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Scaling Limit of Anticommuting Self-adjoint Operators and Nonrelativistic Limit of Dirac Operators†

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Let A and B be two anticommuting self-adjoint operators and $V(\kappa)$ be a symmetric operator in a Hilbert space, where $\kappa > 0$ is a parameter. We prove that, under some conditions for $V(\kappa)$, the resolvents of $\kappa A + \kappa^2 B \pm \kappa^2 |B| + V(\kappa)$ converge as $\kappa \to \infty$. This solves in a most general form the nonrelativistic-limit problem of Dirac operators.

I. INTRODUCTION

Since Vasilescu [13] introduced a proper notion of anticommutativity of (*unbounded*) self-adjoint operators, some progress has been made on the theory of anticommuting self-adjoint operators (A.C.S.O.)[10,7,1-3]. Samoilenko [10] and Pedersen [7] considered conditions equivalent to Vasilescu's original definition of anticommutativity of self-adjoint operators and discussed some fundamental aspects of A.C.S.O. In [1] we investigated more detailed properties of A.C.S.O. including commutation properties of the partial isometries associated with A.C.S.O. and applications to Dirac operators. In [2] we gave a new and "natural" characterization of anticommutativity of self-adjoint operators in connection with Clifford algebra and applied a result obtained as a by-product of the characterization to the self-adjointness problem of operators of Dirac's type. In [3] an application of the theory of A.C.S.O. to supersymmetry is given as well as a summary of the main results in [1,2].

In this paper we consider scaling limits of the sum of two A.C.S.O. A motivation for this work comes from the nonrelativisitic-limit problem of Dirac operators, which has been extensively discussed in both the physics and the mathematics literature (e.g., [5], [12, Chapt.6] and refer-

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ences therein). As is well-known, Dirac operators in relativistic quantum mechanics have a parameter $c > 0$, physically meaning the speed of light. The nonrelativistic limit of these Dirac operators is then obtained by taking the limit $c \to \infty$ in a suitable sense. As already remarked in [1], it is more natural to regard Dirac operators as concrete realizations of the sum of two A.C.S.O. in the abstract form or of its perturbations. In other words, the sum of two A.C.S.O. or its perturbations give a most abstract form of Dirac operators. Thus we are naturally led to consider the nonrelativistic-limit problem of Dirac operators *in the framework of the theory of A.C.S.O.* This view-point, which gives a most general idea for the problem under consideration, has not *explicitly* been taken in the literature.

The present paper is organized as follows. In Section II we present some known facts on A.C.S.O. Let A and B be A.C.S.O. in a Hilbert space $\mathcal{H}$ (Definition 2.1) and $\kappa > 0$ be a constant. Section III concerns the limit $\kappa \to \infty$ of the operator

$$H(\kappa) = \kappa A + \kappa^2 B. \tag{1.1}$$

In applications to Dirac operators, this limit corresponds to the nonrelativistic limit with $\kappa = c$. It follows that $H(\kappa)$ is self-adjoint (see Lemma 2.2(vi)). One finds that, in considering the limit mentioned above, the operator $H(\kappa)$ itself is not a suitable object, but, the "renormalized" versions of $H(\kappa)$ defined by

$$H_{\pm}^{\mathrm{ren}}(\kappa) = H(\kappa) \pm \kappa^2 |B| \tag{1.2}$$

are suitable ones*. It is shown that $H_{\pm}^{\mathrm{ren}}(\kappa)$ are essentially self-adjoint (Lemma 3.1). We consider the limit $\kappa \to \infty$ of these operators in terms of the resolvents $(H_{\pm}^{\mathrm{ren}}(\kappa) - z)^{-1}, z \in \mathbb{C} \setminus \mathbb{R}$. It is shown that this problem is reduced to the case where B is injective. Let

$$m = \inf \sigma(|B|), \tag{1.3}$$

the infimum of the spectrum of $|B|$. We shall see that the topology of the convergence of $(H_{\pm}^{\mathrm{ren}}(\kappa) - z)^{-1}$ has to be taken differently corresponding to the case $m > 0$ and the case $m = 0$. Thus we discuss separately these two cases. We prove that, in the case $m > 0$ $(H_{\pm}^{\mathrm{ren}}(\kappa) - z)^{-1}$ converge in the

*For the *physical reason* of this procedure in the context of the usual Dirac operators, see [12, p.177, Remark 1].

operator-norm topology as $\kappa \to \infty$ (Theorem 3.2), while, in the case $m = 0$, they converge in general only in the strong topology as $\kappa \to \infty$ (Theorem 3.5). The idea of the method of the proof is similar to that of the methods used in [5] and [12, Chapt.6], but, we need some modifications (in particular, in the case $m = 0$) due to the fact that B is not necessarily bounded. We give a counter example, which exemplifies that, in the case $m = 0$, the topology of the convegence of $(H_{\pm}^{\mathrm{ren}}(\kappa) - z)^{-1}$ cannot be strengthened in general to the operator-norm topology. Section IV is devoted to extensions of the limit theorems established in Section III to the case where $H_{\pm}^{\mathrm{ren}}(\kappa)$ are perturbed by a symmetric operator. In the case $m = 0$, we adopt a method different from that of the corresponding case considered in Section III; we use the abstract Foldy-Wouthuysen transformation introduced in [1] to prove a convergence theorem for the perturbed operators.

In the present paper we do not discuss concrete examples of Dirac operators, for which we refer the reader to [12].

II. PRELIMINARIES

In this section we recall some basic facts on A.C.S.O. We denote by $D(A)$ the domain of the operator A. For two linear operators A and B acting in a Hilbert space, $D(A+B)$ is always taken to be $D(A) \cap D(B)$ unless otherwise stated.

DEFINITION 2.1 [7]. Two non-zero self-adjoint operators A and B in a Hilbert space are said to *anticommute* if, for all $t \in \mathbb{R}$, e^{itA} leaves $D(B)$ invariant and

$$e^{itA}B \subset Be^{-itA}.$$

Remark. It can be shown that this definition is symmetric in A and B [7]. For definitions equivalent to the above one, see [13,9,7].

A self-adjoint operator A in a Hilbert space has the polar decomposition

$$A = U_A|A|$$

with U_A a partial isometry uniquely determined by the condition $\ker U_A = \ker A$ [6, p.358]. The self-adjointness of A implies that U_A is self-adjoint.

We say that two self-adjoint operators in a Hilbert space are said to *strongly commute* if their spectral measures commute. The following lemma summarizes some fundamental properties of A.C.S.O.

LEMMA 2.2 [13,7]. *Let A and B be A.C.S.O. in a Hilbert space. Then the following* (i)-(vi) hold:

(i) $U_B A \subset -AU_B$ *and* $U_A B \subset -BU_A$.

(ii) $U_B|A| \subset |A|U_B$ *and* $U_A|B| \subset |B|U_A$.

(iii) $|A|$ *and* $|B|$ *strongly commute.*

(iv) $U_A U_B + U_B U_A = 0$.

(v) A *and* $|B|$ *strongly commute, and* B *and* $|A|$ *strongly commute.*

(vi) $A + B$ *is self-adjoint.*

Let A and B be A.C.S.O. in a Hilbert space $\mathcal{H}$. *In the rest of this section, we assume that B is injective.* Then U_B is unitary and self-adjoint. It follows from Lemma 2.2(i) that $U_B \neq \pm I$, where I denotes identity. Hence we have the orthogonal decomposition

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_+ \bigoplus \mathcal{H}_- \\ &= \left\{ \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \middle| f_\pm \in \mathcal{H}_\pm \right\}\end{aligned} \tag{2.1}$$

with

$$\mathcal{H}_\pm = \ker(U_B \mp I).$$

The orthogonal projections onto $\mathcal{H}_\pm$ are given by

$$P_\pm = \frac{1}{2}(I \pm U_B). \tag{2.2}$$

Relatively to the decomposition (2.1), every linear operator in $\mathcal{H}$ is represented as a 2×2 matrix with entries being linear operators. For example, we have

$$U_B = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad P_+ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \tag{2.3}$$

Since $|B|$ strongly commutes with U_B, it follows that $|B|$ is reduced by $\mathcal{H}_\pm$. Hence

$$B = \begin{pmatrix} |B|_+ & 0 \\ 0 & -|B|_- \end{pmatrix}, \tag{2.4}$$

where $|B|_\pm = |B| \restriction \mathcal{H}_\pm$.

Lemma 2.2(ii) implies that $|A|$ also is reduced by $\mathcal{H}_\pm$, so that

$$|A| = \begin{pmatrix} |A|_+ & 0 \\ 0 & |A|_= \end{pmatrix}. \tag{2.5}$$

Moreover, it follows from Lemma 2.2(iv) that there exists a unique partial isometry W_A with intial domain $(\ker A)^\perp \cap \mathcal{H}_+$ and final domain $(\ker A)^\perp \cap \mathcal{H}_=$ such that

$$U_A = \begin{pmatrix} 0 & W_A^* \\ W_A & 0 \end{pmatrix}. \tag{2.6}$$

By (2.5) and (2.6), we obtain

$$A = \begin{pmatrix} 0 & W_A^*|A|_- \\ W_A|A|_+ & 0 \end{pmatrix}. \tag{2.7}$$

Denoting by $e_\pm(A)$ the orthogonal projections onto $(\ker A)^\perp \cap \mathcal{H}_\pm$, we have

$$e_+(A) = W_A^* W_A, \quad e_-(A) = W_A W_A^*. \tag{2.8}$$

Noting the well-known fact that $(ST)^* = T^*S^*$ for all bounded linear operators S and densely defined linear operators T and using the self-adjointness of A, we have

$$|A|_+ W_A^* = W_A^*|A|_-, \quad |A|_- W_A = W_A|A|_+. \tag{2.9}$$

LEMMA 2.3. *The self-adjoint operator $|A|_+$ (resp. $|A|_-$) strongly commutes with $|B|_+$ (resp. $|B|_-$). Moreover,*

$$W_A|B|_+ \subset |B|_- W_A. \tag{2.10}$$

Proof. The first half of this lemma follows from Lemma 2.2(iii). The relation (2.10) is proven by rewriting Lemma 2.2(i) using (2.4) and (2.6). ∎

Remark. The matrix repesentations (2.4) and (2.7) together with Lemma 2.3 were given in [1, Theorem 4.1] without proof.

Let

$$K = \frac{1}{2} A^2 |B|^{-1}. \tag{2.11}$$

with domain $D_0(K) := D(A^2|B|^{-1}) \cap D(|B|^{-1}A^2)$.

LEMMA 2.4. *The operator K is essentially self-adjoint and nonnegative, satisfying*

$$K = \frac{1}{2}|B|^{-1}A^2 = \frac{1}{2}|A||B|^{-1}|A| \quad \textit{on } D_0(K). \tag{2.12}$$

Moreover, the closure of K is reduced by $\mathcal{H}_\pm$.

Proof. By Lemma 2.2(iii), A^2 and $|B|$ strongly commute, which implies A^2 strongly commutes with $|B|^{-1}$. In general, the product ST of two strongly commuting self-adjoint operators S and T in a Hilbert space is essentially self-adjoint on $D := D(ST) \cap D(TS)$ and $ST = TS$ on D. Thus the assertion on the essential self-adjointness of K follows. Noting that $A^2 = |A|^2$ and employing the functional calculus, we obtain the second equality in (2.12), which implies the nonnegativity of K. The reducibility of K to $\mathcal{H}_\pm$ follows from Lemma 2.2(ii). ∎

We denote the closure K by the same symbol and by $K_\pm$ the reduced parts of K to $\mathcal{H}_\pm$, respectively.

The operators $K_\pm$ can be identified more explicitly as follows. Given a self-adjoint operator S in a Hilbert space, we denote its spectral measure by E_S. By Lemma 2.3, $E_{|A|_+}$ (resp. $E_{|A|_-}$) and $E_{|B|_+}$ (resp. $E_{|B|_-}$) commute. Hence we can define a two-dimensional spectral measure $E_\pm$ by

$$E_\pm = E_{|A|_\pm} \otimes E_{|B|_\pm}. \tag{2.13}$$

LEMMA 2.5. *The following operator equalities hold:*

$$K_\pm = \int \frac{\lambda^2}{2\mu} dE_\pm(\lambda, \mu). \tag{2.14}$$

Proof. Let $\widetilde{K}_\pm$ denote the operators on the right hand side (RHS) of (2.14). By the injectivity of B, $E_\pm(\mathbb{R} \times \{0\}) = 0$. Hence $\widetilde{K}_\pm$ are self-adjoint. By the functional calculus, one easily sees that $K_\pm \restriction D_0(K) \cap \mathcal{H}_\pm \subset \widetilde{K}_\pm$, which, together with Lemma 2.4, implies (2.14). ∎

III. SCALING LIMIT (I)

Let A and B be A.C.S.O. in a Hilbert space $\mathcal{H}$ and $H(\kappa)$ and $H_\pm^{\mathrm{ren}}(\kappa)$ be given by (1.1) and (1.2), respectively.

LEMMA 3.1. *The operators $H_\pm^{\mathrm{ren}}(\kappa)$ are essentially self-adjoint.*

Proof. By the anticommutativity of A and B, we have

$$||H(\kappa)f||^2 = ||\kappa A f||^2 + ||\kappa^2 B f||^2$$

for all $f \in D(A) \cap D(B)$ [13]. Hence

$$||\kappa^2 |B| f|| \le ||H(\kappa)f||^2, \quad f \in D(A) \cap D(B),$$

which, by Wüst's theorem (e.g.,[9, Theorem X.14, p.164]), implies the desired result. ∎

In what follows, we denote the closures of $H_\pm^{\mathrm{ren}}(\kappa)$ by the same symbols, respectively.

Our aim in this section is to investigate the limit $\kappa \to \infty$ of the resolvents of $H_\pm^{\mathrm{ren}}(\kappa)$. We first remark that, it is sufficient to consider only the case where B is injective. To explain this, suppose that $\ker B \neq \{0\}$. Then we have the orthogonal decomposition

$$\mathcal{H} = \widetilde{\mathcal{H}} \oplus \ker B = \widetilde{\mathcal{H}} \oplus \ker |B|$$

with $\widetilde{\mathcal{H}} = (\ker B)^\perp$. One can show that $|B|$ strongly commutes with A, B, and $H_\pm^{\mathrm{ren}}(\kappa)$. Hence A, B and $H_\pm^{\mathrm{ren}}(\kappa)$ are reduced by $\widetilde{\mathcal{H}}$. Let $\widetilde{A}, \widetilde{B}$, and

$\widetilde{H}_{\pm}^{\mathrm{ren}}(\kappa)$ denote the reduced parts of A, B, and $H_{\pm}^{\mathrm{ren}}(\kappa)$ to $\widetilde{\mathcal{H}}$, respectively. Then it follows that $\widetilde{A}$ and $\widetilde{B}$ are A.C.S.O. and

$$\kappa\widetilde{A} + \kappa^2\widetilde{B} \pm \kappa^2|\widetilde{B}| = \widetilde{H}_{\pm}^{\mathrm{ren}}(\kappa)$$

on $D(\widetilde{A}) \cap D(\widetilde{B})$. On the other hand, we have for all $f \in \ker B$

$$(H_{\pm}^{\mathrm{ren}}(\kappa) - z)^{-1} f = (\kappa A - z)^{-1} f.$$

Hence it follows that

$$\mathrm{s} - \lim_{\kappa\to\infty} (H_{\pm}^{\mathrm{ren}}(\kappa) - z)^{-1} = -\frac{e(A)}{z} \quad \text{on } \ker B, \tag{3.1}$$

where $\mathrm{s} - \lim_{\kappa\to\infty}$ means strong limit and $e(A)$ is the orthogonal projection to $\ker A$. Hence we see how the operators $(H_{\pm}^{\mathrm{ren}}(\kappa) - z)^{-1}$ converge in the closed subspace $\ker |B|$ as $\kappa \to \infty$. In the Hilbert space $\widetilde{\mathcal{H}}$, B is injective. Thus, without loss of generality, we can assume that B is injective.

By the reason mentioned in the Introduction, we consider separately the case $m > 0$ and the case $m = 0$, where m is defined by (1.3).

3.1. *The case* $m > 0$

In this case, we have the following result.

THEOREM 3.2. *Let* $m > 0$. *Then, for all* $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\mathrm{n} - \lim_{\kappa\to\infty} (H_{+}^{\mathrm{ren}}(\kappa) + z)^{-1} = -P_{-}(K - z)^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & (-K_{-} + z)^{-1} \end{pmatrix}, \tag{3.2}$$

$$\mathrm{n} - \lim_{\kappa\to\infty} (H_{-}^{\mathrm{ren}}(\kappa) - z)^{-1} = P_{+}(K - z)^{-1} = \begin{pmatrix} (K_{+} - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.3}$$

where $\mathrm{n} - \lim_{\kappa\to\infty}$ *means limit in the operator-norm topology.*

Remark. This theorem and Theorem 3.3 below are generalizations of Corollary 6.2 and Theorem 6.1 in [12, p.179], respectively. In [12] both $|B|$ and $|B|^{-1}$ are assumed to be bounded ($|B| = M, A = Q, U_B = \tau, \kappa = c$, in the notations there) (cf. also [5]). The methods of the proofs of Theorems 3.2 and 3.3 are quite similar to those of the results just mentioned; One is only required to be careful about technical aspects related to the condition that B is not necessarily bounded. We remark that, in the cited work, the notion of anticommuting self-adjoint operators is not explicitly used.

As in Theorem 6.1 in [12], we we first derive useful formulas for the resolvents of $H_{\pm}^{\rm ren}(\kappa)$.

By the present assumption, $|B|^{-1}$ is bounded with $|B|^{-1} \le m^{-1}$. Let $z \in \mathbb{C} \setminus \mathbb{R}$ and suppose that

$$\frac{1}{\kappa} < \frac{\sqrt{2m|\mathrm{Im}\ z|}}{|z|}. \tag{3.4}$$

Then it is easy to see that the operator

$$K_z(\kappa) = K - z - \frac{z^2}{2\kappa^2}|B|^{-1} \tag{3.5}$$

is bijective with

$$K_z(\kappa)^{-1} = \sum_{n=0}^{\infty} \left(\frac{z^2}{2\kappa^2}\right)^n \left((K-z)^{-1}|B|^{-1}\right)^n (K-z)^{-1} \tag{3.6}$$

in the operator-norm topology.

THEOREM 3.3. *Let $m > 0$ and (3.4) be satisfied. Then*

$$(H_{\pm}^{\rm ren}(\kappa) \pm z)^{-1} = \left(\frac{1}{2\kappa^2}(\kappa A \mp z)|B|^{-1} \mp P_{\mp}\right) K_z(\kappa)^{-1}. \tag{3.7}$$

Proof. By Lemma 2.2(iii), we can define a two-dimensional spectral measure E by

$$E = E_{|A|} \otimes E_{|B|}. \tag{3.8}$$

Let

$$F_z(\lambda, \mu; \kappa) = \lambda^2 - 2z\mu - \frac{z^2}{\kappa^2}. \tag{3.9}$$

and define

$$F_z(\kappa) = \int F_z(\lambda, \mu; \kappa) dE(\lambda, \mu).$$

Let

$$\mathcal{D}_{A,B} = D(A^2) \cap D(A|B|) \cap D(|B|A) \cap D(B^2).$$

Then, using Lemma 2.2(ii),(v) and the fact

$$P_+ + P_- = I,$$

we can show that, for all $f \in \mathcal{D}_{A,B}$,

$$\begin{aligned}(H_+^{\mathrm{ren}}(\kappa) + z)(H_-^{\mathrm{ren}}(\kappa) - z)f &= (H_-^{\mathrm{ren}}(\kappa) - z)(H_+^{\mathrm{ren}}(\kappa) + z)f \\ &= \kappa^2 F_z(\kappa) f. \end{aligned} \tag{3.10}$$

By the functional calculus, $\mathcal{D}_{A,B}$ is a core for $F_z(\kappa)$. Hence, by a limiting argument, we obtain from (3.10)

$$(H_+^{\mathrm{ren}}(\kappa) + z)(H_-^{\mathrm{ren}}(\kappa) - z) \supset \kappa^2 F_z(\kappa), \tag{3.11}$$

$$(H_-^{\mathrm{ren}}(\kappa) - z)(H_+^{\mathrm{ren}}(\kappa) + z) \supset \kappa^2 F_z(\kappa). \tag{3.12}$$

Considering the adjoint relation of (3.11), we see that

$$\kappa^2 F_z(\kappa)^* \supset (H_-^{\mathrm{ren}}(\kappa) - z^*)(H_+^{\mathrm{ren}}(\kappa) + z^*). \tag{3.13}$$

The functional calculus gives

$$F_z(\kappa)^* = F_{z^*}(\kappa),$$

which, together with (3.12) and (3.13), implies

$$(H_-^{\mathrm{ren}}(\kappa) - z)(H_+^{\mathrm{ren}}(\kappa) + z) = \kappa^2 F_z(\kappa). \tag{3.14}$$

Similarly we obtain

$$(H_+^{\mathrm{ren}}(\kappa) + z)(H_-^{\mathrm{ren}}(\kappa) - z) = \kappa^2 F_z(\kappa). \tag{3.15}$$

Operator equations (3.14) and (3.15) imply that $F_z(\kappa)$ is bijective and

$$(H_+^{\mathrm{ren}}(\kappa)+z)^{-1} = \frac{1}{\kappa^2}(H_-^{\mathrm{ren}}(\kappa)-z)F_z(\kappa)^{-1}, \tag{3.16}$$

$$(H_-^{\mathrm{ren}}(\kappa)-z)^{-1} = \frac{1}{\kappa^2}(H_+^{\mathrm{ren}}(\kappa)+z)F_z(\kappa)^{-1}. \tag{3.17}$$

Using the easily proven relation

$$F_z(\kappa)f = 2K_z(\kappa)|B|f, \quad f \in \mathcal{D}_{A,B},$$

we can show that

$$F_z(\kappa)^{-1} = \frac{1}{2}|B|^{-1}K_z(\kappa)^{-1}. \tag{3.18}$$

In the present case, we have

$$\mathrm{Ran}\, K_z(\kappa)^{-1} = D(K) = D(A^2),$$

since $|B|^{-1}$ is bounded. Hence it follows that $\mathrm{Ran}\, F_z(\kappa)^{-1} \subset D(A^2) \cap D(B)$. By virtue of this fact, we can rewrite the RHS's of (3.16) and (3.17) to obtain (3.7). ∎

Remark. Introducing the Hermitian matrix

$$T_\kappa(\lambda,\mu) = \begin{pmatrix} 0 & \lambda \\ \lambda & 2\kappa\mu \end{pmatrix},$$

we can write the function $F_z(\lambda,\mu;\kappa)$ given by (3.9)

$$F_z(\lambda,\mu;\kappa) = -\det\left(T_\kappa(\lambda,\mu)+\frac{z}{\kappa}\right),$$

which implies that $F_z(\lambda,\mu;\kappa) \neq 0$ for all $z \in \mathbb{C}\setminus\mathbb{R}, \lambda,\mu \geq 0, \kappa > 0$. Hence, $F_z(\kappa)$ is injective and $F_z(\kappa)^{-1}$ exists as a densely defined closed linear operator. If $m > 0$, then one can show that $F_z(\kappa) \geq C$ with a constant $C = C(m,z,\kappa) > 0$ and hence $F_z(\kappa)^{-1}$ is bounded.

Proof of Theorem 3.2: We have from (3.7)

$$
\begin{aligned}
&(H_+^{\mathrm{ren}}(\kappa)+z)^{-1}+P_-(K-z)^{-1} \\
&=\frac{1}{2\kappa}A|B|^{-1}K_z(\kappa)^{-1}-\frac{z}{2\kappa^2}|B|^{-1}K_z(\kappa)^{-1} \\
&\quad -P_-\left(K_z(\kappa)^{-1}-(K-z)^{-1}\right).
\end{aligned} \tag{3.19}
$$

By the functional calculus, we have for all $f\in\mathcal{H}$

$$
\|A|B|^{-1}K_z(\kappa)^{-1}f\|^2=\int_{\mathbb{R}_+\times[m,\infty)}\psi_\kappa(\lambda,\mu)d\|E(\lambda,\mu)f\|^2,
$$

where $\mathbb{R}_+=[0,\infty)$ and

$$
\psi_\kappa(\lambda,\mu)=\frac{4\lambda^2}{|F_z(\lambda,\mu;\kappa)|^2}
$$

It is easy to see that there exists a positive constant κ_0 depending on z and m such that

$$
C:=\sup_{\kappa>\kappa_0,\lambda\geq 0,\mu\geq m}\psi_\kappa(\lambda,\mu)<\infty.
$$

Hence $A|B|^{-1}K_z(\kappa)^{-1}$ is uniformly bounded in $\kappa>\kappa_0$ with

$$
\|A|B|^{-1}K_z(\kappa)^{-1}\|\leq\sqrt{C},\quad \kappa>\kappa_0.
$$

Hence

$$
\mathrm{n}-\lim_{\kappa\to\infty}\frac{1}{2\kappa}A|B|^{-1}K_z(\kappa)^{-1}=0.
$$

It follows from (3.6) that

$$
\mathrm{n}-\lim_{\kappa\to\infty}K_z(\kappa)^{-1}=(K-z)^{-1}.
$$

Since $|B|^{-1}$ is bounded, we have

$$
\mathrm{n}-\lim_{\kappa\to\infty}\frac{z}{2\kappa^2}|B|^{-1}K_z(\kappa)^{-1}=0.
$$

Thus the RHS of (3.19) conveges to 0 in the operator-norm topology as $\kappa \to \infty$. This proves (3.2). Similarly we can prove (3.3). ∎

Theorem 3.3 and (3.6) give the following result.

COROLLARY 3.4. *Let $m > 0$. Then, for all $\kappa > 0$ and $z \in \mathbb{C} \setminus \mathbb{R}$ satisfying (3.4), we have*

$$(H_\pm^{\mathrm{ren}}(\kappa) \pm z)^{-1} = \sum_{n=0}^{\infty} \frac{R_n^\pm(z)}{\kappa^n}$$

in the operator-norm topology, where

$$R_0^\pm(z) = \mp P_\mp (K - z)^{-1},$$
$$R_{2n}^\pm(z) = \left(\frac{z^2}{2}\right)^{n-1} \left(\mp P_\mp \frac{z^2}{2}(K - z)^{-1} \mp \frac{z}{2}\right) \left(|B|^{-1}(K - z)^{-1}\right)^n,$$
$$R_{2n-1}^\pm(z) = \frac{1}{2}\left(\frac{z^2}{2}\right)^{n-1} A \left(|B|^{-1}(K - z)^{-1}\right)^n, \quad n \geq 1.$$

Corollary 3.4 shows that the resolvents of $H_\pm^{\mathrm{ren}}(\kappa)$ as functions of κ^{-1} can be extended to operator-valued functions holomorphic in a neighborhood of the origin depending on z.

3.2 *The case $m = 0$*

In this case we have a weaker version of Theorem 3.2.

THEOREM 3.5. *Assume that $\bar{B}$ is injective and $m = 0$. Then, for all $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\mathrm{s} - \lim_{\kappa \to \infty} (H_+^{\mathrm{ren}}(\kappa) + z)^{-1} = -P_-(K - z)^{-1}, \tag{3.20}$$
$$\mathrm{s} - \lim_{\kappa \to \infty} (H_-^{\mathrm{ren}}(\kappa) - z)^{-1} = P_+(K - z)^{-1}. \tag{3.21}$$

Remark. To the author's best knowledge, the present case has not been discussed in the literature.

Proof. In the present case also, (3.16) and (3.17) hold. Computing the RHS of (3.16), we obtain

$$(H_+^{\text{ren}}(\kappa)+z)^{-1} = \frac{1}{\kappa}U_A|A|F_z(\kappa)^{-1} - 2P_-|B|F_z(\kappa)^{-1} - \frac{z}{\kappa^2}F_z(\kappa)^{-1}. \tag{3.22}$$

It follows from the injectivity of B that

$$E(\mathbb{R}\times\{0\}) = 0. \tag{3.23}$$

Hence the operators

$$L_1 := \int \frac{\lambda}{\lambda^2 - 2\mu z} dE(\lambda,\mu)$$

and

$$L_2 := \int \frac{1}{\lambda^2 - 2\mu z} dE(\lambda,\mu)$$

are densely defined and closed. Let

$$\mathcal{D} = \cup_{a>0, 0<b<c} \text{Ran } E_{|A|}([0,a])E_{|B|}((b,c]).$$

Then $\mathcal{D}$ is dense in $\mathcal{H}$ and a common core for $L_j, j=1,2$. Let $f \in \mathcal{D}$, so that $f = E_{|A|}([0,a])E_{|B|}((b,c])g$ for some a,b,c and $g \in \mathcal{H}$. Then we have

$$\||A|F_z(\kappa)^{-1}f - L_1 f\|^2 = \int \eta_\kappa(\lambda,\mu) d\|E(\lambda,\mu)f\|^2,$$

with

$$\eta_\kappa(\lambda,\mu) = \chi_{[0,a]}(\lambda)\chi_{(b,c]}(\mu)\lambda^2 \left|\frac{1}{F_z(\lambda,\mu;\kappa)} - \frac{1}{\lambda^2 - 2\mu z}\right|^2,$$

where χ_J is the characteristic function of the set J. Take $\kappa_0 > 0$ such that

$$\alpha := b - \frac{|\text{Re}z|}{\kappa_0^2} > 0.$$

Then we have for all $(\lambda, \mu) \in [0, a] \times (b, c]$

$$|F_z(\lambda, \mu; \kappa)| \geq 2|\mathrm{Im} z|\sqrt{\alpha} > 0.$$

Hence it follows that

$$|\eta_\kappa(\lambda, \mu)| \leq C,$$

where C is a constant only depending on a, b, c, and z. Obviously we have for all $\lambda \geq 0, \mu > 0$

$$\lim_{\kappa \to \infty} \eta_\kappa(\lambda, \mu) = 0.$$

Hence, we can apply the Lebesgue dominated convergence theorem to obtain

$$\lim_{\kappa \to \infty} |||A|F_z(\kappa)^{-1} f - L_1 f||^2 = 0.$$

Similarly we can show that

$$\lim_{\kappa \to \infty} ||2|B|F_z(\kappa)^{-1} f - (K - z)^{-1} f|| = 0,$$
$$\lim_{\kappa \to \infty} ||F_z(\kappa)^{-1} f - L_2 f|| = 0.$$

Hence, by (3.22), we obtain

$$\lim_{\kappa \to \infty} ||(H_+^{\mathrm{ren}}(\kappa) + z)^{-1} f + P_-(K - z)^{-1} f|| = 0.$$

Since $\mathcal{D}$ is dense, an $\varepsilon/3$ argument gives (3.20). Similarly we can prove (3.21). ∎

Remarks. (1) In the present case, $|B|$ is unbounded, so that (3.6) and (3.18) (hence (3.7)) may fail to hold (see the example given below).

(2) Another proof of Theorem 3.5 using a diagonalization of $H(\kappa)$ (cf. Section 4.2) is given in [4].

Under the present condition $m = 0$, the topology of convergence in (3.20) and (3.21) cannot be strengthened to the operator-norm topology as the following simple example shows.

EXAMPLE. Consider the following case :

$$\mathcal{H} = L^2(\mathbb{R}; \mathbb{C}^2) = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}),$$
$$A = \begin{pmatrix} 0 & |x| \\ |x| & 0 \end{pmatrix}, \quad B = \begin{pmatrix} x^2/2 & 0 \\ 0 & -x^2/2 \end{pmatrix},$$

where x denotes the multiplication operator by the variable $x \in \mathbb{R}$. It is easy to check that A and B are anticommuting self-adjoint operators and B is injective with $\inf \sigma(|B|) = 0$. Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then we have

$$H_-^{\mathrm{ren}}(\kappa) - z = \begin{pmatrix} -z & \kappa|x| \\ \kappa|x| & -\kappa^2 x^2 - z \end{pmatrix}$$

and

$$P_+(K - z)^{-1} = \begin{pmatrix} (1-z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that

$$(H_-^{\mathrm{ren}}(\kappa) - z)^{-1} - P_+(K - z)^{-1} = \begin{pmatrix} C_\kappa & D_\kappa \\ D_\kappa & E_\kappa \end{pmatrix},$$

where

$$C_\kappa = \frac{\kappa^2 x^2 + z}{(1-z)\kappa^2 x^2 - z^2} - \frac{1}{1-z}, \quad D_\kappa = \frac{\kappa x}{(1-z)\kappa^2 x^2 - z^2},$$
$$E_\kappa = \frac{z}{(1-z)\kappa^2 x^2 - z^2}.$$

It follows that

$$||E_\kappa|| \leq ||(H_-^{\mathrm{ren}}(\kappa) - z)^{-1} - P_+(K - z)^{-1}||.$$

On the other hand, we have

$$\begin{aligned} ||E_\kappa|| &= \sup_{x \in \mathbb{R}} \frac{|z|}{|(1-z)\kappa^2 x^2 - z^2|} \\ &= \sup_{y \geq 0} \frac{|z|}{|(1-z)y - z^2|}, \end{aligned}$$

i.e., $||E_\kappa||$ is a positive constant independent of κ. Hence $(H_-^{\mathrm{ren}}(\kappa) - z)^{-1}$ does not converge to $P_+(K-z)^{-1}$ in the operator-norm topology as $\kappa \to \infty$. Similarly we can show that the same holds for $(H_+^{\mathrm{ren}}(\kappa)+z)^{-1}$ and its strong limit $-P_-(K-z)^{-1}$.

IV. SCALING LIMIT (II)

Let A and B be A.C.S.O. in $\mathcal{H}$ as before . In this section we consider the limit $\kappa \to \infty$ of perturbations of $H_\pm^{\mathrm{ren}}(\kappa)$. As in the last section, we consider separately the case $m > 0$ and the case $m = 0$.

4.1 *The case* $m > 0$

Let V be a symmetric operator in the Hilbert space $\mathcal{H}$ of the form

$$V \equiv \begin{pmatrix} V_+ & 0 \\ 0 & V_- \end{pmatrix}, \tag{4.1}$$

where $V_\pm$ are symmetric operators in $\mathcal{H}_\pm$, respectively. We assume the following.

ASSUMPTION A.

(A.1) $D(A) \subset D(V)$ and there exist positive constants a, b such that for all $f \in D(A)$

$$||Vf|| \leq a||Af|| + b||f||.$$

(A.2) There exist positive constants c and κ_0 such that for all $\kappa \geq \kappa_0$ and $f \in D(A) \cap D(B)$

$$||Vf|| \leq c||(A + \kappa(B \pm |B|))f|| + d_\kappa||f||,$$

where $d_\kappa \geq 0$ is a constant that may depend on κ.

(A.3) The operators

$$K_\pm(V) = K \pm VP_\pm \tag{4.2}$$

are self-adjoint.

Remarks. (1) If B is bounded, then conditions (A.1) and (A.2) are equivalent.

(2) In the present case, $|B|^{-1}$ is bounded and strongly commutes with A^2 (Lemma 2.2(v)). Hence $D(K) = D(A^2)$, which, together with (A.1), implies

$$D(K_\pm(V)) = D(K) = D(A^2). \tag{4.3}$$

By condition (A.2) we have for all $f \in D(A) \cap D(B)$

$$||Vf|| \leq \frac{c}{\kappa}||H_\pm^{\mathrm{ren}}(\kappa)f|| + d_\kappa||f||.$$

Since $D(A) \cap D(B)$ is a core of $H_\pm^{\mathrm{ren}}(\kappa)$, this inequality extends to all $f \in D(H_\pm^{\mathrm{ren}}(\kappa))$. Hence, by the Kato-Rellich theorem(e.g.,[9, Theorem X.12, p.162]), the operators

$$H_\pm^{\mathrm{ren}}(\kappa; V) := H_\pm^{\mathrm{ren}}(\kappa) + V \tag{4.4}$$

are self-adjoint for all κ satisfying

$$\kappa > r_0 := \max\{c, \kappa_0\}. \tag{4.5}$$

In what follows, we assume (4.5).

THEOREM 4.1. *Let $m > 0$. Then, for all $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\begin{aligned} \mathrm{n}-\lim_{\kappa\to\infty}(H_+^{\mathrm{ren}}(\kappa; V) + z)^{-1} &= -P_-(K_-(V) - z)^{-1} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & (-K_- + V_- + z)^{-1} \end{pmatrix}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \mathrm{n}-\lim_{\kappa\to\infty}(H_-^{\mathrm{ren}}(\kappa; V) - z)^{-1} &= P_+(K_+(V) - z)^{-1} \\ &= \begin{pmatrix} (K_+ + V_+ - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{4.7}$$

This theorem can be proven by extending Theorem 3.3 to the present case. Let

$$K_z^\pm(\kappa; V) = K_z(\kappa) \pm VP_\pm = K_\pm(V) - z - \frac{z^2}{2\kappa^2}|B|^{-1} \tag{4.8}$$

and

$$\kappa > \kappa_1 := \max\left\{ r_0, \frac{|z|}{\sqrt{2m|\operatorname{Im} z|}} \right\}. \tag{4.9}$$

Then $K_z^{\pm}(\kappa; V)$ are bijective and

$$\begin{aligned} K_z^{\pm}(\kappa; V)^{-1} &= (K_{\pm}(V) - z)^{-1} \left(1 - \frac{z^2}{2\kappa^2} |B|^{-1} (K_{\pm}(V) - z)^{-1} \right)^{-1} \\ &= \sum_{n=0}^{\infty} \left(\frac{z^2}{2\kappa^2} \right)^n (K_{\pm}(V) - z)^{-1} \left(|B|^{-1} (K_{\pm}(V) - z)^{-1} \right)^n \end{aligned} \tag{4.10}$$

in the operator-norm topology.

LEMMA 4.2. *Let $m > 0$. Then, for all sufficiently large κ, the operators*

$$L_z^{\pm}(\kappa) := 1 + \frac{1}{2\kappa^2} V(\kappa A \mp z) |B|^{-1} K_z^{\pm}(\kappa; V)^{-1}. \tag{4.11}$$

are bijective and expanded as

$$L_z^{\pm}(\kappa)^{-1} = \sum_{n=0}^{\infty} \frac{T_n^{\pm}(z)}{\kappa^n} \tag{4.12}$$

in the operator-norm topology with $T_n^{\pm}(z)$ being bounded operators on $\mathcal{H}$.

Proof. We prove the assertion only for $L_z^+(\kappa)$. The proof of the other case is similar. Let $\kappa > \kappa_1$. By condition (A.1) and (4.3), the domain of $L_z^+(\kappa)$ is equal to $\mathcal{H}$ and, for all $f \in \mathcal{H}$, we have

$$\begin{aligned} &||V(\kappa A - z)|B|^{-1} K_z^+(\kappa; V)^{-1} f|| \\ &\le 2a\kappa ||K K_z^+(\kappa; V)^{-1} f|| + (a|z| + b\kappa) ||A|B|^{-1} K_z^+(\kappa; V)^{-1} f|| \\ &\quad + \frac{b|z|}{m} ||K_z^+(\kappa; V)^{-1} f||. \end{aligned}$$

For all $\varepsilon > 0$ and $g \in D(A^2)$, we have

$$\begin{aligned} ||Ag|| &\le ||g||^{1/2} ||A^2 g||^{1/2} \\ &\le \varepsilon ||A^2 g|| + \frac{1}{4\varepsilon} ||g||. \end{aligned}$$

Hence

$$||A|B|^{-1}K_z^+(\kappa;V)^{-1}f|| \leq 2\varepsilon||KK_z^+(\kappa;V)^{-1}f|| + \frac{1}{4m\varepsilon}||K_z^+(\kappa;V)^{-1}f||.$$

Thus we obtain

$$\begin{aligned}&||V(\kappa A - z)|B|^{-1}K_z^+(\kappa;V)^{-1}f||\\ &\leq c_1(\kappa)||KK_z^+(\kappa;V)^{-1}f|| + c_2(\kappa)||K_z^+(\kappa;V)^{-1}f||,\end{aligned}$$

where

$$\begin{aligned}c_1(\kappa) &= 2(a+b\varepsilon)\kappa + 2a\varepsilon|z|,\\ c_2(\kappa) &= \frac{b}{4m\varepsilon}\kappa + \frac{|z|}{m}\left(b + \frac{a}{4\varepsilon}\right).\end{aligned}$$

By the closedness of $K_+(V)$ on the domain $D(K)$, there exists a constant $d > 0$ such that

$$||Kg|| \leq d(||K_+(V)g|| + ||g||), \quad g \in D(K).$$

Hence

$$||KK_z^+(\kappa;V)^{-1}f|| \leq d||K_+(V)K_z^+(\kappa;V)^{-1}f|| + d||K_z^+(\kappa;V)^{-1}f||.$$

Using (4.10), we have

$$\begin{aligned}||K_+(V)K_z^+(\kappa;V)^{-1}|| &\leq ||K_+(V)(K_+(V)-z)^{-1}|| \left(1 - \frac{|z|^2}{2\kappa_1^2}u_z\right)^{-1},\\ ||K_z^+(\kappa;V)^{-1}|| &\leq ||(K_+(V)-z)^{-1}|| \left(1 - \frac{|z|^2}{2\kappa_1^2}u_z\right)^{-1},\end{aligned}$$

where

$$u_z = ||\,|B|^{-1}(K_+(V)-z)^{-1}|| \leq \frac{1}{m|\mathrm{Im}\, z|}.$$

Thus we obtain

$$\frac{1}{2\kappa^2}||V(\kappa A - z)|B|^{-1}K_z^+(\kappa;V)^{-1}|| \leq \frac{c_0(c_1(\kappa)+c_2(\kappa))}{\kappa^2}, \tag{4.13}$$

where c_0 is a constant independent of $\kappa > \kappa_1$. The RHS of (4.13) is less than one if κ is sufficiently large. Hence the first half of the lemma follows. The expansion (4.12) for $L_z^+(\kappa)^{-1}$ follows from the geometric expansion

$$L_z^{\pm}(\kappa)^{-1} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2\kappa^2)^n}\left\{V(\kappa A \mp z)|B|^{-1}K_z^{\pm}(\kappa; V)^{-1}\right\}^n$$

and (4.10). ∎

THEOREM 4.3. *Let $m > 0$. Then, for all sufficiently large κ,*

$$(H_{\pm}^{\mathrm{ren}}(V) \pm z)^{-1} = \left(\frac{1}{2\kappa^2}(\kappa A \mp z)|B|^{-1} \mp P_{\mp}\right) K_z^{\mp}(\kappa; V)^{-1} \times \left(1 + \frac{1}{2\kappa^2} V(\kappa A \mp z)|B|^{-1}K_z^{\mp}(\kappa; V)^{-1}\right)^{-1}. \tag{4.14}$$

Proof. Since Lemma 4.2 is established, Theorem 4.2 can be proven in the same way as in the proof of Theorem 6.4 in [12, p.180]. Hence we omit the details. ∎

Proof of Theorem 4.1

Using Theorem 4.3, we can prove Theorem 4.1 in the same way as in the proof of Theorem 3.2. ∎

As in the previous case $V = 0$, we have the following analyticity results on the resolvents of $H_{\pm}^{\mathrm{ren}}(\kappa)$.

COROLLARY 4.4. *Let $m > 0$. Then, for all sufficiently large κ,*

$$(H_{\pm}^{\mathrm{ren}}(V) \pm z)^{-1} = \sum_{n=0}^{\infty} \frac{S_n^{\pm}(z)}{\kappa^n}$$

in the operator-norm topology, where $S_n^{\pm}(z)$ are bounded linear operators on $\mathcal{H}$.

Remark. The coefficients $S_n^{\pm}(z)$ can be explicitly represented in terms of $A, B, V, P_{\pm}$, but, here, we omit the details.

4.2 *The case* $m = 0$

In this case the method of the last subsection does not work. We employ the abstract Foldy-Wouthuysen transformation introduced in [1]. Let $V(\kappa)$ be a symmetric operator in $\mathcal{H}$ of the form

$$V(\kappa) = \begin{pmatrix} V_+(\kappa) & 0 \\ 0 & V_-(\kappa) \end{pmatrix} \tag{4.15}$$

with symmetric operators $V_+(\kappa)$ and $V_-(\kappa)$ acting in $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively. We assume the following:

ASSUMPTION B. Let $\kappa_0 > 0$ be a constant.

(B.1) For all $\kappa \geq \kappa_0$, $D(K) \subset D(V(\kappa))$ and there exist constants $a, b \geq 0$ independent of $\kappa \geq \kappa_0$ such that for all $\kappa \geq \kappa_0$

$$||V(\kappa)f|| \leq a||Kf|| + b||f||, \quad f \in D(K).$$

(B.2) There exist symmetric operators $V_\pm$ in $\mathcal{H}_\pm$ such that $D(K_\pm) \subset D(V_\pm)$ and, for all $f_\pm \in D(K_\pm)$,

$$\lim_{\kappa \to \infty} V_\pm(\kappa) f_\pm = V_\pm f_\pm.$$

(B.3) For all $\kappa \geq \kappa_0$, the operators

$$H_\pm^{\mathrm{ren}}(\kappa; V(\kappa)) := H_\pm^{\mathrm{ren}}(\kappa) + V(\kappa)$$

are essentially self-adjoint on $D(A) \cap D(B) \cap D(K)$ (we denote their closures by the same symbols, respectively).

(B.4) The operators $K_\pm(V)$ given by (4.2) are essentially self-adjoint on $D(K)$ (we denote their closures by the same symbols, respectively).

We define V by (4.1) with $V_\pm$ given in (B.2). We shall prove the following theorem.

THEOREM 4.5. *Let A and B be as in Theorem 3.5. Then, for all $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\mathrm{s} - \lim_{\kappa \to \infty} (H_\pm^{\mathrm{ren}}(\kappa; V(\kappa)) \pm z)^{-1} = \mp P_\mp (K_\mp(V) - z)^{-1}. \tag{4.16}$$

We prove Theorem 4.1 by a series of lemmas. Let A and B be A.C.S.O. in $\mathcal{H}$ and B be injective. Using the spectral measure E defined by (3.8), we introduce

$$\Lambda(A,B) := \arctan(|A||B|^{-1}) := \int \arctan\left(\frac{\lambda}{\mu}\right) dE(\lambda,\mu), \tag{4.17}$$

which is bounded and self-adjoint. It follows from Lemma 2.2(iv) that the operator

$$X(A,B) := i\frac{U_A U_B}{2}$$

is bounded, self-adjoint. Moreover, Lemma 2.2 (ii) implies that $X(A,B)$ commutes with $\Lambda(A,B)$. Hence, putting

$$Y(A,B) = X(A,B)\Lambda(A,B) = \Lambda(A,B)X(A,B),$$

the operator

$$U(A,B) := e^{iY(A,B)}$$

is unitary. In [1, Theorem 4.4], we have proven the following fact, which gives a "diagonalization" of $A+B$.

LEMMA 4.6. *The following operator equality holds:*

$$U(A,B)(A+B)U(A,B)^{-1} = U_B(A^2+B^2)^{1/2}.$$

Remark. The unitary operator $U(A,B)$ is an abstrat version of the usual Foldy-Wouthuysen transformation for Dirac operators (cf. [12, §5.6], [11]). Lemma 4.6 is a more general result than the corresponding results in [11], because we do not impose any condition on the "size" of B.

LEMMA 4.7. *Let*

$$U(\kappa) = U(\kappa A, \kappa^2 B)$$

Then

$$U(\kappa)H(\kappa)U(\kappa)^{-1} = U_B(\kappa^2 A^2 + \kappa^4 B^2)^{1/2},$$

where $H(\kappa)$ *is defined by (1.1).*

Proof. This follows from Lemma 4.6 and the fact that $U_{\lambda B} = U_B$ for all $\lambda > 0$. ∎

LEMMA 4.8. *The operator* $U(\kappa)$ *strongly commutes with* $|B|$*. In particular, we have*

$$U(\kappa)|B|U(\kappa)^{-1} = |B|$$

as an operator equality.

Proof. It follows from the strong commutativity of $|A|$ and $|B|$ (Lemma 2.2(iii)) that $|B|$ strongly commutes with

$$\Lambda(\kappa) := \Lambda(\kappa A, \kappa^2 B) = \arctan\left(\kappa^{-1}|A||B|^{-1}\right).$$

By Lemma 2.2(ii), $|B|$ strongly commutes with

$$X(\kappa A, \kappa^2 B) = X(A, B).$$

Hence $|B|$ strongly commutes with $Y(\kappa A, \kappa^2 B)$, which implies the strong commutativity of $|B|$ and $U(\kappa)$. ∎

Let

$$L_\pm(\kappa) = (\kappa^2|A|_+^2 + \kappa^4|B|_+^2)^{1/2} \pm \kappa^2|B|_+, \tag{4.18}$$

$$M_\pm(\kappa) = -(\kappa^2|A|_-^2 + \kappa^4|B|_-^2)^{1/2} \pm \kappa^2|B|_-. \tag{4.19}$$

LEMMA 4.9. *The operators* $L_\pm(\kappa)$ *and* $M_\pm(\kappa)$ *are essentially self-adjoint.*

Proof. This lemma follows from the strong commutativity of $|A|_\pm$ and $|B|_\pm$. ∎

We denote the closures of $L_\pm(\kappa)$ and $M_\pm(\kappa)$ by the same symbols, respectively.

LEMMA 4.10. *The following operator equalities hold:*

$$U(\kappa)H_{\pm}^{\mathrm{ren}}(\kappa)U(\kappa)^{-1} = \begin{pmatrix} L_{\pm}(\kappa) & 0 \\ 0 & M_{\pm}(\kappa) \end{pmatrix}. \tag{4.20}$$

Proof. By (2.3), (2.4) and (2.5), we have

$$U_B(\kappa^2 A^2 + \kappa^4 B^2)^{1/2}$$
$$= \begin{pmatrix} (\kappa^2|A|_+^2 + \kappa^4|B|_+^2)^{1/2} & 0 \\ 0 & -(\kappa^2|A|_-^2 + \kappa^4|B|_-^2)^{1/2} \end{pmatrix}.$$

Hence, by Lemmas 4.7 and 4.8, (4.20) holds on the domain

$$\mathcal{D} := [D(|A|_+) \cap D(|B|_+)] \oplus [D(|A|_-) \cap D(|B|_-)].$$

By Lemma 4.9, the operators on the RHS of (4.20) are essentially self-adjoint on $\mathcal{D}$. Thus (4.20) extends as operator equalities. ∎

LEMMA 4.11.

$$\mathrm{s}-\lim_{\kappa\to\infty} U(\kappa) = \mathrm{s}-\lim_{\kappa\to\infty} U(\kappa)^{-1} = I. \tag{4.21}$$

Proof. By the functional calculus, we have for all $f \in \mathcal{H}$

$$||\Lambda(\kappa)f||^2 = \int_{\mathbb{R}_+\times\mathbb{R}_+} \phi_\kappa(\lambda,\mu)d||E(\lambda,\mu)f||^2$$

with

$$\phi_\kappa(\lambda,\mu) = \left|\arctan\left(\frac{\lambda}{\kappa\mu}\right)\right|^2.$$

It is obvious that

$$|\phi_\kappa(\lambda,\mu)| \leq \frac{\pi^2}{4},$$
$$\lim_{\kappa\to\infty}\phi_\kappa(\lambda,\mu) = 0, \quad (\lambda,\mu) \in \mathbb{R}_+ \times (0,\infty).$$

Hence, by (3.23), we can apply the Lebesgue dominated convergence theorem to obtain

$$||\Lambda(\kappa)f|| \to 0 \quad (\kappa \to \infty),$$

which, implies that

$$\mathrm{s}-\lim_{\kappa\to\infty} Y(\kappa A, \kappa^2 B) = 0.$$

Thus, by general convergence theorems [8,Theorem VIII.25(a), Theorem VIII.21], we obtain (4.21). ∎

LEMMA 4.12. *The operator $Y(A,B)$ anticommutes with A and B.*

Proof. Let $f \in D(A)$. Then, by the functional calculus, we have $U_A\Lambda(A,B)f \in D(A)$. Hence, by Lemma 2.2(i),

$$U_B A U_A \Lambda(A,B)f = -AU_B U_A \Lambda(A,B)f.$$

Since the left hand side is equal to $U_B U_A \Lambda(A,B)Af$, it follows that $Y(A,B)$ leaves $D(A)$ invariant and

$$Y(A,B)Af = -AY(A,B)f.$$

Noting that $Y(A,B)$ is bounded, we can apply Corollary 1.7 in [13] to conclude that A and $Y(A,B)$ anticommute. Similarly we can show that B and $Y(A,B)$ anticommute. ∎

Since U_B and $\Lambda(A,B)$ commutes, we have

$$\Lambda(A,B) = \begin{pmatrix} \Lambda_+(A,B) & 0 \\ 0 & \Lambda_-(A,B) \end{pmatrix}$$

with

$$\Lambda_+(A,B) = \arctan(|A|_+|B|_+^{-1}), \quad \Lambda_-(A,B) = \arctan(|A|_-|B|_-^{-1}).$$

LEMMA 4.13. *The operator $U(A,B)$ has the following matrix representation:*

$$U(A,B) = \begin{pmatrix} \cos\frac{\Lambda_+(A,B)}{2} & W_A^*\sin\frac{\Lambda_-(A,B)}{2} \\ -W_A\sin\frac{\Lambda_+(A,B)}{2} & \cos\frac{\Lambda_-(A,B)}{2} \end{pmatrix}.$$

Proof. By (2.3) and (2.6), we have

$$Y(A,B) = \frac{i}{2}\begin{pmatrix} 0 & -W_A^*\Lambda_-(A,B) \\ W_A\Lambda_+(A,B) & 0 \end{pmatrix}.$$

Since $Y(A,B)$ is bounded, we can expand $U(A,B)$ as

$$U(A,B) = \sum_{n=0}^{\infty} \frac{i^n Y(A,B)^n}{n!}$$

in the operator-norm topology. It follows from the commutativity of U_A and $\Lambda(A,B)$ that

$$W_A\Lambda_+(A,B) = \Lambda_-(A,B)W_A.$$

Using this fact, we obtain for $n \geq 0$

$$Y(A,B)^{2n} = \frac{1}{2^{2n}}\begin{pmatrix} \Lambda_+(A,B)^{2n} & 0 \\ 0 & \Lambda_-(A,B)^{2n} \end{pmatrix},$$

$$Y(A,B)^{2n+1} = i\frac{1}{2^{2n+1}}\begin{pmatrix} 0 & -W_A^*\Lambda_-(A,B)^{2n+1} \\ W_A\Lambda_+(A,B)^{2n+1} & 0 \end{pmatrix}.$$

Thus the desired result follows. ∎

LEMMA 4.14.
(i) $D(K_+) \subset D(L_-(\kappa))$ and, for all $f \in D(K_+)$,

$$\lim_{\kappa\to\infty} L_-(\kappa)f = K_+f. \tag{4.22}$$

(ii) For all $z \in \mathbb{C}\setminus\mathbb{R}$,

$$\mathrm{s}-\lim_{\kappa\to\infty}(L_+(\kappa) - z)^{-1} = 0. \tag{4.23}$$

(iii) $D(K_-) \subset D(M_+(\kappa))$ and, for all $f \in D(K_-)$,

$$\lim_{\kappa\to\infty} M_+(\kappa)f = -K_-f. \tag{4.24}$$

(iv) For all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\mathrm{s}-\lim_{\kappa\to\infty} (M_-(\kappa) - z)^{-1} = 0. \tag{4.25}$$

Proof. Let $\kappa \geq \kappa_0$. We have

$$L_-(\kappa) \equiv \int_{\mathbb{R}_+\times\mathbb{R}_+} u_\kappa(\lambda,\mu) dE_+(\lambda,\mu)$$

with

$$u_\kappa(\lambda,\mu) = \sqrt{\kappa^2\lambda^2 + \kappa^4\mu^2} - \kappa^2\mu,$$

where E_+ is the spectral measure defined by (2.13). Let

$$u(\lambda,\mu) = \frac{\lambda^2}{2\mu}, \quad (\lambda,\mu) \in \mathbb{R}_+ \times (0,\infty).$$

Then we have

$$0 \leq u_\kappa(\lambda,\mu) \leq u(\lambda,\mu), \quad (\lambda,\mu) \in \mathbb{R}_+ \times (0,\infty), \tag{4.26}$$

which implies that $D(K_+) \subset D(L_-(\kappa))$ (see Lemma 2.5). Moreover, we have for all $f \in D(K_+)$

$$||L_-(\kappa)f - K_+f||^2 = \int |u_\kappa(\lambda,\mu) - u(\lambda,\mu)|^2 d||E_+(\lambda,\mu)f||^2.$$

It is easy to see that

$$\lim_{\kappa\to\infty} u_\kappa(\lambda,\mu) = u(\kappa,\mu), \quad (\lambda,\mu) \in \mathbb{R}_+ \times (0,\infty).$$

Hence, by (4.26) and the fact that $E_+(\mathbb{R}_+ \times \{0\})f = 0$, we can apply the Lebesgue dominated convergence theorem to obtain (4.22). Similarly we can prove (4.23)–(4.25). ∎

The following lemma gives a general convergence theorem on self-adjoint operators acting in the direct sum of two Hilbert spaces.

LEMMA 4.15. *Let $\mathcal{K}_\pm$ be Hilbert spaces and*

$$\mathcal{K} = \mathcal{K}_+ \bigoplus \mathcal{K}_- = \left\{ \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \middle| f_\pm \in \mathcal{K}_\pm \right\}.$$

Let Q and T_κ (resp. Q_κ, S_κ) be self-adjoint (resp. symmetric) operators in $\mathcal{K}_+$ and $\mathcal{K}_-$, respectively, and $R_\kappa^{(\pm)} : \mathcal{K}_\pm \to \mathcal{K}_\mp$ be densely defined linear operators with $\kappa > c > 0$ (c: a constant). Assume that there exist dense subspaces $\mathcal{D}_\pm \subset \mathcal{K}_\pm$ such that, for all $f_\pm \in \mathcal{D}_\pm$, $z \in \mathbb{C} \setminus \mathbb{R}$, and $\kappa > c$, we have $(Q-z)^{-1}f_+ \in D(Q_\kappa) \cap D(R_\kappa^{(+)}), (T_\kappa - z)^{-1}f_- \in D(S_\kappa) \cap D(R_\kappa^{(-)})$ and

$$\lim_{\kappa\to\infty} (Q_\kappa - Q)(Q-z)^{-1}f_+ = 0, \quad \lim_{\kappa\to\infty} R_\kappa^{(+)}(Q-z)^{-1}f_+ = 0,$$
$$\lim_{\kappa\to\infty} R_\kappa^{(-)}(T_\kappa - z)^{-1}f_- = 0, \quad \lim_{\kappa\to\infty} S_\kappa(T_\kappa - z)^{-1}f_- = 0,$$
$$\lim_{\kappa\to\infty} (T_\kappa - z)^{-1}f_- = 0.$$

Let

$$M_\kappa = \begin{pmatrix} Q_\kappa & R_\kappa^{(-)} \\ R_\kappa^{(+)} & S_\kappa \end{pmatrix}, \quad N_\kappa = \begin{pmatrix} 0 & 0 \\ 0 & T_\kappa \end{pmatrix}$$

and suppose that, for all $\kappa > c$, $M_\kappa + N_\kappa$ is essentially self-adjoint (its closure is denoted by the same symbol). Then, for all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\mathrm{s}-\lim_{\kappa\to\infty} (M_\kappa + N_\kappa - z)^{-1} = \begin{pmatrix} (Q-z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \qquad (4.27)$$

Proof. Let

$$M = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}$$

and $z \in \mathbb{C} \setminus \mathbb{R}$. Then

$$R_z := (M-z)^{-1} = \begin{pmatrix} (Q-z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

The operator $M + N_\kappa$ is self-adjoint with

$$(M + N_\kappa - z)^{-1} = \begin{pmatrix} (Q - z)^{-1} & 0 \\ 0 & (T_\kappa - z)^{-1} \end{pmatrix}. \tag{4.28}$$

We write

$$(M_\kappa + N_\kappa - z)^{-1} - R_z = I_\kappa + J_\kappa,$$

where

$$I_\kappa = (M_\kappa + N_\kappa - z)^{-1} - (M + N_\kappa - z)^{-1},$$
$$J_\kappa = (M + N_\kappa - z)^{-1} - R_z = \begin{pmatrix} 0 & 0 \\ 0 & (T_\kappa - z)^{-1} \end{pmatrix}.$$

Let $f_\pm \in \mathcal{D}_\pm$ and $f = (f_+, f_-)$. Then we can write

$$I_\kappa f = (M_\kappa + N_\kappa - z)^{-1}(M - M_\kappa)(M + N_\kappa - z)^{-1} f.$$

Hence we have

$$||I_\kappa f|| \leq \frac{1}{|\mathrm{Im} z|}||(M - M_\kappa)(M + N_\kappa - z)^{-1} f||.$$

By (4.28), we obtain

$$\begin{aligned} &||(M - M_\kappa)(M + N_\kappa - z)^{-1} f|| \\ &\leq ||(Q - Q_\kappa)(Q - z)^{-1} f_+|| + ||R_\kappa^{(-)}(T_\kappa - z)^{-1} f_-|| \\ &+ \quad ||R_\kappa^{(+)}(Q - z)^{-1} f_+|| + ||S_\kappa(T_\kappa - z)^{-1} f_-||. \end{aligned}$$

Hence $||I_\kappa f|| \to 0$ as $\kappa \to \infty$. It is obvious that $\mathrm{s} - \lim_{\kappa \to \infty} J_\kappa = 0$. Thus $(M_\kappa + N_\kappa - z)^{-1} f \to R_z f$ strongly as $\kappa \to \infty$. Since $\mathcal{D}_+ \oplus \mathcal{D}_-$ is dense in $\mathcal{K}$ and

$$||(M_\kappa + N_\kappa - z)^{-1}|| \leq |\mathrm{Im} z|^{-1},$$

we can use an $\varepsilon/3$-argument to obtain (4.27). ∎

Proof of Theorem 4.5: We prove only (4.16) with $(H_-^{\rm ren}(\kappa;V(\kappa))-z)^{-1}$. The other case can be proven similarly. By Lemma 4.13, we have

$$U(\kappa)=\begin{pmatrix} c_+(\kappa) & W_A^* s_-(\kappa) \\ -W_A s_+(\kappa) & c_-(\kappa)\end{pmatrix},$$

where

$$c_\pm(\kappa)=\cos\frac{\Lambda_\pm(\kappa A,\kappa^2 B)}{2},\quad s_\pm(\kappa)=\sin\frac{\Lambda_\pm(\kappa A,\kappa^2 B)}{2}.$$

Hence

$$U(\kappa)V(\kappa)U(\kappa)^{-1}=\begin{pmatrix}\alpha(\kappa) & \beta_-(\kappa)\\ \beta_+(\kappa) & \gamma(\kappa)\end{pmatrix},$$

where

$$\begin{aligned}\alpha(\kappa)&=c_+(\kappa)V_+(\kappa)c_+(\kappa)+W_A^*s_-(\kappa)V_-(\kappa)s_-(\kappa)W_A,\\ \beta_+(\kappa)&=-W_As_+(\kappa)V_+(\kappa)c_+(\kappa)+c_-(\kappa)V_-(\kappa)s_-(\kappa)W_A,\\ \beta_-(\kappa)&=-c_+(\kappa)V_+(\kappa)s_+(\kappa)W_A^*+W_A^*s_-(\kappa)V_-(\kappa)c_-(\kappa),\\ \gamma(\kappa)&=W_As_+(\kappa)V_+(\kappa)s_+(\kappa)W_A^*+c_-(\kappa)V_-(\kappa)c_-(\kappa).\end{aligned}$$

By Lemma 4.10, we obtain

$$H_-^{\rm ren}(\kappa;V(\kappa))=U(\kappa)^{-1}\{C(\kappa)+D(\kappa)\}U(\kappa)$$

on the subspace $F_0:=D(A)\cap D(B)\cap D(K)$, where

$$C(\kappa)=\begin{pmatrix}L_-(\kappa)+\alpha(\kappa) & \beta_-(\kappa)\\ \beta_+(\kappa) & \gamma(\kappa)\end{pmatrix},$$

$$D(\kappa)=\begin{pmatrix}0 & 0\\ 0 & M_-(\kappa)\end{pmatrix}.$$

By the functional calculus, one can easily see that K strongly commutes with U_A, which implies that

$$W_AK_+\subset K_-W_A,\quad W_A^*K_-\subset K_+W_A^*. \tag{4.29}$$

Similarly we have

$$c_\pm(\kappa)K_\pm \subset K_\pm c_\pm(\kappa), \tag{4.30}$$
$$s_\pm(\kappa)K_\pm \subset K_\pm s_\pm(\kappa). \tag{4.31}$$

It follows from these facts that

$$D(K_+) \subset D(L_-(\kappa)) \cap D(\alpha(\kappa)) \cap D(\beta_+(\kappa)), \tag{4.32}$$
$$D(K_-) \subset D(\beta_-(\kappa)) \cap D(\gamma(\kappa)). \tag{4.33}$$

Let $K_+(V_+)$ be the closure of $K_+ + V_+$ and $f_\pm \in D(K_\pm)$. Then, by (4.32) and (4.33), we have for all $z \in \mathbb{C} \setminus \mathbb{R}$

$$g_+ := (K_+(V_+) - z)^{-1} f_+ \in\subset D(L_-(\kappa)) \cap D(\alpha(\kappa)) \cap D(\beta_+(\kappa)),$$
$$g_-(\kappa) := (M_-(\kappa) + z)^{-1} f_- \in D(K_-) \subset D(\beta_-(\kappa)) \cap D(\gamma(\kappa)).$$

We have

$$(L_-(\kappa) + \alpha(\kappa) - K_+(V_+))g_+ = I_1(\kappa) + I_2(\kappa),$$

where

$$I_1(\kappa) = (L_-(\kappa) - K_+)g_+, \quad I_2(\kappa) = (\alpha(\kappa) - V_+)g_+.$$

By Lemma 4.14(i),

$$\lim_{\kappa \to \infty} I_1(\kappa) = 0.$$

We write

$$\begin{aligned} I_2(\kappa) =& c_+(\kappa)V_+(\kappa)(c_+(\kappa) - I)g_+ + c_+(\kappa)(V_+(\kappa) - V_+)g_+ \\ &+ (c_+(\kappa) - I)V_+ g_+ + W_A^* s_-(\kappa)V_-(\kappa)s_-(\kappa)W_A g_+. \end{aligned}$$

It is easy to see that

$$\mathrm{s}-\lim_{\kappa \to \infty} c_\pm(\kappa) = I, \tag{4.34}$$
$$\mathrm{s}-\lim_{\kappa \to \infty} s_\pm(\kappa) = 0. \tag{4.35}$$

Using (B.1), (B.2), (4.29)–(4.31), (4.34), (4.35) and the fact that

$$||c_\pm(\kappa)|| \leq 1, \quad ||s_\pm(\kappa)|| \leq 1,$$

we have the following estimates:

$$\begin{aligned}
||c_+(\kappa)V_+(\kappa)(c_+(\kappa) - I)g_+|| &\leq ||V_+(\kappa)(c_+(\kappa) - I)g_+|| \\
&\leq a||(c_+(\kappa) - I)K_+g_+|| \\
&\quad + b||(c_+(\kappa) - I)g_+|| \\
&\to 0 \quad (\kappa \to \infty), \\
||c_+(\kappa)(V_+(\kappa) - V_+)g_+|| &\leq ||(V_+(\kappa) - V_+)g_+|| \\
&\to 0 \quad (\kappa \to \infty), \\
||(c_+(\kappa) - I)V_+g_+|| &\to 0 \quad (\kappa \to \infty), \\
||W_A^* s_-(\kappa)V_-(\kappa)s_-(\kappa)W_A g_+|| &\leq ||V_-(\kappa)s_-(\kappa)W_A g_+|| \\
&\leq a||s_-(\kappa)K_-W_A g_+|| \\
&\quad + b||s_-(\kappa)W_A g_+|| \\
&\to 0 \quad (\kappa \to \infty).
\end{aligned}$$

Hence $\lim_{\kappa\to\infty} I_2(\kappa) = 0$. Thus we obtain

$$\lim_{\kappa\to\infty} (L_-(\kappa) + \alpha(\kappa) - K_+(V_+))g_+ = 0.$$

Similaly we can show that

$$\lim_{\kappa\to\infty} \beta_+(\kappa)g_+ = 0.$$

Let $h \in D(K_-)$. Then, using (B.1) and (4.29), we have

$$\begin{aligned}
||\beta_-(\kappa)h|| &\leq ||V_+(\kappa)s_+(\kappa)W_A^* h|| + ||V_-(\kappa)c_-(\kappa)h|| \\
&\leq 2a||K_-h|| + 2b||h||.
\end{aligned}$$

Similarly we can show that

$$||\gamma(\kappa)h|| \leq 2a||K_-h|| + 2b||h||.$$

On the other hand, we have for all $f_- \in D(K_-)$

$$\begin{aligned}||K_-(M_-(\kappa) - z)^{-1}f_-|| &= ||(M_-(\kappa) - z)^{-1}K_-f_-|| \\ &\to 0 \quad (\kappa \to \infty).\end{aligned}$$

Hence we obtain

$$\begin{aligned}&||\beta_-(\kappa)(M_-(\kappa) - z)^{-1}f_-|| \\ &\leq 2a||K_-(M_-(\kappa) - z)^{-1}f_-|| + 2b||(M_-(\kappa) - z)^{-1}f_-|| \\ &\to 0 \quad (\kappa \to \infty).\end{aligned}$$

Similarly we have

$$||\gamma(\kappa)(M_-(\kappa) - z)^{-1}f_-|| \to 0 \quad (\kappa \to \infty).$$

It is easy to see that $U(\kappa)$ maps F_0 onto F_0 (note that $D(A) = D(|A|)$, $D(B) = D(|B|)$). Hence, by (B.3), $C(\kappa) + D(\kappa)$ is essentially self-adjoint on F_0. Thus we can apply Lemma 4.15 with $M_\kappa = C(\kappa)$ and $N_\kappa = D(\kappa)$ to obtain

$$\text{s} - \lim_{\kappa \to \infty} (C(\kappa) + D(\kappa) - z)^{-1} = \begin{pmatrix} (K_+(V_+) - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

which, together with Lemma 4.11, gives the desired result. ∎

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