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Toeplitz Operators And Weighted Norm Inequalities

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Abstract. The symbols of invertible Toeplitz operators from $H^p(wd\theta/2\pi)$ to $L^p(wd\theta/2\pi)/e^{-i\theta}\bar{H}^p(wd\theta/2\pi)$ are described completely where $H^p(wd\theta/2\pi)$ denotes a weighted Hardy space. The result is strongly related with a weighted norm inequality. If the weight w satisfies the condition (A_p) then $L^p(wd\theta/2\pi)/e^{-i\theta}\bar{H}^p(wd\theta/2\pi) = H^p(wd\theta/2\pi)$ with equivalent norms.

§ 1. Introduction

Let m be a normalized Lebesgue measure on the unit circle T and let w be a non-negative integrable function on T which does not vanish identically. Suppose $1 \leqq p \leqq \infty$. Let $L^p(w) = L^p(wdm)$ and $L^p(w) = L^p$ when $w = 1$. Let $H^p(w)$ denote the closure of $L^p(w)$ of the set $\mathcal{P}$ of all analytic polynomials. We will write $H^p(w) = H^p$ when $w = 1$ and then this is a usual Hardy space. P denotes the projection from the set $\mathcal{C}$ of all trigonometric polnomials to $\mathcal{P}$. This densely defined operator P may not be extended to a bounded map of $L^p(w)$ onto $H^p(w)$ for some weight w. P can be extended to a bounded map of $L^p(w)$ onto $H^p(w)$ if and only if w satisfies the condition :

$$(A_p) \quad \sup_I \left(\frac{1}{|I|} \int_I w \, dm \right) \left(\frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} \, dm \right)^{p-1} < \infty$$

where the supremum is over all intervals of T. This is the well known theorem of Hunt, Muckenhoupt and Wheeden [6] which is a generalization of the theorem of Helson and Szegő [4] .

For ϕ in L^∞, the Toeplitz operator T_ϕ is defined as a densely defined map from $H^p(w)$ to $H^p(w)$ by

$$T_\phi f = P(\phi f).$$

When w satisfies the condition (A_p), Rochberg [8] gave a nice characterization of the symbol ϕ for the invertible T_ϕ. If $p = 2$ and $w = 1$, this reduces to the theorem of Widom and Devinatz [2] .

In this paper we give a simple proof of the Rochberg's theorem and we study the Toeplitz operators for arbitrary weights w. For ϕ in L^∞, the Toeplitz operator T_ϕ is defined as a bounded map from $H^p(w)$ to $L^p(w)/\bar{H}^p_0(w)$ by

$$T_\phi f = \phi f + \bar{H}^p_0(w),$$

where $H^p_0(w) = zH^p(w)$ and $z = e^{i\theta}$. We give a characterization of the symbol ϕ for the invertible T_ϕ . If w satisfies the condition (A_p) then this shows the Rochberg's theorem. It is known that the symbols of invertible singular integral operators can be described by the Rochberg's theorem. Recently Yamamoto [9] studied singular integral operatore for p = 2 and w without the condition (A_p). He used a lifting theorem of Cotlar and Sadosky [1] . Clearly this argument does not apply to $p \neq 2$. We study singular integral operators for arbitrary p and w as a simple corollary of a previous section.

In this paper, a linear operator on a Banach space is called left invertible when it is bounded below. If V is a real function in L^1 then $\tilde{V}$ denotes the harmonic conjugate function with V(0) = 0.

§ 2. T_ϕ and $L^p(w)/\bar{H}^p(w)$

In this section, we will give a characterization of the symbol ϕ for the invertible Toeplitz operator T_ϕ from $H^p(w)$ onto $L^p(w)/\bar{H}^p_0(w)$. If $\log w$ is not integrable, then by a theorem of Szegő [5, p49] $L^p(w) = \overline{H^p_0(w)}$. Hence for the research of T_ϕ it is reasonable to assume the integrability of $\log w$ and hence $w = |h|^p$ for some outer function h in H^p. We need several lemmas to get the theorem.

Lemma 1. Suppose $1 \leqq p < \infty$ and $w = |h|^p$ for some outer function h in H^p. Let ϕ be a nonzero function in L^∞ and $\phi_0 = \phi\bar{h}/h$.

(1) T_ϕ is a left invertible operator from $H^p(w)$ into $L^p(w)/\bar{H}^p_0(w)$ if and only if T_{ϕ_0} is a left invertible operator from H^p into $L^p/\bar{H}^p_0$.

(2) T_ϕ is an invertible operator from $H^p(w)$ onto $L^p(w)/\bar{H}^p_0(w)$ if and only if T_{ϕ_0} is an invertible operator from H^p onto $L^p/\bar{H}^p_0$.

Proof. (1) Let ε be a positive constant. For any $f \in \mathcal{P}$ and $g \in \mathcal{P}_0 = e^{i\theta}\mathcal{P}$,

$$\int |\phi f + \bar{g}|^p w dm \geqq \varepsilon \int |f|^p w\, dm$$

if and only if

$$\int |\phi_0 hf + \bar{h}\bar{g}|^p dm \geqq \varepsilon \int |hf|^p dm.$$

Since $hH^p(w) = H^p$, this equivalence shows (1).

(2) If T_ϕ is invertible then there exist two functions f and g such that $\phi f = 1 + \bar{g}$, $f \in H^p(w)$ and $g \in H^p(w)$. Hence

$$\phi_0 hf = \bar{h} + \bar{h}\bar{g}.$$

Since h is outer, there exists a function F in H^p such that $T_{\phi_0} F = 1 + \bar{H}^p_0$ This implies that $\phi_0 F = 1 + \bar{G}$ for some function G in H^p_0 and hence

$$\phi_0 zF = z + \overline{\hat{G}(1)} + z\overline{(G - \hat{G}(1)\bar{z})}$$

and $z\overline{(G - G(1)\bar{z})} \in \bar{H}^p_0$. Therefore there exists a function F_1 in H^p such that $T_{\phi_0} F_1 = z + \bar{H}^p_0$ because $1 + \bar{H}^p_0$ is in the range of T_{ϕ_0} . Proceeding similarly, we obtain $T_{\phi_0} H^p \ni z^\ell + \bar{H}^p_0$ for any nonnegative integer ℓ and hence T_{ϕ_0} has *a* dense range. By this and (1), T_{ϕ_0} is invertible. The proof is reversible and hence the converse is valid.

Lemma 2. Suppose $1 \leqq p < \infty$. T_ϕ is a left invertible operator from H^p to $L^p/\bar{H}^p_0$ if and only if $\phi = \phi k$ and T_ϕ is left invertible, where ϕ is unimodular and k is an invertible function in H^∞.

Proof. If T_ϕ is a left invertible operator then for any $f \in H^p$

$$\int |\phi f|^p dm \geqq \varepsilon \int |f|^p dm$$

where ε is a positive constant. This implies that ϕ^{-1} is bounded. Hence ϕ has the form : $\phi = \phi k$ where $|\phi| = |k|$ and k is an invertible function in H^∞. The following inequalities show the 'only if' part. For any $f \in H^p$ and $g \in H^p_0$

$$\int |\phi f + \bar{g}|^p dm = \int |\phi(kf) + \bar{g}|^p dm$$

$$\geqq \varepsilon \int |f|^p dm \geqq \varepsilon \|k\|_\infty^{-p} \int |kf|^p dm.$$

The proof above shows the 'if' part because $\phi = \phi k^{-1}$ and k^{-1} is an invertible function in H^∞.

The following lemma reduces the hard part of the problem to a theory of weighted norm inequalities.

Lemma 3. Suppose $1 \leqq p < \infty$ and $\phi = \bar{h}_0/h_0$ where h_0 is an outer function in H^p. T_ϕ is a left invertible operator from H^p into $L^p/\bar{H}_0^p$ if and only if for any $f \in H^p(w)$ and $g \in H_0^p(w)$

$$\int |f + \bar{g}|^p w \, dm \geqq \varepsilon \int |f|^p w \, dm$$

where $w = |h_0|^p$ and ε is a positive constant.

Proof. If T_ϕ is left invertible then for any $F \in H^p$ and $G \in H_0^p$

$$\int |\phi F + \bar{G}|^p dm \geqq \varepsilon \int |F|^p dm$$

where ε is a positive constant. Hence

$$\int |(h_0^{-1}F) + \overline{h_0^{-1}G}|^p |h_0|^p dm \geqq \varepsilon \int |h_0^{-1}F|^p |h_0|^p dm$$

because $\phi = \bar{h}_0/h_0$, and this implies the 'only if' part. The proof is reversible and the 'if' part follows.

Lemma 4. Suppose $1 < p < \infty$. T_ϕ is an invertible operator from H^p onto $L^p/\bar{H}_0^p$ if and only if $\phi = k\bar{h}_0/h_0$ where k is an invertible function and

h_0 is an outer function in H^p with $|h_0|^p$ satisfing the (A_p) condition.

Proof. It is known [3, p133] that $(L^p/\bar{H}^p_0)^* = H^q$ and $(H^p)^* = L^q/\bar{H}^q_0$ where $1/p + 1/q = 1$. This is a simple result of the Hahn-Banach theorem. T^*_ϕ is a bounded linear operator from H^q to $L^q/\bar{H}^q_0$ and

$$T^*_\phi\, g = \bar{\phi} g + \bar{H}^q_0.$$

In fact, for any $f \in H^p$ and $g \in H^q$

$$\langle f,\ T^*_\phi\, g \rangle = \langle T_\phi f,\ g \rangle = \int \phi f \bar{g} dm = \int f \overline{\bar{\phi} g} dm.$$

We can assume that ϕ is unimodular, by Lemma 2. If T_ϕ is invertible then there exist two functions $f_1 \in H^p$ and $f_2 \in H^p_0$ such that $\phi f_1 = 1 + \bar{f}_2$. Since T^*_ϕ is also invertible, there exist two functions $g_1 \in H^q$ and $g_2 \in H^q_0$ such that $\bar{\phi} g_1 = 1 + \bar{g}_2$. Thus $f_1 g_1 = (1 + \bar{f}_2)(1 + \bar{g}_2) = 1$ and hence both f_1 and $1 + f_2$ are outer. Therefore $\phi = \bar{h}_0/h_0$ for some outer function h_0 in H^p. By Lemma 3, if $w = |h_0|^p$ then for any $f \in H^p(w)$ and $g \in H^p_0(w)$

$$\int |f + \bar{g}|^p w\, dm \geqq \varepsilon \int |f|^p w\, dm$$

where ε is a positive constant. Hence $|h_0|^p$ satisfies the condition (A_p) by a theorem of Hunt, Muckenhoupt and Wheeden [6] . Conversely if $\phi = \bar{h}_0/h_0$ and $|h_0|^p$ satisfies the condition (A_p) then by Lemma 3 T_ϕ is left invertible. We may assume that $\int h_0 dm = 1$. Then $T_\phi h_0 = 1 + \bar{H}^p_0$ and hence by the proof of Lemma 1 T_ϕ has a dense range in $L^p/\bar{H}^p_0$.

Theorem 1. Suppose $1 < p < \infty$ and $w = |h|^p$ for some outer function h in H^p. Then the following conditions on ϕ and w are equivalent.

(1) T_ϕ is an invertible operator from $H^p(w)$ onto $L^p(w)/\bar{H}^p_0(w)$.

(2) $\phi = k \dfrac{\bar{h}_0}{h_0} \dfrac{h}{\bar{h}}$

where k is an invertible function in H^∞ and h_0 is an outer function in H^p with $|h_0|^p$ satisfying the condition (A_p).

(3) $\phi = \gamma \exp(U - i\tilde{V})$

where γ is constant with $|\gamma| = 1$, U is a bounded real function, V is a real function in L^1 and

$$w \exp(\frac{p}{2} V) \text{ satisfies } (A_p).$$

Proof. (1) $\Longrightarrow$ (2). If T_ϕ is an invertible operator from $H^p(w)$ onto $L^p(w)/\bar{H}^p_0(w)$ then by Lemma 1 T_{ϕ_0} is an invertible operator from H^p onto $L^p/\bar{H}^p_0$ where $\phi_0 = \phi\bar{h}/h$. Then Lemma 4 implies (2).

(2) $\Longrightarrow$ (1). By Lemma 4, T_{ϕ_0} is an invertible operator from H^p onto $L^p/\bar{H}^p_0$ where $\phi_0 = \phi\bar{h}/h$. Then Lemma 1 implies (1).

(2) $\Longrightarrow$ (3). Since k, h_0 and h are outer, $k = \gamma_1 \exp(U + i\tilde{U})$, $h_0 = \gamma_2 \exp(v_0 + i\tilde{v}_0)$ and $h = \gamma_3 \exp(v + i\tilde{v})$ where U is bounded, $v_0 \in L^1$, $v \in L^1$ and γ_j is constant with $|\gamma_j| = 1$ $(j = 1, 2, 3)$. Hence $\phi = \gamma \exp(U - i\tilde{V})$ where $\gamma = \gamma_1(\bar{\gamma}_2 \gamma_3)^2$ and $V = -U + 2v_0 - 2v$. Therefore

$$|h|^p \exp\left(\frac{p}{2}V\right) = \exp\left(-\frac{p}{2}U\right)\exp(pv_0) = \exp\left(-\frac{p}{2}U\right)|h_0|^p.$$

Since U is bounded and $|h_0|^p$ satisfies the condition (A_p), $w\exp\left(\frac{-p}{2}V\right)$ satisfies (A_p).

(3) $\Longrightarrow$ (2). Since $w\exp\left(\frac{p}{2}V\right)$ satisfies the condition (A_p), $\exp\left(\frac{-p}{2}U\right) w \exp\left(\frac{p}{2}V\right)$ does the condition, too. Then its logarithm is integrable and hence $\exp\left(\frac{p}{2}U\right) w \exp\left(\frac{-p}{2}V\right) = |h_0|^p$ for some outer function h_0 in H^p. Put $k = \gamma \exp(U + i\tilde{U})$, $h_0 = \exp(v_0 + i\tilde{v}_0)$ and $h =$

$\exp(v + i\tilde{v})$ then $V = -U + 2v_0 - 2v$ because $\exp\left(\frac{U}{2}\right)|h| \exp\left(\frac{V}{2}\right) = |h_0|$. Hence

$$\phi = \gamma \exp(U - i\tilde{V}) = k \frac{\bar{h}_0}{h_0} \frac{h}{\bar{h}}$$

and this implies (2).

§ 3. T_ϕ and $H^p(w)$

In this section, we will show a theorem of Widom, Devinatz and Rochberg (cf. [8]) as a corollary of Theorem 1.

Lemma 5. Suppose $1 < p < \infty$. w satisfies the condition (A_p) if and only if

$$L^p(w)/\bar{H}^p_0(w) \cong H^p(w)$$

with the equivalent norms.

Proof. If w satisfies the condition (A_p) then P is bounded from $L^p(w)$ to $H^p(w)$ by [6] . Hence the mapping : $F + \bar{H}^p(w) \rightarrow PF$ is one-one from $L^p(w)/\bar{H}^p_0(w)$ onto $H^p(w)$. Since P is bounded,

$$\varepsilon^p \int |PF|^p w\, dm \leqq \inf \{\int |F + \bar{g}|^p w\, dm : g \in H^p(w)\}$$
$$\leqq \int |PF|^p w\, dm$$

where $\varepsilon \|P\| = 1$. This implies the 'only if' part. The proof is reversible and the 'if' part follows.

Lemma 6. Suppose $1 < p < \infty$ and w satisfies the condition (A_p). Then T_ϕ is an (left) invertible operator from $H^p(w)$ into $L^p(w)/\bar{H}^p_0(w)$ if and only if T_ϕ is an (left) invertible operator on $H^p(w)$.

Proof. By Lemma 5, for any $f \in H^p(w)$

$$\varepsilon^p \| T_\phi f \|_p \leqq \| T_\phi f \|_p \leqq \| T_\phi f \|_p$$

where $\varepsilon \|P\| = 1$ and $\| \cdot \|_p$ is the norm of $L^p(w)/\bar{H}^p_0(w)$ or $H^p(w)$. This implies the statement about the left invertibility. Moreover $T_\phi f = 1 + \bar{H}^p_0(w)$ if and only if $T_\phi f = 1$. This implies the statement about the invertibility.

Theorem 2. Suppose $1 < p < \infty$ and $w = |h|^p$ satisfies the condition (A_p) where h is an outer function in H^p.

(1) T_ϕ is invertible on $H^p(w)$ if and only if $\phi = \gamma \exp(U - i\widetilde{V})$ where γ is constant with $|\gamma| = 1$, U is a bounded real function, V is a real function in L^1 and $w \exp(\frac{-p}{2} V)$ satisfies (A_p).

(2) T_ϕ is left invertible on $H^p(w)$ if and only if $\phi = k(h/\bar{h})\psi$ where k is an invertible function in H^∞, and T_ψ is left invertible on H^p and ψ is unimodular.

Proof. (1) is a result of Theorem 1 and Lemma 6. (2) is a result of Lemmas 2 and 6.

By Theorem 2, if $\phi = q \exp(U - i\widetilde{V})$ and q is inner then T_ϕ is left invertible on $H^p(w)$. If p = 2 the converse is also true. However unfortunately we don't know whether if the converse is true for $p \neq 2$. By (2) of Theorem 2, it is important to describe a unimodular symbol ψ such that T is left invertible on H^p. Unfortunately we could not do it except p = 2. If p = 2, P is an orthogonal projection and hence T_ψ is left invertible on H^2 if and only if $\| \psi + H^\infty \| < 1$. This is a theorem of Devinatz and Widom (cf. [2]). Then it is not difficult to see that $\psi = q\psi_0$ where q is inner and T_{ψ_0} is invertible on H^2. Now we can conjecture the following.

Conjecture. Suppose ψ is unimodular, $1 < p < \infty$ and $p \neq 2$. T_ψ is left invertible on H^p if and only if $\psi = q\psi_0$ where q is inner and T_{ψ_0} is invertible on H^p.

The 'if' part of this conjecture is true trivially. If this conjecture is true, we can describe the symbol of a left invertible Toeplitz operator on $H^p(w)$ when w satisfies the condition (A_p). It is more interesting to show Theorem without the condition (A_p). Unfortunately we could not do it. However we can give a sufficient condition.

Propositiion 3. Suppose $1 < p < \infty$ and $w = |h|^p$ for some outer function h in H^∞. If ϕ satisfies the condition of (1) in Theorem 2 then T_ϕ is left invertible.

Proof. Note that $H^p \subset H^p(w)$ because w is bounded. If $f \in \mathcal{P}$ then $P(\phi f) \in H^p \subset H^p(w)$ and hence $\phi f - P(\phi f) \in \overline{H}^p_0 \subset \overline{H}^p_0(w)$. Therefore by Theorem 1

$$\int |P(\phi f)|^p w\, dm \geqq \inf_{g \in \mathcal{P}_0} \int |\phi f + \overline{g}|^p w\, dm \geqq \varepsilon \int |f|^p w\, dm$$

for some $\varepsilon > 0$.

§ 4. Singular integral operators

Put $Q = I - P$. For α, β in L^∞, the singular integral operator S is defined as a densely defined map on $L^p(w)$ by

$$S_{\alpha,\beta} f = \alpha Pf + \beta Qf.$$

If w satisfies the condition (A_p) and $1 < p < \infty$, then $S_{\alpha,\beta}$ is bounded and a characterization of symbols α, β of the invertible $S_{\alpha,\beta}$ are known. In fact, this is the easy corollary of a theorem of Devinatz, Widom and Rochberg (cf. [8]). We are interested in the general weight w. In this direction, Yamamoto [9] got a result when p = 2. In this section, we will prove it for arbitrary p with $1 < p < \infty$, using Theorem 1. For f in $\mathcal{P}$ and g in $\mathcal{P}_0$, put

$$\| f + g \|_{p,w} = \left(\int | f |^p w \, dm\right)^{1/p} + \left(\int | g |^p w \, dm\right)^{1/p}$$

then $\| \cdot \|_{p,w}$ is a norm on $\mathcal{C} = \mathcal{P} + \bar{\mathcal{P}}_0$. $L^p(w)$ denotes the completion of $\mathcal{C}$ by this norm $\| \cdot \|_{p,w}$. Then $L^p(w)$ is a Banach space. It is reasonable to assume that log w is integrable as in Section 2. If w satisfies the condition (A_p) and $1 < p < \infty$ then $L^p(w) \cong L^p(w)$ with equivalent norms. The converse is also true. Even if w does not satisfy the condition (A_p) $S_{\alpha,\beta}$ is a bounded operator from $L^p(w)$ into $L^p(w)$. When p = 2, Yamamoto and the author [7] determined when $S_{\alpha,\beta}$ is bounded on $L^p(w)$.

Lemma 7. Suppose $1 < p < \infty$. Then $S_{\phi,1}$ is an (left) invertible operator from $L^p(w)$ to $L^p(w)$ if and only if T_ϕ is an (left) invertible operator from $H^p(w)$ to $L^p(w)/\bar{H}^p_0(w)$.

Proof. By Lemma 2 and the proof, we can assume that ϕ is unimodular. If T_ϕ is left invertible then for any $f \in \mathcal{D}$ and $g \in \mathcal{D}_0$

$$\int |\phi f + \bar{g}|^p w\, dm \geq \varepsilon^p \int |f|^p w\, dm$$

where $\varepsilon > 0$. Since $\bar{z}g \in \mathcal{D}$ and $zf \in \mathcal{D}_0$,

$$\int |\phi g + \bar{f}|^p w\, dm = \int |\phi(\bar{z}g) + \overline{zf}|^p w\, dm$$

$$\geq \varepsilon^p \int |\bar{z}g|^p w\, dm = \varepsilon^p \int |g|^p w\, dm.$$

Note that $|\phi f + \bar{g}| = |\phi g + \bar{f}|$ because ϕ is unimodular. Then by the two inequalities above, for any $f \in \mathcal{D}$ and $g \in \mathcal{D}_0$

$$\left(\int |\phi f + \bar{g}|^p w\, dm\right)^{1/p} \geq \varepsilon \left\{\left(\int |f|^p w\, dm\right)^{1/p} + \left(\int |g|^p w\, dm\right)^{1/p}\right\}$$

and this implies that $S_{\phi,1}$ is left invertible. The converse is trivial. Now we will show the statement of the invertibility.

$$T_\phi H^p(w) \text{ is dense in } L^p(w)/\bar{H}_0^p(w)$$

if and only if

$$\phi H^p(w) + \bar{H}_0^p(w) \text{ is dense in } L^p(w)$$

if and only if

$$S_{\phi,1} L^p(w) \text{ is dense in } L^p(w)$$

by the definition of $L^p(w)$. This completes the proof of the lemma.

Theorem 3. Suppose $1 < p < \infty$ and $w = |h|^p$ for some outer function h in H^p.

(1) $S_{\alpha,\beta}$ is an invertible operator from $L^p(w)$ onto $L^p(w)$ if and only if both α and β are invertible in L^∞ and $\alpha/\beta = \gamma \exp(U - i\tilde{V})$ where γ is constant with $|\gamma| = 1$, U is a bounded real function, V is a real function in L^1 and $w \exp(\frac{p}{2} V)$ satisfies (A_p).

(2) $S_{\alpha,\beta}$ is a left invertible operator from $L^p(w)$ into $L^p(w)$ if and only if both α and β are invertible in L^∞ and $\alpha/\beta = k(h/\bar{h})\phi$ where k is an invertible function in H^∞, ϕ is unimodular and $S_{\phi,1}$ is left invertible on L^p.

Proof. (1) is clear by Lemma 7 and Theorem 1. (2) is clear by Lemmas 1, 2 and 7.

As in [9], we can show that $S_{\alpha,\beta}$ is left invertible if and only if both $S_{\alpha,\beta}$ and $S_{\alpha,-\beta}$ is left invertible.

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