p \leq -2$), and (ii) the system $\pi_*^{-1}(F)$ on P by the system of Pfaffian equations $\omega_p = \omega_-^+ = 0$ ($p \leq -2$).

(2) Conversely let Γ be a normal connection of type $\mathcal{L}$ on a manifold R . Then there is a unique almost pseudo-product structure (E, F) on R such that the almost pseudo-product manifold R expressed by (R, E, F) is of type $\mathcal{L}$ and such that Γ is naturally isomorphic to the normal connection of type $\mathcal{L}$ associated with R .

(3) Let R and R' be two almost pseudo-product manifolds of type $\mathcal{L}$, and let Γ and Γ' be the associated normal connection of type $\mathcal{L}$ respectively. Then an isomorphism of R to R' is an isomorphism of Γ to Γ' and vice versa.

Thus we know that the almost pseudo-product manifolds of type $\mathcal{L}$, $\mathcal{P}$, are naturally in a one-to-one correspondence with the normal connections of type $\circ$ (P, ω) , up to isomorphisms. (*)

Suppose now that $\mathcal{L}$ is a projective FGLA of type (n, r) , and let $\mathcal{P}$ be an almost pseudo-product manifold. Then by Proposition — $\mathcal{P}$ is an almost projective system of type (n, r) if and only if it is of type $\mathcal{L}$. Accordingly Theorem can be applied to the geometry of almost projective systems of type (n, r) . Let us now consider the geometry of single ordinary differential equations of the third order

$$\frac{d^3 y}{dx^3} = F(x, y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}),$$

where the equivalence is considered with respect to contact transformations of the space of variables x, y . Then it can be shown that this geometry may be represented by the geometry of pseudo-product manifolds of type $\mathcal{L}$, where $\mathcal{L}$ is a non-degenerate pseudo-product FGLA of the third kind such that the prolongation $\mathcal{G}$ of $\mathcal{L}$ is a simple graded Lie algebra and the underlying Lie algebra $\mathfrak{g}$ is isomorphic with the simple Lie algebra $O(3, 2)$.

Thus Theorem can be applied to the former geometry through the latter (cf. Cartan [1] and Chern [2]).

(*) It is easy to see that the standard pseudo-product manifold of type $\mathcal{L}$, $\mathcal{P}_0$, just corresponds to the standard normal connection of type $\mathcal{L}$, (G, ω) , on $G/G^{(0)}$, and hence we get

Corollary. Let $\varphi: U \rightarrow V$ be a local transformation of $G/G^{(0)}$. Then φ is a local automorphism of $\mathcal{P}_0$ if and only if it is obtained from the action of G on $G/G^{(0)}$, i.e., there is an $a \in G$ such that $\varphi(x) = ax$, $x \in U$.

2. The normal connections of type $\mathcal{O}$ associated with almost projective systems of type (n, r) . The notations being as above, we assume in this paragraph that $\mathcal{L}$ is a projective FGLA of type (n, r) . Let $\mathcal{P} = \{R; E, F\}$ be an almost projective system of type (n, r) , and $(\mathcal{P}, \omega)$ the corresponding normal connection of type $\mathcal{O}$.

Let us consider the curvature function $K: \mathcal{P} \rightarrow C^2(\mathcal{O})$. The space $C^2(\mathcal{O})$ is decomposed as follows:

$$C^2(\mathcal{O}) = \sum_{p=-1}^5 C^{p,2}(\mathcal{O}),$$

and each $C^{p,2}(\mathcal{O})$ as follows

$$C^{p,2}(\mathcal{O}) = C_{p-3}^{p,2} + C_{p-2}^{p,2} + C_{p-1}^{p,2}.$$

(The notations being as in §III.1, $C_{p-3}^{p,2} = A_{p-3}^{2,0}$, $C_{p-2}^{p,2} = A_{p-2}^{1,1}$ and $C_{p-1}^{p,2} = A_{p-1}^{0,2}$.) Therefore according to the notations given in §III.2, K is decomposed in two ways as follows:

$$K = \sum_{p=-1}^5 K^p = \sum_{j=-2}^2 K_j,$$

each K^p as follows:

$$K^p = K_{p-3}^p + K_{p-2}^p + K_{p-1}^p,$$

and each K_j as follows:

$$K_j = K_j^{j+3} + K_j^{j+2} + K_j^{j+1}$$

Here we remark that

$$K^{-1}(=K_{-2}^{-1})=0, \text{ and } J^* K^p=0, 0 \leq p \leq 5,$$

because the connection (P, ω) is normal.

Thus the structure equation $d\omega + \frac{1}{2}[\omega, \omega] = \frac{1}{2}K(\omega \wedge \omega)$ is decomposed into the five equations as follows:

$$\begin{aligned} (E_j) \quad d\omega_j + \frac{1}{2} \sum_{r+s=j} [\omega_r, \omega_s] \\ = \frac{1}{2} K_j^{j+3}(\omega_{-2} \wedge \omega_{-2}) + K_j^{j+2}(\omega_{-2} \wedge \omega_{-1}) + \frac{1}{2} K_j^{j+1}(\omega_{-1} \wedge \omega_{-1}), \\ -2 \leq j \leq 2. \end{aligned}$$

For example equations (E_{-2}) and (E_{-1}) may be explicitly written as follows:

$$\begin{aligned} (E_{-2}) \quad d\omega_{-2} + \frac{1}{2} [\omega_{-1}, \omega_{-1}] + [\omega_0, \omega_{-2}] \\ = \frac{1}{2} K_{-2}^{-1}(\omega_{-2} \wedge \omega_{-2}) + K_{-2}^0(\omega_{-2} \wedge \omega_{-1}) \end{aligned}$$

$$\begin{aligned} (E_{-1}) \quad d\omega_{-1} + [\omega_0, \omega_{-1}] + [\omega_1, \omega_{-2}] \\ = \frac{1}{2} K_{-1}^{-2}(\omega_{-2} \wedge \omega_{-2}) + K_{-1}^{-1}(\omega_{-2} \wedge \omega_{-1}) + \frac{1}{2} K_{-1}^0(\omega_{-1} \wedge \omega_{-1}). \end{aligned}$$

As an application of the structure equation we shall now prove an integrability theorem on almost projective systems. We first recall that the space of harmonic forms, $H^{0,2}(\mathcal{G})$, is calculated as follows:

$$H^{0,2}(\mathcal{G}) = V_{(1)}^{0,2}(n,r) + V_{(2)}^{0,2}(n,r) + V_{(3)}^{0,2}(n,r)$$

(see Proposition —), noting that $V_{(1)}^{0,2}(n,r) \subset \mathcal{G}_1^+ \otimes \wedge^2(\mathcal{G}_1^-)$, $V_{(2)}^{0,2}(n,r) \subset \mathcal{G}_1^- \otimes \wedge^2(\mathcal{G}_1^+)^*$, and $V_{(3)}^{0,2}(n,r) \subset \mathcal{G}_1^- \otimes (\mathcal{G}_1^+)^* \wedge (\mathcal{G}_1^-)^*$.

Theorem —. Let $\mathcal{P} = \{R; E, F\}$ be an almost projective system of type (n,r) , and (P, ω) the corresponding normal connection of type $\mathcal{I}$ on R . Consider the invariant K^0 .

(1) $\mathcal{P}$ is integrable if and only if K^0 takes values in $V_{(3)}^{0,2}(n,r)$.

(2) Assume that $n-r=1$ or $r=1$. Then $\mathcal{P}$ is integrable if and only if $K^0=0$.

(3) If $n-r \geq 3$, then F is automatically completely integrable (cf. Yamaguchi [Y]).

(4) If $r=1$ and $n \geq 4$, then $\mathcal{P}$ is automatically integrable.

The differential system $\pi_*^{-1}(E)$ (resp. $\pi_*^{-1}(F)$) on P is defined by the equations:

$$\omega_{-2} = \omega_{-1} = 0 \quad (\text{resp. } \omega_{-2} = \omega_{-1}^+ = 0),$$

and it is clear that the differential system E (resp. F) is completely integrable if and only

if so is $\pi_*^{-1}(E)$ (resp. $\pi_*^{-1}(F)$). Furthermore $K^0 = K_{-2}^0 + K_{-1}^0$ takes values in $H^{0,2}(\mathcal{O})$ by Proposition —. (This implies $K_{-2}^0 = 0$.) Therefore the theorem follows immediately from Proposition — and the next

Lemma —. (1) $d\omega_{-2} \equiv 0 \pmod{\omega_{-2}, \omega_{-1}^-}$,

$$d\omega_{-1}^- \equiv \frac{1}{2} K_{-1}^0(\omega_{-1}^+ \wedge \omega_{-1}^+) \pmod{\omega_{-2}, \omega_{-1}^-}.$$

$$(2) \quad d\omega_{-2} \equiv 0 \pmod{\omega_{-2}, \omega_{-1}^+},$$

$$d\omega_{-1}^+ \equiv \frac{1}{2} K_{-1}^0(\omega_{-1}^- \wedge \omega_{-1}^-) \pmod{\omega_{-2}, \omega_{-1}^+}.$$

This fact follows easily from equations (E-2) and (E-1), noting that $\frac{1}{2} [\omega_{-1}, \omega_{-1}] = [\omega_{-1}^+, \omega_{-1}^-]$.

Part I Chapter V

Part I Chapter V

Chapter V. The geometry of systems of ordinary differential equations of the second order

S.V.I.

Infinitesimal normal connections of type $\mathfrak{g}$

P is a manifold M on a fibered manifold, V is the system of vertical vectors of the fibered manifold. For $\mathfrak{g}^{(0)}$ of each $x \in M$, P on the vector field X^* that satisfies the conditions

1) For $X \rightarrow X^*$ is Lie algebra homomorphism, i.e., $[X^*, Y^*] = [X, Y]^*$, $X, Y \in \mathfrak{g}^{(0)}$

2) P of each $x \in M$, for $X \rightarrow X^*$ is $\mathfrak{g}^{(0)}$ from V_x of x to the isomorphism.

ω is P on $\mathfrak{g}$ -valued form. For the conditions of the weak Cartan connection of type $(\mathfrak{g}, \mathfrak{g}^{(0)})$

(IC.1) $X \in T(P)_x$ and $\omega(X) = 0 \Rightarrow X = 0$

(IC.2) $\omega(X^*) = X$, $X \in \mathfrak{g}^{(0)}$

(IC.3) $\mathcal{L}_{X^*} \omega = -[X, \omega]$, $X \in \mathfrak{g}^{(0)}$

Cartan connection of type $\mathfrak{g} \ltimes \mathfrak{g}^{(0)}$ is a weak Cartan connection of type $(\mathfrak{g}, \mathfrak{g}^{(0)})$

(P, ω) (resp. (P', ω')) is M (resp. M') on the weak Cartan connection of type $(\mathfrak{g}, \mathfrak{g}^{(0)})$ and $(z_0, z'_0) \in P \times P'$ and z_0 and z'_0 of the diffeomorphism γ that

$$\gamma^* \omega' = \omega, \quad \gamma(z_0) = z'_0$$

is satisfied, γ is local isomorphism

of (P, ω) to (P', ω') at (z_0, z'_0) とする.

14.5 の local isomorphism ψ は

$$\psi_*(X^*) = X'^*, \quad X \in \mathfrak{g}^{(0)}$$

を満足する (14.1)

Proposition 14.5 の weak Cartan connection of type $(\mathfrak{g}, \mathfrak{g}^{(0)})$ は 1 つの Cartan connection of type $G/G^{(0)}$ によって represent される. 即ち

(P, ω) は M 上の weak Cartan connection of type $(\mathfrak{g}, \mathfrak{g}^{(0)})$, $z_0 \in P$ とする. このとき 1 つの manifold M' 上で定義された Cartan connection of type $G/G^{(0)}$, (P', ω') と z_0 の nbd π^{-1} 上で定義された local isomorphism $\psi: (P, \omega) \rightarrow (P', \omega')$ が存在する.

この Prop. は the Lie algebra of vector fields, $\{X^* | X \in \mathfrak{g}^{(0)}\}$ on P と z_0 の nbd π^{-1} 上で定義されることを示すことによって容易に示される.

Proposition 14.12 (P, ω) (resp. (P', ω')) は M (resp. M') 上で定義された Cartan connection of type $G/G^{(0)}$ とする. ψ と $z_0 \in P$ の nbd π^{-1} 上で定義された local isomorphism $\psi: (P, \omega) \rightarrow (P', \omega')$ as weak Cartan connections とする.

このとき $z_0 = \pi(z_0)$ の nbd π^{-1} 上で定義された

local isomorphism $\phi: (P, \omega) \rightarrow (P', \omega')$ が

存在して, $\psi = \phi$ が z_0 の nbd π^{-1} 上で成立する.

再

No.

The spaces $V^{p,2}(K)$. Let us consider a class K of normal connection of type $\mathcal{G}$, (P, ω) , satisfying certain invariance conditions, for example, such as $K^0 = 0$ or $K^0 = K^1 = 0$, etc. Every connection (P, ω) in the class K satisfies the identities

$$\partial^* K^p = 0, \text{ and } \partial K^p = \Psi^{p-1}, \quad p \geq 0.$$

Now the function Ψ^{p-1} takes values in $C^{p-1,3}(\mathcal{G})$, and we define a subspace $V^{p-1,3}(K)$ of $C^{p-1,3}(\mathcal{G})$ by

$$V^{p-1,3}(K)^\perp = \{ L \in (C^{p-1,3}(\mathcal{G}))^* \mid L \Psi^{p-1} = 0 \text{ for any } (P, \omega) \in K \},$$

where $V^{p-1,3}(K)^\perp$ stands for the subspace of $(C^{p-1,3}(\mathcal{G}))^*$ orthogonal to $V^{p-1,3}(K)$. Clearly every $(P, \omega) \in K$ satisfies the equations

$$L \partial K^p = \partial^* K^p = 0, \quad p > 0, \quad L \in V^{p-1,3}(K)^\perp.$$

This being said, we define a subspace $V^{p,2}(K)$ of $C^{p,2}(\mathcal{G})$ by

$$V^{p,2}(K) = \{ c \in C^{p,2}(\mathcal{G}) \mid L \partial c = \partial^* c = 0 \text{ for any } L \in V^{p-1,3}(K)^\perp \},$$

which is nothing but the smallest subspace of $C^{p,2}(\mathcal{G})$ in which the function K^p corresponding to any $(P, \omega) \in K$ takes values.

Hence we notice that a considerable amount of informations for the determination of $V^{p-1,3}(K)$

the informations on
are obtainable from the spaces $VP^2(K)$, ZKP . We
also notice that $HP^2(\mathcal{G}) \subset VP^2(K)$. As we have
already seen in §IV.3, the determination of $HP^2(\mathcal{G})$
(at least for projective/graded Lie algebras) can be
easily carried out by the use of Proposition,
but we shall see in §V.3 that the works
towards the determination of $VP^2(K)$
require messy and complicated calculations.

4. Some remarks in the case where $\mathcal{G}$ is a simple graded Lie algebra of the first kind.

1.7" simple graded Lie algebra of the first kind は $\square$ にあつた irreducible $\mathfrak{l}$ -system を意味している. よつて $\square$ にあつた結果はこのような graded Lie algebras に對して適用しうる.

さて $\mathcal{G}$ を simple graded Lie algebra of the first kind とし, 付随する groups $G_0 = \text{Aut}(\mathcal{G})$, $G^{(0)} = G_0 \cdot G^{(1)}$, $G = \text{Aut}(\mathcal{G}) \cdot G_0$ とおくと, $G^{(0)}$ は G における $\mathfrak{g}^{(0)}$ の normalizer である (cf. 1.7).

(P, ω) は connection of type $G/G^{(0)}$ on M , K は K の curvature とする. K は $C^2(\mathcal{G}) = \mathfrak{g} \otimes \wedge^2 \mathfrak{g}_1^*$ の値を持つており, 分解 $C^2(\mathcal{G}) = \sum_{p=0}^2 C^{p,2}(\mathcal{G})$ に従つて

$$K = \sum_{p=0}^2 K^p$$

と分解される $C^{p,2}(\mathcal{G}) = \mathfrak{g}_{p-1} \otimes \wedge^2 \mathfrak{g}_1^*$ である. とは注意しよう. よつて K^0 は (P, ω) の torsion である. $(e_0, \dots, e_n)$ は $\mathfrak{g}_1$ の basis とすれば

$$\omega^* K^0 = \dots$$

$$\omega^* K^1 = \dots$$

τ'' あり, $\partial^* K^2 = 0$. よって (P, ω) が "normal connection of type $\mathcal{Q}$ " であることは (P, ω) が $[\]$ の意味で "normal connection τ'' がある" ことを意味していることがわかる.

$GL(\mathfrak{g}_1)$ の subgroup $G_0^\#$ は G_0 に一致しており, $G_0^\#$ -structure of type $\mathcal{M}$ は $\mathfrak{g}$ に G_0 -structure を意味している. よって Th. 5 の次の prop. がある.

Proposition. $\mathcal{Q}$ が $(\mathcal{M}, \mathfrak{g}_0)$ の prolongation τ'' があると仮定する, i.e., $H^0(\mathcal{Q}) = H^1(\mathcal{Q}) = 0$.

このとき

M 上の normal connections of type $\mathcal{Q}$

$\longleftrightarrow M$ 上の G_0 -structures

の結果は $\square$ にあいて既に得られている.

G の closed subgroup H は $H = \exp \mathfrak{g}_1$, G_0 に $\mathfrak{g}$ として定義する. H は $G^{(0)}$ の dual τ'' があり, H の Lie algebra $\mathfrak{h}$ は $\mathfrak{h} = \mathfrak{g}_1 + \mathfrak{g}_0$ τ'' によって与えられる.

$\mathcal{Q}$ は M 上の G_0 -structure, ξ は $\mathcal{Q}$ の basic form とする. X は $\mathcal{Q}$ 上の $\mathfrak{h}$ -valued 1-form とし, X_1 (resp. X_0) は X の $\mathfrak{g}_1$ (resp. $\mathfrak{g}_0$) component とする. $(\mathcal{Q}, X)$ が connection of type H/G_0 τ'' があるから $X_1 = \xi$ τ'' があると ξ , X は $\mathcal{Q}$ にあいて affine connection という.

X は $\mathcal{Q}$ にあいて affine connection, S は ξ の curvature とする: Prop 5 S は

$$dX + \frac{1}{2}[X, X] = \frac{1}{2}S(X_1 X_1)$$

2 満つる unique な function $Q \rightarrow \mathfrak{g} \otimes \wedge^2 \mathfrak{g}_1^*$ である
 分解 $\mathfrak{g} \otimes \wedge^2 \mathfrak{g}_1^* = \mathfrak{g}_1 \otimes \wedge^2 \mathfrak{g}_1^* + \mathfrak{g}_0 \otimes \wedge^2 \mathfrak{g}_1^*$ に従つて, S
 は $S = S_1 + S_0$ と分解される. S_1 は X の
 torsion とよぶ. (S_0 は普通 linear connection X_0
 の curvature とよぶ) である)

Proposition (cf. [7]). 次の条件を満足する
 function $T: Q \rightarrow \mathfrak{g} \otimes \wedge^2 \mathfrak{g}_1^*$ が unique に存在する.

1) $d^*T = 0$.

2) T は Q にあつる affine connection の
 torsion S_1 である.

T は Q の torsion とよぶ. これは Q の
 structure function (cf. [7]) の精細化である.

(P, ω) は M 上の normal connection of type G ,
 Q に対して G_0 -structure, $p: P \rightarrow Q$ は
 自然な homeomorphism とする. $T \in Q$ の torsion
 として, X は Q にあつる affine connection として,
 $T = S_1$ なるものとす.

Proposition (cf. [7]).

(1) $K^0 = p^*T$.

(2) $K^0 = 0$, 従つて $T = S_1 = 0$ と仮定する.

このとき, function $U: Q \rightarrow \mathfrak{g}_1$ が存在して,

$$K'(X \wedge Y) = S_0(X \wedge Y) - [U(X), Y] + [U(Y), X],$$

$$X, Y \in \mathfrak{g}_1,$$

すなわち S_0, U は P 上の functions とおき
 べしである.

connection of type $\mathcal{O}$ とする。このとき $\mathcal{K}$ は $\mathcal{K}(\mathcal{O})$ の制限と見做される, i.e., $E = E(\mathcal{O})$ on $R(\mathcal{O})$. 以下
 の証明から $(P, \omega)_{\mathcal{O}}$ は $(P(\mathcal{O}), \omega(\mathcal{O}))$ の
 R 上への制限 τ であることが
 §III.11 で TH の証明より
 TH と IV.8.1 の議論から大雑ばに

言って,

$\{$ projective systems of type $(n, D, \mathcal{O})$
 (or projective structures of type $(n, 1, \mathcal{O})$ with $HK^2 = 0$)
 $\iff \{$ projective structures in the
 usual sense, $\mathcal{O}$

がわかる。

TH の証明に入ろう

Lemma^{2.1} (P, ω) は M 上の infinitesimal
 normal connection of type $\mathcal{O}$ である
 (この connection $\mathcal{E}(P, \omega)_{\mathcal{O}}$ とかく)

以下 §III.2 の i.c. connections に関する
 結果を使う

$$\begin{array}{ccccc}
 \exists \mathcal{O} \in P & (P, \omega) & (P', \omega') & & (P, \omega)_{\mathcal{O}} \xrightarrow{\psi} (P', \omega')_{\mathcal{O}} \text{ as i.c.} \\
 \downarrow & & \downarrow & & (P, \omega)_{\mathcal{O}} \xrightarrow{\psi} (P', \omega')_{\mathcal{O}} \text{ as i.c.} \\
 P & R & G^*(M) & P' & (P, \omega)_{\mathcal{O}} \xrightarrow{\psi} (P', \omega')_{\mathcal{O}} \text{ as c.} \\
 \downarrow \iota & & \downarrow \iota & & \\
 M & & M' & & \\
 & & & & R \xrightarrow{\psi} G^*(M)
 \end{array}$$

2. $\mathcal{P} = \{P; E, F\}, \dots, (P, \omega)$ など"は
III.9.1 と同様とする

次の Prop. は §III.10 で"証明される
Proposition⁵²¹ $K'=0$ と仮定する このとき

K は $f^{(0)} \otimes \wedge^2 h_1^*$ の値を持つ.

この Prop. を使って 次の Th. を証明しよう 我々
任意の allowable B_0 -structure on N, Q , には N 上
の GP-system $\mathcal{Y}(Q) = \{R(Q); F(Q)\}$ が存在する = と
思い込んでおく.

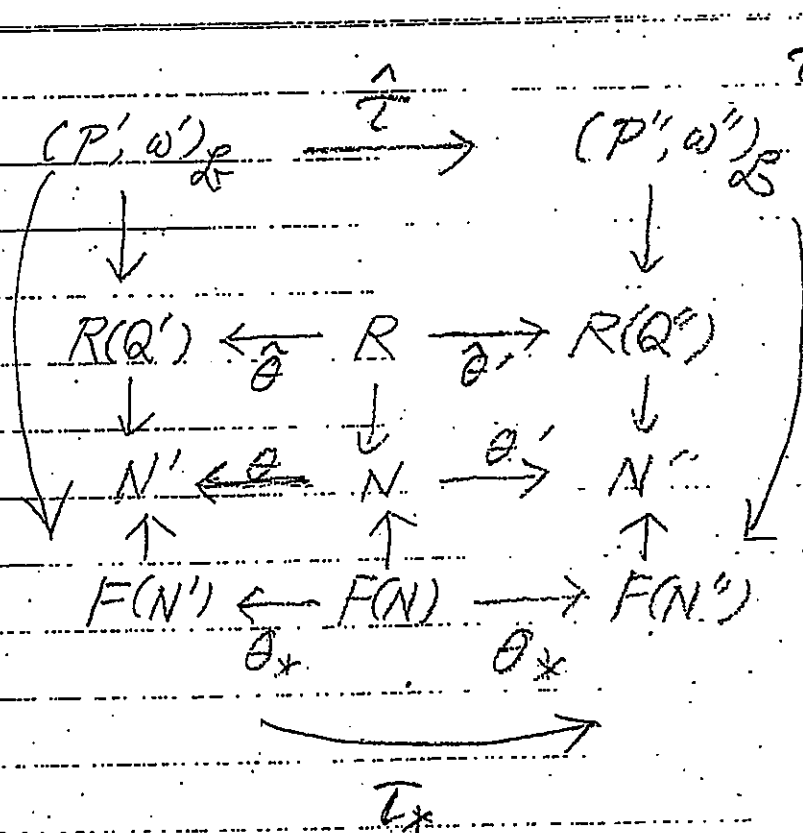
Theorem⁵²² $K'=0$ と仮定する このとき unique
な allowable B_0 -structure on N, Q , が"存在
して, $\mathcal{H}$ の dual $\mathcal{Y}$ は $\mathcal{Y}(Q)$ の $\mathcal{H}$ への制限として与え
れる $\exists p \in i) R \subset R(Q)$, 従って R は $R(Q)$ の open
submanifold であり, ii) E は $E(Q)$ の $R(Q)$
への制限である

$(P(Q), \omega(Q))$ は Q 上に存在する normal
connection of type $\mathcal{F}$ on $R(Q)$ とする このとき
(Th の証明) から $(P, \omega)_\mathcal{Y}$ は $(P(Q), \omega(Q))_\mathcal{Y}$ の $R(Q)$
への制限であることがわかる

Th. と III.8.2 の議論から, 大雑把は
12.17.2

$\{ \text{projective systems of type } (n, 1), \mathcal{P}, \dots \}$
 $\{ \text{or projection structures of type } (n, 1), \mathcal{X} \text{ with } K'=0 \}$
 $\longleftrightarrow_{1,1} \{ \text{allowable } B_0\text{-structures, } \mathcal{Q} \}$

がわかる.



$R \rightarrow M$ a fiber is connected
 $\Rightarrow Q$ is globally defined

Remark. $\mathcal{L}$ is projective FGLA of type (n, r) ,
 $\mathcal{Q}$ is its prolongation of type $\mathcal{F}$. $\mathcal{Q}$ is a simple
 graded Lie algebras $\mathcal{O}_r$, $\mathcal{L}$ is attached to it.
 $\mathcal{P} = \{R; E, F\}$ is projective system of type (n, r) ,
 $(\mathcal{P}, \omega)$ is attached to it normal connection of type $\mathcal{Q}$
 and type $\mathcal{F}$. $r = n-1$, $n \geq 3$ or $r = 1$.

$$K^0 = HK^2 = HK^3 = HK^4 = HK^5 = 0$$

There is. $HK^1 = K^1$ is a basic invariant and
 $H^{1,2}(\mathcal{Q}) \subset \mathcal{L}_0 \otimes \mathcal{L}_1^* \wedge \mathcal{L}_1^*$ of type $\mathcal{F}$. Also
 $2 \leq r \leq n-2$ or $r = 1$.

即ち curvature K
 が $\mathcal{L}_0 \wedge \mathcal{L}_1^*$ または
 $\mathcal{L}_0 \wedge \mathcal{L}_1^* \wedge \mathcal{L}_1^*$ の値を
 取れば,

$$HK^1 = HK^2 = HK^3 = HK^4 = HK^5 = 0$$

There is. HK^0 is a basic invariant and
 $V_{(3)}^{0,2}(n, n-r) \subset \mathcal{L}_1 \otimes \mathcal{L}_1^* \wedge \mathcal{L}_1^*$ of type $\mathcal{F}$.

So if $r = n-1$, $n \geq 3$ or $r = 1$ or $2 \leq r \leq n-2$ the case
 is, $(\mathcal{P}, \omega)$ is IC connection of type $\mathcal{O}_r$
 or of type $\mathcal{F}$ to reduce it and it is, it
 is then $K=0$ and it is $\mathcal{P}$ is a standard
 system $\mathcal{P}$ is locally isomorphic to it.
 It is Theorems 5.2.12, 5.2.22 of the form of Reduction Theorems
 is a natural meaning and it is not.

$n=2$, $r=1$ the case is, Cartan is by the
 Reduction theorems is obtained. (see Appendix)
 Reduction theorems is a independent variable
 is one of the three differential equations system
 is a special phenomenon and it is, $r \geq 3$ or $r=1$ the case

すべての常微分方程式系に対して成立するものと思われる。

3. Further properties of the curvatures of projection systems of type $(n, 1)$ with $K' = 0$.
 (P, ω) は E と同様とする.

$$K^3 = K_0^3 + K_1^3 + K_2^3 \quad K^4 = K_1^4 + K_2^4, \quad K^5 = K_2^5$$

7"あることを思い出してあ.こ.う §IV.4 2"

Prop. より詳しい次の Prop. を証明する
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Proposition $K' = 0$ と仮定する.

(1) $K^1 = K^0 = K' = 0$.

(2) K^2 は $V^{2,2}(n, 1)$ の値を持つ.

(3) i) $K_0^3 = K_2^3 = 0$, ii) K_1^3 は $\mathcal{G}_1^+ \otimes \mathcal{G}_2^* \wedge (\mathcal{G}_1^-)^*$ の値を持つ. iii) $\langle K_1^3([e_n, x] \wedge y), z \rangle$ は variables $x, y, z \in \mathcal{G}_1^-$ について symmetric 7"ある.

(4) i) $K_1^4 = 0$, ii) K_2^4 は $\mathcal{G}_2 \otimes \mathcal{G}_2^* \wedge (\mathcal{G}_1^-)^*$ の値を持つ. iii) $\langle K_2^4([e_n, x] \wedge y), [e_n, z] \rangle$ は variables $x, y, z \in \mathcal{G}_1^-$ について symmetric 7"ある.

(5) $K_2^5 = 0$.

(6) $\delta_{e_n} K_0^2 = 0$, $\delta_{e_n} K_1^3 = -[e_n, K_2^4]$, $\delta_{e_n} K_2^4 = 0$

§V.2 Reduction theorems

1. Reduction theorems, a)

$P = \{R, E, F\}$ is a projection system of type $(n, 1)$ and $\exists$ R is E and F of the two of the regular π and $\exists$ π $M = R/F, N = R/E$ and L , $\pi_M: R \rightarrow M, \pi_N: R \rightarrow N$ is projection and $\exists$ $\gamma_M: R \rightarrow G^*(\pi(M)), \gamma_N: R \rightarrow G^*(\pi(N))$ is canonical embeddings and $\exists$ $R \rightarrow M \times N, \gamma_M, \gamma_N$ is embeddings and $\exists$ π

$X = (R, E)$ は M 上の projective structure
of type $(n, 1)$ である, $Y = (R, F)$ は N 上の
GP system of type $(2n-2, n-1)$ である, X の
dual である $(P, \omega) \in \mathcal{P}$ に対して R 上の
normal connection of type G , $K \in$
その curvature である (この connection は (P, ω) の
で表わす)

次の Prop. は §III 10 で証明される。

K Proposition $HK^2=0$ と仮定する このとき
 K は $\alpha^{(0)} \otimes \wedge^2 \alpha_j^*$ の値を持つ.

この Prop. を使って次の Th. を証明しよう

Theorem $HK^2=0$ と仮定する. このとき

$\mathcal{K}$ is projective structure of the restricted $\mathcal{E}_3$.

(iv) Σ (PGL) of 12 $\tilde{F}_4$ $\tilde{E}_6$ $\tilde{F}_4$ projective structure, in the usual sense, $\mathcal{K}(\mathcal{Q}) = \{G' T(N), E(\mathcal{Q})\}$
 Σ $\mathcal{Q}$ 12 $\tilde{F}_4$ $\tilde{E}_6$ $\tilde{F}_4$ projective structure of the restricted type $\tilde{F}_4$ $\tilde{E}_6$ $\tilde{F}_4$. $\Sigma \subset 12$ (PGL), $\omega(\mathcal{Q})$
 Σ $\mathcal{K}(\mathcal{Q})$ 12 $\tilde{F}_4$ $\tilde{E}_6$ $\tilde{F}_4$ $G' T(N)$ $\pm$ of normal

connection of type $\mathcal{Q}$ とする。このとき $\mathcal{K}$ は $\mathcal{K}(\mathcal{Q})$ の制限と見て得られる, i.e., $E = E(\mathcal{Q})$ on R (§I.1.). 以下の証明から $(P, \omega)_{\mathcal{Q}}$ は $(P(\mathcal{Q}), \omega(\mathcal{Q}))_{\mathcal{Q}}$ の M 上への制限 τ'' であることがわかる (§III.11) τ'' $\mathcal{K}$ の制限と見ると $\mathcal{K}$ と §III.8.1 の議論から大雑ぱくに

言って,

$\{ \text{projective systems of type } (n, \mathcal{U}, \mathcal{P}) \text{ or projective structures of type } (n, \mathcal{U}, \mathcal{X}) \text{ with } HK=0 \}$
 $\longleftrightarrow \{ \text{projective structures in the usual sense, } \mathcal{Q} \}$

がわかる。

$\mathcal{K}$ の証明に使う

Lemma. (P, ω) は M 上の infinitesimal normal connection of type $\mathcal{Q}$ τ'' である (この connection τ $(P, \omega)_{\mathcal{Q}}$ とかく)

以下 §III.2 の i.c. connections に関する結果を使う

$$\begin{array}{ccc}
 \mathcal{Q} \in P & (P, \omega) & (P', \omega') \\
 \downarrow & \downarrow & \downarrow \\
 P & R & G^1T(M) \\
 \downarrow \mathcal{U} & \downarrow & \downarrow \\
 M & & M'
 \end{array}
 \quad
 \begin{array}{l}
 (P, \omega) \xrightarrow{\tau} (P', \omega') \text{ as i.c.} \\
 (P, \omega)_{\mathcal{Q}} \xrightarrow{\tau'} (P', \omega')_{\mathcal{Q}} \text{ as i.c.} \\
 (P, \omega)_{\mathcal{Q}} \xrightarrow{\tau} (P', \omega') \text{ as c.} \\
 R \xrightarrow{\varphi} G^1T(M)
 \end{array}$$

2. $\mathcal{P} = \{R; E, F\}, \dots, (P, \omega)$ などとは
 III.9.1 と同様とする

次の Prop. は §III.10 で証明せよ

Proposition. $K'=0$ と仮定する このとき
 K は $f^{(0)} \otimes \wedge^2 B_1^*$ の値を持つ.

この Prop. を使って 次の Th. を証明しよう 以下
 任意の allowable B_0 -structure on N, Q , 1) は N 上
 の GP-system $\mathcal{Y}(Q) = \{R(Q); F(Q)\}$ が存在する = 1
 と共に述べておく

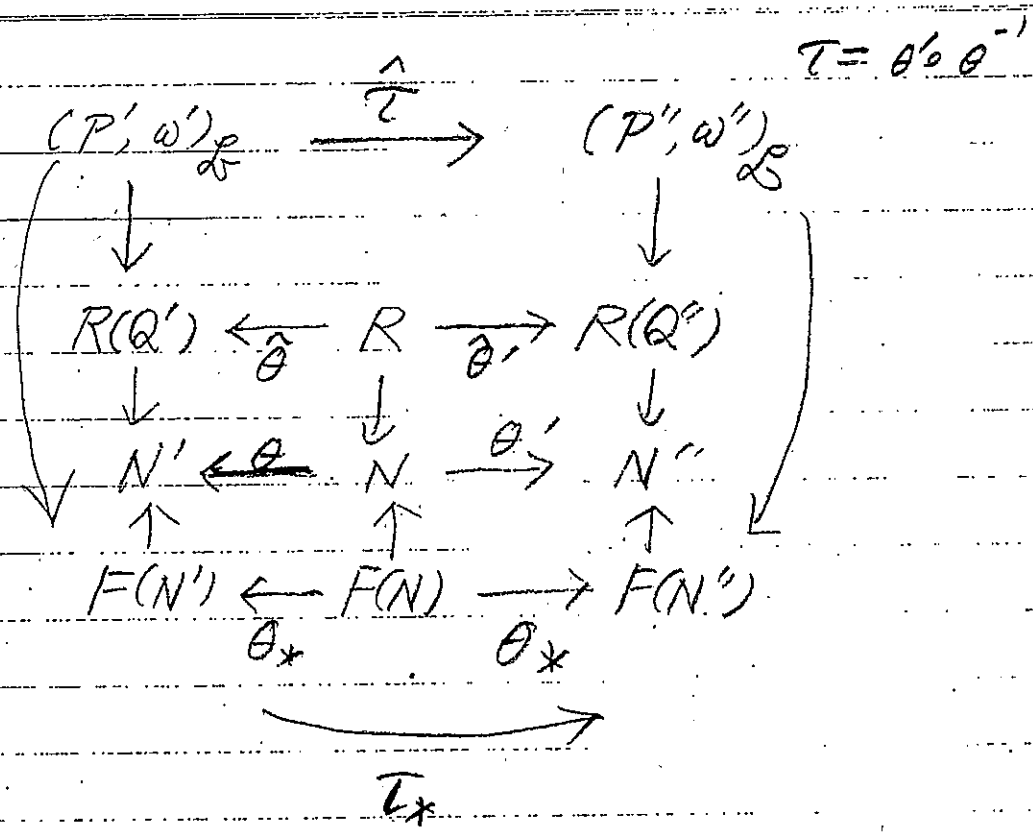
Theorem. $K'=0$ と仮定する このとき unique
 な allowable B_0 -structure on N, Q , が存在
 して, $\mathcal{Y}$ の dual $\mathcal{Y}$ は $\mathcal{Y}(Q)$ の R への制限として得ら
 れる 2) $P \in \mathcal{Y}$, i) $R \subset R(Q)$, 従って R は $R(Q)$ の open
 submanifold であり, ii) E は $E(Q)$ の $R(Q)$
 への制限である

$(P(Q), \omega(Q))$ は Q に存在する normal
 connection of type $\mathcal{F}$ on $R(Q)$ とする このとき
 (Th. の証明) から (P, ω) は $(P(Q), \omega(Q))$ の $R(Q)$
 への制限であることがわかる

Th. と III.8.2 の議論から, 大雑把に
 12.1.2

$\{ \text{projective systems of type } (n, 1), \mathcal{P}, i \}$
 $\leftrightarrow \{ \text{allowable } B_0\text{-structures, } Q \}$

わかる



$R \rightarrow M$ is fiberwise connected
 $\Leftrightarrow Q$ is globally defined

Remark. $\mathcal{L}$ is projective FGLA of type (n, r) ,
 $\mathcal{Q}$ is its prolongation and $\mathcal{Q}$ is a simple
 graded Lie algebras $\mathcal{O}_r$, $\mathcal{L}$ is attached to it.
 $\mathcal{P} = \{R; E, F\}$ is projective system of type (n, r) ,
 $(\mathcal{P}, \omega)$ is attached to it normal connection of type $\mathcal{Q}$
 and $r = n-1, n \geq 3$ is true.

$$K^0 = HK^2 = HK^3 = HK^4 = HK^5 = 0$$

There is, $HK^1 = K^1$ is a basic invariant and
 $H^{1,2}(\mathcal{Q}) \subset \mathcal{G}_0 \otimes \mathcal{G}_1^+ \wedge \mathcal{G}_1^-$ of the value. Also
 $2 \leq r \leq n-2$ is true.

即ち curvature K
 が $\mathcal{G}_0 \otimes \mathcal{G}_1^+ \wedge \mathcal{G}_1^-$ または
 $\mathcal{G}_0 \otimes \mathcal{G}_1^2$ の値を
 取れば,

$$HK^1 = HK^2 = HK^3 = HK^4 = HK^5 = 0$$

There is, HK^0 is a basic invariant and
 $V_{(3)}^{0,2}(n, n-r) \subset \mathcal{G}_1^- \otimes \mathcal{G}_1^+ \wedge \mathcal{G}_1^-$ of the value.

So $r = n-1, n \geq 3$ or $2 \leq r \leq n-2$ the case
 is, $(\mathcal{P}, \omega)$ is IC connection of type $\mathcal{O}_r$
 or of type $\mathcal{L}$ is reduced to be true, and
 then $K=0$ and follow $\mathcal{P}$ is a standard
 system $\mathcal{P}$ is locally isomorphic to be true.
 follow Theorems — the kind of Reduction Theorems
 is a meaningful meaning.

$m=2, r=1$ the case is, Cartan is by the
 Reduction theorems are obtained. (see Appendix)
 Reduction theorems is a independent number of
 1 individual and the differential equation system
 is a special phenomenon and $r \geq 3$ stage above

すべての常微分方程式系に対して成
立つものと思われる。

3. Further properties of the curvatures of projective systems of type $(n, 1)$ with $K' = 0$.
 (P, ω) は Γ と同様とする.

$$K^3 = K_0^3 + K_1^3 + K_2^3 \quad K^4 = K_1^4 + K_2^4, \quad K^5 = K_2^5$$

Γ であることを思い出しておこう. §IV. 4 Γ Prop. より詳しい次の Prop. を証明する
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Proposition $K' = 0$ と仮定する.

$$(1) K^1 = K^0 = K' = 0.$$

(2) K^2 は $V^{2,2}(n, 1)$ の値を持つ.

(3) i) $K_0^3 = K_2^3 = 0$, ii) K_1^3 は $\mathcal{G}_1^+ \otimes \mathcal{G}_2^* \wedge (\mathcal{G}_1^-)^*$ の値を持つ. iii) $\langle K_1^3([e_n, x] \wedge y), z \rangle$ は variables $x, y, z \in \mathcal{G}_1^-$ について symmetric Γ である.

(4) i) $K_1^4 = 0$, ii) K_2^4 は $\mathcal{G}_2 \otimes \mathcal{G}_2^* \wedge (\mathcal{G}_1^-)^*$ の値を持つ. iii) $\langle K_2^4([e_n, x] \wedge y), [e_n, z] \rangle$ は variables $x, y, z \in \mathcal{G}_1^-$ について symmetric Γ である.

$$(5) K_2^5 = 0.$$

$$(6) \delta_{e_n} K_0^2 = 0, \delta_{e_n} K_1^3 = -[e_n, K_2^4], \delta_{e_n} K_2^4 = 0$$

4. Examples

§V.3 Proofs of Propositions —, —, and —, No. —, 0

K_1 (resp. K_2 , resp. K_3) is normal connection of type $\mathcal{G}$, $(P, \omega) \rightarrow K^0$ (resp. $K^0 = HK^2 = 0$, resp. $K^0 = K^1 = 0$) is satisfied for the $\mathcal{G}$ class of $\mathcal{G}$. K_i for $i=1, 2$ $CP^2(\mathcal{G})$ of subspace $VP^2(K_i)$ is satisfied and is thought of (see §IV.1) in this way.

Propositions —, — (resp. Proposition —, resp. Proposition —) is $VP^2(K_1)$ (resp. $VP^2(K_2)$, resp. $VP^2(K_3)$) of the study of the study. For Proposition — is $VP^2(K_3)$ is much determined in this way.

1. The Bianchi identity (P, ω) は
 と同様とする. (P, ω) に関する Bianchi
 identity は 次の identities に分解する;

$$\partial K^P = \Psi^{P-1}$$

(—— を見よ) Ψ^{P-1} は $K^0, \dots, K^{P-1}$ だけに
 depend する量である. $K^0 = 0$ であり, $K^2 = HK^1$
 が $V^{1,2}(n, 1)$ の値を持つことを考慮して Ψ^{P-1} を
 計算して次の Lemma をえる

Lemma $u, v, w \in \mathcal{G}_2$, $x, y, z \in \mathcal{G}_1$ とする

$$\begin{aligned} (1) \quad (\partial K^P)(u \wedge v \wedge z) &= -\delta_u K^{P-2}(v \wedge z) - \delta_v K^{P-2}(z \wedge u) \\ &\quad - \delta_z K^{P-1}(u \wedge v) - \underbrace{K^{P-3}(K^2(u \wedge v) \wedge z)} \\ &\quad - K^{P-2}(K^1(v \wedge z) \wedge u) - K^{P-2}(K^1(z \wedge u) \wedge v) \end{aligned}$$

$$\begin{aligned} (2) \quad (\partial K^P)(u \wedge x \wedge y) &= -\delta_u K^{P-2}(x \wedge y) - \delta_x K^{P-1}(y \wedge u) \\ &\quad - \delta_y K^{P-1}(u \wedge x) - \underbrace{K^{P-2}(K^1(u \wedge x) \wedge y)} \\ &\quad - \underbrace{K^{P-2}(K^1(y \wedge u) \wedge x)} \end{aligned}$$

$$\begin{aligned} (3) \quad (\partial K^P)(u \wedge v \wedge w) &= -\delta_u K^{P-2}(v \wedge w) - \delta_v K^{P-2}(w \wedge u) - \delta_w K^{P-2}(u \wedge v) \\ &\quad - K^{P-3}(K^2(u \wedge v) \wedge w) - K^{P-3}(K^2(v \wedge w) \wedge u) - K^{P-3}(K^2(w \wedge u) \wedge v) \end{aligned}$$

$$(3) (2K^p)(x \wedge y \wedge z) = - \delta_x K^{p-1}(y \wedge z) - \delta_y K^{p-1}(z \wedge x) \\ - \delta_z K^{p-1}(x \wedge y)$$

K^p は $K^p = K_{p-3}^p + K_{p-2}^p + K_{p-1}^p$ と分解する

At the proofs of the propositions from now on, the invariant $K^p, K_{p-3}^p, K_{p-2}^p$ and K_{p-1}^p will be simply denoted by C, C_{p-3}, C_{p-2} and C_{p-1} respectively. $\neq p$

例 2.18 $p=2$ のとき $K_1^2 = C_1$ は $C_1 = C_1^+ + C_1^-$ と分解する (cf. §III.5).

2. Proof of Proposition —. この paragraph

τ'' は invariant $K^2 = C = C_{-1} + C_0 + C_1$ を 2.3

Lemma (1) $(\partial C)(u \wedge v \wedge x) = 0$

(2) $(\partial C)(u \wedge x \wedge y) = 0$ から $(\partial C)(u \wedge x \wedge e_n)$ は δ_1 の値を持つ.

(3) $(\partial C)(x \wedge y \wedge e_n) = 0$

(4) $(\partial_2^* \partial C)(x \wedge e_n) = 0$

Proof Lemma と $K^0 = 0$ と u'' $K' = HK'$ から $V^{1,2}(n,1)$ の値を持つことから, 次の等式を 2.3.

$$(\partial K^2)(u \wedge v \wedge x) = 0$$

$$(\partial K^2)(u \wedge x \wedge y) = 0,$$

$$(\partial K^2)(u \wedge x \wedge e_n) = -\delta_x K'(e_n \wedge u).$$

$$(\partial K^2)(x \wedge y \wedge e_n) = 0$$

2.3 12 $(\partial_2^* K'_{-1}(e_n) = 0)$ より Lemma 2.3

Lemma $c_1^+(x \wedge y) = 0$

Proof $(\partial c)(u \wedge x \wedge y) = 0$ より

$$(\partial_0 c_{-1} + \partial_1 c_0 + \partial_2 c_1)(u \wedge x \wedge y)$$

$$= -[x, c_0(u \wedge y)] + [y, c_0(u \wedge x)] + [u, c_1(x \wedge y)] = 0$$

§ 2.3. これより

$$-[x, c_0(u \wedge y)] + [y, c_0(u \wedge x)] + [u, c_1^-(x \wedge y)] = 0$$

及 u''

$$[u, c_1^+(x \wedge y)] = 0$$

§ 2.3. より Lemma より $c_1^+(x \wedge y) = 0$ が従う。

Lemma (1) $\sum \lambda_i^+ C_{-1}^+(u \wedge v) - [u, a(v)]$

$$- \sum [e^{ni}, [u, C_0(v \wedge e_{ni})]] + [v, C_1^+([e^n u] \wedge e_n)] = 0.$$

$$\Rightarrow \tau'' a(v) = \sum [e^{ni}, C_0(e_{n+1} \wedge v)].$$

$$(2) (\partial_2^* C_{-1}^+)(u) + a(u) + C_1^+([e^n u] \wedge e_n) = 0.$$

$$(3) \partial_1^* \partial_1 C_{-1}^+ + \partial_2 \partial_2^* C_{-1}^+ + 4 \lambda_1^+ C_{-1}^+ = 0.$$

$$(4) C_{-1}^+ = 0.$$

Proof. (1) $(\partial C)(v \wedge x \wedge e_n)$

$$= (\partial_0 C_{-1} + \partial_1 C_0 + \partial_2 C_1)(v \wedge x \wedge e_n)$$

$$= -C_1([x, e_n] \wedge v) - [x, C_0(v \wedge e_n)] + [e_n, C_0(v \wedge x)]$$

$$+ [v, C_1(x \wedge e_n)].$$

$(\partial C)(v \wedge x \wedge e_n)$ は $\mathcal{L}_1^-$ の値を持つから、

$$-C_1^+([x, e_n] \wedge v) + [e_n, C_0(v \wedge x)] + [v, C_1^+(x \wedge e_n)] = 0$$

故に

$$-C_1^+([e^n u, e_n] \wedge v) + [e_n, C_0(v \wedge [e^n u])] + [v, C_1^+([e^n u] \wedge e_n)] = 0$$

$$+ [v, C_1^+([e^n u] \wedge e_n)] = 0$$

$$\S 2.3 \quad [e^n u, e_n] = [E_1^+, u] = -2\lambda_1^+ u \quad \tau'' \text{ に対して}$$

$$[e^n, u] = \sum_i \langle e^n, [u, e^{n+i}] \rangle e_{n+i} \quad \text{したがって}$$

$$\begin{aligned} [e^n, C_0(v \wedge [e^n, u])] &= \sum_i [u, e^{n+i}, C_0(v \wedge e_{n+i})] \\ &= -[u, a(v)] - \sum_i [e^{n+i}, [u, C_0(v \wedge e_{n+i})]] \end{aligned}$$

故に (1) は成り立つ。

(2) $\partial^* C = 0$ より、

$$\begin{aligned} (\partial_2^* C_1 + \partial_1^* C_0 + \partial_0^* C_1)(u) \\ &= \sum_i [e^i, C_1(e_i \wedge u)] + [e^n, C_0(e_n \wedge u)] \\ &\quad + \sum_i [e^{n+i}, C_0(e_{n+i} \wedge u)] + C_1([e^n, u] \wedge e_n) \end{aligned}$$

したがって $C_1([e^n, u] \wedge e_n) = \sum_i C_1([e^{n+i}, u] \wedge e_{n+i})$ である。故に

$$\begin{aligned} \sum_i [e^i, C_1^+(e_i \wedge u)] + \sum_i [e^{n+i}, C_0(e_{n+i} \wedge u)] \\ + C_1^+([e^n, u] \wedge e_n) = 0 \end{aligned}$$

したがって (2) が示される。

(3) (1) と (2) から

$$\begin{aligned} \sum_i [e^{n+i}, [u, C_0(v \wedge e_{n+i})]] \\ &= 2\lambda_1^+ C_1^+(u \wedge v) - [u, a(v)] - [v, a(u)] - [v, (\partial_2^* C_1^+)(u)] \end{aligned}$$

2.2.3. 7. $(\partial C)(u \wedge v \wedge x) = 0$ P_2'' か $\bar{5}$,

$$(\partial_1 C_{-1} + \partial_2 C_0)(u \wedge v \wedge x)$$

$$= [x, C_1^+(u \wedge v)] + [u, C_0(v \wedge x)] - [v, C_0(u \wedge x)] = 0$$

これより

$$(\partial_1^* \partial_1 C_1^+)(u \wedge v) + \sum_i [e^{n+i}, [u, C_0(v \wedge e_{n+i})]] \\ - \sum_i [e^{n+i}, [v, C_0(u \wedge e_{n+i})]] = 0$$

2.2.3. 以上より (3) 2.2.3.

(4) $\lambda_1^+ > 0$ P_2'' か $\bar{5}$, (3) か $\bar{5}$ $C_1^+ = 0$ が従う.

Lemma (1) $-\sum_i [e^{n+i}, c_0(e_i \wedge x)]$

$$-\sum_i [e_n, [e^{n+i}, c_1^-(e_{n+i} \wedge x)]] + (\lambda_1^+ + \mu_1^+ - \lambda_2 - \mu_2) c_1^+(e_n \wedge x) = 0.$$

(2) $\sum_i [e^{n+i}, c_0(e_i \wedge x)] + (\lambda_1^+ + \mu_1^+) c_1^+(e_n \wedge x)$

$$-(\partial_1^* \partial_1 (e_n \lrcorner c_1^+))(x) + \sum_i [e_n, [e^{n+i}, c_1^-(e_{n+i} \wedge x)]] = 0.$$

(3) $\partial_1^* \partial_1 (e_n \lrcorner c_1^+) + (\lambda_2 + \mu_2 - 2\lambda_1^+ - 2\mu_1^+) e_n \lrcorner c_1^+ = 0$

(4) $c_1^+(e_n \wedge x) = 0.$

Proof (1) $c_1^+ = 0$ $\tau'' \nexists \pi, \delta$

$$[e_n, c_0(u \wedge x)] + [u, c_1^+(x \wedge e_n)] = 0$$

$\tau'' \nexists \pi$ (Proof of Lemma, (1)). $\exists \tau \geq 2$

$$\sum_i [[e^i, e_n], c_0(e_i \wedge x)] + \sum_i [e_n, [e^i, c_0(e_i \wedge x)]]$$

$$+ \partial_2^* \partial_2 (c_1^+(x \wedge e_n)) = 0$$

$\tau'' \nexists \pi$ Lemma 1: $\exists \tau$

$$\partial_2^* \partial_2 (c_1^+(x \wedge e_n)) = [E_2, c_1^+(x \wedge e_n)] = (\lambda_2 + \mu_2) c_1^+(x \wedge e_n)$$

$\tau'' \nexists \pi$ $\exists \tau_2$ $\partial^* c = 0$ $\tau'' \nexists \pi, \delta$

$$(\partial_2^* c_0 + \partial_1^* c_1)(x)$$

$$= \sum_i [e_i^i, c_0(e_i \wedge x)] + [e^n, c_1^+(e_n \wedge x)] \\ + \sum_i [e^{n+i}, c_1^-(e_{n+i} \wedge x)] = 0.$$

$\hookrightarrow 2$

$$\sum_i [[e_i^i, e_n], c_0(e_i \wedge x)] - [e_n, [e^n, c_1^+(e_n \wedge x)]] \\ - \sum_i [e_n, [e^{n+i}, c_1^-(e_{n+i} \wedge x)]] + (\lambda_2 + \mu_2) c_1^+(x \wedge e_n) = 0$$

$\exists \exists 3$. Lemma 12 より $[e_i^i, e_n] = -e^{n+i} \tau^i \mathcal{H}$,

$$[e_n, [e^n, c_1^+(e_n \wedge x)]] = -[e_i^i, c_1^+(e_n \wedge x)] \\ = -(\lambda_1^+ + \mu_1^+) c_1^+(e_n \wedge x)$$

$\tau^i \mathcal{H} \exists \mathcal{H} \exists$, (1) $\exists \exists 3$

(2) $(\partial c)(x \wedge y \wedge e_n) = 0 \quad \mathcal{H} \exists$

$$(\partial_0 c_0 + \partial_1 c_1)(x \wedge y \wedge e_n)$$

$$= c_0([x, e_n] \wedge y) - c_0([y, e_n] \wedge x)$$

$$+ [x, c_1^+(y \wedge e_n)] - [y, c_1^+(x \wedge e_n)] + [e_n, c_1^-(x \wedge y)] = 0$$

$\exists \exists 3$. $\hookrightarrow 2$

$$\sum_i [e^{n+i}, c_0([e_{n+i}, e_n] \wedge y)] - \sum_i [e^{n+i}, c_0([y \wedge e_n] \wedge e_{n+i})]$$

$$-(2_1^* 2_1(e_n \perp c_1^+))(y) + \sum_i [e^{n+i}, [e_n, c_1^-(e_{n+i} \wedge y)]] = 0$$

ここで Lemma 1 より $[e^n, [y, e_n]] = [y, E_1^+]$
 $= (\lambda_1^+ + \mu_1^+) y$ であるから Lemma (2) より

$$\sum_i [e^{n+i}, c_0(e_{n+i} \wedge [y, e_n])] + (\lambda_1^+ + \mu_1^+) c_1^+(y \wedge e_n) = 0$$

ここで以上から (2) が従う。

(13) (1) と (2) から

$$-(2_1^* 2_1(e_n \perp c_1^+))(x) + (\lambda_2 + \mu_2 - 2\lambda_1^+ - 2\mu_1^+)(e_n \perp c_1^+)(x) = 0$$

が従う。この式は $x = e_n$ のときにも成り立つ。

よって (3) である。

$$(4) \quad \lambda_2 + \mu_2 - 2\lambda_1^+ - 2\mu_1^+ > 0 \quad \text{であるから}$$

(3) から $e_n \perp c_1^+ = 0$, i.e., $c_1^+(e_n \wedge x) = 0$ である。

11

$a \in A_1^{0,2} \ni a(x \wedge y) = q^-(x \wedge y), a(x \wedge e_n) = 0$ 12 $\hookrightarrow 2$ No.
定義 3.3.

Lemma (1) $\partial_1 \partial_1^* q^- + (\lambda_2 - \mu_2) q^- = 0$

(2) $q^-(x \wedge y) = 0$.

Proof. (1) $[e_n, C_0(u \wedge x)] + [u, q^+(x \wedge e_n)] = 0$
 $\tau'' \nexists$ (Proof of Lemma, (1).) $q^+(x \wedge e_n) = 0$ $\tau'' \nexists$
 $\pi' \nexists$ $[e_n, C_0(u \wedge x)] = 0$ $\tau'' \nexists$ $\hookrightarrow 2$ $[e'', C_0(u \wedge x)] = 0$.
 $\S 2$

$$- [x, C_0(u \wedge y)] + [y, C_0(u \wedge x)] + [u, q^-(x \wedge y)] = 0$$

$\tau'' \nexists$ (Proof of Lemma). $\hookrightarrow 2$

$$- \sum_i [x, [e^i, C_0(e_i \wedge y)]] + \sum_i [y, [e^i, C_0(e_i \wedge x)]]$$

$$+ \partial_2^* \partial_2 (q^-(x \wedge y)) = 0$$

$\S 3$. ($[e^i, x] \in \mathcal{G}_1^-$ $\pi'' \nexists$, 上の $i \partial_2^* \partial_2$ $\hookrightarrow 2$
 $[e^i, x], C_0(e_i \wedge y) = 0$ $\tau'' \nexists$.) Lemma $\pi' \nexists$

$$\partial_2^* \partial_2 (q^-(x \wedge y)) = [E_2, q^-(x \wedge y)] = (\lambda_2 - \mu_2) q^-(x \wedge y)$$

$\tau'' \nexists$, $C_1^+(e_n \wedge x) = 0$ $\pi_2'' \nexists$.

$$\sum_i [e^i, C_0(e_i \wedge x)] + \sum_i [e^{n+i}, C_1^-(e_{n+i} \wedge x)] = 0.$$

(Proof of Lemma). $\hookrightarrow 2$

$$\sum_i [x, [e^{n+i}, c_1^-(e^{n+i} \wedge y)]] - \sum_i [y, [e^{n+i}, c_1^-(e^{n+i} \wedge x)]] \\ + (\lambda_2 - \mu_2) c_1^-(x \wedge y) = 0$$

2.2.3. これは明らか. (1) 3 意味する.

(2) $\lambda_2 - \mu_2 > 0$ 仮定から, (1) から $q=0$, i.e., $c_1^-(x \wedge y) = 0$ 2.2.3.

2.2.

(1) 3

2.2.

$q = K_0^2$ とおけば, Lemma と Prop. (1), (2) から容易に q 2.2.3.

$$\text{Lemma } \partial_2 q = \partial_1 q = \partial_0 q = \partial_1^* q = \partial_2^* q = 0$$

これは $\partial q = \partial^* q = 0$ 3 意味する

Lemma (1) $(\partial_2^* C_1)([x, e_n]) - [x, (\partial_2^* C_0)(e_n)]$

$$- [e^n, C_0([x, e_n] \wedge e_n)] + (\lambda_2 - \mu_2) C_1(x \wedge e_n) = 0.$$

$$(2) (\lambda_2 - \mu_2 - \lambda_1^+ - \mu_1^+) C_1(x \wedge e_n) + [x, (\partial_1^* C_1)(e_n)] = 0$$

$$(3) C_1(x \wedge e_n) = 0.$$

Proof (1) $(\partial C)(u \wedge x \wedge e_n)$

$$= -C_1([x, e_n] \wedge u) - [x, C_0(u \wedge e_n)] + [u, C_1(x \wedge e_n)]$$

2"ある. $\therefore \exists 2'' [e_n, C_0(u \wedge x)] = 0$ $\exists$ $i \in \mathbb{N}$ $\exists 2' (C_0(u \wedge x))$ は 1_0 の値を持つ. $\therefore \exists 2$ Lemma 5)

$$(\partial_2^* \partial C)(x \wedge e_n) = (\partial_2^* C_1)([x, e_n])$$

$$- \sum_i [e^i, [x, C_0(e_i \wedge e_n)]] + \partial_2^* \partial_2(C_1(x \wedge e_n)) = 0$$

$\exists \exists 3$. $\exists 3''$ Lemma 12 5)

$$\partial_2^* \partial_2(C_1(x \wedge e_n)) = [E_2, C_1(x \wedge e_n)] = (\lambda_2 - \mu_2) C_1(x \wedge e_n)$$

$\exists \exists 3$. (Lemma 12 5) $C_1(x \wedge e_n)$ は $\mathfrak{g}_1^-$ の値を持つ)
 $\therefore \exists 12 [e^i, x] = \langle e^i, [x, e_n] \rangle e^n$ $\exists$ 使って

$$\sum_i [e^i, [x, C_0(e_i \wedge e_n)]]$$

$$= [x, (\partial_2^* C_0)(e_n)] + \sum_i [[e^i, x], C_0(e_i \wedge e_n)]$$

$$= [x, (\partial_2^* C_0)(e_n)] + [e^n, C_0([x, e_n] \wedge e_n)]$$

∴ 3. 以上により (1) が示された.

(2) $\partial^* C = 0$ より

$$(\partial_2^* C_{-1} + \partial_1^* C_0 + \partial_0^* C_1)(u) = 0,$$

$$(\partial_2^* C_0 + \partial_1^* C_1)(e_n) = 0$$

∴ 3. $\sum_{i=1}^n [e^{n+i}, C_0(e_{n+i} \wedge u)] = 0$ (Lemma) $\therefore$ 1. 5

$$(\partial_1^* C_0)(u) = [e^n, C_0(e_n \wedge u)]$$

∴ 3. $\therefore$ 2 (1) から.

$$-(\partial_0^* C_1)([x, e_n]) + [x, (\partial_1^* C_1)(e_n)] + (\lambda_2 - \mu_2) C_1(x \wedge e_n) = 0$$

∴ 3. $[e^n, [x, e_n]] = [x, E_1^+] = (\lambda_1^+ + \mu_1^+) x$ $\therefore$ 1. 5,

$$(\partial_0^* C_1)([x, e_n]) = C_1([e^n, [x, e_n]] \wedge e_n)$$

$$= (\lambda_1^+ + \mu_1^+) C_1(x \wedge e_n)$$

∴ 3. 以上より $\therefore$ 2 (2) が示された

(3) (2) から

$$(\lambda_2 - \mu_2 - \lambda_1^+ - \mu_1^+) (\partial_1^* C_1)(e_n) + \sum_i [e^{n+i}, [e_{n+i}, (\partial_1^* C_1)(e_n)]] =$$

∴ 3. $(\lambda_2 - \mu_2 - \lambda_1^+ - \mu_1^+) > 0$ $\therefore$ 1. 5, $\therefore$ 2. 5

直ちに $(\partial_1^* C_1)(e_n) = 0$ が従い $\therefore$ 2 $C_1(x \wedge e_n) = 0$ $\therefore$ 1. 5

3. Proof of Proposition. — この paragraph 2" は invariant $K^3 = C = C_0 + C_1 + C_2$ である。

Lemma. (1) $(\partial C)(x \wedge y \wedge e_n) = 0$

(2) $(\partial_2^* \partial C)(x \wedge y) = 0$

(3) $(\partial_2^* \partial C)(x \wedge e_n) = 0$

Proof. Lemma と 今迄の結果から次の等式を得る

$$(\partial K^3)(u \wedge x \wedge y) = -\delta_x K^2(y \wedge u) - \delta_y K^2(u \wedge x)$$

$$(\partial K^3)(u \wedge x \wedge e_n) = -\delta_x K^2(e_n \wedge u) - \delta_{e_n} K^2(u \wedge x)$$

$$(\partial K^3)(x \wedge y \wedge e_n) = 0$$

である。さらに $(\partial_2^* K_0^2)(x) = 0$ である、 $K_1^2 = 0$ であるから $(\partial_2^* K_0^2)(e_n) = 0$ である。よって Lemma である。

Lemma (1) $-[e^n, q^+([x, e_n] \wedge y)] + [e^n, q^+([y, e_n] \wedge x)]$
 $+ 2\lambda_2 C_2(x \wedge y) = 0$

(2) $q^+([x, e_n] \wedge y) - q^+([y, e_n] \wedge x) + [e_n, C_2(x \wedge y)] = 0$

(3) $C_2(x \wedge y) = 0$

Proof (1) $(\partial C)(u \wedge x \wedge y) = (\partial_0 C + \partial_1 C + \partial_2 C)(u \wedge x \wedge y)$

$= -[x, q^+(u \wedge y)] + [y, q^+(u \wedge x)] + [u, C_2(x \wedge y)]$

τ'' あり. $(\partial_2^* \partial C)(x \wedge y) = 0$ τ_2'' あり.

$-\sum_i [e^i x, q^+(e_i \wedge y)] + \sum_i [e^i y, q^+(e_i \wedge x)]$

$+ \partial_2^* \partial_2 (C_2(x \wedge y)) = 0$

ξ あり. Lemma 12 より

$\partial_2^* \partial_2 (C_2(x \wedge y)) = [E_2, C_2(x \wedge y)] = 2\lambda_2 C_2(x \wedge y)$

τ'' あり, $[e^i, x] = \langle e^i, [x, e_n] \rangle e^n$ τ_2'' あり (1) ξ あり.

(2) $(\partial C)(x \wedge y \wedge e_n) = 0$ τ_2'' あり.

$(\partial_0 C + \partial_1 C)(x \wedge y \wedge e_n)$

$= q^+([x, e_n] \wedge y) - q^+([y, e_n] \wedge x) + [x, C_2(y \wedge e_n)]$

$- [y, C_2(x \wedge e_n)] + [e_n, C_2(x \wedge y)] = 0$

2 2 3 2 2

$$g^+([x, e_n] \wedge y) - g^+([y, e_n] \wedge x) + [e_n, C_2(x \wedge y)] = 0,$$

i.e., (2) 2 2 3

(3) (1) & (2) から

$$[e'', [e_n, C_2(x \wedge y)]] + 2\lambda_2 C_2(x \wedge y) = 0$$

2 2 3

$$[e'', [e_n, C_2(x \wedge y)]] = [E_1^+, C_2(x \wedge y)] = 2\lambda_1^+ C_2(x \wedge y)$$

f_2'' から

$$(\lambda_2 + \lambda_1^+) C_2(x \wedge y) = 0.$$

$$\lambda_2 + \lambda_1^+ > 0 \quad f_2'' \text{ から } C_2(x \wedge y) = 0 \quad 2 \quad 2 \quad 3$$

Lemma

$$(1) - \sum_i [e^{n+i}, C_1(e_{n+i} \wedge [x, e_n])] - \sum_i [e^{n+i}, C_1([e_{n+i}, e_n] \wedge x)] + (2\lambda_2 - \lambda_1^+ - \mu_1^+) C_2(x \wedge e_n) = 0$$

$$(2) \sum_i [e^{n+i}, C_1([e_{n+i}, e_n] \wedge x)] - \sum_i [e^{n+i}, C_1([x, e_n] \wedge e_{n+i})] + (\partial_1^* \partial_1 C_2)(x \wedge e_n) = 0$$

$$(3) (2\lambda_2 - \lambda_1^+ - \mu_1^+)(e_n \lrcorner C_2) + \partial_1^* \partial_1 (e_n \lrcorner C_2) = 0$$

$$(4) C_2(x \wedge e_n) = 0.$$

Proof. (1) $(\partial C)(u \wedge x \wedge e_n)$

$$= (\partial_0 C_0 + \partial_1 C_1 + \partial_2 C_2)(u \wedge x \wedge e_n)$$

$$= -C_0([x, e_n] \wedge u) - [x, C_1(u \wedge e_n)] + [e_n, C_1(u \wedge x)]$$

$$+ [u, C_2(x \wedge e_n)]$$

$$7'' \text{ 及 } 3. (\partial_2^* \partial C)(x \wedge e_n) = 0 \quad 7_2'' \text{ 及 } 5$$

$$(\partial_2^* C_0)([x, e_n]) - \sum_i [e_i^*, x], C_1(e_i \wedge e_n)]$$

$$+ \sum [e_i^*, e_n], C_1(e_i \wedge x)] + \partial_2^* \partial_2 (C_2(x \wedge e_n)) = 0$$

Lemma 1: $\lambda \neq 2$

$$\partial_2^* \partial_2 (C_2(x \wedge e_n)) = [E_2, C_2(x \wedge e_n)] = 2\lambda_2 C_2(x \wedge e_n).$$

$$\neq p_2 \quad [e^i, x] = \langle e^i, [x, e_n] \rangle e^n, \quad [e^i, e_n] = -e^{n+i} \quad p_2 \text{ 的 } \S,$$

$$\sum_i [[e^i, x], C_1(e_n \wedge e_n)] = [e^n, C_1([x, e_n] \wedge e_n)]$$

$$\sum_i [[e^i, e_n], C_1(e_n \wedge x)] = -\sum_i [e^{n+i}, C_1([e_{n+i}, e_n] \wedge x)]$$

p_2 的 $\S$

$$(\partial_2^* C_0)([x, e_n]) = [e^n, C_1([x, e_n] \wedge e_n)]$$

$$-\sum_i [e^{n+i}, C_1([e_{n+i}, e_n] \wedge x)] + 2\lambda_2 C_2(x \wedge e_n) = 0$$

$$\S 2.3 \quad \S 5.12 \quad \partial_2^* C = 0 \quad \S 3$$

$$(\partial_2^* C_0)([x, e_n]) + (\partial_1^* C_1)([x, e_n]) + (\partial_0^* C_2)([x, e_n]) = 0$$

$\S 2.3$,

$$(\partial_1^* C_1)([x, e_n]) = [e^n, C_1(e_n \wedge [x, e_n])]$$

$$+ \sum_i [e^{n+i}, C_1(e_{n+i} \wedge [x, e_n])]$$

$$(\partial_0^* C_2)([x, e_n]) = C_2([e^n, [x, e_n]] \wedge e_n) = C_2([x, E_1^+] \wedge e_n)$$

$$= (\lambda_1^+ + \mu_1^+) C_2(x \wedge e_n)$$

これらの事実から (1) $\exists \exists 3$.

$$(2) \quad (2C)(x \wedge y \wedge e_n) = 0 \quad \text{より}$$

$$(2_0C + 2_1C_2)(x \wedge y \wedge e_n)$$

$$= C([x, e_n] \wedge y) - C([y, e_n] \wedge x) + (2_1C_2)(x \wedge y \wedge e_n) = 0$$

$\exists \exists 3$ より

$$\sum_i [e^{n+i}, C_1([e_{n+i}, e_n] \wedge y)] - \sum_i [e^{n+i}, C([y, e_n] \wedge e_{n+i})]$$

$$+ (2_1^* 2_1 C_2)(y \wedge e_n) = 0$$

$\exists \exists 3$ より (2) が $\exists \exists 3$ である

(3) (1) と (2) から

$$(2\lambda_2 - \lambda_1^+ - \mu_1^+)C_2(x \wedge e_n) + (2_1^* 2_1 C_2)(x \wedge e_n) = 0$$

$\exists \exists 3$ $C_2(x \wedge y) = 0$ より, これは

$$(2\lambda_2 - \lambda_1^+ - \mu_1^+)(e_n \perp C_2) + 2_1^* 2_1(e_n \perp C_2) = 0$$

を意味する $2\lambda_2 - \lambda_1^+ - \mu_1^+ > 0$ より

$e_n \perp C_2 = 0$, i.e., $C_2(x \wedge e_n) = 0 \quad \exists \exists 3$

4. Proof of Proposition. この paragraph
 7" は $HK^2 = K_1^2 = 0$ と仮定する. $K^7 = K^0 = 0$
 7" あり, $K' = HK'$ は $V^{1,2}(u,1) \subset \mathcal{G}_1 \oplus \mathcal{G}_2^* \wedge (\mathcal{G}_1^*)^*$ の
 値を持つ. Prop. — 12 よって $K_1^2(u \wedge v)$ は $\mathcal{G}_1$
 の値を持ち, $K_0^2(u \wedge x) = 0$ (仮定) かつ $K_1^2 = 0$, 5'12
 Prop より $K_2^3 = 0$. よって Prop. — を証明するに
 は次の Lemma を証明すればよい.

$$K_1^3(u \wedge x) = K_2^4(u \wedge x) = 0$$

を証明すればよい

次 is invariant $K^3 = C = C_0 + C_1$ を考える

$$\text{Lemma (1)} \quad (\partial C)(u \wedge x \wedge y) = 0$$

$$(2) \quad (\partial_2^* \partial C)(u \wedge x) + [e^n, (\partial C)(e_n \wedge u \wedge x)] = 0$$

Proof. Lemma と今迄の結果から次式
を導く

$$(\partial K^3)(u \wedge v \wedge x) = -\delta_x K^2(u \wedge v),$$

$$(\partial K^3)(u \wedge x \wedge y) = 0,$$

$$(\partial K^3)(u \wedge x \wedge e_n) = -\delta_x K^2(e_n \wedge u).$$

$$\text{§ 5.12} \quad \partial^* K^2 = 0 \quad \text{より}$$

$$(\partial^* K^2)(u) = (\partial_2^* K_1^2)(u) + (\partial_1^* K_0^2)(u)$$

$$= (\partial_2^* K_1^2)(u) + [e^n, K^2(e_n \wedge u)]$$

§ 5.3. よって Lemma が従う。

$$c_1^+(u \wedge x) = 0.$$

Proof

$$(2C)(u \wedge x \wedge y) = 0 \quad \text{より}$$

$$-[x, c_1^+(u \wedge y)] + [y, c_1^+(u \wedge x)] = 0$$

を z とする. Lemma 12 より, z に対して

$$\langle x, c_1^+(u \wedge y) \rangle z + \langle z, c_1^+(u \wedge y) \rangle x$$

$$= \langle y, c_1^+(u \wedge x) \rangle z + \langle z, c_1^+(u \wedge x) \rangle y$$

を意味する. これより容易に $\langle z, c_1^+(u \wedge y) \rangle = 0$

すなわち, $c_1^+(u \wedge y) = 0$ を得る.

$$a \in \mathcal{Q}_1 \oplus \mathcal{Q}_2^* \wedge (\mathcal{Q}_1^-)^* \mathbb{Z}$$

$$a(u \wedge x) = c_1(u \wedge x)$$

$$a(u \wedge e_n) = 0$$

12 以下定义 7.3.

$$\text{Lemma (1)} \quad (\partial_1 \partial_1^* a)(u \wedge x) + (\partial_2^* \partial_2 a)(u \wedge x)$$

$$+ (\lambda_1^+ - \mu_1^+) a(u \wedge x) = 0$$

$$(2) \quad c_1(u \wedge x) = 0$$

$$\text{Proof (1)} \quad (\partial \partial)(e_i \wedge u \wedge x)$$

$$= (\partial_1 c_0 + \partial_2 c_1)(e_i \wedge u \wedge x)$$

$$= [x, c_0(e_i \wedge u)] + (\partial_2 c_1)(e_i \wedge u \wedge x),$$

$$(\partial \partial)(e_n \wedge u \wedge x) = (\partial_0 c_0 + \partial_1 c_1)(e_n \wedge u \wedge x)$$

$$= c_0([e_n, x] \wedge u) + [e_n, c_1(u \wedge x)] + [x, c_1(e_n \wedge u)]$$

$$(\partial_2^* \partial c)(u \wedge x) + [e^n, (\partial \partial)(e_n \wedge u \wedge x)] = 0$$

7.2'' 例 5.

$$\begin{aligned} & \sum_i [e^i, [x, c_0(e_i \wedge u)]] + (\partial_2^* \partial_2 c_1)(u \wedge x) \\ & + [e^n, c_0([e_n, x] \wedge u)] + [e^n, [e_n, c_1(u \wedge x)]] \\ & + [e^n, [x, c_1(e_n \wedge u)]] = 0 \end{aligned}$$

例 2.3

$$\begin{aligned} & \sum_i [e^i, [x, c_0(e_i \wedge u)]] \\ & = \sum_i [[e^i, x], c_0(e_i \wedge u)] + [x, (\partial_2^* c_0)(u)] \\ & = [e^n, c_0([x, e_n] \wedge u)] + [x, (\partial_2^* c_0)(u)] \end{aligned}$$

$$\partial_2^* c = 0 \quad \text{由}$$

$$(\partial_2^* c_0)(u) + (\partial_1^* c_1)(u) = 0$$

$$\begin{aligned} (\partial_1^* c_1)(u) &= [e^n, c_1(e_n \wedge u)] + \sum_i [e^{n+i}, c_1(e_{n+i} \wedge u)] \\ &= [e^n, c_1(e_n \wedge u)] + (\partial_1^* c_1)(u) \end{aligned}$$

例 2.3 续

$$\begin{aligned} \sum_i [e^i, [x, c_0(e_i \wedge u)]] &= [e^n, c_0([x, e_n] \wedge u)] \\ &\quad - [x, [e^n, c_1(e_n \wedge u)]] - [x, (\partial_1^* c_1)(u)] \end{aligned}$$

より

$$\begin{aligned} [c^n, [c_n, c_1(u \wedge x)]] &= [E_1^+, c_1(u \wedge x)] \\ &= (\lambda_1^+ - \mu_1^+) c_1(u \wedge x) \end{aligned}$$

ここで $c_1(u \wedge x)$ が $\mathcal{G}_1$ の値を持つことを使った
以上から

$$-[x, (\partial_1^* a)(u)] + (\partial_2^* \partial_2 c_1)(u \wedge x) + (\lambda_1^+ - \mu_1^+) c_1(u \wedge x) = 0$$

より、これは

$$(\partial_1 \partial_1^* a)(u \wedge x) + (\partial_2^* \partial_2 a)(u \wedge x) + (\lambda_1^+ - \mu_1^+) a(u \wedge x) = 0$$

より、よって (1) が示された

(2) $\lambda_1^+ - \mu_1^+ > 0$ ならば、この式から $a(u \wedge x) = 0$,
よって、 $c_1(u \wedge x) = 0$ である。

最後 12 invariant $K^4 = c = c_1 + c_2$ を考へよ。

$$\text{Lemma } (\partial_2^* \partial c)(u \wedge x) + [e^n, (\partial c)(e_n \wedge u \wedge x)] = 0$$

Proof Lemma と今迄の結果から

$$(\partial K^4)(u \wedge v \wedge x) = -\delta_x K^3(u \wedge v),$$

$$(\partial K^4)(u \wedge x \wedge e_n) = -\delta_x K^3(e_n \wedge u)$$

$$\text{§ 2.3 } \partial^* K^3 = 0 \text{ から}$$

$$(\partial^* K^3)(u) = (\partial_2^* K^3)(u) + [e^n, K^3(e_n \wedge u)] = 0$$

P_2'' から Lemma § 2.3.

Lemma $C_2(u \wedge x) = 0$

Proof. $(\partial C)(e_i \wedge u \wedge x)$

$$= (\partial_1 C_1 + \partial_2 C_2)(e_i \wedge u \wedge x)$$

$$= [x, C_1(e_i \wedge u)] + (\partial_2 C_2)(e_i \wedge u \wedge x)$$

$$(\partial C)(e_n \wedge u \wedge x) = (\partial_0 C_1 + \partial_1 C_2)(e_n \wedge u \wedge x)$$

$$= C_1([e_n, x] \wedge u) + [e_n, C_2(u \wedge x)] + [x, C_2(e_n \wedge u)]$$

$$\downarrow > 2 \quad (\partial_2^* \partial_2)(u \wedge x) + [e^n, (\partial C)(e_n \wedge u \wedge x)] = 0 \quad \text{Prop. 5}$$

$$\sum_i [[e^i, x], C_1(e_i \wedge u)] + (\partial_2^* \partial_2 C_2)(u \wedge x)$$

$$+ [e^n, C_1([e_n, x] \wedge u)] + [E_1^+, C_2(u \wedge x)] = 0$$

$$[E_1^+, C_2(u \wedge x)] = 2\lambda_1^+ C_2(u \wedge x)$$

$$\sum_i [[e^i, x], C_1(e_i \wedge u)] = [e^n, C_1([x, e_n] \wedge u)]$$

Prop. 5

$$(\partial_2^* \partial_2 C_2)(u \wedge x) + 2\lambda_1^+ C_2(u \wedge x) = 0$$

$$\text{Prop. 3. } \text{if } \lambda_1^+ > 0 \text{ then } C_2(u \wedge x) = 0$$

5. Proof of Proposition

この paragraph

T'' は $K^1=0$ と仮定する。この仮定と $K^0=0$ から $K^2=HK^2$ T'' がある。従って K^2 は $V^{2,2}(n,1) \subset \bigoplus_{i=0}^n \mathcal{H}_i^*(\mathcal{H}_i)^*$ の値を持つ。また Prop. より $K_2^3=0$ 。よって Prop. を証明するに足る。

$$K_1^3(u \wedge e_n) = K_2^4(u \wedge e_n) = 0$$

を証明すればよい。

まず "invariant" $K^3=C=C_0+C_1$ を考える

$$\text{Lemma (1)} \quad (\partial C)(u \wedge v \wedge e_n) = 0 \quad \checkmark$$

$$(2) \quad (\partial_1^* \partial C)(u \wedge e_n) = 0 \quad \checkmark$$

Proof Lemma と仮定から次の等式を得る。

$$(\partial K^3)(u \wedge v \wedge e_n) = 0,$$

$$(\partial K^3)(u \wedge x \wedge e_n) = -\delta_{e_n} K^2(u \wedge x).$$

$$K^2 \text{ は harmonic } T'' \text{ から } (\partial_2^* K^3)(x) = (\partial_1^* K_0^3)(u) = 0$$

よって Lemma を得る。

Lemma $c_1^-(u \wedge v \wedge e_n) = 0$

Proof $(\partial C)(u \wedge v \wedge e_n) = 0 \quad \> 1$

$$(\partial_1 C_0 + \partial_2 C_1)(u \wedge v \wedge e_n)$$

$$= [e_n, C_0(u \wedge v)] + [u, C_1(v \wedge e_n)] - [v, C_1(u \wedge e_n)] = 0$$

$\> 2$

$$[u, c_1^-(v \wedge e_n)] - [v, c_1^-(u \wedge e_n)] = 0$$

$$\> 2 \quad a_j = \langle c_1^-(e_j \wedge e_n), e_n \rangle \quad \> 1 \quad \> 1$$

$$a_j [e_i, e^n] - a_i [e_j, e^n] = 0$$

$$[e_i, e^n] = -(\lambda_i^+ + \mu_i^+) e_{n+i} \quad \> 1 \quad \> 1$$

$$a_j \delta_i^{\> 1} - a_i \delta_j^{\> 1} = 0$$

$$\> 2 \quad a_j = 0, \text{ i.e., } c_1^-(e_j \wedge e_n) = 0 \quad \> 1 \quad \> 1$$

$$\text{Lemma (1)} - \sum_i [e^{n+i}, C_0(e_i \wedge u)] - (\lambda_1^+ + \mu_1^+) C_1(u \wedge e_n)$$

$$+ \sum_i [e^{n+i}, [e_n, C_1(u \wedge e_{n+i})]] + (\partial_2^* \partial_2 C_1)(u \wedge e_n) = 0$$

$$(2) - \sum_i [e^{n+i}, C_0(e_i \wedge u)] - (\lambda_1^- + \mu_1^-) C_1(u \wedge e_n)$$

$$+ \sum_i [e^{n+i}, [e_n, C_1(u \wedge e_{n+i})]] = 0$$

$$(3) (\lambda_1^- + \mu_1^- - \lambda_1^+ - \mu_1^+) C_1(u \wedge e_n) + (\partial_2^* \partial_2 C_1)(u \wedge e_n) = 0$$

$$(4) C_1(u \wedge e_n) = 0$$

$$\text{Proof (1)} \quad (\partial C)(e_i \wedge u \wedge e_n) = 0 \quad \& \&$$

$$(C_1 C_0 + \partial_2 C_1)(e_i \wedge u \wedge e_n)$$

$$= [e_n, C_0(e_i \wedge u)] + (\partial_2 C_1)(e_i \wedge u \wedge e_n)$$

$\& \& 2$

$$\partial^* \sum_i [e^i, [e_n, C_0(e_i \wedge u)]] + (\partial_2^* \partial_2 C_1)(u \wedge e_n) = 0$$

$$\& \& 3. \quad \partial^* C = 0 \quad \& \&$$

$$(\partial_2^* C_0)(u) + (\partial_1^* C_1)(u)$$

$$= \sum_i [e^i, C_0(e_i \wedge u)] + [e^n, C_1(e_n \wedge u)]$$

$$+ \sum_i [e^{n+i}, C_1(e_{n+i} \wedge u)] = 0.$$

$$[e_n, e^i] = e^{n+i} \quad \text{for } i \leq n-1$$

$$\sum_i [e^{n+i}, c_0(e_i \wedge u)] + \sum_i [e^i, [e_n, c_0(e_i \wedge u)]]$$

$$+ [E_1^+, c_1(u \wedge e_n)] + \sum_i [e^{n+i}, [e_n, c_1(e_{n+i} \wedge u)]] = 0$$

$\exists \tilde{z} \in \mathfrak{g}$ $\Rightarrow \tilde{z}''$ $c_1(u \wedge e_n)$ が $\mathfrak{g}_1^+$ の値を持つこと $\exists$
使った

$$[E_1^+, c_1(u \wedge e_n)] = (\lambda_1^+ + \mu_1^+) c_1(u \wedge e_n)$$

$\tilde{z}''$ があるから, (1) $\exists \tilde{z} \in \mathfrak{g}$
(2)

$$(2c)(u \wedge x \wedge e_n)$$

$$= -c_0([x, e_n] \wedge u) - [x, c_1(u \wedge e_n)] + [e_n, c_1(u \wedge x)]$$

$$\exists \tilde{z} \in \mathfrak{g}. \quad (2_1^* 2c)(u \wedge e_n) = \sum_i [e^{n+i}, (2c)(e_{n+i} \wedge u \wedge e_n)] = 0$$

$\mathfrak{g}_2''$ がある

$$- \sum_i [e^{n+i}, c_0([e_{n+i}, e_n] \wedge u)] - [E_1^-, c_1(u \wedge e_n)]$$

$$+ \sum_i [e^{n+i}, [e_n, c_1(u \wedge e_{n+i})]] = 0$$

$\exists \tilde{z} \in \mathfrak{g}$.

$$[E_1^-, c_1(u \wedge e_n)] = (\lambda_1^- + \mu_1^-) c_1(u \wedge e_n)$$

$\tilde{z}''$ があるから (2) $\exists \tilde{z} \in \mathfrak{g}$

(3) (1), (2) より 直ちに (3) が成り立つ。

(4) $\lambda_1^- + \mu_1^- - \lambda_1^+ - \mu_1^+ > 0$ であるから (3) より
 $C(\pi, e_m) = 0$ である。

invariant $K^4 = C = C_1 + C_2$ を考えよう.

$$\text{Lemma } (\partial_2^* \partial C)(u \wedge e_n) + (\partial_1^* \partial C)(u \wedge e_n) = 0.$$

Proof. Lemma と 今迄の結果から 次の等式が成り立つ.

$$(\partial K^4)(u \wedge v \wedge e_n) = -\delta_n K^3(u \wedge v),$$

$$(\partial K^4)(u \wedge x \wedge e_n) = -\delta_n K^3(u \wedge x).$$

$$\partial^* K^3 = 0 \text{ より}$$

$$(\partial_2^* K^3)(u) + (\partial_1^* K^3)(u) = 0$$

よって Lemma が成り立つ.

Lemma (1) $(\partial_2^* \partial_2 C_2)(u \wedge e_n) + 2 \lambda_1^- C_2(u \wedge e_n) = 0$

$$(2) \quad C_2(u \wedge e_n) = 0$$

Proof. (1)

$$(\partial C)(e_i \wedge u \wedge e_n) = (\partial_1 C_1 + \partial_2 C_2)(e_i \wedge u \wedge e_n)$$

$$= [e_n, C_1(e_i \wedge u)] + (\partial_2 C_2)(e_i \wedge u \wedge e_n),$$

$$(\partial C)(e_{n+i} \wedge u \wedge e_n) = (\partial_0 C_1 + \partial_1 C_2)(e_{n+i} \wedge u \wedge e_n)$$

$$= C_1([e_{n+i}, e_n] \wedge u) + [e_{n+i}, C_2(u \wedge e_n)]$$

$$+ [e_n, C_2(e_{n+i} \wedge u)]$$

$$(\partial_2^* \partial C)(u \wedge e_n) + (\partial_1^* \partial C)(u \wedge e_n) = 0 \quad \text{P. 11.5}$$

$$\sum_i [([e_i, e_n], C_1(e_i \wedge u)] + (\partial_2^* \partial_2 C_2)(u \wedge e_n)$$

$$+ \sum_i [e_{n+i}, C_1(e_i \wedge u)] + [E_1^-, C_2(u \wedge e_n)] = 0.$$

$$[E_1^-, C_2(u \wedge e_n)] = 2 \lambda_1^- C_2(u \wedge e_n)$$

从而有, $[e_i, e_n] = -e_{n+i}$ 从而有 P. 11.5 (1) 且 (2).

(2) $\lambda_1^- > 0$ P. 11.5, (1) 从而 $C_2(u \wedge e_n) = 0$ 且 (2).

6. Proof of Proposition 2 of paragraph 7" is $K' = 0$ is fixed.

まず invariant $K^3 = K_0^3 + K_1^3$ を考えよう. Lemma と今迄の結果から次の Lemma を得る.

Lemma (1) $(2K^3)(u \wedge v \wedge x) = (2K^3)(u \wedge v \wedge e_n) = 0$

(2) $(2K^3)(u \wedge x \wedge y) = -\delta_x K^2(y \wedge u) - \delta_y K^2(u \wedge x),$

$(2K^3)(u \wedge x \wedge e_n) = -\delta_{e_n} K^2(u \wedge x),$

(3) $(2K^3)(x \wedge y \wedge z) = (2K^3)(x \wedge y \wedge e_n) = 0.$

Lemma (1) $\delta_{e_n} K_0^2 = 0.$

(2) $K_0^3 = 0$ かつ K_1^3 は $\mathcal{G}_1^+ \oplus \mathcal{G}_2^* \wedge (\mathcal{G}_1^-)^*$ の値を持つ

(3) $\langle K_1^3([e_n, x] \wedge y), z \rangle$ は variables $x, y, z \in \mathcal{G}_1^-$ について symmetric

Proof. (1) $(2K^3)(u \wedge x \wedge e_n)$

$= (2K_0^3 + 2K_1^3)(u \wedge x \wedge e_n)$

$= -K_0^3([x, e_n] \wedge u) + [e_n, K_1^3(u \wedge x)]$

7" より, $(2K^3)(u \wedge x \wedge e_n) = -\delta_{e_n} K_0^2(u \wedge x)$ 7" より 8",

$\delta_{e_n} K_0^2(u \wedge x) = -[e_n, K_1^3(u \wedge x)] + K_0^3(u \wedge [e_n, x])$

例 3. $K_1^3(u \wedge x)$ は

$$K_1^3(u \wedge x) \equiv \alpha(u \wedge x) e'' \pmod{\mathcal{I}_1^+}$$

と表わされる よって

$$\delta_{e_n} K_0^2(u \wedge x) = \alpha(u \wedge x) E_1^+ + K_0^3(u \wedge [e_n, x])$$

例 3. よって α は function $P \rightarrow \mathcal{I}_2^* \wedge (\mathcal{I}_1^-)^*$ である。
 $u = [e_n, x]$ とおき

$$\delta_{e_n} K_0^2([e_n, x] \wedge x) = \alpha([e_n, x] \wedge x) E_1^+$$

例 3. $K_0^2([e_n, x] \wedge x)$ は 1_0 の値を持ち, $\langle 1_0, E_1^+ \rangle = 0$
 P_2^+ かつ $\alpha([e_n, x] \wedge x) = 0$, 従って

$$\delta_{e_n} K_0^2([e_n, x] \wedge x) = 0$$

例 3. $K_0^2([e_n, x] \wedge y)$ は x, y について symmetric
 P_2^+ かつ

$$\delta_{e_n} K_0^2([e_n, x] \wedge y) = 0,$$

よって $\delta_{e_n} K_0^2 = 0$ 例 3.

(2) (1) かつ

$$\alpha(u \wedge x) E_1^+ + K_0^3(u \wedge [e_n, x]) = 0$$

例 3. $(\partial K^3)(u \wedge v \wedge e_n) = 0$ かつ

$$(\partial_1 K_0^3 + \partial_2 K_1^3)(u \wedge v \wedge e_n) = [e_n, K_0^3(u \wedge v)] = 0$$

$\exists \xi \in S$. 仮定

$$\alpha(u \wedge x) [E_1^+, e_n] = 0.$$

$[E_1^+, e_n] = (-\lambda_1^+ + \mu_1^+) e_n$ で $-\lambda_1^+ + \mu_1^+ \neq 0$ だ"から
 $\alpha(u \wedge x) = 0$ $\exists \xi \in S$. よって $K_1^3(u \wedge x)$ が $\mathcal{G}_1^+$ の値
 ξ を持つ $K_0^3 = 0$ がわかった

$$(3) \quad (\partial K^3)(x \wedge y \wedge e_n) = 0 \text{ だ"から}$$

$$(\partial_0 K_1^3)(x \wedge y \wedge e_n) = K_1^3([x, e_n] \wedge y) - K_1^3([y, e_n] \wedge x) = 0$$

$\exists \xi \in S$. $\xi \in S$ に

$$(\partial K^3)(u \wedge x \wedge y) = (\partial_1 K_1^3)(u \wedge x \wedge y)$$

$$= -[x, K_1^3(u \wedge y)] + [y, K_1^3(u \wedge x)]$$

だ"あり, $(\partial K^3)(u \wedge x \wedge y) = -\delta_x K_0^2(y \wedge u) - \delta_y K_0^2(u \wedge x)$
 だ"から

$$-[x, K_1^3(u \wedge y)] + [y, K_1^3(u \wedge x)] = -\delta_x K_0^2(y \wedge u) - \delta_y K_0^2(u \wedge x)$$

$\exists \xi \in S$. 右辺は $\mathcal{I}_0$ の値を持ち, $\langle \mathcal{I}_0, E \rangle = 0$ だ"から,

$$\langle x, K_1^3(u \wedge y) \rangle = \langle y, K_1^3(u \wedge x) \rangle$$

$\exists \xi \in S$. 以上により $\langle K_1^3([e_n, x] \wedge y), \xi \rangle$ が x, y, ξ について

対称であることがわかった。

次に invariant $K^4 = K_1^4 + K_2^4$ を考える。
 Lemma と 今迄の結果から 次の lemma をえる

$$\text{Lemma (1)} \quad (2K^4)(u \wedge v \wedge x) = -\delta_u K^3(v \wedge x) - \delta_v K^3(x \wedge u),$$

$$(2K^4)(u \wedge v \wedge e_n) = 0.$$

$$(2) \quad (2K^4)(u \wedge x \wedge y) = -\delta_x K^3(y \wedge u) - \delta_y K^3(u \wedge x),$$

$$(2K^4)(u \wedge x \wedge e_n) = -\delta_{e_n} K^3(u \wedge x).$$

$$(3) \quad (2K^4)(x \wedge y \wedge z) = (2K^4)(x \wedge y \wedge e_n) = 0.$$

$$\text{Lemma (1)} \quad K_1^4 = 0 \text{ かつ } \delta_{e_n} K_1^3 = -[e_n, K_2^4]$$

(2) $\langle K_2^4([e_n, x] \wedge y), [e_n, z] \rangle$ は x, y, z について
 symmetric である。

$$\text{Proof (1)} \quad (2K^4)(u \wedge x \wedge e_n) = (-\delta_{e_n} K^3(u \wedge x)) \text{ である}$$

$$(2K_1^4 + 2K_2^4)(u \wedge x \wedge e_n)$$

$$= -K_1^4([x, e_n] \wedge u) + [e_n, K_2^4(u \wedge x)] = -\delta_{e_n} K_1^3(u \wedge x)$$

である。よって $K_1^4(u \wedge u)$ が $\mathfrak{g}_1^+$ の値を持つこと
 がわかる。 $(2K^4)(u \wedge v \wedge x) = -\delta_u K_0^2(v \wedge x) - \delta_v K_0^2(x \wedge u)$
 である。

$$(2K_1^4 + 2K_2^4)(u \wedge v \wedge x)$$

$$= [x, K_1^4(u \wedge v)] + [u, K_2^4(v \wedge x)] - [v, K_2^4(u \wedge x)]$$

$$= -\delta_u K_0^2(v \wedge x) - \delta_v K_0^2(x \wedge u)$$

$\exists \exists 3$. 右辺は $\mathcal{I}_0$ の値を持ち, $\langle J, \mathcal{I}_0 \rangle = 0$,
 $[J, \mathcal{I}_2] = 0$ $\mathcal{I}_2''$ から

$$\langle x, K_1^4(u \wedge v) \rangle = 0$$

$\exists \exists 3$. $K_1^4(u \wedge v)$ は $\mathcal{I}_1^+$ の値を持つから
 $K_1^4 = 0$. $\exists \exists 3$. よって $\delta_{e_n} K_1^3 = -[e_n, K_2^4]$ $\exists \exists 3$.

$$(2) \quad [u, K_2^4(v \wedge x)] - [v, K_2^4(u \wedge x)]$$

$$= -\delta_u K_0^2(v \wedge x) - \delta_v K_0^2(x \wedge u)$$

$\mathcal{I}_2''$ あり. 右辺は $\mathcal{I}_0$ の値を持ち, $\langle \mathcal{I}_0, E \rangle = 0$,
 $\mathcal{I}_2''$ から

$$\langle u, K_2^4(v \wedge x) \rangle = \langle v, K_2^4(u \wedge x) \rangle$$

$\exists \exists 3$ さらに $(\partial K^4)(x \wedge y \wedge e_n) = 0$ より

$$(\partial_0 K_2^4)(x \wedge y \wedge e_n) = K_2^4([x, e_n] \wedge y) - K_2^4([y, e_n] \wedge x) = 0$$

$\exists \exists 3$ 以上から, $\langle K_2^4([e_n, x] \wedge y), [e_n, z] \rangle$ が
 x, y, z について symmetric $\mathcal{I}_2''$ あり 2 とがわかるから

最後に invariant $K^5 = K_2^5$ を考へる. Lemma
と今迄の結果から次を證す.

$$\text{Lemma (1)} \quad (\partial K^5)(u \wedge v \wedge x) = -\delta_u K^3(v \wedge x) - \delta_v K^3(x \wedge u)$$

$$(2) \quad (\partial K^5)(u \wedge x \wedge e_n) = -\delta_{e_n} K^4(u \wedge x)$$

$$\text{Lemma (1)} \quad K_2^5 = 0$$

$$(2) \quad \delta_{e_n} K_2^4 = 0$$

Proof (1) $(\partial K^5)(u \wedge v \wedge x) = -\delta_u K^3(v \wedge x) - \delta_v K^3(x \wedge u)$
から

$$(\partial_1 K_2^5)(u \wedge v \wedge x) = [x, K_2^5(u \wedge v)]$$

$$= -\delta_u K^3(v \wedge x) - \delta_v K^3(x \wedge u)$$

を證す. 右辺は $\mathfrak{g}_1^+$ の値を持ち, 左辺は
 $\mathfrak{g}_1^-$ の値を持つから,

$$[x, K_2^5(u \wedge v)] = 0$$

を證す. $[e^i, e_{n+j}] = \delta_j^i e^n$ であるから, $K_2^5 = 0$ を
證す.

(2) $(\partial K^5)(u \wedge x \wedge e_n) = -\delta_{e_n} K_2^4(u \wedge x)$ から $K^5 = K_2^5 = 0$
であるから $\delta_{e_n} K_2^4 = 0$ を證す.

以上で Proposition が完全に証明された.

§V.4 The expressions of the invariants HK^2 and K' in terms of canonical coordinate systems

この § 2" は $\mathcal{P} = \{R: E, F\}$ は projective system of type (1), (P, ω) は $\mathcal{P}$ に対する normal connection of type G , K は $\mathcal{P}$ の curvature とする $\pi: P \rightarrow R$ は projection とする

(1) Fundamental formulas for the covariant derivatives, f は R 上の function とする P 上の function $f \circ \pi$ は $\mathcal{P}$ 上の f と $\nabla_X f = X(f) \circ \pi$ とする

と $\mathcal{P}$ 上の $X \in \mathcal{G}$ に対して, covariant differentiation $\delta_X = \nabla_X(X)$ が $\mathcal{P}$ 上にある R 上の function f と $X_1, \dots, X_r \in \mathcal{G}$ に対して P 上の function $\tilde{f}(X_1, \dots, X_r)$ と

$$\begin{aligned} \tilde{f}(X_1, \dots, X_r) &= \delta_{X_1} \cdots \delta_{X_r} f \\ &= \tilde{\omega}(X_1) \cdots \tilde{\omega}(X_r) f. \end{aligned}$$

と定義する. §.4.1.1

Lemma 1 $a \in G^{(0)}$, $X \in \mathcal{G}^{(0)}$ とする

$$(1) \quad Ra^* \tilde{f}(X_1, \dots, X_r) = \tilde{f}(Ad(a)X_1, \dots, Ad(a)X_r)$$

$$(2) \quad \delta_X \tilde{f}(X_1, \dots, X_r) = \sum_i \tilde{f}(X_1, \dots, [X, X_i], \dots, X_r) \quad \text{1-th place}$$

Proof $a \in G^{(0)}$ とする. このとき

$$(Ra)_* \tilde{\omega}(Y) = \tilde{\omega}(Ad(a)Y), \quad Y \in \mathcal{G}$$

が成立する。さらに

$$R_a^* f = f$$

である。よって

$$R_a^* \tilde{f}(X_1, \dots, X_r) = \tilde{f}(Ad(a)X_1, \dots, Ad(a)X_r)$$

である。よって (1) が従う。 $X \in \mathfrak{g}^{(0)}$ のとき、 $dX = dX^*$ であるから、(1) から直ちに (2) が従う。

space $C^2(\mathfrak{g}) = \mathfrak{g} \oplus \Lambda^2(\mathfrak{m}^*)$ は

$$C^2(\mathfrak{g}) = \mathfrak{m} \oplus \Lambda^2(\mathfrak{m}^*) + \mathfrak{g}^{(0)} \oplus \Lambda^2(\mathfrak{m}^*)$$

と分解する。この分解依りて, curvature K は

$$K = K_- + K_+$$

と分解する。

$$K_- = \sum_{j < 0} K_j, \quad K_+ = \sum_{j \geq 0} K_j$$

であることと注意する。 $X \in \mathfrak{g}$ に対して、
 X で分解 $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}^{(0)}$ に関する $\mathfrak{m}$ -component
 を表わす。

Lemma $z \in P, x_1, x_2 \in \mathcal{G}$ とする。このとき
次式が成立する。

$$[\check{\omega}(x_1), \check{\omega}(x_2)]_z = \check{\omega}([x_1, x_2])_z - \check{\omega}(A)_z - B_z^*.$$

ここで, A (resp. B) は $K_-(x_1 \wedge x_2)$ (resp. $K_+(x_1 \wedge x_2)$) の z での直交である。

Proof. structure equation より 容易に

$$\omega([\check{\omega}(x_1), \check{\omega}(x_2)]) = [x_1, x_2] - K(x_1 \wedge x_2)$$

より。この式より直ちに Lemma^{1,2} が従う。

$$\text{Lemma. } \tilde{f}(x_1, x_2, x_3, \dots, x_R)$$

$$= \tilde{f}(x_2, x_1, x_3, \dots, x_R) + \tilde{f}([x_1, x_2], x_3, \dots, x_R)$$

$$- \tilde{f}(K_-(x_1 \wedge x_2), x_3, \dots, x_R)$$

$$- \sum_{i=3}^R \tilde{f}(x_3, \dots, [K_+(x_1 \wedge x_2), x_i], \dots, x_R)$$

(i-2)-th place

Proof. $z \in P$ とする

$$\tilde{f}(x_1, x_2, x_3, \dots, x_R)(z) = \check{\omega}(x_1)_z \check{\omega}(x_2) \tilde{f}(x_3, \dots, x_R)$$

$$= \check{\omega}(x_2)_z \check{\omega}(x_1) \tilde{f}(x_3, \dots, x_R)$$

$$+ [\check{\omega}(x_1), \check{\omega}(x_2)]_z \tilde{f}(x_3, \dots, x_R).$$

Lemma 2.2

$$[\psi(X_1), \psi(X_2)]_{\mathbb{Z}} \tilde{f}(X_3, \dots, X_R)$$

$$= \psi([X_1, X_2])_{\mathbb{Z}} \tilde{f}(X_3, \dots, X_R) - \psi(A)_{\mathbb{Z}} \tilde{f}(X_3, \dots, X_R)$$

$$- B_{\mathbb{Z}}^* \tilde{f}(X_3, \dots, X_R)$$

$$= \tilde{f}([X_1, X_2], X_3, \dots, X_R)(\mathbb{Z}) - \tilde{f}(A, X_3, \dots, X_R)(\mathbb{Z})$$

$$- \sum_{i=3}^R \tilde{f}(X_3, \dots, [B, X_i], \dots, X_R)(\mathbb{Z})$$

故由 Lemma 2.2

f は R 上の function であり, F の first integral であるとする. $\pi_*^{-1}(F)$ は equation $\omega_{-2} = \omega_{-1}^+ = 0$ で定義されるから, $Z \in \mathcal{G}_1^-$ のとき $\tilde{\omega}(Z)$ は $\pi_*^{-1}(F)$ の cross section である. よって

$$\tilde{f}(Z) = 0, \quad Z \in \mathcal{G}_1^- \quad (5.4.1)$$

である. 次の Lemma は Lemma と Propositions —, — から得られる.

Lemma $X \in \mathcal{G}_1^+, Z \in \mathcal{G}_1^-$ とする. (5.4.4)

$$(1) \quad \tilde{f}(Z, X) = \tilde{f}([Z, X]).$$

$$(2) \quad \tilde{f}(Z, Z, X) = \tilde{f}(Z, [Z, X]) = 0$$

$$(3) \quad \tilde{f}(Z, X, X) = 2 \tilde{f}([Z, X], X)$$

$$(4) \quad \tilde{f}(Z, Z, X, X) = 2 \tilde{f}([Z, X], [Z, X])$$

$$(5) \quad \tilde{f}(Z, Z, Z, X, X)$$

$$= -2 \tilde{f}([K_0(Z \wedge [Z, X]), Z], X, X)$$

$$- 2 \tilde{f}([K_1(Z \wedge [Z, X]), [Z, X]])$$

$$(6) \quad \tilde{f}(Z, X, X, X) = 3 \tilde{f}([Z, X], X, X)$$

$$- 2 \tilde{f}([K_1'(X \wedge [Z, X]), X]) - 2 \tilde{f}([K_0'(X \wedge [Z, X]), X])$$

Proof. この Lemma¹³ と 下の 2つの Lemma は
Lemma — と Propositions — , — から得ら
れる. 54143

(1) Lemma 12 より

$$\tilde{f}(Z, X) = \tilde{f}(X, Z) + \tilde{f}([Z, X]) - \tilde{f}(K_-(Z \wedge X))$$

3 2 3. $\tilde{f}(Z) = 0$, $K(Z \wedge X) = 0$ (Prop. —) T_2'' から,

(1) 3 2 3.

(2) (1) から

$$\tilde{f}(Z, Z, X) = \tilde{f}(Z, [Z, X])$$

3 2 3. Lemma 12 より
5413

$$\begin{aligned} \tilde{f}(Z, [Z, X]) &= \tilde{f}([Z, X], Z) + \tilde{f}([Z, [Z, X]]) \\ &\quad - \tilde{f}(K_-(Z \wedge [Z, X])) \end{aligned}$$

3 2 3. $\tilde{f}(Z) = 0$, $[Z, [Z, X]] = 0$, $K_-(Z \wedge [Z, X]) = 0$
(Prop.) T_2'' から (2) 3 2 3.

(3) $K(Z \wedge X) = 0$ T_2'' から, (1) と 同様 12 12

$$\tilde{f}(Z, X, X) = \tilde{f}(X, Z, X) + \tilde{f}([Z, X], X)$$

5413

3 2 3. (1) と Lemma から

$$\tilde{f}(X, Z, X) = \tilde{f}(X, [Z, X])$$

$$= \tilde{f}([Z, X], X) + \tilde{f}(X, [Z, X])$$

$$- \tilde{f}(K_-(X \wedge [Z, X]))$$

2.2.3. $[X, [Z, X]] = 0$ である。また Prop. 12
より $K_-(X \wedge [Z, X]) = K'_-(X \wedge [Z, X])$ は $\mathfrak{g}_1$ の値
を持つ。よって (3) が従う

(4) (3) から

$$\tilde{f}(Z, Z, X, X) = 2 \tilde{f}(Z, [Z, X], X)$$

2.2.3. Lemma 12 より

$$\tilde{f}(Z, [Z, X], X) = \tilde{f}([Z, X], Z, X) + \tilde{f}([Z, [Z, X]], X)$$

$$= \tilde{f}(K_-(Z \wedge [Z, X]), X) - \tilde{f}([K_+(Z \wedge [Z, X]), X])$$

2.2.3. 7.3 (1) から

$$\tilde{f}([Z, X], Z, X) = \tilde{f}([Z, X], [Z, X])$$

2.2.3. 2.12 $[Z, [Z, X]] = 0$, $K_-(Z \wedge [Z, X]) = 0$

7.3 (1) より

$$K_+(Z \wedge [Z, X]) = K_0(Z \wedge [Z, X]) + K_1(Z \wedge [Z, X]) + K_2(Z \wedge [Z, X])$$

7.3 (1) より $K_0(Z \wedge [Z, X]) = K_0^2(Z \wedge [Z, X])$ は 1.0

値を持つ (Prop. —) から, $[K_0(Z \wedge [Z, X]), X] = 0$.
よって $[K_+(Z \wedge [Z, X]), X]$ は $\mathcal{G}_0 + \mathcal{G}_1(\mathcal{G}^{(0)})$ の値
を持つ. 以上から

$$\tilde{f}(Z, [Z, X], X) = \tilde{f}([Z, X], [Z, X], X)$$

よって (4) が従う
(5) (4) から

$$\tilde{f}(Z, Z, Z, X, X) = 2 \tilde{f}(Z, [Z, X], [Z, X], X)$$

よって (3) の証明と同様にして

$$\begin{aligned} \tilde{f}(Z, [Z, X], [Z, X]) &= \tilde{f}([Z, X], Z, [Z, X]) \\ &\quad - \tilde{f}([K_+(Z \wedge [Z, X]), [Z, X]]) \end{aligned}$$

よって (2) 1. によって $\tilde{f}(Z, [Z, X]) = 0$ である. 次に

$$K_+(Z \wedge [Z, X]) = K_0(Z \wedge [Z, X]) + K_1(Z \wedge [Z, X]) + K_2(Z \wedge [Z, X]) + \dots$$

である, $[K_2(Z \wedge [Z, X]), [Z, X]]$ は $\mathcal{G}_0(\mathcal{G}^{(0)})$ の値を持ち,
 $[K_0(Z \wedge [Z, X]), [Z, X]] = [[K_0(Z \wedge [Z, X]), Z], X]$ である.
よって

$$\begin{aligned} \tilde{f}(Z, [Z, X], [Z, X]) &= - \tilde{f}([K_0(Z \wedge [Z, X]), [Z, X]]) \\ &\quad - \tilde{f}([K_1(Z \wedge [Z, X]), [Z, X]]) \end{aligned}$$

よって (5) が従う.

(6) (1) の証明と同様にして,

$$\tilde{f}(Z, X, X, X) = \tilde{f}(X, Z, X, X) + \tilde{f}([Z, X], X, X)$$

よって (3) から

$$\tilde{f}(X, Z, X, X) = 2 \tilde{f}(X, [Z, X], X)$$

よって Lemma 1.3 から

$$\tilde{f}(X, [Z, X], X) = \tilde{f}([Z, X], X, X)$$

$$= \tilde{f}(K_1(X \wedge [Z, X]), X) - \tilde{f}([K_0(X \wedge [Z, X]), X])$$

よって $K_1(X \wedge [Z, X])$ は $\mathfrak{g}_1$ の値を持つから
(1) を使って

$$\tilde{f}(K_1(X \wedge [Z, X]), X) = \tilde{f}([K_1(X \wedge [Z, X]), X])$$

よって

$$\tilde{f}(X, [Z, X], X) = \tilde{f}([Z, X], X, X)$$

$$= \tilde{f}([K_1(X \wedge [Z, X]), X]) - \tilde{f}([K_0(X \wedge [Z, X]), X])$$

よって (6) が従う.

2. Canonical coordinate systems and the cross sections σ . R の任意の点 a_0 に対して, a_0 の nbd U で定義された coordinate system $\{x, y_1^1, \dots, y^{n-1}, p_1^1, \dots, p^{n-1}\}$ が存在して, F は equation

$$dx = dy^i = 0, \quad 1 \leq i \leq n-1$$

で定義され, E は

$$\begin{cases} dy^i - p^i dx = 0, \\ dp^j - F^j dx = 0, \quad 1 \leq i, j, k \leq n-1. \end{cases}$$

なる n の equation で定義される. ここで F^j は a_0 の nbd で定義された function である.

以下 $U=R$ と仮定する. また indices i, j, k, l は $\{1, \dots, n-1\}$ を動くものとする. $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_1^+, \mathcal{I}_1^-,$ etc. などにはそれぞれ $\text{basis } (e_i), (e_m), (e_{n+j}),$ etc. を ———— のように取っておく.

次の Prop. を証明しよう.

P の点

Proposition. 任意の $a \in R$ に対して, $y = \sigma(a)$ であって, $\pi(y) = a$ かつ次を満足するものが唯一存在する.

$$1) \quad p^i(e_{n+j})(y) = \delta_j^i, \quad \tilde{x}(e_n)(y) = 1,$$

$$2) \quad \tilde{x}(e_n, e_n)(y) = \tilde{x}(\tau_Z, e_n)(y) = 0, \quad Z \in \mathcal{I}_1^-$$

$$3) \quad \tilde{\chi}(Z, e_n, e_n)(y) = 0, \quad Z \in \mathcal{Y}_1$$

§ 5.12. $\tilde{\chi}$ は $a \rightarrow \sigma(a)$ は principal fibre bundle P の differentiable cross section σ である。

χ は F の first integral τ'' から, Lemma 5.11

$$\tilde{\chi}(Z, e_n)(y) = \tilde{\chi}(\tau(Z, e_n, 1))(y) = 0$$

$$\tilde{\chi}(Z, e_n, e_n)(y) = 2 \tilde{\chi}(\tau(Z, e_n, e_n))(y) = 0$$

§ 5.3.

§ 5.3. χ の Lemma を証明する。

Lemma (1) $\det(\tilde{p}^i(e_{n+j})) \neq 0$ everywhere.

5.4.12

(2) $\tilde{\chi}(e_n) \neq 0$ everywhere

Proof. (1) χ, y^i は F の first integrals τ'' があるから

$$\tilde{\chi}(Z) = \tilde{y}^i(Z) = 0, \quad Z \in \mathcal{Y}_1$$

τ'' がある。よって $\tilde{p}^i(Z)(y) = 0$ なる $Z \in \mathcal{Y}_1$ と $y \in P$

があったとすれば, $\pi_* \tilde{\omega}(Z)y = 0$ である。よって

$Z = 0$. この事実とは (1) とも矛盾する

(2) $\pi_*(E)$ は equation $\omega_2 = \omega_1 = 0$ τ''

定義されるから, $\tilde{\omega}(e_n)$ は $\pi_*(E)$ の

cross section τ'' がある。よって

$$\tilde{y}^i(e_n) - p^i \tilde{\chi}(e_n) = 0,$$

$$\tilde{p}^i(e_n) - F^i \tilde{\chi}(e_n) = 0$$

§ 5.3. よって $\tilde{\chi}(e_n)(y) = 0$ なる $y \in P$ があつたと

すなわち $\pi_*(\tilde{\omega}(e_n)) = 0$ となる。これは明らかに矛盾である。よって (2) が従う。

$g \in G^{(0)}$ とする。Prop. 12 より g は

$$g = \theta \exp X_1 \exp X_2$$

と表わされる。ここで $\theta \in G_0$, $X_1 \in \mathfrak{g}_1$, $X_2 \in \mathfrak{g}_2$ 。さらに X_1 は

$$X_1 = X_1^+ + X_1^-$$

と表わされる。ここで $X_1^+ \in \mathfrak{g}_1^+$, $X_1^- \in \mathfrak{g}_1^-$ である。以上の記号の下で次の Lemma を示す。

Lemma $Z \in \mathfrak{g}_1^-$ とする。
5.4.23

$$(1) \quad R_a^* \tilde{\rho}^i(Z) = \tilde{\rho}^i(\text{Ad}(\theta)Z),$$

$$R_a^* \tilde{\chi}(e_n) = \tilde{\chi}(\text{Ad}(\theta)e_n)$$

$$(2) \quad \theta = e \text{ のとき,}$$

$$R_a^* \tilde{\chi}(e_n, e_m) = \tilde{\chi}(e_n, e_m) + \tilde{\chi}([X_1^-, e_n], e_m)$$

$$R_a^* \tilde{\chi}([Z, e_n]) = \tilde{\chi}([Z, e_n]) + \tilde{\chi}([X_1^-, [Z, e_n]])$$

$$(3) \quad \theta = e, X_1 = 0 \text{ のとき}$$

$$R_a^* \tilde{\chi}([Z, e_n], e_n) = \tilde{\chi}([Z, e_n], e_n) + \tilde{\chi}([X_2^-, [Z, e_n]], e_n)$$

Proof (1) $Ad(a)Z \equiv Ad(b)Z \pmod{\mathfrak{g}^{(0)}}$

$$Ad(a)e_n \equiv Ad(b)e_n \pmod{\mathfrak{g}^{(0)}}$$

2" あり. $\gamma \geq 2$ Lemma 12 より

$$R_a^* \tilde{\rho}^i(Z) = \tilde{\rho}^i(Ad(a)Z) = \tilde{\rho}^i(Ad(b)Z),$$

$$R_a^* \tilde{\chi}(e_n) = \tilde{\chi}(Ad(a)e_n) = \tilde{\chi}(Ad(b)e_n)$$

2 2 3. $\gamma \geq 2$ (1) が 1 通り.

(2) $b=c$ と仮定する. このとき

$$Ad(a)e_n \equiv e_n + [X_1^-, e_n] \pmod{\mathfrak{g}_1 + \mathfrak{g}_2},$$

$$Ad(a)[Z, e_n] \equiv [Z, e_n] + [X_1^+, [Z, e_n]] \pmod{\mathfrak{g}^{(0)}}$$

$\gamma \geq 2$ Lemmas 7411, 8413 を使うと,

$$R_a^* \tilde{\chi}(e_n, e_n) = \tilde{\chi}(Ad(a)e_n, Ad(a)e_n)$$

$$= \tilde{\chi}(Ad(a)e_n, e_n) = \tilde{\chi}(e_n, Ad(a)e_n)$$

$$+ \tilde{\chi}([Ad(a)e_n, e_n]) - \tilde{\chi}(K_-(Ad(a)e_n) \cdot e_n)$$

$$= \tilde{\chi}(e_n, e_n) + \tilde{\chi}([X_1^-, e_n], e_n]$$

$$R_q^* \tilde{\chi}([Z, e_n]) = \tilde{\chi}(Ad(a)[Z, e_n])$$

$$= \tilde{\chi}([Z, e_n]) + \tilde{\chi}([X_1, [Z, e_n]])$$

したがって $\tilde{\chi}([X_1, [Z, e_n]]) = \tilde{\chi}([X_1, [Z, e_n]])$ であるから、(2)が成り立つ。

(3) $f=e, X_1=0$ と仮定する。このとき

$$Ad(a)[Z, e_n] \equiv [Z, e_n] + [X_2, [Z, e_n]] \pmod{\mathfrak{g}_1 + \mathfrak{g}_2}$$

したがって $\tilde{\chi}([Z, e_n]) = \tilde{\chi}([Z, e_n] + [X_2, [Z, e_n]])$ であるから、(3)が成り立つ。

$$R_q^* \tilde{\chi}([Z, e_n], e_n) = \tilde{\chi}(Ad(a)[Z, e_n], Ad(a)e_n)$$

$$= \tilde{\chi}(Ad(a)[Z, e_n], e_n) = \tilde{\chi}(e_n, Ad(a)[Z, e_n])$$

$$+ \tilde{\chi}([Ad(a)[Z, e_n], e_n]) - \tilde{\chi}(K_-(Ad(a)[Z, e_n]) \wedge e_n)$$

$$= \tilde{\chi}(e_n, [Z, e_n]) + \tilde{\chi}([X_2, [Z, e_n]], e_n)$$

$$- \tilde{\chi}(K_-([Z, e_n]) \wedge e_n)$$

$$= \tilde{\chi}([Z, e_n], e_n) + \tilde{\chi}([X_2, [Z, e_n]], e_n)$$

したがって (3) が成り立つ。

$$\pi(y) =$$

$$\tilde{\pi}(y)$$

以上の準備の下で, Prop. 2 を証明しよう.
 $a \in R'$ とし, $\pi(y_0) = a$ なる $y_0 \in P$ を一つとる.

I. $f \in G_0$ をとり, $y_1 = f \cdot y_0$ とおき, Lemma 1 により,

$$\tilde{P}^i(e_{n+j})(y_1) = \tilde{P}^i(\text{Ad}(f)e_{n+j})(y_0),$$

$$\tilde{\pi}(e_n)(y_1) = \tilde{\pi}(\text{Ad}(f)e_n)(y_0)$$

である. $G_0 = \text{Aut}(\mathcal{L})$ は $\mathfrak{g}_1 = \mathfrak{g}_1^+ + \mathfrak{g}_1^-$ 上への自然な表現を通じて, product group $GL(\mathfrak{g}_1^+) \times GL(\mathfrak{g}_1^-)$ と同一視される. また Lemma 1 により $\tilde{\pi}(e_n)(y_0) \neq 0$, $\det(\tilde{P}^i(e_{n+j})(y_0)) \neq 0$. よって

$$\tilde{P}^i(e_{n+j})(y_1) = \delta_j^i, \quad \tilde{\pi}(e_n)(y_1) = 1$$

を満足する f が唯一つ存在する.

II. $X_1 \in \mathcal{F}_1$ とし, $a_1 = \exp X_1$, $y_2 = y_1 a_1$ とおき,
 このとき Lemma から

(542)

$$\tilde{\chi}(e_n, e_n)(y_2) = \tilde{\chi}(e_n, e_n)(y_1) + \tilde{\chi}([X_1, e_n], e_n)(y_1),$$

$$\tilde{\chi}([Z, e_n])(y_2) = \tilde{\chi}([Z, e_n])(y_1) + \tilde{\chi}([X_1, [Z, e_n]])(y_1)$$

とえる. $X_1^- = \alpha e^n$ とおけば,

$$[X_1^-, e_n], e_n] = \alpha(-\lambda_1^+ + \mu_1^+)e_n$$

$$[X_1^+, [Z, e_n]] = -(\lambda_1^+ + \mu_1^+) \langle X_1^+, Z \rangle e_n$$

とえる. ように

$$\tilde{\chi}(e_n, e_n)(y_2) = \tilde{\chi}(e_n, e_n)(y_1) + \alpha(-\lambda_1^+ + \mu_1^+)$$

$$\tilde{\chi}([Z, e_n])(y_2) = \tilde{\chi}([Z, e_n])(y_1) - (\lambda_1^+ + \mu_1^+) \langle X_1^+, Z \rangle$$

$= \lambda_1^+ + \mu_1^+ \neq 0$, $\lambda_1^+ + \mu_1^+ \neq 0$ より,

$$\tilde{\chi}(e_n, e_n)(y_2) = 0,$$

$$\tilde{\chi}([Z, e_n])(y_2) = 0$$

を満足する X_1 が唯一つ存在することが
 わかる. Lemma 12 より

(542)

$$\tilde{\rho}^i(e_{n+j})(y_2) = \delta_j^i,$$

$$\tilde{\chi}(e_n)(y_3) = 1$$

2" あり こと 注意す

III. $X_2 \in \mathcal{G}_2$ とし, $q_2 = \exp X_2$, $y_3 = y_2 \cdot q_2$
 とおく. このとき Lemma 12 より

(542)

$$\begin{aligned} \tilde{\chi}([Z, e_n], e_n)(y_3) &= \tilde{\chi}([Z, e_n], e_n)(y_2) \\ &\quad + \tilde{\chi}([X_2, [Z, e_n]], e_n)(y_2) \end{aligned}$$

と なる.

$$[X_2, [Z, e_n]], e_n] = -2\lambda_1^+ \langle X_2, [Z, e_n] \rangle e_n$$

2" あり から

$$\tilde{\chi}([Z, e_n], e_n)(y_3) = \tilde{\chi}([Z, e_n], e_n)(y_2)$$

$$- 2\lambda_1^+ \langle X_2, [Z, e_n] \rangle$$

と なる. $\lambda_1^+ \neq 0$ より 2" から,

$$\tilde{\chi}([Z, e_n], e_n)(y_3) = 0$$

と 満足する X_2 が 唯一つ 存在する こと
 が わかる. Lemma 12 より, この y_3 は 2" 12

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$$\tilde{\rho}^i(e_{n+j})(y_3) = \delta_j^i, \quad \tilde{\chi}(e_n)(y_3) = 1$$

$$\tilde{x}(e_n, e_n)(y_3) = \tilde{x}([Z, e_n])(y_3) = 0$$

であることに注意する。

以上によって y の存在が示された。

一意性は上の議論と Lemma から明らかである。

また対応 $a \rightarrow y$ が P の differentiable cross-section を与えることも上の議論から明らかである。

□

3. The invariant HK^2

今迄の記号を保存する Prop 12 より,
 $HK^2 = K_0^2$ である. P 上の functions K_{jkl}^i を

$$[K_0^2([e_{ntj}, e_n] \wedge e_{n+R}), e_{n+l}]$$

$$= [K_0([e_{ntj}, e_n] \wedge e_{n+R}), e_{n+l}]$$

$$= \sum_i K_{jkl}^i e_{n+i}$$

によって定義する. K_0^2 は $V^{2,2}(n,1)$ の値を持つから, K_{jkl}^i は j, k, l について symmetric である.

$$\sum_i K_{jkl}^i = 0$$

を満足する.

P 上の functions F^i を使って, P 上の functions F_{jkl}^i, F_{jk} を

$$F_{jkl}^i = \frac{\partial^3 F^i}{\partial p^j \partial p^k \partial p^l},$$

$$F_{jk} = \sum_i F_{jkl}^i$$

$$\begin{aligned} P^i &= \sum_{jkl} F_{jkl}^i z^j z^k z^l \\ &\text{と置く} \\ &= \sum_{jkl} \Phi_{jkl}^i z^j z^k z^l \\ &= P^i - \frac{1}{n+1} z^i \sum_j \frac{\partial P^j}{\partial z^j} \end{aligned}$$

によって定義する. Φ_{jkl}^i は P 上の functions Φ_{jkl}^i を

$$\Phi_{jkl}^i = F_{jkl}^i - \frac{1}{n+1} (F_{jk} \delta_l^i + F_{kl} \delta_j^i + F_{lj} \delta_k^i)$$

によって定義する。

以上の準備の下で、次のTh. を示す

Theorem. $\sigma \in \text{Prop. 12}$ における cross section $R \rightarrow P$ とする。このとき

$$\sigma^* K_{jkl}^i = \frac{1}{2} \Phi_{jkl}^i$$

が成立する

Corollary. $HK^2=0$ なるための必要かつ十分な条件は F^i が次の形であることである:

$$F^i = p^i A + B^i,$$

ここで A, B^i は $p_1, \dots, p^{n-1}$ によって高々2次の polynomials である。

Proof. $HK^2=0$ と仮定する。このとき Th. 5.1

$$(*) \quad F_{jkl}^i - \frac{1}{n+1} (F_{jk} \delta_l^i + F_{kl} \delta_j^i + F_{lj} \delta_k^i) = 0$$

である。この式に $\frac{\partial}{\partial p^t}$ を apply して

$$(*)' \quad F_{tjkl}^i - \frac{1}{n+1} (F_{tjk} \delta_l^i + F_{tkl} \delta_j^i + F_{tlj} \delta_k^i) = 0$$

を示す。ここで

$$F_{tjkl}^i = \frac{\partial}{\partial p^t} F_{jkl}^i,$$

$$F_{tjk} = \frac{\partial}{\partial p^t} F_{jk}$$

である。明らかに

$$\sum_i F_{ijkl} = F_{jkl}$$

である。 F_{jkl} は j, k, l について symmetric であるから、(*) は i と k について縮約することによって

$$F_{jkl} - \frac{3}{n+1} F_{jkl} = \frac{n-2}{n+1} F_{jkl} = 0$$

である。よって $F_{jkl} = 0$ 。よって (*) より $F_{ijkl} = 0$ である。これは F^i が $p_1, \dots, p_{n-1}$ について高々 3 次の polynomials であることと意味する。そして F^i が (*) を満足することから、 F^i が Car. にいう形であることがわかる。逆は容易に確かめられる。

さて上の Car. を使って Th. の別証明を与えよう。 R は F に $\mathbb{P}^1$ 上 regular であると仮定し、 $M = R/F$ とおく。 ι は natural immersion $\iota_M: R \rightarrow G'(T(M))$ は imbedding であると仮定する。まず R の coordinate system $\{x, y, p\}$ は $G'(T(M))$ の canonical coordinate system から得られていることと注意しよう。この注意と Car. 及び Prop から、 $HK^2 = 0$ なるための必要かつ十分な条件は $\pi = (R, \iota)$ が projective structure of the restricted type なることであることがわかる。よって Th. の別証明が得られた。

さて Th. 2 を証明しよう.

$$\text{Lemma. } \tilde{y}^i(e_n, e_n) - p^i \tilde{x}(e_n, e_n) = \tilde{x}(e_n)^2 F^i$$

$$\text{Proof. } \tilde{y}^i(e_n) = p^i \tilde{x}(e_n) \text{ であるから,}$$

$$\tilde{y}^i(e_n, e_n) = \tilde{p}^i(e_n) \tilde{x}(e_n) + p^i \tilde{x}(e_n, e_n)$$

$$\text{よって, } \tilde{p}^i(e_n) = F^i \tilde{x}(e_n) \text{ であるから, Lemma 2.2.3,}$$

以下 R の点 q を fix し, $y = \sigma(q)$ とおく. また q_1 の元 Z を fix する.

$$u = \tilde{x}(e_n), \quad u' = \tilde{x}(Z, e_n)$$

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とあけは, Lemma から

$$\delta_Z u = u', \quad \delta_Z u' = 0$$

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よって 2.12 Lemma から $u=1, u'=0$ at y であることは容易に示される.

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Lemma 2.2.3 y において次が成立する

$$(1) \quad \tilde{p}^i(Z) = \tilde{y}^i(Z, e_n),$$

$$(2) \quad \tilde{p}^i(Z, Z) = 0$$

$$\text{Proof. } (1) \quad \tilde{y}^i(e_n) = p^i \tilde{x}(e_n) = p^i u \text{ である}$$

$$\text{Lemma から } \tilde{y}^i(Z, e_n) = \tilde{y}^i(Z, e_n) \text{ である 故に}$$

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$$\tilde{y}^i([z, e_n]) = \tilde{p}^i(z)u + p^i u'$$

である。よって (1) が従う。

(2) Lemma から $\tilde{y}^i(z, [z, e_n]) = 0$ である。よって上式より (44) 4

$$0 = \tilde{p}^i(z, z)u + 2\tilde{p}^i(z)u'$$

が従う。よって (2) が従う。

(44) 4

Lemma. y において次式が成立する。

$$\delta_Z^3 (\tilde{y}^i(e_n, e_n) - p^i \tilde{x}(e_n, e_n))$$

$$= \tilde{y}^i(z, z, z, e_n, e_n) - p^i \tilde{x}(z, z, z, e_n, e_n)$$

$$+ 3\tilde{p}^i(z) \tilde{x}(z, z, e_n, e_n)$$

Proof. まず 3 次の 2 つの式である

$$\delta_Z (p^i \tilde{x}(e_n, e_n)) = \tilde{p}^i(z) \tilde{x}(e_n, e_n) + p^i \tilde{x}(z, e_n, e_n)$$

$$\delta_Z^2 (p^i \tilde{x}(e_n, e_n)) = \tilde{p}^i(z, z) \tilde{x}(e_n, e_n) + 2\tilde{p}^i(z) \tilde{x}(z, e_n, e_n)$$

$$+ p^i \tilde{x}(z, z, e_n, e_n).$$

(C3)

Lemma. γ_{12} おいて次式が成立する.

$$-2 \tilde{p}^i([K_0(Z \wedge [Z, e_n]), Z]) + 3 \tilde{p}^i(Z) \tilde{x}(Z, Z, e_n, e_n)$$

$$= \sum_{j,k,l} F_{jkl}^i \tilde{p}^j(Z) \tilde{p}^k(Z) \tilde{p}^l(Z).$$

Proof. 以下の各式は γ_{12} において評価する
 ことができる. 以下 Lemmas —, —, — から
 次式を得る

$$\tilde{y}^i(Z, Z, Z, e_n, e_n) - p^i \tilde{x}(Z, Z, Z, e_n, e_n)$$

$$+ 3 \tilde{p}^i(Z) \tilde{x}(Z, Z, e_n, e_n).$$

$$= \sum_{j,k,l} F_{jkl}^i \tilde{p}^j(Z) \tilde{p}^k(Z) \tilde{p}^l(Z).$$

§12 Lemma から

$$\tilde{y}^i(Z, Z, Z, e_n, e_n) - p^i \tilde{x}(Z, Z, Z, e_n, e_n)$$

$$= -2 \tilde{y}^i(T^2(Z)) - 2 \tilde{y}^i(T^3(Z))$$

$$+ 2 p^i \tilde{x}(T^2(Z)) + 2 p^i \tilde{x}(T^3(Z))$$

§23. γ_{12}

$$T^2(Z) = [[K_0(Z \wedge [Z, e_n]), Z], e_n],$$

$$T^3(Z) = [K_1(Z \wedge [Z, e_n]), [Z, e_n]]$$

7" あり. Lemma 7.5

$$\tilde{y}^i(T^3(Z)) = \tilde{p}^i([K_0(Z, [Z, e_1]), Z])$$

7" あり,

$$\tilde{x}(T^3(Z)) = 0$$

7" あり. さて $T^3(Z)$ は $\mathcal{G}_1$ の値をとり, R 上の differential system $D = E + F$ は $dy^i - p^i dx = 0$ で定義されるから,

$$\tilde{y}^i(T^3(Z)) - p^i \tilde{x}(T^3(Z)) = 0$$

7" あり. 以上の事実から Lemma 7.5 従う

$Z = \sum z^i e_{n+i}$ とおけば, $z^i = \tilde{p}^i(Z)$ 7" あり (Prop.). z^1, z^2

$$\tilde{x}(Z, Z, e_n, e_n) = \sum_{j, k} A_{jk} z^j z^k, \quad A_{jk} = A_{kj}$$

とおけば, Lemma 7.5 から

$$\begin{aligned} & 2 \sum_{j, k, l} K_{jkl}^i z^j z^k z^l + 3 z^i \sum_{j, k} A_{jk} z^j z^k \\ &= \sum_{j, k, l} F_{jkl}^i z^j z^k z^l \end{aligned}$$

7" あり. $z^1, \dots, z^{n-1}$ は任意 7" あり, K_{jkl}^i は j, k, l について symmetric 7" から

$$2K_{jkl}^i + (A_{jk}\delta_l^i + A_{kl}\delta_j^i + A_{lj}\delta_k^i) \\ = F_{jkl}^i$$

よって, $\sum_i K_{jkl}^i = 0$ となる。

$$(n+1)A_{jk} = F_{jk}$$

よって

$$2K_{jkl}^i = \Phi_{jkl}^i$$

よって, この式は任意の $a \in R$ に対して $\sigma(a)$ で成り立つ。よって TH の証明が完了した。

4. The invariant K^2

この paragraph において今迄の記号を保存する K' は harmonic であり, $V^{1,2}(n,1)$ の値を持つ, K' は $K_1([Z, e_n] \wedge e_n)$, $Z \in \mathcal{Z}_1^-$, 12 によって決定される. $\mathcal{Z} = \mathcal{Z}'$, $\mathcal{P}$ 上の functions K_R^i を

$$K_1([e_{n+R}, e_n] \wedge e_n) = \sum_i K_R^i e_{n+i}$$

12 によって定義する.

$$\sum_i K_i^i = 0$$

が成立する.

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \sum_j p_j^i \frac{\partial}{\partial y_j^i} + \sum_j F_j^i \frac{\partial}{\partial p_j^i}$$

を便に用いる,

$$F_R^i = \frac{d}{dx} \frac{\partial F^i}{\partial p_R} - 2 \frac{\partial F^i}{\partial y_R} - \frac{1}{2} \sum_j \frac{\partial F^i}{\partial p_j^i} \frac{\partial F^j}{\partial p_R}$$

functions F^i を使って, R 上の functions F_R^i を

$$F_R^i = \frac{\partial^2 F^i}{\partial p_R \partial x} + \sum_j \frac{\partial^2 F^i}{\partial p_R \partial y_j^i} p_j^i - 2 \frac{\partial F^i}{\partial y_R} + \sum_j \frac{\partial^2 F^i}{\partial p_R \partial p_j^i} F_j^i - \frac{1}{2} \sum_j \frac{\partial F^i}{\partial p_j^i} \frac{\partial F^j}{\partial p_R}$$

12 によって定義し, R 上の functions Φ_R^i を

$$\Phi_R^i = F_R^i - \frac{\delta_R^i}{n-1} \sum_j F_j^j$$

12 によって定義する.

以上の記号の下で次の Th. を与える.

Theorem 4.1 の Prop. 12 における cross section $R \rightarrow P$ とする。このとき

$$\omega^* K_R^i = \frac{i}{2} \Phi_R^i$$

が成立する。

この Th. の証明に進む前に, $K^i = 0$, $HK^2 \neq 0$ なる projective system of type $(n, 1)$ の例を挙げよう。

R は variables x, y^i, p^i の空間 $\mathbb{R}^{2n-1}$ とする。 R' 上の functions $F_0^i, \dots, F_{n-1}^i$ を次のように定める: $F_0^i = 0$, $2 \leq i \leq n-1$; F_1^i は x と $p_0^i, \dots, p_{n-1}^i$ だけに depend する任意の functions. $P = \{R, E, F\}$ を $F_0^i, \dots, F_{n-1}^i$ によって決まる R 上の projective system of type $(n, 1)$ とする。即ち F が equation

$$dx = dy^i = 0$$

で定義され, E が equation

$$\begin{cases} dy^i - p^i dx = 0, \\ dp^i - F^i dx = 0 \end{cases}$$

で定義される π の π' である。

明らかに $F_R^i = 0$, 従って $\Phi_R^i = 0$ π' である。よって Th. 72 よって $K^i = 0$ π' である。したがって Th. から容易に次の命題が得られる:

Proposition 4.1 $HK^2=0$ なるための必要かつ十分な条件は F' が $p_2^2, \dots, p^{n-1}$ について高々 2 次の polynomial なることである。

故に例えは F' が $p_2^2, \dots, p^{n-1}$ についての 3 次の polynomial ならば, $HK^2 \neq 0$ である。

Th. 12 より $K' = HK^2 = 0$ なるための必要かつ十分な条件は $\mathcal{P}$ が standard projection system of type $(n, 1)$, $\mathcal{P}_0$, 12 locally equivalent なることである。より上の Prop. は次のように言換えられる:
equation

$$\begin{cases} \frac{d^2 y^1}{dx^2} = F'(\frac{dy^2}{dx}, \dots, \frac{dy^{n-1}}{dx}), \\ \frac{d^2 y^k}{dx^2} = 0, \quad 2 \leq k \leq n-1 \end{cases}$$

が standard equation

$$\frac{d^2 y^i}{dx^2} = 0, \quad 1 \leq i \leq n-1$$

12 locally equivalent 12 なるための必要かつ十分な条件は, F' が $p_2^2, \dots, p^{n-1}$ について高々 2 次の polynomial なることである。

さて TH を証明しよう。以下 μ の点 q を
fix し, $y = \sigma(q)$ とおく。

Lemma. y に対する次式が成立する。

$$(1) \quad \tilde{y}^i(e_n) = p^i$$

$$(2) \quad \tilde{p}^i(e_n) = F^i$$

$$(3) \quad \tilde{y}^i(e_{n+R}, e_n) = \tilde{y}^i([e_{n+R}, e_n]) = \delta_R^i$$

$$(4) \quad \tilde{p}^i(e_{n+R}, e_n) = 2 \tilde{p}^i([e_{n+R}, e_n]) = \frac{\partial F^i}{\partial p^R}$$

Proof. (1) and (2).

$$\tilde{y}^i(e_n) = p^i \tilde{x}(e_n),$$

$$\tilde{p}^i(e_n) = F^i \tilde{x}(e_n)$$

つまり, $\tilde{x}(e_n) = 1$ at y かつ (1), (2) が成り立つ。

$$(3) \quad \tilde{y}^i(e_n) = p^i \tilde{x}(e_n) \text{ より}$$

$$\tilde{y}^i(e_{n+R}, e_n) = \tilde{p}^i(e_{n+R}) \tilde{x}(e_n) + p^i \tilde{x}(e_{n+R}, e_n),$$

$$\text{よって (Lemma)} \quad \tilde{y}^i(e_{n+R}, e_n) = \tilde{y}^i([e_{n+R}, e_n]) \text{ かつ}$$

$$\tilde{x}(e_n) = 1, \quad \tilde{p}^i(e_{n+R}) = \delta_R^i, \quad \tilde{x}(e_{n+R}, e_n) = 0 \text{ at } y \text{ かつ (3)}$$

が成り立つ。(cf. the proof of Lemma.)

$$(4) \quad \tilde{p}^i(e_n) = F^i \tilde{x}(e_n) \text{ より}$$

$$\tilde{p}^i(e_{n+R}, e_n) = \left(\sum_j \frac{\partial F^i}{\partial p^j} \tilde{p}^j(e_{n+R}) \right) \tilde{x}(e_n) + F^i \tilde{x}(e_{n+R}, e_n)$$

よって

$$\tilde{p}^i(e_{n+k}, e_n) = \frac{\partial F^i}{\partial p^k} \quad \text{at } y$$

よって 3. $\sum_{i=1}^n \tilde{y}^i(e_n) = p^i \tilde{x}(e_n)$ かつ

$$\tilde{y}^i([e_{n+k}, e_n], e_n) = \tilde{p}^i([e_{n+k}, e_n]) \tilde{x}(e_n) + p^i \tilde{x}([e_{n+k}, e_n], e_n),$$

$$\tilde{y}^i(e_n, e_n) = \tilde{p}^i(e_n) \tilde{x}(e_n) + p^i \tilde{x}(e_n, e_n),$$

$$\tilde{y}^i(e_{n+k}, e_n, e_n) = \tilde{p}^i(e_{n+k}, e_n) \tilde{x}(e_n)$$

$$+ \tilde{p}^i(e_n) \tilde{x}(e_{n+k}, e_n) + \tilde{p}^i(e_{n+k}) \tilde{x}(e_n, e_n) + p^i \tilde{x}(e_{n+k}, e_n, e_n)$$

よって 3. $\tilde{y}^i(e_{n+k}, e_n, e_n) = 2 \tilde{y}^i([e_{n+k}, e_n], e_n)$ かつ

$$\tilde{x}(e_{n+k}, e_n, e_n) = 2 \tilde{x}([e_{n+k}, e_n], e_n) = 0 \quad \text{at } y, \text{ etc.}$$

よって 上式から

$$\tilde{p}^i(e_{n+k}, e_n) = 2 \tilde{p}^i([e_{n+k}, e_n]) \quad \text{at } y$$

が従う。以上で (4) が示された。

Lemma 4.2 より お.い.て 次式 が 成.立.す.

$$\begin{aligned}
 (1) \quad & \delta_{e_{t+R}} \delta_{e_n} (\tilde{y}^i(e_n, e_n) - p^i \tilde{x}(e_n, e_n)) \\
 &= \tilde{y}^i(e_{t+R}, e_n, e_n, e_n) - p^i \tilde{x}(e_{t+R}, e_n, e_n, e_n) \\
 &\quad - \delta_R^i \tilde{x}(e_n, e_n, e_n)
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & \delta_{[e_{t+R}, e_n]} (\tilde{y}^i(e_n, e_n) - p^i \tilde{x}(e_n, e_n)) \\
 &= \tilde{y}^i([e_{t+R}, e_n], e_n, e_n) - p^i \tilde{x}([e_{t+R}, e_n], e_n, e_n)
 \end{aligned}$$

Proof. (1) 33"

$$\delta_{e_n} (p^i \tilde{x}(e_n, e_n)) = \tilde{p}^i(e_n) \tilde{x}(e_n, e_n) + p^i \tilde{x}(e_n, e_n, e_n)$$

33. 317. $\tilde{x}(e_n, e_n) = \tilde{x}(e_{t+R}, e_n, e_n) = 0$, $\tilde{p}^i(e_{t+R}) = \delta_R^i$ at 47" 33. 572 より お.い.て 次式 33

$$\begin{aligned}
 \delta_{e_{t+R}} \delta_{e_n} (p^i \tilde{x}(e_n, e_n)) &= \delta_R^i \tilde{x}(e_n, e_n, e_n) \\
 &\quad + p^i \tilde{x}(e_{t+R}, e_n, e_n, e_n)
 \end{aligned}$$

33. 572 (1) が 従.う

(2) 上と同様に, より お.い.て 次式 33.

$$\delta_{[e_{t+R}, e_n]} (p^i \tilde{x}(e_n, e_n)) = p^i \tilde{x}([e_{t+R}, e_n], e_n, e_n)$$

33. 572 (2) が 従.う

Lemma 4.3 γ は 12 次式 である

$$(1) \delta_{e_{n+R}} \delta_{e_n} (\tilde{x}(e_n)^2 F^i)$$

$$= \frac{\partial^2 F^i}{\partial p^R \partial x} + \sum_j \frac{\partial^2 F^i}{\partial p^R \partial y_j} p^j + \frac{\partial F^i}{\partial y^R} \\ + \sum_j \frac{\partial^2 F^i}{\partial p^R \partial p_j} F^j + \sum_j \frac{\partial F^i}{\partial p^j} \frac{\partial F^j}{\partial p^R}$$

$$(2) \delta_{[e_{n+R}, e_n]} (\tilde{x}(e_n)^2 F^i)$$

$$= \frac{\partial F^i}{\partial y^R} + \frac{1}{2} \sum_j \frac{\partial F^i}{\partial p_j} \frac{\partial F^j}{\partial p^R}$$

Proof (1) である

$$\delta_{e_n} (\tilde{x}(e_n)^2 F^i) = 2 \tilde{x}(e_n) \tilde{x}(e_n, e_n) F^i$$

$$+ \tilde{x}(e_n)^2 \left(\frac{\partial F^i}{\partial x} \tilde{x}(e_n) + \sum_j \frac{\partial F^i}{\partial y_j} \tilde{y}(e_n) + \sum_j \frac{\partial F^i}{\partial p_j} \tilde{p}(e_n) \right)$$

である。Prop. 2.1 と Lemma 3 考慮して、

$\delta_{e_{n+R}} \delta_{e_n} (\tilde{x}(e_n)^2 F^i)$ を計算し、(1) である。

(2) (1) と同様にして (2) である。

Lemma y において次式が成立する

$$-2K_R^i - \delta_R^i \tilde{x}(e_n, e_n, e_n) = F_R^i$$

Proof 以下の各式は y において評価する。
Lemma (b) と Lemmas —, —, — から直ちに
次式を得る:

$$\begin{aligned} & -2\tilde{y}^i(S_R^i) - 2\tilde{y}^i(S_R^2) + 2p^i \tilde{x}(S_R^i) \\ & + 2p^i \tilde{x}(S_R^2) - \delta_R^i \tilde{x}(e_n, e_n, e_n) = F_R^i. \end{aligned}$$

$\tau > \tau''$

$$S_R^i = [K_{-1}(e_n \wedge [e_{n+R}, e_n]), e_n],$$

$$S_R^2 = [K_0(e_n \wedge [e_{n+R}, e_n]), e_n]$$

τ'' 終了. Lemma, etc から

$$\tilde{y}^i(S_R^i) = \tilde{p}^i(K_{-1}(e_n \wedge [e_{n+R}, e_n]) = -K_R^i$$

と

$$\tilde{x}(S_R^i) = 0$$

を得る. さらに S_R^2 は $\mathcal{I}_1^+$ の値を持つから,

$$\tilde{y}^i(S_R^2) - p^i \tilde{x}(S_R^2) = 0$$

②"ある. 以上から Lemma 2.2.3.

$\sum_i K_i^L = 0$ ②"から, Lemma 4.4

$$-(n-1) \tilde{\chi}(e_n, e_n, e_n) = \sum_i F_i^L$$

②.2.3 よって

$$12, K_R^L = \Phi_R^L$$

②.2.3 この式は $\forall a \in R$ に対して,
 $g = \sigma(a)$ で成立する. よって Th. の証明が
 完了した.

Part I Chapter V

(R4 version)

Part I Chapter V

Chapter V The geometry of systems of ordinary differential equations of the second order

Throughout this chapter L is a projective FGLA of type $(n, 1)$, and $\mathcal{Q}$ its prolongation. As to these objects we use the notations as in § III.3 and III.4. Especially we fix bases (d_i) of $\mathcal{L}_2$, e_0 of $\mathcal{L}_1^+$, (e_i) of $\mathcal{L}_1^-$, et in the manner as in § III.4. We always assume that $n \geq 3$.

§ V.1 Normal connections of type $\mathcal{Q}$, type $\mathcal{O}$ and type $\mathcal{L}$

1. The normal connections of type $\mathcal{Q}$ associated with projective systems of type $(n, 1)$. Let $\mathcal{P} = \{R; E, F\}$ be a projective system of type $(n, 1)$, and (P, ω) the corresponding normal connection of type $\mathcal{Q}$ on R . Let us consider the invariants K^P together with their harmonic part HK^P . First of all by Theorem we have

$$K^0 = 0.$$

It follows from Proposition — that

$$HK^1 = K^1.$$

By Proposition — the spaces of harmonic forms, $H^{p,2}(\mathcal{Q})$ are calculated as follows: $H^{p,2}(\mathcal{Q}) = V^{p,2}(n, 1)$ for all p . Since the space $V^{p,2}(n, 1)$ vanishes for $p = 3, 4, 5$, it follows that

$$HK^3 = HK^4 = HK^5 = 0.$$

Therefore we know from Theorem — in — that $K' (= HK')$ and HK^2 form a fundamental system of invariants of the connection, and especially we get the following

Theorem — $K' = HK^2 = 0$ if and only if the given projection system $\mathcal{P}$ is locally isomorphic to the standard projection system of type $(n, 1)$, $\mathcal{P}_0$.

Consider the natural homomorphism K of $G^{(0)}$ onto G_0 . By Proposition — we know that K' has a tensorial property in the following sense:

$$Ra^* K' = (K')^{K(a)}, \quad a \in G^{(0)}.$$

We shall now study the invariants $K^2 = K_{-1}^2 + K_0^2 + K_1^2$ and HK^2 . K_0^2 takes values in $A_0^{h,1}$, and this space is decomposed as follows:

$$A_0^{h,1} = \mathcal{I}_0 \oplus \mathcal{I}_2^* \wedge (\mathcal{I}_1^+)^* + \mathcal{I}_0 \oplus \mathcal{I}_2^* \wedge (\mathcal{I}_1^-)^*$$

Correspondingly K_0^2 is decomposed as follows:

$$K_0^2 = {}^+K_0^2 + {}^-K_0^2.$$

Proposition — (1) K_{-1}^2 takes values in $\mathcal{I}_1^- \oplus \wedge^2 \mathcal{I}_2^*$.

(2) $K_1^2 = 0$.

(3) $2K_0^2 = 2^* K_0^2 = 0$, i.e., ${}^-K_0^2$ takes values in $V^{2,2}(n, 1)$.

This fact together with Proposition — below will be proved in §IV. 3.

We assert that $HK^2 = {}^-K_0^2$. Let us consider the inner product $(,)$ in $C^{2,2}(\mathcal{P})$. Putting $L = K_{-1}^2 + {}^+K_0^2$, we have $K^2 = L + {}^-K_0^2$ and $(L, {}^-K_0^2) = 0$.

Since HK^2 takes values in $V^{2,2}(n,1)$, we also see that $(L, HK^2) = 0$. Since $HK_0^2 = K_0^2$ by (3) of Proposition —, we therefore ^{obtain}
 $(HK^2 - K_0^2, HK^2 - K_0^2) = (HK^2 - K_0^2, K^2 - K_0^2) = (HK^2 - K_0^2, L) = 0$,
 whence $HK^2 = K_0^2$, proving our assertion. Thus we have proved the following

Proposition — $HK^2 = K_0^2$.

From this fact we can derive a tensorial property for HK^2 . Namely we have

Proposition — $R_a^*(HK^2) = (HK^2)^{K(a)}$, $a \in G^{(0)}$

Proof. By Proposition — we have $R_a^* K^2 = (K^2)^a$, $a \in G_0$, whence $R_a^* K_0^2 = (K_0^2)^a$, $a \in G_0$. Let $X_i \in \mathcal{G}_r$, where $r=1$ or 2 . Again by Proposition —, we have $\delta_{X_i} K^2 = (K^2)^{X_i}$. Hence for any $U \in \mathcal{G}_{-2}$ and $Y \in \mathcal{G}_1^-$ we obtain

$$(\delta_{X_i} K^2)(U \wedge Y) = -[X_i, K^{2-r}(U \wedge Y)] + K^{2-r}(P(X_i)U \wedge Y) + K^{2-r}(U \wedge P(X_i)Y).$$

Since $K^0 = 0$ and since $K^1 = HK^1$ takes values in $V^{1,2}(n,1)$, we find that the right hand side of the equality above vanishes. Hence $(\delta_{X_i} K^2)(U \wedge Y) = 0$, meaning that $\delta_{X_i} K_0^2 = 0$. Thus the proposition follows.

Finally concerning the invariant $K^3 = K_0^3 + K_1^3 + K_2^3$, we have the following

Proposition — $K_2^3 = 0$.

We mention that the curvature satisfies further linear relations, which will be stated in Chapter VI. (Cf. Proposition — under the assumption that $HK^2 = 0$; Propositions — and — under the assumption that $K^1 = 0$.)

2. Normal connections of type $\mathcal{O}$ and projective structures in the usual sense

simple graded Lie algebra $\mathcal{O}$ と groups $A_0, A^{(0)}, G \in \mathbb{R} \times \mathbb{R}$ における group は $\mathcal{O}$ に付随する groups である $G/A^{(0)} = P^n$ である A_0 は $G/A^{(0)}$ の原点における linear isotropy group である $GL(\mathcal{O}_{-1})$ に同一視され、従って $\mathcal{O}_0$ は $gl(\mathcal{O}_{-1})$ に同一視される $\mathcal{O}$ は $[\]$ にあける graded Lie algebra $\mathfrak{g} = V + gl(V) + V^*$ と同型であることに注意する また $[\]$ に従えば、 $\mathcal{O}$ は irreducible $\mathfrak{h}$ -module of type $P^n(\mathbb{R})$ である $\mathcal{O}$ は $(\mathcal{O}_{-1}, \mathcal{O}_0)$ の prolongation であることに注意する

容易に $H^{0,2}(\mathcal{O}) = 0$ が示される $n \geq 2$ である normal connection of type $\mathcal{O}$, (P, ω) , ω の形式を持つ。これは (P, ω) が normal projective connection であることと示している。故に次の Prop. がある

Proposition. (cf. [7] and [7]). manifolds M 上の normal connections of type $\mathcal{O}$, (P, ω) , と M 上の projective structures in the usual sense, $\mathcal{P}$, とは自然な仕方によって対応する

(= connection を
 $(P, \omega)_0$ とする.)

(P, ω) を normal connection of type $\mathcal{O}_L$
 on M , K を この curvature とする. K は
 $C^2(\mathcal{O}_L) = \mathcal{F} \otimes \wedge^2 \mathcal{O}_L^*$ の値を持つもの, 分解
 $C^2(\mathcal{O}_L) = \sum_{p=0}^2 C^p(\mathcal{O}_L)$ に従って

$$K = \sum_{p=0}^2 K^p_{\mathcal{O}_L}$$

と分解する. $K^0_{\mathcal{O}_L}$ は $(P, \omega)_0$ の torsion であり, zero
 である. $\mathcal{M} = \mathcal{O}_L + \mathcal{F}$, であるから, $C^2(\mathcal{O}_L) = \mathcal{F} \otimes \wedge^2 \mathcal{O}_L^*$
 は $C^2(\mathcal{F}) = \mathcal{F} \otimes \wedge^2 \mathcal{M}^*$ の subspace とみなさける. よって
 K は $C^2(\mathcal{F})$ の値を持つ function とみなさけ,
 分解 $C^2(\mathcal{F}) = \sum_{p=-1}^5 C^p(\mathcal{F})$ に従って,

$$K = \sum_{p=-1}^5 K^p$$

と分解される.

さて M の frame bundle $F(M) = F(M, \mathcal{O}_L)$ と
 Grassmann bundle $G^1(T(M))$ を考える. $y \in F(M)$
 とする. y は isomorphism $\mathcal{O}_L \rightarrow T(M)_x$ である. x は y の origin である. $\mathcal{F}_x^+$ は $\mathcal{O}_L$ の 1-subspace
 であるから $y\mathcal{F}_x^+$ は M の 1-dim. contact element
 である: $y\mathcal{F}_x^+ \in G^1(T(M))$. 対応 $y \rightarrow y\mathcal{F}_x^+$ は onto
 であり, $F(M)$ は $G^1(T(M))$ 上の fibre manifold である.

$$\begin{array}{ccc} F(M) & \longrightarrow & G^1(T(M)) \\ & \searrow & \swarrow \\ & M & \end{array}$$

$$(*) \begin{cases} a \in A^{(0)}, p_{0*}(a) g_{-1}^+ = g_{-1}^+ \\ \Rightarrow a \in G^{(0)} \end{cases}$$

No.

p_0 は自然な homomorphism $P \rightarrow F(M)$ として
 $(A_0 = GL(n, \mathbb{R}))$ に対して p_0 は surjective である。
 $z \in P, a \in A^{(0)}$ として $p_0(za) g_{-1}^+ = p_0(z) g_{-1}^+$
 $\Leftrightarrow a \in G^{(0)}$ であることが示される。よって
 $G^*(T(M))$ は

$$G^*(T(M)) = P / G^{(0)}$$

と表現される。よって P は $G^*(T(M))$ 上の
 $G^{(0)}$ -principal bundle である。

$$\begin{array}{ccc} P & \xrightarrow{p_0} & G^*(T(M)) \\ & \searrow & \downarrow \\ & & M \end{array}$$

Proposition (1) (P, ω) は normal connection
of type 0 on $G^*(T(M))$ である。 K はその curvature
である。 (この connection は $(P, \omega)_0$ と表す。)

$$(2) K^0 = HK^2 = 0$$

$K^0 = 0$ であるから normal connection
 $(P, \omega)_0$
system of type $(n, 1)$, $P = \{G^*(T(M)); E, F\}$ が
対応する。容易にわかるように F は
the system of vertical vectors of the fibred
manifold $G^*(T(M))$ over M である。よって
 $\pi = (G^*(T(M)), E)$ は M 上の projective system
of type $(n, 1)$ である。 ω は normal connection
of type 0 on M , $(P, \omega)_0$ に対応する
projective structure in the usual sense である。
 $\pi(\omega)$ は ω に対応する projective structure

of the restricted type, $P(\mathcal{V})$ is $\mathcal{X}(\mathcal{V})$ to which
 is a projective system of type $(n, 1)$ and is.

Proposition $P = P(\mathcal{V})$ and $\mathcal{X} = \mathcal{X}(\mathcal{V})$.

この定理の証明は次の事実を使って
 容易に示される. $V \in \mathcal{V}$ を fix する. このとき
 $\gamma(t)$ が V の geodesic ならば, $X \in \mathfrak{a}_{-1}$ と $\tilde{\omega}(X)$ の
 integral curve $\tilde{\gamma}(t)$ があって $\gamma(t) = \tilde{\gamma}(t)$
 as subsets. 反対も成立する.

$$(*) \quad a \in A^{(0)}, \rho_a(a)g_1^+ = g_1^+ \Rightarrow a \in G^{(0)}$$

Proof. $a \in A_0$ としよ. $\rho_a(a)g_1^+ = g_1^+$ は
 $\text{Ad}(a)g_1^+ \subset g_1^+ + \mathfrak{a}^{(0)} = g^{(-1)}$ を意味する. $\langle, \rangle$ に 12.17)
 $g^{(1)}$ の annihilator は g_2^+ である. よって $\text{Ad}(a)g_2 = g_2$
 $g^{(1)} = [g^{(1)}, g_2] + g_2^+$ であるから $\text{Ad}(a)g^{(1)} = g^{(1)}$
 $g^{(0)} = [g^{(-1)}, g^{(1)}] + g^{(1)}$ であるから $\text{Ad}(a)g^{(0)} = g^{(0)}$
 よって $a \in \text{Aut}(\mathfrak{g})$. $a \in \text{Aut}(\mathfrak{g}) \cap G = \text{Aut}(\mathfrak{g})^0 \cap \text{Aut}(\mathfrak{g})(G_0$.

$\mathcal{F} \cong M_2(4)$, Möbius algebra
 B_0 -structure is conformal
 structure

and allowable B_0 -structures
 No.

3. Normal connections of type $\mathcal{F}$. simple
 graded Lie algebra of the first kind, $\mathcal{F}$, &
 groups $B_0, B^{(0)}, G$ 2 考 2 3. これらの group は $\mathcal{F}$
 12 付 随 する groups τ'' あり. $G/B^{(0)} = G^r(P^n) = G^{r+1}(R^{n+1})$
 τ'' あり. B_0 は $G/B^{(0)}$ の 原 上 12 linear isotropy
 group τ'' あり. $\ell(a,b) \in GL(n+1, \mathbb{R}) \times GL(2, \mathbb{R})$ 12 詳 1 2,
 $M_{n+1,2}(\mathbb{R})$ の linear automorphism $\ell(a,b) \in$

$$\ell(a,b)X = aXb^{-1}, X \in M_{n+1,2}(\mathbb{R})$$

12 上 2 定 義 3 4 1 5', B_0 は $\mathcal{F}$ の basis
 $(e_1, \dots, e_{2n-2})$ 12 詳 1 2, $\ell(a,b)$ 12 詳 1 2, $M_{n+1,2}(\mathbb{R})$
 の linear automorphisms 全体 12 作 2 linear
 group と 12 表 現 さ 4 3

容易 12 $\mathcal{F}$ は (b_1, b_0) の prolongation τ''
 あり 2 3, i.e., $H^{1,1}(\mathcal{F}) = H^2(\mathcal{F}) = 0$ なる 2 3 あり
 確 定 あり. よ っ 2 次 の Prop. 3 2 3.

Proposition. manifold N 上 の normal
 connections of type $\mathcal{F}$, (P, ω) と N 上 の
 B_0 -structures, Q , と は 自 然 に 仕 方 2 1:1 12
 対 応 3 3.

(P, ω) 3 normal connection of type $\mathcal{F}$
 on N , K 3 3 の curvature と 3 3. K 3
 $C^2(\mathcal{F}) = \mathcal{F} \otimes \wedge^2 \mathcal{F}^*$ の 値 を 持 っ 2 あり, 分 解
 $C^2(\mathcal{F}) = \sum_{p=0}^2 C^p(\mathcal{F})$ 12 従 っ 2

$$K = \sum_{p=0}^2 K_p$$

と 分 解 と 3 3. $m = b_1 + g_1^+$ τ'' あり 2 3 あり,

$C^2(\mathcal{L}) = \mathcal{L} \otimes \Lambda^2 R_1^*$ は $C^2(\mathcal{Q}) = \mathcal{L} \otimes \Lambda^2 m^*$ の subspace とみなされ、よって K は $C^2(\mathcal{Q})$ の値を持つ function とみなされ、分解 $C^2(\mathcal{Q}) = \sum_{p=-1}^5 C^p(\mathcal{Q})$ に従って

$$K = \sum_{p=-1}^5 K^p$$

と分解される

$(P, \omega)_\mathcal{Q}$ に対して

$\mathcal{Q}$ は normal connection $\wedge N$ 上の B_0 -structure とする $\gamma \in \mathcal{Q}$ とする γ は isomorphism $\mathcal{L}_1 \rightarrow T(N)_x$ を導く $x \in \mathcal{Q}$ は γ の origin である $\mathcal{L}_1$ は $\mathcal{L}$ の $(n-1)$ -dim subspace であるから、 $\gamma \mathcal{L}_1$ は N の $(n-1)$ -dim contact element である $\gamma, \mathcal{L}_1 \in G^{n-1}(T(N))$ 対応 $\gamma \rightarrow \gamma, \mathcal{L}_1$ による $\mathcal{Q}$ の像を $R(\mathcal{Q})$ で表わす $R(\mathcal{Q})$ は $G^{n-1}(T(N))$ の submanifold であり、 $\dim R(\mathcal{Q}) = 2n-1$ である また $R(\mathcal{Q})$ は N 上の、 $\mathcal{Q}$ は $R(\mathcal{Q})$ 上の fibred manifold である

$$a \in B^{(0)}$$

$$p_\mathcal{Q}(a) \gamma_1^+ = \gamma_1^+$$

$$\Rightarrow a \in G^{(0)}$$

$$\mathcal{Q} \longrightarrow R(\mathcal{Q}) \subset G^{n-1}(T(N))$$

$$\searrow \swarrow \downarrow$$

$$N$$

$p_\mathcal{Q}$ は 自然な homomorphism $P \rightarrow \mathcal{Q}$ とする $z \in P, a \in B^{(0)}$ とすれば、 $p_\mathcal{Q}(za) \gamma_1^- = p_\mathcal{Q}(z) \gamma_1^-$
 $\Leftrightarrow a \in G^{(0)}$ であることが容易に示される
 $\therefore R(\mathcal{Q})$ は

$$R(\mathcal{Q}) = P/G^{(0)}$$

と表現される $\therefore P$ は $R(\mathcal{Q})$ 上の $G^{(0)}$ -principal

B_0 -structure Q は $\mathcal{H}$ から決まるということがわかる。
 $n \geq 4$ のとき, $\mathcal{H}$ から $K^0 = 0$ である。

よって Prop. から $K' = 0$ である。また Lemma から $T'_0 = 0$ が従うことに注意する。次に $n = 3$ の場合を考える。このとき $K^0 = HK^0$ であり, K^0 は $V^{0,2}(\mathcal{G})$ -valued である。 $X, Y \in \mathcal{G}_1^-$ のとき, $K^0(X, Y) = \text{the } \mathcal{G}_1^+ \text{-component of } K'_0(X, Y)$ である。よって Lemma から, $K^0 = 0 \iff T' = 0$ である。また $K^0 = 0$ のとき, Prop. から $K' = 0$ が従う。故に次の Prop. がある。

Proposition (1) $n \geq 4$ のとき, $K^0 = K' = 0$
 (2) $n = 3$ のとき, $K^0 = 0 \iff T' = 0$. またこの場合, $T'_0 = 0$ ならば $K' = 0$

$T'_0 = 0$ と仮定する。このとき $K^0 = 0$ である。

よって normal connection

$(P, \omega)_\mathcal{G}$ は projection system of type $(n, 1)$, $P = \{R(Q), E, F\}$ が対応する。容易にわかるように E は the system of vertical vectors of the fibred manifold $R(Q)$ on N である (cf. Th.)
 よって $\mathcal{F} = \{R(Q), F\}$ は N 上の GP system of type $(2n-2, n-1)$ である。さらに Prop. から F は $R(Q)$ から決まるということがわかる。よって Prop. から, Q は $T'_0 = 0$ を満足する任意の B_0 -structure without torsion になる。ことに注意する。

Q を B_0 -structure とする。 Q の torsion T が zero である $T'_0 = 0$ であるとき, Q を allowable B_0 -structure とよぶ。上の議論から任意の allowable (1.1),

B_0 -structure Q on N に対して 自然な仕方にて
 projective system of type $(n, 1)$, $\mathcal{P}$, と
 GP system of type $(2n-2, n-1)$, $\mathcal{Y}$, が対応する
 ことがわかった

Part I Chapter VI

Part I Chapter VI

§. Contact manifolds of order k

1. The Grassmann bundles and their prolongations. Let (n, r) be a pair of integers with $1 \leq r \leq n-1$. Let M be an n -dimensional manifold.

For every $k \geq 1$ we shall construct a manifold $J^k(M, r)$ with a differential system C^k inductively in the following manner: First $J^1(M, r)$ is defined to be the Grassmann bundle $J(M, r)$ over M , and C^1 to be the contact system C on it. Suppose that for some $k \geq 2$ we have constructed $J^l(M, r)$ together with C^l for all $1 \leq l \leq k-1$, so that $J^l(M, r)$ are fibred manifolds over $J^{l-1}(M, r)$, where we set $J^0(M, r) = M$. Let V^{k-1} be the vertical tangent bundle of the fibred manifold $J^{k-1}(M, r)$ over $J^{k-2}(M, r)$. Then $J^k(M, r)$ is defined to be the set of all r -dimensional integral elements Z^k of C^{k-1} such that $Z^k \cap V^{k-1}(Z^{k-1}) = \{0\}$, where Z^{k-1} is the origin of Z^k . Note that $J^k(M, r)$ can be inductively shown to become a fibred submanifold of the Grassmann bundle $J(J^{k-1}(M, r), r)$ over $J^{k-1}(M, r)$. Furthermore C^k is defined to be the contact system on the fibred submanifold $J^k(M, r)$, i.e., the pull-back of the contact system on $J(J^{k-1}(M, r), r)$ to

(see)
 $J^R(M, V)$ completing our inductive definition.
 We have thus obtained a sequence

$$\cdots \rightarrow J^R(M, V) \xrightarrow{p_{R-1}^R} J^{R-1}(M, V) \rightarrow \cdots \rightarrow J^1(M, V) \rightarrow J^0(M, V) = M$$

of fibred manifolds and a differential system C^R on each $J^R(M, V)$, where p_{R-1}^R denotes the projection. The pair $(J^R(M, V), C^R)$ will be called the $(R-1)$ -th prolongation of $(J^1(M, V), C)$, which will be also called the R -th prolongation of the pair (M, V) .

As above we denote by V^R the vertical tangent bundle $J^R(M, V)$ over $J^{R-1}(M, V)$, which is called the symbol system of $J^R(M, V)$.

If $0 \leq l \leq R$, $J^R(M, V)$ becomes naturally a fibred manifold over $J^l(M, V)$, and we denote by p_l^R its projection: $p_l^R = p_{R-1}^R \circ \cdots \circ p_{l+1}^R$ if $l < R$, and p_R^R is the identity. This being said, we define differential systems C_l^R and V_l^R ($1 \leq l \leq R$) on $J^R(M, V)$ as follows:

$$C_l^R = (p_l^R)_*^{-1}(C^l), \quad V_l^R = (p_l^R)_*^{-1}(V^l).$$

Clearly we have $C_R^R = C^R$, $V_R^R = V^R$, and

$$C_l^R \supset C_{l+1}^R, \quad V_l^R \supset V_{l+1}^R, \quad C_l^R \supset V_l^R.$$

Let z_0^R be a point of $J^R(M, r)$. Let $(x^1, \dots, x^r, y^1, \dots, y^{n-r})$ be a coordinate system of M at $p_0^R(z_0^R)$ such that the 1-forms $dx^1, \dots, dx^r$ are linearly independent, restricted to the r -dimensional contact element $p_1^R(z_0^R)$ to M . We denote by U the domain of the coordinate system, and by $\tilde{U}$ the set of all $z^R \in J^R(M, r)$ such that $p_0^R(z^R) \in U$, and $dx^1, \dots, dx^r$ are linearly independent, restricted to $p_1^R(z^R)$. It is clear that $\tilde{U}$ is an open set of $J^R(M, r)$, $z_0^R \in \tilde{U}$, and $p_0^R(\tilde{U}) = U$. In the following the indices $i, j, i_1, i_2, \dots$ will range over the integers $1, \dots, r$, and the index α over the integers $1, \dots, n-r$.

Lemma. There are unique functions $p_{i_1 \dots i_\ell}^\alpha$ ($1 \leq \ell \leq R$) on $\tilde{U}$ satisfying the following conditions:

1) $p_{i_1 \dots i_\ell}^\alpha$ are symmetric with respect to the indices $i_1, \dots, i_\ell$.

2) The functions $x^i (= x^i \circ p_0^R)$, $y^\alpha (= y^\alpha \circ p_0^R)$ and $p_{i_1 \dots i_\ell}^\alpha$ ($i_1 \leq \dots \leq i_\ell$, $1 \leq \ell \leq R$) together give a coordinate system of $J^R(M, r)$ on $\tilde{U}$.

3) V_ℓ^R is defined by the Pfaffian equations:

$$dx^i = dy^\alpha = dp_{i_1 \dots i_m}^\alpha = 0 \quad (1 \leq m \leq \ell-1),$$

and C_ℓ^R by the Pfaffian equations:

$$\begin{cases} \int dy^\alpha - \sum_i p_i^\alpha dx^i = 0, \\ dp_{i_1 \dots i_m}^\alpha - \sum_i p_{i_1 \dots i_m i}^\alpha dx^i = 0 \quad (1 \leq m \leq l-1) \end{cases}$$

The coordinate system $(x^i, y^\alpha, p_{i_1 \dots i_\ell}^\alpha) (1 \leq \ell \leq R)$ will be called a canonical coordinate system of $J^R(M, \nu)$ at z_0^R .

Let S be an r -dimensional submanifold of M . Then S is prolonged to the r -dimensional submanifold $p(S)$ of $J^1(M, r)$, and in turn this is prolonged to the r -dimensional submanifold $p^2(S) = p(p(S))$ of $J^2(M, r)$, which is clearly in $J^2(M, r)$. In this way S is prolonged to the r -dimensional submanifold $p^k(S)$ of $J^k(M, r)$ for every k . Note that $p^k(S)$ gives an integral manifold of the contact system C^k .

Lemma. Let $\tilde{S}$ be an r -dimensional integral manifold of C^k . Assume that the projection p_0^k maps $\tilde{S}$ diffeomorphically onto an r -dimensional submanifold S of M . Then $\tilde{S} = p^k(S)$.

Let us now consider a canonical coordinate system $(x^i, z^{\alpha}, p_{i\alpha}^{\beta})$ ($1 \leq i \leq r$) of $J^k(M, r)$ at a point z_0^k . Let S be an r -dimensional submanifold of M . Suppose that $p^k(S)$ is in the domain $\bar{U}$ of the coordinate system, and S is defined by the equations:

$$z^{\alpha} = u^{\alpha}(x^1, \dots, x^r).$$

Then we remark that $p^k(S)$ is defined by the equations:

$$\begin{cases} y^{\alpha} = u^{\alpha}(x^1, \dots, x^r), \\ p_{i_1 \dots i_\ell}^{\alpha} = \frac{\partial^{\ell} u^{\alpha}}{\partial x^{i_1} \dots \partial x^{i_\ell}}(x^1, \dots, x^r) \quad (1 \leq \ell \leq k). \end{cases}$$

For simplicity we set as follows:

$$R = J^R(M, r), \quad D = C^R = \mathbb{C}^R,$$

$$D_\ell = C_\ell^R, \quad F_\ell = V_\ell^R \quad (1 \leq \ell \leq R)$$

Lemma^A. (1) $D = D_R$ is regular, and D_ℓ is the $(R-\ell)$ -th derived system of D_R for $0 \leq \ell \leq R-1$, where D_0 is given by $D_0 = T(R)$. Furthermore D_ℓ is the first derived system of $D_{\ell+1}$ for $0 \leq \ell \leq R-1$.

$$(2) \quad D_\ell = \text{ch}(D_{\ell-1}), \quad 2 \leq \ell \leq R,$$

$$F_\ell \subset D_\ell, \quad 1 \leq \ell \leq R,$$

$$F_\ell = D_\ell \cap F_{\ell-1}, \quad 2 \leq \ell \leq R$$

$$(3) \quad D_R \text{ is non-degenerate, i.e., } \text{ch}(D_R) = 0$$

$$(4) \quad \text{rank}(D_\ell/D_{\ell+1}) = (n-r)_r H_\ell, \quad 0 \leq \ell \leq R-1,$$

$$\text{rank}(D_\ell/F_\ell) = r, \quad 1 \leq \ell \leq R,$$

$$\text{rank}(F_\ell/F_{\ell+1}) = (n-r)_r H_\ell, \quad 1 \leq \ell \leq R-1,$$

$$\text{rank}(F_R) = (n-r)_r H_R$$

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Recall that $r H_p = \binom{r+l-1}{l}$.

By virtue of Lemma we may speak of the symbol algebra $F = (f, (\mathcal{F}_p)_{p \leq 0})$ of (R, D) at any point $z \in R$:

$$\mathcal{I}_1 = D_R(z),$$

$$\mathcal{F}_p = D_{R+p+1}(z) / D_{R+p+2}(z), \quad p \leq -2,$$

where we set $D_l = T(R)$ if $l \leq 0$. Let f denote the subspace $F_R(z)$ of $\mathcal{I}_1$.

Lemma^B. (1) The FGLA, F , is of the $(R+1)$ -th kind and non-degenerate

(2) $[\mathcal{F}_p, \mathcal{F}_q] = \{0\}$ for $p, q \leq -2$, $[\mathcal{F}_p, f] = \{0\}$ for $p \leq -2$, and $[f, f] = \{0\}$.

(3) If $p \leq -2$, $X \in \mathcal{F}_p$, and $[X, Y] = 0$ for all $Y \in \mathcal{I}_1$, then $X = 0$.

(4) $\dim \mathcal{I}_{-(R+1)} = n-r$, $\dim \mathcal{F}_p = (n-r) r H_{R+p+1}$ for $-R \leq p \leq -2$, $\dim (\mathcal{I}_1 / f) = r$, and $\dim f = (n-r) r H_R$.

(5) There is a subspace e of $\mathcal{I}_1$ such that $\mathcal{I}_1 = e + f$ (direct sum) and $[e, e] = \{0\}$.

This can be easily derived from Lemma together with Lemma.

Here we remark that if $R \geq 2$, f consists of all $X \in \mathcal{G}_1$ such that $[X, \mathcal{G}_2] = \{0\}$ showing that f is naturally associated with F (or rather naturally constructed from F). In the case where $R=1$ we have the following

Lemma. If $R=1$ and $r \leq n-2$, then f is naturally associated with F .

Indeed, it is known that f is spanned by all the elements $X \in \mathcal{G}_1$ such that $\text{rank}(\text{ad}(X)) = 1$.

It should be noted that if $R=1$ and $r=n-1$, F is a contact FGLA, i.e., $\dim \mathcal{G}_2 = 1$, and hence f is not naturally associated with F .

By Lemma we know that the systems D_ℓ ($1 \leq \ell \leq R$) and F_ℓ ($2 \leq \ell \leq R$) are all naturally associated with the system D or covariant systems of C^R in E . Cartan's terminology. As for the system F_1 we have the following

Lemma. If $r \leq n-2$, then F_1 is naturally associated with D .

Indeed this lemma can be easily reduced to the case where $R=1$, and in this case the lemma follows immediately from Lemma. (Another more direct proof can be given by

considering the symbol algebra of F_1 at any $z \in R$ and by using Lemma.)

It should be noted that $(J^2(M, n-1), C')$ is a contact manifold of degree n , and hence V' is not naturally associated with C' . It follows that if $r = n-1$, F_1 is not naturally associated with D .

Now, let us consider another n -dimensional manifold M' and the prolongations $J^R(M', r)$ of the pair (M', r) .

Assuming that M is diffeomorphic with M' , let φ be a diffeomorphism of M onto M' . Then φ is prolonged to the diffeomorphism $p^1(\varphi)$ of $J^1(M, r)$ onto $J^1(M', r)$, and in turn this is prolonged to the diffeomorphism $p^2(\varphi) = p(p^1(\varphi))$ of $J(J^1(M, r), r)$ onto $J(J^1(M', r), r)$, which clearly sends $J^2(M, r)$ onto $J^2(M', r)$. In this way φ is prolonged to the diffeomorphism $p^R(\varphi)$ of $J^R(M, r)$ onto $J^R(M', r)$ for every R . We note that $p^R(\varphi)$ gives an isomorphism of $(J^R(M, r), C^R)$ onto $(J^R(M', r), C^R)$.

By Lemma we have

Lemma. Let $\tilde{\varphi}$ be any local isomorphism of $(J^R(M, r), C^R)$ onto $(J^R(M', r), C^R)$ at any $(z, z') \in J^R(M, r) \times J^R(M', r)$. If $r \leq n-2$, there is a unique local diffeomorphism φ of M to M' at $(p_0^R(z), p_0^R(z'))$ such that $\tilde{\varphi} = p^R(\varphi)$.

Proof of Lemma. Let us consider the symbol algebra $\bar{F} = \sum_{p=0}^{\infty} \bar{F}_p$ of C_1^R at any point $z \in J^R(M, V)$: $\bar{F}_p = \{0\}$ for $p \leq -2$, $\bar{F}_{-2} = T_z(J^R(M, V)/C_1^R(z))$, and $\bar{F}_{-1} = C_1^R(z)$. Let $\bar{F}$ and $\bar{F}_0$ denote the subspaces $V_1^R(z)$ and $V_2^R(z)$ of $\bar{F}_{-1}$ respectively. Clearly $\bar{F}_0$ consists of all $X \in \bar{F}_{-1}$ such that $[X, \bar{F}_{-1}] = \{0\}$, and hence it is an graded ideal of $\bar{F}$. Therefore $\bar{F}/\bar{F}_0 (\cong \bar{F}_{-2} + (\bar{F}_{-1}/\bar{F}_0))$ becomes naturally a graded Lie algebra. Let us now consider the symbol algebra $F' = \sum_{p=0}^{\infty} F'_p$ of C' at $x = P_1^R(z)$: $F'_p = \{0\}$ for $p \leq -2$,

$F'_{-2} = T_x(J^R(M, V)/C'(x))$, and $F'_{-1} = C'(x)$. Let F' denote the subspace $V'(x)$ of F'_{-1} . Then we see that there is a natural isomorphism of $\bar{F}/\bar{F}_0$ onto F' (as graded Lie algebras) which sends the subspace $\bar{F}/\bar{F}_0$ of $\bar{F}/\bar{F}_0$ onto the subspace F' of F'_{-1} . Consequently it follows from Lemma that $\bar{F}$ is naturally associated with the graded Lie algebra $\bar{F}$, which clearly implies that V_1^R is naturally associated with C_1^R and hence C^R .

2. Contact manifolds and their prolongation

Let (M, C) be a contact manifold of degree n .

We denote by $L(M, C)$ the set of all $(n-1)$ -dimensional integral elements of C , which can be shown to become naturally a fibred submanifold of the Grassmann bundle $J(M, n-1)$ over M . $L(M, C)$ will be called the Lagrange-Grassmann bundle over (M, C) .

We denote by $\hat{C}$ the contact system on $L(M, C)$.

..... $L^R(M, C)$

For every $R \geq 1$ we shall now construct a manifold $\Gamma^R(M, C)$ with a differential system C^R inductively in the following manner: First $\Gamma^1(M, C)$ is defined to be the manifold M , and C^1 to be the contact structure C . Second $\Gamma^2(M, C)$ is defined to be the Lagrange-Grassmann bundle $L(M, C)$, and C^2 to be the contact system $\hat{C}$ on it. Suppose that for some $R \geq 3$ we have constructed $\Gamma^{R-1}(M, C)$ together with C^{R-1} for all $1 \leq l \leq R-1$, so that $\Gamma^l(M, C)$ are fibred submanifolds over $\Gamma^{l-1}(M, C)$ for all $2 \leq l \leq R-1$. Let V^{R-1} be the vertical tangent bundle of the fibred manifold $\Gamma^{R-1}(M, C)$ over $\Gamma^{R-2}(M, C)$. Then $\Gamma^R(M, C)$ is defined to be the set of all $(n-1)$ -dimensional integral elements z^R of C^{R-1} such that $z^R \cap V^{R-1}(z^{R-1}) = \{0\}$, where z^{R-1} is the origin of z^R . Note that $\Gamma^R(M, C)$ can be inductively shown to become a fibred:

submanifold of the Grassmann bundle $J(\Gamma^{R-1}(M, C), n-1)$ over $\Gamma^{R-1}(M, C)$. Furthermore C^R is defined to be the contact system on the fibred submanifold $\Gamma^R(M, C)$.

We have thus obtained a sequence

$$\dots \Gamma^R(M, C) \xrightarrow{p_{R-1}^R} \Gamma^{R-1}(M, C) \rightarrow \dots \rightarrow \Gamma^1(M, C) = M$$

of fibred manifolds and a differential system C^R on each $\Gamma^R(M, C)$, where p_{R-1}^R denotes the projection. The pair $(\Gamma^R(M, C), C^R)$ will be called the $(R-1)$ -th prolongation of the contact manifold (M, C) .

As above we denote by V^R the vertical tangent bundle of the fibred manifold $\Gamma^R(M, C)$ over $\Gamma^{R-1}(M, C)$, which is called the symbol system of $\Gamma^R(M, C)$.

Remark: Let us consider the prolongations $(J^R(\bar{M}, n-1), \bar{C}^R)$ of the pair $(\bar{M}, n-1)$, where $\bar{M}$ is an n -dimensional manifold. Then $(M, C) = (J^1(\bar{M}, n-1), \bar{C}^1)$ is a contact manifold of degree n , and for every $R \geq 1$ $(J^R(\bar{M}, n-1), \bar{C}^R)$ may be naturally identified with an open submanifold of $(\Gamma^R(M, C), C^R)$ (as manifolds with differential system).

If $1 \leq l \leq R$, $\Gamma^R(M, C)$ becomes naturally a fibred manifold over $\Gamma^l(M, C)$, and we denote by p_l^R the projection of $\Gamma^R(M, C)$ onto $\Gamma^l(M, C)$. As above let V^l ($l \geq 2$) denote the vertical tangent bundle of the fibred manifold $\Gamma^l(M, C)$ over $\Gamma^{l-1}(M, C)$. Then we define differential systems

C_l^R ($1 \leq l \leq R$) and V_l^R ($2 \leq l \leq R$) on $\Gamma^R(M, C)$ as follows:

$$C_l^R = (\rho_l^R)^{-1}(C^l), \quad V_l^R = (\rho_l^R)^{-1}(V^l).$$

Let z' be any point of M . By Darboux's theorem there is a coordinate system $(x_1, \dots, x_{n-1}, y, p_1, \dots, p_{n-1})$ of M at z' such that C is defined by the Pfaffian equation

$$dy - \sum_{i=1}^{n-1} p_i dx_i = 0.$$

Such a coordinate system will be called a canonical coordinate system of (M, C) .

Now, let z_0^R be a point of $\Gamma^R(M, C)$. Let $(x_1, \dots, x_{n-1}, y, p_1, \dots, p_{n-1})$ be a canonical coordinate system of (M, C) at $\rho_1^R(z_0^R)$ such that the 1-forms $dx_1, \dots, dx_{n-1}$ are linearly independent, restricted to the $(n-1)$ -dimensional contact element $\rho_2^R(z_0^R)$ to M . (Since $\rho_2^R(z_0^R) \in L(M, C)$, we easily see that such a canonical coordinate system exists for the given z_0^R .) We denote by U the domain of the coordinate system, and by $\tilde{U}$ the set of all $z^R \in \Gamma^R(M, C)$ such that $\rho_1^R(z^R) \in U$, and $dx_1, \dots, dx_{n-1}$ are linearly independent, restricted to $\rho_2^R(z^R)$. It is clear that $\tilde{U}$ is an open set of $\Gamma^R(M, C)$, $z_0^R \in \tilde{U}$, and $\rho_1^R(\tilde{U}) = U$. In the following the indices $i, i_1, i_2, \dots$ will range over the integers $1, \dots, n-1$.

Lemma. There are unique functions $p_{i_1 \dots i_l}$ ($2 \leq l \leq k$) on $\tilde{U}$ satisfying the following conditions:

1) $p_{i_1 \dots i_l}$ are symmetric with respect to the indices $i_1, \dots, i_l$.

2) The functions $x^i (= x^i \circ \rho_1^k)$, $y (= y \circ \rho_1^k)$, $p_i (= p_i \circ \rho_1^k)$ and $p_{i_1 \dots i_l}$ ($i_1 \leq \dots \leq i_l$, $2 \leq l \leq k$) together give a coordinate system of $\Gamma^k(M, C)$ on $\tilde{U}$.

3) V_l^k is defined by the Pfaffian equations:

$$dx^i = dy = dp_{i_1 \dots i_m} = 0 \quad (1 \leq m \leq l-1),$$

and C_l^k by the Pfaffian equations

$$\begin{cases} dy - \sum_i p_i dx^i = 0, \\ dp_{i_1 \dots i_m} - \sum_i p_{i_1 \dots i_m i} dx^i = 0 \quad (1 \leq m \leq l-1). \end{cases}$$

The coordinate system $(x^i, y, p_{i_1 \dots i_l})$ ($1 \leq l \leq k$) will be called a canonical coordinate system of $\Gamma^k(M, C)$ at z_0^k .

By a Lagrangean submanifold of the contact manifold (M, C) we mean an $(n-1)$ -dimensional integral manifold S of C . Let S be a Lagrangean submanifold of (M, C) . Then S is prolonged to the $(n-1)$ -dimensional submanifold $p(S)$ of $T(M, n-1)$, which is clearly in $\Gamma^2(M, C)$. In this way S is prolonged to the $(n-1)$ -dimensional submanifold $p^{R-1}(S)$ of $\Gamma^R(M, C)$ for every R . Note that $p^{R-1}(S)$ gives an integral manifold of the contact system C^R .

Lemma. Let S be an $(n-1)$ -dimensional integral manifold of C^R . Assume that the projection p_1^R maps S diffeomorphically onto an $(n-1)$ -dimensional submanifold $\tilde{S}$ of M . Then $\tilde{S}$ is a Lagrangean submanifold of (M, C) , and $\tilde{S} = p^{R-1}(S)$.

Let us now consider a canonical coordinate system $(x^i, y, p_i, \dots, p_r)$ ($1 \leq i \leq r$) of $\Gamma^R(M, C)$ at a point z_0^R . Let S be a Lagrangean submanifold of (M, C) . Suppose that $p^{R-1}(S)$ is in the domain $\tilde{U}$ of the coordinate system, and S is defined by the equations:

$$\begin{cases} y = u(x^1, \dots, x^{n-1}) \\ p_i = \frac{\partial u}{\partial x^i}(x^1, \dots, x^{n-1}) \end{cases}$$

Then we remark that $p^{R-1}(S)$ is defined by the equations

$$\begin{cases} y = u(x_1', \dots, x_{n-1}'), \\ p_{i_1, \dots, i_l} = \frac{\partial^l u}{\partial x_{i_1}' \dots \partial x_{i_l}'}(x_1', \dots, x_{n-1}') \quad (1 \leq l \leq R). \end{cases}$$

For simplicity we set as follows:

$$R = \Gamma^R(M, C), \quad D = C_R^R = C^R,$$

$$D_l = C_l^R \quad (1 \leq l \leq R), \quad F_l = V_l^R \quad (2 \leq l \leq R).$$

Then we notice that Lemma holds true in our present situation, where, of course, the assertions concerning the system F , should be neglected, and r should be replaced by $n-1$.

Therefore we may speak of the symbol algebra $F = (F, (\mathbb{Z}_p)_{x_0})$ of (R, D) at any point $z \in R$. We then notice that Lemma holds true in addition, where r should be replaced by $n-1$.

Now, let us consider another contact manifold (M', C') of degree n and their prolongations $\Pi^k(M', C')$. Assuming that (M, C) is isomorphic with (M', C') , let φ be a (contact) isomorphism of (M, C) onto (M', C') . Then φ is prolonged to the diffeomorphism $p(\varphi)$ of $J(M, n-1)$ onto $J(M', n-1)$ which clearly sends $\Pi^2(M, C)$ onto $\Pi^2(M', C')$. In this way φ is prolonged to the diffeomorphism $p^{k-1}(\varphi)$ of $\Pi^{k-1}(M, C)$ onto $\Pi^{k-1}(M', C')$ for every k . Note that $p^{k-1}(\varphi)$ gives an isomorphism of $(\Pi^{k-1}(M, C), C^k)$ onto $(\Pi^{k-1}(M', C'), C'^k)$.

Lemma. Let $\tilde{\varphi}$ be any local isomorphism of $(\Pi^k(M, C), C^k)$ to $(\Pi^k(M', C'), C'^k)$ at any $(z, z') \in \Pi^k(M, C) \times \Pi^k(M', C')$. Then there is a unique local (contact) isomorphism φ of (M, C) to (M', C') at $(p_1^k(z), p_1^k(z'))$ such that $\tilde{\varphi} = p^{k-1}(\varphi)$.

3. Contact manifolds of order k . Following Yamaguchi, we shall introduce the notion of a contact manifold of order k of bidegree (n, r) , where $k \geq 1$, and $1 \leq r \leq n-1$. Hereafter "of bidegree $(n, n-1)$ " will be also called "of degree n ".

(A) The case where $1 \leq r \leq n-2$. Let R be a manifold, and D a differential system on it. Then the pair (R, D) is called a contact manifold of order k of bidegree (n, r) , if there are differential systems D_l and F_l ($1 \leq l \leq k$) on R satisfying the following conditions:

1) $D_k = D$, and D_l is the first derived system of D_{l+1} for $1 \leq l \leq k-1$.

2) $\text{rank}(T(R)/D_1) = n-r$.

3) $F_1 = \text{ch}(D_{k-1}) \subset D_k$, and $\text{rank}(D_l/F_l) = r$ for $2 \leq l \leq k$.

4) F_1 is completely integrable, $F_2 \subset F_1 \subset D_1$, $\text{rank}(D_1/F_1) = r$, and $F_k = D_k \cap F_1$.

5) D_k is non-degenerate.

6) $\dim R = n + (n-r) \sum_{l=1}^k r_l H_l$.

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 (F_ℓ) family of auxiliary systems

Now, let M be an n -dimensional manifold. Then we see from Lemma that the R -th prolongation $(J^R(M, \gamma), C^R)$ of the pair (M, γ) becomes a contact manifold of order R of bidegree (n, r) . The next lemma implies that every contact manifold (R, D) of order R of bidegree (n, r) is locally isomorphic with $(J^R(M, \gamma), C^R)$. Therefore we know that the system F_1 together with the systems D_ℓ ($1 \leq \ell \leq R$) and F_ℓ ($2 \leq \ell \leq R$) is naturally associated with D (Lemma). (The differential systems D_ℓ and F_ℓ ($1 \leq \ell \leq R$) will be called associated with D , (*))

Let (R, D) be a contact manifold of order R of bidegree (n, r) . Assume that R is regular with respect to the completely integrable system F_1 . Set $M = R/F_1$, and let p_0 be the projection of R onto M . Clearly we have $\dim M = n$.

Lemma. There is a unique open immersion ι of R to $J^R(M, \gamma)$ such that $p_0 = p_0^R \circ \iota$, and $D = \iota_*^{-1}(C^R)$.

Note that $D_\ell = \iota_*^{-1}(C_\ell^R)$, and $F_\ell = \iota_*^{-1}(V_\ell^R)$ for $1 \leq \ell \leq R$.

(*) In particular the system F_R will be called the symbol system of (R, D) , which corresponds to the symbol system V^R of $J^R(M, \gamma)$. We also remark that we may speak of a canonical coordinate system of (R, D) , which corresponds to a canonical coordinate system of $J^R(M, \gamma)$.

(B) The case where $r = n-1$. Let R be a manifold, and D a differential system on it. Then the pair (R, D) is called a contact manifold of order k of degree n , if there are differential systems D_ℓ ($1 \leq \ell \leq k$) and F_ℓ ($2 \leq \ell \leq k$) on R satisfying the following conditions:

- 1) $D_k = D$, and D_ℓ is the first derived system of $D_{\ell+1}$ for $1 \leq \ell \leq k-1$.
- 2) $\text{rank}(T(R)/D_1) = 1$, and $\text{rank}(D_1/D_2) = n-1$.
- 3) $F_\ell = \text{ch}(D_{\ell-1}) \subset D_\ell$, and $\text{rank}(D_\ell/F_\ell) = n-1$ for $2 \leq \ell \leq k$.
- 4) $F_k = D_k \cap F_2 \dots$
- 5) D_k is non-degenerate.
- 6) $\dim R = n + \sum_{\ell=1}^k n-1 H_\ell$.

Let (M, C) be a contact manifold of degree n . Then we remark that the $(k-1)$ -th prolongation $(\Gamma^k(M, C), C^k)$ of (M, C) becomes a contact manifold of order k of degree n (cf. Lemma). The next lemma implies that every contact manifold (R, D) of order k of degree n is locally isomorphic with

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$(\Gamma^R(M, C), C^R)$. As before the differential systems D_ℓ ($1 \leq \ell \leq k$) and F_ℓ ($2 \leq \ell \leq k$) will be called associated, with D . In particular the system F_k will be called the symbol system of (R, D) . We further remark that we may speak of a canonical coordinate system of (R, D) .

Let (R, D) be a contact manifold of order k of degree n . Assume that R is regular with respect to the completely integrable system $F_2 = \text{ch}(D_1)$. Set $M = R/F_2$, and let p_1 be the projection of R onto M . Then there is a unique differential system C on M such that $F_2 = p_{1*}^{-1}(C)$. Clearly the pair (M, C) gives a contact manifold of degree n .

Lemma. There is a unique open immersion ι of R to $\Gamma^R(M, C)$ such that $p_1 = p_1^R \circ \iota$, and $D = \iota_*^{-1}(C^R)$.

Note that $D_\ell = \iota_*^{-1}(C_\ell^R)$ ($1 \leq \ell \leq k$), and $F_\ell = \iota_*^{-1}(V_\ell^R)$ ($2 \leq \ell \leq k$).

We have thus introduced the notion of a contact manifold of order k .

Incidentally we shall define the prolongation of a contact manifold (M, C) of order $k-1$ of bidegree (n, r) , where $k \geq 2$, and $1 \leq r \leq n-1$. Let V be the symbol system of (M, C) , which exists unless $k=2$, $r=n-1$. We denote by $L(M, C)$ the set of all r -dimensional integral elements Z of C such that $Z \cap V(x) = \{0\}$, x being the origin of Z . Then $L(M, C)$ can be shown to become a fibred submanifold of the Grassmann bundle $J(M, r)$ over M . Let $\hat{C}$ denote the contact system on the fibred submanifold $L(M, C)$. It is easy to see that the pair $(L(M, C), \hat{C})$ becomes a contact manifold of order k of bidegree (n, r) , which will be called the prolongation of (M, C) .

Let S be an r -dimensional integral manifold of C which is transversal to V : $T_x(S) \cap V(x) = \{0\}$ at each $x \in M$. Then S is prolonged to the r -dimensional submanifold $p(S)$ of $J(M, C)$, which is clearly in $L(M, C)$. Note that $p(S)$ gives an integral manifold of $\hat{C}$.

Let us now consider another contact manifold (M', C') of order $k-1$ of bidegree (n, r) and its prolongation $(L(M', C'), \hat{C}')$. Assuming that (M, C) is isomorphic with (M', C') , let φ be an isomorphism of (M, C) onto (M', C') . Then φ is

prolonged to the diffeomorphism, $p(\varphi)$ of $J(M, r)$ onto $J(M', r')$, which clearly sends $L(M, C)$ onto $L(M', C')$. Note that $p(\varphi)$ gives an isomorphism of $(L(M, C), \hat{C})$ onto $(L(M', C'), \hat{C}')$.

Now, let M be an n -dimensional manifold, and let us consider the prolongations $(J^R(M, r), C^R)$ of the pair (M, r) . Then we remark that $J^R(M, r)$ are inductively defined as follows:

$$J^1(M, r) = J(M, r), \quad J^R(M, r) = L(J^{R-1}(M, r), C^{R-1}), \quad R \geq 2.$$

Furthermore let (M, C) be a contact manifold of degree n , and let us consider its prolongations $(\Gamma^R(M, C), C^R)$. Then we remark that $\Gamma^R(M, C)$ are inductively defined as follows

$$\Gamma^1(M, C) = M, \quad \Gamma^R(M, C) = L(\Gamma^{R-1}(M, C), C^{R-1}), \quad R \geq 2.$$

§. Pseudo-projective systems of order k
 $\rightarrow$ FD (differential equation of finite type)

1. $\odot$ structures of order k . In this paragraph we shall introduce the notion of an OD structure of order k of bidegree (n, r) , where $k \geq 2$, and $1 \leq r \leq n-1$. "OD", of course, means "ordinary differential".

(A) The case where $k=2$ or $k \geq 3$, $r \leq n-2$. Let M be an n -dimensional manifold, and let us consider the $(k-1)$ -th prolongation $(J^{k-1}(M, r), C^{k-1})$ of the pair (M, r) . Let R be an open fibred submanifold of the fibred manifold $J^{k-1}(M, r)$ over M . We denote by D and F the restrictions to R of the contact system C^{k-1} and the symbol system V^{k-1} respectively. Let E be a differential system on R . Then the pair $\mathcal{X} = (R, E)$ is called an OD structure of order k of bidegree (n, r) on M , if it satisfies the following conditions:

1) E is completely integrable,

2) $D = E + F$ (direct sum).

Let $\mathcal{X} = (R, E)$ (resp. $\mathcal{X}' = (R', E')$) be an OD structure of order k of bidegree (n, r) on a manifold M (resp. on M'). By an isomorphism of $\mathcal{X}$ to $\mathcal{X}'$ we mean a

diffeomorphism φ of M onto M' such that the prolongation $p^{R-1}(\varphi)$ of φ gives an isomorphism of (R, E) onto (R', E') (as manifolds with differential system). Let (z, z') be a point of $R \times R'$. By a local isomorphism of $\mathcal{K}$ to $\mathcal{K}'$ at (z, z') we mean a local diffeomorphism φ of M to M' at $(p_0^{R-1}(z), p_0^{R'-1}(z'))$ such that $p^{R-1}(\varphi)$ gives a local isomorphism of (R, E) to (R', E') at (z, z') .

Clearly an OD structure of order 2 of bidegree (n, r) means a pseudo-projective structure of bidegree (n, r) in the sense of Chapter I.

Let $\mathcal{K} = (R, E)$ be an OD structure of order k of bidegree (n, r) on a manifold M . Then we say that an r -dimensional submanifold S of M is a solution of $\mathcal{K}$, if the prolongation $p^k(S)$ of S is in R , and is an integral manifold of E . (Note that $\text{rank}(E) = r$). In this way the structure $\mathcal{K}$ may be regarded as a differential equation.

Let z_0 be any point of R , and let $(x^i, y^i, p_{ij}^{x,i})$ ($1 \leq i \leq k-1$) be any canonical coordinate system of $J^k(M, r)$ at z_0 . Let $\tilde{U}$ denote the domain of the coordinate system. By Lemma we easily see that there are unique functions $f_{ij}^{x,i}$ on the intersection $R \cap \tilde{U}$ such that E is defined by the Pfaffian equations:

$$(*) \quad \begin{cases} dy^\alpha - \sum_i p_i^\alpha dx^i = 0, \\ d p_{i_1 \dots i_l}^\alpha - \sum_i p_{i_1 \dots i_{l-1} i}^\alpha dx^i = 0 \quad (1 \leq l \leq R-2), \\ d p_{i_1 \dots i_{R-1}}^\alpha - \sum_i f_{i_1 \dots i_{R-1} i}^\alpha dx^i = 0. \end{cases}$$

Now, let S be an r -dimensional submanifold of M . Suppose that $p^{R-1}(S) \subset \tilde{U}$, and S is defined by the equations $y^\alpha = u^\alpha(x^1, \dots, x^r)$. Then it is easy to see that S is a solution of $\mathcal{K}$, if and only if (u^α) gives a solution of the system of partial differential equations:

$$(*)' \quad \frac{\partial^R y^\alpha}{\partial x^{i_1} \dots \partial x^{i_R}} = f_{i_1 \dots i_R}^\alpha((x^i), (y^\alpha), (\frac{\partial^l y^\alpha}{\partial x^{i_1} \dots \partial x^{i_l}})) \quad (1 \leq l \leq R-1).$$

We note that this system is involutive or satisfies the integrability conditions. We also note that if $r=1$, $(*)'$ simply means the system of ordinary differential equations:

$$\frac{d^R y^\alpha}{dx^R} = f^\alpha(x, (y^\alpha), (\frac{dy^\alpha}{dx}), \dots, (\frac{d^{R-1} y^\alpha}{dx^{R-1}})).$$

The system $(*)'$ will be called a local expression of $\mathcal{K}$ at z_0 . Conversely it is clear that any involutive system of the form $(*)'$ can be obtained as a local expression of an OD structure $\mathcal{K}$ of order R of bidegree (n, r) .

(B) The case where $k \geq 3$ and $r = n-1$. Let (M, C) be a contact manifold of degree n , and let us consider the $(k-2)$ -th prolongation $(\Gamma^{k-1}(M, C), C^{k-1})$ of (M, C) . Let R be an open fibred submanifold of the fibred manifold $\Gamma^{k-1}(M, C)$ over M . We denote by D and F the restrictions to R of the contact system C^{k-1} and the symbol system V^{k-1} respectively. Let E be a differential system on R . Then the pair $\mathcal{X} = (R, E)$ is called an OD structure of order k of degree n on (M, C) , if it satisfies the conditions 1) and 2) in (A).

Let $\mathcal{X} = (R, E)$ (resp. $\mathcal{X}' = (R', E')$) be an OD structure of order k of degree n on a contact manifold (M, C) (resp. on (M', C')). By an isomorphism of $\mathcal{X}$ to $\mathcal{X}'$ we mean a (contact) isomorphism φ of (M, C) onto (M', C') such that the prolongation $p^{k-2}(\varphi)$ of φ gives an isomorphism of (R, E) onto (R', E') (as manifolds with differential system). Let (z, z') be a point of $R \times R'$. By a local isomorphism of $\mathcal{X}$ to $\mathcal{X}'$ at (z, z') we mean a (contact) isomorphism φ of (M, C) to (M', C') at $(p_1^{k-1}(z), p_1^{k-1}(z'))$ such that $p^{k-2}(\varphi)$ gives a local isomorphism of (R, E) to (R', E') at (z, z') .

Let $\mathcal{X} = (R, E)$ be an OD structure k of degree n on a contact manifold (M, C) . Then we say that a Lagrangian submanifold S of (M, C) is

a solution of $\mathcal{K}$, if the prolongation $p^{k-2}(S)$ of S is in R , and is an integral manifold of E .

In this way the structure $\mathcal{K}$ may be regarded as a differential equation.

Let z_0 be any point of R , and let (x^i, y, p_{i-1}) ($1 \leq i \leq k-1$) be any canonical coordinate system of $\Gamma^{k-1}(M, C)$ at z_0 . Let $\tilde{U}$ denote the domain of the coordinate system. By Lemma we see that there are unique functions f_{i-1} on the intersection $R \cap \tilde{U}$ such that E is defined by the Pfaffian equations of the form (*), where the index α should be taken off everywhere. Now, let S be a Lagrangian submanifold of (M, C) . Suppose that $p^{k-2}(S) \subset \tilde{U}$, and S is defined by the equations

$$\begin{cases} y = u(x^1, \dots, x^{n-1}), \\ p_i = \frac{\partial u}{\partial x^i}(x^1, \dots, x^{n-1}). \end{cases}$$

Then we see that S is a solution of the equation $\mathcal{K}$, if and only if u gives a solution of the system of partial differential equations:

$$(*)'' \quad \frac{\partial^2 u}{\partial x^i \partial x^{i+k}} = f_{i-1} \left((x^i), y, \left(\frac{\partial u}{\partial x^i} \right) \right) \quad (1 \leq i \leq k-1),$$

We note that this system is involutive.

We also note that if $r=1$, this system simply means the single ordinary differential equation:

$$\frac{d^R y}{dx^R} = f(x, y, \frac{dy}{dx}, \dots, \frac{d^{R-1}y}{dx^{R-1}}).$$

The system $(*)'$ will be called a local expression of $\mathcal{F}$ at z_0 . Conversely any involutive system of the form $(*)'$ can be obtained as a local expression of an OD structure $\mathcal{F}$ of order R of degree n .

2. Pseudo-projective systems of order k .
 Let (k, n, r) be a triplet of integers with $k \geq 2$ and $1 \leq r \leq n-1$. Let $P = (R; E, F)$ be a pseudo-product manifold, and let us consider the differential system $D = E + F$ on R . Then P is called a pseudo-projective system of order k of bidegree (n, r) , if it satisfies the following conditions:

1) (R, D) is a contact manifold of order $k-1$ of bidegree (n, r) .

2) F coincides with the symbol system F_{k-1} of (R, D) , except that $k=2$ and $r=n-1$.

Clearly a pseudo-projective system of order 2 of bidegree (n, r) means a pseudo-projective system of bidegree (n, r) in the sense of Chapter I.

Let $X = (R, E)$ be an OD structure of order k of bidegree (n, r) on a manifold M or a contact manifold (M, C) . Then it is clear that the triplet $P \equiv (R; E, F)$ gives a pseudo-projective system of order k of bidegree (n, r) , which is called associated with X . Here F is the restriction of the symbol system of $L^k(M, r)$ or $\Gamma^k(M, C)$ to R .

(*) By Lemmas we have the following.

Lemma. Let $\mathcal{X}$ (resp. $\mathcal{X}'$) be an OD-structure of order k of bidegree (n, r) on M or (M, C) (resp. on M' or (M', C')), and let $\mathcal{P}$ (resp. $\mathcal{P}'$) be the associated pseudo-projective system of order k of bidegree (n, r) . If φ is a local isomorphism of $\mathcal{X}$ to $\mathcal{X}'$ at (z, z') , then the prolongation $p^{k+1}(\varphi)$ or $p^{k-2}(\varphi)$ of φ gives a local isomorphism of $\mathcal{P}$ to $\mathcal{P}'$ at (z, z') . Conversely if $\tilde{\varphi}$ is a local isomorphism of $\mathcal{P}$ to $\mathcal{P}'$ at (z, z') , then there is a unique local isomorphism φ of $\mathcal{X}$ to $\mathcal{X}'$ at (z, z') such that $\tilde{\varphi} = p^{k+1}(\varphi)$ or $\tilde{\varphi} = p^{k-2}(\varphi)$.

Furthermore we know from Lemmas that any pseudo-projective system $\mathcal{P}$ of order k of bidegree (n, r) is locally associated with an OD structure $\mathcal{X}$ of order k of bidegree (n, r) . We have thereby seen that in the local point of view there is a natural one-to-one correspondence between the OD structures of order k of bidegree (n, r) and the pseudo-projective systems of order k of bidegree (n, r) .

(*) By Lemmas we see that the assignment $\mathcal{X} \rightarrow \mathcal{P}$ is compatible with the respective isomorphisms.

In connection with OD structures and pseudo-projective systems of order k we shall now introduce the notion of a pseudo-projective structure of order k of bidegree (n, r) . In the case where $k=2$, this is defined to be an OD structure of order 2 of bidegree (n, r) or a pseudo-projective structure of bidegree (n, r) in the sense of Chapter I.

Let us consider the case where $k \geq 3$. Let (M, C) be a contact manifold of order $k-2$ of bidegree (n, r) , and $(L(M, C), \hat{C})$ its prolongation. Let R be an open fibred submanifold of $L(M, C)$ over M . We denote by D the restriction of the contact system $\hat{C}$ to R , and by F the vertical tangent bundle of the fibred manifold R over M . Let E be a differential system on R . Then the pair $\mathcal{F} = (R, E)$ is called a pseudo-projective structure of order k of bidegree (n, r) on (M, C) , if it satisfies the conditions 1) and 2) in (A), the preceding paragraph...

Concerning pseudo-product structures of order $k \geq 3$, we remark the following.

(1) A pseudo-projective structure $\mathcal{F} = (R, E)$ of order k of bidegree (n, r) on (M, C) is a generalized pseudo-projective structure of bidegree (n, r) on M in the sense of Chapter I. Therefore, $\mathcal{F}$ may be regarded as a differential equation, which is locally expressed by an involutive

system of partial differential equations of the second order. Moreover the dual $\mathcal{X}^*$ of $\mathcal{X}$ can be considered under suitable regularity conditions on $\mathcal{X}$.

(2) A pseudo-projective structure of order 3 of degree n means an OD structure of order 3 of degree n .

(3) Let $\mathcal{X} = (R, E)$ be a pseudo-projective structure of order k of bidegree (n, r) on (M, C) . Then the triplet $\mathcal{P} = (R; E, F)$ gives a pseudo-projective system of order k of bidegree (n, r) .

(4) Let $\mathcal{X}_0 = (R, E)$ be an OD structure of order k of bidegree (n, r) on a manifold $\bar{M}$ or a contact manifold $(\bar{M}, \bar{C})$. We denote by M the image of R by the projection of R to $J^{k-2}(\bar{M}, \bar{C})$ or $\pi^{k-2}(\bar{M}, \bar{C})$, and by C the restriction of the contact system $\bar{C}^{k-2}$ on $J^{k-2}(\bar{M}, \bar{C})$ or $\pi^{k-2}(\bar{M}, \bar{C})$ to R . Then the pair (M, C) gives a contact manifold of order $k-2$ of bidegree (n, r) , and the pair (R, E) gives a pseudo-projective structure of order k of bidegree (n, r) on (M, C) , which we denote by $\mathcal{X}$. The structure $\mathcal{X}$ will be called associated with $\mathcal{X}_0$.

(5) In the local point of view, the equations $\mathcal{X}_0$ and $\mathcal{X}$ describe essentially the same differential equation.

(6) In the local point of view, there is a natural one-to-one correspondence between the OD structures of order k of bidegree (n, r) and the

pseudo-projective structures of order k
of bidegree (m, r) .

3. Pseudo-projective FGLA's of order R .
 Let $\mathcal{P} = (R; E, F)$ be a pseudo-projective system of order R of $\text{fdgree}(n, R)$. Then D is regular, and therefore we may speak of the symbol algebra of $\mathcal{P}$ at any point $z \in R$.
 Let $\mathcal{F} = (\mathcal{F}, (\mathcal{F}_p)_{p \leq 0})$ be the symbol algebra of the contact manifold (R, D) of order $R-1$ at z :

$$\mathcal{F}_{-1} = D(z),$$

$$\mathcal{F}_p = D_{R+p}(z) / D_{R+p+1}(z), \quad p \leq -2,$$

where D_ℓ ($1 \leq \ell \leq R-1$) denote the differential systems associated with D , and we set $D_\ell = T(R)$ for $\ell \leq 0$. Let e and f denote the subspaces $E(z)$ and $F(z)$ of $\mathcal{F}_{-1}$ respectively. Then the triplet $\mathcal{L} = (\mathcal{F}; e, f)$ gives the symbol algebra of $\mathcal{P}$ at z , which is a pseudo-product FGLA.

By Lemma we know that $\mathcal{L}$ satisfies the following conditions:

1) $\mathcal{L}$ or $\mathcal{F}$ is of the R -th kind and non-degenerate,

2) $[\mathcal{F}_p, \mathcal{F}_q] = \{0\}$ for $p, q \leq -2$, and $[\mathcal{F}_p, f] = \{0\}$ for $p \leq -2$.

3) If $p \leq -2$, $X \in \mathcal{G}_p$, and $[X, Y] = 0$ for all $Y \in e$, then $X = 0$.

4) $\dim \mathcal{G}_R = n - r$, $\dim \mathcal{G}_p = (n - r)r + H_{R+p}$ for $-R+1 \leq p \leq -2$, $\dim e = r$, and $\dim f = (n - r)r + H_{R-1}$.

In general, consider a pseudo-product FGPA, $\mathcal{L}$. Then $\mathcal{L}$ is called a pseudo-projective FGPA of order R of bidegree (n, r) , if it satisfies the four conditions above.

Now, let $\mathcal{L}$ be a pseudo-projective FGPA of order R of bidegree (n, r) . Let us consider the tensor products $\mathcal{G}_R \otimes S^l(e^*)$, where $S^l(e^*)$ denotes the l -th symmetric product of the dual space e^* of e . Note that $\mathcal{G}_R \otimes S^l(e^*)$ may be naturally regarded as the space of all $\mathcal{G}_R$ -valued symmetric l -linear forms on e . For any $X \in \mathcal{G}_R \otimes S^l(e^*)$ and $Y \in e$, we define an element $Y \lrcorner X \in \mathcal{G}_R \otimes S^{l-1}(e^*)$ by

$$(Y \lrcorner X)(Y_1, \dots, Y_{l-1}) = X(Y, Y_1, \dots, Y_{l-1}),$$

Here and in the following $Y_1, Y_2, \dots$ denote any elements of e . Now, let p be any integer with $-R+1 \leq p \leq -2$. For any $X \in \mathcal{G}_p$ we define a $\mathcal{G}_R$ -valued $(R+p)$ -linear form $\tilde{X}$ on e by

$$\tilde{X}(Y_1, \dots, Y_{R+p}) = [[[[X, Y_1], Y_2], \dots], Y_{R+p}].$$

Since e is abelian, we see that $\tilde{X}$ is symmetric, and hence $\tilde{X} \in \mathcal{I}_R \otimes S^{R+1}(e^*)$.

From the conditions 3) and 4) we also see that the assignment $X \rightarrow \tilde{X}$ gives an isomorphism of $\mathcal{I}_p$ onto $\mathcal{I}_R \otimes S^{R+1}(e^*)$.

For any $X \in f$ we now define an element $\tilde{X} \in \mathcal{I}_R \otimes S^{R-1}(e^*)$ in the same manner as above. Then we see from the conditions 1), 3) and 4) that the assignment $X \rightarrow \tilde{X}$ gives an isomorphism of f onto $\mathcal{I}_R \otimes S^{R-1}(e^*)$.

Hereafter we shall identify $\mathcal{I}_p$ ($-R+1 \leq p \leq -2$) with $\mathcal{I}_R \otimes S^{R+1}(e^*)$, and f with $\mathcal{I}_R \otimes S^{R-1}(e^*)$, by the isomorphism $X \rightarrow \tilde{X}$. Then we clearly have

$$[X, Y] = Y \lrcorner X,$$

where $X \in \mathcal{I}_p$ ($-R+1 \leq p \leq -2$) or $X \in f$, and $Y \in e$.

From the discussion above we know that there is a unique pseudo-projective FGLA of order R of bidegree (n, r) up to isomorphism.

Accordingly let $\mathcal{L}$ be a pseudo-projective FGLA of order R of bidegree (n, r) . Then it follows that every pseudo-projective system $\mathcal{P}$ of order R of bidegree (n, r) is of type $\mathcal{L}$.

The prolongation of $\mathcal{L}$ will be called a pseudo-projective GLA of order R of bidegree (n, r) .

Chapter VII. Geometry of systems of
ordinary differential equations of
higher order, (I)

§ Pseudo-projective GLAs of order 3 of degree n 1 Contact projective geometry

n is integer ≥ 2 & L , $m = 2n-1$ & $\leq$. P^m is m -dim. projective space over $\mathbb{R}$ & $\exists$. $\mathbb{R}_*^{2n} = \mathbb{R}^{2n} - \{0\}$ & $\exists$, $p: \mathbb{R}_*^{2n} \rightarrow P^m$ is projection & $\exists$. $(z_0, \dots, z_m)$, $\in P^m$ of canonical homogeneous coordinate system or canonical coordinate system of $\mathbb{R}_*^{2n}$ & $\exists$. $\mathbb{R}_*^{2n}$ 上の 1-form ω is

$$\omega = z_0 dz_m - z_m dz_0 + \sum_{i=1}^{n-1} (z_i dz_{n+i} - z_{n+i} dz_i)$$

by defining C' as Pfaffian equation $\omega = 0$ by defining $\mathbb{R}_*^{2n}$ on differential system & $\exists$. 容易に C' の Cauchy characteristic system $ch(C')$ is the vertical tangent bundle of the fibred manifold $\mathbb{R}_*^{2n}$ over P^m が示される(*) 故に C' は自然に P^m 上の differential system C に導き, C は contact structure となる.

α is $(n-1)$ -dim projective subspace of P^m & $\exists$. α is contact manifold (P^m, C) の Lagrangian submanifold である & $\exists$, α is Lagrangian projective subspace & $\exists$. $\tilde{\alpha} = p^{-1}(\alpha) \cup \{0\}$ & $\exists$ & $\exists$, $\tilde{\alpha}$ is $\mathbb{R}^{2n}$ の n -dim. linear subspace である. 容易に

(*) $\lambda \in \mathbb{R}_*$ $R_\lambda(x) = \lambda x$, $x \in \mathbb{R}_*^{2n}$, $R_\lambda^* \omega = \lambda^2 \omega$ が必要...

α Lagrangian projective subspace

$\Leftrightarrow \alpha$ is symplectic form

$$d\omega = 2(dz_0 \wedge dz_m + \sum_{i=1}^{m-1} dz_i \wedge dz_{n+i})$$

12 例 33 Lagrangian subspace

が示される。

$\Lambda(P^n, \mathbb{C})$ is the set of Lagrangian projective subspaces of P^n である。

$\Lambda(P^n, \mathbb{C})$ is compact connected manifold である, $\dim \Lambda(P^n, \mathbb{C}) = \frac{1}{2}n(n+1)$ が示される。

product manifold $P^n \times \Lambda(P^n, \mathbb{C})$ がある relation $R = R(n)$.

$$R = \{(p, \alpha) \in P^n \times \Lambda(P^n, \mathbb{C}) \mid p \subset \alpha\}$$

によって定義する。 R は $P^n \times \Lambda(P^n, \mathbb{C})$ の compact connected submanifold である, $\dim R = 2n+1 + \frac{n}{2}(n-1)$ であることが示される。

$\pi: R \rightarrow P^n$, $\rho: R \rightarrow \Lambda(P^n, \mathbb{C})$ は projections である。 R は fibered manifold over P^n with projection π である。 同時に fibered manifold over $\Lambda(P^n, \mathbb{C})$ with projection ρ である。

Let contact manifold (P^m, C) , 1-attached Lagrange-Grassmann bundle $L(P^m, C)$ is considered. $(p, \alpha) \in R$ is a point, $T_p(\alpha) \in L(P^m, C)$ is a point. So for each $(p, \alpha) \rightarrow T_p(\alpha)$ is a point in $L(P^m, C)$ on the diffeomorphism $\pi: L(P^m, C) \rightarrow P^m$ is projection and it is, $\pi \circ \tau = \pi$ is true.

F (resp. E) is fiber manifold R over P^m (resp. over $L(P^m, C)$). The vertical tangent bundle is $D = E + F$. So D is the image of D is $L(P^m, C)$, the contact system is true so that (2) is true R and $L(P^m, C)$ are identified. $\mathcal{K}_0 = (R, E)$ is P^m on the pseudo-projective structure of order 3 of degree n is given. So triplet $\mathcal{P} = (R, E, F)$ is $\mathcal{K}_0$ 1-attached pseudo-projective system of order 3 of degree n is given. So S is P^m of $(n-1)$ -dim. submanifold is true. This is true.

S is a solution of $\mathcal{K}_0$

$\Leftrightarrow \alpha$ is part of a Lagrangian projective subspace α

It is true. So $\mathcal{K}_0$ is Lagrangian projective subspaces is defined differential equation is true and it is true.

U を $z_0 \neq 0$ で定義される P^m の open set とする. U は $\mathbb{R}^m$ に自然に同一視され, $z_\lambda = z_\lambda / z_0$ ($1 \leq \lambda \leq m$) : とおけば, (x_λ) は $U = \mathbb{R}^m$ の canonical coordinate system を与える. 以下 i, j は $1, \dots, n-1$ を動くものとする.

$$\omega' = dx_m + \sum_{i=1}^n (x_i dx_{n+i} - x_{n+i} dx_i)$$

とおけば, $\omega = z_0^2 \omega'$ となる. さらに,

$$y = x_m + \sum_i x_i x_{n+i}, \quad p_i = 2x_{n+i}$$

とおけば, $\omega' = dy - \sum p_i dx_i$ となる. 故に (x_i, y, p_i) は contact manifold (P^m, C) に関する canonical coordinate system を与える.

さて, α は equation

$$\begin{cases} z_{n-1+i} = a_i z_0 + \sum_j a_{ij} z_j, \\ z_m = a_0 z_0 + \sum_i b_i z_i \end{cases}$$

で定義された P^m の $(n-1)$ -dim. projective subspace となる coordinate system (x_i, y, p_i) に関して $d\alpha U$ は

$$\begin{cases} y = q_0 + \sum_i (a_i + b_i) x_i + \sum_{i,j} a_{ij} x_i x_j, \\ p_i = 2a_i + 2 \sum_j a_{ij} x_j. \end{cases}$$

τ 定義され, $d \cap U$ 上 τ ,

$$dy - \sum_i p_i dx_i = \sum_i (b_i - a_i) dx_i + \sum_{i,j} (a_{ji} - a_{ij}) x_i dx_j.$$

が成り立つ。故に,

d が Lagrangian projective subspace

$$\Leftrightarrow b_i = a_i, a_{ij} = a_{ji}$$

がわかる。よって, d が Lagrangian projective subspace ならば, $d \cap U$ は equation

$$\begin{cases} y = q_0 + 2 \sum_i a_i x_i + \sum_{i,j} a_{ij} x_i x_j, \\ p_i = 2a_i + 2 \sum_j a_{ij} x_j \quad (a_{ij} = a_{ji}) \end{cases}$$

τ 定義された。以上より τ equation $\mathcal{L}_0 = (R, E)$ が local に

$$\frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = 0$$

なる π_3 の equation によって表現されたことがわかった。(Note that in the discussion

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above $(z_0, \dots, z_m)$ may be replaced by any homogeneous coordinate system $(z'_0, \dots, z'_m)$ such that

$$z'_0 dz'_m - z'_m dz'_0 + \sum_i (z'_i dz'_{n+i} - z'_{n+i} dz'_i)$$

$$= C \cdot \omega,$$

where C is a constant $\neq 0$.

2. Pseudo-projective graded Lie algebras of order 3 of degree n

以下 "pseudo-projective graded Lie algebra of order 3 of degree n の matricial representation としよう。

$2n$ 次 skew-symmetric matrix I_n と

$$I_n = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1_{n-1} & 0 \\ 0 & -1_{n-1} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

によって定義する。ここで 1_{n-1} は $n-1$ 次 unit matrix である。 $\mathfrak{gl}(2n, \mathbb{R})$ の subalgebra $\mathfrak{g}$ と

$$\mathfrak{g} = \{ X \in \mathfrak{gl}(2n, \mathbb{R}) \mid X I_n + I_n X = 0 \}$$

によって定義する。 $\mathfrak{g}$ は simple Lie algebra $\mathfrak{sp}(n, \mathbb{R})$ に isomorphic である。

" = n = 1

$n_1 = n_4 = 1, n_2 = n_3 = n-1$ とある。このとき, matrix $X \in \mathfrak{gl}(2n, \mathbb{R})$ は $X = (X_{ij})_{1 \leq i, j \leq 4}$ と表わされる。ここで, X_{ij} は $n_i \times n_j$ -matrix である。

$X \in \mathcal{G}$ なるための条件は:

$$X_{42} = {}^t X_{31}, X_{43} = -{}^t X_{21}, {}^t X_{32} = X_{32}, X_{44} = -{}^t X_{11},$$

$$X_{33} = -{}^t X_{22}, {}^t X_{23} = X_{23}, X_{34} = -{}^t X_{12}, X_{24} = {}^t X_{13}$$

である。

$\mathcal{G}$ の subspaces の family $(\mathcal{G}_p) \in \mathcal{G}$.
 $= \mathcal{G} \cap \mathcal{G}_p(1, n-1, n-1, 1; \mathbb{R}) = \{X \in \mathcal{G} \mid X_{ij} = 0 \text{ if } i \neq j\}$ と定義すれば,
 $\mathcal{G} = (\mathcal{G}, (\mathcal{G}_p))$ は $\mathcal{G}(1, n-1, n-1, 1; \mathbb{R})$ の graded
 subalgebra になる. $\mathcal{F} = \sum_{p \geq 0} \mathcal{G}_p$ とお
 けよう, $\mathcal{F} = (\mathcal{F}, (\mathcal{G}_p)_{p \geq 0})$ は FGLA of the third
 kind である, 従って, $\mathcal{G}$ は NGLA of the third
 kind になる.

$\mathcal{F}_1$ の subspaces e, f をそれぞれ

$$e = \{X \in \mathcal{F}_1 \mid X_{32} = 0\},$$

$$f = \{X \in \mathcal{F}_1 \mid X_{21} = 0\}$$

によって定義すれば, $\mathcal{L} = (\mathcal{F}; e, f)$ は
 pseudo-projective FGLA of order 3 of degree n
 になり, $\mathcal{G}_0$ は $\mathcal{L}$ の derivation algebra
 $\text{Der}(\mathcal{L})$ に同一視される. 従って, Proposition
 1-5', $\mathcal{G}$ は $\mathcal{L}$ の prolongation になる. この
 $\mathcal{G}$ は the standard pseudo-projective graded Lie
 algebra of order 3 of degree n である.

E を

$$X_{11} = \frac{3}{2}, X_{22} = \frac{1}{2} 1_{n-1}$$

で "定義される $\mathcal{G}_0$ の centre の元とすれば", E は $\mathcal{G}$ の characteristic element である。

J を

$$X_{11} = -\frac{1}{2}, X_{22} = \frac{1}{2} 1_{n-1}$$

で "定義される $\mathcal{G}_0$ の centre の元とすれば",

$$[J, \mathcal{G}_p] = \{0\} \text{ if } p \text{ is even.}$$

$$[J, [J, X]] = X \text{ for all } X \in \mathcal{G}_p \text{ if } p \text{ is odd}$$

が確かめられる。故に odd integer $p \geq 1$ に対して,
 $\mathcal{G}_p$ の subspaces $\mathcal{G}_p^+, \mathcal{G}_p^-$ を

$$\mathcal{G}_p^\pm = \{X \in \mathcal{G}_p \mid [J, X] = \pm X\}$$

によって定義すれば,

$$\mathcal{G}_p = \mathcal{G}_p^+ + \mathcal{G}_p^- \text{ (direct sum)}$$

2.2.5. $\mathcal{G}_1^+ = e, \mathcal{G}_1^- = f; \mathcal{G}_3^+ = \mathcal{G}_{-3}, \mathcal{G}_3^- = \mathcal{G}_3$ 及び

$$\mathcal{G}_1^+ = \{X \in \mathcal{G}_1 \mid X_{12} = 0\},$$

$$\mathcal{G}_1^- = \{X \in \mathcal{G}_1 \mid X_{23} = 0\}$$

が確かめられる.

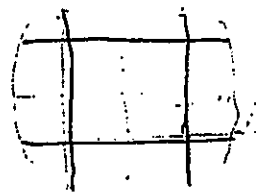
$\mathcal{G}_0$ の centre の $\bar{e}$ $E_{\mathcal{G}}$ を

$$E_{\mathcal{G}} = \frac{1}{2}(E - J)$$

によって定義し, $\mathcal{G}$ の subspaces の family (σ_p) を

$$\sigma_p = \{X \in \mathcal{G} \mid [E_{\mathcal{G}}, X] = pX\}$$

によって定義する このとき,



$$\sigma_p = \mathcal{G}_{2p-1}^- + \mathcal{G}_{2p} + \mathcal{G}_{2p+1}^+$$

とある. 従って, $\sigma_p = \{0\}$ if $p < -2$ or $p > 2$ である,

$$\sigma_{-2} = \mathcal{G}_{-3}, \quad \sigma_{-1} = \mathcal{G}_{-2} + \mathcal{G}_{-1}^+, \quad \sigma_0 = \mathcal{G}_{-1}^- + \mathcal{G}_0 + \mathcal{G}_1^+,$$

$$\sigma_1 = \mathcal{G}_1^- + \mathcal{G}_2, \quad \sigma_2 = \mathcal{G}_3.$$

とある. 故に, $\mathcal{G} = \{\mathcal{G}, (\sigma_p)\}$ is SGLA of the second kind である, $\mathcal{G}(1, 2(n-1), 1; \mathbb{R})$ の graded subalgebra である. $\mathcal{G}$ is projective contact graded Lie algebra of degree n

1. isomorphic である。

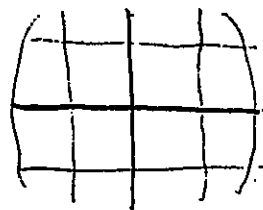
$\mathfrak{g}_0$ の centre の元 $E_{\mathfrak{g}}$ を

$$E_{\mathfrak{g}} = \frac{1}{2}(E+J)$$

によって定義し, $\mathfrak{g}$ の subspaces の family $(\mathfrak{g}_p)$ を

$$\mathfrak{g}_p = \{X \in \mathfrak{g} \mid [E_{\mathfrak{g}}, X] = pX\}$$

によって定義する。このとき,



$$\mathfrak{g}_p = \mathfrak{g}_{2p-1}^+ + \mathfrak{g}_{2p} + \mathfrak{g}_{2p+1}^-$$

をえる。従って, $\mathfrak{g}_p = \{0\}$ if $p < -1$ or $p > 1$ である,

$$\mathfrak{g}_{-1} = \mathfrak{g}_{-3} + \mathfrak{g}_{-2} + \mathfrak{g}_{-1}^-, \quad \mathfrak{g}_0 = \mathfrak{g}_1^+ + \mathfrak{g}_0 + \mathfrak{g}_1^-,$$

$$\mathfrak{g}_1 = \mathfrak{g}_1^+ + \mathfrak{g}_2 + \mathfrak{g}_3$$

をえる。故に $\mathfrak{L} = (\mathfrak{g}, (\mathfrak{g}_p))$ は SGLA of the first kind である, $\mathfrak{g}(n, n; \mathbb{R})$ の graded subalgebra である。[] の記号の下, $\mathfrak{L}$ は $\mathfrak{D}_p(n, \mathbb{R})$ に isomorphic である。特に, $n=2$ のとき, $\mathfrak{L}$ は $\mathfrak{M}_4(3)$ に isomorphic である。

3. The homogeneous spaces $G/G^{(0)}$, $G/A^{(0)}$ and $G/B^{(0)}$
 G is the group of projective transformations
 of P^m leaving the contact structure C
 invariant とする. G は P^m 上の transitive
 に作用する. G は自然に $\Lambda(P^m, C)$ 上に
 作用し, その作用は transitive である. さらに
 G は diagonal map $G \rightarrow G \times G$ を通じて
 product manifold $P^m \times \Lambda(P^m, C)$ 上に作用し,
 その作用による orbits は $R = R(n)$ と
 その complement によって与えられる.

$p_0 \in Z_1 = \dots = Z_m = 0$ によって与えられる P^m
 の点, $\alpha_0 \in Z_1 = \dots = Z_m = 0$ によって与えら
 れる P^m の Lagrangian projective subspace
 とする. 明らかに $(p_0, \alpha_0) \in R$ である. $A^{(0)}$ は
 p_0 における G の isotropy group, $B^{(0)}$ は α_0
 における G の isotropy group, $G^{(0)}$ は
 (p_0, α_0) における G の isotropy group, と
 する. このとき, P^m , $\Lambda(P^m, C)$, R はそれぞれ
 homogeneous space $G/A^{(0)}$, $G/B^{(0)}$,
 $G/G^{(0)}$ によって表現される.

$GL(2n, \mathbb{R})$ の subgroup G' は

$$G' = \{A \in GL(2n, \mathbb{R}) \mid A I_n A = \lambda I_n \text{ with some } \lambda \in \mathbb{R}_*\}$$

によって定義する. 明らかに, $G' = \{A \in GL(2n, \mathbb{R}) \mid$
 $A^* d\omega = \lambda d\omega \text{ with some } \lambda \in \mathbb{R}_*\}$ である. 故に group
 G' は factor group $G'/C(G')$ によって表わされる.
 ここで $C(G')$ は G' の centre であり, $\lambda \cdot 1_{2n-1} (\lambda \in \mathbb{R}_*)$ なる.

$$\underline{A^* d\omega = \lambda d\omega \iff A^* \omega = \lambda \omega}$$

形の diagonal matrices 全体からなる.

G の Lie algebra は $\mathfrak{g}$ に同一視され, G は $\text{Aut}(\mathfrak{g})$ の open subgroup に同一視される. G' の subgroups $G_0', G^{(1)'}$ を

$$G_0' = G' \cap G_0(1, n-1, n-1, 1; \mathbb{R}),$$

$$G^{(1)'} = G' \cap G^{(1)}(1, n-1, n-1, 1; \mathbb{R})$$

によって定義し, $G_0 = G_0' / C(G')$ とおく. 明らかに $G^{(0)} = G_0 \cdot G^{(1)}$ である. こゝで $G^{(1)}$ は自然に G の subgroup とみなすべきである. さらに $G_0 = \text{Aut}(\mathcal{L})$ であり, $G = \text{Aut}(\mathfrak{g}) \cdot G_0$ であり, $G^{(1)}$ は $\mathfrak{g}$ の nilpotent subalgebra $\mathfrak{g}^{(1)} = \sum_{p \geq 1} \mathfrak{g}^p$ によって生成される G の Lie subgroup であることが示される. 以上から groups $G_0, G^{(0)}, G$ が $\mathcal{L}$ に付随する group であることがわかった (実は $\text{Aut}(\mathcal{L}) = \text{Aut}(\mathfrak{g})$ が示される. ように $G_0, G^{(0)}, G$ は $\mathfrak{g}$ に付随する groups にもなっている.)

P^m 上の pseudo-product structure (E, F) は $\mathfrak{g}$ の subalgebras $\mathfrak{g}^{(0)}$ と $\mathfrak{g}^{(1)}$ から導かれる $G/G^{(0)}$ 上の invariant pseudo-product structure と一致する $\mathfrak{g}_1^+ + \mathfrak{g}^{(0)}$ $\mathfrak{g}_1^- + \mathfrak{g}^{(0)}$

(*) $\forall a \in \text{Aut}(\mathfrak{g})$ に対して $\text{Ad}(a)\mathfrak{g}_1^+ \neq \mathfrak{g}_1^-$ である. 実際 $\text{Ad}(a)\mathfrak{g}_1^+ = \mathfrak{g}_1^-$ とすれば, $\mathfrak{g}_1^- = [\mathfrak{g}_2, \mathfrak{g}_1^+] = [\mathfrak{g}_2, \mathfrak{g}_1^-] = \{0\}$ となり, 矛盾が生ずる. $\mathfrak{g}_0$ の $\mathfrak{g}_1^+, \mathfrak{g}_1^-$ 上への表現は irreducibly T_2 から, $a \in \text{Aut}(\mathfrak{g})$ ならば $\text{Ad}(a)\mathfrak{g}_1^+ = \mathfrak{g}_1^+, \text{Ad}(a)\mathfrak{g}_1^- = \mathfrak{g}_1^-$, i.e., $a \in \text{Aut}(\mathcal{L})$ が従う.

G' の subgroups A'_0 と $A^{(1)}$ をそれぞれ

$$A'_0 = G' \cap G_0(1, 2(n-1), 1; \mathbb{R}),$$

$$A^{(1)} = G' \cap G^{(1)}(1, 2(n-1), 1; \mathbb{R})$$

によって定義し, $A_0 = A'_0 / C(G')$ とおく. 明らかに $A^{(0)} = A_0 \cdot A^{(1)}$ である. さらに $A_0 = \text{Aut}(\mathcal{O})$ である, $G = \text{Aut}(\mathfrak{g}) \cdot A_0$ である, $A^{(1)}$ は $\mathfrak{g}$ の nilpotent subalgebra $\mathcal{O}^{(1)} = \sum_{p \geq 1} \mathcal{O}_p$ によって生成される G の Lie subgroup であることが示される. 故に groups $A_0, A^{(0)}, G$ は $\mathcal{O}$ に付随する groups であることがわかる.

P^m 上の contact structure C は $\mathfrak{g}$ の subspace $\mathcal{O}^{(1)} = \sum_{p \geq 1} \mathcal{O}_p$ から導かれる $G/A^{(0)}$ 上の invariant differential system と一致する.

$$\mathfrak{g}^{(-2)} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \dots$$

$k-R \leq p \leq -1$ に対して, S_p を次の条件を満足する $X \in S$ 全体の作る S の subspace とする:

$$X(\overbrace{e_0, \dots, e_0}^{-p-1}, Y_1, \dots, Y_{R+p}) = 0.$$

$i > r$, $i \neq p$, $Y_i \in \mathcal{G}_1^+$. 明らかに
 $S = \sum_p S_p$ (direct sum) である.

$\forall X \in S$ に対して, $X^{(p)} \in \mathcal{G}_R \oplus S^{R+p}((\mathcal{G}_1^+)^*)$ と

$$X^{(p)}(Y_1, \dots, Y_{R+p}) = \frac{(R-1)!}{(p-1)!} X(\overbrace{e_0, \dots, e_0}^{-p-1}, Y_1, \dots, Y_{R+p})$$

によって定義する. $i > r$, $Y_i \in \mathcal{G}_1^+$. 故して,
 linear map $\Phi: S \rightarrow \mathcal{G}_{-1}$ を

$$\Phi(X) = \sum_{p=-R}^{-1} X^{(p)}$$

によって定義する. $X \in S_p$ ならば, $\Phi(X) = X^{(p)}$ であり, Φ は S_p から $\mathcal{G}_R \oplus S^{R+p}((\mathcal{G}_1^+)^*)$ の上への isomorphism を与える. 故に Φ は S から $\mathcal{G}_{-1}$ の上への isomorphism を与える.

Lemma $Z \in L, X \in S$

$$\Phi(Z \cdot X) = [\Phi(Z), \Phi(X)].$$

$$D \in GL(\mathcal{G}_1^+) \text{ 12 } \tilde{D} \in GL(V) \text{ 12}$$

$$\tilde{D} = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$$

によって定義する. $\tilde{L}$ の subgroup L_0 12

$$L_0 = \{ (\tilde{D}, B) \mid D \in GL(\mathcal{G}_1^+), B \in GL(\mathcal{G}_2) \}$$

によって定義する. L_0 は自然に $L = \tilde{L}/U$ の subgroup と見做される.

group $G_0 = GL(\mathcal{G}_1^+) \times GL(\mathcal{G}_2)$ 12 12
group isomorphism $\Phi_0: L_0 \rightarrow G_0$ 12

$$\Phi_0((\tilde{D}, B)) = (D, B)$$

によって定義する.

Lemma. $\forall g \in L_0, X \in \mathfrak{g} \text{ 12 } \tilde{D} \in GL(V) \text{ 12}$,

$$\Phi(g \cdot X) = \text{Ad}(\Phi(g)) \Phi(X).$$

Lemma There is a unique Lie group isomorphism $\Phi: L \rightarrow B_0$ such that $\Phi(a) = \Phi_0(a)$, $a \in L_0$, and such that

$$\Phi(a \cdot X) = \text{Ad}(\Phi(a))\Phi(X),$$

where $a \in L$, and $X \in \mathfrak{S}$.

Proof. $L = L^0 \cdot L_0$, $\text{Aut}(L) = \text{Aut}(L^0) \cdot G_0$.
 L^0 (resp. $\text{Aut}(L^0)$) is 1 (resp. $\text{Aut}(L)$) の単位元 の connected component. 故に, この Lemma は Lemma から 直ちに 従う.

5. A lemma on the graded Lie algebra $\mathfrak{g}$ ¹⁰

$\mathfrak{g}_1^+$ と $\mathfrak{g}_{-k}$ に inner products $(,)$ を入れる。
 V_0 の inner product $(,)$ は $(e_0, e_0) = 1$ となるよう
 に定義する。 $\mathfrak{g}_1^+$ と V_0 の inner products $(,)$
 は自然に V の inner product $(,)$ を導く。

S の inner product $(,)$ は次のように
 定義する。 $(\bar{e}_i)_{1 \leq i \leq r}$ は V の orthonormal basis
 とする。このとき、 $X, X' \in V$ に対して

$$(X, X') = \sum_{\lambda_0, \dots, \lambda_{r-1}} (X(\bar{e}_{\lambda_0}, \dots, \bar{e}_{\lambda_{r-1}}), X'(\bar{e}_{\lambda_0}, \dots, \bar{e}_{\lambda_{r-1}}))$$

明らかに (X, X') は $(\bar{e}_i)$ のとり方に依らない。
 $(\bar{e}_i)_{1 \leq i \leq r}$ は $\mathfrak{g}_1^+$ の orthonormal basis とする。
 $(\bar{e}_i)$ と $(\bar{e}_j)$ とをとりこえに依り, decomposition
 $S = \sum_{i=1}^r S_i$ が orthogonal decomposition であることがわかる。

$\mathcal{L}$ の inner product $(,)$ を次のように
 定義する。 $Z = (A, B), Z' = (A', B') \in \mathcal{L}$ に対して,

$$(Z, Z') = \text{tr}(AA') + \text{tr}(BB')$$

によって定義する。ここで ${}^t A$ (resp. ${}^t B$) は V
 (resp. $\mathfrak{g}_{-k}$) の inner product $(,)$ に関する
 A (resp. B) の transpose である。

map $\hat{\sigma}: \tilde{L} \rightarrow \tilde{L}$ を

$$\hat{\sigma}((A, B)) = (\hat{A}^-, \hat{B}^-), \quad (A, B) \in \tilde{L}$$

によって定義する. 明らかに $\hat{\sigma}$ は $\tilde{L}$ の involutive automorphism である. $\hat{\sigma}(U) = U$ であるから, $\hat{\sigma}$ は $\tilde{L}$ の involutive automorphism $\hat{\sigma}$ を導く. さらにこの $\hat{\sigma}$ は Lie algebra $\mathfrak{L}$ の involutive automorphism σ を導く: 明らかに

$$\sigma((A, B)) = (-\hat{A}, -\hat{B}), \quad (A, B) \in \mathfrak{L}$$

である.

明らかに

$$(aX, X') = (X, \hat{\sigma}(a)^{-1} X'), \quad a \in L, X, X' \in \mathfrak{S}$$

及び

$$(aZ, Z') = (Z, \hat{\sigma}(a)^{-1} Z'), \quad a \in L, Z, Z' \in \mathfrak{L}$$

が成り立つ. 更に, $\hat{\sigma}(L_0) = L_0$ を注意する.

$\mathfrak{g}_- \text{ (resp. } \mathfrak{g}_0)$ の inner product $(,)$ と isomorphism $\Phi: \mathfrak{g}_- \rightarrow \mathfrak{g}_+ \text{ (resp. } \Phi: \mathfrak{g}_- \rightarrow \mathfrak{g}_0)$ が isometry になるように定義する. $\exists B \in L \text{ (resp. } \mathbb{R})$ の involutive automorphism $\hat{\sigma} \text{ (resp. } \sigma)$ と isomorphism $\Phi: L \rightarrow B_0 \text{ (resp. } \Phi: \mathfrak{g}_- \rightarrow \mathfrak{g}_0)$ は $B_0 \text{ (resp. } \mathfrak{g}_0)$ の involutive automorphism $\hat{\sigma} \text{ (resp. } \sigma)$ を導く: $\Phi(\hat{\sigma}(a)) = \hat{\sigma}(\Phi(a)), a \in B_0 \text{ (resp. } \Phi(\sigma(Z)) = \sigma(\Phi(Z)), Z \in \mathfrak{g}_0$.

さて $\mathfrak{g}_+$ と $\mathfrak{g}_0$ の inner products $(,)$ は $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_0$ の inner product $(,)$ を導く.

Lemma $\mathfrak{g}$ の inner product $(,)$ と B_0 の involutive automorphism $\hat{\sigma}$ と $\mathfrak{g}_0$ の involutive automorphism σ は 次の条件, を満足する:

1) decomposition $\mathfrak{g} = \sum_p \mathfrak{g}_p, \mathfrak{g}_1 = \mathfrak{g}_1^+ + \mathfrak{g}_1^-, \mathfrak{g}_0 = \mathfrak{g}_0' + \mathfrak{g}_0''$ は すべて orthogonal decompositions である.

2) $\hat{\sigma}(\mathfrak{g}_0) = \mathfrak{g}_0, \hat{\sigma}(\mathfrak{g}_0') = \mathfrak{g}_0''$ である. したがって $\sigma(\mathfrak{g}_0') = \mathfrak{g}_0', \sigma(\mathfrak{g}_0'') = \mathfrak{g}_0'', \sigma(\mathfrak{g}_1^+) = \mathfrak{g}_1^-$ である.

3) $(\text{Ad}(a)X, X') = (X, \text{Ad}(\hat{\sigma}(a))X'), a \in B_0, X, X' \in \mathfrak{g}$. 従って,

$$(\text{ad}(Z)X, X') = -(X, \text{ad}(\sigma(Z))X'), Z \in \mathfrak{g}_0, X, X' \in \mathfrak{g}.$$

Remark. $\mathcal{G}$ の inner product $(\cdot, \cdot)$ が
 $\mathcal{G}_p$ ($-R \leq p \leq -2$), $\mathcal{G}_1^-$, $\mathcal{G}_1^+$, $\mathcal{G}_0$, $\mathcal{G}_1$ によつて inner
 products は $\mathcal{K}$ のように与えられる。

1) $\mathcal{G}_1^+$, $\mathcal{G}_1^-$ の induced inner products
 は与えられたものと一致する。

2) $X_p, X_{p'} \in \mathcal{G}_p$ ($-R \leq p \leq -2$), $X_i, X_{i'} \in \mathcal{G}_1^-$
 とする。 (e_i) を $\mathcal{G}_1^+$ の orthonormal basis とする。この
 とき, $\forall -R \leq p \leq -1$ に対して,

$$(X_p, X_{p'}) = \frac{(-p-1)!}{(R-1)!(R+p)!} \sum_{i_0 \dots i_{R+p}} (X_p(e_{i_0} \dots e_{i_{R+p}}), X_{p'}(e_{i_0} \dots e_{i_{R+p}}))$$

3) $X_0 = (D, \frac{R-1}{R+1} \text{Tr} D + B)$, $X_0' = (D', \frac{R-1}{R+1} \text{Tr} D' + B) \in \mathcal{G}_0$
 とする。このとき,

$$(X_0, X_0') = \text{Tr}(D^* D') - \frac{1}{R+1} \text{Tr} D \cdot \text{Tr} D' + \text{Tr}(B^* B)$$

4) $\mathcal{G}_1 = (\mathcal{G}_1^+)^*$ の induced inner product
 は $\mathcal{G}_1^+$ の与えられた inner product から自然に
 得られる。即ち

$$(X_i, X_{i'}) = \sum_j X_i(e_j) X_{i'}(e_j), \quad X_i, X_{i'} \in \mathcal{G}_1$$

序 12,

$$\hat{\sigma}(\mathcal{D}, B) = (\hat{\mathcal{D}}^{-1}, \hat{B}^{-1}), \quad (\mathcal{D}, B) \in \mathcal{G}_0$$

$$\sigma(\mathcal{D}, B) = (-\hat{\mathcal{D}}, -\hat{B}), \quad (\mathcal{D}, B) \in \mathcal{G}_0$$

$$(\mathcal{G}X)(Y) = -(X, Y), \quad X, Y \in \mathcal{G}_1^+$$

等を注意してあく。

記号変更 ($\lambda \rightarrow -\lambda$) 後

①

$$Z, Z' \in \mathcal{Q} = \hat{\mathcal{Q}}/\hbar$$

$$[Z, Z'] =$$

$$\left[\begin{pmatrix} -\frac{1}{\hbar} \text{Tr} D & \\ 0 & D - \frac{1}{\hbar} \text{Tr} D \end{pmatrix}, \begin{pmatrix} \frac{1}{\hbar} \text{Tr} D' & \\ 0 & D' - \frac{1}{\hbar} \text{Tr} D' \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & [D, D'] \end{pmatrix}$$

$$[Z_{(0)}, Z'_{(0)}] = ([D, D'], 0)$$

$$\left[\begin{pmatrix} -\frac{1}{\hbar} \text{Tr} D & 0 \\ 0 & D - \frac{1}{\hbar} \text{Tr} D \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ Z'_{(-1)} & 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 & 0 \\ D Z'_{(-1)} & 0 \end{pmatrix}$$

$$[Z_{(0)}, Z'_{(1)}] = \left[\left(D, \frac{R-1}{R+1} \overline{\text{Tr}} D \right), Z'_{(1)} \right]$$

$$= D Z'_{(1)}.$$

$$= \left[\begin{pmatrix} -\frac{1}{R+1} \overline{\text{Tr}} D & 0 \\ 0 & D \frac{1}{R+1} \overline{\text{Tr}} D \end{pmatrix}, \begin{pmatrix} 0 & Z'_{(1)} \\ 0 & 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 & -Z'_{(1)} D \\ 0 & 0 \end{pmatrix}$$

$$[Z_{(0)}, Z'_{(0)}]$$

$$= \left[\left(D, \frac{R-1}{R+1} \overline{\text{Tr}} D \right), Z'_{(0)} \right]$$

$$[Z_{(0)}, Z'_{(1)}], Y] \quad Y \in \mathfrak{g}_1^+$$

$$= [[Z_{(0)}, Y], Z'_{(1)}] + [Z_{(0)}, [Z'_{(1)}, Y]]$$

$$= [DY, Z'_{(1)}] - [Z_{(0)}, (Y \otimes Z'_{(1)} + Z'_{(1)}(Y), (R-1)Z'_{(1)}(Y))]$$

$$= \oplus (DY \otimes Z'_{(1)} + Z'_{(1)}(DY), (R-1)Z'_{(1)}(DY))$$

$$+ \underbrace{([D, Y \otimes Z'_{(1)} + Z'_{(1)}(Y)], 0)}$$

$$= -(DY \otimes Z'_{(1)} + Y \otimes (Z'_{(1)}D))$$

$$= (Y \otimes (Z'_{(1)}D) + Z'_{(1)}(DY), (R-1)Z'_{(1)}(DY))$$

$$\underbrace{[Z_{(0)}, Z'_{(1)}] = -Z'_{(1)}D}$$

$$\left[\begin{pmatrix} 0 & 0 \\ Z_{(1)} & 0 \end{pmatrix}, \begin{pmatrix} 0 & Z_{(1)}' \\ 0 & 0 \end{pmatrix} \right] /$$

$$= \begin{pmatrix} -Z_{(1)}'(Z_{(1)}) & 0 \\ 0 & Z_{(1)}Z_{(1)}' \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{n+1} \text{Tr} D & 0 \\ 0 & D - \frac{1}{n+1} \text{Tr} D \end{pmatrix}$$

$$D = Z_{(1)}Z_{(1)}' + Z_{(1)}'Z_{(1)}$$

$$[Z_{(1)}, Z_{(1)}'] = (Z_{(1)} \otimes Z_{(1)}' + Z_{(1)}'Z_{(1)}, (R-1)Z_{(1)}'Z_{(1)})$$

$Z \in \mathcal{L}$ with $Z_0 = Z_1 = 0$.

$$A(t) = \begin{pmatrix} 1 & 0 \\ tZ_{(-1)} & 1 \end{pmatrix}$$

$X \in \mathcal{P}$

$$\cdot (A(t), 1) X^{(p)} (Y_1, \dots, Y_{R+p})$$

$$= \lambda_p X(A(t)^T e_0, \dots, A(t)^T e_0, A(t)^T Y_1, \dots, A(t)^T Y_{R+p})$$

$$= \lambda_p X(e_0 - tZ_{(-1)}, \dots, e_0 - tZ_{(-1)}, Y_1, \dots, Y_{R+p})$$

これは t の次数が 1, $t=0$ とおくと

$$(Z, X)^{(p)} (Y_1, \dots, Y_{R+p})$$

$$= \lambda_p (-p-1) X(e_0, \dots, e_0, Z_{(-1)} Y_1, \dots, Y_{R+p})$$

$$= -\lambda_p (-p-1) \frac{1}{\lambda_{p+1}} (Z_{(-1)}, X^{(p+1)}) (Y_1, \dots, Y_{R+p})$$

$$= \frac{\lambda_p}{\lambda_{p+1}} (-p-1) [Z_{(-1)}, X^{(p+1)}] (Y_1, \dots, Y_{R+p})$$

$$(Z, X)^{(p)} = \frac{\lambda_p}{\lambda_{p+1}} (-p-1) [Z_{(-1)}, X^{(p+1)}]$$

$(-R \leq p \leq -2)$

$Z \in \mathcal{I}$ with $Z_{(-1)} = Z_{(0)} = 0$.

$$A(t) = \begin{pmatrix} 1 & t Z_0 \\ 0 & 1 \end{pmatrix}$$

$X \in \mathcal{P}$

$$((A(t), 1) \cdot X)^{(p)}(Y_1, \dots, Y_{R+p})$$

$$= \lambda_p X(A(t)^{-1} e_0, \dots, A(t)^{-1} e_0, A(t)^{-1} Y_1, \dots, A(t)^{-1} Y_{R+p})$$

$$= \lambda_p X(e_0, \dots, e_0, -t Z_0(e_0 + Y_1, \dots), -t Z_0(Y_{R+p} e_0 + Y_{R+p}))$$

これを t で微分し, $t=0$ とおく.

$$(Z_0 \cdot X)^{(p)}(Y_1, \dots, Y_{R+p})$$

$$= \lambda_p \sum_{i=1}^{R+p} Z_0(Y_i) X(e_0, \dots, e_0, \hat{Y}_1, \dots, \hat{Y}_{R+p})$$

$$= -\frac{\lambda_p}{\lambda_{p+1}} (R+p) (Z_0 \otimes X^{(p+1)})(Y_1, \dots, Y_{R+p})$$

$$= \frac{\lambda_p}{\lambda_{p+1}} \frac{R+p}{(R+p)(-p)} [Z_0, X^{(p+1)}](Y_1, \dots, Y_{R+p})$$

$$(Z_0 \cdot X)^{(p)} = \frac{\lambda_p}{\lambda_{p+1}(-p)} [Z_0, X^{(p+1)}]$$

$$(-R+1 \leq p \leq 0)$$

$$\frac{\lambda_p}{\lambda_{p+1}} \cdot (-p-1) = 1 \quad (-R \leq p \leq -2)$$

$$\frac{\lambda_p}{\lambda_{p-1} \cdot (-p)} = 1 \quad (-R+1 \leq p \leq 0)$$

$$\frac{\lambda_{p+1}}{\lambda_p} = (-p-1)$$

$$\frac{\lambda_{-R+1}}{\lambda_{-R}} = (R-1)$$

$$\frac{\lambda_{-R+2}}{\lambda_{-R+1}} = (R-2)$$

$$\frac{\lambda_{p+1}}{\lambda_p} = (-p-1)$$

$$\frac{\lambda_{p+1}}{\lambda_{-R}} = \frac{(R-1)!}{(-p-2)!}$$

$$\lambda_{-R} = 1 \quad \text{且 } 1, 2$$

$$\lambda_{p+1} = \frac{(R-1)!}{(-p-2)!}$$

$$\lambda_p = \frac{(R-1)!}{(-p-1)!}$$

$Z \in \mathcal{L}$ with $Z(t) = Z_0 = 0$.

$$Z = (\tilde{D}, B) \quad Z_0 = (D, \frac{R-1}{R+1} \text{Tr} D + B)$$

$$A(t) = \exp t \tilde{D}, \quad B(t) = \exp t B$$

$$((A(t), B(t))X)^{(p)}(Y_1, \dots, Y_{R+p})$$

$$= \lambda_p B(t) X(A(t)^{-1} e_0, \dots, A(t)^{-1} e_0, A(t)^{-1} Y_1, \dots, A(t)^{-1} Y_{R+p})$$

$$= \lambda_p e^{t \frac{R-1}{R+1} \text{Tr} D} B(t) X(e_0, \dots, e_0, \exp(-tD) Y_1, \dots, \exp(-tD) Y_{R+p})$$

$$(Z \cdot X)^{(p)}(Y_1, \dots, Y_{R+p})$$

$$= \lambda_p B X(e_0, \dots, e_0, Y_1, \dots, Y_{R+p})$$

$$+ \lambda_p \frac{R-1}{R+1} \text{Tr} D X(e_0, \dots, e_0, Y_1, \dots, Y_{R+p})$$

$$- \lambda_p \sum X(e_0, \dots, e_0, Y_1, \dots, D Y_1, \dots, Y_{R+p})$$

$$= [Z_0, X^{(p)}](Y_1, \dots, Y_{R+p})$$

$\square \square \square$

$$(Z \cdot X)^{(p)} = [Z_0, X^{(p)}]$$

記号変更 ($\lambda \rightarrow -\lambda$) : 変

No. ①

$$① \quad [X_1, X_p] = (R+p+1)(p+1) X_1 \otimes X_p$$

$$② \quad ((X_p, X_p')) = \frac{1}{(R+p)!} \sum_{1 \leq i_1, \dots, i_{R+p} \leq r} (X_p(e_{i_1}, \dots, e_{i_{R+p}}), X_p'(e_{i_1}, \dots, e_{i_{R+p}}))$$

$$Z \in \mathfrak{g}_1^+$$

$$((Z \lrcorner X_{p+1}, X_p')) = - ((X_{p+1}, \sigma Z \otimes X_p'))$$

$$③ \quad (X_p, X_p') = \lambda_p ((X_p, X_p'))$$

$$([\sigma Z, X_p], X_{p+1}') = \lambda_{p+1} (([\sigma Z, X_p], X_{p+1}'))$$

$$= \lambda_{p+1} \cdot (R+p+1)(p+1) ((\sigma Z \otimes X_p, X_{p+1}'))$$

$$= \lambda_{p+1} \cdot (R+p+1)(-p-1) ((X_p, Z \lrcorner X_{p+1}'))$$

$$= - \frac{\lambda_{p+1}}{\lambda_p} \cdot (R+p+1)(-p-1) (X_p, [Z, X_{p+1}'])$$

$$\frac{\lambda_{p+1}}{\lambda_p} = \frac{1}{(R+p+1)(-p-1)}$$

$$\frac{\lambda_{-R+1}}{\lambda_{-R}} = \frac{1}{1(R-1)}, \quad \frac{\lambda_{-R+2}}{\lambda_{-R+1}} = \frac{1}{2(R-2)}, \dots$$

$$\frac{\lambda_{p+1}}{\lambda_p} = \frac{1}{(R+p+1)(-p-1)} \quad \frac{\lambda_{p+1}}{\lambda_{-R}} = \frac{(-p-2)!}{(R+p+1)!(R-1)!}$$

$$\lambda_{-R} = 1, \quad \lambda_{p+1} = \frac{(-p-2)!}{(R+p+1)!(R-1)!}$$

$$\lambda_p = \frac{(-p-1)!}{(R+p)!(R-1)!}$$

$$J > 2$$

$$(X_p, X_p') = \frac{(-p-1)!}{(R-1)!(R+p)!} ((X_p, X_p'))$$

$$\textcircled{c} \quad \Phi(\bar{X}_p) = X_p$$

$$(X_p, X_p') = (\bar{X}_p, \bar{X}_p') = \sum_{0 \leq i_1, \dots, i_{R-1} \leq r} (\bar{X}_p(e_{i_1}, \dots, e_{i_{R-1}}), \bar{X}_p'(e_{i_1}, \dots, e_{i_{R-1}}))$$

$$= \binom{R-1}{p-1} \sum_{1 \leq i_1, \dots, i_{R+p} \leq r} (\bar{X}_p(e_{i_1}, \dots, e_{i_p}, e_{i_{p+1}}, \dots, e_{i_{R+p}}), \bar{X}_p'(e_{i_1}, \dots, e_{i_p}, e_{i_{p+1}}, \dots, e_{i_{R+p}}))$$

$$= \frac{\binom{R-1}{p-1} (-p-1)!^2}{(-p-1)!(R-1)!^2} \sum_{1 \leq i_1, \dots, i_{R+p} \leq r} (X_p(e_{i_1}, \dots, e_{i_{R+p}}), X_p'(e_{i_1}, \dots, e_{i_{R+p}}))$$

$$= \frac{(-p-1)!}{(R-1)!(R+p)!} \sum_{1 \leq i_1, \dots, i_{R+p} \leq r} (X_p(e_{i_1}, \dots, e_{i_{R+p}}), X_p'(e_{i_1}, \dots, e_{i_{R+p}}))$$

$$= \frac{(-p-1)!}{(R-1)!(R+p)!} ((X_p, X_p'))$$

$$\textcircled{1} \quad Z \in \mathcal{G}_1^+, \quad \bar{\Phi}(\bar{Z}) = Z$$

$$([Z, X_{p+1}], X_p') = (\bar{Z} \cdot \bar{X}_{p+1}, \bar{X}_p')$$

$$\equiv -(\bar{X}_{p+1}, \delta \bar{Z} \cdot \bar{X}_p') = -(\bar{X}_{p+1}, \delta \bar{Z} \cdot \bar{X}_p')$$

$$= -(X_{p+1}, [\delta Z, X_p'])$$

$$X_p(Y_1, \dots, Y_{R+p}) = \frac{(R-1)!}{(-p-1)!} \bar{X}_p(e_0, \dots, e_0, Y_1, \dots, Y_{R+p})$$

$$Z \in \mathfrak{g}_1^+$$

$$\bar{Z} = \left(\begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix}, 0 \right)$$

$$A(t) = \begin{pmatrix} 1 & 0 \\ tZ & 1 \end{pmatrix}$$

$$(\exp t \bar{Z} \cdot \bar{X}_{p+1})^{(p)}(Y_1, \dots, Y_{R+p})$$

$$= \frac{(R-1)!}{(-p-1)!} \bar{X}_{p+1}(A(t)^{-1}e_0, \dots, A(t)^{-1}e_0, A(t)^{-1}Y_1, \dots, A(t)^{-1}Y_{R+p})$$

$$= \frac{(R-1)!}{(-p-1)!} \bar{X}_{p+1}(e_0 - tZ, \dots, e_0 - tZ, Y_1, \dots, Y_{R+p})$$

$$(\bar{Z} \cdot \bar{X}_{p+1})^{(p)}(Y_1, \dots, Y_{R+p})$$

$$= \frac{(R-1)!}{(-p-1)!} (p+1) \bar{X}_{p+1}(e_0, \dots, e_0, Z, Y_1, \dots, Y_{R+p})$$

$$= - \frac{(R-1)!}{(-p-2)!} \frac{(-p-2)!}{(R-1)!} X_{p+1}(Z, Y_1, \dots, Y_{R+p})$$

$$= [Z, X_{p+1}](Y_1, \dots, Y_{R+p})$$

$$\Phi(\bar{Z} \cdot \bar{X}_{p+1}) = [Z, X_{p+1}]$$

$$\sigma \bar{Z} = \left(\begin{pmatrix} 0 & -tZ \\ 0 & 0 \end{pmatrix}, 0 \right)$$

$$A(t) = \begin{pmatrix} 1 & -tZ \\ 0 & 1 \end{pmatrix}$$

$$((\exp t \sigma \bar{Z}) \cdot \bar{X}_p)^{(p+1)}(Y_1, \dots, Y_{R+p+1})$$

$$= \frac{(R-1)!}{(-p-2)!} \bar{X}_p(A(t)^{-1}e_0, \dots, A(t)^{-1}e_0, A(t)^{-1}Y_1, \dots, A(t)^{-1}Y_{R+p+1})$$

$$= \frac{(R-1)!}{(-p-2)!} \bar{X}_p(e_0, \dots, e_0, t^t Z Y_1 e_0 + Y_1, \dots, t^t Z Y_{R+p+1} e_0 + Y_{R+p+1})$$

$$(\sigma \bar{Z} \cdot \bar{X}_p)^{(p+1)}(Y_1, \dots, Y_{R+p+1})$$

$$= \frac{(R-1)!}{(-p-2)!} \sum_i t^t Z Y_i \cdot \bar{X}_p(e_0, \dots, e_0, Y_1, \dots, \hat{Y}_i, \dots, Y_{R+p+1})$$

$$= \frac{(R-1)!}{(-p-2)!} \frac{(-p-1)!}{(R-1)!} \sum_i t^t Z Y_i \bar{X}_p(Y_1, \dots, \hat{Y}_i, \dots, Y_{R+p+1})$$

$$= (-p-1)(R+p+1) \sum_i t^t (Z \otimes X_p)(Y_1, \dots, Y_{R+p+1})$$

$$= -[tZ, X_p](Y_1, \dots, Y_{R+p+1})$$

$$= [\sigma Z, X_p](Y_1, \dots, Y_{R+p+1})$$

$$\Phi(\sigma \bar{Z} \cdot \bar{X}_p) = [\sigma Z, X_p]$$

$$(\overline{Z}, \overline{X}_{p+1}, \overline{X}_p') = -(\overline{X}_{p+1}, \sigma \overline{Z}, \overline{X}_p')$$

$$\parallel$$

$$(\overline{[Z, X_{p+1}]}, \overline{X}_p') = -(\overline{X}_{p+1}, \overline{[\sigma Z, X_p']})$$

$$\parallel$$

$$\parallel$$

$$([Z, X_{p+1}], X_p') = -(X_{p+1}, [\sigma Z, X_p'])$$

$$([Z, X_{p+1}], X_p') = -(X_{p+1}, [\sigma Z, X_p'])$$

$$-(Z \lrcorner X_{p+1}, X_p') = (R+p+1)(p+1)(X_{p+1}, \overset{t}{Z} \otimes X_p')$$

$$-\lambda_p([Z \lrcorner X_{p+1}, X_p']) = \lambda_{p+1}(R+p+1)(p+1)(X_{p+1}, \overset{t}{Z} \otimes X_p')$$

Part II

On the integration problems derived from two remarkable classes of systems of ordinary differential equations of the second order.

Preliminary remarks

By a foliated manifold we mean a manifold R equipped with a completely integrable system E or the pair (R, E) .

Let (R, E) be a foliated manifold.

1°. A function f on R is called a first integral of E if it satisfies the differential equation:

$$Xf = 0 \text{ for any } X \in E.$$

2°. Let p be any point of R . If we set $n = \dim R - \text{rank } E$, we see from the Frobenius theorem that there are n first integrals $u^1, \dots, u^n$ of E defined on a neighborhood of p such that the differentials $du^1, \dots, du^n$ are linearly independent at p .

2

It can be shown that any first integral f of E defined on a neighborhood of p may be described as a function of $u^1, \dots, u^n$:
$$f = F(u^1, \dots, u^n).$$

3° By the integration of E at a point p we mean to construct a fundamental system $(u^1, \dots, u^n)$ of first integrals of E at p .

Let p be a mapping of a manifold R to a manifold M . Then R is called a fibred manifold over the base space M with projection p , if p is surjective, and for each point p the differential p_{*p} is surjective. A vector $x \in T_p(R)$ is called a vertical vector if $p_{*p}(x) = 0$, and the

totality of vertical vectors is called the vertical tangent bundle of the fibred manifold.

§1. Transverse Cartan connections and Cartan systems

1.1. Transverse Cartan connections.

Let G/G' be a coset of a Lie group G over its closed subgroup G' . Let $\mathfrak{g}$ and $\mathfrak{g}'$ be the Lie algebras of $\mathfrak{g}$ and $\mathfrak{g}'$ respectively.

Let us consider a foliated manifold (R, E) . Suppose that R becomes a fibred manifold over a manifold R' with projection $p: R \rightarrow R'$ so that the vertical

tangent bundle coincides with E .

Furthermore suppose that there is given

a Cartan connection $\omega': (P', \omega')$ of model coset space G/G' on R' . Then we denote by P the p.f.b. over R induced from the p.f.b. P' by the mapping ρ , and by $\hat{\rho}$ the natural homomorphism of P to P' .

We also denote by ω the $\mathfrak{g}$ -valued 1-form $\hat{\rho}^* \omega'$ on P . Then it is clear that the pair (P, ω) satisfies the following conditions:

$$(C.1) \quad R_a^* \omega = \text{Ad}(a^{-1}) \omega, \quad a \in G'$$

$$(C.2) \quad \omega(A^*) = A, \quad A \in \mathfrak{g}'$$

Let π' (resp. π) be the projection of P' (resp. P) onto R' (resp. R). Since $\pi' \circ \hat{\rho} = \rho \circ \pi$ and $E = \rho_*^{-1}(0)$, we also see that the pair satisfies the third condition:

(C.3) Let X be a tangent vector to P .
Then, $\omega(X) \in \mathfrak{g}'$, if and only if $\pi_*(X) \in E$.

The observation above leads to the following

Definition. A transverse Cartan connection Γ ^[of model coset space G/G'] on a foliated manifold (R, E) is the pair (P, ω) formed by a p.f.b. P over R with structure group G' with projection π and a $\mathfrak{g}$ -valued 1-form ω on P satisfying conditions (C.1), (C.2) and (C.3).

Let $\Gamma: (P, \omega)$ be as in the definition. We once for all fix a complementary subspace m of $\mathfrak{g}'$ in $\mathfrak{g}$, and denote by Θ the m -component of ω with respect to

the decomposition: $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}'$. We also denote by $E(\Gamma)$ the differential system (possibly with singularities) by the equation $\omega = 0$, and by $V(\Gamma)$ the vertical tangent bundle of the p.f.b. P , or simply $V(\Gamma) = \pi_*^{-1}(0)$. Then we have the following

Proposition 1.1. (1) $\theta^{-1}(0) = \pi_*^{-1}(E)$

$$= E(\Gamma) \oplus V(\Gamma).$$

(2) For each $z \in P$ the differential π_* of π maps the fibre $E(\Gamma)_z$ of $E(\Gamma)$ at z isomorphically onto the fibre $E_{\pi(z)}$ of E .

(3) $E(\Gamma)$ is right invariant.

(4) There is a unique function

$K: P \rightarrow \mathfrak{g} \otimes \Lambda^2 \mathfrak{m}^*$ such that

$$d\omega + \frac{1}{2} [\omega, \omega] = \frac{1}{2} K(\theta \wedge \theta).$$

Proof. (1) ~ (3) can be easily verified

If we set $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$, we have $A^* \lrcorner \Omega = 0$ for any $A \in \mathcal{F}'$, and $\Omega(X, Y) = 0$ for any crosssections X, Y of $E(\Gamma)$. Since $\theta^{-1}(0) = E(\Gamma) + V(\Gamma)$, it follows that $X \lrcorner \Omega = 0$ for any $X \in \theta^{-1}(0)$. Therefore (4) follows.

The equality in (4) will be called the structure equation, and the function K the curvature. We know from the existence of the structure equation that the system $E(\Gamma)$ is completely integrable.

1.2. A reduction proposition for the integration of completely integrable systems. Let $\Gamma: (P, \omega)$ be a transverse Cartan connection of model coset space G/G' on a foliated manifold (R, E) .

Proposition 1.2. Let x_0 be any point of P and set $p_0 = \pi(x_0)$. Then, once the system $E(\Gamma)$ was integrated at x_0 , the integration of the system E at p_0 can be carried out by elementary operations.

By elementary ^{operations} we here mean a collection of operations formed by rational operations, differentiations and the uses of the implicit function theorem.

Proof. We set $r = \text{rank } E = \text{rank } E(\Gamma)$,
 $(= \dim G)$ $(= \dim G/G')$
 $m = \dim P - r$ and $n = \dim R - r$. Now, let

$(u^1, \dots, u^m)$ be a fundamental system of first integrals of $E(\Gamma)$ at x_0 .

(1) We take r functions $u^{m+1}, \dots, u^{m+r}$ defined on a neighborhood of x_0 so that $u^1, \dots, u^m, u^{m+1}, \dots, u^{m+r}$ form a coordinate system of P at x_0 . Let P_0 be a small cubic neighborhood of x_0 . We then construct a mapping φ of P_0 to $\mathbb{R}^m$ by

$$\varphi(z) = (u^1(z), \dots, u^m(z)), \quad z \in P_0,$$

and set $Q_0 = \varphi(P_0)$. Then P_0 becomes a fibred manifold over the base space Q_0 with projection φ so that the vertical tangent bundle coincides with the system $E(\Gamma)$, restricted to P_0 . We set $z_0 = \varphi(z_0)$.

(2) Let us now construct a local right

action of the group G' on Q_0 . For this purpose we first construct a mapping φ of Q_0 to U such that $\varphi(q_0) = z_0$ and $\varphi \circ \varphi^{-1}$ is the identity mapping. Then the local action is constructed by the following rule:

$$y \cdot q = \varphi(\varphi(y)q),$$

where $(y, q) \in Q_0 \times G'$ with $\varphi(y)q \in P_0$.

Since the system $E(\pi)$ is right invariant, we see that this rule really gives a local action of G' on Q_0 and that φ is compatible with the respective actions of G' .

(3) For any $A \in \mathfrak{g}'$ we have the (vertical) vector field A^* on P_0 . Let us denote by the

same symbol A^* the vector field on Q_0 induced from the local 1-parameter group $y \rightarrow y \cdot \exp tA$ of local right translations on Q_0 . Clearly these two vector fields are φ -related: $\varphi_* A_z^* = A_{\varphi(z)}^*$, $z \in U$. Therefore the condition " $A_{\varphi(z)}^* = 0$ " implies $A_z^* \in E(\Gamma)_z$. By Proposition 1.1 it follows that $A_z^* = 0$ or $A = 0$. We have thereby proved the following:

Lemma 1.3. The local action of G' on Q_0 is infinitesimally free, that is, the condition " $A_y^* = 0$ " for some $A \in \mathfrak{g}'$ and for some $y \in Q_0$ implies $A = 0$.

(4) We now take an n -dimensional small submanifold, say M_0 , of Q_0 through y_0 which is transversal to the G' -orbit through y_0 as well as a small neighborhood, say N_0 , of the identity element e of G' . Then we define a mapping Φ of $M_0 \times N_0$ to Q_0 by $\Phi(y, a) = y \cdot a$, $(y, a) \in M_0 \times N_0$. By Lemma 1.3 we easily see that the differential of Φ at (y_0, e) gives an isomorphism. Thus Φ gives a diffeomorphism of a neighborhood of (y_0, e) onto a neighborhood of y_0 so that $\Phi(y_0, e) = y_0$, and we construct the inverse, say Ψ , of Φ .

The matter being highly of local character, we may assume that Ψ give a diffeomorphism of Q_0 onto $M_0 \times U_0$. This being said, we denote by ω the projection of $M_0 \times U_0$ onto M_0 , and set $\varepsilon = \omega \circ \Phi$. Then Q_0 becomes a fibred manifold over M_0 with projection ε . Furthermore if we set $R_0 = \pi(P_0)$, R_0 is an open submanifold of R , and P_0 becomes a fibred manifold over R_0 with projection π .

(5) We set $p_0 = \pi(z_0)$, and take a local cross section σ of the fibred manifold P_0 over R_0 defined on a neighborhood V_0 of P_0 and such that $\sigma(p_0) = z_0$. Then we define a mapping ρ of V_0 to M_0 by $\rho = \varepsilon \circ \varphi \circ \gamma$.

Lemma 1.4. There is a neighborhood U_0 of z_0 such that $\pi(U_0) \subset V_0$ and the two mappings $\rho \circ \pi$ and $\varepsilon \circ \varphi$ coincide on U_0 .

Proof. Let $z \in \pi^{-1}(V_0) \cap P_0$. Then we have $(\rho \circ \pi)(z) = (\varepsilon \circ \varphi)(\gamma(\pi(z)))$. Since $\pi(\gamma(\pi(z))) = \pi(z)$, there is an element $a(z)$ of G' with $\gamma(\pi(z)) = z \cdot a(z)$. It follows $(\rho \circ \pi)(z) = (\varepsilon \circ \varphi)(z \cdot a(z)) = \varepsilon(\varphi(z) a(z))$. In view

of the structure of Q_0 , φ may be expressed as follows: $\varphi(z) = \varepsilon(\varphi(z)) \cdot b(z)$, $z \in P_0$, where $b(z) \in N_0$. Since $\varphi(z_0) = y_0$ and $\varepsilon(y_0) = y_0$, we have $b(z_0) = e$. Similarly we have $\psi(p) = z_0 = z_0 \cdot a(z_0)$, whence $a(z_0) = e$. Therefore it follows that there is a neighborhood U_0 of z_0 such that $\pi(U_0) \subset V_0$ and $b(z) \cdot a(z) \in N_0$ for any $z \in U_0$. Consequently for any $z \in U_0$ we obtain

$$\varepsilon(\varphi(z) \cdot a(z)) = \varepsilon(\varepsilon(\varphi(z)) \cdot b(z) a(z)) = \varepsilon(\varphi(z)),$$

whence $(p \circ \pi)(z) = \varepsilon(\varphi(z))$, proving the lemma.

(6) By Proposition 2.1 we have

$$\pi_*^{-1}(E) = E(\Gamma) + V(\Gamma).$$

By Lemma 1.4 we have

$$(p \circ \pi)_*^{-1}(0) = (E \circ \varphi)_*^{-1}(0) \text{ on } U_0.$$

Furthermore we can easily verify that

$$(E \circ \varphi)_*^{-1}(0) = E(\Gamma) + V(\Gamma) \text{ on } U_0.$$

Consequently it follows that

$$(p \circ \pi)_*^{-1}(0) = \pi_*^{-1}(E) \text{ on } U_0,$$

which clearly means that

$$p_*^{-1}(0) = E \text{ on } p(U_0).$$

Now, let

Now, let $(\xi^1, \dots, \xi^n)$ be a coordinate system of M , at $p(p_0)$. Then this last fact implies that $(\xi^1 p, \dots, \xi^n p)$ gives a fundamental system of first integrals of E at p_0 .

We have thus completed the proof of 1.2.

Proposition 1.2, because all the constructions

above can be carried out by elementary operations. We have thus completed the proof of the proposition.

§2. Cartan systems

As for a detailed account of the theory of Cartan systems, see the appendix.

2.1. Cartan systems. At the outset we give the following

Definition (cf. Cartan [1]). A Cartan system Σ (of degree n) is the pair (P, ω) formed by a manifold P and an $\mathbb{R}^n$ -valued 1-form ω or a system of 1-forms $\omega^1, \dots, \omega^n$ on P satisfying the following conditions:

1) $\omega^1, \dots, \omega^n$ are linearly independent at each point of P ,

2) There are functions c_{jk}^i ($1 \leq i, j, k \leq n$) on P such that

$$d\omega^i + \frac{1}{2} \sum_{j, k} c_{jk}^i \omega_j \wedge \omega_k = 0, \quad c_{jk}^i + c_{kj}^i = 0.$$

Given a Cartan system Σ , the equality

above will be called the structure equation, and the system (G_{jk}^i) of functions the structure functions.

We also denote by $E(\Sigma)$ the differential system on P defined by the equation $\omega = 0$, which is clearly completely integrable.

Now, a local vector field X on P is called a local infinitesimal automorphism of Σ , if X leaves ω invariant or $L_X \omega = 0$. In view of the structure equation we know that a local cross section of $E(\Sigma)$ is an infinitesimal automorphism of Σ and that if γ is a local infinitesimal automorphism of Σ and

γ is a local cross section of $E(\Sigma)$, then the bracket $[X, \gamma]$ is a local cross section of $E(\Sigma)$. This being said, we denote by A the sheaf of Lie algebras of local infinitesimal automorphisms of Σ , and by E the sheaf of local cross sections of $E(\Sigma)$, being a subsheaf of ideals of A . Furthermore for each $x \in P$ the factor Lie algebra A_x / E_x is denoted by $\text{aut}(\Sigma)_x$, and is called the reduced Lie algebra of infinitesimal automorphisms of Σ at x , where A_x (resp. E_x) denotes the stalk at x of A (resp. of E). We shall see that under a suitable regularity condition on Σ $\text{aut}(\Sigma)_x$ is naturally isomorphic

with the stalk at x of the factor Lie algebra sheaf A/E .

For example let us consider a transverse Cartan connection $\Gamma: (P, \omega)$ of model coset space $G/G^{(0)}$ on a foliated manifold (R, E) . Then the pair (P, ω) may be regarded as a Cartan system, say Σ , which is called associated.

Proposition 1.2 indicates that the integration problem in the foliated manifold (R, E) can be reduced to the integration problem in the associated Cartan system Σ .

2.2. Non-degenerate Cartan systems.

A Cartan system of degree $n: \Sigma: (P, \omega)$ is called non-degenerate, if $\dim P = n$ or in other words ω gives an absolute parallelism on P . A non-degenerate Cartan system no more possesses the meaning of a differential equation, but is a geometric object.

Proposition 2.1. Let $\Sigma: (P, \omega)$ be a Cartan system of degree n , and let x_0 be any point of P .

(1) There are a neighborhood V of x_0 and a non-degenerate Cartan system $\Sigma': (V', \omega')$ satisfying the following conditions: 1) V becomes a fibred manifold over V' with projection κ so that the

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vertical tangent bundle coincides with the system $E(\Sigma)$, restricted to V , and $\kappa^* \omega$ coincides with ω , restricted to V .

(2) The projection κ naturally induces an isomorphism of $\text{aut}(\Sigma)_{x_0}$ onto $\text{aut}(\Sigma')_{\kappa(x_0)}$.

Corollary 2.2. $\dim \text{aut}(\Sigma)_x \leq n$, $x \in P$.

Proof (of the proposition). (1) We set $r = \text{rank } E(\Sigma)$, whence $\dim P = n+r$. In the following the letters i, j will range over the integers $1, \dots, n$, and the letters λ, μ over the integers $n+1, \dots, n+r$.

(1) Let x_0 be any point of P , and let $(u^1, \dots, u^{n+r})$ be a coordinate system of P at x_0 such that (u^i) gives a fundamental

system of first integrals of the system $E(\Sigma)$ at x_0 .

Let V be a cubic neighborhood of x_0 with respect to this coordinate system, and let us consider the moving frame $(\partial/\partial u^1, \dots, \partial/\partial u^{n+r})$ of the tangent bundle $T(V)$ of V . For simplicity we set $Y_\lambda = \partial/\partial u^\lambda$. Then we see that (Y_λ) gives a moving frame of the system $E(\Sigma)$.

ii. We define a mapping κ of V to $\mathbb{R}^n$ by $p(x) = (u^i(x))$ for any $x \in V$, and set $V' = \kappa(V)$. Then we see that V becomes a fibred manifold over V' with projection κ so that the vertical tangent bundle coincides with the system $E(\Sigma)$, restricted

to V .

Now, the $\mathbb{R}^n$ -valued 1-form ω may be described as follows:

$$\omega = \sum_i f_i du^i + \sum_\mu f_\mu du^\mu,$$

where f_i, f_μ are $\mathbb{R}^n$ -valued functions on V .

Since $\omega(Y_\lambda) = 0$ and $Y_\lambda u^\mu = \delta_\lambda^\mu$, we have $f_\lambda = 0$.

Furthermore since $L_{Y_\lambda} \omega = 0$ and $Y_\lambda u^i = 0$, we obtain $Y_\lambda f_i = 0$. It follows that f_i are

first integrals of $E(\Sigma)$, and hence may

be described as follows: $f_i = \hat{f}_i(u^1, \dots, u^n)$,

$\hat{f}_i$ being functions on V' .

Let (v^i) be the canonical coordinate system of V' . Then we have $u^i = v^i \circ \kappa$

and $f_i = \hat{f}_i \circ \kappa$. If we set

$$\omega' = \sum_i f_i dv^i,$$

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we find that $\omega = \kappa^* \omega'$, and the pair (V', ω') gives a non-degenerate Cartan system, say Σ' , of degree n , proving the first assertion.

(2) Let X be any local infinitesimal automorphism of Σ defined on a neighborhood of x_0 , which may be described as follows:

$$X = \sum_i g^i \partial / \partial u^i + \sum_\mu g^\mu Y_\mu,$$

If we set $\bar{X} = \sum g^i \partial / \partial u^i$, we see that

$\bar{X}$ is a local infinitesimal automorphism of Σ .

Lemma 2.3. The vector fields $[\gamma_\lambda, \bar{X}]$ and $[\gamma_\lambda, \partial/\partial u^i]$ are local cross sections of $E(\Sigma)$.

For the proof of this fact we remark that i) $L_{\bar{X}}\omega = 0$ and $\omega(\gamma_\lambda) = 0$, and ii) $du^j(\partial/\partial u^i) = \delta_i^j$ and $\gamma_\lambda u^j = 0$.

Now, we have

$$[\gamma_\lambda, \bar{X}] = \sum_i \gamma_\lambda g^i \partial/\partial u^i + \sum_i g^i [\gamma_\lambda, \partial/\partial u^i].$$

By Lemma 2.3 it follows that $\gamma_\lambda g^i = 0$.

Therefore g^i are first integrals of $E(\Sigma)$ and hence, may be described as follows:

$g^i = \hat{g}^i(u^1, \dots, u^n)$, $\hat{g}^i$ being functions defined on a neighborhood of $p(\alpha_0)$.

If we set

$$X' = \sum_i \hat{g}^i \partial/\partial u^i,$$

we know that i) X' is a local infinitesimal

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automorphism of Σ' , 2) X and X' are κ -related, and 3) the assignment $X \rightarrow X'$ induces an isomorphism of $\text{aut}(\Sigma)_{x_0}$ with $\text{aut}(\Sigma')_{\kappa(x_0)}$, proving the second assertion

2.2. The regularity condition. Let $\Sigma: (P, \omega)$ be a Cartan system of degree n . First of all we remark that a function f on P is a first integral of $E(\Sigma)$, if and only if

$$df \equiv 0 \pmod{\omega^1, \dots, \omega^n}.$$

Therefore if f is a first integral of $E(\Sigma)$, the differential df may be described as follows:

$$df = \sum_i f_i \omega^i.$$

The coefficients f_i will be denoted by $\delta_i f$ or $\partial f / \partial \omega^i$.

By using the structure equation we can easily verify the following

Lemma 2.4. (1) The structure functions c_{jk}^i are first integrals of $E(\Sigma)$.

(2) If f is a first integral of $E(\Sigma)$, then the coefficients $\delta_i f$ of df are first integrals of $E(\Sigma)$.

This being said, let us denote by $I'(\Sigma)$ the family of all (global) first integrals of $E(\Sigma)$. Then we denote by $I(\Sigma)$ the smallest subfamily among the subfamilies I of $I'(\Sigma)$ satisfying the following conditions:

- 1) $c_{jk}^i \in I$,
- 2) If $f \in I$, then $\delta_i f \in I$.

Namely $I(\Sigma)$ consists of all functions of the following form:

$$\delta_{i_1} \cdots \delta_{i_l} c_{jk}^i \quad (1 \leq i, j, k, i_1, \dots, i_l \leq n, l \geq 0).$$

Now, for each point $x \in P$ we denote by $\Delta(\Sigma)_x$ the subspace of $T_x(P)$ consisting of all tangent vectors $X \in T_x(P)$ such that

$$Xf = 0 \quad \text{for all } f \in I(\Sigma),$$

and set $\Delta(\Sigma) = \bigcup_x \Delta(\Sigma)_x$, being a differential system (possibly with singularities) on P . Clearly we have $E(\Sigma) \subset \Delta(\Sigma)$.

We say that the Cartan system Σ is regular, if $\Delta(\Sigma)$ gives a differential system in the usual sense, restricted to each connected component of P , or equivalently $\dim \Delta(\Sigma)_x$ is locally constant.

It can be shown that in the real analytic category any Cartan system Σ is necessarily regular.

2.3. The Cartan's integration theorem for Cartan systems. We are now in a position to state the following

Theorem 2.5. (E. Cartan [1]). Let Σ :
 (P, ω) be a Cartan system, and let x_0 be any point of P . If Σ is regular,

then the integration of the completely integrable system $E(\Sigma)$ can be carried out by elementary operations,

quadratures and the integrations of the Lie systems associated with the simple components of the Lie algebra $\text{aut}(\Sigma)_{x_0}$.

§3. The integration problems associated with pseudo-projective systems of the restricted types

3.1. The transverse Cartan connections associated with pseudo-projective systems of the restricted types.

(1) Let us consider the graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ as well as the coset space $G/B^{(0)}$ defined in §I.2. We denote by P_2 the linear isotropy representation of the group $B^{(0)}$ on the space $\mathfrak{g}_{-1}$ identified with the tangent space to $G/B^{(0)}$ at the origin o :

$$P_2(a)X \equiv \text{Ad}(a)X \pmod{\mathfrak{g}^{(0)}},$$

where $a \in B^{(0)}$ and $X \in \mathfrak{g}_{-1}$. Furthermore

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the group $B^{(0)}$ acts on the space $\mathfrak{g} \otimes \wedge^2 \mathfrak{b}_{-1}^*$ by the following rule:

$$L^a(X \wedge Y) = \text{Ad}(a^{-1})L(P_2(a)X \wedge P_2(a)Y),$$

where $X, Y \in \mathfrak{b}_{-1}$, $L \in \mathfrak{g} \otimes \wedge^2 \mathfrak{b}_{-1}^*$ and $a \in B^{(0)}$.

Clearly $\mathfrak{b}^{(0)} \otimes \wedge^2 \mathfrak{b}_{-1}^*$ is an invariant subspace in this action.

Now, let $\mathcal{R}: (R, E, F)$ be a pseudo-projective system of degree n and of the second restricted type, and let $\Gamma: (P, \omega)$ be the associated pseudo-projective connection, being a Cartan connection of model coset space $G/G^{(0)}$ on R . $G^{(0)}$ being a subgroup of $B^{(0)}$, let P_2 denote the extension of the p.f.b. P to the group $B^{(0)}$. It is easy to see that the connection form ω of Γ

is naturally extended to a $\mathfrak{g}$ -valued 1-form, say $\bar{\omega}$, on P_2 so that the pair $(P_2, \bar{\omega})$ satisfies conditions (C.1) and (C.2) in the definition of a transverse Cartan connection.

By Theorem I.4.1 we know that the curvature K of Γ takes values in $\mathfrak{g}^{(0)} \otimes \wedge^2 \mathfrak{g}_{-1}^*$ and hence the structure equation for Γ takes the following form:

$$d\omega + \frac{1}{2} [\omega, \omega] = \frac{1}{2} K(\beta_{-1} \wedge \beta_{-1}),$$

where β_{-1} denotes the $\mathfrak{g}_{-1}$ -component of ω with respect to the decomposition:

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}^{(0)}. \quad \text{It follows that}$$

$$R_a^* K = K^a, \quad a \in \mathfrak{g}^{(0)}.$$

Therefore K is naturally extended to a

function, say $\bar{K}$, on P_2 taking values in $\mathfrak{g}^{(0)} \otimes \wedge^2 \mathfrak{g}_1^*$ so that

$$R_a^* \bar{K} = \bar{K}^a, \quad a \in B^{(0)}$$

Then we have the following

Proposition 3.1. The pair $(P_2, \bar{\omega})$ gives a transverse Cartan connection, say $\bar{\Gamma}_2$, of model coset space $G/B^{(0)}$ on the foliated manifold (P, E) , and its curvature (with respect to the decomposition $\mathcal{F} = \mathfrak{g}_1 + \mathfrak{g}^{(0)}$) coincides with the natural extension $\bar{K}_1$ of the curvature K of Γ .

Proof. To prove the first assertion we must show that the pair $(P_2, \bar{\omega})$ satisfies condition (C.3). Namely, let $x \in P_2$ and $X \in T_x(P_2)$. Then we must show

that $\bar{\omega}(X) \in \mathfrak{f}^{(0)}$, if and only if $\pi_{2*}(X) \in E$,
 where π_2 denotes the projection of P_2 onto P .

Clearly it suffices to deal with the
 case where $x \in P$. Then there are a
 vector $Y \in T_x(P)$ and an element $A \in \mathfrak{f}^{(0)}$
 such that $X = Y + A_x^*$. Since $\omega_{-1}^+(Y) \in \mathfrak{f}^{(0)}$,
 we have

$$\begin{aligned}\bar{\omega}(X) &= \omega(Y) + A \\ &\equiv \omega_{-}(Y) \pmod{\mathfrak{f}^{(0)}} \\ &\equiv \omega_{-2}(Y) + \omega_{-1}^-(Y) \pmod{\mathfrak{f}^{(0)}}.\end{aligned}$$

Therefore it follows that $\bar{\omega}(X) \equiv 0 \pmod{\mathfrak{f}^{(0)}}$,
 if and only if $\omega_{-2}(Y) + \omega_{-1}^-(Y) = 0$. On the
 other hand we have

$$\bar{\pi}_*(X) = \pi_*(Y) = p(x) \cdot \omega_{-}(Y)$$

$$\begin{aligned}\bar{\pi}_*(X) &= \pi_*(Y) = \rho(x) \cdot \omega_-(Y) \\ &= \rho(x) \cdot (\omega_{-2}(Y) + \omega_{-1}^-(Y) + \omega_{-1}^+(Y)),\end{aligned}$$

where π denotes the projection of P onto R , and ρ the natural homomorphism of P to the frame bundle $F(R, \mathfrak{g}_-)$.

Therefore it follows that $\bar{\pi}_*(X)$ is in E , if and only if $\omega_{-2}(Y) + \omega_{-1}^-(Y) = 0$, proving the first assertion.

Let us now prove the second assertion.

Let L be the curvature of Γ_2 . Then the structure equation for $\bar{\Gamma}$ takes the following form:

$$d\bar{\omega} + \frac{1}{2}[\bar{\omega}, \bar{\omega}] = \frac{1}{2}L(\bar{\beta}_{-1} \wedge \bar{\beta}_{-1}),$$

where $\bar{\beta}_{-1}$ denotes the $\mathfrak{t}_{-1}$ -component of $\bar{\omega}$ with respect to the decomposition:

$\mathcal{I} = \mathcal{I}_1 + \mathcal{I}^{(0)}$. It follows that

$$R_a^* L = L^a, \quad a \in B^{(0)}$$

Furthermore it is clear that the restriction of the equation above to P yields the structure equation for Γ , whence the restriction of L to P coincides with K . Therefore we obtain $L = \bar{K}$, proving the second assertion.

(2) Let us now consider the graded Lie algebra $\mathcal{A}: \mathcal{A} = \mathcal{A}_{-1} + \mathcal{A}_0 + \mathcal{A}_1$, as well as the coset space $G/A^{(0)}$ defined in §I, 2. We denote by P_1 the linear isotropy representation of the group $A^{(0)}$ on the space $\mathcal{A}_{-1}$:

$$P_1(a)X \equiv \text{Ad}(a)X \pmod{\mathcal{A}^{(0)}},$$

where $a \in A^{(0)}$ and $X \in \mathcal{A}_{-1}$. Furthermore the group $A^{(0)}$ acts on the space $\mathcal{G} \otimes \wedge^2 \mathcal{A}_{-1}^*$ by the following rule:

$$L^a(X \wedge Y) = \text{Ad}(a^{-1})L(P_1(a)X \wedge P_1(a)Y),$$

where $X, Y \in \mathcal{A}_{-1}$, $L \in \mathcal{G} \otimes \wedge^2 \mathcal{A}_{-1}^*$ and $a \in B^{(0)}$.

Now, let $R: (R, E, F)$ be a pseudo-projective system of degree n and of the first restricted type, and let $\Gamma: (P, \omega)$ be the

associated pseudo-projective connection.

$G^{(0)}$ being a subgroup of $A^{(0)}$, let P_1 denote the extension of the p.f.b. P to the group $A^{(0)}$. Then we see that the connection form ω of Γ is naturally extended to a $\mathfrak{g}$ -valued 1-form, say $\hat{\omega}$, on P_1 so that the pair $(P_1, \hat{\omega})$ satisfies conditions (C.1) and (C.2).

By Theorem I.4.1 we know that the curvature K of Γ takes values in $\alpha^{(0)} \otimes \wedge^2 \alpha_{-1}^*$, and hence the structure equation for Γ takes the following form:

$$d\omega + \frac{1}{2} [\omega, \omega] = \frac{1}{2} K(\alpha_{-1} \wedge \alpha_{-1}),$$

where α_{-1} denotes the α_{-1} -component of ω with respect to the decomposition:

$\mathfrak{g} = \alpha_{-1} + \alpha^{(0)}$. Therefore, as before K is

naturally extended to a function, say $\hat{K}$, on P , taking values in $\alpha^{(0)} \otimes \Lambda^2 \alpha^*$, so that

$$R_a^* \hat{K} = \hat{K}^a, \quad a \in A^{(0)}$$

Reasoning just in the same manner as in the proof of the preceding proposition, we can prove the following

Proposition 3.2. The pair $(P, \hat{\omega})$ gives a transverse Cartan connection, say Γ , of model coset space $G/A^{(0)}$ on the foliated manifold (P, F) , and its curvature (with respect to the decomposition $\mathfrak{g} = \alpha_{-1} + \alpha^{(0)}$) coincides with the natural extension $\hat{K}$ of the curvature K of Γ .

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Finally we add that both the transverse Cartan connections Γ_2 and Γ_1 can be constructed from the pseudo-projective system $\mathcal{R}$ by differentiations, because so is the Cartan connection Γ .

3.2. The Lie algebras $\text{aut}(\mathcal{R})_p$, $\text{aut}(\Sigma)_x$, $\text{aut}(\Sigma_2)_x$ and $\text{aut}(\Sigma_1)_x$.

(0) Let $\mathcal{R}: (R, E, F)$ be a pseudo-projective system of degree n , and let $\Gamma: (P, \omega)$ be the associated pseudo-projective connection on R . We denote by $\text{aut}(\mathcal{R})$ the sheaf of germs of local infinitesimal automorphisms of $\mathcal{R}$, which coincides with the sheaf of germs of local infinitesimal automorphisms of Γ .

Let $\Sigma: (P, \omega)$ be the Cartan system associated with the connection Γ . We take any point x of P and set $p = \pi(x)$, π being the projection P onto R . Then

we know from Lemma 3.3 just below that the projection π naturally induces a (Lie algebra) isomorphism of $\text{aut}(\Sigma)_x$ with $\text{aut}(\mathbb{R})_p$.

Lemma 3.3. Let U be a simply connected neighborhood of x . Then any germ of $\text{aut}(\Sigma)_x$ can be represented by a right invariant infinitesimal automorphism of Σ defined on the whole of $\pi^{-1}(U)$.

For the proof of this fact we only remark that if γ is a local infinitesimal automorphism of Σ , then $[\gamma, A^*] = 0$ for any $A \in \mathfrak{g}^{(0)}$, which can be easily obtained from the structure equation for Γ .

(1) Now, assume that R is of the second restricted type, and consider the transverse Cartan connection $\Gamma_2: (P_2, \bar{\omega})$ on the foliated manifold (P, E) as well as the Cartan system $\mathbb{Z}_2: (P_2, \bar{\omega})$. Then we easily see that any right invariant vector field X on $\pi^{-1}(U)$ is extended to a right invariant vector field, say $\bar{X}$, on $\pi_2^{-1}(U)$, π_2 being the projection of P_2 onto R .

Lemma 3.4. Let X and Y be any right invariant vector fields on $\pi^{-1}(U)$. Then the following hold:

$$(\bar{L}_{\bar{X}} \bar{\omega})(\bar{Y}_{y_a}) = \text{Ad}(a^{-1})(L_X \omega)(Y_g),$$

$$(\bar{L}_{\bar{X}} \bar{\omega})(A_{y_a}^*) = 0,$$

where $y \in P$, $a \in B^{(0)}$ and $A \in \mathfrak{g}^{(0)}$.

It follows from Lemma 2.4 that if X is a right invariant infinitesimal automorphism of Σ , then $\bar{X}$ is a right invariant infinitesimal automorphism of Σ_2 , and hence the assignment $X \rightarrow \bar{X}$ yields an (injective) homomorphism, say ι_2 of $\text{aut}(\Sigma)_X$ to $\text{aut}(\Sigma_2)_X$.

Then we have the following

Proposition 3.5. (1) ι_2 gives an isomorphism of $\text{aut}(\Sigma)_X$ with $\text{aut}(\Sigma_2)_X$.

(2) The injection, say ι_2 , of P to P_2 induces an isomorphism of $\Delta(\Sigma)_X$ with $\Delta(\Sigma_2)_X / E(\Sigma_2)_X$.

Corollary 3.6. If the Cartan system Σ is regular, so is the Cartan system Σ_2 .

For the proof of this fact we note that the system $\Delta(\Sigma_2)$ together with the system $E(\Sigma_2)$ is right invariant.

Proof of the proposition.

(1) By Proposition 2.1 we know that there are a neighborhood V of x and a non-degenerate Cartan system $\Sigma': (V', \omega')$ satisfying the following conditions:

1) V becomes a fibred manifold over V' with projection κ so that the vertical tangent bundle coincide with the system $E(\Sigma_2)$, restricted to V , and 2) $\kappa^* \omega'$ coincides with ω_2 , restricted to V .

Now, both Σ and Σ' are non-degenerate, and $\dim P = \dim V'$. Furthermore, $\omega = \iota_2^* \omega_2$ and $\kappa^* \omega' = \omega_2$. Therefore, if we set $\varphi = \kappa \circ \iota_2$, we see that $\varphi^* \omega' = \omega$, meaning that φ gives a local isomorphism

of Σ with Σ' defined on a neighborhood of x . Accordingly φ induces an isomorphism of $\text{aut}(\Sigma)_x$ with $\text{aut}(\Sigma')_{k(x)}$. Furthermore we know from the second assertion of Proposition 2.1 that there is a natural isomorphism of $\text{aut}(\Sigma_2)_x$ with $\text{aut}(\Sigma')_{k(x)}$. Since the homomorphism $\nu_2: \text{aut}(\Sigma)_x$ to $\text{aut}(\Sigma_2)_x$ is injective, we have thereby shown that ν_2 is an isomorphism, completing the proof of the first assertion.

(3) From the proof of Proposition 2.1 we know that κ induces an isomorphism of $\Delta(\Sigma_2)_x / E(\Sigma)_x$ with $\Delta(\Sigma')_x$. Furthermore it is clear that φ induces an isomorphism of

$A(\Sigma)_\alpha$ with $A(\Sigma')_\alpha$. Consequently the second assertion follows. $\square$

(3) Next, assume that R is of the first restricted type, and consider the transverse Cartan connection $\Gamma_1: (P_1, \hat{\omega})$ on the foliated manifold (R, F) as well as the Cartan system $\Sigma_1: (P_1, \hat{\omega})$. Then just as above we have a natural (injective) homomorphism, say τ_1 , of $\text{aut}(\Sigma)_x$ to $\text{aut}(\Sigma_1)_x$ as well as the following

Proposition 3.7. (1) τ_1 gives an isomorphism of $\text{aut}(\Sigma)_x$ with $\text{aut}(\Sigma_1)_x$.

(2) The injection, say ι_1 , of P to P_1 induces an isomorphism of $\Delta(\Sigma)_x$ with $\Delta(\Sigma_1)_x / E(\Sigma_1)_x$.

Corollary 3.8. If the Cartan system Σ is regular, so is the Cartan system Σ_1 .

2.3. An integration theorem for the system E or F associated with a pseudo-projective system $\mathcal{R}$ of $\mathcal{R}$ of the second or the first restricted type.

A pseudo-projective system $\mathcal{R}$ will be called regular, if the associated Cartan system $\Sigma: (P, \omega)$ is regular.

Theorem 3.8. Let $\mathcal{R}(R, E, F)$ be a pseudo-projective system, and let p_0 be any point of R .

(1) Assume that R is regular and of the second restricted type. Then the integration of the system E at the point p_0 can be carried out by elementary operations, quadratures and the integrations of the Lie systems associated with the simple components of the Lie algebra $\text{aut}(R)_{p_0}$.

(2) Assume that R is regular and of the first restricted type. Then the integration of the system F at the point p_0 can be carried out by the same operations as above.

Above all we notice that the first assertion settles the integration problem for systems of ordinary differentiations of the second order and of the second restricted type:

$$(X) \quad \frac{d^2 y^j}{dx^2} = f^j(x, (y^k), (\frac{dy^k}{dx})).$$

Proof of Theorem 3.8. To prove the first assertion let us consider the following objects:

$$\Gamma: (P, \omega), \Sigma: (P, \omega), \Gamma_2: (P_2, \bar{\omega}), \Sigma_2: (P_2, \bar{\omega}).$$

As we have remarked, the transverse Cartan connection Γ_2 together with the Cartan connection Γ can be constructed from the system R by differentiations.

Now, we take a point x_0 of P with $\pi(x_0) = p_0$. Then we have the following:

1°. Since Γ_2 is a transverse Cartan connection on the foliated manifold (R, E) , we know from Proposition 1.2 that the integration problem for the system E at p_0 can be reduced to the

integration problem for the system $E(\Sigma_2)$ at x_0 .

2°. By Proposition 3.5 we know that

$$\text{aut}(R)_p \cong \text{aut}(\Sigma)_{x_0} \cong \text{aut}(\Sigma_2)_{x_0}.$$

3°. Since the system R or the Cartan system Σ is regular by the assumption, we know from Corollary 3.6 that the Cartan system Σ_2 as well is regular.

Therefore the first assertion follows immediately from Theorem 2.5. Here, we mention that the second assertion can be dealt with just in the same manner as above, thus completing the proof of the theorem.

