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**VANISHING THEOREM FOR THE
TEMPERED DISTRIBUTIONS**

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VANISHING THEOREM FOR THE TEMPERED DISTRIBUTIONS

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0. Introduction

The functor $\mathrm{RH}(\cdot)$ which gives the Riemann-Hilbert correspondence was constructed and deeply studied by Kashiwara [K 3]. This functor plays an important role in the construction of the tempered micro-localization functor due to Andronikof ([A 1, A 2]). The tempered micro-localization functor in Gevrey classes is also constructed by the author of this paper [H 2] using $\mathrm{RH}^{(s)}(\cdot)$. It is an important problem to study the vanishing of the higher cohomology groups of $\mathrm{RH}(\mathbb{C}_\Omega)$ ($\mathrm{RH}^{(s)}(\mathbb{C}_\Omega)$) with Ω a pseudo-convex open set, because it is deeply connected with the concentration of cohomology groups of the tempered micro-localization functor (see section 3). We also need the vanishing of cohomology groups to study the tempered (Gevrey class) micro-localization functor with Čech representation.

Let $\Omega \subset X = \mathbb{C}^n$ be an open set and $s \in (1, \infty)$ or ϕ . We denote by $\mathbb{C}_\Omega$ a constructible sheaf whose stalk is $\mathbb{C}$ in Ω and 0 outside of Ω .

Theorem 0.1. *Assume Ω is a relative compact pseudo-convex open set. Then the complex of sheaves $\mathrm{RH}^{(s)}(\mathbb{C}_\Omega)$ is concentrated degree 0, and we have*

$$H^k(\mathrm{R}\Gamma(X, \mathrm{RH}^{(s)}(\mathbb{C}_\Omega))) = 0 \quad k \neq 0.$$

We give the definitions and notations of Theorem 0.1 in the next section, and some applications in section 3. The proof of the main theorem is given in section 2.

1. Notations

Let $M = \mathbb{R}^n$ with a coordinate (x_i) and $s \in (1, \infty)$ or ϕ . We denote by $\mathcal{E}(\mathcal{D})$ the sheaf of infinitesimal differentiable functions (with compact support). We set

$$\|f\|_K^{l,s} = \sup_{x \in K, \alpha} \frac{|D^\alpha f|}{|\alpha|!^{(s)} l^{|\alpha|}} \quad (s \neq \phi)$$

for a compact set $K \subset M$ and $l > 0$.

We define the sheaf $\mathcal{E}^{(s)}$ ($\mathcal{D}^{(s)}$) of ultradifferentiable functions of Beurling class (s) as follows. For an open set U and $s \neq \phi$,

$$\mathcal{E}^{l,s}(U) := \{f \in \mathcal{E} : \|f\|_K^{l,s} < \infty \text{ for any } K \subset\subset U\},$$

$$\mathcal{D}^{l,s}(U) := \mathcal{E}^{l,s} \cap \mathcal{D}(U),$$

and

$$\mathcal{E}^{(s)} := \lim_{\overline{l}} \mathcal{E}^{l,s},$$

$$\mathcal{D}^{(s)} := \mathcal{E}^{(s)} \cap \mathcal{D}.$$

We set $\mathcal{E}^{(\phi)} := \mathcal{E}$ and so on.

The sheaf of ultradistributions $\mathcal{D}_b^{(s)}$ of Beurling class (s) is the topological dual of $\mathcal{D}^{(s)}$. When $s = \phi$, this is nothing but the sheaf of distributions by definition. For the details, refer to Komatsu [Ko 2, Ko 3].

Let U be an open set of M and $\mathbb{C}_U$ a constructible sheaf whose stalk is $\mathbb{C}$ in U and 0 outside of U . We define the sheaf $\text{TH}^{(s)}(\mathbb{C}_U)$ of tempered ultradistributions as follows. For any open set $V \subset M$

$$\text{TH}^{(s)}(\mathbb{C}_U)(V) := \{u \in \mathcal{D}_b^{(s)}(U \cap V) : \text{there exists a } u' \in \mathcal{D}_b^{(s)}(V) \text{ satisfying } u = u'|_{U \cap V}\}.$$

We remark that $\text{TH}(\cdot) := \text{TH}^{(\phi)}(\cdot)$ was constructed by Kashiwara [K 3] for general constructible sheaves.

Proposition 1.1 (cf. [K 3; Lemma 3.3]). *Let $U \subset M$ be a relative compact open set. Then we have the following equivalent conditions (1), (2) and (3) for any $s \in (1, \infty)$ and $u \in \Gamma(U, \mathcal{D}_b^{(s)})$.*

- (1) *There exists a $v \in \Gamma(M, \mathcal{D}_b^{(s)})$ satisfying $u = v$ in U . (i.e. $u \in \Gamma(M, \text{TH}^{(s)}(\mathbb{C}_U))$).*
- (2) *There exist $C, l > 0$ satisfying*

$$|\langle u, \psi \rangle| \leq C \| \psi \|_U^{l,s}$$

for any $\psi \in \mathcal{D}_b^{l,(s)}(U)$.

- (3) *There exist positive constants C, l and L satisfying*

$$|\langle u, \psi \rangle| \leq C \sup_{x \in U, \alpha} \frac{\exp(L \text{dist}(CU, x)^{\frac{-1}{s-1}}) |D^\alpha f|}{|\alpha|!^{(s)} l^{|\alpha|}}$$

for any $\psi \in \mathcal{D}_b^{l,(s)}(U)$.

In the case of $s = \phi$, we have the same proposition for the tempered distributions [K 3; Lemma 3.3].

Finally we introduce the functor $\text{RH}^{(s)}(\cdot)$. Let X be a complex manifold. We denote by $\bar{X}$ the complex conjugate space of X .

$$\text{RH}^{(s)}(\mathbb{C}_U) := \mathbb{R}\text{Hom}_{\mathcal{D}_{\bar{X}}}(\mathcal{O}_{\bar{X}}, \text{TH}^{(s)}(\mathbb{C}_U))$$

where $\mathcal{D}_{\bar{X}}$ is the sheaf of linear differential operators with holomorphic coefficients in $\bar{X}$ and $\mathcal{O}_{\bar{X}}$ is the sheaf of holomorphic functions in $\bar{X}$ (i.e. Cauchy-Riemann system of $X^{\mathbb{R}}$ as $\mathcal{D}_{X \times \bar{X}}$ module).

2. Proof of the main theorem

In this section, we give the proof of Theorem 0.1 in the case of $s \neq \phi$. Since the proof of the vanishing of $\text{RH}^{\phi}(\mathbb{C}_{\Omega})$ goes in the similar way, we omit it.

We make a several preparation. Let $s > 1$ and $Q_l(\zeta)$ ($l > 0$) entire functions in $\mathbb{C} = (\zeta)$ satisfying the following conditions. There exists a function $l' : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $l'(l) \rightarrow \infty$ ($l \rightarrow \infty$), and there exist positive constants B_l and C_l , we have estimates

$$|Q_l(\zeta)| \leq B_l \exp(l|\zeta|^{\frac{1}{2s}}) \quad \zeta \in \mathbb{C},$$

and

$$|Q_l(\zeta)| \geq C_l \exp(l'(l)|\zeta|^{\frac{1}{2s}}) \quad \text{Re } \zeta \geq 0.$$

We can concretely construct a family of entire functions satisfying the above conditions (see [K 1; Theorem 10.1]). Our first step is to construct a differential operator of infinite order which possesses an elliptic property. Let $X = \mathbb{C}^n = (\zeta_1, \dots, \zeta_n)$ with a coordinate $(\xi_1 + i\eta_1, \dots, \xi_n + i\eta_n)$. We set

$$P_l(\zeta) := Q_l(\zeta_1^2 + \dots + \zeta_n^2).$$

Then $P_l(\zeta)$ satisfies the following estimations.

$$|P_l(\zeta)| \leq B_l \exp(l|\zeta|^{\frac{1}{2s}}) \quad \zeta \in \mathbb{C}^n,$$

and

$$|P_l(\zeta)| \geq C_l \exp(l'(l)|\zeta|^{\frac{1}{2s}}) \quad \frac{1}{\sqrt{2}}|\zeta| \geq |\eta|.$$

We consider a kernel function in $X = \mathbb{C}^n = (z_1 = x_1 + iy_1, \dots, z_n = x_n + iy_n)$ as

$$K_l(z) := \int_{\mathbb{R}^n} \frac{e^{-iz\zeta}}{P_l(\zeta)} d\zeta \quad z \in \mathbb{R}^n.$$

It is easy to see $K_l(x)$ is an ultradifferentiable function of Gevery order (s) in $\mathbb{R}^n$. Moreover we have the following lemma.

Lemma 2.1. *There exist $\sigma > 0$ and a function $l' : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $l'(l) \rightarrow \infty$ ($l \rightarrow \infty$), $K_l(z)$ is holomorphic in $T_\sigma := \{z; \sigma|x| > |y|\}$ and satisfies the estimate*

$$\sup_{z \in T_\sigma} \left| \frac{l'(\sigma)^{|\alpha+\beta|} D_x^\alpha D_y^\beta K_l(z)}{|\alpha + \beta|!^s} \right| < \infty.$$

Proof. The first assertion of the lemma is easily shown by changing the path of the integral as

$$\eta = \frac{-x}{\sqrt{2}|x|} |\xi|.$$

To obtain the estimate of the lemma, we remark the inequality

$$t^n \exp(-lst^{\frac{1}{s}}) \leq l^{-sn} n!^s,$$

for any $t > 0$ and $n \in \mathbb{N}$. Since $K(z)$ is holomorphic in T_σ , it is enough to show the estimates for $(\frac{\partial}{\partial z})^\alpha K_l(z)$, and this is a direct calculation.

We remark that $P_l(\frac{\partial}{\partial z})$ is a partial differential operator of infinite order with Gevrey growth (s) and satisfies

$$P_l(\frac{\partial}{\partial x}) K_l(x) = \delta(x).$$

Let $U \subset \mathbb{R}^n$ be a relative compact open set and $u \in \mathcal{D}_b^{(s)}(\mathbb{R}^n)$ with $\text{supp}(u) \subset \partial U$. We define an ultradistribution in $\mathbb{R}^n$ as

$$g(x) := K_l * u \in \mathcal{D}_b^{(s)}(\mathbb{R}^n).$$

Lemma 2.2. *If we take a sufficiently large l , we have*

- (1) $g(x)$ is an ultradifferentiable function of Gevrey class (s) in $\mathbb{R}^n$,
- (2) and, g is an analytic function outside of ∂U and can be holomorphically extended over $W_\sigma = \{z \in X : |y| < \text{dist}(\partial U, x)\}$ for some $\sigma > 0$. Moreover any derivative of $g(z)$ is bounded in W_σ .

Proof. On account of the estimation of K_l in Lemma 2.1, we easily obtain the first assertion of the lemma. Since for any point $(x, y) \in W_\sigma$

$$\text{supp}(u) - (x, y) \subset \subset T_\sigma,$$

and $K_l(z)$ is holomorphic in T_σ , g can be holomorphically extended over W_σ . Moreover $K_l(z)$ can be extended over $\mathbb{C}^n$ as an ultradifferentiable function preserving Gevrey growth order. It is easy to see g is bounded in W_σ .

Let Ω a relative compact pseudo-convex open set and $\phi = |z|^2$. We denote by $L_{(p,q)}^2(\Omega)$ (resp. $C_{(p,q)}^\infty(\Omega)$, $\text{TH}_{(p,q)}^{(s)}(\mathbb{C}_\Omega)(X)$) the space of (p, q) forms with coefficients in $L^2(\Omega)$ (resp. $C^\infty(\Omega)$, $\text{TH}^{(s)}(\mathbb{C}_\Omega)(X)$). $L_{(p,q)}^2(\Omega)$ is Hilbert space equipped with a norm

$$\|f\|_{L^2(\Omega)}^2 := \sum_{I,J} \int_{\mathbb{C}^n} |f_{I,J}|^2 dz \wedge d\bar{z}.$$

We denote by $\theta : C_{(p,q+1)}^\infty(\Omega) \rightarrow C_{(p,q)}^\infty(\Omega)$ the formal adjoint operator of $\bar{\partial}$ which has the concrete expression as

$$\theta f := \sum_{I,J} \sum_{j \in [1,n]} \frac{\partial}{\partial z_j} f_{I,jk} dz^I \wedge d\bar{z}^J,$$

for any $f \in C_{(p,q+1)}^\infty(\Omega)$. From now on, we fix $\phi = |z|^2$.

Proposition 2.3. *Let $f \in C_{(p,q+1)}^\infty(\Omega)$ and $u \in C_{(p,q)}^\infty(\Omega)$. We assume u satisfies*

$$\bar{\partial}u = f,$$

and

$$\theta(e^{-\phi}u) = 0.$$

Then we have the following estimate.

$$\|\delta(z)^i \sum_{|\alpha|=i} D^\alpha u\|_{L^2(\Omega)}^2 \leq C^i (\|u\|_{L^2(\Omega)}^2 + \sum_{|\alpha|=i-1} \|\delta(z)^{|\alpha|} D^\alpha f\|_{L^2(\Omega)}^2) \quad (i \geq 1)$$

where C is a constant which depends only on Ω and $\delta(z) := \text{Min}(\text{dist}(C\Omega, z), 1)$.

Proof. We assume $q \geq 1$. In the case of $q = 0$, we can prove the proposition in the similar way as the following proof. Set

$$U_k = \{z \in \Omega; \quad \text{dist}(C\Omega, z) \geq \frac{1}{2^k}\} \quad k \geq 0.$$

We choose C^1 functions $\{\chi_k(z)\}_{k \geq 0}$ satisfying

$$0 \leq \chi_k \leq 1,$$

$$\chi_k(z) = 1 \quad z \in U_k,$$

$$\text{supp}(\chi_k) \subset U_{k+1}^-$$

and, there exists a positive constant C

$$\sum_{i=0}^n (|\frac{\partial}{\partial z_i} \chi_k| + |\frac{\partial}{\partial \bar{z}_i} \chi_k|) \delta(z) \leq C$$

for any k . By the assumption, we have

$$\bar{\partial}u = f,$$

$$\theta u = au$$

where a is a differential operator of order 0 whose coefficients are polynomials of degree 1. We have for any multi index α

$$\bar{\partial}(\chi_k D^\alpha u) = \bar{\partial}\chi_k \wedge D^\alpha u + \chi_k D^\alpha f,$$

$$\theta(\chi_k D^\alpha u) = b D^\alpha u + \chi_k D^\alpha (au)$$

where b is a differential operator of degree 0. Remark that

$$\delta(z)b : L_{(p,q+1)}^2(\Omega) \rightarrow L_{(p,q)}^2(\Omega)$$

is a bounded operator. We have for any multi index α ($|\alpha| \geq 1$),

$$\begin{aligned} \|\delta^{|\alpha|+1} \bar{\partial} \chi_k D^{\alpha+1} u\|_{L^2(\Omega)}^2 &\leq \|\delta^{|\alpha|+1} \bar{\partial} \chi_k \wedge D^\alpha u\|_{L^2(\Omega)}^2 + \|\delta^{|\alpha|+1} \chi_k D^\alpha f\|_{L^2(\Omega)}^2 \\ &\leq C_1 (\|\delta^{|\alpha|} D^\alpha u\|_{L^2(\Omega)}^2 + \|\delta^{|\alpha|} D^\alpha f\|_{L^2(\Omega)}^2). \end{aligned}$$

Similary we have

$$\begin{aligned} \|\delta^{|\alpha|+1} \theta(\chi_k D^\alpha u)\|_{L^2(\Omega)}^2 &\leq C_2 (\|\delta^{|\alpha|} D^\alpha u\|_{L^2(\Omega)}^2 \\ &\quad + \sum_{|\beta|=|\alpha|-1, \beta \prec \alpha} \|\delta^{|\beta|} D^\beta u\|_{L^2(\Omega)}^2 + \|\delta^{|\alpha|} D^\alpha u\|_{L^2(\Omega)}^2). \end{aligned}$$

Therefore we obtain the following estimation.

$$\begin{aligned} &\sum_{1 \leq i \leq n} (\|\delta^{|\alpha|+1} \frac{\partial}{\partial z_i} \chi_k D^{\alpha+1} u\|_{L^2(\Omega)}^2 + \|\delta^{|\alpha|+1} \frac{\partial}{\partial \bar{z}_i} \chi_k D^{\alpha+1} U\|_{L^2(\Omega)}^2) \\ &\leq 4 (\|\delta^{|\alpha|+1} \partial \chi_k D^{\alpha+1} u\|_{L^2(\Omega)}^2 + \|\delta^{|\alpha|+1} \bar{\partial} \chi_k D^{\alpha+1} u\|_{L^2(\Omega)}^2) \\ &\leq C_3 (\|\delta^{|\alpha|} D^\alpha u\|_{L^2(\Omega)}^2 + \|\delta^{|\alpha|} D^\alpha f\|_{L^2(\Omega)}^2 \\ &\quad + \sum_{|\beta|=|\alpha|-1, \beta \prec \alpha} \|\delta^{|\beta|} D^\beta u\|_{L^2(\Omega)}^2) \end{aligned}$$

where C_3 is a positive constat depending only on Ω . Taking limit $k \rightarrow \infty$, we have for any multi index α

$$\begin{aligned} &\sum_{1 \leq i \leq n} (\|\delta^{|\alpha|+1} \frac{\partial}{\partial x_i} D^{\alpha+1} u\|_{L^2(\Omega)}^2 + \|\delta^{|\alpha|+1} \frac{\partial}{\partial \bar{x}_i} D^{\alpha+1} u\|_{L^2(\Omega)}^2) \\ &\leq C_4 (\|\delta^{|\alpha|} D^\alpha u\|_{L^2(\Omega)}^2 + \|\delta^{|\alpha|} D^\alpha f\|_{L^2(\Omega)}^2 \\ &\quad + \sum_{|\beta|=|\alpha|-1, \beta \prec \alpha} \|\delta^{|\beta|} D^\beta u\|_{L^2(\Omega)}^2). \end{aligned}$$

By induction and the above inequality, we obtain the disired estimation.

Now we show the proof of the main theorem.

proof. Consider the complex

$$\mathrm{TH}_{(p,q)}^{(s)}(\mathbb{C}_\Omega)(X) \xrightarrow{\bar{\partial}} \mathrm{TH}_{(p,q+1)}^{(s)}(\mathbb{C}_\Omega)(X) \xrightarrow{\bar{\partial}} \mathrm{TH}_{(p,q+2)}^{(s)}(\mathbb{C}_\Omega)(X).$$

Given $f \in \mathrm{TH}_{(p,q+1)}^{(s)}(\mathbb{C}_\Omega)(X)$ with $\bar{\partial} f = 0$. We take $f' \in \mathcal{D}_b^{(s)}(X)$ satisfying

$$f'|_U = f|_U,$$

$$\mathrm{supp}(f') \subset \bar{U},$$

and

$$\mathrm{supp}(\bar{\partial} f') \subset \partial U.$$

For a sufficiently large $l > 0$, we set

$$h := K_l * \bar{\partial} f',$$

where $K(z, \bar{z})$ is a kernel function associated with $P_l(\zeta, \bar{\zeta})$ constructed in the beginning of this section. Remark that, on account of Lemma 2.1, $K_l(z, \bar{z})$ is a real analytic function in the underlying space $X^R = \mathbb{R}^{2n}$ except for the origin and can be holomorphically extended over the domain $T_\sigma \subset X \times \bar{X}$ described in the same lemma. Therefore h satisfies the following.

- (1) $\bar{\partial} h = k_l * \bar{\partial} \bar{\partial} f' = 0$ in X^R .
- (2) h is a $(p, q+2)$ form with coefficients in ultradifferentiable functions of Gevrey order (s) in X^R .
- (3) There exists a positive constant C , we have the estimate

$$\left| \frac{\partial^\alpha}{\partial z^\alpha} \frac{\partial^\beta}{\partial \bar{z}^\beta} h \right| \leq C \frac{|\alpha + \beta|!}{\text{dist}(C\Omega, z)^{|\alpha + \beta|}} \quad (z, \bar{z}) \in X^R \subset X \times \bar{X}$$

on account of Lemma 2.2.

We fix a relative compact convex open set $W \subset X^R$ with $\Omega \subset\subset W$. By the theorem of Hörmander [Hö; chapter 4], we can find $g \in C_{p,q+1}^\infty(W)$ satisfying

$$\bar{\partial} g = h \quad \text{in } W,$$

and

$$\theta(e^{-\phi} u)|_U = 0$$

where $\phi = |z|^2$. Then by proposition 2.3, we have

$$\|\delta(z)^{|\alpha + \beta|} \frac{\partial^\alpha}{\partial z^\alpha} \frac{\partial^\beta}{\partial \bar{z}^\beta} g\|_{L^2(\Omega)} \leq C^{|\alpha + \beta|} |\alpha + \beta|! \quad (z \in \Omega)$$

with a positive constant C . Since

$$P_l\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) = \Sigma C_{\alpha\beta} \frac{\partial^\alpha}{\partial z^\alpha} \frac{\partial^\beta}{\partial \bar{z}^\beta}$$

is a differential operator of infinite order with Gevrey class (s) , there exists a positive constants l satisfying

$$|C_{\alpha\beta}| \leq \frac{l^{|\alpha + \beta|}}{|\alpha + \beta|!^s}.$$

Therefore we obtain

(2.1)

$$\begin{aligned} \left\| \left(P_l\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) g \right) \exp(-l' \delta(z)^{\frac{-1}{s-1}}) \right\|_{L^2(\Omega)} &\leq \Sigma \frac{(Cl)^{|\alpha + \beta|} |\alpha + \beta|!}{|\alpha + \beta|!^s} \sup_{z \in \Omega} |\delta(z)^{-|\alpha + \beta|} \exp(-l' \delta(z)^{\frac{-1}{s-1}})| \\ &\leq C_1 \Sigma \frac{(Cl)^{|\alpha + \beta|}}{|\alpha + \beta|!^{(s-1)}} \frac{|\alpha + \beta|!^{(s-1)}}{(l')^{|\alpha + \beta|}} \\ &\leq C_2 \Sigma \left(\frac{Cl}{l'}\right)^{|\alpha + \beta|} < \infty \end{aligned}$$

for a sufficiently large l' .

We set $f''(z) := P_l(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})g$. Since W is convex and

$$\bar{\partial}(f' - f'') = \bar{\partial}f' - P_l\bar{\partial}g = \bar{\partial}f' - P_lh = \bar{\partial}f' - P_l(K_l * \bar{\partial}f') = 0$$

on account of Palamndov [P; p.300 Theorem 1], we can find $u' \in \mathcal{D}_b^{(s)}(W)$ with $\bar{\partial}u' = f' - f''$. On the other hand, since $\bar{\partial}f''|_\Omega = \bar{\partial}f'|_\Omega = 0$ and f'' satisfies the esitimation (2.1), we can find $u'' \in L_{(p,q)}^{loc}(\Omega)$ satisfying

$$f'' = \bar{\partial}u'' \quad \text{in } \Omega$$

and the estimation

$$\|u'' \frac{\exp(-l'\delta(z)^{\frac{-1}{s-1}})}{(1+|z|^2)}\|_{L^2(\Omega)} < \infty$$

due to [Hö; Theorem 4.4.2]. The above inequality means, in particular, $u'' \in \text{TH}_{(p,q)}^{(s)}(\mathbb{C}_\Omega)(X)$. Thus $u = u' + u'' \in \text{TH}_{(p,q)}^{(s)}(\mathbb{C}_\Omega)(X)$ satisfies

$$\bar{\partial}(u' + u'')|_U = ((f' - f'') + f'')|_U = f|_U.$$

3. The application of the main theorem

One of the application of the main theorem, we give the vanishing of distribution solutions along the submanifolds which are micro-locally weakly pseudo-convex.

Let M be a real analytic submanifold in a complex manifold X , and $\pi : T^*X \rightarrow X$ (resp. $\pi : T_M^*X \rightarrow X$) a cotangent bundle of X (resp. conormal bundle of M). We set for $p \in T^*X$,

$$\begin{aligned} E(p) &= T_p(T^*X), \\ \lambda_M(p) &= T_p(T_M^*X), \\ \lambda_0(p) &= T_p(\pi^{-1}\pi(p)), \\ \rho_M(p) &= \lambda_M(p) \cap i\lambda_M(p), \\ \delta(p) &= \dim_{\mathbb{C}}(\lambda_M(p) \cap i\lambda_M(p) \cap \lambda_0(p)). \end{aligned}$$

The vector space $(E(p), \sigma)$ has a natural complex symplectic structure. For any real Lagrangian vector space $\lambda \in E(p)^{\mathbb{R}}$, we denote by λ^{ρ_M} the quotient space

$$((\lambda \cap \rho(p)_M^\perp) + \rho_M(p)) / \rho_M(p).$$

We also define the bilinear form γ_{λ_M} on $\lambda_0^{\rho_M}(p)$ by setting for $(u, v) \in \lambda_0^{\rho_M}(p) \times \lambda_0^{\rho_M}(p)$

$$\gamma_{\lambda_M}(u, v) := \sigma_{\rho_M}(u, \bar{v})$$

where $\bar{v}$ is the complex conjugate of v with respect to $\lambda_M^{\rho_M}(p)$. We denote by $S_+(M, p)$ (resp $S_-(M, p)$) the number of positive (resp. negative) eigen values of the hermitian form γ_{λ_M} , and define $S(M, p) = S_+(M, p) - S_-(M, p)$.

Theorem 3.1 (cf. [K-S 1; Theorem 11.3.5]). *Let $p_0 \in T_M^*X \setminus T_X^*X$. We assume the following conditions.*

- (1) $\dim_{\mathbb{R}}(\lambda_M(p_0) \cap \nu(p_0)) = 1$ where $\nu(p_0)$ is a complex line generated by the Euler vector field.
- (2) $S_-(M, p) - \delta(p)$ is locally constant in a neighborhood of p_0 .

Then the complex $T\text{-}\mu_M(\mathcal{O}_X)_{p_0}$ is concentrated degree $k = \text{codim } M + S_-(M, p_0) - \delta(p_0)$, and the canonical morphism

$$H^k T\text{-}\mu_M(\mathcal{O}_X)_{p_0} \rightarrow H^k \mu_M(\mathcal{O}_X)_{p_0}$$

is injective.

When X is the complexification of a real analytic manifold M , the above theorem has been shown by Maltineau [Ma 1]. Andronikof, in his papers [A 1, A 2], has obtained the edge of the wedge with tempered growth order, and proved the same results in the complex case (i.e. M is a complex manifold). The first assertion replaced the functor $T\text{-}\mu_M$ with a micro-localization functor μ_M was established by Kashiwara-Schapira [K-S 1]. They proved this to construct a quantized contact transformation cohomologically and reduce the problem to the case M is a weakly pseudo-convex hypersurface. Therefore the important point is to construct a quantized contact transformation cohomologically for the tempered micro-localization functor. We first study how to construct the quantized contact transformation. Let X, Y be complex manifolds and $N \subset Y$, $M \subset X$ real analytic submanifolds, $\Phi : T^*X \rightarrow T^*Y$ a complex contact transformation which interchange (T_M^*X, p_0) and $(T_N^*Y, \Phi(p_0))$ in a neighborhood of $p_0 \in T_M^*X \setminus T_X^*X$. We denote by Q_1, Q_2 the first and the second projection from $X \times Y$ respectively and by P_1, P_2 the first and the second projection from $T^*(X \times Y)$. We set $P_1^a = P_1 \cdot a$ where a is the anti-podal map on T^*X . Let $\Lambda \subset T^*(X \times Y)$ be a complex conic Lagrangian submanifold in a neighborhood of $(p_0, -\Phi(p_0))$ obtained by taking the image of graph of Φ by the anti-podal map on T^*X . Moreover we may assume $Z = \pi(\Lambda) \subset X \times Y$ is a smooth complex hypersurface. We denote by K a pure sheaf $\mathbb{C}_Z[-1]$. And if necessarily, we cut the support of K in a neighborhood of $\pi(p, -\Phi(p))$ satisfying that $Q_2^a|_{\text{supp}(K)}$ is proper over $\text{supp}(C_N)$. This does not affect on the following arguments because we consider the problem in the micro-local point of view. To construct the quantized contact transformation, we use direct image and restriction of the tempered micro-localization functor due to Andronikof [A 1, A 2]. We consider the following diagram.

$$\begin{array}{ccc} X & \xleftarrow{Q_1} & X \times (X \times Y) \\ & \uparrow \wr & \\ \Delta_X \times Y \simeq X \times Y & \xrightarrow{Q_2} & Y. \end{array}$$

$$\begin{aligned}
& (\mathrm{T}\text{-}\mu\mathrm{hom}(K, \mathcal{O}_{X \times Y})_{\mathcal{O}_{X \times Y}} \otimes \Omega_{X \times Y/Y})(-p_0, \Phi(p_0)) \boxtimes \mathrm{T}\text{-}\mu\mathrm{hom}(\mathbb{C}_M, \mathcal{O}_X)_{p_0} \\
& \rightarrow \mathrm{T}\text{-}\mu\mathrm{hom}(\mathbb{C}_M \boxtimes K, \mathcal{O}_{X \times X \times Y})(p_0, -p_0, \Phi(p_0))_{\mathcal{O}_{X \times Y}} \otimes \Omega_{X \times Y/Y} \\
& \rightarrow \mathcal{D}_{X \times Y \rightarrow X \times X \times Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_{X \times X \times Y}} \mathrm{T}\text{-}\mu\mathrm{hom}(\mathbb{C}_M \boxtimes K, \mathcal{O}_{X \times X \times Y})(p_0, -p_0, \Phi(p_0))_{\mathcal{O}_{X \times Y}} \otimes \Omega_{X \times Y/Y} \\
& \xrightarrow{i^*} \mathrm{T}\text{-}\mu\mathrm{hom}(i^{-1}(\mathbb{C}_M \boxtimes K), \mathcal{O}_{X \times Y})(-p_0, \Phi(p_0))_{\mathcal{O}_{X \times Y}} \otimes \Omega_{X \times Y/Y} \\
& \rightarrow \mathcal{D}_{Y \rightarrow X \times Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_{X \times Y}} \mathrm{T}\text{-}\mu\mathrm{hom}(i^{-1}(\mathbb{C}_M \boxtimes K), \mathcal{O}_{X \times Y})(-p_0, \Phi(p_0)) \\
& \xrightarrow{f_*} \mathrm{T}\text{-}\mu\mathrm{hom}(\mathbb{R}Q_{2*}i^{-1}(\mathbb{C}_M \boxtimes K), \mathcal{O}_Y)_{\Phi(p_0)}[-\dim_{\mathbb{C}} X].
\end{aligned}$$

The second map is given by choosing a canonical section $1_{X \times Y \rightarrow X \times X \times Y} \in \mathcal{D}_{X \times Y \rightarrow X \times X \times Y}$. Since

$$\mathrm{T}\text{-}\mu\mathrm{hom}(K, \mathcal{O}_{X \times Y}) = \mathcal{E}_{X \times Y}^{\mathbb{R}, f} \otimes_{\mathcal{D}_{X \times Y}} \mathcal{B}_{Z|X \times Y}$$

and $\mathcal{B}_{Z|X \times Y}$ is a simple holonomic system, we can take a section $s \in \mathcal{E}_{Z|X \times Y}^{\mathbb{R}, f}$ which generates $\mathcal{E}_{Z|X \times Y}^{\mathbb{R}, f}$ over $\mathcal{E}_{X \times Y}^{\mathbb{R}, f}$. Then we have the morphism

$$\mathrm{T}\text{-}\mu\mathrm{hom}(\mathbb{C}_M, \mathcal{O}_X)_{p_0} \rightarrow (\mathrm{T}\text{-}\mu\mathrm{hom}(K, \mathcal{O}_{X \times Y})_{\mathcal{O}_{X \times Y}} \otimes \Omega_{X \times Y/Y})(-p_0, \Phi(p_0)) \boxtimes \mathrm{T}\text{-}\mu\mathrm{hom}(\mathbb{C}_M, \mathcal{O}_X)_{p_0},$$

by $u \rightarrow s \boxtimes u$. Thus we have the cohomological quantized contact transformation map

$$\mathrm{T}\text{-}\mu\mathrm{hom}(\mathbb{C}_M, \mathcal{O}_{X \times Y})_{p_0} \rightarrow \mathrm{T}\text{-}\mu\mathrm{hom}(\mathbb{R}Q_{2*}i^{-1}(\mathbb{C}_M \boxtimes K), \mathcal{O}_Y)_{\Phi(p_0)}[-\dim_{\mathbb{C}} X].$$

Since $\mathbb{R}Q_{2*}i^{-1}(\mathbb{C}_M \boxtimes K)$ is a simple sheaf along T_N^*Y with the shift $-\frac{1}{2} \dim M - \frac{1}{2}(S(N, p) - S(M, \Phi(p)))$ in a neighborhood p_0 , we have

$$\mathrm{T}\text{-}\mu\mathrm{hom}(\mathbb{C}_M, \mathcal{O}_{X \times Y})_{p_0} \rightarrow \mathrm{T}\text{-}\mu\mathrm{hom}(\mathbb{C}_N, \mathcal{O}_Y)_{\Phi(p_0)}[d]$$

where $d := \frac{1}{2}(\dim M - \dim N + S(M, p_0) - S(N, \Phi(p_0)))$. We can easily check the map constructed above satisfies the composition law, and is isomorphism. Moreover we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{T}\text{-}\mu\mathrm{hom}(\mathbb{C}_M, \mathcal{O}_X)_{p_0} & \longrightarrow & \mathrm{T}\text{-}\mu\mathrm{hom}(\mathbb{C}_N, \mathcal{O}_Y)_{\Phi(p_0)}[d] \\
\downarrow & & \downarrow \\
\mathbb{R}\mu\mathrm{hom}(\mathbb{C}_M, \mathcal{O}_X)_{p_0} & \longrightarrow & \mathbb{R}\mu\mathrm{hom}(\mathbb{C}_N, \mathcal{O}_Y)_{\Phi(p_0)}[d].
\end{array}$$

proof of Theorem 3.1. In the same way of the proof of [K-S 1, Theorem 11.3.5], we can find a hypersurface N which is the boundary of a pseudo-convex open

set $j : \Omega \rightarrow X$ (in particular, $S_-(N, q) = 0$) and a contact transformation ϕ interchanging (T_M^*X, p_0) and (T_N^*Y, q_0) in a neighborhood of p_0 . We obtain

$$\mathrm{T-}\mu\mathrm{hom}(\mathbb{C}_M, \mathcal{O}_X)_{p_0}[\mathrm{codim} M + S_-(M, p_0) - \delta(p_0)] = \mathrm{T-}\mu\mathrm{hom}(\mathbb{C}_N, \mathcal{O}_Y)_{q_0}[1].$$

Since there is a triangle

$$\mathrm{T-}\mu\mathrm{hom}(\mathbb{C}_N, \mathcal{O}_Y)_{q_0} \rightarrow \mathcal{O}_{\pi(q_0)} \rightarrow \mathrm{RH}(\mathbb{C}_\Omega)_{\pi(q_0)} \xrightarrow{+1},$$

we obtain the first assertion of Theorem 3.1 due to Theorem 0.1. Moreover

$$H^0(\mathrm{RH}(\mathbb{C}_\Omega))/\mathcal{O} \rightarrow j_*\mathcal{O}/\mathcal{O}$$

is clearly injective, we have injectivity of $H^k \mathrm{T-}\mu_M(\mathcal{O}_X) \rightarrow H^k \mu_M(\mathcal{O}_X)$.

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