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**SINGULAR DEGENERATE PARABOLIC
EQUATIONS WITH APPLICATIONS
TO GEOMETRIC EVOLUTIONS**

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SINGULAR DEGENERATE PARABOLIC EQUATIONS WITH APPLICATIONS TO GEOMETRIC EVOLUTIONS

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ABSTRACT. We prove a comparison theorem for viscosity solutions of degenerate parabolic equations which is singular at finite directions of derivatives. We apply our theorem to construct a global generalized evolution for interfaces equations with a certain class of the interface energy not necessarily C^2 .

1. Introduction. We are concerned with a degenerate parabolic equation of form

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in } Q = (0, T) \times \Omega, \quad (1.1)$$

where Ω is a bounded domain in R^n and $T > 0$. The function $F(p, X)$ is allowed to have singularities when p belongs to finitely many half lines ℓ_i of the form

$$\ell_i = \{\eta q_i ; \eta \geq 0\}, \quad q_i \in R^n \setminus \{0\}, \quad i = 1, \dots, m.$$

As explained later such an F naturally arises in a level set approach of motion of phase boundaries. Here $u_t = \partial u / \partial t$, ∇u and $\nabla^2 u$ denote, respectively, the time derivative of u , the gradient of u and the Hessian of u in space variables.

Our first goal is to establish a comparison principle for viscosity solutions of (1.1). If F has singularities only for $p = 0$, a comparison principle is established in [5] assuming that F can be extended continuously at $(p, X) = (0, O)$; See [11] for simplification of the proof. (The paper [6] includes corrections of technical errors in [5], [11]).

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Although we still appeal to Crandall-Ishii's lemma [7], the method in [11] or [5] does not apply to our setting because F has singularities other than $p = 0$. By a clever choice of "test function" we shall prove a comparison principle under assumptions on the value of semicontinuous envelope of F at $(\mu q_i, \nu q_i \otimes q_i)$, $\mu > 0$, $\nu \in \mathbb{R}$, where $\otimes$ denotes the tensor product.

Our second goal is to apply our comparison results to geometric evolutions. Let Γ_t denote the hypersurface expressed as the boundary of a bounded open set D_t in $\mathbb{R}^n$ ($n \geq 2$) at time t . Let n denote the unit exterior normal vector field on $\Gamma_t = \partial D_t$. Let $V = V(t, x)$ denote the speed of Γ_t at $x \in \Gamma_t$ in the exterior normal direction. The geometric evolution of Γ_t studied in [2], [3] is of the form

$$V = \frac{1}{\beta(n)} \left(- \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial H}{\partial p_i}(n) \right) + c \right), \quad (1.2)$$

where H is positively homogeneous of degree one, β is a positive function on a unit sphere S^{n-1} in $\mathbb{R}^n$ and c is a constant.

A level set approach is to regard Γ_t as the zero-level set of an auxiliary function $u : (0, T) \times \Omega \rightarrow \mathbb{R}$ of the evolution equation

$$\begin{aligned} u_t - \text{trace} \left(A \left(\frac{\nabla u}{|\nabla u|} \right) \left(I - \frac{\nabla u \otimes \nabla u}{|\nabla u|^2} \right) \nabla^2 u \left(I - \frac{\nabla u \otimes \nabla u}{|\nabla u|^2} \right) \right) + B(\nabla u) &= 0, \\ A(\bar{p}) &= \frac{1}{\beta(-\bar{p})} \left(\frac{\partial^2 H}{\partial p_i \partial p_j}(-\bar{p}) \right), \quad \bar{p} = \frac{p}{|p|}, \\ B(p) &= -\frac{1}{\beta(-\bar{p})} |p|. \end{aligned} \quad (1.3)$$

Here Ω is taken so that Γ_t stays in Ω for $t \in (0, T)$ and u is taken so that $u > 0$ on D_t and $u < 0$ outside $\Gamma_t \cup D_t$.

A fundamental analytic question related to (1.2) and (1.3) is to construct a global in time unique generalized solution $\{\Gamma_t\}_{t \geq 0}$ for a given initial data Γ_0 . Chen, Giga and Goto [5] have adapted the theory of viscosity solutions to construct unique global generalized solutions to the equation (1.3) when β is continuous and $H \in C^2(\mathbb{R}^n \setminus \{0\})$ is convex not necessarily strictly convex. Moreover, they proved the zero-level set Γ_t of u of (1.3) is determined by Γ_0 and independent of initial value of u . This yields a global unique

generalized evolution to (1.2). Nearly at the same time Evans and Spruck [9] carried out this programme in a slightly different way and only for the mean curvature flow equation.

For the history of level set approach as well as its recent development we refer to [1], [4] and references therein.

In physics there is also the possibility that H is not convex as studied in [2], [3]. If H is not convex, the equation (1.3) is no longer parabolic and not well-posed. It seems to be natural to consider the convexification $\hat{H}$ when H is not convex. The problem is that the convexification $\hat{H}$ of a function H may be no longer C^2 away from zero even if H is smooth. So the equation (1.3) may have singularities other than at $\nabla u = 0$. Our comparison theory does apply to (1.3) with $H = \hat{H}$ provided that $\hat{H}$ is singular at most finitely many directions and that the derivative of $\hat{H}$ is locally Lipschitz outside zero. Once the comparison principle is established for (1.3) with $H = \hat{H}$, we can adapt the theory in [5] of constructing global unique generalized solutions of (1.2) with $H = \hat{H}$.

Angenent and Gurtin [3] solved such an equation (1.2) with $H = \hat{H}$ for $n = 2$ at least locally if each normal of initial curve (with corners) lies in the direction that the curvature of $\hat{H} = 1$ is positive. Our theory applies to their setting. Moreover we allow that normal of initial curve lies in the direction that the curvature of $\hat{H} = 1$ is zero.

In Section 2 we shall establish a comparison principle on a bounded domain for the equation (1.1). We remark the case when Ω is an unbounded domain. In Section 3 we show that the theorem in Section 2 applies to the evolution equation (1.2) so that we get a unique global solution for a given initial data Γ_0 .

During this work is prepared we learned that a comparison theorem for nonsmooth interfacial energy is obtained by Giga [12] when the interface is a graph of a function on R . After this work was completed, we learned a recent work of Gurtin, Soner and Souganidis [13] closely related to ours. They also proved a comparison principle for (1.3) with $H = \hat{H}$, but the proof differs from ours. They also proved that generalized solution is consistent with solutions of Angenent and Gurtin [3].

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2. Comparison theorem. Let Ω be a bounded domain in R^n and let T be a positive number. We consider a degenerate parabolic equation of form

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in } Q = (0, T) \times \Omega. \quad (2.1)$$

For $i = 1, \dots, m$ let ℓ_i be a half line in R^n of the form

$$\ell_i = \{\eta q_i; \eta \geq 0\}, \quad \text{where } q_1, \dots, q_m \in R^n \setminus \{0\}.$$

We list assumptions on $F = F(p, X)$.

$$(F1) \quad F : (R^n \setminus \bigcup_{i=1}^m \ell_i) \times S^n \longrightarrow R \text{ is continuous,}$$

where S^n denotes the space of real $n \times n$ symmetric matrices.

$$(F2) \quad \begin{aligned} &F \text{ is degenerate elliptic, i.e.,} \\ &F(p, X + Y) \leq F(p, X) \quad \text{for all } Y \geq 0. \end{aligned}$$

$$(F3) \quad \begin{aligned} &-\infty < F_*(\mu q_i, \nu q_i \otimes q_i) = F^*(\mu q_i, -\nu q_i \otimes q_i) < +\infty \\ &\mu > 0, \nu > 0 \quad i = 1, \dots, m, \end{aligned}$$

$$(F4) \quad -\infty < F_*(0, O) = F^*(0, O) < +\infty,$$

where F_* and F^* are the lower and upper semicontinuous relaxation (envelope) of F on $R^n \times S^n$, respectively, i.e.,

$$F_*(p, X) = \liminf_{\varepsilon \downarrow 0} \{F(r, Y); r \in (R^n \setminus \bigcup_{i=1}^m \ell_i), |p - r| < \varepsilon, |X - Y| < \varepsilon\}$$

and $F^* = -(-F)_*$. Here $|X|$ denotes the operator norm of X as a self-adjoint operator on R^n ; $\otimes$ denotes a tensor product of vector in R^n .

The assumption (F1) allows the possibility that (2.1) is singular at $\nabla u = \eta q_i$ ($i = 1, \dots, m$). The equation (2.1) is called degenerate parabolic if (F2) holds.

We recall one of equivalent definitions of viscosity sub- and supersolutions of (2.1) (cf. [8]). A function $u : Q \rightarrow \mathbb{R}$ is called a *viscosity sub-(super)solution* of (2.1) in Q if $u^* < \infty$ (resp. $u_* > -\infty$) in $\overline{Q}$ and

$$\tau + F_*(p, X) \leq 0 \quad \text{for all } (\tau, p, X) \in \mathcal{P}_Q^{2,+} u^*(t, x), (t, x) \in Q$$

(resp. $\tau + F^*(p, X) \geq 0$ for all $(\tau, p, X) \in \mathcal{P}_Q^{2,-} u_*(t, x), (t, x) \in Q$). Here $\mathcal{P}_Q^{2,+}$ denotes the *parabolic super 2-jet* in Q , i.e., $\mathcal{P}_Q^{2,+} u(t, x)$ is the set of $(\tau, p, X) \in \mathbb{R} \times \mathbb{R}^n \times S^n$ such that

$$\begin{aligned} u(s, y) \leq u(t, x) + \tau(s - t) + \langle p, y - x \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle \\ + o(|s - t| + |y - x|^2) \quad \text{as } (s, y) \rightarrow (t, x) \text{ in } Q, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product; similarly, $\mathcal{P}_Q^{2,-} u = -\mathcal{P}_Q^{2,+}(-u)$. For $U = (0, T) \times D$, the set

$$\partial_p U = \{0\} \times D \cup [0, T] \times \partial D$$

is often called the parabolic boundary of U . We are now in position to state our main comparison theorem. We often suppress the word "viscosity", except in statements of theorems.

Theorem 2.1. *Suppose that Ω is bounded domain in $\mathbb{R}^n$ and that F satisfies (F1)-(F4). Let u and v be, respectively, viscosity sub- and supersolutions of (2.1) in $Q = (0, T) \times \Omega$. If $u^* \leq v_*$ on $\partial_p Q$, then $u^* \leq v_*$ in Q .*

We shall prove Theorem 2.1 in several steps.

The basic strategy of the proof of Theorem 2.1 is similar to the case when $F(p, X)$ has singularities only on $\{p = 0\}$ (cf. [11]). We argue by contradiction. Roughly speaking we shall find a parabolic super 2-jet of

$$w(t, x, y) = u(t, x) - v(t, y)$$

at a point (t, x, y) where $u^*(t, x) - v_*(t, y) > 0$ and x is close to y .

We should find a nice parabolic super 2-jet of w . For this purpose we introduce a test function $\Psi(t, x, y)$ and study the maximum of $w - \Psi$. When F has singularities only on $\{p = 0\}$, a suitable choice of Ψ is

$$\Psi(t, x, y) = \frac{|x - y|^4}{4\varepsilon} + \frac{\sigma}{T - t}$$

with small $\varepsilon, \sigma > 0$ (cf. [11]). This choice is not appropriate in our present situation because of singularities of F on half lines. We shall construct a suitable test function Ψ .

For vectors $\{q_i\}_{i=1}^m$ ($q_i \in \mathbb{R}^n \setminus \{0\}$) we take convex set M satisfying the following properties.

$$M \text{ is closed convex set in } \mathbb{R}^n \text{ and contains neighborhood of zero;} \quad (2.2a)$$

$$\text{the boundary } \partial M \text{ is } C^2; \quad (2.2b)$$

$$\text{if } n(x) = q_i/|q_i| \text{ at } x \in \partial M \text{ (} i = 1, \dots, m \text{), then } \langle \tau(x), \nabla \rangle n(x) = 0. \quad (2.2c)$$

Here n is a unit exterior normal C^1 vector field on ∂M and $\tau(x)$ is a unit tangent vector at $x \in \partial M$. We can easily construct a convex set M satisfying (2.2a)-(2.2c).

For this convex set M we define the Minkowski function

$$P_M(x) = \inf\{\alpha ; \alpha > 0, \alpha^{-1}x \in M\}.$$

We note that P_M has C^2 regularity outside of origin. From now on we shall suppress the subscript M . Let ε and σ be positive constants and we shall use

$$\Psi(t, x, y) = \frac{1}{4\varepsilon}(P(x - y))^4 + \frac{\sigma}{T - t}$$

as a test function. We note that $c_1|x| \leq P(x) \leq c_2|x|$ with $0 < c_1 \leq c_2$ by (2.2a). This implies that Ψ is C^2 even at $x = y$. Since Ψ depends on x and y through $x - y$ the following identities are trivially obtained.

Lemma 2.2. *Let P be as above. Then*

$$\nabla_x(P(x - y))^4 = -\nabla_y(P(x - y))^4, \quad (2.3)$$

$$\begin{aligned} \nabla_{xx}^2 \Psi(t, x, y) &= \nabla_{yy}^2 \Psi(t, x, y) \\ &= -\nabla_{xy}^2 \Psi(t, x, y) = -\nabla_{yx}^2 \Psi(t, x, y), \end{aligned} \quad (2.4)$$

where $\nabla_{xx}^2, \nabla_{yy}^2, \nabla_{xy}^2, \nabla_{yx}^2$ denote the Hessian operator in space variables $(x, x), (y, y), (x, y), (y, x)$, respectively.

We set

$$w(t, x, y) = u(t, x) - v(t, y)$$

$$\text{for } (t, x, y) \in \bar{U} \text{ with } U = (0, T) \times \Omega \times \Omega.$$

Proposition 2.3. Suppose that w is upper semicontinuous (u.s.c) in $\bar{U}$, $w < \infty$ in $\bar{U}$ and that

$$\alpha = \limsup_{\theta \downarrow 0} \{w(t, x, y); |x - y| < \theta, (t, x, y) \in \bar{U}, t < T\} > 0. \quad (2.5)$$

Set $\Phi(t, x, y) = w(t, x, y) - \Psi(t, x, y)$, then there is a positive constant σ_0 such that

$$\sup_{\bar{U}} \Phi(t, x, y) > \frac{\alpha}{2} \quad (2.6)$$

holds for all $0 < \sigma < \sigma_0$, $\varepsilon > 0$.

Proof. Since w is u.s.c and $\bar{U}$ is compact, we see $\alpha < \infty$. Moreover we easily see $\sup_{\bar{U}} w(t, x, x) = \alpha$. By (2.5) there is a point (t_0, x_0, x_0) ($t_0 < T$) such that $w(t_0, x_0, x_0) > 3\alpha/4$ and $\sigma/(T - t_0) < \alpha/4$ if σ is sufficiently small. We now observe that $\Phi(t_0, x_0, x_0) > \alpha/2$. $\square$

Let $(\hat{t}, \hat{x}, \hat{y}) \in \bar{U}$ be a maximum point of Φ , i.e.,

$$\sup_{\bar{U}} \Phi(t, x, y) = \Phi(\hat{t}, \hat{x}, \hat{y}).$$

Proposition 2.4. Let σ_0 be as in Proposition 2.3. Suppose that w is u.s.c in $\bar{U}$.

(i) $(P(\hat{x} - \hat{y}))^4$ tends to zero as $\varepsilon \rightarrow 0$; the convergence is uniform in $0 < \sigma < \sigma_0$.

(ii) $|\hat{x} - \hat{y}|$ tends to zero as $\varepsilon \rightarrow 0$; the convergence is uniform in $0 < \sigma < \sigma_0$.

Proof.

(i) From (2.6) it follows $\Phi(\hat{t}, \hat{x}, \hat{y}) > 0$ for $0 < \sigma < \sigma_0$, $\varepsilon > 0$. This yields

$$\begin{aligned} w(\hat{t}, \hat{x}, \hat{y}) &\geq \frac{1}{4\varepsilon}(P(\hat{x} - \hat{y}))^4 + \frac{\sigma}{T - \hat{t}} \\ &\geq \frac{1}{4\varepsilon}(P(\hat{x} - \hat{y}))^4. \end{aligned}$$

Since U is a bounded domain and since w is u.s.c, there is a positive constant M such that

$$u(t, x) - v(t, y) \leq M \quad \text{in } \overline{U}.$$

We now observe

$$\frac{1}{4\varepsilon}(P(\hat{x} - \hat{y}))^4 \leq M, \quad (2.7)$$

which yields (i) as $\varepsilon \rightarrow 0$.

(ii) Since $P(x)$ is comparable with $|x|$,

$$\text{if } (P(x - y))^4 \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \text{ then } |\hat{x} - \hat{y}| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad \square$$

Proposition 2.5. *Let σ_0 be as in Proposition 2.3 and σ be $0 < \sigma < \sigma_0$. Suppose that w is u.s.c in $\overline{U}$. Then*

$$\frac{1}{4\varepsilon}(P(\hat{x} - \hat{y})) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (2.8)$$

Proof. By (2.7) we observe that

$$\frac{1}{4\varepsilon}(P(\hat{x}(\varepsilon) - \hat{y}(\varepsilon)))^4 \rightarrow \xi \quad \text{as } \varepsilon \rightarrow 0 \quad (2.9)$$

for some non-negative number ξ if we take a subsequence. By Proposition 2.4 (ii) and boundedness of Ω

$$\hat{t}(\varepsilon) \rightarrow \bar{t}, \quad \hat{x}(\varepsilon), \quad \hat{y}(\varepsilon) \rightarrow \bar{z} \quad \text{as } \varepsilon \rightarrow 0 \quad (2.10)$$

for some $\bar{t} \in [0, T]$, $\bar{z} \in \overline{\Omega}$ if we take a subsequence $\varepsilon = \varepsilon_j \rightarrow 0$. By the definition of the point $(\hat{t}, \hat{x}, \hat{y})$ we have

$$\Phi(t, x, y) \leq \Phi(\hat{t}_j, \hat{x}_j, \hat{y}_j),$$

where $\hat{t}_j = \hat{t}(\varepsilon_j)$ and so on. Plunging $t = \bar{t}$, $x = y = \bar{z}$ in this inequality, we obtain

$$\begin{aligned} u^*(\bar{t}, \bar{z}) - v_*(\bar{t}, \bar{z}) - \frac{\sigma}{T - \bar{t}} \\ \leq u^*(\hat{t}_j, \hat{x}_j) - v_*(\hat{t}_j, \hat{y}_j) - \frac{1}{4\varepsilon}(P(\hat{x}_j - \hat{y}_j))^4 - \frac{\sigma}{T - \hat{t}_j}. \end{aligned} \quad (2.11)$$

From (2.9) letting $\varepsilon_j \rightarrow 0$ in (2.11) yields

$$\begin{aligned} u^*(\bar{t}, \bar{z}) - v_*(\bar{t}, \bar{z}) - \frac{\sigma}{T - \bar{t}} \\ \leq \overline{\lim}_{\varepsilon_j \rightarrow 0} \left(u^*(\hat{t}_j, \hat{z}_j) - v_*(\hat{t}_j, \hat{y}_j) - \frac{\sigma}{T - \hat{t}_j} \right) - \xi. \end{aligned}$$

Since $u^* - v_*$ is upper semicontinuous, from (2.10) it follows $\xi \leq 0$. Since the limit in (2.9) is independent of the choice of subsequence, the convergence (2.9) now yields (2.8). $\square$

Proposition 2.6. *Assume the hypotheses of Proposition 2.4. Suppose that $u^* \leq v_*$ on $\partial_p Q$. There is $\varepsilon_0 > 0$ such that Φ attains a maximum over $\bar{U}$ at an interior point $(\hat{t}, \hat{z}, \hat{y})$ of U , i.e., $(\hat{t}, \hat{z}, \hat{y}) \in (0, T) \times \Omega \times \Omega$ for all $0 < \varepsilon < \varepsilon_0$ and $0 < \sigma < \sigma_0$.*

Proof. Suppose that the conclusion were false. By the properties of barrier function $\sigma/(T - t)$ we see $\hat{t} < T$. There would exist sequence $\{\varepsilon_j\}$ with $\varepsilon_j \rightarrow 0$, $\{\sigma_j\} \subset (0, \sigma_0)$ such that $\partial_p U$ contains a maximum point $(\hat{t}_j, \hat{z}_j, \hat{y}_j)$ of Φ for the value $\varepsilon = \varepsilon_j, \sigma = \sigma_j$. By (2.6) we see

$$\frac{\alpha}{2} \leq \Phi(\hat{t}_j, \hat{z}_j, \hat{y}_j) \leq w(\hat{t}_j, \hat{z}_j, \hat{y}_j).$$

By the definition of sub- (super- resp.) solution and the boundedness of Ω , replacing u (v resp.) by $\{\max(u(t, x), -L)\}^*$, $(\{\min(v(t, x), L)\}_* \text{ resp.})$ for sufficiently large L we may assume that u (v resp.) is bounded u.s.c (l.s.c resp.) on $\bar{Q}$. Since U is bounded, the assumption $u^* \leq v_*$ on $\partial_p Q$ implies that there is a modulus function m (i.e., $m : [0, \infty) \rightarrow [0, \infty)$ is continuous, nondecreasing and $m(0) = 0$) such that $u(t, x) - v(t, y) \leq m(|x - y|)$ on $\partial_p U$. We have

$$w(\hat{t}_j, \hat{z}_j, \hat{y}_j) \leq m(|\hat{z}_j - \hat{y}_j|).$$

Since $\varepsilon_j \rightarrow 0$, applying Proposition 2.4 (ii) yields $|\hat{z}_j - \hat{y}_j| \rightarrow 0$, which leads a contradiction $0 < \alpha/2 \leq 0$. $\square$

The following is a variant of Crandall-Ishii's lemma [7].

Lemma 2.7. *Let u_i be a viscosity solution of*

$$u_t + F_i(\nabla u, \nabla^2 u) = 0 \tag{2.12}$$

in a neighborhood of $(s, z_i) \in (0, T) \times \mathbb{R}^{N_i}$ for $i = 1, 2, \dots, k$, where $F_i : \mathbb{R}^{N_i} \times S^{N_i} \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous. Let w be a function in $(0, T) \times \mathbb{R}^N$ given by

$$w(s, z) = \sum_{i=1}^k u_i(s, z_i) \quad \text{for } z = (z_1, \dots, z_k) \in \mathbb{R}^N,$$

where $N = N_1 + \dots + N_k$. Let

$$(\tau, p, A) \in \mathcal{P}^{2,+} w(s, z),$$

where $p = (p_1, \dots, p_k)$, $z = (z_1, \dots, z_k)$. Then for each $\lambda > 0$ there exists $X_i \in S^{N_i}$ such that

$$\tau + \sum_{i=1}^k F_i(p_i, X_i) \leq 0$$

and

$$-\left(\frac{1}{\lambda} + |A|\right)I \leq \begin{pmatrix} X_1 & & O \\ & \ddots & \\ O & & X_k \end{pmatrix} \leq A + \lambda A^2,$$

where I denotes the identity matrix.

Remark 2.8. This lemma is Lemma 2.10 in [11]. Here and hereafter the subscript of $\mathcal{P}^{2,+}$ is suppressed. The bar over $\mathcal{P}^{2,+}$ means the closure. Although the domain considered here is $\mathbb{R}^{N_i}$, it is easily seen that the result is local and that $\mathbb{R}^{N_i}$ may be replaced by a neighborhood of $z_i \in \mathbb{R}^{N_i}$.

Proof of Theorem 2.1. We may assume that u and v are, respectively, upper and lower semicontinuous so that

$$w(t, x, y) = u(t, x) - v(t, y)$$

is upper semicontinuous in $\overline{U}$. We will deduce contradiction supposing that

$$\alpha = \limsup_{\varepsilon \downarrow 0} \{w(t, x, y); |x - y| < \theta, (t, x, y) \in \overline{U}, t < T\} > 0.$$

By (2.5) we see all conclusions Proposition 2.3 - 2.6 would hold for $\Phi = w - \Psi$. Proposition 2.6 says that Φ attains a maximum over $\overline{U}$ at $(\hat{t}, \hat{x}, \hat{y}) \in (0, T) \times \Omega \times \Omega$ for small ε, σ . In particular

$$w(t, x, y) \leq w(\hat{t}, \hat{x}, \hat{y}) + \Psi(t, x, y) - \Psi(\hat{t}, \hat{x}, \hat{y}) \quad \text{in } U.$$

Expanding Ψ at $(\hat{t}, \hat{x}, \hat{y})$ yields

$$\left(\hat{\Psi}_t, \begin{pmatrix} \hat{\Psi}_x \\ \hat{\Psi}_y \end{pmatrix}, A \right) \in \mathcal{P}^{2,+} w(\hat{t}, \hat{x}, \hat{y}) \quad (2.13)$$

with $A = \begin{pmatrix} \hat{\Psi}_{xx} & \hat{\Psi}_{xy} \\ \hat{\Psi}_{yx} & \hat{\Psi}_{yy} \end{pmatrix}$, where $\hat{\Psi}_t = \partial_t \Psi(\hat{t}, \hat{x}, \hat{y})$, $\hat{\Psi}_x = \nabla_x \Psi(\hat{t}, \hat{x}, \hat{y})$, $\hat{\Psi}_{xx} = \nabla_{xx}^2 \Psi(\hat{t}, \hat{x}, \hat{y})$ and so on.

We shall apply Lemma 2.7 with $k = 2$, $u_1 = u$, $u_2 = -v$, $s = \hat{t}$, $z = (\hat{x}, \hat{y})$. Since u and v are, respectively, sub- and supersolution of (2.1) and since $(\hat{t}, \hat{x}, \hat{y})$ is an interior point of U , by Remark 2.8 we now apply Lemma 2.7 and conclude that for each $\lambda > 0$ there are $X, Y \in S^n$ such that

$$\hat{\Psi}_t + F_*(\hat{\Psi}_x, X) - F^*(-\hat{\Psi}_y, -Y) \leq 0 \quad (2.14)$$

and

$$-\left(\frac{1}{\lambda} + |A|\right)I \leq \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq A + \lambda A^2 \quad (2.15)$$

where $\hat{\Psi}_t = \partial_t \Psi(\hat{t}, \hat{x}, \hat{y})$, $\hat{\Psi}_x = \nabla_x \Psi(\hat{t}, \hat{x}, \hat{y})$, etc. Calculating $\hat{\Psi}_x$, $\hat{\Psi}_y$, $\hat{\Psi}_{xx}$, $\hat{\Psi}_{xy}$, $\hat{\Psi}_{yy}$, $\hat{\Psi}_t$ by using Lemma 2.2, (2.13) becomes

$$\left(\frac{\sigma}{(T-\hat{t})^2}, \begin{pmatrix} \frac{1}{\varepsilon} \hat{P}^3 \hat{P}_x \\ -\frac{1}{\varepsilon} \hat{P}^3 \hat{P}_x \end{pmatrix}, \frac{1}{\varepsilon} \begin{pmatrix} J & -J \\ -J & J \end{pmatrix} \right) \in \mathcal{P}^{2,+} w(\hat{t}, \hat{x}, \hat{y}) \quad (2.16)$$

with $J = 3\hat{P}^2(\hat{P}_x \otimes \hat{P}_x) + \hat{P}^3 \hat{P}_{xx}$, where $\hat{P} = P(\hat{x} - \hat{y})$, $\hat{P}_x = (\nabla_x P)(\hat{x} - \hat{y})$, $\hat{P}_{xx} = (\nabla_{xx}^2 P)(\hat{x} - \hat{y})$. We shall study (2.14). By (2.16) we have $\hat{\Psi}_x = -\hat{\Psi}_y$. We observe

$$0 \geq \frac{\sigma}{(T-\hat{t})^2} + F_*(\hat{\Psi}_x, X) - F^*(\hat{\Psi}_x, -Y).$$

Moreover, we obtain

$$0 \geq \frac{\sigma}{T^2} + F_*(\hat{\Psi}_x, X) - F^*(\hat{\Psi}_x, -Y). \quad (2.17)$$

Since F is singular where $\hat{\Psi}_x = \eta q_i$ ($i = 1, \dots, m$), we divide the situation in several cases.

Case I. $\hat{\Psi}_x = \eta q_i$ ($i = 1, \dots, m$).

Case Ia. $\hat{P} > 0$ where $\hat{P} = P(\hat{x} - \hat{y})$.

Since P is positively homogeneous of degree one, we see, by (2.2c), P is linear on a neighborhood of $\{\eta q_i\}$ ($\eta \neq 0$). This implies that $\hat{P}_{zz} = 0$ near ηq_i . Then

$$A = \frac{3\hat{P}^2}{\varepsilon} \begin{pmatrix} \hat{P}_z \otimes \hat{P}_z & -\hat{P}_z \otimes \hat{P}_z \\ -\hat{P}_z \otimes \hat{P}_z & \hat{P}_z \otimes \hat{P}_z \end{pmatrix} \quad \text{and} \\ A^2 = \frac{18\hat{P}^4}{\varepsilon^2} \begin{pmatrix} (\hat{P}_z \otimes \hat{P}_z)^2 & -(\hat{P}_z \otimes \hat{P}_z)^2 \\ -(\hat{P}_z \otimes \hat{P}_z)^2 & (\hat{P}_z \otimes \hat{P}_z)^2 \end{pmatrix}.$$

Moreover, since $(\hat{P}_z \otimes \hat{P}_z)^2 = |\hat{P}_z|^2 (\hat{P}_z \otimes \hat{P}_z)$, we obtain

$$A^2 = \frac{18\hat{P}^4 |\hat{P}_z|^2}{\varepsilon^2} \begin{pmatrix} \hat{P}_z \otimes \hat{P}_z & -\hat{P}_z \otimes \hat{P}_z \\ -\hat{P}_z \otimes \hat{P}_z & \hat{P}_z \otimes \hat{P}_z \end{pmatrix}.$$

We take $\lambda = \varepsilon/18\hat{P}^2 |\hat{P}_z|^2$ in (2.15) to get

$$A + \lambda A^2 = \frac{4\hat{P}^2}{\varepsilon} \begin{pmatrix} \hat{P}_z \otimes \hat{P}_z & -\hat{P}_z \otimes \hat{P}_z \\ -\hat{P}_z \otimes \hat{P}_z & \hat{P}_z \otimes \hat{P}_z \end{pmatrix}$$

and

$$\begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq \frac{4\hat{P}^2}{\varepsilon} \begin{pmatrix} \hat{P}_z \otimes \hat{P}_z & -\hat{P}_z \otimes \hat{P}_z \\ -\hat{P}_z \otimes \hat{P}_z & \hat{P}_z \otimes \hat{P}_z \end{pmatrix},$$

which yields

$$X, Y \leq \frac{4\hat{P}^2}{\varepsilon} \hat{P}_z \otimes \hat{P}_z.$$

Since $\hat{P} > 0$ we see $\hat{\Psi}_z \neq 0$. From $\hat{\Psi}_z = \frac{\hat{P}^3}{\varepsilon} \hat{P}_z = \mu q_i$, we obtain

$$\hat{P}_z = \frac{\varepsilon \mu}{\hat{P}^3} q_i \quad \text{and} \quad \hat{P}_z \otimes \hat{P}_z = \frac{\varepsilon^2 \mu^2}{\hat{P}^6} q_i \otimes q_i,$$

which yields

$$X, Y \leq Z \quad \text{with} \quad Z = \frac{4\varepsilon \mu^2}{\hat{P}^4} q_i \otimes q_i.$$

We shall study (2.17). By (F2) we obtain

$$0 \geq \frac{\sigma}{T^2} + F_*(\mu q_i, Z) - F^*(\mu q_i, -Z).$$

By (F3) this yields $\sigma \leq 0$, which contradicts $\sigma > 0$.

Case Ib. $\hat{P} = 0$ (i.e., $\hat{z} = \hat{y}$).

In this case $\hat{\Psi}_x = 0$ and $A = O$. From (2.15) we have

$$\begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq \begin{pmatrix} O & O \\ O & O \end{pmatrix},$$

which yields $X \leq O, Y \leq O$. We prove similarly as Case Ia using (F4) instead of (F3).

Case II. $\hat{\Psi}_x \neq \eta q_i \quad (i = 1, \dots, m)$.

This can be treated by a standard argument. We give a proof for completeness.

From (2.16) we have

$$A = \frac{1}{\varepsilon} \begin{pmatrix} J & -J \\ -J & J \end{pmatrix}$$

with $J = 3\hat{P}^2(\hat{P}_x \otimes \hat{P}_x) + \hat{P}^3\hat{P}_{xx}$. We take $\lambda = \varepsilon$ in (2.15) to get

$$A + \lambda A^2 = \frac{1}{\varepsilon} \begin{pmatrix} K & -K \\ -K & K \end{pmatrix}$$

with $K = J + 2J^2$. Moreover, we obtain

$$\begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq \frac{1}{\varepsilon} \begin{pmatrix} K & -K \\ -K & K \end{pmatrix},$$

which yields $X + Y \leq O$. We shall study (2.17). By (F2) we obtain

$$0 \geq \frac{\sigma}{T^2} + F(\hat{\Psi}_x, X) - F(\hat{\Psi}_x, X).$$

This yields $\sigma \leq 0$, which contradicts $\sigma > 0$. We thus prove Theorem 2.1. $\square$

Remark 2.9. When Ω is an unbounded domain, we suppose additional assumption on F

For every $R > 0$,

$$(F5) \quad C_R = \sup\{|F(p, X)|; |p| \leq R, |X| \leq R, p \in \mathbb{R}^n \setminus \bigcup_{i=1}^m \ell_i\} < +\infty.$$

Since $c_1|x - y| \leq P(x - y) \leq c_2|x - y|$ for some positive constants c_1 and c_2 , we can prove the following theorem on an unbounded domain in the same way as in [11].

Theorem 2.10. Suppose that F satisfies (F1)-(F5). Let u and v be, respectively, viscosity sub- and supersolutions of (2.1) in $Q = (0, T) \times \Omega$. Assume that

$$(A1) \quad \begin{aligned} &u(t, x) \leq K(|x| + 1), \quad v(t, x) \geq -K(|x| + 1) \text{ for some } K > 0 \\ &\text{independent of } (t, x) \in Q; \end{aligned}$$

$$(A2) \quad \begin{aligned} &\text{there is a modulus } m_T \text{ such that} \\ &u^*(t, x) - v_*(t, y) \leq m_T(|x - y|) \text{ for all } (x, t, y) \in \partial_p U, \\ &\text{where } U = (0, T) \times \Omega \times \Omega; \end{aligned}$$

$$(A3) \quad \begin{aligned} &u^*(t, x) - v_*(t, y) \leq K(|x - y| + 1) \text{ on } \partial_p U \text{ for some } K > 0 \\ &\text{independent of } (t, x, y) \in \partial_p U. \end{aligned}$$

Then there is a modulus m such that

$$u^*(t, x) - v_*(t, y) \leq m(|x - y|) \quad \text{in } U.$$

In particular $u^* \leq v_*$ in Q .

Remark 2.11. When Ω is unbounded the choice of γ, δ in [11, Proposition 2.4] may actually depend on ε , although it is not explicitly mentioned. But the proof of [11, Theorem 2.1] works even if γ, δ depends on ε .

3. Application. Let D_t be a bounded open set in $R^n (n \geq 2)$ at time t . Let Γ_t denote the hypersurface of the boundary of D_t . Let n denote the unit exterior normal vector field on $\Gamma_t = \partial D_t$. We extend n to a vector field (still denote by n) on a tubular neighborhood of Γ_t so that n is constant in the normal direction. Let $V = V(t, x)$ denote the speed of Γ_t at $x \in \Gamma_t$ in the exterior normal direction. We are concerned with the evolution equation for Γ_t :

$$V = \frac{1}{\beta(n)} \left(- \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial H}{\partial p_i}(n) \right) + c \right), \quad (3.1)$$

where H is positively homogeneous of degree one, i.e.,

$$H(\lambda p) = \lambda H(p) \quad \text{for } \lambda > 0, p \in R^n \setminus \{0\},$$

β is a positive function on a unit sphere S^{n-1} in R^n and c is a constant.

This evolution equation is derived by Gurtin describing motion of phase-boundaries [2,3]. When H is convex, the equation (3.1) is a degenerate parabolic equation. However, if H is not convex, (3.1) is no longer parabolic and not well-posed. In this case we should consider the convexification of H so that (3.1) is parabolic. Angenent and Gurtin [3] solve such evolution equation for $n = 2$ at least locally if each normal of initial curve (with corners) lies in the direction that the curvature of $H = 1$ is positive.

We aim at constructing global generalized solution of (3.1) by a level set approach when H is the convexification. The problem is that the convexification H of a function h may be no longer C^2 even if h is smooth. The theory of [5] does not apply because their theorem needs C^2 regularity of H . As we discuss below, our comparison theorem in Section 2 does apply to $H \in C^1(R^n \setminus \{0\})$ provided that $H \in C^2(R^n \setminus \bigcup_{i=1}^m \ell_i)$ and ∇H is locally Lipschitz on $R^n \setminus \{0\}$, where ℓ_i ($i = 1, \dots, m$) is a half line. Although not all h give such regularity for H , our theory applies to the equation (3.1) studied in [3] for $n = 2$.

We shall show the comparison theorem in Section 2 applies to degenerate parabolic equations associated with (3.1) through a level set approach when H is convex C^2 outside $\bigcup_{i=1}^m \ell_i$ and ∇H is locally Lipschitz on $R^n \setminus \{0\}$. As a result we can apply the level set approach to (3.1) and construct a unique global generalized solution.

Suppose that the hypersurface Γ_t is expressed as a zero-level of an auxiliary function $u = u(t, x)$ and that $u(t, x) > 0$ if and only if $x \in D_t$. As in [10] (3.1) is equivalent to

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad (3.2)$$

on Γ_t with

$$\begin{aligned} F(p, X) &= -\text{trace}(A(\bar{p})R_{\bar{p}}X R_{\bar{p}}) + B(p), \\ A(\bar{p}) &= \frac{1}{\beta(-\bar{p})} \left(\frac{\partial^2 H}{\partial p_i \partial p_j}(-\bar{p}) \right), R_{\bar{p}} = I - \bar{p} \otimes \bar{p}, \\ \bar{p} &= \frac{p}{|p|}, B(p) = -\frac{c}{\beta(-\bar{p})}|p|. \end{aligned} \quad (3.3)$$

By the definition of A and B we have the following proposition.

Proposition 3.1.

(i) If $H \in C^2(R^n \setminus \bigcup_{i=1}^m \ell_i)$, then $A \in C(S^{n-1} \setminus \bigcup_{i=1}^m \bar{q}_i)$,

where $\ell_i = \{\eta \bar{q}_i \in R^n; \eta \geq 0\}$ and $|\bar{q}_i| = 1$.

(ii) If ∇H of $H \in C^1(R^n \setminus \{0\})$ is locally Lipschitz on $R^n \setminus \{0\}$, then A is bounded.

(iii) If $\beta > 0$ is continuous, so is B on R^n .

Lemma 3.2. Let $H \in C^2(R^n)$ be positively homogeneous of degree one. Then

$$pA(\bar{p}) = 0, \quad A(\bar{p})^t p = 0,$$

provided that H is C^2 near $-\bar{p} = -p/|p|$ and $p \neq 0$, where p is a row vector.

Proof. From the homogeneity of H we have

$$\sum_{i=1}^n \frac{\partial H}{\partial p_i}(\lambda p) = H(p).$$

Plugging $\lambda = 1$ and differentiating this identity in p_j , we obtain

$$\sum_{i=1}^n \frac{\partial^2 H}{\partial p_i \partial p_j}(p) p_i = 0 \quad j = 1, \dots, m.$$

which easily yields $pA(\bar{p}) = 0$ by replacing p by $-\bar{p}$. Since $A(\bar{p})$ is a symmetric matrix, we have

$${}^t(pA(\bar{p})) = {}^tA(\bar{p})^t p = A(\bar{p})^t p = 0. \quad \square$$

Corollary 3.3. Assume the hypotheses of Lemma 3.2. Then

$$A(\bar{p})p \otimes p = O, \quad p \otimes pA(\bar{p}) = O.$$

Proof. By Lemma 3.2, we have

$$A(\bar{p})p \otimes p = A(\bar{p})^t pp = O,$$

$$p \otimes pA(\bar{p}) = {}^t ppA(\bar{p}) = O. \quad \square$$

Since $\text{trace}(A(\bar{p})R_{\bar{p}}X R_{\bar{p}}) = \text{trace}(R_{\bar{p}}A(\bar{p})R_{\bar{p}}X)$, Corollary 3.3 yields

$$\text{trace}(A(\bar{p})R_{\bar{p}}X R_{\bar{p}}) = \text{trace}(A(\bar{p})X).$$

This yields

$$F(p, X) = -\text{trace}(A(\bar{p})X) + B(p). \quad (3.4)$$

Applying Corollary 3.3 to (3.4), we obtain the following lemma.

Lemma 3.4. *Let F be defined by (3.4). Suppose that $A \in C(S^{n-1} \setminus \bigcup_{i=1}^m \bar{q}_i)$ is bounded and that B is continuous on R^n . Then*

$$F(p, \sigma p \otimes p) = B(p),$$

where $p \in R^n \setminus \bigcup_{i=1}^m \ell_i$ and $\sigma \in R$.

Theorem 3.5. *Let F be defined by (3.4). Suppose that $A \in C(S^{n-1} \setminus \bigcup_{i=1}^m \bar{q}_i)$ is bounded and that B is continuous on R^n . Then*

$$F^*(\mu q_i, \nu q_i \otimes q_i) = B(\mu q_i), \quad (3.5)$$

$$F_*(\mu q_i, \nu q_i \otimes q_i) = B(\mu q_i) \quad (3.6)$$

$$\mu > 0, \nu \in R.$$

Proof. Without the lost of generality we may assume $\mu = 1$ by setting μq_i as a new vector $\tilde{q}_i$. Since B is continuous,

$$F^*(q_i, \nu q_i \otimes q_i) - B(q_i) = G^*(q_i, \nu q_i \otimes q_i)$$

with $G(p, X) = F(p, X) - B(p)$. We shall show $G^*(q_i, \nu q_i \otimes q_i) = 0$. By the definition of G^* we have

$$\begin{aligned} G^*(q_i, \nu q_i \otimes q_i) &= \limsup_{\varepsilon \downarrow 0} \{G(\xi, Y); |\xi - q_i| \leq \varepsilon, |Y - \nu q_i \otimes q_i| \leq \varepsilon, (\xi, Y) \in (R^n \setminus \bigcup_{i=1}^m \ell_i) \times S^n\}. \end{aligned}$$

From Lemma 3.4 and (3.4) it follows that

$$\begin{aligned} G(\xi, Y) &= G(\xi, \nu \xi \otimes \xi) + G(\xi, Y) - G(\xi, \nu \xi \otimes \xi) \\ &= -\text{trace}(A(\bar{\xi})(Y - \nu \xi \otimes \xi)) \\ &= -\text{trace}(A(\bar{\xi})(Y - \nu q_i \otimes q_i)) - \text{trace}(A(\bar{\xi})(\nu q_i \otimes q_i - \nu \xi \otimes \xi)). \end{aligned}$$

Since $|Y - \nu q_i \otimes q_i| \leq \varepsilon$ and $|\xi - q_i| \leq \varepsilon$, we obtain

$$\begin{aligned} |G(\xi, Y)| &\leq |-\text{trace}(A(\bar{\xi})(Y - \nu q_i \otimes q_i))| + |-\text{trace}(A(\bar{\xi})(\nu q_i \otimes q_i - \nu \xi \otimes \xi))| \\ &\leq \varepsilon \sup_{|\bar{\xi}|=1} |\text{trace} A(\bar{\xi})| + c|\nu|\varepsilon \sup_{|\bar{\xi}|=1} |\text{trace} A(\bar{\xi})| \rightarrow 0 \end{aligned}$$

as $\varepsilon \rightarrow 0$, which yields (3.5), where c is a constant depending only on n . The proof of (3.6) parallels (3.5). $\square$

Theorem 3.6. Assume that $H \in C^1(\mathbb{R}^n \setminus \{0\})$ is convex and positively homogeneous of degree one. Assume that $H \in C^2(\mathbb{R}^n \setminus \bigcup_{i=1}^m \ell_i)$ and that ∇H is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$. Assume that β is positive continuous. Let F be defined by (3.4). Then F satisfies (F3) and (F4) in Section 2.

Proof. Applying Proposition 3.1 and Theorem 3.5, we observe that F satisfies (F3). It remains to prove

$$F_*(0, O) = F^*(0, O) (= 0).$$

By the definition of F^*

$$\begin{aligned} F^*(0, O) \\ = \limsup_{\varepsilon \downarrow 0} \{F(\xi, Y) ; |\xi| \leq \varepsilon, |Y| \leq \varepsilon, (\xi, Y) \in (\mathbb{R}^n \setminus \bigcup_{i=1}^m \ell_i) \times S^n\} \end{aligned}$$

From (3.4) it follows that

$$F(\xi, Y) = -\text{trace}(A(\bar{\xi})Y) + B(\xi).$$

Since $|Y| \leq \varepsilon$ and $|\xi| \leq \varepsilon$, we obtain

$$\begin{aligned} |F(\xi, Y)| &\leq |-\text{trace}(A(\bar{\xi})Y)| + |B(\xi)| \\ &\leq \varepsilon \sup_{|\bar{\xi}|=1} |\text{trace} A(\bar{\xi})| + |B(\xi)| \end{aligned}$$

$\rightarrow 0$ as $\varepsilon \rightarrow 0$, which yields $F^*(0, O) = 0$. Similarly we prove $F_*(0, O) = 0$. Thus we show that F satisfies (F3) and (F4). $\square$

Remark 3.7. We shall show another short proof of Theorem 3.5 observing that F defined by (3.4) is geometric in the sense of [5], i.e.,

$$\begin{aligned} F(\lambda p, \lambda X + \sigma p \otimes p) &= \lambda F(p, X) \\ \text{for all } \lambda > 0, \sigma \in \mathbb{R}, p \in \mathbb{R}^n \setminus \bigcup_{i=1}^m \ell_i, X \in S^n. \end{aligned}$$

Indeed it is easy to check that F^* and F_* are geometric provided that F is geometric. Note that values of F^* and F_* are finite. Thus we observe that

$$F^*(\mu q_i, \nu q_i \otimes q_i) = F^*(\mu q_i, O) = B(\mu q_i),$$

which yields (3.5). The proof of (3.6) is the same.

We apply our theory to (3.1) to construct a global weak solution. The equation (3.2) is clearly geometric. We can see that $\theta(u)$ is viscosity sub(super)solution of (3.2) if u is sub(super)solution of (3.2), where θ is a continuous nondecreasing function (cf. [5, Theorem 5.6]). So we can construct viscosity solution of (3.2) with initial data similarly as in [5]. Similarly to [5] we define a weak solution $\{(\Gamma_t, D_t)\}_{t \geq 0}$ of (3.1) through a viscosity solution of (3.2) with initial data $u(0, x) = a(x)$.

Definition 3.8. D_0 denotes a bounded open set and $\Gamma_0(\subset \mathbb{R}^n \setminus D_0)$ denotes a compact set containing ∂D_0 . $\{(\Gamma_t, D_t)\}_{t \geq 0}$ denotes a family of compact sets and bounded open sets in $\mathbb{R}^n$. Suppose that for some $\alpha > 0$ there is a viscosity solution $u \in C_\alpha([0, T] \times \mathbb{R}^n)$ for (3.2) with initial data $u(0, x) = a(x)$ in $(0, \infty) \times \mathbb{R}^n$ such that zero-level surface of $u(t, \cdot)$ at time $t \geq 0$ equals Γ_t and that the set D_t where $u > 0$ is bounded open. If $(\Gamma_t, D_t)|_{t=0} = (\Gamma_0, D_0)$, we say $\{(\Gamma_t, D_t)\}_{t \geq 0}$ is a weak solution of (3.1) with initial data (Γ_0, D_0) . Here $T > 0$ is arbitrary and $v \in C_\alpha(A)$ means $v - \alpha$ is continuous and has compact support in A .

Similarly as in [10, §3], applying the comparison theorem 2.1 yields:

Theorem 3.9. Suppose that $\beta > 0$ is continuous and that $H \in C^1(\mathbb{R}^n \setminus \{0\})$ is convex and positively homogeneous of degree one. Suppose that $H \in C^2(\mathbb{R}^n \setminus \bigcup_{i=1}^m \ell_i)$ and that ∇H is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$. Let D_0 be a bounded open set in $\mathbb{R}^n$ and let $\Gamma_0(\subset \mathbb{R}^n \setminus D_0)$ be a compact set containing ∂D_0 . Then there is a unique global solution $\{(\Gamma_t, D_t)\}_{t \geq 0}$ of (3.1) with initial data (Γ_0, D_0) (cf. [5, Theorem 7.3], [10, Proposition 3.3]).

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