Face number inequalities for matroid complexes and Cohen-Macaulay types of Stanley-Reisner rings of distributive lattices

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Face number inequalities for matroid complexes and Cohen-Macaulay types of Stanley-Reisner rings of distributive lattices

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Abstract. We discuss two topics related with combinatorial study of canonical modules of Stanley-Reisner rings, viz., (i) some linear inequalities on the number of faces of a matroid complex and (ii) a formula to compute the Cohen-Macaulay type of the Stanley-Reisner ring of a finite distributive lattice.

Introduction

We study the following two problems in the field of commutative algebra and combinatorics :

(i) What can be said about the number of faces of a matroid complex ?

(ii) How can we calculate the Cohen-Macaulay type of the Stanley-Reisner ring of the order complex of a finite distributive lattice ?

Recently, some topics on Hilbert functions of noetherian graded algebras have been studied by several authors, e.g., [Sta3], [Sta7], [G-M-R], [R-R] and [H5] from viewpoints of commutative algebra, algebraic geometry and combinatorics. In the first half of the present paper, we are concerned with Hilbert functions of Stanley-Reisner rings of matroid complexes. Via well-known facts [H-K], [Sta3] on canonical modules of Cohen-Macaulay graded integral domains, Stanley [Sta7] found certain linear inequalities for the Hilbert function of a Cohen-Macaulay graded integral domain. Based on an idea of J. Herzog (cf. Corollary (1.5)), we see that the same linear inequalities as in [Sta7] hold for the Hilbert function of the Stanley-Reisner ring of a matroid complex (cf. Theorem (1.8)).

On the other hand, it would be of interest to find a combinatorial formula to compute the Cohen-Macaulay type (i.e., the minimal number of generators of the canonical module) of the Stanley-Reisner ring of a Cohen-Macaulay complex, e.g., [H7]. In the latter half of this paper, we find a formula for the computation of the Cohen-Macaulay type of the Stanley-Reisner ring of the order complex of a finite distributive lattice. In fact, our main result (cf. Theorem (2.10)) guarantees that the Cohen-Macaulay type of the Stanley-Reisner ring of the order complex of a finite distributive lattice is equal to the number of distinct equivalence classes of a certain equivalence relation (cf. (2.8)) on the set of linear extensions of a finite partially ordered set associated with the distributive lattice.

§1. Level rings and matroid complexes

(1.1) Let k be a field and A a *semi-standard* k-algebra, that is, A is a commutative graded ring $\oplus_{n\geq 0} A_n$ satisfying (i) $A_0 = k$, (ii) A is finitely generated as a k-algebra, and (iii) A is integral over the subalgebra $k[A_1]$ of A generated by A_1. The *Hilbert function* of A is defined to be

$$H(A,n) := \dim_k A_n \qquad \text{for } n = 0, 1, \ldots,$$

while the *Hilbert series* of A is given by

$$F(A,\lambda) := \sum_{n=0}^{\infty} H(A,n)\, \lambda^n .$$

Since A is finitely generated as a $k[A_1]$-algebra and is integral over $k[A_1]$, it follows that A is finitely generated as a $k[A_1]$-module. Hence, well-known properties on Hilbert series, e.g., [Mat, pp.94-95] guarantee that

$$F(A,\lambda) = (h_0 + h_1\lambda + \ldots + h_s\lambda^s)/(1-\lambda)^d$$

for some integers $h_0, h_1, \ldots, h_s$ with $h_s \neq 0$. Here d is the Krull dimension of A. We say that the vector $h(A) := (h_0, h_1, \ldots, h_s)$ is the *h-vector* of A.

(1.2) Suppose that a semi-standard k-algebra $A = \oplus_{n\geq 0} A_n$ is Cohen-Macaulay. Let K_A be the canonical module [H-K] of A. It is known [H-K, Corollary (6.7)] that there exists a graded ideal I of A with $I \cong K_A$ (as graded modules over A, up to shift in grading) if and only if A is generically Gorenstein, i.e., the localization A_q is Gorenstein for every minimal prime ideal q of A. Also, see [H_3, Lemma (1.7)].

(1.3) PROPOSITION. Let a Cohen-Macaulay semi-standard k-algebra $A = \oplus_{n\geq 0} A_n$ be generically Gorenstein, and let $I = \oplus_{n\geq a} (I \cap A_n)$, $I \cap A_a \neq (0)$, be a graded ideal of of A with $I \cong K_A$. Suppose that there exists a non-zero divisor $\theta \in I \cap A_a$ on A. Then the h-vector $h(A) = (h_0, h_1, \ldots, h_s)$ of A satisfies the linear inequality

$$h_0 + h_1 + \ldots + h_i \leq h_s + h_{s-1} + \ldots + h_{s-i} \qquad (*)$$

for every $0 \leq i \leq s$.

Proof. Since $\theta \in I \cap A_a$ is a non-zero divisor on A, the dimension of $I/\theta A$ as an A-module is less than the Krull dimension of A if $\theta A \neq I$. Thus the proof of [Sta_7, Theorem (2.1)] is valid in our situation without modification. Q.E.D.

(1.4) We say that a Cohen-Macaulay semi-standard k-algebra $A = \oplus_{n\geq 0} A_n$ is *level* [Sta_2] if the canonical module $K_A = \oplus_{n\geq a} (K_A)_n$ with $(K_A)_a \neq (0)$, $a \in \mathbb{Z}$, of A is generated by $(K_A)_a$ as an A-module. In other words, A is level if and only if the Cohen-Macaulay type of A coincides with the last component of the h-vector of A. Consult, e.g., [H_2, pp.343-345].

(1.5) COROLLARY. Suppose that a Cohen-Macaulay semi-standard k-algebra $A = \oplus_{n\geq 0} A_n$ is both generically Gorenstein and level. Then the h-vector $h(A) = (h_0, h_1, \ldots, h_s)$ of A satisfies the linear inequality (*) for every $0 \leq i \leq s$.

Proof. A routine technique enables us to assume that k is an infinite field. Let $I = \oplus_{n\geq a} (I \cap A_n)$, $I \cap A_a \neq (0)$, be a graded ideal of A with $I \cong K_A$. Thanks to Proposition (1.3), what we must show is the existence of a non-zero divisor $\theta \in I \cap A_a$ on A. Let $\mathfrak{N}_A$ be the set of prime ideals of A which belong to the ideal (0). Since A is Cohen-Macaulay, we know that the Krull dimension of A/q equals that of A for each $q \in \mathfrak{N}_A$. We write $\mathfrak{U}$ for the (set-theoretic) union of all prime ideals $q \in \mathfrak{N}_A$.

Recall (e.g., [Mat, p.38]) that the set $\mathfrak{U}$ coincides with the set of zero-divisors on A. If $I \cap A_a \subset \mathfrak{U}$, then $I \cap A_a \subset q$ for some $q \in \mathfrak{N}_A$ since k is infinite (see, e.g., [Her, Problem 21, p.136]). Now, A is level, thus I is generated by $I \cap A_a$ as an A-module. Hence, if $I \cap A_a \subset q$ then $I \subset q$, thus the Krull dimension of A/I is equal to that of A, which contradicts [H-K, Corollary (6.13)].

Q.E.D.

The author is grateful to Professor Jürgen Herzog for suggesting the above proof. We remark that Corollary (1.5) is false if we drop the assumption that A is generically Gorenstein.

(1.6) Let V be a finite set, called the *vertex set*, and Δ a *simplicial complex* on V. Thus Δ is a collection of subsets of V such that (i) $\{x\} \in \Delta$ for every $x \in V$ and (ii) $\sigma \in \Delta$, $\tau \subset \sigma$ imply $\tau \in \Delta$. Each element of Δ is called a *face* of Δ. Set $d := \max\{\#(\sigma) ; \sigma \in \Delta\}$. Here $\#(\sigma)$ is the cardinality of σ as a set. Then the *dimension* of Δ is defined to be $\dim \Delta := d - 1$. We say that Δ is *pure* if every maximal face has the same cardinality. We write $f_i = f_i(\Delta)$, $0 \le i < d$, for the number of faces σ of Δ with $\#(\sigma) = i + 1$. Thus $f_0 = \#(V)$. We say that $f(\Delta) := (f_0, f_1, \ldots, f_{d-1})$ is the *f-vector* of Δ. Define the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of Δ by the formula

$$\sum_{i=0}^{d} f_{i-1} (\lambda - 1)^{d-i} = \sum_{i=0}^{d} h_i \lambda^{d-i}$$

with $f_{-1} = 1$. Consult, e.g., [Sta_4] and [Hoc] for further information.

(1.7) A simplicial complex Δ on the vertex set V is called a *matroid complex* (or G-complex [Sta_2]) if the following conditions are satisfied :

(i) If $\sigma, \tau \in \Delta$ and $\#(\sigma) < \#(\tau)$, then there exists $x \in \tau$ such that $x \notin \sigma$ and $\sigma \cup \{x\} \in \Delta$.

(ii) $\dim(\Delta - x) = \dim \Delta$ for every $x \in V$. Here $\Delta - x$ is the subcomplex $\{\sigma \in \Delta ; x \notin \sigma\}$ of Δ on $V - \{x\}$.

We remark that the above condition (ii) is required only to avoid the inessential case ; if $\dim(\Delta - x) < \dim \Delta$ then Δ is a cone over $\Delta - x$ with apex x, thus we should study $\Delta - x$ rather than Δ.

For example, let V be a finite set of non-zero vectors of a vector space over a field and suppose that the dimension of the subspace spanned by V is equal to the dimension of the subspace spanned by $V - \{x\}$ for every $x \in V$. Then the set Δ of linearly independent subsets of V is a matroid complex.

Now, what can be said about the h-vector of an arbitrary matroid complex ?

(1.8) THEOREM. Suppose that $h(\Delta) = (h_0, h_1, \ldots, h_d)$ is the h-vector of a matroid complex Δ of dimension $d - 1$. Then we have the linear inequality

$$h_0 + h_1 + \ldots + h_i \le h_d + h_{d-1} + \ldots + h_{d-i}$$

for every $0 \le i \le d$.

Proof. Let $V = \{X_1, X_2, \ldots, X_t\}$ be the vertex set of Δ and $k[\Delta] = k[X_1, X_2, \ldots, X_t]/I_\Delta$ the *Stanley-Reisner ring* ([Sta1], [Rei]) of Δ over a field k with the standard grading, i.e., each $\deg(X_i) = 1$. Then the Krull dimension of $k[\Delta]$ is d, and the Hilbert series of $k[\Delta]$ is just

$$F(k[\Delta], \lambda) = (h_0 + h_1\lambda + \ldots + h_d\lambda^d)/(1 - \lambda)^d,$$

see, e.g., [Sta4, pp.62-68]. It is known (and, in fact, not difficult to prove) that a matoroid complex is "doubly" Cohen-Macaulay in the sense of [Bac]. In other words, $k[\Delta]$ is a level ring with $h_d \neq 0$. See also [Sta2]. Moreover, $k[\Delta]$ is generically Gorenstein [Sta4, p.80]. Hence Corollary (1.5) enables us to obtain the required inequality. Q.E.D.

(1.9) CONJECTURE. Work in the same notation as in Theorem (1.8). Then we have the following linear inequalities :

(i) $h_i \le h_{d-i}$ for every $0 \le i \le [d/2]$, and

(ii) $h_0 \le h_1 \le \ldots \le h_{[d/2]}$.

Consult [H_4] for further information on the inequalities in the above Conjecture (1.9). We easily see the inequality $h_1 \le h_2$ when $d \ge 3$. Also, note that, thanks to [H_4], the above conjecture is weaker than that of [Sta_2, p.59].

On the other hand, a log-concavity conjecture on f-vectors of matroid complexes is presented by Mason [Mas]. Some partial results on this conjecture are obtained by Dowling [Dow] and by Mahoney [Mah].

It would, of course, be of great interest to find a combinatorial characterization of the h-vectors of matroid complexes.

The f-vectors (or h-vectors) of various classes of simplicial complexes have been studied by several combinatorialists. We refer the reader to, e.g., [B-K] for a survey of the topic.

§2. Cohen-Macaulay types of distributive lattices

(2.1) Given a finite partially ordered set (*poset* for short) P we write $\mathcal{J}(P)$ for the poset which consists of all *poset ideals* (or *order ideals* [Sta$_6$, p.100]) of P, ordered by inclusion. Then $\mathcal{J}(P)$ is a *distributive lattice* [Sta$_6$, p.105]. On the other hand, the fundamental theorem for finite distributive lattices, e.g., [Sta$_6$, Theorem (3.4.1)] guarantees that, for every finite distributive lattice L, there exists a unique poset P for which $L \cong \mathcal{J}(P)$.

(2.2) Let $\rho(P;\ell)$ be the number of *chains* [Sta$_6$, p.99]

$$\mathfrak{M} : \emptyset = I_0 \subsetneqq I_1 \subsetneqq \dots \subsetneqq I_{\ell+1} = P$$

of *length* $\ell + 1$ (cf. [Sta$_6$, p.99]) in the distributive lattice $\mathcal{J}(P)$ such that

(i) $I_{i+1} - I_i$ is a *clutter* [Sta$_6$, p.100] in P for each $0 \le i \le \ell$, and

(ii) for every $1 \le i \le \ell$, there exist $y \in I_{i+1} - I_i$ and $x \in I_i - I_{i-1}$ with $x < y$ in P.

Then $\rho(P;\ell) = 0$ if $\ell < \mathrm{rank}(P)$ and $\rho(P;\mathrm{rank}(P)) \neq 0$. Here $\mathrm{rank}(P)$ is the *rank* [Sta$_6$, p.99] of P.

(2.3) We now study the Stanley-Reisner ring $k[\Delta(L)] = k[X_\alpha;\alpha\in L]/I_{\Delta(L)}$, with each $\deg(X_\alpha) = 1$, of the *order complex* $\Delta(L)$ (cf. [Sta$_6$, p.120]) of a finite distributive lattice L over a field k. It is well known, e.g., [B-G-S] that $k[\Delta(L)]$ is Cohen-Macaulay. We are interested in the Cohen-Macaulay type $\mathrm{type}(k[\Delta(L)])$ of $k[\Delta(L)]$, i.e., the minimal number of generators of the canonical module $K_{k[\Delta(L)]}$ of $k[\Delta(L)]$ as a $k[\Delta(L)]$-module. We refer the reader to, e.g., [B-G-S] and [Sta$_6$, Chap.4, Sect.5] for the information on the h-vector of the order complex of a finite distributive lattice. Also, consult [H$_1$], [H$_3$] and [H$_6$] for some topics on commutative algebra related with distributive lattices.

(2.4) PROPOSITION. The Cohen-Macaulay type $\mathrm{type}(k[\Delta(L)])$ of the Stanley-Reisner ring $k[\Delta(L)]$ of the order complex $\Delta(L)$ of a finite distributive lattice $L = \mathfrak{J}(P)$ is

$$\mathrm{type}(k[\Delta(L)]) = \rho(P;\mathrm{rank}(P)) + \rho(P;\mathrm{rank}(P)+1) + \ldots . \qquad (**)$$

Proof. Suppose that $\#(P) = n$, say $P = \{p_1,p_2,\ldots,p_n\}$, and we write $e(I) = (e_1,e_2,\ldots,e_n) \in \mathbb{R}^n$ for the incident vector of a poset ideal I of P, i.e., $e_i = 1$ if $p_i \in I$ and $e_i = 0$ otherwise. Thus in particular $e(\emptyset) = (0,0,\ldots,0)$ and $e(P) = (1,1,\ldots,1)$. If $\mathfrak{M}$ is a chain in L of the form (☆) $(\emptyset \subset)\ I_0 \subsetneqq I_1 \subsetneqq \ldots \subsetneqq I_\ell\ (\subset P)$ with each $I_i \in \mathfrak{J}(P)$, then we write $[\mathfrak{M}]$ for the convex hull of $\{e(I_0),e(I_1),\ldots,e(I_\ell)\}$ in $\mathbb{R}^n$. Thus $[\mathfrak{M}]$ is an ℓ-simplex in $\mathbb{R}^n$. Let $\mathfrak{C} = \mathfrak{C}(L)$ be the set of chains in $L = \mathfrak{J}(P)$ and $\mathcal{P} = \mathcal{P}(L)$ the convex hull of $\{ e(I) ; I \in \mathfrak{J}(P) \}$ in $\mathbb{R}^n$. Hence $\mathcal{P} \subset \mathbb{R}^n$ is a convex polytope of dimension n. We identify $\{ [\mathfrak{M}] ; \mathfrak{M} \in \mathfrak{C} \}$ with the order complex $\Delta(L)$ of L. It is known, e.g., [Sta5, p.17] that $\{ [\mathfrak{M}] ; \mathfrak{M} \in \mathfrak{C} \}$ is a triangulation of $\mathcal{P}$, hence $\mathcal{P}$ is a geometric realization of $\Delta(L)$.

Now, let $\mathfrak{I}$ be the ideal of the Stanley-Reisner ring $k[\Delta(L)] = k[X_\alpha;\alpha\in L]/I_{\Delta(L)}$ generated by those square-free monomials $\prod_{\alpha\in\mathfrak{M}} X_\alpha$ with $[\mathfrak{M}] \in \Delta(L) - \partial\Delta(L)$. Here $\partial\Delta(L)$ is the boundary of $\Delta(L)$. Then, by virtue of [Sta4, Theorem (7.3), p.81], $\mathfrak{I}$ is isomorphic to the canonical module $K_{k[\Delta(L)]}$ of $k[\Delta(L)]$. On the other hand, thanks to [Sta5, p.10], if $[\mathfrak{M}] \in \mathfrak{C}$ is of the form (☆), then $[\mathfrak{M}] \in \Delta(L) - \partial\Delta(L)$ if and only if the following conditions are satisfied : (i) $I_0 = \emptyset$, (ii) $I_\ell = P$, and (iii) each $I_{i+1} - I_i$ is a clutter. Hence, it follows immediately that the minimal number of generators of $\mathfrak{I}$ as a $k[\Delta(L)]$-module is just (**) as required.

Q.E.D.

We should remark that the ideal $\mathfrak{I}$ in the above proof is generated by $\{ [\mathfrak{M}] ;\ [\mathfrak{M}] \in \Delta(L) - \partial\Delta(L),\ \#(\mathfrak{M}) = \mathrm{rank}(P) + 2 \}$ as a $k[\Delta(L)]$-module if and only if $\rho(P;\ell) = 0$ for every $\ell \neq \mathrm{rank}(P)$. In other words, (2.5) **COROLLARY.** The Stanley-Reisner ring $k[\Delta(L)]$ of the order complex $\Delta(L)$ of a finite distributive lattice $L = \mathcal{J}(P)$ is level if and only if $\rho(P;\ell) = 0$ for every $\ell \neq \mathrm{rank}(P)$.

(2.6) Let $\mathbb{N}$ be the set of non-negative integers and P a finite poset. We say that a map $\sigma : P \to \mathbb{N}$ is *strictly order-preserving* if $x < y$ in P implies $\sigma(x) < \sigma(y)$ in $\mathbb{N}$. We write $\mathcal{B}(P;\ell)$ for the set of strictly order-preserving maps $\sigma : P \to \mathbb{N}$ such that (i) $\sigma(P) = \{0,1,\ldots,\ell\}$ and (ii) $\sigma^{-1}(\{i-1,i\})$ is *not* a clutter in P for every $1 \le i \le \ell$.

(2.7) **LEMMA.** $\rho(P;\ell) = \#(\mathcal{B}(P;\ell))$.

Proof. Given a chain $\mathcal{M} : \emptyset = I_0 \subsetneqq I_1 \subsetneqq \ldots \subsetneqq I_{\ell+1} = P$ in the distributive lattice $\mathcal{J}(P)$ which satisfies the conditions (i) and (ii) in (2.2), we can define a map $\sigma : P \to \mathbb{N}$ in $\mathcal{B}(P;\ell)$ by $\sigma(x) = i$ if $x \in I_{i+1} - I_i$. On the other hand, if $\sigma \in \mathcal{B}(P;\ell)$, then $\emptyset \subsetneqq \sigma^{-1}(\{0\}) \subsetneqq \sigma^{-1}(\{0,1\}) \subsetneqq \ldots \subsetneqq \sigma^{-1}(\{0,1,\ldots,\ell-1\}) \subsetneqq P$ is a chain in $\mathcal{J}(P)$ with the properties (i) and (ii) in (2.2). Q.E.D.

(2.8) We recall that a *linear extension* [Sta_6, p.110] of a finite poset P is a strictly order-preserving map $\sigma : P \to \mathbb{N}$ such that $\sigma(P) = \{1,2,\ldots,\#(P)\}$. If σ is a linear extension of P , then there exists a unique sequence $\mathcal{D}(\sigma) = (d_1,d_2,\ldots,d_\ell) \in \mathbb{Z}^\ell$, $0 \le \ell = \ell(\sigma) \in \mathbb{Z}$, with $1 \le d_1 < d_2 < \ldots < d_\ell < \#(P)$ such that

(i) $\sigma^{-1}(\{d_i+1,d_i+2,\ldots,d_{i+1}\})$ is a clutter in P for each $0 \le i \le \ell$, where we set $d_0 = 0$ and $d_{\ell+1} = \#(P)$, and

(ii) for every $1 \le i \le \ell$, there exists $x \in \sigma^{-1}(\{d_{i-1}+1,\ldots,d_i\})$ with $x < \sigma^{-1}(d_i+1)$ in P .

We say that two linear extensions σ and τ of P are *equivalent* (written as $\sigma \sim \tau$) if $\mathcal{D}(\sigma) = \mathcal{D}(\tau)$ ($= (d_1,d_2,\ldots,d_\ell)$) and $\sigma^{-1}(\{1,2,\ldots,d_{i+1}\}) = \tau^{-1}(\{1,2,\ldots,d_{i+1}\})$ for every $0 \le i \le \ell$.

(2.9) Given a linear extension σ of a finite poset P with $\mathcal{D}(\sigma) = (d_1, d_2, \ldots, d_\ell)$, we write $I_i(\sigma)$ for the poset ideal $\sigma^{-1}(\{1,2,\ldots,d_i\})$ of P for each $1 \le i \le \ell + 1$, where $d_{\ell+1} = \#(P)$. Also, set $I_0(\sigma) = \emptyset$. Then the chain

$$\mathcal{M}(\sigma) : \emptyset = I_0(\sigma) \subsetneqq I_1(\sigma) \subsetneqq \ldots \subsetneqq I_{\ell+1}(\sigma) = P$$

in the distributive lattice $\mathcal{J}(P)$ possesses the properties (i) and (ii) in (2.2).

On the other hand, for each chain $\mathcal{M}$ in (2.2), there exists a linear extension σ of P with $\mathcal{M} = \mathcal{M}(\sigma)$. Moreover, $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ if and only if σ and τ are equivalent.

We now come to the main result of this section in consequence of Proposition (2.4) with Lemma (2.7) and (2.9).

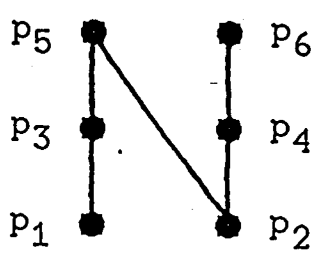
(2.10) THEOREM. The following quantities on a finite poset P are equal :

(a) the Cohen-Macaulay type $\mathrm{type}(k[\Delta(L)])$ of the Stanley-Reisner ring $k[\Delta(L)]$ of the order complex $\Delta(L)$ of the finite distributive lattice $L = \mathcal{J}(P)$,

(b) the number of strictly order preserving maps $\sigma : P \to \mathbb{N}$ such that $\sigma^{-1}(\{i-1,i\})$ is not a clutter in P for every $i \in \sigma(P)$ with $i \ge 1$,

(c) the number of distinct equivalence classes of the equivalence relation " $\sim$ " in (2.8) on the set of linear extensions of the poset P.

(2.11) EXAMPLE. Let $P = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ be the following finite poset



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and we employ the notation, e.g., 214635 for denoting the linear extension σ of P with $\sigma(p_2) = 1$, $\sigma(p_1) = 2$, $\sigma(p_4) = 3$, $\sigma(p_6) = 4$, $\sigma(p_3) = 5$ and $\sigma(p_5) = 6$. Then the equivalence classes of the equivalence relation " $\sim$ " in (2.8) on the set of linear extensions of the poset P are

{123456, 123465, 124356, 124365,
213456, 213465, 214356, 214365},
{123546, 213546},
{124635, 214635},
{132546, 132456},
{132465},
{241635, 241365},
{246135}, and
{241356}.

Hence the Cohen-Macaulay type type(k[Δ(L)]) of the Stanley-Reisner ring k[Δ(L)] of the order complex Δ(L) of the distributive lattice L = $\mathfrak{J}$(P) is equal to eight. Note that the h-vector of k[Δ(L)] is h(k[Δ(L)]) = (1,8,9,1).

It might be of interest to find a "nice" formula to compute the number of distinct equivalence classes of the equivalence relation " $\sim$ " in (2.8) on the set of linear extensions of P when P is, e.g., a *rooted tree* [Sta_6, p.294]).

We here turn to the problem of finding a chain condition of P for the Stanley-Reisner ring k[Δ(L)] to be level.

(2.12) The *altitude* of a finite poset P, written as alt(P), is defined to be the maximal number $\ell \geq 0$ for which there exists a finite sequence $C_0, C_1, \ldots, C_r$ of chains in P such that

(i) every $y \in C_j$ is neither less than nor equal to each $x \in C_i$ if $0 \leq i < j \leq r$, and

(ii) the sum of the cardinalities of C_i's is $\ell + r + 1$.

Obviously, we have rank(P) $\leq$ alt(P).

(2.13) LEMMA. $\rho(P;\mathrm{alt}(P)) \neq 0$.

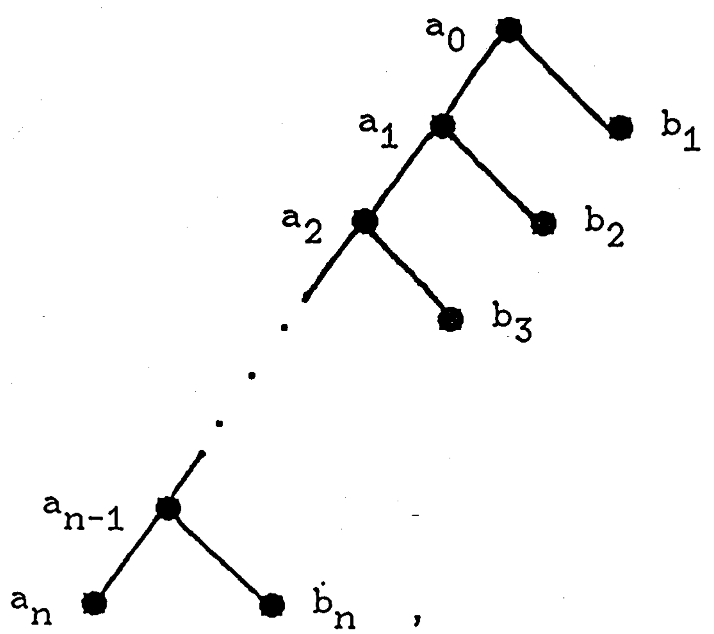
Proof. Work in the same notation as in (2.12) with $\ell = \mathrm{alt}(P)$. We write Q for the subposet $C_0 \cup C_1 \cup \ldots \cup C_r$ of P. Then we have $\mathrm{alt}(P) = \mathrm{alt}(Q)$. On the other hand, there exists a unique $\tau \in \mathcal{B}(Q;\mathrm{alt}(Q))$ such that $\tau(\alpha) \le \tau(\beta)$ if $\alpha \in C_i$ and $\beta \in C_j$ with $0 \le i < j \le r$. Let I_i, $0 \le i \le \mathrm{alt}(P)$, be the poset ideal of P which consists of those elements $x \in P$ such that $x < \alpha$ for some $\alpha \in Q$ with $\tau(\alpha) \le i$. In particular $I_0 = \emptyset$. Also, we set $I_{\mathrm{alt}(P)+1} = P$. Then, the chain $\emptyset = I_0 \subsetneqq I_1 \subsetneqq \ldots \subsetneqq I_{\mathrm{alt}(P)+1} = P$ in the distributive lattice $\mathfrak{J}(P)$ satisfies the conditions (i) and (ii) in (2.2). Thus $\rho(P;\mathrm{alt}(P)) \neq 0$ as desired.

Q.E.D.

Hence, we have $\rho(P;\ell) = 0$ if either $\ell < \mathrm{rank}(P)$ or $\ell > \mathrm{alt}(P)$ and $\rho(P;\mathrm{rank}(P)) \neq 0$, $\rho(P;\mathrm{alt}(P)) \neq 0$. Thus, thanks to Corollary (2.5), we immediately obtain

(2.14) COROLLARY. The Stanley-Reisner ring $k[\Delta(L)]$ of the order complex $\Delta(L)$ of a finite distributive lattice $L = \mathfrak{J}(P)$ is level if and only if $\mathrm{rank}(P) = \mathrm{alt}(P)$.

(2.15) EXAMPLE. If C_n is the following finite poset



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then the Stanley-Reisner ring $k[\Delta(L)]$ of the order complex $\Delta(L)$ of the finite distributive lattice $L = \mathfrak{J}(C_n)$ is level with the Cohen-Macaulay type $\mathrm{type}(k[\Delta(L)]) = n\,!$.

(2.16) Recall that the *height* (resp. *depth*) $\mathrm{height}_P(\alpha)$ (resp. $\mathrm{depth}_P(\alpha)$) of an element α of a finite poset P is the maximal number $\ell \geq 0$ for which there exists a chain in P of the form $\alpha_\ell < \alpha_{\ell-1} < \ldots < \alpha_0 = \alpha$ (resp. $\alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_\ell$). Thus we have $\mathrm{height}_P(\alpha) + \mathrm{depth}_P(\alpha) \leq \mathrm{rank}(P)$ for every element $\alpha \in P$. On the other hand, if α and β are incomparable elements of P, then $\mathrm{height}_P(\alpha) + \mathrm{depth}_P(\beta) \leq \mathrm{alt}(P)$. We write $P^{(+)}$ for the subposet of P which consists of all elements $\alpha \in P$ with $\mathrm{height}_P(\alpha) + \mathrm{depth}_P(\alpha) = \mathrm{rank}(P)$.

(2.17) COROLLARY. Suppose that the Stanley-Reisner ring $k[\Delta(L)]$ of the order complex $\Delta(L)$ of a finite distributive lattice $L = \mathfrak{J}(P)$ is level. If α and β are incomparable elements of the poset P, then we have the inequality $\mathrm{height}_P(\alpha) + \mathrm{depth}_P(\beta) \leq \mathrm{rank}(P)$. Thus, in particular, the subposet $P^{(+)}$ of P is the ordinal sum [Sta_6, p.100] of clutters.

(2.18) We say that a finite poset P satisfies the *λ-chain condition* [Sta_6, p.219] if $P = P^{(+)}$. It is known, e.g., [Sta_6, Corollary (4.5.17)] that a poset P satisfies the λ-chain condition if and only if the last non-zero component of the h-vector of the order complex $\Delta(L)$ of the distributive lattice $L = \mathfrak{J}(P)$ is equal to one.

(2.19) COROLLARY. The Stanley-Reisner ring $k[\Delta(L)]$ of the order complex $\Delta(L)$ of a finite distributive lattice $L = \mathfrak{J}(P)$ is Gorenstein, i.e., $\mathrm{type}(k[\Delta(L)]) = 1$, if and only if the poset P is the ordinaly sum of clutters.

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