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Canonical modules and Cohen-Macaulay types of partially ordered sets

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Every partially ordered set (or "poset" for short) to be studied is finite. We write $\#(X)$ for the cardinality of a finite set X.

(1.1) Let P be a poset. Then we set $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ with $\hat{0} < \alpha < \hat{1}$ for every $\alpha \in P$. A chain C of the poset $\hat{P}$ is a totally ordered subset of $\hat{P}$. From now on, we assume that $\hat{P}$ is graded of rank $d+1$ [6, p.99], that is to say, there exists a unique rank function $\rho : \hat{P} \to \{0,1,\ldots,d+1\}$ such that $\rho(\hat{0}) = 0$, $\rho(\hat{1}) = d+1$, and $\rho(\beta) = \rho(\alpha)+1$ if $\alpha < \beta$ and $\alpha < \gamma < \beta$ for no $\gamma \in \hat{P}$.

(1.2) Fix a field k. Let $A = k[\,x\,;\, x \in P\,]$ be the polynomial ring over k with the "standard grading," i.e., each $\deg x = 1$, whose variables are the elements of P. We write I for the ideal of A generated by those quadratic monomials xy such that x and y are incomparable in P. The quotient algebra $k[P] = A/I$ is called the *Stanley-Reisner ring* of P over k. Define $\theta_i \in k[P]$ by $\theta_i = \Sigma_{\rho(x)=i}\, x$ for each $1 \le i \le d$. Then the sequence $\theta_1, \theta_2, \ldots, \theta_d$ is a system of parameters for $k[P]$, in other words, (i) $\theta_1, \ldots, \theta_d$ are algebraically independent over k and (ii) $k[P]$ is finitely generated as a module over the subalgebra $k[\theta] = k[\theta_1, \theta_2, \ldots, \theta_d]$. We say that $k[P]$ is *Cohen-Macaulay* (or P is Cohen-Macaulay over k) if $k[P]$ is a free module over $k[\theta]$.

For example, if $L = P\hat{}$ is a semimodular lattice then $k[P]$ is Cohen-Macaulay. We refer the reader to, e.g., [2] for basic information about Cohen-Macaulay posets.

(1.3) Suppose that the Stanley-Reisner ring $k[P]$ of a poset P is Cohen-Macaulay. First, the *canonical module* $\Omega(k[P])$ of $k[P]$ is defined to be the graded module

$$\Omega(k[P]) = \mathrm{Hom}_{k[\theta]}(k[P], k[\theta])$$

over $k[P]$. Secondly, the *socle* $\mathrm{Soc}(k[P]/(\theta))$ of $k[P]/(\theta)$ is

$$\mathrm{Soc}(k[P]/(\theta)) = \{\, y \in k[P]/(\theta)\, ;\, xy = 0 \text{ for every } x \in P \,\},$$

where (θ) is the parameter ideal $(\theta_1, \theta_2, \ldots, \theta_d)$ of $k[P]$. The dimension $\dim_k \mathrm{Soc}(k[P]/(\theta))$ of $\mathrm{Soc}(k[P]/(\theta))$ as a vector space over k is called the *Cohen-Macaulay type* of $k[P]$.

It is known, e.g., [3, Corollary 6.11] that the Cohen-Macaulay type of $k[P]$ is equal to the minimal number of generators of $\Omega(k[P])$ as a module over $k[P]$. Moreover, in the language of homological algebra, the Cohen-Macaulay type of $k[P]$ is equal to $\dim_k \mathrm{Tor}_{v-d}{}^A(k[P],k)$ with $v = \#(P)$.

(1.4) Now, if $\alpha < \beta$ in $P\hat{}$ with $\rho(\beta) - \rho(\alpha) = r + 1$, then the $(r-1)$-th reduced homology group $H^{\sim}{}_{r-1}((\alpha,\beta);k)$ of the open interval $(\alpha, \beta) = \{\, x \in P\hat{}\, ;\, \alpha < x < \beta \,\}$ of $P\hat{}$ can be imbedded in $k[P]$. Given a chain $C : \hat{0} = x_0 < x_1 < \ldots < x_s < x_{s+1} = \hat{1}$ of $P\hat{}$ with each $\rho(x_{i+1}) - \rho(x_i) = r(i) + 1$, we write $\mathcal{R}(C)$ for the subspace of $k[P]$ spanned by those polynomials

$$f_0 x_1{}^2 f_1 x_2{}^2 \ldots f_{s-1} x_s{}^2 f_s$$

with $f_i \in H^{\sim}{}_{r(i)-1}((x_i, x_{i+1});k)$ for every $0 \le i \le s$. Here we employ the convention that each monimial of f_i is of the form $\alpha_1\alpha_2 \ldots \alpha_{r(i)}$ with $x_i < \alpha_1 < \alpha_2 < \ldots < \alpha_{r(i)} < x_{i+1}$. Moreover, we define $\mathcal{Q}(C)$ to be the subspace of $k[P]/(\theta)$ which is the image of $x_1^{-2}x_2^{-2} \ldots x_s^{-2}\mathcal{R}(C)$ in $k[P]/(\theta)$.

(1.5) Let $\mu = \mu_{P\hat{}}$ be the Möbius function of $P\hat{}$ as defined in, e.g., [6, p.116]. If $C : 0\hat{} = x_0 < x_1 < \ldots < x_s < x_{s+1} = 1\hat{}$ is a chain of $P\hat{}$, then we set

$$\mu(C) = \mu(x_0,x_1)\mu(x_1,x_2) \ldots \mu(x_s,x_{s+1}).$$

Thus, in particular,

$$\dim_k(\mathcal{R}(C)) = |\mu(C)|.$$

We say that a chain $C : 0\hat{} = x_0 < x_1 < \ldots < x_s < x_{s+1} = 1\hat{}$ of $P\hat{}$ is *essential* if $\mu(C) \neq 0$. Let $\mathcal{E}(P\hat{})$ be the set of essential chains of $P\hat{}$ and $\mathcal{E}^*(P\hat{})$ ($\subset \mathcal{E}(P\hat{})$) the set of minimal essential chains (or *fundamental chains* [4]) of $P\hat{}$.

We write $\mathcal{J}^*(k[P])$ for the ideal of $k[P]$ generated by all $\mathcal{R}(C)$ with $C \in \mathcal{E}^*(P\hat{})$. Moreover, we define $\mathfrak{J}^*(k[P])$ to be the ideal of $k[P]$ which is generated by those square-free monomials $x_1x_2 \ldots x_s$ such that $x_i < x_{i+1}$ in P for every $1 \leq i \leq s$ and $0\hat{} = x_0 < x_1 < \ldots < x_s < x_{s+1} = 1\hat{} \in \mathcal{E}^*(P\hat{})$.

We are now in the position to state our main result in the paper.

(1.6) THEOREM. Suppose that the Stanley-Reisner ring $k[P]$ of a poset P is Cohen-Macaulay. Then the ideal $\mathcal{J}^*(k[P])$ is isomorphic to the canonical module $\Omega(k[P])$ of $k[P]$, up to shift in grading, if and only if the Cohen-Macaulay type of $k[P]$ is equal to $\Sigma_{C \in \mathcal{E}^*(P\hat{})} |\mu(C)|$.

Proof. Let $\mathcal{J}(k[P])$ be the ideal of $k[P]$ generated by all $\mathcal{R}(C)$ with $C \in \mathcal{E}(P\hat{})$. The monomorphism ψ in [1, Theorem 1] enables us to define a monomorphism $\psi : \mathcal{I}(k[P]) \to \Omega(k[P])$ of graded modules over $k[P]$ in the analogous way. Note that the lowest degree of a non-zero homogeneous element of $\Omega(k[P])$ (resp. $\mathcal{I}(k[P])$) is $-d + \min\{\#(C); C \in \mathcal{E}(P\hat{\ })\}$ (resp. $d + \min\{\#(C); C \in \mathcal{E}(P\hat{\ })\}$), while ψ has degree $-2d$. We write $\mathcal{A}(k[P])$ for the subspace of $k[P]$ which is spanned by those polynomials $f_0 x_1^{n_1} f_1 x_2^{n_2} \ldots f_{s-1} x_s^{n_s} f_s$ such that $\hat{0} = x_0 < x_1 < \ldots < x_s < x_{s+1} = \hat{1} \in \mathcal{E}(P\hat{\ })$ with $\rho(x_{i+1}) - \rho(x_i) = r(i) + 1$, $f_i \in \tilde{H}_{r(i)-1}((x_i, x_{i+1}); k)$ for every $0 \le i \le s$, and each $n_i \ge 2$ is an integer. Since $\mathcal{A}(k[P]) \subset \mathcal{I}(k[P])$, it follows from [5, Theorem 7.1, p.80] that the above monomorphism $\psi : \mathcal{I}(k[P]) \to \Omega(k[P])$ is an isomorphism and $\mathcal{A}(k[P]) = \mathcal{I}(k[P])$. Moreover, each subspace $\mathcal{R}(C)$ with $C \in \mathcal{E}^*(P\hat{\ })$ is contained in the subspace of $k[P]$ spanned by an arbitrary system of generators of the ideal $\mathcal{I}(k[P])$. Hence $\mathcal{I}^*(k[P]) = \mathcal{I}(k[P])$ if and only if the minimal number of generators of $\Omega(k[P])$, i.e., the Cohen-Macaulay type of $k[P]$, is equal to $\sum_{C \in \mathcal{E}^*(P\hat{\ })} |\mu(C)|$ as desired. Q.E.D.

When $\mu(\hat{0}, \hat{1}) \neq 0$, the above Theorem (1.6) essentially coincides with [1, Theorem 2]. Now, it is shown in [4] that, for a poset P such that $L = P\hat{\ }$ is a modular lattice, the Cohen-Macaulay type of $k[P]$ is equal to $\sum_{C \in \mathcal{E}^*(P\hat{\ })} |\mu(C)|$.

(1.7) COROLLARY. If $L = P\hat{\ }$ is a modular lattice, then $\mathcal{I}^*(k[P])$ is isomorphic to the canonical module $\Omega(k[P])$ of $k[P]$.

REMARK. Every subspace $\mathcal{Q}(C)$ of $k[P]/(\theta)$ with $C \in \mathcal{E}^*(P\hat{\ })$ is contained in $\mathrm{Soc}(k[P]/(\theta))$. However, even though the Cohen-Macaulay type of $k[P]$ is equal to $\sum_{C \in \mathcal{E}^*(P\hat{\ })} |\mu(C)|$, the subspace of $\mathrm{Soc}(k[P]/(\theta))$ spanned by all $\mathcal{Q}(C)$ with $C \in \mathcal{E}^*(P\hat{\ })$ does not necessarily coincide with $\mathrm{Soc}(k[P]/(\theta))$.

It might be of interest to give a similar result to Hochster's theorem [5, Theorem7.3, p.81]. We refer the reader to, e.g., [5, p.28] for the definition of an orientable pseudo-manifold with boundary. It follows easily that the order complex $\Delta(P)$

(see, e.g., [6, p.120]) of a Cohen-Macaulay poset P possesses the connectivity property [5, Definition 3.15 (c), p.28]. Moreover, if the order complex $\Delta(P)$ of a Cohen-Macaulay poset P over a field k is an orientable pseudo-manifold with boundary, then the poset P satisfies (☆) $|\mu(x,y)| \le 1$ for every $\hat{0} \le x \le y \le \hat{1}$ in $\hat{P}$.

(1.8) PROPOSITION. Let P be a Cohen-Macaulay poset over k. Then there exists a homogeneous non-zero divisor $\Theta \in k[P]$ of degree d with $\mathfrak{J}^*(k[P])\cdot\Theta = \mathfrak{J}(k[P])$ ($= \mathfrak{a}(k[P])$) if and only if $\Delta(P)$ is an orientable pseudo-manifold with boundary.

Proof. Let $\mathfrak{M}(\hat{P})$ be the set of maximal chains of $\hat{P}$. Given a function $\varepsilon : \mathfrak{M}(\hat{P}) \to k - \{0\}$, for each $F : \hat{0} = y_0 < y_1 < \ldots < y_d < y_{d+1} = \hat{1} \in \mathfrak{M}(\hat{P})$, we set $m(F;\varepsilon) = \varepsilon(F) \prod_{1 \le i \le d} y_i \in k[P]$, and define $\Theta(\varepsilon) \in k[P]$ by $\Theta(\varepsilon) = \sum_{F \in \mathfrak{M}(\hat{P})} m(F;\varepsilon)$, which is a homogeneous non-zero divisor on $k[P]$ of degree d.

Now, let P be a Cohen-Macaulay poset over k and suppose the existence of a homogeneous non-zero divisor $\Theta \in k[P]$ of degree d with $\mathfrak{J}^*(k[P])\cdot\Theta = \mathfrak{a}(k[P])$. Then Θ must be of the form $\Theta(\varepsilon)$ for some ε. Hence, the poset P satisfies the above condition (☆), in particular, $\Delta(P)$ is a pseudo-manifold with boundary. If $F : \hat{0} < \alpha_1 < \ldots < \alpha_p < \gamma < \beta_1 < \ldots < \beta_q < \hat{1}$, $F' : \hat{0} < \alpha_1 < \ldots < \alpha_p < \delta < \beta_1 < \ldots < \beta_q < \hat{1} \in \mathfrak{M}(\hat{P})$ with $\gamma \ne \delta$, then $\varepsilon(F) = -\varepsilon(F')$ since $C : \hat{0} < \alpha_1 < \ldots < \alpha_p < \beta_1 < \ldots < \beta_q < \hat{1} \in \mathfrak{C}(\hat{P})$ and $\alpha_1^2 \ldots \alpha_p^2 (\gamma - \delta) \beta_1^2 \ldots \beta_q^2 \in \mathfrak{R}(C)$. Thus $0 \ne \Theta \in H_{d-1}(\Delta(P),\partial\Delta(P);k)$, hence $\Delta(P)$ is orientable.

On the other hand, suppose that the order complex $\Delta(P)$ of a Cohen-Macaulay poset P over k is an orientable pseudo-manifold with boundary. Since $H_{d-1}(\Delta(P),\partial\Delta(P);k) \ne (0)$, there exists $\varepsilon : \mathfrak{M}(\hat{P}) \to k - \{0\}$ such that $\varepsilon(F) = -\varepsilon(F')$ if $\#(F \cap F') = d - 1$. Then $0 \ne x_1 x_2 \ldots x_s \Theta(\varepsilon) \in \mathfrak{R}(C)$ for every $C : \hat{0} = x_0 < x_1 < \ldots < x_s < x_{s+1} = \hat{1} \in \mathfrak{C}(\hat{P})$. Thus, since $\dim_k \mathfrak{R}(C) = 1$, the subspace $\mathfrak{R}(C)$ is spanned by $x_1 x_2 \ldots x_s \Theta$ over k. Hence $\mathfrak{J}^*(k[P])\cdot\Theta = \mathfrak{a}(k[P])$ as required. Q.E.D.

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