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Throughout this paper A will be a division ring with non-trivial valuation v , and C will be the center of A . A has the completion with respect to the v -topology, which is also a division ring. We will denote it by A^* . For each division subring B of A the closure of B in A^* with respect to the v -topology will be also denoted by B^* . B^* is isomorphic to the completion of B as topological ring.

The aim of this paper is to show that B^* coincides with the double centralizer of B in A^* for each division C -subalgebra B of A , in the case where A is finite over C and A^* is a local field with respect to v .

In this paper we will use the same terminology as [4] and [6]. In particular for each division subring B of A we write

$$O(B) = \{x \in B : v(x) \leq 1\}, \quad P(B) = \{x \in B : v(x) < 1\} \text{ and}$$

Since v is non-trivial, A is locally compact if and only if $O(A)$ is compact (See § 17.5 [4]). In this case A is complete (See the proof of Proposition 17.5 b [4]).

Let v be non-archimedian and B an arbitrary division subring of A . $O(B)$ is a local ring with the maximal ideal $P(B)$. Hence $O(B)/P(B)$ is a division ring, which will be denoted by $E(B)$. Write $E(B) = K$ and $E(A) = E$. Then K is a division subring of E . We will write $f_r(A/B) = [E:K]_r$, $f_l(A/B) = [E:K]_l$ and $e(A/B) = [v(A^\circ) : v(B^\circ)]$, where A° and B° are the unit groups of A and B , respectively. In the case where $[E:K]_l = [E:K]_r$, we will write $f(A/B)$ in stead of $f_r(A/B)$ or $f_l(A/B)$. The next lemma is well known in the case where A is a commutative field.

Lemma 1. Let A, B and v be as above, and assume that both $e(A/B)$ and $f_r(A/B)$ are finite. Then we have $e(A/B)f_r(A/B) \leq [A:B]_r$.

Proof. Since $v(ab) = v(a)v(b) = v(b)v(a) = v(ba)$ for any $a, b \in A$, we can follow the same lines as the proof of Theorem 4.5 Chap. 2 [2].

Now suppose that A is finite dimensional over C . Then for any division C -subalgebra B of A we have $B = V_A(V_A(B))$, and $V_A(B)$ is finite dimensional division C -subalgebra (See e.g., Theorem 12.7 [4]). Therefore A is an H -separable extension of every division C -subalgebra B of A by Theorem 1 [5]. Moreover $\{A^*/A, B^*/B\}$ have the centralizer property in the sense of [6], and we have $[A:B] = [A^*:B^*]$ and $B^* \supset B^*$, where $B^* = V_A \cdot (V_A \cdot (B))$ by Theorem 1.3 (4) [6]. In particular C^* is contained in the center of A^* . C is an infinite field, otherwise A must be commutative by Wedderburn's Theorem. Therefore, if furthermore v is non-archimedian and $O(A^*)$ is compact in the v -topology, for any division C -subalgebra B of A we have $e(B^*/C^*)f(B^*/C^*) = [B^*:C^*] < \infty$ by Proposition 17.7 [4], since $O(B^*)$ is also compact and C^* is infinite and closed in B^* .

Now we are ready to have our main theorem.

Theorem 1. Let A be a division ring with non-trivial non-archimedian valuation v . Assume A is finite dimensional over its center C , and $O(A^*)$ is compact in the v -topology. Then we have

(1) The closure C^* of C in A^* coincides with the center of A^* , and we have $[A^*:C^*] = [A:C]$

(2) For any division C -subalgebra B of A , we have $B^* = V_A \cdot (V_A \cdot (B))$, $A^* = B^*A = AB^*$ and $B^* \cap A = B$.

Proof. (1). Write $C^* = V_A \cdot (V_A \cdot (C))$. C^* coincides with the center of A^* by Proposition 1.3 [6]. Since v is non-archimedian, we have

$$e(C^*/C) = e(A^*/A) = 1 \text{ and } f(A^*/A) = f(C^*/C) = 1$$

(See § 17.4 [4]). Since

$$e(A^*/C) = e(A^*/C^*)e(C^*/C) = e(A^*/A)e(A/C),$$

we have $e(A/C) = e(A^*/C^*)$. Similarly we have $f(A^*/C^*) = f(A/C)$. Then by Proposition 17.7 (1) [4] and Lemma 1 we have

$$[A^*:C^*] = e(A^*/C^*)f(A^*/C^*) = e(A/C)f(A/C) \leq [A:C].$$

But by Theorem 1.7 (3) [6] we have $[A:C] = [A^*:C^*] \leq [A^*:C^*]$. Hence we have $[A^*:C^*] = [A^*:C^*]$ and $C^* = C^*$.

(2). Put $B^* = V_A \cdot (V_A \cdot (B))$. B^* is complete, and $[A^*: B^*] = [A: B]$ by Theorem 1.7 (3) and (4) [6]. On the other hand since $e(B^*/B) = f(B^*/B) = 1$, we have

$$e(A^*/B^*)f(A^*/B^*) = e(A/B)f(A/B) \leq [A: B]$$

$$e(B^*/C^*)f(B^*/C^*) = e(B/C)f(B/C) \leq [B: C]$$

where the inequalities $\leq$'s are due to Lemma 1. Then we have

$$\begin{aligned} [A^*: C^*] &= e(A^*/C^*)f(A^*/C^*) = e(A^*/B^*)e(B^*/C^*)f(A^*/B^*)f(B^*/C^*) \\ &\leq [A: B][B: C] = [A: C] = [A^*: C^*] \end{aligned}$$

This means that $e(A/B)f(A/B) = [A: B]$ and $e(B/C)f(B/C) = [B: C]$. The latter equality shows $[B: C] = e(B^*/C^*)f(B^*/C^*) = [B^*: C^*]$. Then $[A^*: B^*] = [A: B] = [A^*: B^*]$ and we have $B^* = B$. The proof of the remaining part is due to Theorem 1.7 [6].

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