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**UNFOLDINGS OF CODIMENSION
ONE COMPLEX ANALYTIC
FOLLATION SINGULARITIES**

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UNFOLDINGS OF CODIMENSION ONE COMPLEX ANALYTIC FOLIATION SINGULARITIES

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Dedicated to Professor Heisuke Hironaka on his sixtieth birthday

ABSTRACT. We discuss the unfolding and determinacy theories for codimension one complex analytic foliation germs. The first order unfoldings of such a germ are classified and the versality theorem which asserts that an infinitesimally versal unfolding is versal is stated together with a proof and various applications. Some types of finite determinacy results are also explained.

The unfolding and determinacy theories play fundamental roles in the study of singularities of functions. A codimension one foliation is defined by an "integrable" 1-form and is considered as a natural generalization of the notion of a function as a function defines a codim 1 foliation whose leaves are the level sets and whose singular set is its critical set. In this article, we explain the unfolding and determinacy theories for codim 1 complex analytic local foliations (foliation germs) summarizing mainly the contents of [27]-[34] and supplementing them with a little more material. The theories are developed so that they generalize the corresponding theories in the function germ case. The article is organized as follows:

Part I

1. Review of the function germ case
2. Codimension 1 local singular foliations
3. An example
4. Unfoldings of codim 1 foliation germs
5. Some special cases
6. Rigidity of some foliation germs
7. Deformations and unfoldings
8. Determinacy of foliation germs

Part II

9. Proof of the versality theorem (1. formal solution)
10. Malgrange privileged neighborhood theorem
11. Proof of the versality theorem (2. convergence)

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

In section 1, we review some of the basic results in the function case which are to be generalized to the foliation case in the subsequent sections. After recalling the (generalized) de Rham division theorem, we introduce codim 1 local foliations (with singularities) in section 2. The singular set of a form is the set of its zeros and a codim 1 foliation is defined by a 1-form satisfying the integrability condition, i.e., the ordinary integrability condition outside of the singular set. Before we start the general unfolding theory, we give an example in section 3. The unfolding theory is explained in section 4. The idea of unfolding is to extend the given integrable form to some integrable form on a larger space in a proper manner. We first try to classify the first order (infinitesimal) unfoldings by introducing various algebraic objects associated to a given foliation germ. Then we state versality theorems which assert that an infinitesimally versal unfolding is versal. Thus an algebraic condition on the first order term actually implies some analytic consequence. For the readers who might be interested in how the theorems are proved, we give a proof together with some relevant materials in Part II. In section 5, we apply the versality theorems to foliations which are defined by forms of some special types. It is shown that our theory includes, as special cases, not only the unfolding theory for holomorphic functions but also those for meromorphic functions and multiform functions. It has been proved by several authors using various methods that some forms are rigid, i.e., every unfolding of them is trivial. We give, in section 6, systematic proofs of these results as another application of the versality theorem. In section 7, we consider the relation between the deformation theory and the unfolding theory. In the both theories, we extend the given form to a larger space. The essential difference between the two is that in the unfolding theory, we require the integrability of the extended form and thus we obtain a new foliation on the larger space, while in the deformation theory we require only the integrability for each fixed value of the parameter and we have a family of foliations on the same dimensional spaces. We see that the distinction is caused by the presence of integrating factors. We quickly review the determinacy theory for codim 1 complex analytic foliation germs in section 8. It is closely related to the unfolding theory and is hoped to be useful in the analysis of the moduli space of such a germ.

The ideas of the proofs for the versality theorems and the determinacy theorem are basically the same. In each case, we prove the existence of a certain holomorphic map germ. In order to do so, we first show that the infinitesimal condition guarantees the existence of required map germs as formal power series in the variables of the parameter space and then we show the existence of a convergent one among them. For the latter part, we combine the method of Kodaira and Spencer [14] with the privileged neighborhood theorem of Malgrange [15]. Here we only give the proof of the versality theorem for the ordinary morphism to illustrate how it is done (sections 9 and 11). The proof is originally in [27] but is rewritten here so as to make it a little easier to read.

I would like to thank the referee for valuable suggestions.

Part I

1. Review of the function germ case

We refer to Gunning-Rossi [11] or Kodaira [13] for fundamental facts on analytic functions of several complex variables and analytic sets. In what follows, we let $\mathcal{O}_n$ denote the ring of germs of holomorphic functions at the origin 0 in the n -dimensional complex number space $\mathbb{C}^n = \{(z_1, z_2, \dots, z_n)\}$. For a germ f in $\mathcal{O}_n$, we denote by (∂f) the ideal in $\mathcal{O}_n$ generated by $\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \dots, \frac{\partial f}{\partial z_n}$ and call it the Jacobian ideal of f . Also, we denote by $C(f)$ the critical set of f . Thus $C(f) = V((\partial f))$, the germ of analytic set defined by the ideal (∂f) . Note that $\mathcal{O}_n/(\partial f)$ is finite dimensional as a $\mathbb{C}$ -vector space if and only if $C(f) \subset \{0\}$ (Hilbert Nullstellen Satz).

First, we recall the unfolding theory for function germs (Cf. Martinet [16], Mather [17] and [18] and Wassermann [36]). Here we consider only the complex analytic case. However similar results hold in the C^∞ case as well.

(1.1) Definition. An unfolding of a germ f in $\mathcal{O}_n$ is a germ $\tilde{f}$ in $\mathcal{O}_{n+m}$ such that $\iota^* \tilde{f} = f$, where ι denotes the embedding of $\mathbb{C}^n = \{z\}$ into $\mathbb{C}^{n+m} = \{(z, t)\}$ given by $\iota(z) = (z, 0)$. We call $\mathbb{C}^m$ the parameter space of $\tilde{f}$.

Let $\tilde{f}$ be an unfolding of f with parameter space $\mathbb{C}^m$. For each t near 0 in $\mathbb{C}^m$, we define the embedding $\iota_t : \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+m}$ by $\iota_t(z) = (z, t)$ and set $f_t = \iota_t^* \tilde{f}$. Then we may think of f_t as a deformation of f and $\{f_t\}$ a complex analytic family of deformations of f .

(1.2) Definition. Let $\tilde{f}$ be an unfolding of f with parameter space $\mathbb{C}^m$ and g another unfolding of f with parameter space $\mathbb{C}^l = \{s\}$.

(I) A strict right morphism from g to $\tilde{f}$ is a pair (Φ, ψ) satisfying the following conditions.

(a) Φ and ψ are holomorphic map germs making the diagram

$$\begin{array}{ccc} (\mathbb{C}^n \times \mathbb{C}^l, 0) & \xrightarrow{\Phi} & (\mathbb{C}^n \times \mathbb{C}^m, 0) \\ \downarrow & & \downarrow \\ (\mathbb{C}^l, 0) & \xrightarrow{\psi} & (\mathbb{C}^m, 0) \end{array}$$

commutative, where the vertical maps are the projections.

(b) $\Phi(z, 0) = (z, 0)$.

(c) $g = \Phi^* \tilde{f}$ (the pull-back of f by Φ).

(I₁) A right morphism from g to $\tilde{f}$ is a pair (Φ, ψ) satisfying (a) and (b) above and

(c₁) $g(z, s) = \Phi^* \tilde{f}(z, s) + \alpha(s)$
for some germ α in $\mathcal{O}_l$.

(II) A right-left morphism from g to $\tilde{f}$ is a pair (Φ, ψ) satisfying (a) and (b) above and

$$(c') \quad g(z, s) = \lambda(\tilde{f}(\Phi(z, s)), s) \\ \text{for some germ } \lambda \text{ in } \mathcal{O}_{1+l} \text{ with } \lambda(w, 0) = w, \text{ where } w \text{ denotes a coordinate} \\ \text{on } \mathbb{C}.$$

Let $\tilde{f}$ and g be two unfoldings of f with the same parameter space $\mathbb{C} = \{t\}$. A holomorphic map germ $\Phi : (\mathbb{C}^n \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^n \times \mathbb{C}, 0)$ satisfying the conditions, in (1.2) Definition, (a) with $l = m = 1$ and $\psi = id$. and (b) is called a strict right, right or right-left first order (infinitesimal) isomorphism from g to $\tilde{f}$ if it satisfies, in addition, (c), (c₁) or (c'), respectively, with equality holding modulo terms of order ≥ 2 in $s = t$. For an unfolding $\tilde{f}$ of f with parameter space $\mathbb{C} = \{t\}$, we may write

$$\tilde{f}(z, t) = f(z) + h(z)t + \text{terms of order } \geq 2 \text{ in } t.$$

Thus the unfolding $\tilde{f}$ determines a germ h in $\mathcal{O}_n$, which we call the first order term in $\tilde{f}$. Conversely, each element h in $\mathcal{O}_n$ determines an unfolding of f up to the first order. It is shown that two unfoldings corresponding to two germs h and h' in $\mathcal{O}_n$ are strict right, right or right-left first order isomorphic if and only if $h' - h$ is in (∂f) , $(\partial f) + \mathbb{C}$ or $(\partial f) + f^*\mathcal{O}_1$, respectively. Thus the sets of strict right, right and right-left first order isomorphism classes are given, respectively, by $\mathcal{O}_n/(\partial f)$, $\mathcal{O}_n/(\partial f) + \mathbb{C}$ and $\mathcal{O}_n/(\partial f) + f^*\mathcal{O}_1$. Note that they are complex vector spaces and if one of them is finite dimensional, then so are the other two ([36]Corollary 2.17). Since (∂f) is an ideal in $\mathcal{O}_n$, $\mathcal{O}_n/(\partial f)$ has also a structure of a $\mathbb{C}$ -algebra and is called the Milnor algebra of f . Its dimension as a $\mathbb{C}$ -vector space, if it is a finite, is called the Milnor number of f . For various interpretations of this number, see [24].

For a germ h in $\mathcal{O}_n$, we denote by $[h]$ its class in one of the three spaces above. If $\tilde{f}$ is an unfolding of f with parameter space $\mathbb{C}^m = \{(t_1, t_2, \dots, t_m)\}$, we may write

$$\tilde{f}(z, t) = f(z) + \sum_{k=1}^m h_k(z)t_k + \text{terms of order } \geq 2 \text{ in } t.$$

(1.3) **Definition.** An unfolding $\tilde{f}$ of f is said to be infinitesimally strict right, right or right-left versal, respectively, if the classes $[h_1]$, $[h_2]$, $\dots$, $[h_m]$ span the vector space $\mathcal{O}_n/(\partial f)$, $\mathcal{O}_n/(\partial f) + \mathbb{C}$ or $\mathcal{O}_n/(\partial f) + f^*\mathcal{O}_1$.

(1.4) **Definition.** An unfolding $\tilde{f}$ of f is said to be strict right, right or right-left versal, respectively, if for every unfolding g of f , there is a strict right, right or right-left morphism from g to $\tilde{f}$.

The proof of the following theorem can be found in Mather [18] and Wassermann [36].

(1.5) **Theorem (Versality theorem).** *Let $\tilde{f}$ be an unfolding of f . If it is infinitesimally strict right, right or right-left versal, respectively, then it is strict right, right or right-left versal.*

Thus if one of the spaces $\mathcal{O}_n/(\partial f)$, $\mathcal{O}_n/(\partial f) + \mathbb{C}$ and $\mathcal{O}_n/(\partial f) + f^*\mathcal{O}_1$ is finite dimensional, then we can construct a versal unfolding $\tilde{f}$ of f of a certain type, by choosing generators $[h_1], [h_2], \dots, [h_m]$ of the corresponding space and setting

$$\tilde{f}(z, t) = f(z) + \sum_{k=1}^m h_k(z) t_k.$$

Later we generalize the above versality theorem to the foliation case.

Next we review the determinacy results for function germs. We denote by H_n the set of germs of local biholomorphic maps φ about 0 in $\mathbb{C}^n$ with $\varphi(0) = 0$. Note that H_n form a group with group multiplication the composition of such map germs. We also denote by $\mathfrak{m}$ the maximal ideal in $\mathcal{O}_n$. For a germ f in $\mathcal{O}_n$ and a non-negative integer k we denote by $j^k f$ the k -jet of f , i.e., the k -th Taylor polynomial of f .

(1.6) **Definition.** We say that f is (right) k -determined if for any germ g in $\mathcal{O}_n$ with $j^k g = j^k f$, there exists a germ φ in H_n such that $g = \varphi^* f$. The germ f is finitely determined if it is k -determined for some k .

Thus if f is k -determined, after a local change of coordinates of $\mathbb{C}^n$, f becomes a polynomial of degree k (its k -th Taylor polynomial).

The proof of the following theorem can be found in Mather [17] and Wassermann [36].

(1.7) **Theorem.** *For a germ f in $\mathcal{O}_n$, we have the implications*

$$(a) \iff (a') \implies (b) \implies (c) \implies (d)$$

among the following statements:

- (a) $\mathfrak{m}^k \subset \mathfrak{m}(\partial f) + \mathfrak{m}^{k+1}$.
- (a') $\mathfrak{m}^k \subset \mathfrak{m}(\partial f)$.
- (b) f is k -determined.
- (c) $\mathfrak{m}^{k+1} \subset \mathfrak{m}(\partial f)$.
- (d) f is "locally k -determined", i.e., if $\{f_t\}$ is a complex analytic family of deformations of f with $j^k f_t = j^k f$ for all t , there exists a germ φ_t , for each t near 0, in H_n such that $f_t = \varphi_t^* f$.

Now the group H_n acts on $\mathcal{O}_n$ from the right:

$$\mathcal{O}_n \times H_n \longrightarrow \mathcal{O}_n$$

by $(f, \varphi) \mapsto \varphi^* f$. The set $m(\partial f)$ can be interpreted as the "tangent space" at f of the orbit $H_n f$ of f by this action and m^{k+1} the "tangent space" at f of the set of germs g in $\mathcal{O}_n$ with $j^k g = j^k f$.

In view of this, the condition (c) above may be interpreted as the "infinitesimal k -determinacy", which implies the "local k -determinacy" (d). Later we generalize the implication (c) $\implies$ (d) to the foliation case.

We quote some factorization theorems for later use. First, we say that a germ f in $\mathcal{O}_n$ is a power if $f = f_0^m$ for some positive integer m and a non-unit f_0 in $\mathcal{O}_n$. Note that if $\text{codim } C(f) \geq 2$, then f is not a power.

(1.8) Theorem (Mattei-Moussu [22]). *Let f be a germ in $\mathcal{O}_n$ which is not a power or a unit. If g is a germ in $\mathcal{O}_n$ with $dg \wedge df = 0$, then there exists a germ λ in $\mathcal{O}_1$ such that $g = f^* \lambda$.*

Because of the above theorem, we may write, for a germ f with the properties above,

$$\{g \in \mathcal{O}_n \mid dg \wedge df = 0\} = f^* \mathcal{O}_1.$$

If $\tilde{f}$ is a germ in $\mathcal{O}_{n+l}$, we may think of $\tilde{f}$ as an unfolding of f in $\mathcal{O}_n$ given by $f(z) = \tilde{f}(z, 0)$ with parameter space $\mathbb{C}^l = \{s\}$. We denote by d_z the exterior derivation with respect to z ; $d_z \tilde{f} = \sum_{i=1}^n \frac{\partial \tilde{f}}{\partial z_i} dz_i$.

(1.9) Theorem ([34]). *Let $\tilde{f}$ be a germ in $\mathcal{O}_{n+l}$ such that the germ f in $\mathcal{O}_n$ defined by $f(z) = \tilde{f}(z, 0)$ is not a power or a unit. If g is a germ in $\mathcal{O}_{n+l}$ with $d_z g \wedge d_z \tilde{f} = 0$, then there exists a germ λ in $\mathcal{O}_{1+l}$ such that $g(z, s) = \lambda(\tilde{f}(z, s), s)$.*

(1.10) Remark. The germ λ is determined uniquely by g (and f). If we further assume that g is an unfolding of f , then $\lambda(w, 0) = w$.

2. Codimension 1 local singular foliations

We list Kodaira [13] and Matsushima [19] as references for manifolds, differential forms, vector fields, the Frobenius theorem and so forth. We denote by Ω_n^p and Θ_n , respectively, the $\mathcal{O}_n$ -modules of germs of holomorphic p -forms and of holomorphic vector fields. We set $\Omega_n^0 = \mathcal{O}_n$ and $\Omega_n^p = 0$ for $p < 0$ or $p > n$. For a germ ω in $\Omega_n = \Omega_n^1$, we denote by $S(\omega)$ (the germ at 0 of) the set $\{z \mid \omega(z) = 0\}$ and call it the singular set of ω . If we write $\omega = \sum_{i=1}^n f_i dz_i$ with f_i in $\mathcal{O}_n$, we have $S(\omega) = V(f_1, f_2, \dots, f_n)$, the germ of analytic set defined by the ideal in $\mathcal{O}_n$ generated by $f_1, f_2, \dots, f_n$.

Before we introduce codim 1 local singular foliations, we quote the generalized de Rham division theorem. For a fixed ω in Ω_n , we consider the complex

$$0 \rightarrow \mathcal{O}_n \xrightarrow{\alpha} \Omega_n \xrightarrow{\alpha} \Omega_n^2 \xrightarrow{\alpha} \dots \xrightarrow{\alpha} \Omega_n^n \rightarrow 0,$$

where $\alpha : \Omega_n^p \rightarrow \Omega_n^{p+1}$ is defined by $\alpha(\theta) = \theta \wedge \omega$ for θ in Ω_n^p . If we set

$$Z^p(\omega) = \{ \theta \in \Omega_n^p \mid \theta \wedge \omega = 0 \} \quad \text{and} \\ B^p(\omega) = \{ \theta \in \Omega_n^p \mid \theta = \eta \wedge \omega \text{ for some } \eta \in \Omega_n^{p-1} \},$$

then $B^p(\omega)$ is a sub- $\mathcal{O}_n$ module of $Z^p(\omega)$. Hence we may consider the cohomology group

$$H^p(\omega) = Z^p(\omega)/B^p(\omega).$$

For the proof of the following theorem, we refer to Malgrange [15] or Saito [25].

(2.1) Theorem ((Generalized) de Rham division theorem). *Let ω be a germ in Ω_n . We have $H^p(\omega) = 0$ for $p < \text{codim } S(\omega)$.*

Thus for any p -form θ with $p < \text{codim } S(\omega)$ and $\theta \wedge \omega = 0$, there exists a $(p-1)$ -form η such that $\theta = \eta \wedge \omega$. Especially, if $S(\omega) = \emptyset$, namely, if ω is non-singular, then for a form θ of arbitrary degree p with $\theta \wedge \omega = 0$, there exists a $(p-1)$ -form η such that $\theta = \eta \wedge \omega$. We call (2.1) Theorem simply the de Rham division theorem hereafter.

(2.2) Definition. A codimension 1 local foliation (or foliation germ) at 0 in $\mathbb{C}^n$ is a rank 1 free submodule $F = (\omega)$ of Ω_n with a generator ω satisfying the integrability condition

$$(2.3) \quad d\omega \wedge \omega = 0.$$

(2.4) Remarks. 1°. If one of the generators of F satisfies the integrability condition (2.3), so does any other. Also the singular set $S(\omega)$ is independent of the choice of the generator ω . We denote it by $S(F)$ and call it the singular set of F .

2°. Let ω be a germ of holomorphic 1-form and suppose that it has a representative on a neighborhood U of 0 in $\mathbb{C}^n$. If ω satisfies (2.3), then for any point z in $U - S(\omega)$, there exists a 1-form η_z in a neighborhood of z such that $d\omega = \eta_z \wedge \omega$ in a neighborhood of z . Since the converse of this statement is obviously true, the condition (2.3) is equivalent to the (classical) integrability of ω outside of its singular set.

3°. An arbitrary 1-form on $\mathbb{C}^2$ is integrable.

4°. Especially if $\omega = df$ for some f in $\mathcal{O}_n$, it always satisfies (2.3) and thus $F = (df)$ is a codim 1 foliation germ whose singular set $S(F)$ coincides with the critical set $C(f)$ of f . In what follows we try to construct theories for foliation germs so that they reduce to the corresponding results for function germs when $\omega = df$.

5°. As we see in the sequel, it is often more convenient to consider the module generated by a form than to consider the form itself.

(2.5) **Lemma.** *Let ω be a germ in Ω_n . We have $\text{codim } S(\omega) = 1$ if and only if there exist a non-unit f in $\mathcal{O}_n$ and a germ ω' in Ω_n such that $\omega = f\omega'$.*

Proof. Suppose $\text{codim } S(\omega) = 1$ and let S be an irreducible component of $S(\omega)$ with $\text{codim } S = 1$. There exists a non-unit f in $\mathcal{O}_n$ such that $S = V(f)$, the germ of analytic set defined by the ideal (f) . If we write $\omega = \sum_{i=1}^n f_i dz_i$ with f_i in $\mathcal{O}_n$, then each f_i is in the ideal $I(V(f))$ of function germs vanishing on $V(f)$, which is equal to the radical $\sqrt{(f)}$ of (f) . Since f is irreducible, we have $\sqrt{(f)} = (f)$. Thus for each i , there exists f'_i in $\mathcal{O}_n$ such that $f_i = f'_i f$. Hence, if we set $\omega' = \sum_{i=1}^n f'_i dz_i$, then we have $\omega = f\omega'$. The converse is obvious. $\square$

In view of the above lemma, we have foliation singularities essentially when the codimension of the singular set is greater than or equal to two. In what follows we only consider this case. Moreover we have the following theorem due to B. Malgrange ([15]).

(2.6) **Theorem (Frobenius with singularities).** *Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^n$. If $\text{codim } S(F) \geq 3$, Then there exist germs f and u in $\mathcal{O}_n$ with u a unit, such that $\omega = udf$, i.e., F is generated by an exact form df .*

The idea of the proof of this theorem fits naturally into the framework of the unfolding theory for foliation germs, which we develop in the forthcoming sections. Roughly speaking, to unfold a given integrable form means to extend it to some integrable form on a larger space (Cf. (4.1) Definition). By choosing an appropriate unfolding, the situation may become easier to handle. Thus in our present case, we try to unfold the given form ω to obtain an integrable form $\tilde{\omega}$ without singularities in a larger space and we apply the classical Frobenius theorem to $\tilde{\omega}$. Namely, first we introduce a new variable t and try to construct a germ of 1-form $\tilde{\omega}$ at 0 in $\mathbb{C}^{n+1} = \{(x, t)\}$ of the form

$$(2.7) \quad \tilde{\omega} = \omega + \sum_{p \geq 1} \omega^{(p)} t^p + dt, \quad \omega^{(p)} \in \Omega_n$$

so that it satisfies the integrability condition $d\tilde{\omega} \wedge \tilde{\omega} = 0$. If we apply the de Rham division theorem under the given codimension condition of $S(\omega)$, we see the existence of the forms $\omega^{(p)}$ such that the series (2.7), as a formal series in t , satisfies the integrability condition. In Malgrange [15], it is proved that if we choose the $\omega^{(p)}$'s properly, the series can be made to converge. Once we have $\tilde{\omega}$ as above, it is a non-singular integrable 1-form on $\mathbb{C}^{n+1}$ and, as we mentioned above, the classical Frobenius theorem asserts the existence of germs $\tilde{f}$ and $\tilde{u}$ in $\mathcal{O}_{n+1}$ with $\tilde{u}$ a unit, such that $\tilde{\omega} = \tilde{u}d\tilde{f}$. Thus if we let f and u be germs in $\mathcal{O}_n$ obtained from $\tilde{f}$ and $\tilde{u}$, respectively, by restricting to $t = 0$, then they have the desired properties.

(2.8) **Remarks.** 1°. Knowing that the above result holds, it is quite natural to look for a form of the form (2.7), since if $\omega = udf$, the isomorphism classes of first order

unfoldings (see section 4) of the foliation $F = (\omega)$ will be equal to $\mathcal{O}_n/(\partial f)$ and we may look for an unfolding of (ω) corresponding to the class [1] in $\mathcal{O}_n/(\partial f)$, i.e., an unfolding such that the coefficient of dt is 1.

2°. For the convergence part of the above proof, Malgrange uses his privileged neighborhood theorem, which we take up again in section 10, since we also use it in the proof of our versality theorem.

The above (2.5) Lemma and (2.6) Theorem show that we have singularities of foliations essentially when the codimension of the singular set is equal to two. Thus if we have a foliation singularity ω in higher dimension, its restriction ω_0 to a suitable two dimensional subspace admits only isolated singularity (Cf. (5.1) Theorem) and ω may be thought of as being obtained from ω_0 by unfolding.

3. An example

Before we start the general theory of unfoldings of codim 1 local foliation singularities, we give a simple example to illustrate the effect of "unfolding" foliation singularities.

We consider $\mathbb{C}^2$ with coordinates (x, y) and let ω be the 1-form on $\mathbb{C}^2$ given by

$$(3.1) \quad \omega = (y^2 + 2xy)dx - (x^2 + 2xy)dy.$$

The singular set of ω is given by $S(\omega) = \{0\}$ and its integral curves are obtained by solving the differential equation

$$(3.2) \quad \frac{dy}{dx} = \frac{y^2 + 2xy}{x^2 + 2xy}.$$

Advanced calculus text books teach us that this is a homogeneous equation and thus we may solve it by setting $u = y/x$. Then the equation reduces to a separable equation in x and u . This is so to speak the method of solving by "blowing up". We may also solve this as well as the inhomogeneous equations obtained from this by deformation by the method of "unfolding".

We first extend ω to a form $\tilde{\omega}_1$ on $\mathbb{C}^3 = \{(x, y, t)\}$ given by

$$\tilde{\omega}_1 = (y^2 + 2xy - yt)dx - (x^2 + 2xy + xt)dy + 3xydt.$$

This form satisfies the integrability condition $d\tilde{\omega}_1 \wedge \tilde{\omega}_1 = 0$ and its singular set consists of three lines:

$$S(\tilde{\omega}_1) = \{(x, y, t) \mid x = y = 0, \quad x = y - t = 0 \quad \text{or} \quad x - t = y = 0\}.$$

For each fixed t , we set

$$\omega_t = (y^2 + 2xy - yt)dx - (x^2 + 2xy + xt)dy.$$

Then this is the form obtained by restricting $\tilde{\omega}_1$ to the plane $\mathbb{C}^2 \times \{t\}$ and is considered as a deformation of ω . The singular set of ω_t consists of four points $0 = (0,0), (0,t), (-t,0)$ and $(t,-t)$. Note that the last singularity is the result of the tangency of the plane $\mathbb{C}^2 \times \{t\}$ and one of the leaves (integral submanifolds) of $\tilde{\omega}_1$. We further "unfold" $\tilde{\omega}_1$ and consider the 1 form $\tilde{\omega}$ on $\mathbb{C}^4 = \{(x,y,t,s)\}$ given by

$$\tilde{\omega} = (y^2 + 2xy - yt - 3s)dx - (x^2 + 2xy + xt - 3s)dy + 3(xy - s)dt + (x - y + t)ds.$$

This form is also integrable and its singular set is given by

$$S(\tilde{\omega}) = \{(x,y,t,s) \mid x - y + t = xy - s = 0\}.$$

We recover $\tilde{\omega}_1$ by setting $s = 0$ and ω by $s = t = 0$. Later we see that we do not get essentially new forms by further unfolding this form, i.e., this is a "universal unfolding" of ω . This form actually turns out to be a very simple form. If we introduce new coordinates (x', y', t', s') on $\mathbb{C}^4$ by $x' = x, y' = y, t' = t + x - y$ and $s' = s - xy$, then in this coordinate system, we have

$$\tilde{\omega} = t'ds' - 3s'dt'.$$

Thus the integral submanifolds of $\tilde{\omega}$ are given by $t'^3 = Cs'$, or in terms of the old coordinate system,

$$(x - y + t)^3 = C(xy - s).$$

Hence we get the integral curves of ω_t by setting $s = 0$ and those of ω by $s = t = 0$. We come back to this example in section 5 ((5.23) Example).

In the coming sections we consider the following

- (3.3) *Problems.* 1°. How can one unfold a given form?
2°. How many unfoldings does a given form have?

4. Unfoldings of codim 1 foliation germs

In this section and in sections 5 and 6, we summarize the unfolding theory for codim 1 local foliations developed mainly in [27] - [31] and [33]. The article [28] contains considerations for more general foliations. We start with the following

(4.1) **Definition.** Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^n$. An unfolding of F is a codim 1 foliation germ $\mathcal{F} = (\tilde{\omega})$ at 0 in $\mathbb{C}^n \times \mathbb{C}^m = \{(z,t)\}$ with a generator $\tilde{\omega}$ such that $\iota^*\tilde{\omega} = \omega$, where ι denotes the embedding $(\mathbb{C}^n, 0) \hookrightarrow (\mathbb{C}^n \times \mathbb{C}^m, 0)$ given by $\iota(z) = (z, 0)$. We call $\mathbb{C}^m$ the parameter space of $\mathcal{F}$.

(4.2) **Remark.** Let $\mathcal{F} = (\tilde{\omega})$ be an unfolding of $F = (\omega)$ with parameter space $\mathbb{C}^m = \{t\}$. For t near 0, let ι_t denote the embedding $\mathbb{C}^n \hookrightarrow \mathbb{C}^n \times \mathbb{C}^m$ given by $\iota_t(z) = (z, t)$ and set $\omega_t = \iota_t^*\tilde{\omega}$. Then we may think of $\{\omega_t\}$ as a family of deformations of ω .

(4.3) **Definition.** Let $\mathcal{F}$ be an unfolding of F with parameter space $\mathbb{C}^m$ and $\mathcal{F}'$ another unfolding of F with parameter space $\mathbb{C}^l = \{(s_1, s_2, \dots, s_l)\}$.

(I) A morphism from $\mathcal{F}'$ to $\mathcal{F}$ is a pair (Φ, ψ) satisfying the following conditions.

(a) Φ and ψ are holomorphic map germs making the diagram

$$\begin{array}{ccc} (\mathbb{C}^n \times \mathbb{C}^l, 0) & \xrightarrow{\Phi} & (\mathbb{C}^n \times \mathbb{C}^m, 0) \\ \downarrow & & \downarrow \\ (\mathbb{C}^l, 0) & \xrightarrow{\psi} & (\mathbb{C}^m, 0) \end{array}$$

commutative, where the vertical maps are the projections.

(b) $\Phi(z, 0) = (z, 0)$.

(c) For any generator $\tilde{\omega}$ of $\mathcal{F}$, $\Phi^*\tilde{\omega}$ generates $\mathcal{F}'$.

(II) An RL-morphism from $\mathcal{F}'$ to $\mathcal{F}$ is a pair (Φ, ψ) satisfying (a) and (b) above and

(c') For any generator $\tilde{\omega}$ of $\mathcal{F}$, there is a germ $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l)$ in $\mathcal{O}_{n+l}^l$ such that the germ

$$\Phi^*\tilde{\omega} + \sum_{k=1}^l \alpha_k ds_k$$

generates $\mathcal{F}'$.

(4.4) *Remarks.* 1°. In the both cases above, we may replace "any" by "some".

2°. From the integrability condition, we see that, for α in (c'), each $\alpha_k(z) = \alpha_k(z, 0)$ is an integrating factor of $\omega = \iota^*\tilde{\omega}$, i.e., $\alpha_k d\omega = d\alpha_k \wedge \omega$.

Next we show that a morphism is a generalization of a strict right morphism in the unfolding theory for function germs and an RL-morphism that of a right-left morphism. Note first that if a germ $\tilde{f}$ in $\mathcal{O}_{n+m}$ is an unfolding of some germ f in $\mathcal{O}_n$, then $\mathcal{F} = (d\tilde{f})$ is an unfolding of $F = (df)$ with the same parameter space $\mathbb{C}^m$.

(4.5) **Theorem.** Let $\tilde{f}$ be an unfolding of f with parameter space $\mathbb{C}^m$ and g another unfolding of f with parameter space $\mathbb{C}^l$. A pair (Φ, ψ) is, respectively, a strict right morphism or a right-left morphism from g to $\tilde{f}$ if and only if it is a morphism or an RL-morphism from $\mathcal{F}' = (dg)$ to $\mathcal{F} = (d\tilde{f})$.

Proof. The statement is obvious for the equivalence of strict right morphism and morphism. If (Φ, ψ) is a right-left morphism, then by definition, there is a germ λ in $\mathcal{O}_{1+l}$ with $\lambda(w, 0) = w$ and $g(z, s) = \lambda(\Phi^*\tilde{f}(z, s), s)$. We have $dg = \frac{\partial \lambda}{\partial w} d\Phi^*\tilde{f} + \sum_{k=1}^l \frac{\partial \lambda}{\partial s_k} ds_k$. By the condition $\lambda(w, 0) = w$, we see that $\frac{\partial \lambda}{\partial w}(\Phi^*\tilde{f}(z, s), s)$ is a unit in $\mathcal{O}_{n+l}$. Thus (Φ, ψ) is an RL-morphism from (dg) to $(d\tilde{f})$. Conversely, suppose (Φ, ψ) is an RL-morphism. Then there is a unit u in $\mathcal{O}_{n+l}$ and germs α_k in $\mathcal{O}_{n+l}^l$ such that $udg = d(\Phi^*\tilde{f}) + \sum_{k=1}^l \alpha_k ds_k$. Hence we have $ud_z g = d_z(\Phi^*\tilde{f})$ and

therefore, $d_z(\Phi^* \tilde{f}) \wedge d_z g = 0$. Thus by (1.9) Theorem, there exist λ in $\mathcal{O}_{1+l}$ such that $g(z, s) = \lambda(\Phi^* \tilde{f}(z, s), s)$. It remains to show that $\lambda(w, 0) = w$. If we set $s = 0$ in the above, we get $g(z, 0) = \lambda(\Phi^* \tilde{f}(z, 0), 0)$. We have $g(z, 0) = \Phi^* \tilde{f}(z, 0) = f(z)$, which is not a constant. Hence $\lambda(w, 0) = w$. $\square$

In the next section, we see that every unfolding of a foliation germ F generated by df admits a generator of the form $d\tilde{f}$ with $\tilde{f}$ an unfolding of f and thus the unfolding theory for F with respect to morphisms or RL-morphisms is equivalent to the unfolding theory for f with respect to strict right morphisms or right-left morphisms, respectively.

In order to investigate the problems stated at the end of the previous section, we first look at the "first order terms" in the unfoldings of a given foliation germ and try to get some algebraic control of the set of unfoldings.

Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^n$.

(4.6) Definition. A first order (infinitesimal) unfolding of F is a rank 1 free submodule $\mathcal{F}^{(1)} = (\tilde{\omega})$ of Ω_{n+1} with a generator $\tilde{\omega}$ such that $\iota^* \tilde{\omega} = \omega$ and that

$$(4.7) \quad d\tilde{\omega} \wedge \tilde{\omega} \equiv 0 \quad \text{mod. } t^2, tdt.$$

Let $\mathcal{F}^{(1)} = (\tilde{\omega})$ be a first order unfolding of $F = (\omega)$. If we write

$$\tilde{\omega} \equiv \omega + \omega^{(1)}t + hdt \quad \text{mod. } t^2, tdt$$

with $\omega^{(1)}$ in Ω_n and h in $\mathcal{O}_n$, then the first order integrability (4.7) is equivalent to

$$\begin{aligned} d\omega \wedge \omega^{(1)} + d\omega^{(1)} \wedge \omega &= 0 \quad \text{and} \\ h d\omega + (\omega^{(1)} - dh) \wedge \omega &= 0. \end{aligned}$$

It is not difficult to show that the second equation above implies the first. Also, the germ h does not depend on the choice of the generator $\tilde{\omega}$ of $\mathcal{F}^{(1)}$ with $\iota^* \tilde{\omega} = \omega$. Hence if we set

$$I(\omega) = \{ h \in \mathcal{O}_n \mid h d\omega = \eta \wedge \omega \text{ for some } \eta \text{ in } \Omega_n \},$$

then each first order unfolding of $F = (\omega)$ determines an element in $I(\omega)$. Conversely, if we have an element h in $I(\omega)$, by setting $\tilde{\omega} = \omega + \omega^{(1)}t + hdt$, $\omega^{(1)} = dh - \eta$, we have a first order unfolding of $F = (\omega)$. We also set

$$\begin{aligned} J(\omega) &= \{ h \in \mathcal{O}_n \mid h = \langle v, \omega \rangle \text{ for some } v \text{ in } \Theta_n \}, \\ K(\omega) &= \{ h \in \mathcal{O}_n \mid h d\omega = dh \wedge \omega \} \quad \text{and} \\ \Omega(\omega) &= \{ \theta \in \Omega_n \mid \theta \wedge \omega = dh \wedge \omega - h d\omega \text{ for some } h \text{ in } \mathcal{O}_n \}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing of a vector field and a 1-form. Obviously, $I(\omega)$ is an ideal in $\mathcal{O}_n$. If we write $\omega = \sum_{i=1}^n f_i dz_i$ with f_i in $\mathcal{O}_n$, then $J(\omega)$ is the ideal in $\mathcal{O}_n$ generated by $f_1, f_2, \dots, f_n$. We call it the Jacobian ideal of ω . The set $K(\omega)$ is contained in $J(\omega)$ and is the $\mathbb{C}$ -vector space of integrating factors of ω . The set $\Omega(\omega)$ has also a structure of a $\mathbb{C}$ -vector space. Note that the dimension of $\mathcal{O}_n/J(\omega)$ is finite if and only if $S(\omega) \subset \{0\}$.

The ideals $I(\omega)$ and $J(\omega)$ do not depend on the choice of the generator ω of F . If ω' is another generator of F , then $\omega' = u\omega$ for some unit u in $\mathcal{O}_n$. The multiplication by u gives an isomorphism of $K(\omega)$ onto $K(\omega')$ and that of $\Omega(\omega)$ onto $\Omega(\omega')$.

(4.8) **Lemma.** For a germ v in Θ_n , we have

$$\langle v, \omega \rangle d\omega + (L_v \omega - d\langle v, \omega \rangle) \wedge \omega = 0.$$

Proof. Apply the interior product ι_v to the identity $d\omega \wedge \omega = 0$ and use the identity $L_v = di_v + i_v d$. $\square$

(4.9) **Corollary.** We have

$$J(\omega) \subset I(\omega).$$

For an ideal I in $\mathcal{O}_n$, we set $L_I(\omega) = \{ L_v \omega \mid v \in I \cdot \Theta_n \}$. We denote $L_{\mathcal{O}_n}(\omega)$ simply by $L(\omega)$.

(4.10) **Corollary.** For any germ v in Θ_n , $L_v \omega$ is in $\Omega(\omega)$. In particular,

$$L_I(\omega) + F \subset \Omega(\omega)$$

for any ideal I in $\mathcal{O}_n$.

(4.11) **Proposition.** For any ideal I in $\mathcal{O}_n$, there is an isomorphism of $\mathbb{C}$ -vector spaces

$$I(\omega)/(I \cdot J(\omega) + K(\omega)) \simeq \Omega(\omega)/(L_I(\omega) + F).$$

Proof. If θ is in $\Omega(\omega)$, there is a germ h in $\mathcal{O}_n$ such that $\theta \wedge \omega = dh \wedge \omega - h d\omega$. Such an h is determined uniquely modulo elements in $K(\omega)$. Thus the correspondence $\theta \mapsto [h]$ defines a surjective $\mathbb{C}$ -linear map

$$\Omega(\omega) \longrightarrow I(\omega)/(I \cdot J(\omega) + K(\omega)).$$

Moreover, if h is in $I \cdot J(\omega) + K(\omega)$, we may write $h = \langle v, \omega \rangle + \alpha$ for v in $I \cdot \Theta_n$ and α in $K(\omega)$. Then by (4.8) Lemma, we have

$$\theta \wedge \omega = L_v \omega \wedge \omega.$$

Hence $\theta = L_v \omega + g\omega$ for some g in $\mathcal{O}_n$. $\square$

(4.12) **Definition.** Let $\mathcal{F}^{(1)} = (\tilde{\omega})$ and $\mathcal{F}'^{(1)} = (\theta)$ be two first order unfoldings of $F = (\omega)$.

(I) $\mathcal{F}^{(1)}$ and $\mathcal{F}'^{(1)}$ are isomorphic if there is a pair (Φ, u) satisfying the following conditions.

(a) Φ is a holomorphic map germ making the diagram

$$\begin{array}{ccc} (\mathbb{C}^n \times \mathbb{C}, 0) & \xrightarrow{\Phi} & (\mathbb{C}^n \times \mathbb{C}, 0) \\ \downarrow & & \downarrow \\ (\mathbb{C}, 0) & \xrightarrow{id.} & (\mathbb{C}, 0) \end{array}$$

commutative, where the vertical maps are the projections. u is a unit in $\mathcal{O}_{n+1}$.

(b) $\Phi(z, 0) = (z, 0)$ and $u(z, 0) = 1$.

(c) $u\theta \equiv \Phi^*\tilde{\omega} \pmod{t^2, tdt}$.

(II) $\mathcal{F}^{(1)}$ and $\mathcal{F}'^{(1)}$ are RL-isomorphic if there is a triple (Φ, u, α) , where Φ and u are germs satisfying (a) and (b) above and α is a germ in $\mathcal{O}_{n+1}$. Instead of (c), we require

(c') $u\theta \equiv \Phi^*\tilde{\omega} + \alpha dt \pmod{t^2, tdt}$.

(4.13) *Remark.* In (II) above, we may assume that α is in $\mathcal{O}_n$. Moreover, the first order integrability of $\Phi^*\tilde{\omega}$ and $\Phi^*\tilde{\omega} + \alpha dt$ implies that α is in $K(\omega)$.

We denote by $U(F)$ and $U_{RL}(F)$, respectively, the sets of isomorphism classes and of RL-isomorphism classes of first order unfoldings of $F = (\omega)$.

(4.14) **Proposition.** The set $U(F)$ is naturally identified with the vector space $I(\omega)/J(\omega)$ and the set $U_{RL}(F)$ with the vector space $I(\omega)/(J(\omega) + K(\omega))$, which is isomorphic to $\Omega(\omega)/(L(\omega) + F)$.

Proof. We prove the statement for $U_{RL}(F)$. If we set $\alpha = 0$ in the following arguments, we have a proof for $U(F)$. As we noted earlier, each first order unfolding of $F = (\omega)$ determines a unique germ in $I(\omega)$ and conversely, each germ in $I(\omega)$ determines a first order unfolding, although it is not unique. Let $\mathcal{F}^{(1)} = (\tilde{\omega})$ and $\mathcal{F}'^{(1)} = (\theta)$ be two first order unfoldings of $F = (\omega)$. We write

$$\tilde{\omega} \equiv \omega + \omega^{(1)}t + hdt \quad \text{and} \quad \theta \equiv \omega + \theta^{(1)}t + edt \pmod{t^2, tdt}$$

with $\omega^{(1)}$ and $\theta^{(1)}$ in Ω_n and h and e in $\mathcal{O}_n$. Suppose $\mathcal{F}^{(1)}$ and $\mathcal{F}'^{(1)}$ are RL-isomorphic and let (Φ, u, α) be as in (4.12)(II). We may assume that α is in $K(\omega)$ ((4.13) Remark). By (a) in (4.12), we may write $\Phi(z, t) = (\varphi(z, t), t)$ for some holomorphic map germ

$$\varphi : (\mathbb{C}^n \times \mathbb{C}, 0) \longrightarrow (\mathbb{C}^n, 0).$$

On the other hand, by (b), we may also write

$$\varphi(z, t) \equiv z + \varphi^{(1)}(z)t \quad \text{and} \quad u(z, t) \equiv 1 + u^{(1)}(z)t \quad \text{mod. } t^2$$

with $\varphi^{(1)}$ in $\mathcal{O}_n^n$ and $u^{(1)}$ in $\mathcal{O}_n$. Then we have

$$\Phi^* \tilde{\omega} \equiv \omega + (\omega^{(1)} + L_v \omega)t + (h + \langle v, \omega \rangle)dt \quad \text{mod. } t^2, tdt,$$

where v is the germ of vector field given by $v = \sum_{i=1}^n \varphi_i^{(1)} \frac{\partial}{\partial z_i}$. Thus from (4.12)(II), (c'), we get

$$(4.15) \quad e = h + \langle v, \omega \rangle + \alpha,$$

which shows that h and e determine the same element in $I(\omega)/(J(\omega) + K(\omega))$.

Conversely, suppose we have the relation (4.15) between h and e for some $v = \sum_{i=1}^n \xi_i \frac{\partial}{\partial z_i}$ in Θ_n and α in $K(\omega)$. Then from the identities

$$hd\omega + (\omega^{(1)} - dh) \wedge \omega = 0, \quad ed\omega + (\omega^{(1)} - de) \wedge \omega = 0 \quad \text{and} \quad \alpha d\omega - d\alpha \wedge \omega = 0$$

and (4.8) Lemma, we have

$$\theta^{(1)} \wedge \omega = (\omega^{(1)} + L_v \omega) \wedge \omega.$$

Hence we see that there is a germ g in $\mathcal{O}_n$ such that

$$\theta^{(1)} + g\omega = \omega^{(1)} + L_v \omega.$$

Hence if we set

$$\begin{aligned} \Phi(z, t) &= (\varphi(z, t), t), \quad \varphi(z, t) = z + \xi(z) \quad \text{and} \\ u(z, t) &= 1 + g(z)t, \end{aligned}$$

then the triple (Φ, u, α) gives an RL-isomorphism between $\mathcal{F}^{(1)}$ and $\mathcal{F}'^{(1)}$. $\square$

(4.16) *Remark.* If $\omega = df$ for some f in $\mathcal{O}_n$, then $I(\omega) = \mathcal{O}_n$, $J(\omega) = (\partial f)$ (the Jacobian ideal of f). Thus $I(\omega)/J(\omega) = \mathcal{O}_n/(\partial f)$ is the Milnor algebra of f . We also have $K(\omega) = \{ \alpha \in \mathcal{O}_n \mid d\alpha \wedge df = 0 \}$. If we assume that f is a non-unit then by the factorization theorem (1.8), we may write

$$K(df) = f^* \mathcal{O}_1,$$

since the condition $\text{codim } S(df) \geq 2$ implies that f is not a power. Thus in this case we have

$$I(\omega)/(J(\omega) + K(\omega)) = \mathcal{O}_n/((\partial f) + f^* \mathcal{O}_1).$$

For a germ h in $I(\omega)$, we denote by $[h]$ the class of h in $I(\omega)/J(\omega)$ or in $I(\omega)/(J(\omega) + K(\omega))$.

Let $\mathcal{F} = (\tilde{\omega})$ be an unfolding of $F = (\omega)$ with parameter space $\mathbb{C}^m = \{(t_1, t_2, \dots, t_m)\}$. If we write

$$\tilde{\omega} = \omega + \sum_{k=1}^m \omega^{(1_k)} t_k + \sum_{k=1}^m h_k dt_k + \text{terms of order } \geq 2 \text{ in } t$$

with $\omega^{(1_k)}$ in Ω_n and h_k in $\mathcal{O}_n$ for $k = 1, 2, \dots, m$, then from the integrability of $\tilde{\omega}$, we see that the h_k 's are in $I(\omega)$.

(4.17) **Definition.** An unfolding $\mathcal{F}$ of F is said to be infinitesimally versal or infinitesimally RL-versal, respectively, if the classes $[h_1], [h_2], \dots, [h_m]$ span the vector space $I(\omega)/J(\omega)$ or $I(\omega)/(J(\omega) + K(\omega))$.

(4.18) **Definition.** An unfolding $\mathcal{F}$ of F is said to be versal or RL-versal, respectively, if for every unfolding $\mathcal{F}'$ of F , there is a morphism or an RL-morphism from $\mathcal{F}'$ to $\mathcal{F}$.

(4.19) **Theorem (Versality theorem [27]).** *Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^n$. If an unfolding $\mathcal{F}$ of F is infinitesimally versal, then it is versal.*

(4.20) **Theorem (RL-versality theorem [33]).** *Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^n$ with $\dim K(\omega) < +\infty$. If an unfolding $\mathcal{F}$ of F is infinitesimally RL-versal, then it is RL-versal.*

In sections 9 and 11, we give a proof of (4.19) Theorem, which was originally proved in [27]. For the proof of (4.20) Theorem, we refer to [33]. In both cases, we prove the versality by constructing a morphism or an RL-morphism as a power series in the variables of the parameter space of an arbitrarily given unfolding. The infinitesimal versality condition guarantees the existence of a formal solution. We prove the existence of a convergent solution by combining the method of Kodaira and Spencer in [13] and [14] with the Malgrange privileged neighborhood theorem in [15] which we discuss in section 10. The condition $\dim K(\omega) < +\infty$ in the RL case is needed for the convergence.

We finish this section by quoting the following "stability theorem" for integrating factors.

(4.21) **Theorem ([6], [33]).** *Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^n$. If α is a germ in $K(\omega)$, then for any unfolding $\mathcal{F} = (\tilde{\omega})$ of F , there exists a unique germ $\tilde{\alpha}$ in $K(\tilde{\omega})$ which is an unfolding of α .*

5. Some special cases

In this section, we apply the versality theorem to some special cases. We first quote the following "transversality theorem".

(5.1) **Theorem (Mattei-Moussu [21]).** *Let ω be a germ in Ω_n with $\text{codim } S(\omega) \geq 2$. There is an embedding $\iota : (\mathbb{C}^2, 0) \hookrightarrow (\mathbb{C}^n, 0)$ such that $S(\iota^*\omega) \subset \{0\}$.*

(I) The case $F = (\omega)$, $\omega = df$, $f \in \mathcal{O}_n$. As we have seen before, in this case we have $S(F) = C(f)$, $I(\omega) = \mathcal{O}_n$, $J(\omega) = (\partial f)$ and $K(\omega) = f^*\mathcal{O}_1$.

(5.2) **Theorem.** *If $\dim \mathcal{O}_n/(\partial f) < +\infty$, then $F = (df)$ admits a versal unfolding $\mathcal{F}$ with a generator of the form $d\tilde{f}$, where $\tilde{f}$ is an unfolding of f . In fact, if*

$[h_1], [h_2], \dots, [h_m]$ is a basis of $\mathcal{O}_n/(\partial f)$, then the unfolding $\mathcal{F} = (\tilde{\omega})$ of F with parameter space $\mathbb{C}^m = \{(t_1, t_2, \dots, t_m)\}$ generated by

$$\tilde{\omega} = d\tilde{f}, \quad \tilde{f} = f + \sum_{k=1}^m h_k t_k$$

is universal.

Proof. Since $d\tilde{f} = df + \sum_{k=1}^m dh_k \cdot t_k + \sum_{k=1}^m h_k dt_k$, this is a direct consequence of the versality theorem. $\square$

In the above, we say that an unfolding is *universal* if it is versal and if the classes of the first order terms are linearly independent in the set of isomorphism classes of first order unfoldings.

From this we recover the extension theorem for strong first integrals in Mattei-Moussu [22].

(5.3) Corollary. *If $F = (\omega)$ is a codim 1 foliation germ at 0 in $\mathbb{C}^n$ generated by a germ ω of the form $\omega = df$ for some f in $\mathcal{O}_n$ with $\text{codim } C(f) \geq 2$, then every unfolding of F admits a generator $\tilde{\omega}$ of the form $\tilde{\omega} = d\tilde{f}$, where $\tilde{f}$ is an unfolding of f .*

Proof. We take an embedding $\iota : (\mathbb{C}^2, 0) \hookrightarrow (\mathbb{C}^n, 0)$ as in (5.1) Theorem and set $f_1 = \iota^* f$. Since $\mathcal{O}_2/(\partial f_1)$ is finite dimensional, by the above theorem the foliation $F_1 = (df_1)$ admits a universal unfolding $\mathcal{F}_1$ generated by an exact form $d\tilde{f}_1$. By the implicit function theorem, the given unfolding $\mathcal{F}$ of F can be thought of as an unfolding of F_1 . Therefore, there exists a morphism $\Phi : \mathcal{F} \rightarrow \mathcal{F}_1$ and $\mathcal{F}$ is generated by $\Phi^* d\tilde{f}_1 = d(\Phi^* \tilde{f}_1)$. Set $\tilde{f} = \Phi^* \tilde{f}_1$. $\square$

As we noted in the previous section, (4.5) Theorem together with (5.3) Corollary shows that the unfolding theory for a foliation of the form (df) , $f \in \mathcal{O}_n$, with respect to morphisms or RL-morphisms is equivalent to the unfolding theory for f with respect to strict right morphisms or right-left morphisms, respectively. Thus, in the complex analytic case, we recover the versality theorem for (strict) right morphism of Mather [18] and a part of the versality theorem for right-left morphism of Wassermann [36] (Cf. [33] §3).

(II) The case $F = (\omega)$, $\omega = gdf - fdg$, $f, g \in \mathcal{O}_n$. We assume that $\text{codim } S(\omega) \geq 2$. This implies that neither f nor g is a power. Moreover, if f or g is a unit, we may take $d(g/f)$ or $d(f/g)$ as a generator of F and it is of the type considered in (I). So we also assume that f and g are both non-units. First we express the algebraic objects $I(\omega)$, $J(\omega)$ and $K(\omega)$ in terms of f and g . We denote by (f, g) the ideal in $\mathcal{O}_n$ generated by f and g and by $[f^2, fg, g^2]_{\mathbb{C}}$ the vector subspace of $\mathcal{O}_n$ spanned by f^2 , fg and g^2 . For the proof of the following lemma we refer to [29](2.1)Lemma and [33](4.1)Lemma.

(5.4) Lemma. We have

$$I(\omega) = (f, g) \quad \text{and} \quad K(\omega) = [f^2, fg, g^2]_{\mathbb{C}}.$$

We denote by $(g\partial f - f\partial g)$ the ideal in $\mathcal{O}_n$ generated by $g\frac{\partial f}{\partial z_i} - f\frac{\partial g}{\partial z_i}$, $i = 1, 2, \dots, n$. Thus $J(\omega) = (g\partial f - f\partial g)$. The following is a direct consequence of the versality theorems.

(5.5) Theorem. Let F be a codim 1 foliation germ generated by a 1-form ω of the form $\omega = gdf - fdg$, $f, g \in \mathcal{O}_n$. If the vector space $(f, g)/(g\partial f - f\partial g)$ or $(f, g)/(g\partial f - f\partial g) + [f^2, fg, g^2]_{\mathbb{C}}$ is finite dimensional, then F admits a versal or an RL-versal unfolding, respectively. In fact, if $[u_1g - v_1f], [u_2g - v_2f], \dots, [u_mg - v_mf]$, $u_i, v_i \in \mathcal{O}_n$, is a $\mathbb{C}$ -basis of either of the above vector space, then the unfolding $\mathcal{F} = (\tilde{\omega})$ of F with parameter space $\mathbb{C}^m = \{(t_1, t_2, \dots, t_m)\}$ generated by

$$\tilde{\omega} = \tilde{g}d\tilde{f} - \tilde{f}d\tilde{g}$$

is universal or RL-universal, respectively, where $\tilde{f}$ and $\tilde{g}$ are germs in $\mathcal{O}_{n+m}$ given by

$$\tilde{f} = f + \sum_{j=1}^m u_j t_j, \quad \tilde{g} = g + \sum_{j=1}^m v_j t_j.$$

The proof for the following is similar to that of (5.3) Corollary.

(5.6) Corollary. Let $F = (\omega)$ be as in the above theorem. Every unfolding $\mathcal{F}$ of F admits a generator $\tilde{\omega}$ of the form

$$\tilde{\omega} = \tilde{g}d\tilde{f} - \tilde{f}d\tilde{g},$$

where $\tilde{f}$ and $\tilde{g}$ are unfoldings of f and g , respectively.

In the following, we show that the unfolding theory for foliations F generated by forms of the form $gdf - fdg$ is equivalent to the unfolding theory for the meromorphic function f/g .

Let μ be a meromorphic function germ at 0 in $\mathbb{C}^n$. We write $\mu = f/g$ with f and g relatively prime germs in $\mathcal{O}_n$. We set $\omega = gdf - fdg$ and define the critical set $C(\mu)$ of μ by $C(\mu) = S(\omega)$. An unfolding of μ is a germ $\tilde{\mu}$ of meromorphic function at 0 in $\mathbb{C}^n \times \mathbb{C}^m = \{(z, t)\}$ which can be written as $\tilde{\mu} = \tilde{f}/\tilde{g}$ for some unfoldings $\tilde{f}$ and $\tilde{g}$ of f and g , respectively, in $\mathcal{O}_{n+m}$. We call $\mathbb{C}^m$ the parameter space of $\tilde{\mu}$.

(5.7) Definition. Let $\tilde{\mu} = \tilde{f}/\tilde{g}$ and $\tilde{\mu}_1 = \tilde{f}_1/\tilde{g}_1$ be unfoldings of $\mu = f/g$ with parameter spaces $\mathbb{C}^m$ and $\mathbb{C}^l$, respectively. A (strict right) morphism from $\tilde{\mu}_1$ to $\tilde{\mu}$ is a pair (Φ, ψ) satisfying (a) and (b) in (1.2) Definition and $\tilde{\mu}_1 = \Phi^*\tilde{\mu}$, i.e., $\tilde{f}_1/\tilde{g}_1 = \Phi^*\tilde{f}/\Phi^*\tilde{g}$.

If $\tilde{\mu} = \tilde{f}/\tilde{g}$ is an unfolding of $\mu = f/g$, then $\mathcal{F} = (\tilde{\omega})$, $\tilde{\omega} = \tilde{g}d\tilde{f} - \tilde{f}d\tilde{g}$ is an unfolding of $F = (\omega)$, $\omega = gdf - fdg$ with the same parameter space as that of $\tilde{\mu}$. Conversely, we have seen that every unfolding of $F = (gdf - fdg)$ admits a generator of the form $\tilde{g}d\tilde{f} - \tilde{f}d\tilde{g}$ with $\tilde{f}$ and $\tilde{g}$ unfoldings of f and g , respectively. In what follows we see that the unfolding theory for the foliation $(gdf - fdg)$ with respect to morphisms is equivalent to the unfolding theory for the meromorphic function germ f/g with respect to (strict right) morphisms. More precisely we have

(5.8) Lemma. *Let $\tilde{\mu} = \tilde{f}/\tilde{g}$ and $\tilde{\mu}_1 = \tilde{f}_1/\tilde{g}_1$ be unfoldings of $\mu = f/g$ with parameter spaces $\mathbb{C}^m$ and $\mathbb{C}^l$, respectively. A pair (Φ, ψ) is a morphism from $\tilde{\mu}_1$ to $\tilde{\mu}$ if and only if it is a morphism from $\mathcal{F}_1 = (\tilde{\omega}_1)$ to $\mathcal{F} = (\tilde{\omega})$, where $\tilde{\omega}_1 = \tilde{g}_1d\tilde{f}_1 - \tilde{f}_1d\tilde{g}_1$ and $\tilde{\omega} = \tilde{g}d\tilde{f} - \tilde{f}d\tilde{g}$.*

For the proof we refer to [29](3.3)Lemma.

(5.9) Definition. Let $\tilde{\mu} = \tilde{f}/\tilde{g}$ be an unfolding of $\mu = f/g$ with parameter space $\mathbb{C}^m$. We set

$$u_k(z) = \frac{\partial \tilde{f}}{\partial t_k}(z, 0) \quad \text{and} \quad v_k(z) = -\frac{\partial \tilde{g}}{\partial t_k}(z, 0), \quad k = 1, 2, \dots, m.$$

We say that $\tilde{\mu}$ is infinitesimally (strictly right) versal if the classes $[u_1g - v_1f]$, $[u_2g - v_2f]$, $\dots$, $[u_mg - v_mf]$ span the vector space $(f, g)/(g\partial f - f\partial g)$.

(5.10) Definition. We say that an unfolding $\tilde{\mu}$ of μ is (strictly right) versal if for any unfolding $\tilde{\mu}_1$ of μ , there is a (strict right) morphism from $\tilde{\mu}_1$ to $\tilde{\mu}$.

The following is a direct consequence of the versality theorem and the above consideration.

(5.11) Theorem. *If $\tilde{\mu}$ is an infinitesimally strictly right versal unfolding of μ , then it is strictly right versal.*

(5.12) Example. Let f and g be defined by

$$f(x, y) = x - y \quad \text{and} \quad g(x, y) = xy$$

on $\mathbb{C}^2 = \{(x, y)\}$ and set $\omega = gdf - fdg = y^2dx - x^2dy$. We have $I(\omega) = (x - y, xy)$, $J(\omega) = (x^2, y^2)$ and $K(\omega) = [(x - y)^2, (x - y)xy, x^2y^2]_{\mathbb{C}}$. Hence we see that $\mathcal{O}_2/I(\omega) = \mathbb{C}^2$ and we may choose $[1]$ and $[x]$ as its basis, $\mathcal{O}_2/J(\omega) = \mathbb{C}^4$ and we may choose $[1]$, $[x]$, $[x - y]$ and $[xy]$ as its basis and that $\mathcal{O}_2/(J(\omega) + K(\omega)) = \mathbb{C}^3$ and we may choose $[1]$, $[x]$ and $[x - y]$ as its basis. From the exact sequences

$$0 \rightarrow I(\omega)/J(\omega) \rightarrow \mathcal{O}_2/J(\omega) \rightarrow \mathcal{O}_2/I(\omega) \rightarrow 0 \quad \text{and}$$

$$0 \rightarrow I(\omega)/(J(\omega) + K(\omega)) \rightarrow \mathcal{O}_2/(J(\omega) + K(\omega)) \rightarrow \mathcal{O}_2/I(\omega) \rightarrow 0,$$

we see that $I(\omega)/J(\omega) = \mathbb{C}^2$ and we may choose $[x - y] = [0 \cdot g - (-1) \cdot f]$ and $[xy] = [1 \cdot g - 0 \cdot f]$ as its basis and that $I(\omega)/(J(\omega) + K(\omega)) = \mathbb{C}$ and we may choose $[x - y] = [0 \cdot g - (-1) \cdot f]$ as its basis. Thus if we set $\tilde{f} = f + t = x - y + t$ and $\tilde{g} = g - s = xy - s$, then, by (5.5) Theorem, the unfolding $\mathcal{F} = (\tilde{\omega})$ with parameter space $\mathbb{C}^2 = \{(t, s)\}$ generated by

$$\tilde{\omega} = \tilde{g}d\tilde{f} - \tilde{f}d\tilde{g} = (y^2 - ty - s)dx - (x^2 + tx - s)dy + (x - y + t)ds + (xy - s)dt$$

is a universal unfolding of $F = (\omega)$. Thus $(x - y + t)/(xy - s)$ is a universal unfolding of $(x - y)/xy$ ((5.11) Theorem). Also, if we set $\tilde{f}' = f = x - y$ and $\tilde{g}' = \tilde{g} = xy - s$, then the unfolding $\mathcal{F}' = (\tilde{\omega}')$ with parameter space $\mathbb{C} = \{s\}$ generated by

$$\tilde{\omega}' = \tilde{g}'d\tilde{f}' - \tilde{f}'d\tilde{g}' = (y^2 - s)dx - (x^2 - s)dy + (x - y)ds$$

is an RL-universal unfolding of $F = (\omega)$.

(III) The case $F = (\omega)$, $\omega = f_1 f_2 \cdots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}$, $f_i \in \mathcal{O}_n$ and $\lambda_i \in \mathbb{C}$. We assume that $p \geq 2$, $\lambda_i \neq \lambda_j \neq 0$, if $i \neq j$, the f_i 's are non-units and that $\text{codim } S(\omega) \geq 2$. The latter condition implies that each f_i is reduced and that f_i and f_j are relatively prime if $i \neq j$. We set $F_i = f_1 \cdots \hat{f}_i \cdots f_p$ and denote by $(F_1, F_2, \dots, F_p)$ the ideal in $\mathcal{O}_n$ generated by these germs. We denote by $\text{ht } I$ the height of an ideal I in $\mathcal{O}_n$. The following is proved in [30].

(5.13) **Lemma.** Suppose that (a) $\text{ht}(f_i, f_j, f_k) = 3$, if i, j and k are distinct and that (b) $df_1 \wedge \cdots \wedge df_p \neq 0$. Then we have

$$I(\omega) = (F_1, F_2, \dots, F_p).$$

(5.14) *Remark.* In the case $p = 2$, we may think that the condition (a) above is automatically satisfied. Moreover, if $df_1 \wedge df_2 = 0$, then both $I(\omega)$ and (F_1, F_2) are equal to $\mathcal{O}_n$ (see the proof of [29](2.1) Lemma), which implies that either f_1 or f_2 is a unit and F is of type (I). Thus the condition (b) is redundant in this case.

We denote by $(\sum_{i=1}^p \lambda_i F_i \partial f_i)$ the ideal in $\mathcal{O}_n$ generated by $\sum_{i=1}^p \lambda_i F_i \frac{\partial f_i}{\partial z_1}, \dots, \sum_{i=1}^p \lambda_i F_i \frac{\partial f_i}{\partial z_n}$. Thus $J(\omega) = (\sum_{i=1}^p \lambda_i F_i \partial f_i)$. The following is a direct consequence of the versality theorem.

(5.15) **Theorem.** Let F be a codim 1 foliation germ generated by a 1-form ω of the form $\omega = f_1 f_2 \cdots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}$, $f_i \in \mathcal{O}_n$ and $\lambda_i \in \mathbb{C}$. Suppose that $\lambda_i \neq \lambda_j \neq 0$, if $i \neq j$, the f_i 's are non-units, $\text{codim } S(\omega) \geq 2$, $\text{ht}(f_i, f_j, f_k) = 3$ if i, j, k are distinct and that $df_1 \wedge \cdots \wedge df_p \neq 0$. If the vector space

$$(F_1, F_2, \dots, F_p) / \left(\sum_{i=1}^p \lambda_i F_i \partial f_i \right)$$

is finite dimensional, then F admits a versal unfolding. In fact, if

$$[\sum_{i=1}^p \lambda_i u_i^{(1)} F_i], \dots, [\sum_{i=1}^p \lambda_i u_i^{(m)} F_i],$$

$\sum_{i=1}^p \lambda_i u_i^{(j)} F_i \in \mathcal{O}_n$, is a $\mathbb{C}$ -basis of the above vector space, then the unfolding $\mathcal{F} = (\tilde{\omega})$ of F with parameter space $\mathbb{C}^m = \{(t_1, t_2, \dots, t_m)\}$ generated by

$$\tilde{\omega} = \tilde{f}_1 \tilde{f}_2 \cdots \tilde{f}_p \sum_{i=1}^p \lambda_i \frac{d\tilde{f}_i}{\tilde{f}_i}$$

is universal, where $\tilde{f}_i$ are germs in $\mathcal{O}_{n+m}$ given by

$$\tilde{f}_i = f_i + \sum_{j=1}^m u_i^{(j)} t_j.$$

(5.16) *Remark.* In the case $p = 2$, the condition $df_1 \wedge df_2 \neq 0$ is redundant and this case is basically the same as the case (II).

The proof for the following is similar to that of (5.3) Corollary.

(5.17) *Corollary.* Let $F = (\omega)$ be as in the above theorem. Every unfolding $\mathcal{F}$ of F admits a generator $\tilde{\omega}$ of the form

$$\tilde{\omega} = \tilde{f}_1 \tilde{f}_2 \cdots \tilde{f}_p \sum_{i=1}^p \lambda_i \frac{d\tilde{f}_i}{\tilde{f}_i},$$

where each $\tilde{f}_i$ is an unfolding of f_i .

Foliation germs of this type include the germs with holomorphic or meromorphic first integrals. For more details we refer to [30]. As an application of this, we may prove the stability of certain holomorphic or meromorphic first integrals.

In the following, we show that the unfolding theory for foliations F generated by forms of the form $f_1 f_2 \cdots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}$ is equivalent to the unfolding theory for the multiform function $f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_p^{\lambda_p}$.

A germ of multiform function at 0 in $\mathbb{C}^n$ is an expression $f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_p^{\lambda_p}$ for some germs f_i in $\mathcal{O}_n$ and non-zero complex numbers λ_i . Let $f = f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_p^{\lambda_p}$ be a multiform function. By regrouping the factors of the f_i 's, if necessary, we may always assume that $\lambda_i \neq \lambda_j \neq 0$, if $i \neq j$, each f_i is reduced and that f_i and f_j are relatively prime, if $i \neq j$. Then the expression $f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_p^{\lambda_p}$ is uniquely determined up to the order of the f_i 's and multiplication by units in $\mathcal{O}_n$. The critical set $C(f)$ of $f = f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_p^{\lambda_p}$ is defined to be the singular set $S(\omega)$ of the 1-form $\omega = f_1 f_2 \cdots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}$. Here we only consider multiform functions f with $\text{codim } C(f) \geq 2$. An unfolding of $f = f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_p^{\lambda_p}$ is a germ $\tilde{f}$ of multiform function at 0 in $\mathbb{C}^n \times \mathbb{C}^m = \{(z, t)\}$ which can be written as $\tilde{f} = \tilde{f}_1^{\lambda_1} \tilde{f}_2^{\lambda_2} \cdots \tilde{f}_p^{\lambda_p}$ with $\tilde{f}_i$ an unfolding of f_i in $\mathcal{O}_{n+m}$. We call $\mathbb{C}^m$ the parameter space of $\tilde{f}$.

(5.18) Definition. Let $\tilde{f} = \tilde{f}_1^{\lambda_1} \tilde{f}_2^{\lambda_2} \dots \tilde{f}_p^{\lambda_p}$ and $g = g_1^{\lambda_1} g_2^{\lambda_2} \dots g_p^{\lambda_p}$ be unfoldings of $f = f_1^{\lambda_1} f_2^{\lambda_2} \dots f_p^{\lambda_p}$ with parameter spaces $\mathbb{C}^m$ and $\mathbb{C}^l$, respectively. A (strict right) morphism from g to $\tilde{f}$ is a pair (Φ, ψ) satisfying (a) and (b) in (1.2) Definition and $g = \Phi^* \tilde{f}$, i.e., $g_1^{\lambda_1} g_2^{\lambda_2} \dots g_p^{\lambda_p} = (\Phi^* \tilde{f}_1)^{\lambda_1} (\Phi^* \tilde{f}_2)^{\lambda_2} \dots (\Phi^* \tilde{f}_p)^{\lambda_p}$.

If $\tilde{f} = \tilde{f}_1^{\lambda_1} \tilde{f}_2^{\lambda_2} \dots \tilde{f}_p^{\lambda_p}$ is an unfolding of $f = f_1^{\lambda_1} f_2^{\lambda_2} \dots f_p^{\lambda_p}$, then $\mathcal{F} = (\tilde{\omega})$, $\tilde{\omega} = \tilde{f}_1 \tilde{f}_2 \dots \tilde{f}_p \sum_{i=1}^p \lambda_i \frac{d\tilde{f}_i}{\tilde{f}_i}$ is an unfolding of $F = (\omega)$, $\omega = f_1 f_2 \dots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}$ with the same parameter space as that of $\tilde{f}$. Conversely, we have seen that every unfolding of $F = (f_1 f_2 \dots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i})$ admits a generator of the form $\tilde{f}_1 \tilde{f}_2 \dots \tilde{f}_p \sum_{i=1}^p \lambda_i \frac{d\tilde{f}_i}{\tilde{f}_i}$ with $\tilde{f}_i$ an unfolding of f_i . In what follows we see that the unfolding theory for the foliation $(f_1 f_2 \dots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i})$ with respect to morphisms is equivalent to the unfolding theory for the multiform function germ $f_1^{\lambda_1} f_2^{\lambda_2} \dots f_p^{\lambda_p}$ with respect to (strict right) morphisms. More precisely we have

(5.19) Lemma. Let $\tilde{f} = \tilde{f}_1^{\lambda_1} \tilde{f}_2^{\lambda_2} \dots \tilde{f}_p^{\lambda_p}$ and $g = g_1^{\lambda_1} g_2^{\lambda_2} \dots g_p^{\lambda_p}$ be unfoldings of $f = f_1^{\lambda_1} f_2^{\lambda_2} \dots f_p^{\lambda_p}$ with parameter spaces $\mathbb{C}^m$ and $\mathbb{C}^l$, respectively. A pair (Φ, ψ) is a morphism from g to $\tilde{f}$ if and only if it is a morphism from $\mathcal{F}' = (\theta)$ to $\mathcal{F} = (\tilde{\omega})$, where $\theta = g_1 g_2 \dots g_p \sum_{i=1}^p \lambda_i \frac{dg_i}{g_i}$ and $\tilde{\omega} = \tilde{f}_1 \tilde{f}_2 \dots \tilde{f}_p \sum_{i=1}^p \lambda_i \frac{d\tilde{f}_i}{\tilde{f}_i}$.

For the proof we refer to [30](2.4)Lemma.

(5.20) Definition. Let $\tilde{f} = \tilde{f}_1^{\lambda_1} \tilde{f}_2^{\lambda_2} \dots \tilde{f}_p^{\lambda_p}$ be an unfolding of $f = f_1^{\lambda_1} f_2^{\lambda_2} \dots f_p^{\lambda_p}$ with parameter space $\mathbb{C}^m$. We set, for each $i = 1, 2, \dots, p$ and $k = 1, 2, \dots, m$,

$$u_i^{(k)}(z) = \frac{\partial \tilde{f}_i}{\partial t_k}(z, 0).$$

We say that $\tilde{f}$ is infinitesimally (strictly right) versal if the classes

$$[\sum_{i=1}^p \lambda_i u_i^{(1)} F_i], \dots, [\sum_{i=1}^p \lambda_i u_i^{(m)} F_i]$$

span the vector space $(F_1, F_2, \dots, F_p) / (\sum_{i=1}^p \lambda_i F_i \partial f_i)$.

(5.21) Definition. We say that an unfolding $\tilde{f}$ of f is (strictly right) versal if for any unfolding g of f , there is a (strict right) morphism from g to $\tilde{f}$.

The following is a direct consequence of the versality theorem and the above consideration.

(5.22) Theorem. If $\tilde{f}$ is an infinitesimally strictly right versal unfolding of f , then it is strictly right versal.

We finish this section by examining some examples.

(5.23) **Example.** This is the example we considered in section 3. Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^2 = \{(x, y)\}$ generated by

$$\omega = (y^2 + 2xy)dx - (x^2 + 2xy)dy.$$

We construct a universal unfolding of F in three ways.

(I) By computation, we have $I(\omega) = (xy, x - y)$. In fact, we have $3xyd\omega = 4d(xy) \wedge \omega$ and $(x - y)d\omega = 4d(x - y) \wedge \omega$. In what follows we take $3xy$ instead of xy as one of the generators and we set $h_1 = 3xy$, $h_2 = x - y$, $\eta_1 = 4d(xy)$ and $\eta_2 = 4d(x - y)$. As in (5.12) Example, we see that $I(\omega)/J(\omega) = \mathbb{C}^2$ and we may choose $[h_1]$ and $[h_2]$ as its basis. We may construct a first order unfolding $\tilde{\omega}_1$ of ω corresponding to $[h_1]$ by setting $\tilde{\omega}_1 = \omega + \omega_1^{(1)}t + h_1dt$, $\omega_1^{(1)} = dh_1 - \eta_1 = -d(xy)$. Thus

$$\tilde{\omega}_1 = (y^2 + 2xy - yt)dx - (x^2 + 2xy + xt)dy + 3xydt.$$

We see that this form is already integrable (without adding higher order terms) and is the unfolding of ω described in section 3. An unfolding $\tilde{\omega}_2$ corresponding to $[h_2]$ can also be constructed similarly and it is given by

$$\tilde{\omega}_2 = (y^2 + 2xy - 3s)dx - (x^2 + 2xy - 3s)dy + (x - y)ds.$$

If we combine these two unfoldings, then we obtain the unfolding $\tilde{\omega}$ described in section 3. By the versality theorem, $\mathcal{F} = (\tilde{\omega})$ is a universal unfolding of $F = (\omega)$.

(II) We start with the form $\tilde{\omega}_1$ constructed in (I). By computation, we have $I(\tilde{\omega}_1) = (xy, x - y + t)$. In fact, we have

$$3xyd\tilde{\omega}_1 = 4d(xy) \wedge \tilde{\omega}_1 \quad \text{and} \quad (x - y + t)d\tilde{\omega}_1 = 4d(x - y + t) \wedge \tilde{\omega}_1.$$

Since $J(\tilde{\omega}_1) = (y^2 + 2xy - yt, x^2 + 2xy + xt, xy) = (y^2 - yt, x^2 + xt, xy)$, we see that $I(\tilde{\omega}_1)/J(\tilde{\omega}_1) = \mathbb{C}$ and we may take $[x - y + t]$ as its basis. We set $h = x - y + t$ and $\eta = 4d(x - y + t)$. If we construct a first order unfolding $\tilde{\omega}$ of $\tilde{\omega}_1$ corresponding to $[h]$ by setting $\tilde{\omega} = \tilde{\omega}_1 + \omega^{(1)}s + hds$, $\omega^{(1)} = dh - \eta = -3d(x - y + t)$, then $\tilde{\omega}$ turns out to be integrable and coincides with the form in (I). We note that $I(\tilde{\omega}_2)/J(\tilde{\omega}_2) = 0$, since $d\tilde{\omega}_2(0) \neq 0$ and the form $\tilde{\omega}_2$ (as well as $\tilde{\omega}$) is "rigid" (see (6.6) Theorem).

(III) If we know that ω admits a meromorphic first integral, then we may apply (5.15) Theorem to construct a universal unfolding. In our case, if we set $f(x, y) = x - y$ and $g(x, y) = xy$, then ω admits f/g as a meromorphic first integral, i.e., $\omega \wedge d(f/g) = 0$ and we may write ([30] p.111)

$$\omega = f_1 f_2 \sum_{i=1}^2 \lambda_i \frac{df_i}{f_i}$$

with $f_1 = x - y$, $f_2 = xy$, $\lambda_1 = 3$ and $\lambda_2 = -1$. We have $I(\omega) = (F_1, F_2)$, $F_1 = f_2$, $F_2 = f_1$ ((5.13) Lemma). In order to apply (5.15) Theorem, we set $u_1^{(1)} = 1$, $u_2^{(1)} = 0$, $u_1^{(2)} = 0$, $u_2^{(2)} = -1$. Then $\sum_{i=1}^2 \lambda_i u_i^{(1)} F_i = 3xy$ and $\sum_{i=1}^2 \lambda_i u_i^{(2)} F_i = x - y$ and the classes of these functions form a basis of $I(\omega)/J(\omega)$. Thus the unfolding $\mathcal{F}$ generated by

$$\tilde{\omega} = \tilde{f}_1 \tilde{f}_2 \sum_{i=1}^2 \lambda_i \frac{d\tilde{f}_i}{\tilde{f}_i}$$

with $\tilde{f}_1 = f_1 + t = x - y + t$ and $\tilde{f}_2 = f_2 - s = xy - s$ is a universal unfolding of F . This is identical with the one constructed previously. Note that $\tilde{f}_1/\tilde{f}_2$ is a meromorphic first integral of $\tilde{\omega}$ and thus the meromorphic first integral f_1/f_2 of ω is "stable".

(5.24) Example. Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^2 = \{(x, y)\}$ generated by

$$\omega = (2y^2 + x^3)dx - 2xydy,$$

which is a form studied by D. Cerveau, J.-F. Mattei and R. Moussu ([5], [12]). If we set $f(x, y) = x^2$ and $g(x, y) = y^2 - x^3$, then ω admits f/g as a meromorphic first integral, i.e., $\omega \wedge d(f/g) = 0$ and we may write ([30] p.111)

$$\omega = f_1 f_2 \sum_{i=1}^2 \lambda_i \frac{df_i}{f_i}$$

with $f_1 = x$, $f_2 = y^2 - x^3$, $\lambda_1 = 2$ and $\lambda_2 = -1$. By (5.13) Lemma, we have $I(\omega) = (F_1, F_2) = (f_2, f_1) = (x, y^2)$. Thus $\mathcal{O}_2/I(\omega) = \mathbb{C}^2$ and we may choose $[1]$ and $[y]$ as its basis. On the other hand, we have $\mathcal{O}_2/J(\omega) = \mathbb{C}^5$ and we may choose $[1]$, $[x]$, $[y]$, $[x^2]$ and $[x^3]$ as its basis. Hence $I(\omega)/J(\omega) = \mathbb{C}^3$ and we may choose $[x]$, $[x^2]$ and $[x^3]$ as its basis. If we set $u_1^{(1)} = 0$, $u_2^{(1)} = -1$, $u_1^{(2)} = 0$, $u_2^{(2)} = -x$, $u_1^{(3)} = 0$, $u_2^{(3)} = -x^2$, then $\sum_{i=1}^2 \lambda_i u_i^{(1)} F_i = x$, $\sum_{i=1}^2 \lambda_i u_i^{(2)} F_i = x^2$ and $\sum_{i=1}^2 \lambda_i u_i^{(3)} F_i = x^3$. Therefore, by (5.15) Theorem, the unfolding $\mathcal{F}$ with parameter space $\mathbb{C}^3 = \{(t_1, t_2, t_3)\}$ generated by

$$\tilde{\omega} = \tilde{f}_1 \tilde{f}_2 \sum_{i=1}^2 \lambda_i \frac{d\tilde{f}_i}{\tilde{f}_i}$$

with $\tilde{f}_1 = f_1 = x$ and $\tilde{f}_2 = f_2 - t_1 - xt_2 - x^2 t_3 = y^2 - x^3 - t_1 - xt_2 - x^2 t_3$ is a universal unfolding of $F = (\omega)$. The form $\tilde{\omega}$ is explicitly given by

$$\tilde{\omega} = (2y^2 + x^3 - 2t_1 - xt_2)dx - 2xydy + xdt_1 + x^2 dt_2 + x^3 dt_3.$$

Note that $\tilde{f}_1^2/\tilde{f}_2$ is a meromorphic first integral of $\tilde{\omega}$ and thus the meromorphic first integral $f/g = f_1^2/f_2$ of ω is "stable". Also, an RL-universal unfolding of F is

given as follows. First by computation, we have $K(\omega) = \mathbb{C}x^3$ and $J(\omega) \cap K(\omega) = 0$. Hence we have $I(\omega)/(J(\omega) + K(\omega)) = \mathbb{C}^2$ and we may choose $[x]$ and $[x^2]$ as its basis. Thus, by (4.20) Theorem, the unfolding $\mathcal{F}'$ with parameter space $\mathbb{C}^2 = \{(t_1, t_2)\}$ generated by

$$\tilde{\omega}' = (2y^2 + x^3 - 2t_1 - xt_2)dx - 2xydy + xdt_1 + x^2dt_2$$

is an RL-universal unfolding of F (Cf. (7.12) Example).

6. Rigidity of some foliation germs

It is known that certain foliation germs are rigid, i.e., any of their unfoldings is trivial. We give systematic proofs for these facts as another application of the versality theorem.

(6.1) Definition. Let F be a codim 1 foliation germ at 0 in $\mathbb{C}^n$. An unfolding $\mathcal{F}$ of F is trivial if there is a morphism from $\mathcal{F}$ to F .

Thus if we let ω be a generator of F and $\mathbb{C}^m$ the parameter space of $\mathcal{F}$, then $\mathcal{F}$ is trivial if and only if there is a germ $\Phi : (\mathbb{C}^n \times \mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0)$ of submersion such that $\mathcal{F}$ is generated by $\Phi^*\omega$. We may also define the RL-triviality (and the RL-rigidity) by replacing "a morphism" in (6.1) Definition by "an RL-morphism". The following is a direct consequence of the versality theorem.

(6.2) Theorem. Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^n$. If $I(\omega)/J(\omega) = 0$, then every unfolding of F is trivial, i.e., F is rigid.

Proof. If $I(\omega)/J(\omega) = 0$, then, by (4.19) Theorem, F itself is a versal unfolding of F . $\square$

Thus we may prove the rigidity of some foliation germs simply by showing that $I(\omega)/J(\omega) = 0$. We give such examples subsequently.

(6.3) Theorem (Cerveau - Lins Neto [4]). Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^n$ generated by

$$\omega = z_1 z_2 \cdots z_n \sum_{i=1}^n \lambda_i \frac{dz_i}{z_i}$$

with λ_i mutually distinct non-zero complex numbers. Then F is rigid.

Proof. In this case it is not difficult to see that $I(\omega) = J(\omega) = (z_1 \cdots \hat{z}_i \cdots z_n)$ (see [31]). $\square$

(6.4) Remark. In [4], a stronger result is proved, namely the rigidity is proved for a form ω' whose $(n-1)$ -jet (n -jet in our definition, see section 8) is equal to ω above. It is also proved that, under some conditions, such a form ω' is transformed to ω by some local change of coordinates of $(\mathbb{C}^n, 0)$.

For a germ ω in Ω_n , we denote by $S(d\omega)$ (the germ at 0 of) the set $\{z \mid d\omega(z) = 0\}$. If $S(d\omega)$ is "sufficiently small", then the foliation germ $F = (\omega)$ tends to be rigid as is shown in the following

(6.5) Theorem (Camacho - Lins Neto [1]). *Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^3$ generated by ω with $S(d\omega) \subset \{0\}$. Then F is rigid.*

Proof. In this case we may also show that $I(\omega) = J(\omega)$. See [8] for details. $\square$

In general, we have

(6.6) Theorem. *Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^n$ generated by ω with $S(d\omega) = \phi$, i.e. $d\omega(0) \neq 0$. Then F is rigid.*

Proof. The condition $d\omega(0) \neq 0$ implies that at least one of the components of $d\omega$ must be a unit in $\mathcal{O}_n$. Hence if h is in $I(\omega)$, then it is also in $J(\omega)$. $\square$

(6.7) Remark. It is known that a germ of 1-form ω satisfying the condition in the above theorem is obtained as the pull-back of a germ of 1-form on $\mathbb{C}^2$ by a submersion, i.e., ω involves essentially only two variables (Darboux). The above theorem is well-known (Kupka-Reeb phenomena, see [5] p.31). Lemma 5.6 of Gómez-Mont [10] is a form of this.

As a special case of (6.6) Theorem, we discuss the rigidity of "simple" foliation singularities. We say that a codim 1 foliation germ F at 0 in $\mathbb{C}^2$ has a simple singularity at 0 if F is generated by a germ ω which can be written as, by a suitable choice of coordinates (x, y) of $\mathbb{C}^2$ near 0,

$$\omega = adx + bdy$$

with a and b germs in $\mathcal{O}_2$ satisfying

$$(6.8) \quad j^1 a = \lambda y \quad \text{and} \quad j^1 b = x$$

for a complex number λ which is not a negative rational number. It is known that an arbitrary foliation singularity in $\mathbb{C}^2$ is reduced to simple singularities after a finite number of successive blowing ups ([26], [35], see also [22] Appendice I). The following is a direct consequence of (6.6) Theorem.

(6.9) Theorem. *Let F be a codim 1 foliation germ with a simple singularity at 0 in $\mathbb{C}^2$. If the number λ in (6.8) is different from 1, then F is rigid.*

Proof. We have $d\omega(0) = (1 - \lambda)dx \wedge dy$. $\square$

Next, we examine the case $\lambda = 1$.

(6.10) **Lemma.** Let $\omega = adx + bdy$ be a germ of 1-form at 0 in $\mathbb{C}^2 = \{(x, y)\}$ such that $j^1a = y$ and $j^1b = x$. Then we have $I(\omega) = \mathcal{O}_2$ and $J(\omega) = \mathfrak{m}$, thus $I(\omega)/J(\omega) \simeq \mathbb{C}$.

Proof. We may write $a = y(1 + a_1) + xa_2$ and $b = x(1 + b_1) + yb_2$ with a_1, a_2, b_1 and b_2 in $\mathfrak{m}$. Hence we see that $J(\omega) = (a, b) = \mathfrak{m}$. On the other hand, we have $d\omega = (\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y})dx \wedge dy$, where $\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} = \frac{\partial(xb_1 + yb_2)}{\partial x} - \frac{\partial(ya_1 + xa_2)}{\partial y}$ is in $\mathfrak{m} = J(\omega)$. Thus $I(\omega) = \mathcal{O}_2$. $\square$

(6.11) **Theorem.** Let F be a codim 1 foliation germ at 0 in $\mathbb{C}^2$ generated by a germ ω as in (6.10) Lemma. If F admits an unfolding corresponding to a non-zero element in $I(\omega)/J(\omega) \simeq \mathbb{C}$, then after a suitable change of coordinates (x, y) of $\mathbb{C}^2$ near 0, F is generated by $d(xy)$.

Proof. The condition implies that F admits an unfolding generated by a form without singularity. Thus, as in the proof of the Frobenius with singularity, there exist germs f and u in $\mathcal{O}_2$ with u a unit such that $\omega = udf$. Thus we may choose df as a generator of F . Moreover, since $(\partial f) = J(\omega) = \mathfrak{m}$, f is 2-determined ((1.7) Theorem). We may assume that $f(0) = 0$. Moreover, from $\omega = udf$, we see that $j^2f = cxy$, $c \neq 0$. Hence by a suitable change of coordinates (x, y) of $\mathbb{C}^2$ near 0, we may write $f(x, y) = xy$. $\square$

(6.12) *Remarks.* 1°. The unfolding generated by $d(xy + t)$ is a universal unfolding of F as in the above theorem. By (6.9) Theorem and the following (6.13) Theorem, this is the only case a simple singularity in $\mathbb{C}^2$ can have nontrivial unfoldings.

2°. The unfolding $(d(xy + t))$ of $(d(xy))$ is RL-trivial. Thus every simple singularity is RL-rigid.

The following is essentially in Cerveau-Mattei [5] Ch.II, Théorème 4.2.

(6.13) **Theorem.** Let F be a codim 1 foliation germ at 0 in $\mathbb{C}^2$ generated by a germ ω as in (6.10) Lemma. If F does not admit unfoldings corresponding to any non-zero element in $I(\omega)/J(\omega) \simeq \mathbb{C}$, then F is rigid.

Proof. Let $\mathcal{F} = (\tilde{\omega})$ be an unfolding of $F = (\omega)$. If $\text{codim } S(\tilde{\omega}) \geq 3$, then $\mathcal{F}$ would be generated by an exact form. Hence F would also be generated by an exact form, which would contradict the assumption. Thus $\text{codim } S(\tilde{\omega}) = 2$ and as in the proof of [5] Ch.II, Théorème 4.2, $\mathcal{F}$ is trivial. $\square$

(6.14) *Remarks.* 1°. The cases of (6.11) and (6.13) are divided according as F admits an exact form as a generator or not.

2°. In the situation of (6.13) Theorem, we have the "obstructed" case, i.e., for some elements in $I(\omega)/J(\omega)$, there are no corresponding unfoldings. The foliation germ on $\mathbb{C}^2$ generated by $\omega = ydx + (x + x^2y)dy$ is an example of this.

3°. In his theory of equisingular unfoldings [20], J.-F. Mattei also considers unfoldings of codim 1 foliation germs (in $\mathbb{C}^2$). His definition of an unfolding is stronger

than ours. Namely, if $\mathcal{F} = (\tilde{\omega})$ is an unfolding of $F = (\omega)$ with parameter space $\mathbb{C}^m$ in our sense, he requires also that the leaves of $\mathcal{F}$ are transverse to the projection $\mathbb{C}^n \times \mathbb{C}^m \rightarrow \mathbb{C}^m$. If we write $\tilde{\omega} = \sum_{i=1}^n \tilde{f}_i(z, t) dz_i + \sum_{j=1}^m h_j(z, t) dt_j$ with $\tilde{f}_i, h_j \in \mathcal{O}_{n+m}$, this transversality condition can be expressed as

$$h_j \in \sqrt{(\tilde{f}_1, \dots, \tilde{f}_n)}, \quad j = 1, \dots, m,$$

since $(\tilde{f}_1(z, t), \dots, \tilde{f}_n(z, t), h_1(z, t), \dots, h_m(z, t))$ is a vector normal to the leaf of $\mathcal{F}$ through the point (z, t) . Thus our unfolding $d(xy + t)$ of $d(xy)$ is excluded in his theory and he proves that every unfolding (in his sense) of an arbitrary simple singularity is trivial. His theory is developed based on this fact. See also [23] for related work.

4°. For the reduction of three dimensional foliation singularities, see Cano [2] and its list of references.

7. Deformations and unfoldings

In this section, we take up the deformation theory for complex analytic singular foliations in the codim 1 local case and discuss its relation with the unfolding theory. Note that in the function germ case, there is no essential difference between the two theories. We refer to Gómez-Mont [9] for a general theory of deformations of complex analytic singular foliations.

If we denote by $\mathcal{O}_{n+m}$ the ring of germs of holomorphic functions at 0 in $\mathbb{C}^n \times \mathbb{C}^m = \{(z, t)\}$, then $\mathcal{O}_{n+m} \otimes_{\mathcal{O}_n} \Omega_n$ is considered as a sub- $\mathcal{O}_{n+m}$ -module of Ω_{n+m} and its element $\tilde{\omega}$ is written as $\tilde{\omega} = \sum_{i=1}^n \tilde{f}_i dz_i$ with $\tilde{f}_i \in \mathcal{O}_{n+m}$. For t near 0 in $\mathbb{C}^m$, we denote by ι_t the embedding $\mathbb{C}^n \hookrightarrow \mathbb{C}^n \times \mathbb{C}^m$ given by $\iota_t(z) = (z, t)$. We set $\iota = \iota_0$.

(7.1) Definition. Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^n$. A complex analytic family of deformations (simply a deformation) of F is a rank 1 free submodule $\mathcal{F} = (\tilde{\omega})$ of $\mathcal{O}_{n+m} \otimes_{\mathcal{O}_n} \Omega_n$ with a generator $\tilde{\omega}$ such that $\iota^* \tilde{\omega} = \omega$ and that

$$d_z \tilde{\omega} \wedge \tilde{\omega} = 0.$$

We call $\mathbb{C}^m$ the parameter space of $\mathcal{F}$.

(7.2) Remark. Let $\mathcal{F} = (\tilde{\omega})$ be a deformation of $F = (\omega)$ with parameter space $\mathbb{C}^m = \{t\}$. If we consider $\omega_t = \iota_t^* \tilde{\omega}$ as a germ in Ω_n , for each t near 0, we have the integrability $d\omega_t \wedge \omega_t = 0$. Hence it defines a codim 1 foliation germ $F_t = (\omega_t)$ at 0 in $\mathbb{C}^n$.

(7.3) Definition. Let $\mathcal{F}$ and $\mathcal{F}'$ be two deformations of F with parameter spaces $\mathbb{C}^m$ and $\mathbb{C}^l = \{(s_1, s_2, \dots, s_l)\}$, respectively. A morphism from $\mathcal{F}'$ to $\mathcal{F}$ is a pair (Φ, ψ) satisfying the conditions (I)(a), (b) of (4.3) Definition and

- (d) For any generator $\tilde{\omega}$ of $\mathcal{F}$, $\Phi_z^* \tilde{\omega}$ generates $\mathcal{F}'$, where $\Phi_z^* \tilde{\omega}$ denotes the germ in $\mathcal{O}_{n+m} \otimes_{\mathcal{O}_n} \Omega_n$ obtained from $\Phi^* \tilde{\omega}$ by dropping the terms involving $ds_k, k = 1, 2, \dots, l$.

(7.4) *Remark.* In the above, we may replace "any" by "some".

As in the case of the unfolding theory, we first look at the "first order terms" in deformations of a given foliation germ and try to get some algebraic control of the set of deformations.

Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^n$.

(7.5) **Definition.** A first order (infinitesimal) deformation of F is a rank 1 free submodule $\mathcal{F}^{(1)} = (\tilde{\omega})$ of $\mathcal{O}_{n+m} \otimes_{\mathcal{O}_n} \Omega_n$ with a generator $\tilde{\omega}$ such that $\iota^* \tilde{\omega} = \omega$ and that

$$(7.6) \quad d_z \tilde{\omega} \wedge \tilde{\omega} \equiv 0 \quad \text{mod. } t^2.$$

Let $\mathcal{F}^{(1)} = (\tilde{\omega})$ be a first order deformation of $F = (\omega)$. If we write

$$\tilde{\omega} \equiv \omega + \omega^{(1)} t \quad \text{mod. } t^2$$

with $\omega^{(1)}$ in Ω_n , then (7.6) is equivalent to

$$d\omega \wedge \omega^{(1)} + d\omega^{(1)} \wedge \omega = 0.$$

If we set

$$\Omega'(\omega) = \{ \theta \in \Omega_n \mid d\omega \wedge \theta + d\theta \wedge \omega = 0 \},$$

then $\Omega'(\omega)$ is a $\mathbb{C}$ -vector space containing $\Omega(\omega)$ (section 4) as a subspace. Each first order deformation of $F = (\omega)$ determines an element $\omega^{(1)}$ in $\Omega'(\omega)$ uniquely up to elements in F . Conversely, each element in $\Omega'(\omega)$ defines a first order deformation of $F = (\omega)$. If ω' is another generator of F , then $\omega' = u\omega$ for some unit u in $\mathcal{O}_n$. The multiplication by u gives an isomorphism of $\Omega'(\omega)$ onto $\Omega'(\omega')$.

An isomorphism between two first order deformations is defined as in (4.12) Definition, except we replace the condition (c) by

$$(d) \quad u\theta \equiv \Phi_z^* \tilde{\omega} \quad \text{mod. } t^2.$$

We denote by $D(F)$ the sets of isomorphism classes of first order deformations of $F = (\omega)$. The following is proved as (4.14) Proposition.

(7.7) **Proposition.** The set $D(F)$ is naturally identified with the vector space $\Omega'(\omega)/(L(\omega) + F)$.

From Propositions (4.14) and (7.7), we see that there is an exact sequence (see also (4.11) Proposition)

$$(7.8) \quad 0 \rightarrow K(\omega)/J(\omega) \cap K(\omega) \rightarrow U(F) \rightarrow D(F)$$

for a codim 1 foliation $F = (\omega)$. If $\mathcal{F} = (\tilde{\omega})$ is an unfolding of $F = (\omega)$ with parameter space $\mathbb{C}^m = \{(t_1, t_2, \dots, t_m)\}$, then it naturally defines a deformation of F by taking the submodule of $\mathcal{O}_{n+m} \otimes_{\mathcal{O}_n} \Omega_n$ generated by the form obtained from $\tilde{\omega}$ by dropping the terms involving $dt_k, k = 1, \dots, m$. We denote it by $\mathcal{F}_d$. The last map in (7.8) gives this correspondence at the infinitesimal level. Note that its image is isomorphic to $U_{RL}(F)$.

(7.9) **Definition.** A deformation $\mathcal{F}$ of F is trivial if there is a morphism from $\mathcal{F}$ to F .

The following easily follows from the definitions.

(7.10) **Proposition.** Let $\mathcal{F}$ be an unfolding of F . Then $\mathcal{F}_d$ is a trivial deformation if and only if $\mathcal{F}$ is an RL-trivial unfolding.

As is suggested by the sequence (7.8), if ω admits an integrating factor which is not in $J(\omega)$, then F may admit a nontrivial unfolding which is trivial as a deformation.

(7.11) **Example.** Let F be a foliation germ at 0 in $\mathbb{C}^2 = \{(x, y)\}$ generated by $d(xy)$. Then the unfolding $\mathcal{F}$ of F generated by $d(xy + t)$ is nontrivial, while as a deformation it is trivial, i.e., $\mathcal{F}_d$ is trivial (Cf. Remarks (6.12) and (6.14) 3°).

(7.12) **Example.** Let

$$\omega = (2y^2 + x^3)dx - 2xydy$$

be the form on $\mathbb{C}^2$ considered in (5.24) Example. The unfolding $\mathcal{F}$ of $F = (\omega)$ with parameter space $\mathbb{C} = \{t\}$ generated by

$$\tilde{\omega} = (2y^2 + x^3)dx - 2xydy + x^3dt$$

is also an unfolding in the sense of Mattei [20], since $x^3 \in \sqrt{(2y^2 + x^3, xy)}$ ((6.14) 3°). Obviously $\mathcal{F}_d$ is trivial. However, since $x^3 \in K(\omega)$, $x^3 \notin J(\omega)$, this is a nontrivial unfolding, although it is an RL-trivial unfolding. The unfolding $\mathcal{F}$ is known as an example of unfolding which is equireducible but is not equisingular ([12] Ch.IV, [20] (1.1.7) Remarque).

8. Determinacy of foliation germs

In this section, we recall some results on the finite determinacy of codim 1 foliation germs. It is shown that if a foliation germ F is infinitesimally k -determined, then it is locally k -determined. The local determinacy is closely related to the unfolding theory and it is shown that for a foliation germ $F = (\omega)$ without formal integrating factors, the finite dimensionality of the space $I(\omega)/J(\omega)$ of isomorphism classes of first order unfoldings implies the local finite determinacy.

For a codim 1 foliation germ $F = (\omega)$, we further associate the following algebraic objects. For a positive integer k , we set

$$I^{(k)}(\omega) = \{ h \in \mathcal{O}_n \mid h d\omega + (\theta - dh) \wedge \omega = 0 \text{ for some } \theta \text{ in } \mathfrak{m}^{k-1} \Omega_n \},$$

where $\mathfrak{m}$ denotes the maximal ideal in $\mathcal{O}_n$. By setting $I^{(\infty)}(\omega) = K(\omega)$ we have a decreasing sequence of $\mathbb{C}$ -vector subspaces of $I(\omega)$:

$$I(\omega) = I^{(1)}(\omega) \supset I^{(2)}(\omega) \supset \dots \supset I^{(k)}(\omega) \supset \dots \supset I^{(\infty)}(\omega) = K(\omega).$$

If ω' is another generator of F , then $\omega' = u\omega$ for some unit u in $\mathcal{O}_n$. The correspondence $h \mapsto uh$ gives an isomorphism of $I^{(k)}(\omega)$ onto $I^{(k)}(\omega')$ for each $k = 1, 2, \dots, \infty$.

For a germ g in $\mathcal{O}_n$, we denote by $j^k g$ the k -jet of g and for a germ $\theta = \sum_{i=1}^n g_i dx_i$ in Ω_n , we define the k -jet $j^k \theta$ of θ by

$$j^k \theta = \sum_{i=1}^n j^{k-1} g_i \cdot dx_i.$$

(8.1) Definition. An unfolding $\mathcal{F}$ of a codim 1 foliation germ $F = (\omega)$ with parameter space $\mathbb{C}^m = \{t\}$ is k -trivial if it has a generator $\tilde{\omega}$ such that $\iota^* \tilde{\omega} = \omega$ and that $j^k \omega_t = j^k \omega$ for all t near 0 in $\mathbb{C}^m$.

We say that two germs ω and ω' in Ω_n are holomorphically equivalent and write $\omega \sim \omega'$ if there exist a germ at 0 of biholomorphic map φ of $\mathbb{C}^n$ into itself with $\varphi(0) = 0$ and a unit u in $\mathcal{O}_n$ such that $\omega' = u\varphi^* \omega$. If ω is integrable and if $\omega' \sim \omega$, ω' is also integrable and ω and ω' define isomorphic foliations.

(8.2) Definition. A codim 1 foliation germ $F = (\omega)$ is locally k -determined if for every k -trivial unfolding $\mathcal{F} = (\tilde{\omega})$ of F , we have $\omega_t \sim \omega$ for all t near 0. It is locally finitely determined if it is locally k -determined for some k .

(8.3) Definition. A codim 1 foliation germ $F = (\omega)$ is infinitesimally k -determined if

$$I^{(k+1)}(\omega) \subset \mathfrak{m}J(\omega) + K(\omega).$$

It is infinitesimally finitely determined if it is infinitesimally k -determined for some k .

Note that if we set

$$\Omega^{(k)}(\omega) = \{ \theta \in \Omega(\omega) \mid j^k \theta = 0 \},$$

then the above condition is equivalent to

$$\Omega^{(k)}(\omega) \subset L_{\mathfrak{m}}(\omega) + F.$$

As in section 1, we denote by H_n the group of germs of local biholomorphic maps φ about 0 in $\mathbb{C}^n$ with $\varphi(0) = 0$. We also denote by U_n the multiplicative group of units in $\mathcal{O}_n$. The group H_n acts on U_n from the right by pull-back and thus we may form the semi-direct product $G_n = U_n \rtimes H_n$. Then the group G_n acts naturally on Ω_n and for a germ ω in Ω_n , its orbit $G_n\omega$ is the set of germs holomorphically equivalent to ω . We may think of the space $L_m(\omega) + F$ as the tangent space to the orbit $G_n\omega$ at ω ([32](1.6) Remark). Also the space $\Omega^{(k)}(\omega)$ can be thought of as the tangent space at ω to the set

$$\{ \omega' \in \Omega_n \mid \text{there is a } k\text{-trivial unfolding } \tilde{\omega} \text{ such that } \omega' = \omega_t \text{ for some } t \}$$

(see [32] p.437).

We quote the following

(8.4) Theorem ([32]). *Let $F = (\omega)$ be a codim 1 foliation germ with*

$$(*) \quad \dim K(\omega)/\mathfrak{m}J(\omega) \cap K(\omega) < +\infty.$$

If F is infinitesimally k -determined, then it is locally k -determined.

(8.5) Remark. Actually, in [32] it is shown that, under the assumption (*), if F is infinitesimally k -determined, then every k -trivial (one parameter) unfolding of F is (strongly) RL -trivial. Moreover, when $K(\omega) = 0$, every k -trivial (one parameter) unfolding of F is (strongly) trivial.

A formal integrating factor of an integrable 1-form ω is a formal power series $\hat{h}$ in $z = (z_1, \dots, z_n)$ such that $\hat{h}d\omega = d\hat{h} \wedge \omega$.

(8.6) Theorem ([8]). *Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^n$ without (non-zero) formal integrating factors. If the $\mathbb{C}$ -vector space $I(\omega)/J(\omega)$ is finite dimensional, then F is locally finitely determined.*

When $n = 2$, by the assumption $\text{codim } S(\omega) \geq 2$, $\mathcal{O}_2/J(\omega)$ is always finite dimensional. Thus we have

(8.7) Corollary. *Every codim 1 foliation germ at 0 in $\mathbb{C}^2$ without formal integrating factors is locally finitely determined.*

(8.8) Remarks. 1°. Let $F = (\omega)$ be a codim 1 foliation germ without formal integrating factors. In [8], it is shown that if $I(\omega)/J(\omega)$ is finite dimensional, then $F = (\omega)$ is infinitesimally finitely determined and thus by (8.4) Theorem, we have (8.6) Theorem. Since $K(\omega) = 0$ in this case, the infinitesimal finite determinacy of F also implies the finite dimensionality of $I(\omega)/J(\omega)$, (see [32] (4.3) Remark).

2°. Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^2$. If ω is “non-real” ([5] p.136) and if $K(\omega) = 0$, then ω has no non-zero formal integrating factors ([5] p.136, Proposition 3.2). Note that for general ω , we have $K(\omega) = 0$ ((8.12) Remark).

3°. Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^n$. If $K(\omega) \neq 0$, then ω admits a multiform first integral ([5] Théorème d'intégration).

4°. As for the determinacy problem for multiform functions see [5] Sixième partie and [32] section 5.

Here is a case when we have the finite dimensionality of $I(\omega)/J(\omega)$. Let $F = (\omega)$ be a codim 1 foliation germ. If we let $S(d\omega)$ be (the germ at 0 of) the zero set of $d\omega$, we easily see that $S(\omega) \cap S(d\omega)$ does not depend on the choice of the generator ω of F .

(8.9) Proposition ([8]). *Let $F = (\omega)$ be a codim 1 foliation germ. If $S(\omega) \cap S(d\omega) \subset \{0\}$, then the vector space $I(\omega)/J(\omega)$ is finite dimensional.*

In what follows, we consider foliation germs at 0 in $\mathbb{C}^2 = \{(x, y)\}$ and give some estimates of the order of determinacy. We denote by Ω_k^p the vector space of homogeneous p -forms of degree k on $\mathbb{C}^2$. The space Ω_k^0 of homogeneous polynomials of degree k in (x, y) is especially denoted by $\mathcal{O}_k$. Thus we may write $\Omega_k^1 = \mathcal{O}_{k-1}dx \oplus \mathcal{O}_{k-1}dy$ and $\Omega_k^2 = \mathcal{O}_{k-2}dx \wedge dy$. Note that our definition of the degree of a form is different from the conventional one.

Let ω_{k_0} be an element in $\Omega_{k_0}^1$. Denoting by R the radial vector field;

$$R = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y},$$

we set

$$P_{k_0} = i_R \omega_{k_0}.$$

Hereafter we only consider the case $P_{k_0} \neq 0$, i.e., ω_{k_0} is non-dicritic. We easily see that P_{k_0} is an integrating factor of ω_{k_0} . Following [5] (see also [3] and [7]), we give the following

(8.10) Definition. We say that ω_{k_0} is generic if

- (a) $S(\omega_{k_0}) \subset \{0\}$,
- (b) P_{k_0} is reduced and
- (c) ω_{k_0} does not admit meromorphic first integrals, i.e., there are no non-constant meromorphic function germs μ with $d\mu \wedge \omega_{k_0} = 0$.

Suppose ω_{k_0} is generic. Since the quotient of two integrating factors of ω_{k_0} is a meromorphic first integral of ω_{k_0} , the condition (c) implies that

$$K(\omega_{k_0}) = \mathbb{C}P_{k_0}.$$

We set

$$V(\omega_{k_0}) = \{ \theta \in \Omega_{k_0+1}^1 \mid P_{k_0}d\theta - dP_{k_0} \wedge \theta = df \wedge \omega_{k_0} - f d\omega_{k_0} \text{ for some } f \text{ in } \mathcal{O}_{k_0+1} \}.$$

Then we have

(8.11) *Theorem*([8]). Let ω be a germ of integrable 1-form at 0 in $\mathbb{C}^2$ and let ω_{k_0} be the first non-zero jet of ω . Further, we let $\omega_{k_0+1} = j^{k_0+1}\omega - \omega_{k_0}$. Suppose $k_0 \geq 3$, ω_{k_0} is generic and ω_{k_0+1} is not in $V(\omega_{k_0})$. Then

$$I^{(k)}(\omega) = \mathfrak{m}^k \cap I(\omega)$$

for $k > k_0 + 1$.

(8.12) *Remark*. The above theorem holds also for $k = \infty$, i.e., for a germ ω as in (8.11) Theorem, we have $K(\omega) = 0$. We also have $K(j^k\omega) = 0$ for $k > k_0$.

(8.13) *Corollary*. Let ω be as in (8.11) Theorem. If

$$\mathfrak{m}^{k+1} \cap I(\omega) \subset \mathfrak{m}J(\omega)$$

for some $k > k_0$, then $F = (\omega)$ is locally k -determined.

(8.14) *Remark*. By the Artin-Rees theorem, there is always a k for which the condition in (8.13) is satisfied.

Part II

9. Proof of the versality theorem (1. formal solution)

We only prove (4.19) Theorem. The proof of (4.20) Theorem is a little more complicated and we refer to [33].

Let $F = (\omega)$ be a codim 1 foliation germ at 0 in $\mathbb{C}^n$ and write $\omega = \sum_{i=1}^n f_i dz_i$ with f_i in $\mathcal{O}_n$. Also, let $\mathcal{F} = (\tilde{\omega})$ be an unfolding of F with parameter space $\mathbb{C}^m = \{(t_1, t_2, \dots, t_m)\}$. We write

$$\tilde{\omega} = \sum_{i=1}^n \tilde{f}_i(z, t) dz_i + \sum_{j=1}^m \tilde{h}_j(z, t) dt_j$$

with $\tilde{f}_i$ and $\tilde{h}_j$ in $\mathcal{O}_{n+m}$. We have $\tilde{f}_i(z, 0) = f_i(z)$ for $i = 1, 2, \dots, n$. We set $h_j(z) = \tilde{h}_j(z, 0)$ for $j = 1, 2, \dots, m$. If we set $\omega_t = \sum_{i=1}^n \tilde{f}_i(z, t) dz_i$, the integrability of $\tilde{\omega}$ is equivalent to the following four conditions.

$$(9.1) \quad d_z \omega_t \wedge \omega_t = 0,$$

$$(9.2) \quad \tilde{h}_j d_z \omega_t + \left(\frac{\partial}{\partial t_j} \omega_t - d_z \tilde{h}_j \right) \wedge \omega_t = 0, \quad 1 \leq j \leq m,$$

$$(9.3) \quad \tilde{h}_j \left(\frac{\partial}{\partial t_k} \omega_t - d_z \tilde{h}_k \right) - \tilde{h}_k \left(\frac{\partial}{\partial t_j} \omega_t - d_z \tilde{h}_j \right) - \tilde{h}_{jk} \omega_t = 0, \quad 1 \leq j, k \leq m,$$

$$(9.4) \quad \tilde{h}_i \tilde{h}_{jk} + \tilde{h}_j \tilde{h}_{ki} + \tilde{h}_k \tilde{h}_{ij} = 0, \quad 1 \leq i, j, k \leq m,$$

where d_z denotes the exterior derivative in $z = (z_1, z_2, \dots, z_n)$,

$$\frac{\partial}{\partial t_j} \omega_i = \sum_{i=1}^n \frac{\partial \tilde{f}_i}{\partial t_j}(z, t) dz_i \quad \text{and} \quad \tilde{h}_{jk} = \frac{\partial \tilde{h}_j}{\partial t_k} - \frac{\partial \tilde{h}_k}{\partial t_j}.$$

Note that (9.3) implies (9.4) and that if there exists j such that $\tilde{h}_j \neq 0$, then (9.2) implies (9.1).

Now we assume that $\mathcal{F}$ is infinitesimally versal, i.e.,

$$(9.5) \quad [h_1], [h_2], \dots, [h_m] \quad \text{span the vector space} \quad I(\omega)/J(\omega).$$

Let $\mathcal{F}' = (\theta)$ be an arbitrarily given unfolding of F generated by θ and let $\mathbb{C}^l = \{(s_1, s_2, \dots, s_l)\}$ be its parameter space. We write

$$\theta = \sum_{i=1}^n g_i(z, s) dz_i + \sum_{k=1}^l \tilde{e}_k(z, s) ds_k$$

with g_i and $\tilde{e}_k$ in $\mathcal{O}_{n+l}$. We have $g_i(z, 0) = f_i(z)$ for $i = 1, 2, \dots, n$. We set $e_k(z) = \tilde{e}_k(z, 0)$ for $k = 1, 2, \dots, l$. We also set

$$\theta_s = \sum_{i=1}^n g_i(z, s) dz_i.$$

From the integrability condition of θ , we get

$$(9.6) \quad e_1, e_2, \dots, e_l \quad \text{are in} \quad I(\omega).$$

Our purpose is to find Φ, ψ and u which satisfy the following conditions (9.7), (9.8) and (9.9).

(9.7) Φ and ψ are holomorphic map germs making the diagram

$$\begin{array}{ccc} (\mathbb{C}^n \times \mathbb{C}^l, 0) & \xrightarrow{\Phi} & (\mathbb{C}^n \times \mathbb{C}^m, 0) \\ \downarrow & & \downarrow \\ (\mathbb{C}^l, 0) & \xrightarrow{\psi} & (\mathbb{C}^m, 0) \end{array}$$

commutative. Thus we may write $\Phi(z, s) = (\varphi(z, s), \psi(s))$ for some holomorphic map germ

$$\varphi : (\mathbb{C}^n \times \mathbb{C}^l, 0) \rightarrow (\mathbb{C}^n, 0).$$

u is a unit in $\mathcal{O}_{n+l}$.

$$(9.8) \quad \Phi(z, 0) = (z, 0) \quad \text{and} \quad u(z, 0) = 1.$$

Note that the first condition in the above is equivalent to $\varphi(z, 0) = z$ and $\psi(0) = 0$.

$$(9.9) \quad u\theta = \Phi^*\tilde{\omega}.$$

We show that there exist formal power series $\Phi(z, s)$, $\psi(s)$ and $u(z, s)$ in s satisfying (9.7), (9.8) and (9.9). We write

$$\varphi(z, s) = \sum_{|\nu| \geq 0} \varphi^{(\nu)}(z) s^\nu, \quad \psi(s) = \sum_{|\nu| \geq 0} c^{(\nu)} s^\nu \quad \text{and} \quad u(z, s) = \sum_{|\nu| \geq 0} u^{(\nu)}(z) s^\nu,$$

where $\varphi^{(\nu)} \in \mathcal{O}_n^n$, $c^{(\nu)} \in \mathbb{C}^m$ and $u^{(\nu)} \in \mathcal{O}_n$, $\nu = (\nu_1, \nu_2, \dots, \nu_l)$ is an l -tuple of non-negative integers, $s^\nu = s_1^{\nu_1} s_2^{\nu_2} \cdots s_l^{\nu_l}$ and $|\nu| = \nu_1 + \nu_2 + \cdots + \nu_l$. In general, for a series $\sigma = \sum_{|\nu| \geq 0} \sigma^{(\nu)} s^\nu$ with $\sigma^{(\nu)} \in \mathcal{O}_n^r$ and a non-negative integer p , we set

$$[\sigma]_\nu = \sigma^{(\nu)} s^\nu, \quad [\sigma]_p = \sum_{|\nu|=p} [\sigma]_\nu \quad \text{and} \quad \sigma^{[p]} = \sum_{|\nu|=0}^p [\sigma]_\nu.$$

If we set $\Phi^{[p]} = (\varphi^{[p]}, \psi^{[p]})$, we have

$$(\Phi^{[p]})^* \tilde{\omega} = \Theta(p) + \sum_{k=1}^l E(p)_k ds_k,$$

where

$$(9.10) \quad \Theta(p) = \sum_{i=1}^n (\Phi^{[p]})^* \tilde{f}_i \cdot d_z \varphi_i^{[p]} \quad \text{and}$$

$$(9.11) \quad E(p)_k = \sum_{i=1}^n (\Phi^{[p]})^* \tilde{f}_i \frac{\partial \varphi_i^{[p]}}{\partial s_k} + \sum_{j=1}^m (\Phi^{[p]})^* \tilde{h}_j \frac{\partial \psi_j^{[p]}}{\partial s_k}.$$

Then the condition (9.8) is equivalent to

$$(9.12) \quad \varphi^{(0)}(z) = 0, \quad c^{(0)} = 0 \quad \text{and} \quad u^{(0)}(z) = 1,$$

where $(0) = (0, 0, \dots, 0)$. Also, (9.9) is equivalent to the congruences

$$(9.13)_p \quad u^{[p]} \theta_s \equiv_p \Theta(p) \quad \text{for } p \geq 0 \quad \text{and}$$

$$(9.14)_p \quad u^{[p-1]} \tilde{e}_k \equiv_{p-1} E(p)_k \quad \text{for } p \geq 1, \quad 1 \leq k \leq l,$$

where $\equiv_p$ denotes the identity modulo $s^p, |\nu| = p + 1$. In what follows we show the existence of $\varphi^{(\nu)}, c^{(\nu)}$ and $u^{(\nu)}$ for all ν satisfying (9.12), (9.13)_p and (9.14)_p for all p . We do this by induction on p , which is indicated as

- (i) (9.12), (9.13)₀ and (9.14)₁ hold ((9.15) Lemma).
- (ii) For $p \geq 1$, (9.13)_{p-1} and (9.14)_p imply (9.13)_p and (9.14)_{p+1} ((9.16) Lemma).

First, if we set $\varphi^{(0)}(z) = z, c^{(0)} = 0$ and $u^{(0)}(z) = 1$, then we have (9.12). Also we have (9.13)₀, the 0-th order term in s of each side being equal to ω .

(9.15) Lemma. *For each ν with $|\nu| = 1$, there exist $\varphi^{(\nu)}$ and $c^{(\nu)}$ satisfying (9.14)₁.*

Proof. The congruence (9.14)₁ is equivalent to

$$e_k = \sum_{i=1}^n \varphi_i^{(1_k)} f_i + \sum_{j=1}^m c_j^{(1_k)} h_j, \quad 1 \leq k \leq l,$$

where $(1_k) = (0, \dots, 0, 1, 0, \dots, 0)$ (1 in the k -th entry). By (9.5) and (9.6), there exist c_{kj} in $\mathbb{C}$ and $v_k = \sum_{i=1}^n \xi_{ki} \frac{\partial}{\partial z_i}$ in Θ_n such that

$$e_k = \sum_{j=1}^m c_{kj} h_j + \langle v_k, \omega \rangle.$$

Thus, if we set $\varphi_i^{(1_k)} = \xi_{ki}$ and $c_j^{(1_k)} = c_{kj}$, then (9.14)₁ is satisfied. $\square$

(9.16) Lemma. *Let p be a positive integer. If there exist $\varphi^{[p]}, \psi^{[p]}$ and $u^{[p-1]}$ satisfying (9.13)_{p-1} and (9.14)_p, then there exist $[\varphi]_{p+1}, [\psi]_{p+1}$ and $[u]_p$ satisfying (9.13)_p and (9.14)_{p+1}.*

Proof. By assumption we have $\Phi^{[p]}$ and $u^{[p-1]}$, hence $\Theta(p)$ and $E(p)_k$. The congruence (9.13)_p is equivalent to

$$[u]_p \omega + u^{[p-1]} \theta_s \equiv_p \Theta(p).$$

Thus in order to show the existence of $[u]_p$ satisfying the above, by the de Rham division theorem (2.1), it suffices to show that

$$(9.17) \quad [\Theta(p) - u^{[p-1]} \theta_s]_p \wedge \omega = 0.$$

First, from the integrability of $(\Phi^{[p]})^*\tilde{\omega}$ and $u^{[p-1]}\theta$, we get

$$(9.18)_p \quad E(p)_k d_z \Theta(p) + \left(\frac{\partial}{\partial s_k} \Theta(p) - d_z E(p)_k \right) \wedge \Theta(p) = 0 \quad \text{and}$$

$$(9.19)_{p-1} \quad (u^{[p-1]}\tilde{e}_k) d_z (u^{[p-1]}\theta_s) + \left(\frac{\partial}{\partial s_k} (u^{[p-1]}\theta_s) - d_z (u^{[p-1]}\tilde{e}_k) \right) \wedge (u^{[p-1]}\theta_s) = 0.$$

Subtracting $(9.19)_{p-1}$ from $(9.18)_p$ and using $(9.13)_{p-1}$ and the congruence $\Theta(p-1) \equiv_{p-1} \Theta(p)$, we get

$$\begin{aligned} & (E(p)_k - u^{[p-1]}\tilde{e}_k) d_z \Theta(p) \\ & + \left(\frac{\partial}{\partial s_k} (\Theta(p) - u^{[p-1]}\theta_s) - d_z (E(p)_k - u^{[p-1]}\tilde{e}_k) \right) \wedge \Theta(p) \equiv_{p-1} 0. \end{aligned}$$

Thus using $(9.14)_p$, we have

$$\frac{\partial}{\partial s_k} (\Theta(p) - u^{[p-1]}\theta_s) \wedge \Theta(p) \equiv_{p-1} 0.$$

By $(9.13)_{p-1}$, $\Theta(p) - u^{[p-1]}\theta_s$ has no terms of order in s less than p . Thus comparing the terms of order $p-1$ in the above congruence, we get

$$\frac{\partial}{\partial s_k} [\Theta(p) - u^{[p-1]}\theta_s]_p \wedge \omega = 0, \quad 1 \leq k \leq l.$$

Therefore we get (9.17).

Next, $(9.14)_{p+1}$ reads

$$(9.20) \quad u^{[p]}\tilde{e}_k \equiv_p E(p)_k + \sum_{i=1}^n \frac{\partial[\varphi_i]_{p+1}}{\partial s_k} \cdot f_i + \sum_{j=1}^m \frac{\partial[\psi_j]_{p+1}}{\partial s_k} \cdot h_j.$$

We look for $[\varphi_i]_{p+1}$ and $[\psi_j]_{p+1}$ satisfying (9.20). From the integrability of $u^{[p]}\theta$, we get $(9.19)_p$. Subtracting $(9.18)_p$ from $(9.19)_p$ and using $(9.13)_p$, we have

$$(u^{[p]}\tilde{e}_k - E(p)_k) d_z \Theta(p) + \left(\frac{\partial}{\partial s_k} (u^{[p]}\theta_s - \Theta(p)) - d_z (u^{[p]}\tilde{e}_k - E(p)_k) \right) \wedge \Theta(p) \equiv_p 0.$$

By $(9.13)_p$, $u^{[p]}\theta_s - \Theta(p)$ has no terms of order less than $p+1$ and $u^{[p]}\tilde{e}_k - E(p)_k$ has no terms of order less than p . Hence we get

$$[u^{[p]}\tilde{e}_k - E(p)_k]_p d\omega + \left(\frac{\partial}{\partial s_k} [u^{[p]}\theta_s - \Theta(p)]_{p+1} - d_z [u^{[p]}\tilde{e}_k - E(p)_k]_p \right) \wedge \omega = 0.$$

This shows that the coefficients of s^ν , $|\nu| = p$, in $[u^p \tilde{e}_k - E(p)_k]_p$ are all in $I(\omega)$. Thus to show the existence of $[\varphi_i]_{p+1}$ and $[\psi_j]_{p+1}$, it suffices to show

$$(9.21) \quad \frac{\partial}{\partial s_j} [u^p \tilde{e}_k - E(p)_k]_p = \frac{\partial}{\partial s_k} [u^p \tilde{e}_j - E(p)_j]_p$$

for $j, k = 1, 2, \dots, l$. From the integrability of $(\Phi^p)^* \tilde{\omega}$ and $u^p \theta$, we have ((9.3))

$$(9.22) \quad \begin{aligned} E(p)_j \left(\frac{\partial}{\partial s_k} \Theta(p) - d_z E(p)_k \right) - E(p)_k \left(\frac{\partial}{\partial s_j} \Theta(p) - d_z E(p)_j \right) \\ - \left(\frac{\partial E(p)_j}{\partial s_k} - \frac{\partial E(p)_k}{\partial s_j} \right) \Theta(p) = 0 \quad \text{and} \end{aligned}$$

$$(9.23) \quad \begin{aligned} u^p \tilde{e}_j \left(\frac{\partial}{\partial s_k} (u^p \theta_s) - d_z (u^p \tilde{e}_k) \right) - u^p \tilde{e}_k \left(\frac{\partial}{\partial s_j} (u^p \theta_s) - d_z (u^p \tilde{e}_j) \right) \\ - \left(\frac{\partial}{\partial s_k} (u^p \tilde{e}_j) - \frac{\partial}{\partial s_j} (u^p \tilde{e}_k) \right) u^p \theta_s = 0. \end{aligned}$$

Subtracting (9.22) from (9.23) and using (9.13)_p and (9.14)_p, we see that

$$\left(\frac{\partial}{\partial s_k} (u^p \tilde{e}_j - E(p)_j) - \frac{\partial}{\partial s_j} (u^p \tilde{e}_k - E(p)_k) \right) \Theta(p) \equiv_{p-1} 0.$$

By (9.14)_p, $u^p \tilde{e}_j - E(p)_j$ and $u^p \tilde{e}_k - E(p)_k$ have no terms of order less than p . Hence we have

$$\left(\frac{\partial}{\partial s_k} [u^p \tilde{e}_j - E(p)_j]_p - \frac{\partial}{\partial s_j} [u^p \tilde{e}_k - E(p)_k]_p \right) \omega = 0,$$

which implies (9.21). $\square$

10. Malgrange privileged neighborhood theorem

In this section we quote a "privileged neighborhood theorem" due to B. Malgrange [15], which is used in the proof of the versality theorem.

For an n -tuple $(\rho_1, \rho_2, \dots, \rho_n)$ of positive real numbers ρ_i , we denote by $P(\rho)$ the closed polydisk with center 0 and radius ρ ;

$$P(\rho) = \{ z \in \mathbb{C}^n \mid |z_i| \leq \rho_i, 1 \leq i \leq n \}.$$

If f is a germ in $\mathcal{O}_n$, we write $f(z) = \sum_{|\nu| \geq 0} a_\nu z^\nu$ and define its ρ -norm by

$$|f|_\rho = \sum_{|\nu| \geq 0} |a_\nu| \rho^\nu,$$

which may possibly be $+\infty$. For a germ $f = (f_1, f_2, \dots, f_r)$ in $\mathcal{O}_n^r$, we set $|f|_\rho = \sum_{i=1}^r |f_i|_\rho$. It is not difficult to see that for f and g in $\mathcal{O}_n$, we have

$$|f + g|_\rho \leq |f|_\rho + |g|_\rho \quad \text{and} \quad |fg|_\rho \leq |f|_\rho \cdot |g|_\rho.$$

We also have ([15])

(10.1) **Lemma.** *There is a positive constant c_1 such that for any f in $\mathcal{O}_n$ and for any real numbers a and b with $\frac{1}{2} \leq a < b \leq 1$, we have*

$$\left| \frac{\partial f}{\partial z_i} \right|_{a\rho} \leq \frac{c_1}{b-a} |f|_{b\rho}.$$

If $u : \mathcal{O}_n^q \rightarrow \mathcal{O}_n^p$ is an $\mathcal{O}_n$ -linear map, then u is represented by a matrix (u_{ij}) whose entries u_{ij} are germs in $\mathcal{O}_n$. If we choose ρ sufficiently small, we may assume that $|u_{ij}|_\rho < +\infty$. Hence we see that there is a positive constant c such that for any f in $\mathcal{O}_n$, $|u(f)|_\rho \leq c|f|_\rho$. What is important is that we have the "converse inequality".

(10.2) **Theorem (Malgrange privileged neighborhood theorem).** *Let*

$$u_i : \mathcal{O}_n^{q_i} \rightarrow \mathcal{O}_n^{p_i}, \quad i = 1, 2, \dots, r,$$

be a finite number of $\mathcal{O}_n$ -linear maps. Then the closed polydisks $P(\rho)$ with the following property form a fundamental system of neighborhoods of 0 in $\mathbb{C}^n$:

There exists a positive constant K such that for all r -tuple $(f_1, f_2, \dots, f_r)$ with f_i in the image of u_i , there exists $(g_1, g_2, \dots, g_r)$ with g_i in $\mathcal{O}_n^{q_i}$ satisfying

$$f_i = u_i(g_i) \quad \text{and} \quad |g_i|_{a\rho} \leq K|f_i|_{a\rho} \quad \text{for } a \text{ with } \frac{1}{2} \leq a \leq 1.$$

(10.3) **Remark.** The above theorem without a is the privileged neighborhood theorem of H. Cartan and H. Grauert. We need to put a in order to be able to get estimates involving the partial derivatives (Cf. (10.1) Lemma).

11. Proof of the versality theorem (2. convergence)

Note that the $\mathcal{O}_n$ -modules Ω_n and Θ_n may be identified with $\mathcal{O}_n^n$ by identifying $\theta = \sum_{i=1}^n g_i dz_i$ with $(g_1, g_2, \dots, g_n)$ and $v = \sum_{i=1}^n \xi_i \frac{\partial}{\partial z_i}$ with $(\xi_1, \xi_2, \dots, \xi_n)$. Thus for germs in Ω_n and Θ_n and an n -tuple ρ of positive numbers, we may define their ρ -norm. We choose open neighborhoods U , V and W of the origins in $\mathbb{C}^n$, $\mathbb{C}^m$ and $\mathbb{C}^l$, respectively, so that the germs ω , $\tilde{\omega}$ and θ have representatives on U , $U \times V$ and $U \times W$, respectively. Thus the germs f_i , $1 \leq i \leq n$, h_j , $1 \leq j \leq m$, and e_k , $1 \leq k \leq l$, have representatives on U . We consider the $\mathcal{O}_n$ -homomorphisms

$$\lambda : \Theta_n \rightarrow \mathcal{O}_n \quad \text{and} \quad \mu : \mathcal{O}_n \rightarrow \Omega_n$$

defined by $\lambda(v) = \langle v, \omega \rangle$ for v in Θ_n and $\mu(g) = g\omega$ for g in $\mathcal{O}_n$. Thus $\text{Im } \lambda = J(\omega)$ and $\text{Im } \mu = F$. If we apply the Malgrange privileged neighborhood theorem to λ and μ , we get

(11.1) **Lemma.** *There exist an n -tuple of positive numbers ρ such that $P(\rho) \subset U$ and a positive constant K with the following properties.*

(a) *For all e in $J(\omega)$, there exists v in Θ_n such that $e = \langle v, \omega \rangle$ and*

$$|v|_{a\rho} \leq K|e|_{a\rho} \quad \text{for } a \quad \text{with} \quad \frac{1}{2} \leq a \leq 1.$$

(b) *For all η in F , there exists g in $\mathcal{O}_n$ such that $\eta = g\omega$ and*

$$|g|_{a\rho} \leq K|\eta|_{a\rho} \quad \text{for } a \quad \text{with} \quad \frac{1}{2} \leq a \leq 1.$$

We choose a ρ with the properties in (11.1) Lemma and fix it once for all. We also set

$$K_0 = \max\{|f|_\rho, |h|_\rho, |e|_\rho\},$$

where $f = (f_1, f_2, \dots, f_n)$, $h = (h_1, h_2, \dots, h_m)$ and $e = (e_1, e_2, \dots, e_l)$.

(11.2) **Definition.** For a series $\sigma = \sum \sigma^{(\nu)} s^\nu$ with $\sigma^{(\nu)} \in \mathcal{O}_n^r$ for some r and a series $\sum a^{(\nu)} s^\nu$ with $a^{(\nu)}$ positive real numbers, we say $\sum a^{(\nu)} s^\nu$ dominates σ in $P(\rho)$ if $|\sigma^{(\nu)}|_\rho \leq a^{(\nu)}$ for all ν . In this case, we write

$$\sum \sigma^{(\nu)} s^\nu \ll \sum a^{(\nu)} s^\nu \quad \text{in } P(\rho).$$

We consider the series

$$A(s) = \frac{1}{16c^{\frac{1}{2}}} \sum_{p \geq 1} \frac{c^p}{p^2} (s_1 + s_2 + \dots + s_l)^p,$$

where c is a constant to be determined later. We write by $A'(s)$ the series obtained by differentiating $A(s)$ with respect to the variable $s_1 + s_2 + \dots + s_l$ and set $B(s) = c^{-\frac{1}{2}} A'(s)$. Thus

$$B(s) = \frac{1}{16} \sum_{p \geq 0} \frac{c^p}{p+1} (s_1 + s_2 + \dots + s_l)^p.$$

Note that the series $A(s)$ and $B(s)$ converges absolutely for s with $|s| = \max\{|s_k| \mid 1 \leq k \leq l\} < \frac{1}{lc}$. The series $A(s)$ has the property that ([13], [14])

$$(11.3) \quad A(s)^p \ll \left(\frac{1}{c^{\frac{1}{2}}}\right)^{p-1} A(s) \quad \text{for } p \geq 1.$$

From this we have $A(s)A'(s) \ll \frac{1}{2c^{\frac{1}{2}}}A'(s)$. Hence we get

$$(11.4) \quad A(s)B(s) \ll \frac{1}{2c^{\frac{1}{2}}}B(s)$$

We prove the existence of series φ, ψ and u in s satisfying (9.12), (9.13)_p and (9.14)_p for all p such that if we choose c sufficiently large, then the following estimates hold for all $p \geq 0$.

$$(11.5)_p \quad u^{|p} \ll \frac{8}{1-a}B\left(\frac{s}{1-a}\right),$$

$$(11.6)_p \quad \varphi_i^{|p} \ll z_i + A\left(\frac{s}{1-a}\right), \quad 1 \leq i \leq n, \quad \text{and}$$

$$(11.7)_p \quad \psi_j^{|p} \ll A\left(\frac{s}{1-a}\right), \quad 1 \leq j \leq m,$$

in $P(a\rho)$ for all a with $\frac{1}{2} \leq a \leq 1$, where $\frac{s}{1-a} = (\frac{s_1}{1-a}, \frac{s_2}{1-a}, \dots, \frac{s_l}{1-a})$.

(11.8) **Lemma.** *There exists a positive constant K_1 such that for every e in $I(\omega)$, there exist a germ v in Θ_n and constants $c_j, 1 \leq j \leq m$, such that*

$$e = \langle v, \omega \rangle + \sum_{j=1}^m c_j h_j \quad \text{and} \quad |v|_{a\rho} \leq K_1 |e|_{a\rho}, \quad |c_j| \leq K_1 |e|_{a\rho}, \quad \frac{1}{2} \leq a \leq 1.$$

Proof. First we show the existence of a constant K' such that for every e in $I(\omega)$, there exist constants c_j such that

$$(11.9) \quad [e] = \sum_{j=1}^m c_j [h_j] \quad \text{in} \quad I(\omega)/J(\omega) \quad \text{and} \quad |c_j| \leq K' |e|_{\frac{1}{2}\rho}.$$

This is done as in [14]. Thus for e in $I(\omega)$, set

$$\iota(e) = \inf \left\{ \max_{1 \leq j \leq m} |c_j| \mid [e] = \sum_{j=1}^m c_j [h_j] \right\}.$$

It suffices to show the existence of a constant K' such that $\iota(e) < K' |e|_{\frac{1}{2}\rho}$ for all e in $I(\omega)$. Suppose this is false. Then for every natural number N , there exists $e^{(N)}$ in $I(\omega)$ such that

$$\iota(e^{(N)}) = 1 \quad \text{and} \quad |e^{(N)}|_{\frac{1}{2}\rho} < \frac{1}{N}.$$

By definition of $\iota(e^{(N)})$, there exist $c_j^{(N)} \in \mathbb{C}, 1 \leq j \leq m$, such that $[e^{(N)}] = \sum_{j=1}^m c_j^{(N)}[h_j]$ and $|c_j^{(N)}| < 2$. For each j , $\{c_j^{(N)}\}$ is a bounded sequence. Hence replacing $\{e^{(N)}\}$ by a suitable subsequence, we may assume that the sequence $\{c_j^{(N)}\}$ converges. We set

$$c_j = \lim_{N \rightarrow \infty} c_j^{(N)}.$$

Since $\sum_{j=1}^m c_j[h_j] = [0]$, if we set $c'_j = c_j^{(N)} - c_j$, then for sufficiently large N , we have

$$\sum_{j=1}^m c'_j[h_j] = [e^{(N)}], \quad |c'_j| < \frac{1}{2}.$$

This contradicts with $\iota(e^{(N)}) = 1$.

Next we show the estimate for v . Take e in $I(\omega)$ and choose c_j satisfying (11.9). Since $e - \sum_{j=1}^m c_j h_j$ is in $J(\omega)$, by (11.1) Lemma (a), there exists v in Θ_n such that

$$e = \sum_{j=1}^m c_j h_j + \langle v, \omega \rangle \quad \text{and} \quad |v|_{a\rho} \leq K|e - \sum_{j=1}^m c_j h_j|_{a\rho}, \quad \frac{1}{2} \leq a \leq 1.$$

We have

$$|e - \sum_{j=1}^m c_j h_j|_{a\rho} \leq (1 + K' \sum_{j=1}^m |h_j|_{a\rho})|e|_{a\rho} \leq (1 + K' \sum_{j=1}^m |h_j|_{\rho})|e|_{a\rho}.$$

Thus if we set $K_1 = \max\{K', K(1 + K' \sum_{j=1}^m |h_j|_{\rho})\}$, then we have the desired estimate. $\square$

(11.10) Lemma. (a) We have

$$A\left(\frac{s}{1-a}\right) \ll \frac{1}{c^{\frac{1}{2}}} \frac{1}{1-a} B\left(\frac{s}{1-a}\right) \quad \text{for } a \text{ with } \frac{1}{2} \leq a < 1.$$

(b) If, for a series $\sigma = \sum \sigma^{(\nu)} s^{\nu}$, we have $\sigma \ll \frac{1}{b-a} A\left(\frac{s}{1-b}\right)$ in $P(a\rho)$ for a and b with $\frac{1}{2} \leq a < b < 1$, then

$$\sigma \ll \frac{4e}{c^{\frac{1}{2}}} \frac{1}{1-a} B\left(\frac{s}{1-a}\right).$$

Proof. (a) Straightforward.

(b) This is done as in Malgrange [15]. We have

$$\frac{1}{b-a} A\left(\frac{s}{1-b}\right) = \frac{1}{b-a} \frac{1}{16c^{\frac{1}{2}}} \sum_{p \geq 1} \frac{c^p}{p^2} \left(\frac{s_1 + s_2 + \cdots + s_l}{1-b} \right)^p.$$

The function $\frac{1}{b-a} \frac{1}{(1-b)^p}$ of b in $\frac{1}{2} \leq a < b < 1$ is minimized when $1-b = \frac{p}{p+1}(1-a)$, in which case we have

$$\frac{1}{b-a} \frac{1}{(1-b)^p} = (1 + \frac{1}{p})^p \frac{p+1}{(1-a)^{p+1}} < \frac{(p+1)e}{(1-a)^{p+1}}.$$

□

First, if we set $\varphi^{(0)}(z) = z, c^{(0)} = 0$ and $u^{(0)}(z) = 1$, then we have (9.12) and (9.13)₀. Also, since the constant term of $\frac{8}{1-a} B(\frac{s}{1-a})$ is $\frac{1}{2} \frac{1}{1-a}$, for a with $\frac{1}{2} \leq a < 1$, we have $\frac{1}{2} \frac{1}{1-a} \geq 1$. Hence we have (11.5)₀.

(11.11) Lemma. *If we take c sufficiently large, then for each ν with $|\nu| = 1$, there exist $\varphi^{(\nu)}$ and $c^{(\nu)}$ satisfying (9.14)₁, (11.6)₁ and (11.7)₁.*

Proof. As in the proof of (9.15) Lemma, (9.14)₁ is equivalent to

$$e_k = \sum_{i=1}^n \varphi_i^{(1_k)} f_i + \sum_{j=1}^m c_j^{(1_k)} h_j, \quad 1 \leq k \leq l.$$

By (9.6) and (11.8) Lemma, there exist v_k in Θ_n and constants c_{kj} such that

$$e_k = \langle v_k, \omega \rangle + \sum_{j=1}^m c_{kj} h_j, \quad |v_k|_{a\rho} \leq K_1 |e_k|_{a\rho} \quad \text{and} \quad |c_{kj}| \leq K_1 |e_k|_{a\rho}$$

for a with $\frac{1}{2} \leq a \leq 1$.

From this we see that $|v_k|_\rho$ is bounded. Writing $v_k = \sum_{i=1}^n \xi_{ki} \frac{\partial}{\partial z_i}$, we set $\varphi_i^{(1_k)} = \xi_{ki}$ and $c_j^{(1_k)} = c_{kj}$. Then we have (9.14)₁. Moreover, since

$$A(\frac{s}{1-a}) = \frac{c^{\frac{1}{2}}}{16(1-a)} (s_1 + \cdots + s_l) + \text{terms of order } \geq 2 \text{ in } s,$$

if we take c sufficiently large, we have the estimates (11.6)₁ and (11.7)₁. □

(11.12) Lemma. *If we take c sufficiently large, then for any positive integer p , the existence of $\varphi|_p$, $\psi|_p$ and $u|^{p-1}$ satisfying (9.13)_{p-1}, (9.14)_p, (11.5)_{p-1}, (11.6)_p and (11.7)_p, implies the existence of $[\varphi]_{p+1}$, $[\psi]_{p+1}$ and $[u]_p$ satisfying (9.13)_p, (9.14)_{p+1}, (11.5)_p, (11.6)_{p+1} and (11.7)_{p+1}.*

Proof. By assumption, we have $\Phi|_p$ and $u|^{p-1}$, hence $\Theta(p)$ and $E(p)_k$. As is shown in the proof of (9.16) Lemma, each coefficient of $[\Theta(p) - u|^{p-1}\theta_s]_p$ is in F . Thus there exists $[u]_p$ such that

$$[u]_p \omega \equiv_p \Theta(p) - u|^{p-1} \theta_s.$$

In order to estimate $\Theta(p)$, we consider the power series expansion of $\tilde{f}_i(z+w, t)$ in $(w, t) = (w_1, \dots, w_n, t_1, \dots, t_m)$. Note that, since $\tilde{f}_i(z, t)$ is holomorphic in $U \times V$, the coefficient functions (in z) of the power series (in (w, t)) are holomorphic in an open set containing $P(\rho)$. Hence by taking suitable constants b_0 and c_0 , we may assume that

$$(11.13) \quad \tilde{f}_i(z+w, t) - f_i(z) \ll A_0(w, t) \quad \text{and}$$

$$(11.14) \quad \tilde{h}_j(z+w, t) - h_j(z) \ll A_0(w, t) \quad \text{in } P(\rho),$$

where

$$A_0(w, t) = \frac{b_0}{c_0} \sum_{p \geq 1} c_0^p (w_1 + \dots + w_n + t_1 + \dots + t_m)^p.$$

We first estimate

$$\tilde{f}_i(\varphi^{lp}(z, s), \psi^{lp}(s)) - f_i(z).$$

We set $w = \varphi^{lp}(z, s) - z$ and $t = \psi^{lp}(s)$ in (11.13) and use the estimates (11.6)_p and (11.7)_p to get

$$\tilde{f}_i(\varphi^{lp}(z, s), \psi^{lp}(s)) - f_i(z) \ll \frac{b_0}{c_0} \sum_{q \geq 1} (n+m)^q c_0^q A\left(\frac{s}{1-a}\right)^q$$

$$\text{in } P(a\rho), \frac{1}{2} \leq a \leq 1.$$

We choose c large enough so that

$$\frac{(n+m)c_0}{c^{\frac{1}{2}}} < \frac{1}{2}.$$

Then using (11.3), we have

$$\begin{aligned} \frac{b_0}{c_0} \sum_{q \geq 1} (n+m)^q c_0^q A\left(\frac{s}{1-a}\right)^q &\ll \frac{b_0}{c_0} \sum_{q \geq 1} (n+m)^q c_0^q \left(\frac{1}{c^{\frac{1}{2}}}\right)^{q-1} A\left(\frac{s}{1-a}\right) \\ &= (n+m)b_0 \sum_{q \geq 0} \left(\frac{(n+m)c_0}{c^{\frac{1}{2}}}\right)^q A\left(\frac{s}{1-a}\right) \ll 2(n+m)b_0 A\left(\frac{s}{1-a}\right). \end{aligned}$$

Hence by setting $K_2 = 2(n+m)b_0$, we obtain

$$(11.15) \quad \tilde{f}_i(\varphi^{lp}(z, s), \psi^{lp}(s)) - f_i(z) \ll K_2 A\left(\frac{s}{1-a}\right) \quad \text{in } P(a\rho), \quad \frac{1}{2} \leq a < 1.$$

Next we estimate $\frac{\partial \varphi_i^{lp}}{\partial z_j}$. By (10.1) Lemma and (11.6)_p, there exists a constant c_1 such that

$$(11.16) \quad \frac{\partial \varphi_i^{lp}}{\partial z_j}(z, s) \ll 1 + \frac{c_1}{b-a} A\left(\frac{s}{1-b}\right) \quad \text{in } P(a\rho).$$

Using

$$A\left(\frac{s}{1-a}\right) A\left(\frac{s}{1-b}\right) \ll A\left(\frac{s}{1-b}\right)^2 \ll \frac{1}{c^{\frac{1}{2}}} A\left(\frac{s}{1-b}\right)$$

and $1-b < 1-a \leq \frac{1}{2}$, we get

$$\left[\sum_{i=1}^n \tilde{f}_i(\varphi^{lp}, \psi^{lp}) \frac{\partial \varphi_i^{lp}}{\partial z_j} \right]_p \ll n \left\{ K_2 A\left(\frac{s}{1-a}\right) + (K_0 + \frac{K_2}{c^{\frac{1}{2}}}) \frac{c_1}{b-a} A\left(\frac{s}{1-b}\right) \right\} \text{ in } P(a\rho).$$

Thus by (11.10) Lemma (b), we have

$$(11.17) \quad \left[\sum_{i=1}^n \tilde{f}_i(\varphi^{lp}, \psi^{lp}) \frac{\partial \varphi_i^{lp}}{\partial z_j} \right]_p \ll n \left(\frac{K_2 + 4ec_1 K_0}{c^{\frac{1}{2}}} + \frac{4ec_1 K_2}{c} \right) \frac{1}{1-a} B\left(\frac{s}{1-a}\right).$$

Now we may assume that

$$\theta_s - \omega \ll A\left(\frac{s}{1-a}\right) \quad \text{in } P(a\rho), \quad \frac{1}{2} \leq a < 1.$$

Hence using (11.5)_{p-1} and (11.4), we get

$$(11.18) \quad \begin{aligned} [u^{lp-1} \theta_s]_p &= [u^{lp-1} (\theta_s - \omega)]_p \ll \frac{8}{1-a} B\left(\frac{s}{1-a}\right) A\left(\frac{s}{1-a}\right) \\ &\ll \frac{4}{c^{\frac{1}{2}}} \frac{1}{1-a} B\left(\frac{s}{1-a}\right) \quad \text{in } P(a\rho). \end{aligned}$$

The estimates (11.17) and (11.18) together with (11.1) Lemma (b) imply the existence of $[u]_p$ such that

$$[u]_p \omega + u^{lp-1} \theta_s \equiv_p \Theta(p) \quad \text{and} \quad [u]_p \ll \left(\frac{K_3}{c^{\frac{1}{2}}} + \frac{K_4}{c} \right) \frac{1}{1-a} B\left(\frac{s}{1-a}\right)$$

for some constants K_3 and K_4 which do not depend on p . Thus if c is sufficiently large, we have the estimate (11.5)_p as well as the congruence (9.13)_p.

Next, in order to obtain the estimates for $[\varphi]_{p+1}$ and $[\psi]_{p+1}$, we set

$$\sigma_k = u^{lp} \tilde{e}_k - E(p)_k$$

and estimate it (cf. (9.20)). First, from (9.11), we have

$$[E(p)_k]_p = \left[\sum_{i=1}^n (\tilde{f}_i(\varphi^{lp}, \psi^{lp}) - f_i(z)) \frac{\partial \varphi_i^{lp}}{\partial s_k} + \sum_{j=1}^m (\tilde{h}_j(\varphi^{lp}, \psi^{lp}) - h_j(z)) \frac{\partial \psi_j^{lp}}{\partial s_k} \right]_p.$$

Using (11.15) and a similar one for $\tilde{h}_j$, we get, from (11.6)_p, (11.7)_p, (11.4) and $1 - a \leq \frac{1}{2}$,

$$\begin{aligned} [E(p)_k]_p &\ll (n+m)K_2 A\left(\frac{s}{1-a}\right) \frac{1}{1-a} A'\left(\frac{s}{1-a}\right) \\ &= (n+m)K_2 \frac{c^{\frac{1}{2}}}{1-a} A\left(\frac{s}{1-a}\right) B\left(\frac{s}{1-a}\right) \\ &\ll \frac{1}{2}(n+m)K_2 \frac{1}{1-a} B\left(\frac{s}{1-a}\right) \quad \text{in } P(a\rho). \end{aligned}$$

Second, since we may assume that

$$\tilde{e}_k \ll K_0 + A\left(\frac{s}{1-a}\right),$$

we get

$$[u^{lp}\tilde{e}_k] \ll \frac{8}{1-a} B\left(\frac{s}{1-a}\right) (K_0 + A\left(\frac{s}{1-a}\right)) \ll \frac{1}{1-a} (8K_0 + \frac{4}{c^{\frac{1}{2}}}) B\left(\frac{s}{1-a}\right).$$

Hence we have

$$(11.19) \quad [\sigma_k]_p \ll (K_5 + \frac{4}{c^{\frac{1}{2}}}) \frac{1}{1-a} B\left(\frac{s}{1-a}\right) = (\frac{K_5}{c^{\frac{1}{2}}} + \frac{4}{c}) \frac{1}{1-a} A'\left(\frac{s}{1-a}\right) \quad \text{in } P(a\rho),$$

where $K_5 = \frac{1}{2}(n+m)K_2 + 8K_0$.

Take ν with $|\nu| = p+1$. In (9.16) Lemma, we have shown that for k with $\nu_k \neq 0$, $\frac{1}{\nu_k} \sigma^{(\nu-1_k)}$ is in $I(\omega)$ and is independent of k . We set

$$\tau^{(\nu)} = \frac{1}{\nu_k} \sigma^{(\nu-1_k)}.$$

Then (9.20) reads

$$(11.20) \quad \sum_{|\nu|=p+1} \tau^{(\nu)} s^\nu = \sum_{i=1}^n [\varphi_i]_{p+1} f_i + \sum_{j=1}^m [\psi_j]_{p+1} h_j$$

and from (11.19), we have

$$\sum_{|\nu|=p+1} \tau^{(\nu)} s^\nu \ll (\frac{K_5}{c^{\frac{1}{2}}} + \frac{4}{c}) A\left(\frac{s}{1-a}\right) \quad \text{in } P(a\rho).$$

By (11.8) Lemma, there exist ν_ν in Θ_n and constants $c_{\nu j}$ such that

$$\tau^{(\nu)} = \sum_{j=1}^m c_{\nu j} h_j + \langle v_\nu, \omega \rangle$$

and that

$$|v_\nu|_{a\rho} \ll K_1 |\tau^{(\nu)}|_{a\rho}, \quad |c_{\nu j}| \leq K_1 |\tau^{(\nu)}|_{a\rho}.$$

Writing $v_\nu = \sum_{i=1}^n \xi_{\nu i} \frac{\partial}{\partial z_i}$, we set $\varphi_i^{(\nu)} = \xi_{\nu i}$ and $c_j^{(\nu)} = c_{\nu j}$. Then we have (11.20) and

$$[\varphi_i]_{p+1}, \quad [\psi_j]_{p+1} \ll K_1 \left(\frac{K_5}{c^{\frac{1}{2}}} + \frac{4}{c} \right) A \left(\frac{s}{1-a} \right).$$

Thus if we choose c sufficiently large, we have the estimates (11.6)_{p+1} and (11.7)_{p+1}.
□

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