REGULARLY VARYING CORRELATIONS

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REGULARLY VARYING CORRELATIONS

AKIHIKO INOUE

1. *Introduction*

We consider a Langevin equation with infinite delay

$$\dot{X}(t) = -\int_{-\infty}^{t} \gamma(t-s)\dot{X}(s)ds + \alpha\dot{B}(t), \tag{1.1}$$

where $\alpha > 0$ and $\dot{B}$ is a Gaussian white noise. The delay kernel γ is in the form

$$\gamma(t) = \int_0^\infty e^{-t\lambda}\rho(d\lambda) \qquad (t > 0) \tag{1.2}$$

with a Borel measure ρ on $(0, \infty)$ which satisfies appropriate conditions. We are concerned with existence, uniqueness and properties of a stationary process X which satisfies (1.1) in the sense of stationary random distribution. We are especially interested in the long-time behaviour of the correlation function of X.

One advantage of the assumption (1.2) is that the canonical representation kernel E of X turns out to have a similar integral representation

$$E(t) = \int_0^\infty e^{-t\lambda}\nu(d\lambda) \qquad (t > 0) \tag{1.3}$$

with a Borel measure ν on $(0, \infty)$. Conversely the problem of constructing a solution of (1.1) is reduced to that of constructing such a measure ν, which is more manageable. We propose a set D_0^{10} as an appropriate class of measures for ρ in (1.2). The definition of D_0^{10} needs some preliminaries and cannot be stated here, but it contains, for instance, the measures $\lambda^{q-1}d\lambda/\Gamma(q)$ with $0 < q < 1/2$; in this case, we have $\gamma(t) = 1/t^q$.

The most remarkable feature of solution X of (1.1) is that it has *reflection positivity*, that is, the correlation function R of X is in the form

$$R(t) = \int_0^\infty e^{-|t|\lambda}\sigma(d\lambda) \qquad (t \in \mathbb{R}) \tag{1.4}$$

with a bounded Borel measure σ on $(0, \infty)$. We see that when the pair (α, ρ) moves over the set $(0, \infty) \times D_0^{10}$, the corresponding measure σ ranges over all Borel

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measures on $(0,\infty)$ such that $m_{-1}(\sigma)=\infty$ and $m_1(\sigma)<\infty$, where $m_k(\sigma)$ denotes the k-th moment of σ:

$$m_{-1}(\sigma):=\int_0^\infty \frac{1}{\lambda}\sigma(d\lambda), \qquad m_1(\sigma):=\int_0^\infty \lambda\sigma(d\lambda).$$

Note that the number $m_{-1}(\sigma)$ is nothing but the *diffusion coefficient* D of X defined by $D=\int_0^\infty R(t)dt$. So the solution of (1.1) has always a divergent diffusion coefficient. Conversely, any reflection positive, stationary Gaussian process X with $m_{-1}(\sigma)=\infty$ and $m_1(\sigma)<\infty$ satisfies (1.1) for some appropriate $(\alpha,\rho)\in(0,\infty)\times D_0^{10}$ together with its canonical Brownian motion B.

Now we recall the results of Okabe [**12**] which are closely related to ours. He considers the following delay Langevin equation

$$\dot{X}(t)=-\beta X(t)-\int_{-\infty}^t \gamma(t-s)\dot{X}(s)ds+\alpha\dot{B}(t), \tag{1.5}$$

where $\alpha>0$, $\beta>0$ and γ is in the form (1.2) with a Borel measure ρ on $(0,\infty)$ such that $\int_0^\infty \rho(d\lambda)/(1+\lambda)<\infty$. Equation (1.5) has a unique stationary solution X with correlation function R in the form (1.4). However, in this case, σ ranges over all Borel measures on $(0,\infty)$ such that $m_{-1}(\sigma)<\infty$ and $m_1(\sigma)<\infty$, that is, (1.5) describes all processes with finite diffusion coefficients in the class $\int_0^\infty \lambda\sigma(d\lambda)<\infty$. So (1.1) well supplement (1.5). In [**12**], an equation which describes processes with $m_{-1}(\sigma)<\infty$ and $m_1(\sigma)=\infty$ is also considered. It is in the following form

$$\beta X(t)+\int_{-\infty}^t \gamma(t-s)\dot{X}(s)ds=\alpha\dot{B}(t), \tag{1.6}$$

where $\alpha>0$, $\beta>0$ and γ is in the form (1.2) with appropriate ρ. In this paper, we also consider an equation which describes processes with $m_{-1}(\sigma)=\infty$ and $m_1(\sigma)=\infty$. It is in the following form

$$\int_{-\infty}^t \gamma(t-s)\dot{X}(s)ds=\alpha\dot{B}(t), \tag{1.7}$$

where $\alpha>0$ and γ is in the form (1.2) with appropriate ρ. Any stationary Gaussian process with correlation function R in the form (1.4) with some bounded Borel measure σ on $(0,\infty)$ turns out to be described by one of the equations (1.1), (1.5), (1.6) and (1.7) according to the values of $m_{-1}(\sigma)$ and $m_1(\sigma)$.

Originally, delay Langevin equations as above first appeared in statistical physics to explain the non-Markovian Brownian motion discovered through the computer simulations by Alder and Wainwright [**1**]. The correlation function R of one component of the velocity of the non-Markovian Brownian particle decays like $R(t)\sim ct^{-3/2}$ as $t\to\infty$ for some $c>0$. We refer to [**1**], Oobayashi et al. [**18**], Kubo et al. [**19**] and Okabe [**13**] for the physical backgrounds. Delay Langevin equations are studied by Okabe [**12,13,14,15,16**], Inoue [**6,7,8**], and Okabe and Inoue [**17**] as equations describing reflection positive, stationary Gaussian processes.

We note that (1.1) formally coincides with (1.5) if we put $\beta = 0$. In fact, (1.5) with $\alpha > 0$ and $\gamma(t) = 1/t^q$ $(0 < q < 1/2)$ has always a unique stationary solution for $\beta = 0$ as well as for $\beta > 0$. Here we come across an interesting phenomenon about the long-time behaviour of the correlation function R of the solution X. In view of [**16**, Theorem 3.2], if $\beta > 0$, then $R(t) \sim \alpha^2\beta^{-3}qt^{-(q+1)}$ as $t \to \infty$. On the other hand, we see in this paper that if $\beta = 0$, then $R(t) \sim c\alpha^2 t^{-(1-2q)}$ as $t \to \infty$ for some $c > 0$. Therefore we observe a kind of critical phenomenon when we make $\beta \downarrow 0+$.

The main tool to study such long-time behaviour of R is the regular variation theory (cf. Bingham et al. [3]). In fact, our delay Langevin equations supply another field where the theory works well. However the techniques used in the previous works such as [**16**], [6], [7] and [8] are mostly the classical ones; approximately contained in the first chapter of [3]. A technical novelty of our methods here is to use de Haan theory (cf. [**3**, Ch. 3]). As a result, we can describe the boundary cases well; in particular, we can characterize the behaviour such as $R(t) \sim \text{const.} \times t^{-1}$ as $t \to \infty$.

This paper is organized as follows. In sections 2 and 3, we characterize the index-p regular variation of the correlation R of a stationary process X with diffusion coefficient $D = \infty$ in terms of the canonical representation kernel E of X. Section 2 treats the case $0 < p < 1$, while section 3 deals with the boundary cases corresponding to $p = 0$ and 1. The same problem is discussed by [**16**] and [6] for reflection positive, stationary processes with $D < \infty$. In that case, E is integrable over $(0, \infty)$, and the following simple relation holds:

$$R(t) \sim \left\{ \frac{1}{2\pi} \int_0^\infty E(s)ds \right\} E(t) \qquad (t \to \infty).$$

However, in our case $D = \infty$, E is not integrable over $(0, \infty)$, and so such a simple relation cannot be expected. Especially the implication from the long-time behaviour of R to that of E needs a delicate analysis. The crucial idea is to go by way of the spectral density and the outer function of X. Even more, when dealing with the boundary cases $p = 0$ and 1 in section 3, we use de Haan theory as the appropriate language and methods of analysis. The results of sections 2 and 3 are applied to Equations (1.1) and (1.7) in section 5. In section 4, we consider the delay Langevin equations (1.1) and (1.5), and establish uniqueness, existence and basic properties of their solutions. We use the results of [8] which also deals with reflection positive, stationary Gaussian processes with $D = \infty$. In section 5, we characterize long-time behaviour of the correlation function R of the solution X of (1.1) or (1.5) in terms of the delay kernel γ. In the boundary cases $p = 0$ and 1, de Haan's Abel-Tauber Theorem is crucial.

Our analysis of long-time behaviour of correlations is based on the theory of regular variation, and the author owes much to the book by Bingham, Goldie and Teugels [3] for the theory. The author also wishes to thank Professor Y. Okabe for helpful discussions.

2. *The canonical representation kernel*

In this section, we are concerned with a correlation function R of a purely non-deterministic, stationary process X. Our goal in this section is to characterize the regular variation of R with index $p \in (0,1)$ in terms of the canonical representation kernel E of X under the assumption that E is non-increasing. Our course of the proof below is to go by way of the spectral density Δ and the outer function h of X.

First we recall some notations and facts of stationary processes. For the details, we refer to Gihman and Skorohod [4]. In this paper when we say *stationary process*, it always means a real, mean continuous, weakly stationary process with expectation zero, defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $X = (X(t) : t \in \mathbb{R})$ be a stationary process, and let R be the correlation function of X: $R(t) = E(X(t)X(0))$ for $t \in \mathbb{R}$. Let δ be the *spectral measure of* X:

$$R(t) = \int_{-\infty}^{\infty} e^{-it\xi} \delta(d\xi) \qquad (t \in \mathbb{R}).$$

If δ is absolutely continuous with respect to the Lebesgue measure, we call the density Δ the *spectral density* of X: $\delta(d\xi) = \Delta(\xi)d\xi$. The spectral density Δ is a non-negative, even, and integrable function on $\mathbb{R}$. Let X be *purely nondeterministic.* Here in general a stationary random distribution Y is called purely nondeterministic if

$$\bigcap_{t\in\mathbb{R}} M_t(X) = \{0\},$$

where $M_t(Y)$ denotes the closed linear hull of

$$\{Y(\phi) : \phi \in \mathcal{D}(\mathbb{R}),\ \text{supp}\ \phi \subset (-\infty, t]\}$$

in $L^2(\Omega)$, $\mathcal{D}(\mathbb{R})$ being the space of test functions consisting of functions of class C^∞ on $\mathbb{R}$ with compact support. Then X has a spectral density Δ which satisfies

$$\int_{-\infty}^{\infty} \frac{|\log \Delta(\xi)|}{1+\xi^2} d\xi < \infty.$$

The *outer function* h of X is an outer function in the Hardy class H^2 defined by

$$h(\zeta) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1+\lambda\zeta}{\lambda-\zeta} \cdot \frac{\log\Delta(\lambda)}{1+\lambda^2} d\lambda\right\} \qquad (\text{Im}\ \zeta > 0).$$

Since h belongs to H^2, there exists a $\text{l.i.m.}_{\eta\downarrow 0} h(\cdot + i\eta)$ in $L^2(\mathbb{R})$, which we also denote by h. The *canonical representation kernel* E of X is defined by $E = \hat{h}$, where $\hat{h}$ denotes the Fourier transform of h:

$$\hat{h}(t) = \underset{M\to\infty}{\text{l.i.m.}} \int_{-M}^{M} e^{-it\xi} h(\xi) d\xi \qquad (t \in \mathbb{R}).$$

It holds that $E(t) = 0$ a.e. in $(-\infty, 0)$. Further we have

$$h(\zeta) = \frac{1}{2\pi} \int_0^\infty e^{i\zeta t} E(t) dt \qquad (\operatorname{Im} \zeta > 0), \tag{2.1}$$

$$R(t) = \frac{1}{2\pi} \int_0^\infty E(|t| + s) E(s) ds \qquad (t \in \mathbb{R}), \tag{2.2}$$

We begin by proving some facts on regular variation which are crucial in the proofs of the main theorems in this and next sections. In what follows, we faithfully follow the notations of Bingham et al. [3] on regular variation. Especially we denote by R_0 the whole class of slowly varying functions at infinity, that is, R_0 is the class of positive measurable f, defined on some neighborhood of infinity, satisfying

$$\forall \lambda > 0, \qquad \lim_{x \to \infty} f(\lambda x)/f(x) = 1.$$

PROPOSITION 2.1. *Let $q < 1 < p+q$, and $l_1, l_2 \in R_0$. Let g be locally integrable on $[0, \infty)$, and let f be measurable on $(0, \infty)$. If $f(t) \sim t^{-p} l_1(t)$ as $t \to \infty$ and $g(t) \sim t^{-q} l_2(t)$ as $t \to \infty$, then $f(t + \cdot) g(\cdot)$ is integrable on $(0, \infty)$ for sufficiently large t, and $U(t) := \int_0^\infty f(t+s) g(s) ds$ satisfies*

$$U(t) \sim t^{-(p+q-1)} l_1(t) l_2(t) B(p+q-1, 1-q) \qquad (t \to \infty).$$

Proof. If $f(t) \sim t^{-p} l_1(t)$ and $g(t) \sim t^{-q} l_2(t)$ as $t \to \infty$, then $\int_X^\infty |f(t+s) g(s)| ds < \infty$ for sufficiently large X. Since $p > 0$, we have $\sup_{0<s<\infty} |f(t+s)| < \infty$ for sufficiently large t, and so $\int_0^X |f(t+s) g(s)| ds < \infty$. Thus $U(t)$ exists for t large enough. Choose M so large that, on $[M, \infty)$, f and g are both positive and g is locally bounded. If we set $g_1(s) := 1$ on $(0, M)$, and $:= g(s)$ on $[M, \infty)$, then

$$U(t) = \int_0^M f(t+s)\{g(s) - 1\} ds + \int_0^\infty f(t+s) g_1(s) ds.$$

If t is large enough, then by the Uniform Convergence Theorem (cf. [**3**, Theorem 1.5.2]) $|f(t+s)/f(t)| \le 2$ for $s \in [0, M]$. Further since $1 - q > 0$, we have $\lim_{t\to\infty} t g(t) = \infty$. Thus

$$\lim_{t \to \infty} \frac{1}{t f(t) g(t)} \int_0^M f(t+s)\{g(s) - 1\} ds = 0.$$

Therefore in order to prove the proposition, we may assume that, in $[0, \infty)$, f and g are both positive, and g is locally bounded.

Choose δ_i $(i = 1, 2, 3)$ so that $q < \delta_1 < 1$, $\delta_2 < p$, $\delta_3 < q$, and $1 < \delta_2 + \delta_3$. Set $F_1(u) := u^p f(u)$, $G_1(u) := u^{\delta_1} g(u)$, $F_2(u) := u^{\delta_2} f(u)$, and $G_2(u) := u^{\delta_3} g(u)$. Then $U(t)/\{t f(t) g(t)\} = C(t) + D(t)$, where

$$C(t) = \int_0^1 \frac{F_1(t(u+1))}{F_1(t)} \cdot \frac{G_1(tu)}{G_1(t)} \cdot \frac{1}{(u+1)^p u^{\delta_1}} du,$$
$$D(t) = \int_1^\infty \frac{F_2(t(u+1))}{F_2(t)} \cdot \frac{G_2(tu)}{G_2(t)} \cdot \frac{1}{(u+1)^{\delta_2} u^{\delta_3}} du.$$

In $C(t)$, by the Uniform Convergence Theorem, $F_1(t(u+1))/F_1(t)$ converges to 1 as $t \to \infty$, and $G_1(tu)/G_1(t)$ to $u^{\delta_1 - q}$, both uniformly in $u \in (0,1)$. Therefore $C(t)$ converges to $\int_0^1 (u+1)^{-p} u^{-q} du$ as $t \to \infty$. In the same way, $D(t)$ converges to $\int_1^\infty (u+1)^{-p} u^{-q} du$ as $t \to \infty$. Thus we obtain

$$\lim_{t\to\infty} \frac{U(t)}{tf(t)g(t)} = \int_0^\infty \frac{1}{(u+1)^p u^q} du = B(p+q-1, 1-q).$$

PROPOSITION 2.2. *Let $l \in R_0$ and $p \in \mathbb{R}$. Let f be a positive, even and measurable function on $\mathbb{R}$ such that $\{\log f(\xi)\}/(1+\xi^2)$ is integrable on $\mathbb{R}$. If $f(\xi) \sim \xi^p l(1/\xi)$ as $\xi \to 0+$, then*

$$\exp\left\{\frac{1}{\pi}\int_{-\infty}^\infty \frac{y}{y^2+\xi^2}\log f(\xi) d\xi\right\} \sim y^p l(1/y) \qquad (y \to 0+).$$

Proof. If we put

$$L(x) := x^p f(1/x) \qquad (x > 0),$$

then L is slowly varying and $\{\log L(1/|\xi|)\}/(1+\xi^2)$ is integrable on $\mathbb{R}$. The Representation Theorem (cf. [**3**, Theorem 1.3.1]) yields

$$\log L(x) = \eta(x) + \int_a^x \epsilon(u) du/u \qquad (x \geq a) \tag{2.3}$$

for some $a > 0$, where $\eta(x)$, $\epsilon(x)$ are bounded and measurable on $[a, \infty)$, $\eta(x) \to c \in \mathbb{R}$, $\epsilon(x) \to 0$ as $x \to \infty$. On taking $\eta(x) = \log L(x)$, $\epsilon(x) \equiv 0$ on $(0, a)$, (2.3) holds for $x > 0$; ϵ is bounded on $(0, \infty)$ again, but may not η. As for η, we have the estimate

$$\int_{-\infty}^\infty \frac{|\eta(1/|\xi|)|}{1+\xi^2} d\xi \leq 2\sup_{x\geq a}|\eta(x)| \cdot \int_0^{1/a} \frac{1}{1+\xi^2} d\xi + 2\int_{1/a}^\infty \frac{|\log L(1/\xi)|}{1+\xi^2} d\xi < \infty.$$

For $y > 0$, making use of the change of variables $t = \xi/y$, $s = yu$, and the identity

$$\frac{1}{\pi}\int_{-\infty}^\infty \frac{y}{y^2+\xi^2}\log|\xi| d\xi = \log y,$$

we are led to

$$\frac{1}{\pi}\int_{-\infty}^\infty \frac{y}{y^2+\xi^2}\log f(\xi) d\xi - \log f(y) = \mathrm{I}_1(y) + \mathrm{I}_2(y), \tag{2.4}$$

where

$$\mathrm{I}_1(y) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{y}{y^2+\xi^2}\eta(1/|\xi|) d\xi - \eta(1/y),$$

$$\mathrm{I}_2(y) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{1}{1+t^2}\left\{\int_1^{1/|t|} \epsilon(s/y) ds/s\right\} dt.$$

We claim that $I_1(y)$ and $I_2(y)$ both converge to 0 as $y \to 0+$. In fact, since $\eta(1/|\xi|)/(1+\xi^2)$ is integrable on $\mathbb{R}$ and $\eta(1/|\xi|) \to c$ as $\xi \to 0$, a well known property of Poisson integral yields $I_1(y) \to 0$ as $y \to 0+$. As for I_2, the dominated convergence theorem shows that $I_2(y) \to 0$ as $y \to 0+$ because we have

$$\left|(1+t^2)^{-1}\int_1^{1/|t|}\epsilon(s/y)ds/s\right| \le \sup_{x>0}|\epsilon(x)|\cdot\frac{|\log|t||}{1+t^2}.$$

Thus the left-hand side of (2.4) converges to 0 as $y \to 0+$, whence

$$\exp\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{y^2+\xi^2}\log f(\xi)d\xi\right\} \sim f(y) \sim y^p l(1/y) \qquad (y \to 0+),$$

and the result follows.

In what follows R denotes the correlation function of a stationary process X. At the first, we assume conditions on R only, and collect their consequences. We denote by $\int^{\infty-}$ the improper integral: $\int_a^{\infty-} f(t)dt = \lim_{M\to\infty}\int_a^M f(t)dt$.

PROPOSITION 2.3. *Assume that the correlation function R is eventually monotone on $(0,\infty)$ and satisfies $\lim_{t\to\infty} R(t) = 0$. Then the spectral measure of X is absolutely continuous with respect to the Lebesgue measure. The spectral density Δ is a continuous function in $\mathbb{R}\setminus\{0\}$ given by*

$$\Delta(\xi) = \frac{1}{\pi}\int_0^{\infty-} R(t)\cos\xi t dt \qquad (\xi \in \mathbb{R}\setminus\{0\}).$$

Proof. We refer to Titchmarsh [21, Theorems 6, 7] for the Fourier transforms of monotone functions. By the assumption, R is even, continuous, and either eventually positive and non-increasing, or eventually negative and non-decreasing. Let δ be the spectral measure of X. By Lévy's inversion formula, if a and b are continuity points of δ such that $0 < a < b$, then

$$\delta(a,b) = \lim_{M\to\infty}\frac{1}{2\pi}\int_{-M}^{M}\frac{e^{itb}-e^{ita}}{it}R(t)dt = F(b) - F(a),$$

where

$$F(\xi) := \frac{1}{\pi}\int_0^{\infty-}\frac{R(t)}{t}\sin\xi t dt \qquad (\xi > 0).$$

Since the improper integral $\int_0^{\infty-} R(t)\cos\xi t dt$ converges uniform in $\xi > \epsilon$ for any $\epsilon > 0$, we see that F is of C^1-class in $(0,\infty)$ and satisfies

$$F'(\xi) = \frac{1}{\pi}\int_0^{\infty-} R(t)\cos\xi t dt \qquad (\xi > 0).$$

Therefore δ is absolutely continuous in $(0,\infty)$, and the density there is equal to F'. If we put $\Delta(\xi) := F'(\xi)$ for $\xi > 0$, then we obtain

$$R(t) = \delta\{0\} + 2\int_0^{\infty}\Delta(\xi)\cos\xi t d\xi \qquad (t \in \mathbb{R}).$$

By the Riemann-Lebesgue Lemma, the second term on the right converges to zero as $t \to \infty$, so that $\delta\{0\} = 0$. This completes the proof.

PROPOSITION 2.4. *Let $0 < p < 1$, and $l \in R_0$. Assume that R is eventually non-increasing and satisfies $\lim_{t\to\infty} R(t) = 0$. Let Δ be the spectral density of R. Then the following are equivalent:*

$$R(t) \sim t^{-p} l(t) \quad (t \to \infty), \tag{2.5}$$

$$\Delta(\xi) \sim \xi^{-(1-p)} l(1/\xi) \cdot \frac{\Gamma(1-p)\sin\frac{1}{2}\pi p}{\pi} \quad (\xi \to 0+). \tag{2.6}$$

Proof. Choose $M > 0$ so large that $R(t)$ is non-increasing on $[M, \infty)$. We set $R_1(t) := R(M)$ on $[0, M]$, and $:= R(t)$ on (M, ∞). Then

$$\int_0^{\infty -} R(t)\cos\xi t dt = \int_0^M \{R(t) - R(M)\}\cos\xi t dt + \int_0^\infty R_1(t)\cos\xi t dt \qquad (\xi > 0).$$

Since the first part on the right converges as $\xi \to 0+$, Proposition 2.3 shows that (2.6) is equivalent to

$$\int_0^\infty R_1(t)\cos\xi t dt \sim \xi^{-(1-p)} l(1/\xi) \cdot \Gamma(1-p)\sin\tfrac{1}{2}\pi p \qquad (\xi \to 0+).$$

Therefore by virtue of the theorem of Pitman and Soni-Soni (cf. [**3**, Theorem 4.10.3]), we obtain the result.

In what follows, we assume that X is purely nondeterministic. Let Δ, h and E be the spectral density, the outer function and the canonical representation kernel of X, respectively. Here is our main result in this section.

THEOREM 2.5. *Let $0 < p < 1$ and $l \in R_0$. Assume that E is non-increasing on $(0, \infty)$. Then (2.5), (2.6) and the following are equivalent:*

$$h(iy) \sim y^{-(1-p)/2}(l(1/y))^{1/2} \cdot \frac{(\Gamma(1-p)\sin\frac{1}{2}\pi p)^{1/2}}{\pi^{1/2}} \qquad (y \to 0+), \tag{2.7}$$

$$E(t) \sim t^{-(1+p)/2}(l(t))^{1/2} \cdot \frac{(2\pi)^{1/2}}{(B(\frac{1-p}{2}, p))^{1/2}} \qquad (t \to \infty). \tag{2.8}$$

Proof. Since E is non-increasing and square integrable on $(0, \infty)$, $E(t)$ is non-negative and tends to zero as $t \to \infty$. So by (2.2), $R(t)$ is also non-increasing and tends to zero as $t \to \infty$. Hence the implication (2.5)$\Leftrightarrow$(2.6) follows from Proposition 2.4. Since Δ is even, we have

$$h(iy) = \exp\left\{\frac{1}{2\pi}\int_{-\infty}^\infty \frac{y}{y^2 + \xi^2}\log\Delta(\xi)d\xi\right\} \qquad (y > 0),$$

so that, by Proposition 2.2, (2.6) implies (2.7). The equality

$$h(iy) = \frac{1}{2\pi}\int_0^\infty e^{-yt}E(t)dt \qquad (y > 0)$$

and Karamata's Tauberian Theorem (cf. [**3**, Theorem 1.7.6]) show that (2.7) and (2.8) are equivalent. Finally that (2.8) implies (2.5) follows from Proposition 2.1.

3. *Boundary cases*

In this section, we are concerned with two boundary cases which correspond to $p = 0$ and 1 in Theorem 2.5. These are subtle cases where the slowly varying parts become important. However if we restrict ourselves to reflection positive, stationary processes with divergent diffusion coefficients, then by virtue of de Haan theory we can handle them very well. Here a stationary process X is called *reflection positive* if the correlation function R of X is in the form (1.4) with some bounded Borel measure σ on $(0, \infty)$, and the *diffusion coefficient* D of X is defined by

$$D := \int_0^\infty R(t)dt = \int_0^\infty \frac{1}{\lambda}\sigma(d\lambda).$$

In what follows we shall always assume that X is a stationary process whose correlation function R is in the form (1.4) with a bounded Borel measure σ on $(0, \infty)$ such that $D = \infty$, that is,

$$\int_0^\infty \frac{1}{\lambda}\sigma(d\lambda) = \infty.$$

We recall some basic properties of X. By Okabe [**11**], X is purely nondeterministic, and the spectral density Δ of X is given by

$$\Delta(\xi) = \frac{1}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2 + \xi^2}\sigma(d\lambda) \qquad (\xi \in \mathbb{R} \setminus \{0\}).$$

By [**8**, Theorem 2.6], the canonical representation kernel E of X is in the form (1.3) with a uniquely determined Borel measure ν on $(0, \infty)$ such that

$$\int_0^\infty \int_0^\infty \frac{1}{\lambda + \lambda'}\nu(d\lambda)\nu(d\lambda') < \infty. \tag{3.1}$$

Some of the advantages of the representations above are that R, Δ and E are all of $C^1(0, \infty)$ class, that they and their derivatives $\dot{R}$, $\dot{E}$ and $\dot{\Delta}$ are all monotone on $(0, \infty)$, and that they have the following representations:

$$R(t) = R(0) + \int_0^t \dot{R}(s)ds = \int_t^\infty \{-\dot{R}(s)\}ds \qquad (t \geq 0), \tag{3.2}$$

$$E(t) = E(1) + \int_1^t \dot{E}(s)ds \qquad (t \geq 1), \tag{3.3}$$

$$\Delta(1/x) = \Delta(1) + \int_1^x \{-\dot{\Delta}(1/y)/y^2\}dy \qquad (x > 1). \tag{3.4}$$

We recall some notations of regular variation from Bingham et al. [**3**]. Let $l \in R_0$ such that $\int^\infty l(s)ds/s = \infty$. Choose M so large that l is locally integrable on $[M, \infty)$. We put

$$\tilde{l}(t) := \int_M^t l(s)ds/s \qquad (t \geq M).$$

Then $\tilde{l}$ is also slowly varying, and the asymptotic behaviour of $\tilde{l}$ does not depend on the special choice of M. For $l \in R_0$, the *de Haan class* Π_l is the class of measurable f satisfying

$$\forall \lambda \geq 1, \qquad \lim_{x \to \infty} \{f(\lambda x) - f(x)\}/l(x) = c \log \lambda$$

for some constant c called the *l-index* of f.

First we consider the case $p = 0$.

THEOREM 3.1. *Let $l_1 \in R_0$. Then $R \in \Pi_{l_1}$ with index -1 if and only*

$$E(t) \sim (2\pi)^{1/2} t^{-1/2} (l_1(t))^{1/2} \qquad (t \to \infty). \tag{3.5}$$

Proof. Since $\dot{R}$ is locally integrable on $[0, \infty)$ and $-\dot{R}(t) \downarrow 0$ as $t \to \infty$, we have by integration by parts,

$$\dot{\Delta}(\xi) = \frac{1}{\pi \xi} \int_0^{\infty -} \{-\dot{R}(t)\} \sin t\xi dt \qquad (\xi > 0).$$

By (3.2) and de Haan's theorem (cf. [**3**, Theorem 3.6.8]), $R \in \Pi_{l_1}$ with index -1 if and only if $-\dot{R}(t) \sim l_1(t)/t$ as $t \to \infty$. Then by Pitman [**19**, Theorem 1], we have $\dot{\Delta}(\xi) \sim l_1(1/\xi)/2$ as $\xi \to 0+$, and so by (3.4), $\Delta(\xi) \sim l_1(1/\xi)/(2\xi)$ as $\xi \to 0+$. Therefore in the same way as Theorem 2.5, we obtain (3.5).

Conversely by using the representation (1.3), we can easily justify the equality

$$-\dot{R}(t) = \frac{1}{2\pi} \int_0^\infty \{-\dot{E}(t+s)\} E(s) ds \qquad (t > 0).$$

In view of (3.3) and the Monotone Density Theorem (cf. [**3**, Theorem 1.7.2]), (3.5) implies $-\dot{E}(t) \sim \pi^{1/2} (l_1(t))^{1/2} / (2^{1/2} t^{3/2})$ as $t \to \infty$, hence by Proposition 2.1 we have $-\dot{R}(t) \sim l_1(t)/t$ as $t \to \infty$. Therefore by (3.2) we have $R \in \Pi_{l_1}$ with index -1, and the result follows.

Next we consider the case $p = 1$. We need some results on the asymptotic behaviour of $\Delta(\xi)$ as $\xi \to 0+$.

LEMMA 3.2. *Let $l_1 \in R_0$. If $R(t) \sim l_1(t)/t$ as $t \to \infty$, then $\Delta(\xi) \sim \tilde{l}_1(1/\xi)/\pi$ as $\xi \to 0+$, and $-\dot{\Delta}(\xi) \sim l_1(1/\xi)/(\pi\xi)$ as $\xi \to 0+$.*

Proof. By virtue of Soni and Soni [**20**, Theorem 18(a)], the first assertion follows immediately from Proposition 2.3. So we prove the second half. By [**19**, Theorem 7(iii)] and Proposition 2.3, we have

$$\Delta(1/x) - \frac{1}{\pi} \int_0^x R(t) dt \sim -\frac{\gamma}{\pi} l_1(x) \qquad (x \to \infty),$$

where γ is Euler's constant. Since the function $\int_0^t R(s)ds$ in t is in Π_{l_1} with index 1, we have for any $\lambda \geq 1$,

$$\begin{aligned}\frac{\Delta(1/\lambda x) - \Delta(1/x)}{l_1(x)} &= \frac{\Delta(1/\lambda x) - \pi^{-1}\int_0^{\lambda x} R(t)dt}{l_1(x)} \\ &+ \frac{\int_0^{\lambda x} R(t)dt - \int_0^x R(t)dt}{\pi l_1(x)} - \frac{\Delta(1/x) - \pi^{-1}\int_0^x R(t)dt}{l_1(x)} \\ &\to \frac{1}{\pi}\log\lambda \qquad (x \to \infty).\end{aligned}$$

Thus $\Delta(1/\cdot) \in \Pi_{l_1}$ with index $1/\pi$. By the representation

$$-\dot{\Delta}(\xi) = \frac{2\xi}{\pi}\int_0^\infty \frac{\lambda}{(\lambda^2+\xi^2)^2}\sigma(d\lambda) \qquad (\xi > 0), \tag{3.6}$$

$-\dot{\Delta}(1/\xi)$ is positive and $\log\{-\dot{\Delta}(1/\xi)/\xi^2\}$ is slowly decreasing on $(0,\infty)$. Therefore by applying [**3**, Theorem 3.6.10] to (3.4), we obtain the result.

Here is our main result for $p = 1$.

THEOREM 3.3. *If $l_1 \in R_0$ and $R(t) \sim l_1(t)/t$ as $t \to \infty$, then $E(t) \sim l_2(t)/t$ as $t \to \infty$, where $l_2(t) = \pi^{1/2} l_1(t)/(\widetilde{l_1}(t))^{1/2}$. Conversely if $l_2 \in R_0$ and $E(t) \sim l_2(t)/t$ as $t \to \infty$, then $R(t) \sim l_1(t)/t$ as $t \to \infty$, where $l_1(t) = l_2(t)\widetilde{l_2}(t)/(2\pi)$.*

Proof. First we prove the second half. By slightly modifying Okabe [**12**, Lemma 2.6], we obtain $\int_0^\infty E(t)dt = \infty$, so that $\widetilde{l_2}$ is well defined. Since $\int_0^s E(u)du \sim \widetilde{l_2}(s)$ as $s \to \infty$, we have

$$\lim_{s\to\infty} E(t+s)\int_0^s E(u)du = 0 \qquad (t > 0),$$

which with integration by parts implies

$$R(t) = \frac{1}{2\pi}\int_0^\infty \left\{\int_0^s E(u)du\right\}\left\{-\dot{E}(t+s)\right\}ds \qquad (t > 0).$$

By the Monotone Density Theorem, $E(t) \sim l_2(t)/t$ as $t \to \infty$ implies $-\dot{E}(t) \sim l_2(t)/t^2$ as $t \to \infty$. Therefore by Proposition 2.1 we obtain the result.

Next we prove the first half. We set

$$A(y) := \frac{1}{2\pi}\int_{-\infty}^\infty \frac{y}{y^2+\xi^2}\log\Delta(\xi)d\xi \qquad (y > 0).$$

By (3.6) we have

$$|\dot{\Delta}(\xi)/\Delta(\xi)| \leq 2/\xi \qquad (\xi > 0), \tag{3.7}$$

and so, for any $y > 0$, $\dot{A}(y) = -B(y)/\pi$, where

$$B(y) = \int_0^\infty \frac{\xi}{1+\xi^2}\cdot\frac{\{-\dot{\Delta}(y\xi)\}}{\Delta(y\xi)}d\xi \qquad (y > 0).$$

By differentiating in y the both sides of the equality

$$\frac{1}{2\pi}\int_0^\infty e^{-yt}E(t)dt = \exp A(y) \qquad (y > 0),$$

we obtain

$$\int_0^\infty e^{-yt}tE(t)dt = 2B(y)\exp A(y) \qquad (y > 0).$$

By Lemma 3.2, $R(t) \sim l_1(t)/t$ as $t \to \infty$ implies $\Delta(\xi) \sim \widetilde{l_1}(1/\xi)/\pi$ as $\xi \to 0+$, so that by Proposition 2.2,

$$\exp A(y) \sim \frac{1}{\pi^{1/2}}(\widetilde{l_1}(1/y))^{1/2} \qquad (y \to 0+).$$

Now we set $f(x) := -\dot{\Delta}(1/x)/\{x\Delta(1/x)\}$ for $x > 0$. By Lemma 3.2, $f(x) \sim l_1(x)/\widetilde{l_1}(x)$ as $x \to \infty$. Further (3.7) implies that $x^{1/2}f(x)$ is locally bounded in $[0, \infty)$. Therefore by the Uniform Convergence Theorem,

$$\begin{aligned}\frac{yB(y)}{f(1/y)} &= \int_0^1 \frac{1}{(1+\xi^2)\xi^{1/2}} \cdot \frac{(\xi/y)^{1/2}f(\xi/y)}{(1/y)^{1/2}f(1/y)}d\xi + \int_1^\infty \frac{\xi^{1/2}}{1+\xi^2} \cdot \frac{(\xi/y)^{-1/2}f(\xi/y)}{(1/y)^{-1/2}f(1/y)}d\xi \\ &\to \int_0^\infty \frac{1}{1+\xi^2}d\xi = \frac{\pi}{2} \qquad (y \to 0+),\end{aligned}$$

that is,

$$B(y) \sim \frac{\pi}{2}y^{-1}\frac{l_1(1/y)}{\widetilde{l_1}(1/y)} \qquad (y \to 0+).$$

Therefore we have

$$\int_0^\infty e^{-yt}tE(t)dt \sim \pi^{1/2}y^{-1}\frac{l_1(1/y)}{(\widetilde{l_1}(1/y))^{1/2}} \qquad (y \to 0+).$$

Since $\log\{tE(t)\}$ is slowly increasing, we can apply Karamata's Tauberian Theorem to this and obtain the result.

COROLLARY 3.4. *Let $\alpha > 0$. Then the following are equivalent:*

$$R(t) \sim \alpha^2/t \qquad (t \to \infty),$$
$$E(t) \sim \pi^{1/2}\alpha/\{t(\log t)^{1/2}\} \qquad (t \to \infty).$$

Proof. Since

$$\int_1^t \frac{1}{s(\log s)^{1/2}}ds = 2(\log t)^{1/2},$$

the corollary follows immediately from Theorem 3.3.

4. *Delay Langevin equations*

We consider the delay Langevin equation (1.1). Here $\alpha > 0$ and $\dot{B}$ is a Gaussian white noise. The kernel γ of the delay term is in the form (1.2) with some Borel measure ρ on $(0, \infty)$. In this section, we show uniqueness and existence of solution of (1.1) for a suitable class of ρ. It turns out that the solution of (1.1) is a reflection positive, stationary Gaussian process with $D = \infty$, D being the diffusion coefficient. Similar delay Langevin equations are discussed in Okabe [**12**] and [**8**]. The diffusion coefficients of solutions of Langevin equations in [**12**] are all finite. The delay Langevin equation studied in [**8**] also describes reflection positive, stationary Gaussian processes with $D = \infty$ but it is different from ours because the random noise of it is not white. However the machinery developed in [**8**] is useful in our discussions, too. We also consider the delay Langevin equation (1.6).

We begin with some correspondences between measures on $(0, \infty)$. We set

$$\begin{aligned}\Sigma_0 &= \{\sigma : \text{ a non-zero bounded Borel measure on } (0, \infty)\},\\ N_0 &= \{\nu : \text{ a non-zero Borel measure on } (0, \infty) \text{ which satisfies } (3.1)\}.\end{aligned}$$

We define a map $S : N_0 \ni \nu \mapsto \sigma = S(\nu) \in \Sigma_0$ by

$$\sigma(d\lambda) = \frac{1}{2\pi}\left\{\int_0^\infty \frac{1}{\lambda + \lambda'}\nu(d\lambda')\right\}\nu(d\lambda).$$

By [**8**, Theorem 2.5], S is a bijection from N_0 onto Σ_0. For any Borel measure μ on $(0, \infty)$, we denote by $m_k(\mu)$ its k-th moment:

$$m_k(\mu) = \int_0^\infty \lambda^k \mu(d\lambda) \quad (k \in \mathbb{Z}).$$

Then we have

LEMMA 4.1. (Okabe [**12**, Lemma 2.6]) *Let* $\nu \in N_0$ *and* $\sigma = S(\nu)$. *Then* $m_{-1}(\sigma) = m_{-1}(\nu)^2/(4\pi)$ *and* $m_1(\sigma) = m_0(\nu)^2/(4\pi)$.

Proof. Both follows from direct calculations.

We introduce several subclasses of Σ_0 and N_0 as follows:

$$\begin{aligned}\Sigma_0^1 &= \{\sigma \in \Sigma_0 : \ m_{-1}(\sigma) = \infty\},\\ \Sigma_0^{10} &= \{\sigma \in \Sigma_0 : \ m_{-1}(\sigma) = \infty, \ m_1(\sigma) < \infty\},\\ \Sigma_0^{11} &= \{\sigma \in \Sigma_0 : \ m_{-1}(\sigma) = \infty, \ m_1(\sigma) = \infty\},\\ N_0^1 &= \{\nu \in N_0 : \ m_{-1}(\nu) = \infty\},\\ N_0^{10} &= \{\nu \in N_0 : \ m_{-1}(\nu) = \infty, \ m_0(\nu) < \infty\},\\ N_0^{11} &= \{\nu \in N_0 : \ m_{-1}(\nu) = \infty, \ m_0(\nu) = \infty\}.\end{aligned}$$

Note that $\Sigma_0^1 = \Sigma_0^{10} \cup \Sigma_0^{11}, N_0^1 = N_0^{10} \cup N_0^{11}$, both being disjoint unions. The next proposition follows easily from Lemma 4.1.

PROPOSITION 4.2. $S(N_0^{10}) = \Sigma_0^{10}$, $S(N_0^{11}) = \Sigma_0^{11}$.

We set

$$M_0 = \{\mu : \text{ a non-zero Borel measure on } (0,\infty) \text{ such that } \int_0^\infty \frac{1}{1+\lambda}\mu(d\lambda) < \infty\}.$$

For any $\mu \in M_0$, we define

$$K_\mu(t) := \chi_{(0,\infty)}(t) \int_0^\infty e^{-t\lambda}\mu(d\lambda) \qquad (t \in \mathbb{R}),$$
$$F_\mu(\zeta) := \int_0^\infty \frac{1}{\lambda - i\zeta}\mu(d\lambda) \qquad (\text{Im } \zeta > 0).$$

We note that $F_\mu(\zeta) = \int_0^\infty e^{i\zeta t} K_\mu(t)dt$ for Im $\zeta > 0$. We set

$$M_0^1 = \{\mu \in M_0 : \ m_{-1}(\mu) = \infty\}.$$

Then by [8, Theorem 3.1], the following relation determines a bijection $\nu \mapsto L_0^1(\nu) = (\alpha, \rho)$ from Σ_0^1 onto $(0,\infty) \times M_0^1$:

$$F_\nu(\zeta)\{-i\zeta - i\zeta F_\rho(\zeta)\} = (2\pi)^{1/2}\alpha \quad (\text{Im } \zeta > 0). \tag{4.1}$$

Since $N_0^{10} = \{\nu \in \Sigma_0^1 : \nu \text{ satisfies } (3.1)\} \subset \Sigma_0^1$, the image $L_0^1(N_0^{10})$ forms a subset of $(0,\infty) \times M_0^1$. By (4.1), for any $\nu \in N_0^{10}$ and $c > 0$, we have $L_0^1(c\nu) = (c\alpha, \rho)$, where $(\alpha, \rho) = L_0^1(\nu)$, whence there is a subset $D_0^{10} \subset M_0^1$ such that

$$L_0^1(N_0^{10}) = (0,\infty) \times D_0^{10}.$$

It will turn out that D_0^{10} is the appropriate class of ρ for (1.1).

To formulate (1.1) precisely, we recall some notations of stationary random distributions. Let $\mathcal{D}(\mathbb{R})$ be the space of all functions of class C^∞ on $\mathbb{R}$ with compact support. We introduce the same topology on $\mathcal{D}(\mathbb{R})$ as in the theory of distributions. By a *stationary random distribution*, we mean a system of random variables $\{Y(\phi) : \phi \in \mathcal{D}(\mathbb{R})\}$ on a probability space $(\Omega, \mathcal{F}, P)$ such that

(1) the map $\phi \mapsto Y(\phi)$ is linear, and continuous in the L^2-sense;
(2) $\overline{Y(\phi)} = Y(\overline{\phi})$ for any $\phi \in \mathcal{D}(\mathbb{R})$;
(3) $E(Y(\tau_h\phi)\overline{Y(\tau_h\psi)}) = E(Y(\phi)\overline{Y(\psi)})$ for any $\phi, \psi \in \mathcal{D}(\mathbb{R})$ and $h \in \mathbb{R}$;
(4) $E(Y(\phi)) = 0$.

Here $\overline{a}$ denotes the complex conjugate of a, and τ_s denotes the shift transformation $\tau_s\phi(t) = \phi(t+s)$. The condition (4) implies that we always assume Y to have zero expectation. We refer to Ito [9] for the details. Let $\rho \in M_0$, and let $\gamma = K_\rho$. Then for any stationary random distribution Y, the convolution $\gamma * \dot{Y}$ denotes another stationary random distribution defined by the following improper integral: for any $\phi \in \mathcal{D}(\mathbb{R})$,

$$(\gamma * \dot{Y})(\phi) = \underset{M\to\infty}{\text{l.i.m.}} \int_0^M \gamma(s)\dot{Y}(\tau_s\phi)ds.$$

We refer to [8] for the details.

The precise form of (1.1) is then the following:

$$\dot{X} = -\gamma * \dot{X} + \alpha \dot{B}. \tag{4.2}$$

We search for a stationary random distribution X which satisfies (4.2) as well as the causality condition:

$$\forall t \in \mathbb{R}, \qquad M_t(X) = M_t(\dot{B}). \tag{4.3}$$

We now state our main theorem in this section.

THEOREM 4.3. *Let $\alpha > 0$ and $\rho \in D_0^{10}$. We put $\gamma = K_\rho$. Let $B = (B(t) : t \in \mathbb{R})$ be a one-dimensional Brownian motion. Then there is a unique stationary random distribution X which satisfies both (4.2) and (4.3). The solution X is actually a stationary Gaussian process given by*

$$X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} K_\nu(t-s)dB(s) \qquad (t \in \mathbb{R}), \tag{4.4}$$

where $\nu = (L_0^1)^{-1}(\alpha, \rho)$.

Proof. Let X be any stationary random distribution which satisfies (4.3). Then the general theory of stationary random distributions shows that X is in the form

$$X(\phi) = \sqrt{2\pi} \int_{-\infty}^{\infty} (g \cdot \hat{\phi})\check{\ }(t)dB(t)$$

with some measurable g which satisfies

$$\int_{-\infty}^{\infty} \frac{|g(\xi)|^2}{(1+\xi^2)^k} d\xi < \infty$$

for some $k \in \mathbb{N} \cup \{0\}$. Here $\check{f}$ denotes the inverse Fourier transform of f. The relation $\nu = (L_0^1)^{-1}(\alpha, \rho)$ implies

$$i\xi + i\xi F_\rho(\xi) + \frac{(2\pi)^{1/2}\alpha}{F_\nu(\xi)} = 0 \qquad (\xi \in \mathbb{R} \setminus \{0\}),$$

so that by [8, Proposition 5.1] we have for any $\phi \in \mathcal{D}(\mathbb{R})$,

$$\dot{X}(\phi) + (\gamma * \dot{X})(\phi) - \alpha \dot{B}(\phi) = \alpha \int_{-\infty}^{\infty} \{(\frac{2\pi g}{F_\nu} - 1)\hat{\phi}\}\check{\ }(t)dB(t).$$

Hence X satisfies (4.2) if and only if $g = F_\nu/(2\pi)$, so that the stationary Gaussian process given by (4.4) is the unique solution of (4.2) with (4.3). This completes the proof.

For any $\sigma \in \Sigma_0$, we define

$$R_\sigma(t) := \int_0^\infty e^{-|t|\lambda}\sigma(d\lambda) \qquad (t \in \mathbb{R}).$$

The following corollary follows easily from Theorem 4.3.

COROLLARY 4.4. *Let $\alpha > 0$, $\rho \in D_0^{10}$, $\nu = (L_0^1)^{-1}(\alpha, \rho)$ and $\sigma = S(\nu)$. We set $\gamma = K_\rho$. Let X be the solution of (4.2) with (4.3), and let E and R be the canonical representation kernel and the correlation function of X, respectively. Then*

(1) $E = K_\nu$;
(2) *for any $y > 0$,*

$$\int_0^\infty e^{-yt} E(t)dt \cdot \left\{ y + y \int_0^\infty e^{-yt} \gamma(t) dt \right\} = (2\pi)^{1/2} \alpha;$$

(3) $R = R_\sigma$;
(4) $\int_0^\infty R(t)dt = \infty$.

According to [**12**, Definition 4.3], we give the following definition.

DEFINITION 4.5. We call (4.2) *the first KMO-Langevin equation.*

REMARK 4.6. Some sufficient conditions for $\rho \in M_0^1$ to be in D_0^{10} are given in the next section.

REMARK 4.7. Let $\sigma \in \Sigma_0^{10}$, and let X be any stationary Gaussian process with correlation function R_σ. We set $(\alpha, \rho) = L_0^1(S^{-1}(\sigma))$ and $\gamma = K_\rho$. Let $B = (B(t))$ be the canonical Brownian motion of X, that is, B is the unique one-dimensional Brownian motion which satisfies $B(0) = 0$, (4.3) and $X(t) = (2\pi)^{-1} \int_{-\infty}^t E(t - s)dB(s)$ for $t \in \mathbb{R}$, E being the canonical representation kernel of X. Then we can show that X satisfies (4.2) together with B. The proof is almost the same with Theorem 4.3.

REMARK 4.8. If we omit the condition (4.3), then the uniqueness does not hold. See [**8**, Remark 5.8].

In the above discussions, the following stationary Gaussian processes are excluded: stationary Gaussian processes whose correlations R are in the form $R = R_\sigma$ with $\sigma \in \Sigma_0^{11}$. In what follows, we supplement the results above by considering the Langevin equation (1.7) which describes such processes. We refer to [**12**] for the similar Langevin equation (1.6) which describes processes with $D < \infty$.

We define

$$\Sigma_1^1 = \{\nu \in M_0 : m_0(\nu) = \infty, \ m_{-1}(\nu) = \infty\}.$$

The next theorem corresponds to (4.1).

THEOREM 4.9. *The relation*

$$-i\zeta F_\rho(\zeta) F_\nu(\zeta) = \sqrt{2\pi} \quad (\mathrm{Im}\ \zeta > 0) \tag{4.5}$$

determines a bijection $\nu \mapsto L_1^1(\nu) = \rho$ from Σ_1^1 onto Σ_1^1.

Proof. Let $\nu \in \Sigma_1^1$, and set $\nu_n(d\lambda) = \chi_{(0,n)}(\lambda)\nu(d\lambda)$ on $(0, \infty)$ for $n = 1, 2, \cdots$. Then for any n, we have $m_0(\nu_n) < \infty$ and $m_{-1}(\nu_n) = \infty$. In view of Theorem 3.1 of [8], there exists a positive number ϵ_n and a Borel measure $\rho_n \in M_0$ such that

$$F_{\nu_n}(\zeta)\{-i\epsilon_n\zeta - i\zeta F_{\rho_n}(\zeta)\} = \sqrt{2\pi} \qquad (\mathrm{Im}\ \zeta > 0). \tag{4.6}$$

Since we have $\epsilon_n = \sqrt{2\pi}/m_0(\nu_n)$ by [8, Theorem 3.2], ϵ_n tends to 0 as $n \to \infty$. On the other hand, by taking $\zeta = i$ in (4.6), we get the estimate $\sup_n \int_0^\infty (1+\lambda)^{-1}\rho_n(d\lambda) < \infty$, so that there exists a subsequence $n_1 < n_2 < \cdots$ and a bounded Borel measure $\widetilde{\rho}$ on $[0,\infty]$ such that the measure $(1+\lambda)^{-1}\rho_{n_k}(d\lambda)$ weakly converges to $\widetilde{\rho}$ on $[0,\infty]$ as $k \to \infty$. Transition to the limit $k \to \infty$ yields

$$F_\nu(\zeta)\{c_1 - ic_2\zeta - i\zeta F_\rho(\zeta)\} = \sqrt{2\pi} \qquad (\mathrm{Im}\,\zeta > 0), \tag{4.7}$$

where $\rho(d\lambda) = \chi_{(0,\infty)}(\lambda)(1+\lambda)\widetilde{\rho}(d\lambda)$, $c_1 = \widetilde{\rho}(\{0\})$, $c_2 = \widetilde{\rho}(\{\infty\})$. Take $\zeta = iy$ in (4.7); let $y \downarrow 0$ to see $c_1 = 0$, and let $y \uparrow \infty$ to get $c_2 = 0$; let $y \downarrow 0$ again to see $m_{-1}(\rho) = \infty$. Thus we have shown the existence of a Borel measure ρ in Σ_1^1 which satisfies (4.5) with ν. The uniqueness of Stieltjes transform implies the well-definedness and injectivity of the map L_1^1. The surjectivity of L_1^1 follows in the same way, since (4.5) is symmetric in ν and ρ. This completes the proof.

Since $N_0^{11} = \{\nu \in \Sigma_1^1 : \nu \text{ satisfies } (3.1)\} \subset \Sigma_1^1$, the image $L_1^1(N_0^{11})$ forms a subset of Σ_1^1. We set

$$D_0^{11} = L_1^1(N_0^{11}).$$

Let $\alpha > 0$, $\rho \in D_0^{11}$ and $\gamma = K_\rho$. Then the precise form of (1.7) is the following:

$$\gamma * \dot{X} = \alpha \dot{B}. \tag{4.8}$$

THEOREM 4.10. *Let $\alpha > 0$ and let $\rho \in D_0^{11}$. We put $\gamma = K_\rho$. Let $B = (B(t) : t \in \mathbb{R})$ be a one-dimensional Brownian motion. Then there is a unique stationary random distribution X which satisfies both (4.8) and (4.3). The solution X is actually a stationary Gaussian process given by*

$$X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t K_\nu(t-s)dB(s) \qquad (t \in \mathbb{R}),$$

where $\nu = (L_1^1)^{-1}(\rho/\alpha)$.

The proof of Theorem 4.10 is similar to that of Theorem 4.3, and so omitted. We also call (4.8) the first KMO-Langevin equation.

The following corollary follows easily from Theorem 4.10.

COROLLARY 4.11. *Let $\alpha > 0$, $\rho \in D_0^{11}$, $\nu = (L_1^1)^{-1}(\rho/\alpha)$ and $\sigma = S(\nu)$. We set $\gamma = K_\rho$. Let X be the solution of (4.8) with (4.3), and let E and R be the canonical representation kernel and the correlation function of X, respectively. Then*

(1) $E = K_\nu$;
(2) *for any* $y > 0$,

$$y\left\{\int_0^\infty e^{-yt}E(t)dt\right\} \cdot \left\{\int_0^\infty e^{-yt}\gamma(t)dt\right\} = (2\pi)^{1/2}\alpha;$$

(3) $R = R_\sigma$;
(4) $\int_0^\infty R(t)dt = \infty$.

REMARK 4.12. Some sufficient conditions for $\rho \in \Sigma_1^1$ to be in D_0^{11} are given in the next section.

REMARK 4.13. Let $\sigma \in \Sigma_0^{11}$, and let X be any stationary Gaussian process with correlation function R_σ. We set $\rho = L_1^1(S^{-1}(\sigma))$ and $\gamma = K_\rho$. Let $B = (B(t))$ be the canonical Brownian motion of X. Then we can show that X satisfies (4.8) together with B.

5. *The delay kernel*

In this section, we consider the delay Langevin equation (4.2). By applying the results of sections 2 and 3, we characterize long-time tails of the correlation function R of the solution in terms of the delay kernel γ. Such a correspondence between the decays of R and γ is studied by Okabe [16] and [6] for Equation (1.6). The special case $R(t) \sim \text{const.} \times t^{-3/2}$ as $t \to \infty$ corresponds to the phenomena discovered by Alder and Wainwright through computer simulations (see [1], Kubo et al. [10] and Oobayashi et al. [18]). Discrete parameter versions of these results are discussed in [7]. In Okabe and Inoue [17], long-time behaviour of exponential type is discussed. In [8], a different type of correspondence between the decays of R and γ is obtained for delay Langevin equations describing processes with $D = \infty$.

In what follows, let $\alpha >$, $\rho \in D_0^{10}$ and $\gamma = K_\rho$, let X be the solution of (4.2) with (4.3), and let R and E be the correlation function and canonical representation kernel of X, respectively. The common course of our discussions below is to relate a long-time tail of R with that of γ through that of E.

We begin with the following proposition.

PROPOSITON 5.1. *Let $l \in R_0$, and let $0 < q \leq 1/2$. Then $\gamma(t) \sim t^{-q}l(t)$ as $t \to \infty$ if and only if*

$$E(t) \sim \frac{t^{-(1-q)}}{l(t)} \cdot \frac{2^{1/2}\alpha \sin(q\pi)}{\pi^{1/2}} \qquad (t \to \infty).$$

Proof. In the same way as the proof of [8, Theorem 6.1], we obtain the theorem by applying Karamata's Tauberian Theorem (cf. Bingham et al. [3, Theorem 1.7.6]) to Corollary 4.4(2).

First we consider the case $0 < p < 1$.

THEOREM 5.2. *Let $l_1 \in R_0$ and $0 < p < 1$. Then $R(t) \sim \alpha^2 t^{-p} l_1(t)$ as $t \to \infty$ if and only if*

$$\gamma(t) \sim \frac{t^{-(1-p)/2}}{(l_1(t))^{1/2}} \cdot \frac{(B(\frac{1-p}{2}, p))^{1/2} \cos(\frac{p}{2}\pi)}{\pi} \qquad (t \to \infty).$$

Proof. The theorem follows from Proposition 5.1 and Theorem 2.5.

Next we consider the case $p = 0$.

THEOREM 5.3. *Let $l_1 \in R_0$. Then $R \in \Pi_{l_1}$ with index $-\alpha^2$ if and only if*

$$\gamma(t) \sim \frac{t^{-1/2}}{(l_1(t))^{1/2}} \cdot \frac{1}{\pi} \qquad (t \to \infty).$$

Proof. The theorem follows from Theorem 3.1 and Proposition 5.1.

For $l \in R_0$ such that $\int^\infty l(t)dt/t < \infty$, we set

$$\bar{l}(t) := \int_t^\infty l(s)ds/s.$$

Then $\bar{l}$ is also slowly varying (see [**3**, Theorem 1.5.9b]).

REMARK 5.4. Let $l \in R_0$ and $R \in \Pi_l$ with index $-\alpha^2$. Then by (3.2) and de Haan's Theorem, we have $\dot{R}(t) \sim -\alpha^2 l(t)/t$ as $t \to \infty$. Therefore it follows from (3.2) that $\int^\infty l(t)dt/t < \infty$ and $R(t) \sim \alpha^2 \bar{l}(t)$ as $t \to \infty$.

Finally, we consider the case $p = 1$. If $f : (0,\infty) \to (0,\infty)$ is measurable, we define its *Laplace-Stieltjes transform (LS transform)*

$$\hat{f}(x) := x \int_0^\infty e^{-tx} f(t)dt$$

for all $x > 0$ for which the integral is finite. If f is non-decreasing, right-continuous, and $f(0+) = 0$, we have

$$\hat{f}(x) = \int_{[0,\infty)} e^{-tx} df(t).$$

In the discussion below, the Abel-Tauber theorem of de Haan [**5**] is crucial. The following version of this theorem is also useful.

THEOREM 5.5. (de Haan's Abel-Tauber Theorem; a version). *Let $c > 0$ and $l \in R_0$. Let f be a positive, non-increasing and right-continuous function on $(0,\infty)$. We assume that f is locally integrable on $[0,\infty)$. Then $f \in \Pi_l$ with index $-c$ if and only if $\hat{f}(1/\cdot) \in \Pi_l$ with index $-c$.*

Proof. We define $g(t) = 0$ on $(0,1)$ and $= f(1) - f(t)$ on $[1,\infty)$. Then $f \in \Pi_l$ with index $-c$ is equivalent to $g \in \Pi_l$ with index c. Since g is bounded, non-decreasing, right-continuous and $g(0+) = 0$, by de Haan's Abel-Tauber Theorem (cf. [**3**, Theorem 3.9.1]) $g \in \Pi_l$ with index c is equivalent to $\hat{g}(1/\cdot) \in \Pi_l$ with index c. By integration by parts we have

$$\hat{f}(x) = e^{-x} f(1) + x \int_0^1 e^{-tx} f(t)dt - \hat{g}(x) \qquad (x > 0).$$

For any $\lambda \geq 1$,

$$(e^{-1/x} - e^{-1/\lambda x})/l(x) \leq (1 - \lambda^{-1})/\{xl(x)\} \to 0 \qquad (x \to \infty).$$

Further we have

$$\frac{1}{xl(x)}\int_0^1 e^{-t/x}f(t)dt \leq \frac{1}{xl(x)}\int_0^1 f(t)dt \to 0 \qquad (x \to \infty).$$

Therefore $\hat{g}(1/\cdot) \in \Pi_l$ with index c is equivalent to $\hat{f}(1/\cdot) \in \Pi_l$ with index $-c$. Thus the theorem follows.

REMARK 5.6. The above version of de Haan's theorem is far more generalized as Abel-Tauber theorems for Mellin convolutions by Bingham and Teugels [2].

We set $F(t) = \int_0^t E(s)ds$ for $t \geq 0$. Then

$$\hat{F}(x) = \int_0^\infty e^{-xt}E(t)dt \qquad (x > 0),$$

and so by Corollary 4.4(2), we have

$$\hat{F}(1/x)\{(1/x) + \hat{\gamma}(1/x)\} = (2\pi)^{1/2}\alpha \qquad (x > 0). \tag{5.1}$$

We begin by showing a correspondence between long-time behaviours of γ and E.

PROPOSITION 5.7. *If $l_2 \in R_0$ and $E(t) \sim \alpha l_2(t)/t$ as $t \to \infty$, then $\gamma \in \Pi_{l_3}$ with index -1, where*

$$l_3(t) = (2\pi)^{1/2} l_2(t)/(\widetilde{l_2}(t))^2. \tag{5.2}$$

Conversely if $l_3 \in R_0$ and $\gamma \in \Pi_{l_3}$ with index -1, then $E(t) \sim \alpha l_2(t)/t$ as $t \to \infty$, where

$$l_2(t) = (2\pi)^{1/2} l_3(t)/(\overline{l_3}(t))^2. \tag{5.3}$$

Proof. Since $\int_0^\infty E(t)dt = \infty$, $E(t) \sim \alpha l_2(t)/t$ as $t \to \infty$ implies $F(t) \sim \alpha\widetilde{l_2}(t)$ as $t \to \infty$, and so by Karamata's Tauberian Theorem (cf. [**3**, Theorem 1.7.1]), $\hat{F}(1/x) \sim \alpha\widetilde{l_2}(x)$ as $x \to \infty$. Further since $F \in \Pi_{l_2}$ with index α, it follows from de Haan's Abel-Tauber Theorem that $\hat{F}(1/\cdot) \in \Pi_{l_2}$ with index α. Therefore by (5.1) we have for any $\lambda \geq 1$,

$$\begin{aligned}\frac{\hat{\gamma}(1/\lambda x) - \hat{\gamma}(1/x)}{\{l_2(x)/(\widetilde{l_2}(x))^2\}} &= -(2\pi)^{1/2}\alpha\frac{\widetilde{l_2}(x)}{\hat{F}(1/\lambda x)}\cdot\frac{\widetilde{l_2}(x)}{\hat{F}(1/x)}\cdot\frac{\{\hat{F}(1/\lambda x) - \hat{F}(1/x\})}{l_2(x)} \\ &\quad - \frac{(\widetilde{l_2}(x))^2}{xl_2(x)}(\lambda^{-1} - 1) \\ &\to -(2\pi)^{1/2}\log\lambda \qquad (x \to \infty).\end{aligned}$$

Hence by Theorem 5.5, $\gamma \in \Pi_{l_3}$ with index -1, where l_3 is defined by (5.2).

Conversely, in the same way as Remark 5.4, we see that $\gamma \in \Pi_{l_3}$ with index -1 implies $\int^\infty l_3(t)dt/t < \infty$ and $\gamma(t) \sim \overline{l_3}(t)$ as $t \to \infty$. So Karamata's Tauberian

Theorem yields $\hat{\gamma}(1/x) \sim \overline{l_3}(x)$ as $x \to \infty$. Further Theorem 5.5 implies $\hat{\gamma} \in \Pi_{l_3}$ with index -1. Therefore by (5.1) we have for any $\lambda \geq 1$,

$$\begin{aligned}\frac{\hat{F}(1/\lambda x) - \hat{F}(1/x)}{\{l_3(x)/(\overline{l_3}(x))^2\}} &= -(2\pi)^{1/2}\alpha \frac{\overline{l_3}(x)}{\{(1/\lambda x) + \hat{\gamma}(1/\lambda x)\}} \cdot \frac{\overline{l_3}(x)}{\{(1/x) + \hat{\gamma}(1/x)\}} \\ &\quad \times \frac{1}{l_3(x)} \left[(\lambda^{-1} - 1)\, x^{-1} + \{\hat{\gamma}(1/\lambda x) - \hat{\gamma}(1/x)\} \right] \\ &\to (2\pi)^{1/2}\alpha \log \lambda \qquad (x \to \infty).\end{aligned}$$

Hence by de Haan's Abel-Tauber Theorem, $F \in \Pi_{l_2}$ with index α, where l_2 is defined by (5.3), whence $E(t) \sim \alpha l_2(t)/t$ as $t \to \infty$. This completes the proof.

We are now ready to characterize the regular variation of R with index -1 in terms of γ.

THEOREM 5.8. *If $l_1 \in R_0$ and $R(t) \sim \alpha^2 l_1(t)/t$ as $t \to \infty$, then $\gamma \in \Pi_{l_3}$ with index -1, where $l_3(t) = l_1(t)/(2\widetilde{l_1}(t))^{3/2}$. Conversely if $l_3 \in R_0$ and $\gamma \in \Pi_{l_3}$ with index -1, then $R(t) \sim \alpha^2 l_1(t)/t$ as $t \to \infty$, where $l_1(t) = l_3(t)/(\overline{l_3}(t))^3$.*

Proof. We set $l_2(t) = \pi^{1/2} l_1(t)/(\widetilde{l_1}(t))^{1/2}$ and choose M large enough. Then

$$\begin{aligned}\widetilde{l_2}(t) &= 2\pi^{1/2} \int_M^t \{(\widetilde{l_1}(s))^{1/2}\}' ds = 2\pi^{1/2}\{(\widetilde{l_1}(t))^{1/2} - (\widetilde{l_1}(M))^{1/2}\} \\ &\sim 2\pi^{1/2}(\widetilde{l_1}(t))^{1/2} \qquad (t \to \infty).\end{aligned}$$

Therefore by Theorem 3.3 and Proposition 5.7, we obtain the first half.

Conversely, we set $l_2(t) = (2\pi)^{1/2} l_3(t)/(\overline{l_3}(t))^2$ and choose M large enough. Then

$$\begin{aligned}\widetilde{l_2}(t) &= (2\pi)^{1/2} \int_M^t \{(\overline{l_3}(s))^{-1}\}' ds = (2\pi)^{1/2} \left\{(\overline{l_3}(t))^{-1} - (\overline{l_3}(M))^{-1}\right\} \\ &\sim (2\pi)^{1/2}/\overline{l_3}(t) \qquad (t \to \infty).\end{aligned}$$

So by Theorem 3.3 and Proposition 5.7 we obtain the second half.

COROLLARY 5.9. *We define $l(t) = 1/(2\log t)^{3/2}$. Then $R(t) \sim \alpha^2/t$ as $t \to \infty$ if and only if $\gamma \in \Pi_l$ with index -1.*

Proof. Since

$$\int_t^\infty \frac{1}{s(\log s)^{3/2}} ds = \frac{2}{(\log t)^{1/2}},$$

the corollary follows immediately from Theorem 5.5.

REMARK 5.10. Let $\rho \in D_0^{11}$ and $\gamma = K_\rho$. Then all of Theorems 5.2, 5.3, 5.8, Propositions 5.1, 5.7 and Corollary 5.9 also hold for the solution of the equation (4.8) with (4.2). The proofs are almost the same as the above.

So far we have considered the equation (4.2) with $\gamma = K_\rho$ under the condition $\rho \in D_0^{10}$. But for given $\rho \in M_0^1$, it is not easy to check the condition $\rho \in D_0^{10}$ directly from the definition. So we give some sufficient conditions.

LEMMA 5.11. *Let $l \in R_0$. Let $\rho \in M_0^1$ and $\gamma = K_\rho$.*

(1) *Let $0 < q < 1/2$. If $\gamma(t) \sim t^{-q}l(t)$ as $t \to \infty$, then $\rho \in D_0^{10}$.*
(2) *If $\gamma \in \Pi_l$ with index -1, then $\rho \in D_0^{10}$.*
(3) *If $\int^\infty dt/\{t(l(t))^2\} < \infty$ and $\gamma(t) \sim t^{-1/2}l(t)$ as $t \to \infty$, then $\rho \in D_0^{10}$.*

Proof. Since K_ν is bounded in $[0, \infty)$ for $\nu \in \Sigma_0^1$, we have

$$N_0^{10} = \{\nu \in \Sigma_0^1 : K_\nu \in L^2[1, \infty)\}.$$

We set $\nu := (L_0^1)^{-1}(1, \rho) \in \Sigma_0^1$. Then in order to prove the assertions, it is enough to show that $K_\nu \in L^2[1, \infty)$. If $0 < q < 1/2$ and $\gamma(t) \sim t^{-q}l(t)$ as $t \to \infty$, then in the same way as Proposition 5.1, $K_\nu(t) \sim ct^{-(1-q)}/l(t)$ as $t \to \infty$ for some $c > 0$. Thus $K_\nu \in L^2[1, \infty)$, which proves (1). The second assertion follows easily from the same argument as the proof of Proposition 5.7, and the third from Proposition 5.1.

EXAMPLE 5.12. Let $0 < q < 1/2$ and let $\rho(d\lambda) = \lambda^{q-1}d\lambda/\Gamma(q)$ on $(0, \infty)$. Then $\gamma(t) = t^{-q}$ for $t > 0$, and so by Lemma 5.11 we see that $\rho \in D_0^{10}$.

Similar criteria also hold for the condition $\rho \in D_0^{11}$.

LEMMA 5.13. *Let $l_1, l_2 \in R_0$ and $1/2 < r < 1$. Let $\rho \in \Sigma_1^1$ and $\gamma = K_\rho$. We assume $\gamma(t) \sim t^{-r}l_2(1/t)$ as $t \to 0+$.*

(1) *Let $0 < q < 1/2$. If $\gamma(t) \sim t^{-q}l_1(t)$ as $t \to \infty$, then $\rho \in D_0^{11}$.*
(2) *If $\gamma \in \Pi_{l_1}$ with index -1, then $\rho \in D_0^{11}$.*
(3) *If $\int^\infty dt/\{t(l_1(t))^2\} < \infty$ and if $\gamma(t) \sim t^{-1/2}l_1(t)$ as $t \to \infty$, then $\rho \in D_0^{11}$.*

Proof. We easily see that

$$N_0^{11} = \{\nu \in \Sigma_1^1 : K_\nu \in L^2(0, \infty)\}.$$

We set $\nu = (L_1^1)^{-1}(\rho) \in \Sigma_1^1$. Then by applying a version of [**3**, Theorem 1.7.6] with ∞ and $0+$ replaced by each other, we see that $\gamma(t) \sim t^{-r}l_2(1/t)$ as $t \to 0+$ implies $K_\nu(t) \sim ct^{-(1-r)}/l_2(1/t)$ as $t \to 0+$ for some $c > 0$. Thus $K_\nu \in L^2[0, 1]$. The rest of the proof is almost the same as that of Lemma 5.11, and so omitted.

We conclude this section with one reason why we think Theorem 5.2 interesting. Let $\alpha > 0$, $\rho \in M_0^1$ and $\gamma = K_\rho$. Let $0 < q < 1/2$ and $l \in R_0$. We assume that $\gamma(t) \sim t^{-q}l(t)$ as $t \to \infty$. We consider the following Langevin equation

$$\dot{X} = -\beta X - \gamma * \dot{X} + \alpha\dot{B}. \tag{5.4}$$

under the condition (4.3). If $\beta > 0$, then by Theorem 4.2 of Okabe [**12**], (5.4) with (4.3) has a unique stationary solution. Furthermore even if $\beta = 0$, Lemma 5.11 and Theorem 4.3 show that (5.4) with (4.3) has a unique stationary solution. But if $\beta > 0$, by Theorem 3.2 of Okabe [**16**] we see that $R(t) \sim \alpha^2\beta^{-3}qt^{-(q+1)}l(t)$ as $t \to \infty$, while if $\beta = 0$, by Theorem 5.2 we see that $R(t) \sim ct^{-(1-2q)}/(l(t))^2$ as $t \to \infty$ for some $c > 0$. Therefore we observe a distinct difference between the cases $\beta > 0$ and $\beta = 0$. In this sense, we may say that there occurs a kind of critical phenomenon as $\beta \to 0+$, and that zero is the critical value of β.

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