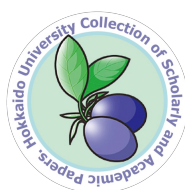




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Title	Existence of periodic solutions for equations of evolving curves
Author(s)	Giga, Y.; Mizoguchi, N.
Citation	Hokkaido University Preprint Series in Mathematics, 216, 1-45
Issue Date	1993-10
DOI	https://doi.org/10.14943/83360
Doc URL	https://hdl.handle.net/2115/68962
Type	departmental bulletin paper
File Information	re216.pdf



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Series #216. October 1993

HOKKAIDO UNIVERSITY
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Existence of periodic solutions for equations of evolving curves

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In the memory of Professor Peter Hess

Abstract

Evolution equations of curvatures of convex curves are considered by the Gauss map parametrization. A time periodic unstable solution is constructed for a reasonable class of time periodic data. Our solution is arranged to satisfy a constraint so that it yields *closed*, embedded, convex curves moving periodically in time (up to translation) whose normal speed equals the curvature minus a given time periodic function depending on curves only through its normals. For curvatures of periodically evolving curves *a priori* lower and upper bounds depending only on periodic data are obtained. A new penalty method is introduced so that our solution satisfies the constraint. Solutions of penalized equations are constructed by adapting the degree theory.

1990 Mathematics Subject Classification:

*Partly supported by the Inamori foundation

1 Introduction.

We are concerned with positive, periodic solutions of a quasilinear parabolic equation for curvatures u of evolving curves:

$$(1.1) \quad u_t = u^2(u_{xx} + u - f) \quad \text{in } K.$$

with $K = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/T\mathbb{Z})$ for given $T > 0$. Here $f = f(x, t)$ is a given continuous function on K , i.e., $f \in C(K)$. In particular, f is 2π -periodic in space, i.e., $f(x + 2\pi, t) = f(x, t)$, and T -periodic in time i.e., $f(x, t + T) = f(x, t)$ for all $(x, t) \in \mathbb{R}^2$. Our goal is to construct a positive solution of (1.1) satisfying a constraint

$$(1.2) \quad \int_0^{2\pi} \frac{e^{ix}}{u(x, t)} dx = 0 \quad \text{for all } t \in \mathbb{R} \quad (i = \sqrt{-1}).$$

1.1. Main Existence Theorem. *Suppose that f is in $C(K)$ together with its time derivative f_t . Then there is a positive function u in $\bigcap_{p>1} W_p^{2,1}(K) \subset C(K)$ (which implies $u_x \in C(K)$) such that u solves (1.1) with the constraint (1.2) if f is positive on K and satisfies*

$$(1.3) \quad \int_0^{2\pi} f(x, t) e^{ix} dx = 0 \quad \text{for all } t \in \mathbb{R}.$$

If f is smooth, so is u .

Here $W_p^{2,1}(K)$ denotes the space of all functions g in $L^p(K)$ together with g_x , g_{xx} and g_t and $L^p(K)$ denotes the space of locally L^p functions in $\mathbb{R}^2$ which is 2π -periodic in x and T -periodic in t . The Main Existence Theorem asserts the existence of a positive solution u , 2π -periodic in x and T -periodic in t .

The restriction (1.3) is necessary if u satisfies (1.2). Indeed, dividing both sides of (1.1) by u^2 yields

$$(1.4) \quad (-u^{-1})_t = u_{xx} + u - f \quad \text{in } K.$$

Multiplying e^{ix} with (1.4) and integrating by parts over $(0, 2\pi)$ yields

$$(1.5) \quad -\frac{d}{dt} \int_0^{2\pi} \frac{e^{ix}}{u} dx = - \int_0^{2\pi} f e^{ix} dx$$

since $(e^{ix})_{xx} + e^{ix} = 0$. If u satisfies the constraint (1.2), f must satisfy (1.3).

The positivity assumption on f cannot be dropped completely even if f is smooth so

that u is smooth. For example let $f \leq 0$ on K , then at the minimizer $(x_0, t_0) \in K$ of u the equation (1.1) yields

$$0 = u_t(x_0, t_0) = u^2(x_0, t_0)(u_{xx}(x_0, t_0) + u(x_0, t_0) - f(x_0, t_0)) \geq u^3(x_0, t_0) > 0,$$

so there are no positive solutions of (1.1) for nonpositive (smooth) f .

Our main result yields the existence of a periodic-in-time solution (up to translation) for an evolution equation of curves whose normal speed equals the curvature minus a given time periodic function depending on curves through its normals. Let $\{\Gamma_t\}$ be a smooth one parameter family of closed, embedded curves bounding a domain in the plane. Let $\mathbf{n}$ denote the inward unit normal vector field on Γ_t . Let V denote the normal velocity of Γ_t in the direction of $\mathbf{n}$. We consider an equation for Γ_t of the form

$$(1.6) \quad V = k - q(\mathbf{n}, t),$$

where k is the inward curvature and q is a given function. The equation (1.6) is an example of curvature flow equation with anisotropy [Gu]. If Γ_t is convex, one can parametrize Γ_t by a Gauss map by introducing θ , $0 \leq \theta \leq 2\pi$ such that $\mathbf{n} = (\cos \theta, \sin \theta)$. The evolution of curvature k is expressed as

$$k_t = k^2(V_{\theta\theta} + V)$$

if we use θ -coordinate [Gu]. Applying this identity to (1.6) yields an evolution equation of curvature

$$(1.7) \quad k_t = k^2(k_{\theta\theta} + k - (Q_{\theta\theta} + Q)) \quad \text{with} \quad Q(\theta, t) = q(\cos \theta, \sin \theta, t),$$

where k and Q are 2π -periodic in θ . We next recover (1.6) from (1.7). For k a curve parametrized by the Gauss map is given by

$$Z(\theta, t) = \left(\int_0^\theta \frac{\sin \sigma}{k(\sigma, t)} d\sigma, - \int_0^\theta \frac{\cos \sigma}{k(\sigma, t)} d\sigma \right).$$

If k solves (1.7), then integrating by parts yields

$$\frac{\partial Z}{\partial t} = ((k-Q) \cos \theta - (k_\theta - Q_\theta) \sin \theta - (k-Q)|_{\theta=0}, (k-Q) \sin \theta + (k_\theta - Q_\theta) \cos \theta - (k_\theta - Q_\theta)|_{\theta=0}).$$

Translate Z by

$$X_0(t) = \left(\int_0^t (k-Q)(0, \tau) d\tau, \int_0^t (k_\theta - Q_\theta)(0, \tau) d\tau \right),$$

so that new curve $X(\theta, t) = Z(\theta, t) + X_0(t)$ fulfills

$$V = n \cdot \frac{\partial X}{\partial t} = (\cos \theta, \sin \theta) \cdot \frac{\partial X}{\partial t} = k - q.$$

We thus obtained the curve

$$\Gamma_t = \{X(\theta, t); 0 \leq \theta \leq 2\pi\}$$

satisfying (1.6). The equation (1.6) and (1.7) are equivalent through X . However to be Γ_t is closed we need $X(0, t) = X(2\pi, t)$ which is equivalent to the constraint

$$\int_0^{2\pi} \frac{e^{i\theta}}{k(\theta, t)} d\theta = 0.$$

If we set $u = k$, $x = \theta$, this is nothing but the constraint (1.2). Since the condition (1.3) is automatically satisfied for $f = Q_{\theta\theta} + Q$, the Main Existence Theorem yields a periodic-in-time solution Γ_t (up to translation in space) of (1.6).

1.2. Existence of periodically evolving curves. *Suppose that Q is in $C(K)$ together with $Q_{\theta\theta}$, Q_t , $Q_{\theta\theta t}$ and that*

$$(1.8) \quad Q_{\theta\theta} + Q > 0 \quad \text{on } K.$$

Then there are a constant vector $c \in \mathbb{R}^2$ and a closed curve evolution Γ_t solving

$$(1.9) \quad V = k - q(n, t)$$

$$(1.10) \quad \Gamma_{t+T} = \Gamma_t + c$$

for all $t \in \mathbb{R}$. The curvature of Γ_t is always positive and the quantities in (1.9) are continuous. If Q is smooth, so is Γ_t .

In general translation by c is necessary. For example, take

$$Q(\theta, t) = \cos \theta + 1$$

so that $Q_{\theta\theta} + Q = 1$. Suppose that Γ_t solves (1.9), (1.10) for some c . Then it is easy to see that $\hat{\Gamma}_t = \Gamma_t + e_1 t$ with $e_1 = (1, 0)$ solves $V = k - 1$ with T -periodic k . If $f = 1$ in (1.1), by a uniqueness result summarized in Remark 1.3 the unique solution of (1.1) with (1.3) is $u \equiv 1$. Thus $\hat{\Gamma}_t$ is a circle C of radius 1 and $V = k - 1 = 0$. We now observe that

under the relation $Q(\theta, t) = \cos \theta + 1$ if Γ_t solves (1.9), (1.10) (with positive curvature) then $\Gamma_t = C - e_1 t$ up to a constant vector (independent of time.) Thus $\Gamma_{t+T} = \Gamma_t - e_1 T$ does not agree with Γ_t so translation is necessary in (1.10).

1.3. Remark on time independent solutions. *Suppose that f in (1.1) is time independent. Then so is a positive solution u . If f is time independent there is at most one positive solution u of (1.1) satisfying (1.2).*

We shall prove this uniqueness result in the Appendix. The condition (1.8) on Q is equivalent to say that the curvature of Frank diagram

$$\mathcal{F} = \{(p_1, p_2) \in \mathbb{R}^2; r = 1/Q(\theta), (p_1, p_2) = r(\cos \theta, \sin \theta)\}$$

is positive. This is also discussed in the Appendix.

There seems to be no literature on the existence of nontrivial periodic solutions of quasilinear equations, in particular of equations of evolving curves like (1.6), although there are several results for semilinear equations. Besides, it turns out that the periodic solution we seek is (Lyapunov) unstable. In fact all periodic, convex solutions for a general parabolic surface evolution equation are unstable unless the equation depends on location of surface explicitly. This was recently proved by the first author and Yama-uchi [GY]. Even for semilinear equations several methods for constructing periodic solutions only work for stable ones. For example in [AHM] stable periodic solutions were constructed by using sub- and supersolutions. For a class of semilinear equations a possibly unstable periodic solution was first constructed in [Am] as the “third” solution. Existence of unstable periodic solutions are recently proved by Hirano and the second author [HM1, HM2] for a class of semilinear parabolic equations. Their method is based on the Leray-Schauder degree theory. Since our equation is quasilinear and not semilinear, their theory does not apply directly to our setting. However we adapt the idea to construct solutions for approximate equations.

Since the equation (1.1) is degenerate at $u = 0$, we wonder whether or not positive solutions are bounded away from zero. Fortunately, under the constraint (1.2) we have *a priori* positive lower bounds for u (or curvature.)

1.4. Theorem on bounds for curvature. *There are positive constants δ and M depending only on T , $\|f\|_\infty$, $\|f_t\|_\infty$ and $\min_K f > 0$ such that if $u \in C^\infty(K)$ satisfies (1.1),*

(1.2) with $u > 0$ and $f, f_t \in C(K)$ with $f > 0$ on K , then $\delta \leq u \leq M$ on K . The constant M is independent of $\min_K f$. Here $\|f\|_\infty$ denotes the maximum norm of f in $C(K)$.

To obtain *a priori* bounds we derive Harnack type inequalities in our setting. It roughly asserts that

(i) $u(x, t)$ and $u(x, s)$ are comparable for all $t, s \in \mathbb{R}$ provided that u has an upper *a priori* bound.

(ii) $u(x, t_0)$ is not small compared with the maximum $u(x_0, t_0)$ of u over K .

Such estimates are derived by a differential identity as in [Ga2]. The derivation of the Harnack inequality from differential identities stems from [LY]. For further development of the Harnack inequality and its applications especially for geometric evolution equations the reader is referred to [Ha], a recent article of Andrews [And] and references therein. We note that a higher dimensional version of (1.1) with $f = 0$ is found in [And] as the harmonic mean curvature flow equation.

The equation (1.1) admits a couple of integral bounds of solutions. If maximum of u is large, the Harnack inequality (ii) says its integral must be also large. Thus integral bounds yield an upper bound for solutions. The constraint is not invoked to get an upper bound but without (1.2) we do not get a lower bound. Indeed, suppose that $f = 1$. Then by Remark 1.3 every positive solution of (1.1) is independent of t and it solves

$$u_{xx} + u = 1 \quad \text{in } \mathbb{R}/2\pi\mathbb{Z}.$$

Of course, $u_\sigma(x) = (1 - \sigma)\cos x + 1$ ($2 > \sigma > 0$) is a positive solution but $\{u_\sigma\}$ is not bounded away from zero.

To get a lower bound we integrate (1.4) over $(0, T)$ and obtain

$$U_{xx} + U > 0 \quad \text{in } \mathbb{R}/2\pi\mathbb{Z} \quad \text{with} \quad U(x) = \int_0^T u(x, t) dt.$$

The constraint (1.2) implies

$$\int_0^{2\pi} \sin(x - \zeta) u^{-1} dx = 0 \quad \text{for all } \zeta \in \mathbb{R}.$$

The key observation is that the integrals where the integrand is negative and positive are balanced or both integrals have the same growth order as u tends to zero. By the Harnack inequality (i) such a growth balance is true for U . We shall prove that if U satisfies

$U_{xx} + U > 0$ together with integral balance, then U must have a lower bound away from zero. This in turn gives a lower bound for u by (i). A rigorous proof is given in §4. Although bounds obtained here do not directly apply to construct solutions, the idea of proof is fundamental to find a uniform bounds for solutions of approximate equations.

1.5. Penalty method and approximate equations. We seek a solution of (1.1) satisfying the nonlinear constraint (1.2). Since not all solution satisfies (1.2), we shall select desired one by introducing a kind of penalty method. It is heuristically explained as follows. For small $\varepsilon > 0$ we consider a penalized equation:

$$(1.11) \quad u_t = u^2(u_{xx} + u + \frac{\varepsilon}{u} - f) \quad \text{in } K$$

For a positive solution u^ε of this equation we observe that (1.3) implies

$$-\frac{d}{dt} \int_0^{2\pi} \frac{e^{ix}}{u^\varepsilon} dx = \varepsilon \int_0^{2\pi} \frac{e^{ix}}{u^\varepsilon} dx$$

in the same way to derive (1.5). Since u^ε is periodic in time, this implies that our approximate solution $u^\varepsilon > 0$ satisfies the constraint (1.2). It is possible to construct a positive solution of (1.11) mainly because $1/u$ has a strong stabilizing effect near $u = 0$; the derivative of $1/u$ tends to minus infinity as $u \rightarrow 0$. However, there is a serious drawback for (1.11). Since we do not know that $f - \varepsilon/u$ is positive, it seems to be impossible to derive a uniform bound from below for u^ε by the method sketched in the previous section. To overcome this difficulty a naive idea may be to replace (1.11) by

$$(1.12) \quad u_t = u^2(u_{xx} + u + \frac{\varepsilon}{u \vee m\varepsilon} - f), \quad 0 < \frac{1}{m} < \min f$$

so that $f - \varepsilon(u \vee m\varepsilon)^{-1}$ is positive, where $a \vee b = \max(a, b)$. The solution u^ε of this equation has *a priori* bounds and it is not difficult to prove that $\{u^\varepsilon\}$ has a convergent subsequence and the limit satisfies (1.1) and (1.2). However, it seems to be difficult to construct a positive solution of (1.12) because new term $\varepsilon(u \vee m\varepsilon)^{-1}$ is constant near $u = 0$ and has no stabilizing effect for small u .

We adapt the method of finding unstable solutions for semilinear equations developed by Esteban [E1], [E2] Hirano and the second author [HM1], [HM2]. Unfortunately, their method does not apply to solve (1.12). We explain the reason by sketching our method. For fixed $b > 0$ we consider a uniformly parabolic equation

$$(1.13) \quad u_t = a(u)(u_{xx} - bu + h) \quad \text{in } K$$

where u has now no sign conditions and h is for the moment a given data. If a is continuous, one can prove that the solution operator $S : h \mapsto u$ is well-defined in $C(K)$. If $h = H(u)$ depend on u , a solution of (1.13) with $h = H(u)$ is interpreted as a fixed point of $S \circ H$. If we prove that

$$\deg(I - S \circ H, 0, B_R(0)) = 0$$

$$\deg(I - S \circ H, 0, B_r(0)) = 1$$

for large ball $B_R(0)$ and small ball $B_r(0)$ centered at zero in $C(K)$, then there is a solution of (1.13) with $h = H(u)$ in $B_R(0) \setminus B_r(0)$. This method is used by Esteban [E1], [E2] for semilinear parabolic equations with superlinear nonlinearity. To implement this programme for our equation we should at least modify our equation (1.12) so that it is uniformly parabolic for all $u \in C(K)$. However, not all modification of u^2 term is good to derive Harnack type inequalities. Also we should be careful to choose approximate equations so that degree of $I - S \circ H$ has desired properties. We should construct a homotopy of equations in a clever way. For example to prove the degree in a large ball, equations appeared in the homotopy should be arranged so that solutions have an upper bound uniformly in the homotopy parameter. Our choice of approximate equation is

(1.14)

$$u_t = (u + \varepsilon^2)^2(u_{xx} - bu + h) \quad \text{with} \quad h = H(u) = bu + \frac{u^2}{(u + \varepsilon^2)^2} \left(u + \frac{\varepsilon}{\xi(u + \varepsilon^2)} - f \right).$$

Here ξ is a mollified function of $u \vee m\varepsilon$. The multiplier u^2 in H is necessary to calculate local degree near zero since it vanishes faster than u . The multiplier $(u + \varepsilon^2)^2$ is good both for Harnack inequalities and uniform parabolicity. The shift term ε^2 should vanish faster than the parameter ε of penalization so that the limit of approximate solution satisfies the constraint (1.2).

A penalty method is adapted in various evolution equations to introduce constraints of solutions. Rather than to present a huge list we point out one typical example for the harmonic gradient flow equations for mapping into a unit sphere. The requirement of values of mapping u is considered as the constraint $|u| = 1$. A global weak solution was constructed independently by Chen [Ch], Keller, Rubinstein and Sternberg [KRS] and Shatah [S] by using a penalty method; see also [CS] for a generalization.

The initial value problem for (1.7) with $q = 0$ was derived in [Ga1] and extensively

studied by Gage and Hamilton [GH] for the curve shortening problem. Since a circle shrinks to a point in a finite time for the curve shortening equation ((1.6) with $q = 0$), the curvature may blow up in a finite time. Blow up profiles for convex immersed curves were classified by Angenent [An] based on results of [AL] and [EW] under the self-similar growth assumption for curvatures. There may happen that curvature growth is faster than self-similar rate. Its asymptotic profile is studied in [An] via (1.7) (with $q = 0$). Recently, more precise profile is obtained by Angenent and Velázquez [AV] by studying (1.6) itself. The initial-boundary value problem for higher dimensional version of (1.1) (with $f = 0$):

$$u_t = u^2(\Delta u + u), \quad u > 0$$

in a bounded domain with zero boundary data was studied in [FM] and [Ga2] for positive initial data. The existence of blow up phenomena now depends on the first eigenvalue of the Laplace operator with zero boundary condition. These authors studied whether a solution blows up and they estimated the size of blow up sets. Recently, Wiegner [W] extended the first part for

$$u_t = u^\gamma(\Delta u + u)$$

for $\gamma > 2$. There are interesting nonuniqueness results for $\gamma = 1$ by [U] (for one dimensional case) and [DL]. Among other results these authors proved that weak solutions may not be unique if initial data takes zero in some open set of the domain because of degeneracy of the equation. They discussed a class of solutions so that uniqueness holds.

This paper is organized as follows. In section 2 we derive the Harnack type inequalities. In section 3 an upper bound of solutions is obtained for a general equation so that it applies to our approximate equations. In section 4 we establish a lower bound and prove Theorem 1.4. In section 5 a lower bound is derived for a general equation. Section 6 is devoted to constructing solutions for approximate equations based on a degree theory. The Main Theorem is proved there. In Section 7 we study the property of the solution operator S defined through (1.12). For the existence of S it may be possible to apply *a priori* estimate for quasilinear parabolic equations in [LSU]. However, since the space dimension is one, we present a simple proof based only on L^p estimates for linear parabolic equations. In the Appendix we give a geometric interpretation of the assumption (1.8); we also give a proof of Remark 1.3.

2 Harnack type inequalities.

We consider a positive smooth solution u of

$$(2.1) \quad u_t = u^\gamma(u_{xx} + g(u, x, t)) \quad \text{in } Q = I \times (a, b]$$

with $\gamma \in \mathbb{R}$ and smooth g , where I denotes an open interval. We study the behavior of the function $z = (\log u)_t$ to show that the growth of $\log u$ does not become too negative. Such an analysis was done by [Ga2, §2] when $\gamma = 2$ and $g = u$. We adapt his method in our setting. As in [Ga2] our results also extend to the case of several space dimensions, where u_{xx} is replaced with Δu , see §2.5.

For $z = (\log u)_t = u_t/u$ a straightforward calculation yields

$$\begin{aligned} z_x &= \frac{u_{tx}}{u} - \frac{u_t u_x}{u^2} \\ z_{xx} &= \frac{u_{txx}}{u} - \frac{u_x u_{tx}}{u^2} - \frac{u_{tx} u_x + u_t u_{xx}}{u^2} + 2 \frac{u_t u_x^2}{u^3} \\ &= \frac{u_{txx}}{u} - \frac{2u_x z_x}{u} - \frac{u_t u_{xx}}{u^2}. \end{aligned}$$

We thus observe that

$$u_{txx} = u z_{xx} + 2u_x z_x + u_{xx} z.$$

Differentiating $z = u^{\gamma-1}(u_{xx} + g)$ in t yields

$$\begin{aligned} z_t &= u^{\gamma-1}(u_{xxt} + g_u u_t + g_t) + (\gamma-1)u^{\gamma-2}u_t(u_{xx} + g) \\ &= u^{\gamma-1}(u_{xxt} + g_u u z + g_t) + (\gamma-1)u^{\gamma-2}u z u^{1-\gamma} z. \end{aligned}$$

Using the expression of u_{xxt} by z 's, we end up with a parabolic differential equation for z .

2.1. Differential identity. Suppose that $u > 0$ solves $u_t = u^\gamma(u_{xx} + g)$ in Q . Then $z = u_t/u$ solves

$$z_t = u^\gamma z_{xx} + 2u^{\gamma-1}u_x z_x + \gamma z^2 + u^{\gamma-1}(g_u u - g)z + g_t u^{\gamma-1} \quad \text{in } Q.$$

2.2. Estimate of minimum of z . Suppose that $u > 0$ solves (2.1) in Q with $\gamma > 0$. Suppose that z takes its minimum over Q at $(x_0, t_0) \in Q$. Then

$$z \geq -\gamma^{-1}u^{\gamma-1}(g_u u - g)_+ - \gamma^{-1/2}u^{(\gamma-1)/2}|g_t|^{1/2} \quad \text{at } (x_0, t_0),$$

where f_+ denotes the positive part of f i.e. $f_+ = f \vee 0$.

Proof. Applying the maximum principle for the differential identity of z yields

$$\gamma z^2 + u^{\gamma-1} G z + g_t u^{\gamma-1} \leq 0 \quad \text{at } (x_0, t_0) \quad \text{with } G = g_u u - g.$$

Since $\gamma > 0$, this quadratic inequality for z implies

$$\begin{aligned} z &\geq -(2\gamma)^{-1} \{u^{\gamma-1} G + (u^{2(\gamma-1)} G^2 - 4\gamma u^{\gamma-1} g_t)^{1/2}\} \\ &\geq -(2\gamma)^{-1} \{u^{\gamma-1} 2G_+ + 2\gamma^{1/2} u^{(\gamma-1)/2} |g_t|^{1/2}\}. \quad \square \end{aligned}$$

We apply the estimate of $\min z$ to get Harnack type inequalities. We consider a positive, smooth solution of

$$(2.2) \quad u_t = u^\gamma (u_{xx} + g(u, x, t)) \quad \text{in } K,$$

where $K = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/T\mathbb{Z})$ and g is a smooth function in each variables.

2.3. Harnack's inequality in time direction. Assume that $\gamma \geq 1$ and $\alpha \geq 0$. Assume that there are constants $c_0, c_1 > 0$ such that

$$(2.3) \quad v g_v(v, x, t) - g(v, x, t) \leq c_0, \quad |g_t(v, x, t)|^{1/2} \leq c_1$$

for all $(v, x, t) \in (\alpha, \infty) \times K$. Assume that there is a constant $c_2 > 0$ such that $\max_K u \geq c_2$ for all solutions u of (2.2) with $u > \alpha$. Then there is a positive constant $C = C(c_0, c_1, c_2, \gamma) > 0$ such that any solution u of (2.2) with $u > \alpha$ fulfills

$$(2.4) \quad u(x, t) \leq u(x, t_0) \exp(-CM^{\gamma-1}(t - t_0)), \quad M = \max_K u$$

for all $(x, t_0), (x, t) \in K$ with $t_0 - T \leq t \leq t_0$.

Proof. Since K is compact, the minimum of z in K is attained. Using the estimate for $\min_K z$, together with (2.3), we observe that

$$(2.5) \quad \min_K z \geq -\gamma^{-1} c_0 M^{\gamma-1} - \gamma^{-1/2} c_1 M^{(\gamma-1)/2}$$

if $\gamma \geq 1$. Since $M \geq c_2 > 0$, this yields

$$\min_K z \geq -CM^{\gamma-1}$$

with $C = \gamma^{-1} c_0 + \gamma^{-1/2} c_1 c_2^{-(\gamma-1)/2}$. We now obtain a differential inequality

$$-(u^{-1})_t = u_t/u^2 = z/u \geq -CM^{\gamma-1}/u \quad \text{in } K$$

which becomes

$$(u^{-1})_t / u^{-1} \leq CM^{\gamma-1} \quad \text{in } K.$$

For fixed $(x, t_0) \in K$ integrating the differential inequality over (t, t_0) with $t_0 - T \leq t \leq t_0$ yields

$$\log u^{-1}(x, t_0) - \log u^{-1}(x, t) \leq CM^{\gamma-1}(t_0 - t)$$

which is the same as (2.4).

2.4. Harnack's inequality in space direction. Assume that $\gamma \geq 1$, $\alpha \geq 0$ and (2.3) for g . Let u be a solution of (2.2) with $u > \alpha$. Let $(x_0, t_0) \in K$ be a maximizer of u . Then

$$(2.6) \quad u(x, t_0)^\gamma \geq M^\gamma - \gamma C_M (x - x_0)^2 / 2$$

with

$$C_M = \frac{c_0}{\gamma} M^{\gamma-1} + \frac{c_1}{\gamma^{\frac{1}{2}}} M^{(\gamma-1)/2} + M^{\gamma-1} g_M, \quad M = \max_K u = u(x_0, t_0),$$

$$g_M = \max\{(g(v, x, t))_+; \alpha < v \leq M, (x, t) \in K\}.$$

Proof. Since $u^{\gamma-1} u_{xx} = z - u^{\gamma-1} g$, it follows from (2.5) that

$$(2.7) \quad u^{\gamma-1} u_{xx} \geq -C_M \quad \text{in } K.$$

This, in particular, implies

$$(2.8) \quad (u^\gamma)_{xx} = (\gamma u^{\gamma-1} u_x)_x = \gamma(\gamma-1)u^{\gamma-2} u_x^2 + \gamma u^{\gamma-1} u_{xx} \geq -\gamma C_M.$$

Integrating over (x_0, x) yields

$$(u^\gamma)_x(x, t_0) \geq (u^\gamma)_x(x_0, t_0) - \gamma C_M (x - x_0) = -\gamma C_M (x - x_0)$$

since (x_0, t_0) is a critical point of u^γ . Integrating again over (x_0, x) now yields (2.6).

2.5. Remarks. Results in §2.1 - §2.4 can be extended to a multi-dimensional problem. Indeed §2.1 and §2.2 are still valid for

$$(2.9) \quad u_t = u^\gamma (\Delta u + g(u, x, t)) \quad \text{in } Q = \Omega \times (a, b], \quad \Omega \subset \mathbb{R}^n$$

if the first and second derivative in x is replaced by the gradient and the Laplacian, respectively. We consider T -periodic positive solution u of (2.9) with $a = 0$, $b = T$. If u is periodic in space, the same technique yields Harnack's inequality (2.4) provided that $W = \Omega \times (\mathbb{R}/T\mathbb{Z})$ replaces K , where Ω is an n -dimensional torus. However, Harnack's inequality (2.6) should be altered for W . Instead of (2.7), (2.8) we obtain

$$(2.10) \quad u^{\gamma-1} \Delta u \geq -C_M, \quad \Delta u^{\gamma} \geq -\gamma C_M \quad \text{in } W.$$

Contrary to one-dimensional problem, (2.10) does not yield (2.6). Instead of (2.6) we observe that

$$(2.11) \quad (u^{\gamma})_1(r, t) \geq M^{\gamma} - \gamma C_M r^2 / 2n, \quad r = |x - x_0|$$

for the maximizer (x_0, t_0) of u over W . Here $f_1(r)$ denotes the mean of f on the sphere of radius r centered at x_0 , i.e.,

$$f_1(r) = \int_{|\omega|=1} f(r\omega + x_0) d\omega a^{-1}, \quad a = \text{area of the unit sphere}.$$

Indeed, using the identity

$$(2.12) \quad \Delta_x(f_1(r(x))) = \int_{|\omega|=1} (\Delta f)(r(x)\omega + x_0) d\omega a^{-1}, \quad r(x) = |x - x_0|$$

we observe that (2.10)₂ yields

$$(r^{n-1}((u^{\gamma})_1)')' \geq -\gamma C_M r^{n-1}$$

since $\Delta_x g(r(x)) = r^{1-n}(r^{n-1}g')'$. Here, the left hand side of (2.12) is the Laplacian of $f_1(r(x))$ as a function of x . Since (x_0, t_0) is a critical point of $(u^{\gamma})_1$, integrating over $(0, r)$ yields $((u^{\gamma})_1)' \geq -\gamma C_M r/n$. Integrating over $(0, r)$ again yields (2.11).

It remains to prove (2.12). We may assume $x_0 = 0$. A direct calculation shows

$$(2.13) \quad \Delta_x(f(r(x)\omega)) = ((\omega \cdot \nabla)(\omega \cdot \nabla)f)(r(x)\omega)$$

$$(2.14) \quad [\nabla - \omega(\omega \cdot \nabla)] \cdot ([\nabla - \omega(\omega \cdot \nabla)]f) = [\Delta - (\omega \cdot \nabla)(\omega \cdot \nabla)]f.$$

for a given unit n -vector ω . If the left hand side of (2.14) is evaluated at $r(x)\omega$, it agrees with $\Delta_x g(\omega)$, $g(\omega) = f(r(x)\omega)$, where Δ_x is the Laplace-Beltrami operator on the unit spheres. By the divergence theorem, integrating (2.14) over the sphere yields

$$0 = \int_{|\omega|=1} [(\Delta - (\omega \cdot \nabla)(\omega \cdot \nabla))f](r(x)\omega) d\omega.$$

This together with (2.13) yields (2.12).

3 Upper bounds.

We shall derive an *a priori* upper bound for a positive smooth solution $u > \alpha > 0$ of

$$(3.1) \quad u_t = u^\gamma \{u_{xx} + \varphi(u)(u + \psi(u) - f(x, t))\} \quad \text{in } K.$$

Here φ and ψ are smooth functions on (α, ∞) and f is smooth on K i.e. $f \in C^\infty(K)$ with $f > 0$; $\alpha \geq 0$ and $\gamma \in \mathbb{R}$.

3.1. Lemma on integral bounds. Assume that $\psi \geq 0$, $f \geq 0$ and that $0 \leq \varphi \leq c_3$, $v - \varphi(v)v \leq c_4$ on (α, ∞) , ($\alpha \geq 0$) with some positive constants c_3 and c_4 . Then

$$\begin{aligned} \iint_K u dx dt &\leq 2\pi T(c_3 \|f\|_\infty + c_4) \equiv C_1 \\ \iint_K \frac{u_t^2}{u^\gamma} dx dt &\leq c_3 C_1 \|f\|_\infty \equiv C_2 \end{aligned}$$

holds for all solutions $u \in C^\infty(K)$ of (3.1) with $u > \alpha$ on K .

Proof. Multiplying $u^{-\gamma}$ with (3.1) and integrating by parts on K yields

$$\begin{aligned} \iint_K \frac{u_t}{u^\gamma} dx dt &= \iint_K \{u_{xx} + \varphi(u)(u + \psi(u) - f)\} dx dt \\ &= \iint_K \{\varphi(u)u - \varphi(u)(f - \psi(u))\} dx dt. \end{aligned}$$

Since u is T -periodic, we see

$$\int_0^T \frac{u_t}{u^\gamma} dt = (1 - \gamma)^{-1} \int_0^T (u^{1-\gamma})_t dt = 0 \quad \text{for all } x \in \mathbb{R}.$$

We thus observe that

$$\iint_K \varphi(u)u dx dt = \iint_K \varphi(u)(f - \psi(u)) dx dt \leq \iint_K \varphi(u)f dx dt \leq 2\pi T c_3 \|f\|_\infty$$

since $\psi \geq 0$ and $\varphi \geq 0$. From $v - \varphi(v)v \leq c_4$ it follows that

$$\iint_K u dx dt \leq \iint_K (\varphi(u)u + c_4) dx dt \leq C_1.$$

It remains to prove the second inequality of Lemma 3.1. Multiplying u_t/u^γ with (3.1) and integrating over K yields

$$\iint_K \frac{u_t^2}{u^\gamma} dx dt = \iint_K \{u_{xx}u_t + \varphi(u)(u + \psi(u) - f)u_t\}.$$

Integrating by parts, we see

$$\iint_K u_{xx}u_t dx dt = - \iint_K u_x u_{xt} dx dt = -\frac{1}{2} \int_0^T \frac{d}{dt} \int_0^{2\pi} u_x^2 dx dt = 0$$

by T -periodicity of u_x . We also observe that

$$\int_0^T \varphi(u)(u + \psi(u))u_t dt = 0 \quad \text{for all } x \in \mathbb{R}.$$

We thus conclude that by integration by parts in t

$$\iint_K \frac{u_t^2}{u^\gamma} dx dt = - \iint_K \varphi(u) f u_t dx dt = \iint_K \Phi(u) f_t dx dt$$

by T -periodicity of u , where Φ is a primitive of φ defined by

$$\Phi(v) = \int_0^v \varphi(r) dr$$

so that $\Phi(v)_t = \varphi(v)v_t$ and $\Phi(v) \leq c_3 v$. It now follows that

$$\iint_K \frac{u_t^2}{u^\gamma} dx dt \leq \|f_t\|_\infty \cdot \iint_K c_3 u dx dt \leq c_3 C_1 \|f_t\|_\infty.$$

This is the same as the second inequality of Lemma 3.1.

3.2. Upper bound Theorem. Assume that $1 \leq \gamma < 3$ and that $\alpha \geq 0$. Assume the hypotheses in Lemma 3.1 on f , ψ , φ . Assume furthermore that

$$(3.2) \quad \varphi'(v)(\psi(v) - f) + \varphi(v)\psi'(v) \leq 0$$

$$(3.3) \quad 0 \leq \varphi'(v)v^2 \leq c_5, \quad \varphi(v)(\psi(v) - \min_K f) \leq c_6(v + 1)$$

on (α, ∞) with constants c_5 and $c_6 > 0$. Then there is a positive constant $M_0 = M_0(c_j, 3 \leq j \leq 6, T, \|f\|_\infty, \|f_t\|_\infty, \gamma)$ such that if $u \in C^\infty(K)$ with $\min_K u > \alpha$ solves (3.1), then $\max_K u \leq M_0$.

Proof. We shall combine Harnack's inequality (2.6) with integral bounds. We set

$$(3.4) \quad g(v, x, t) = \varphi(v)(v + \psi(v) - f(x, t))$$

so that

$$\begin{aligned} g_v v - g &= \{\varphi'(v)(v + \psi(v) - f) + \varphi(v)(1 + \psi'(v))\}v - \varphi(v)(v + \psi(v) - f) \\ &\leq \varphi'(v)v^2 + \varphi(v)(f - \psi(v)) \quad \text{by (3.2)} \\ &\leq c_5 + c_3 \|f\|_\infty \equiv c_0 \quad \text{for } v > \alpha \end{aligned}$$

since $\psi \geq 0$. Moreover,

$$|g_t| \leq \varphi(v)\|f_t\|_\infty \leq c_3\|f_t\|_\infty \equiv c_1^2 \quad \text{for } v > \alpha.$$

Applying Harnack's inequality (2.6), we have

$$(3.5) \quad u(x, t_0)^\gamma \geq M^\gamma - \gamma C_M (x - x_0)^2 / 2 \quad \text{for all } x \in \mathbb{R}$$

with $M = \max_K u = u(x_0, t_0)$ and

$$C_M = \frac{c_0}{\gamma} M^{\gamma-1} + \frac{c_1}{\gamma^{1/2}} M^{(\gamma-1)/2} + M^{\gamma-1} g_M.$$

Here

$$\begin{aligned} g_M &= \max\{g(v, x, t)_+; \alpha < v \leq M, (x, t) \in K\} \\ &\leq c_3 M + c_6 (M + 1) \quad \text{by (3.3)}_2. \end{aligned}$$

It follows that

$$C_M \leq 2c' M^\gamma, \quad (c' = c_3 + c_6)$$

for large M , say, $M \geq M_1 = M_1(c_0, c_1, c_3, c_6, \gamma)$.

We now integrate (3.5) (assuming $M \geq M_1$) to get

$$\begin{aligned} \int_0^{2\pi} u(x, t_0)^\gamma dx &= \int_{x_0-\pi}^{x_0+\pi} u(x, t_0)^\gamma dx \geq \int_{x_0}^{x_0+\sigma} (M^\gamma - \gamma c' M^\gamma (x - x_0)^2) dx \\ &= M^\gamma \sigma (1 - \gamma c' \sigma^2 / 3) \quad \text{for } 0 < \sigma < \pi \\ &\geq M^\gamma \sigma / 2 \quad \text{for small } \sigma, \text{ say, } \sigma \leq (3/2\gamma c')^{1/2} \equiv \sigma_0. \end{aligned}$$

We shall fix $\sigma = \sigma_0$ in the rest of our argument. By Lemma 3.1 there is t_1 such that

$$\begin{aligned} \int_0^{2\pi} u(x, t_1)^\gamma dx &\leq \frac{1}{T} \int_0^T \int_0^{2\pi} u(x, t)^\gamma dx dt \\ &\leq \frac{M^{\gamma-1}}{T} \iint_K u dx dt \leq \frac{C_1 M^{\gamma-1}}{T}. \end{aligned}$$

We now observe that

$$\begin{aligned} \int_0^{2\pi} u(x, t)^\gamma dx &= \int_0^{2\pi} u(x, t_1)^\gamma dx + \int_0^{2\pi} \int_{t_1}^t \gamma u^{\gamma-1} u_t(x, \tau) d\tau dx \\ &\leq \int_0^{2\pi} u(x, t_1)^\gamma dx + \gamma \left(\iint_K u^{3\gamma-2} dx dt \right)^{1/2} \left(\iint_K \frac{u_t^2}{u^\gamma} dx dt \right)^{1/2} \\ &\leq \frac{C_1}{T} M^{\gamma-1} + \gamma C_1^{1/2} M^{(3\gamma-3)/2} C_2^{1/2} \quad \text{for all } t \in \mathbb{R} \end{aligned}$$

since $\iint u_t^2 / u^\gamma$ is estimated by Lemma 3.1. Setting $t = t_0$ now yields

$$\frac{M^\gamma \sigma_0}{2} \leq \int_0^{2\pi} u(x, t_0)^\gamma dx \leq \frac{C_1}{T} M^{\gamma-1} + \gamma C_1^{1/2} C_2^{1/2} M^{(3\gamma-3)/2}.$$

Since $\gamma < 3$ i.e. $(3\gamma-3)/2 < \gamma$, this inequality for M yields a bound for M , say $M \leq M_2 = M_2(\sigma_0, C_1, C_2, T, \gamma)$. We thus conclude that $M \leq M_0$ by setting $M_0 = \max(M_1, M_2)$.

3.3. Remark. The assumptions of Theorem 3.2 are clearly satisfied for $u_t = u^\gamma(u_{xx} + u - f)$ with $f > 0$ which is a particular example of (3.1) by setting $\varphi \equiv 1$ and $\psi \equiv 0$. We study general equations because we need to have a bound of solutions for solutions of equations which appear in homotopy of approximate equations. These technically looking assumptions are really useful in the sequel.

3.4. Remark. We obtain an upper bound as in §3.2 for multi-dimensional problem where K is replaced by W as in §2.5. Note that the integral bounds in §3.1 hold for W with no changes. The only place to be altered in the proof of Theorem 3.2 is the estimate of space integral of u^γ from below since (2.6) should be replaced by (2.11). By (2.11) it holds

$$\begin{aligned} \int_{\Omega} u^\gamma(x, t_0) dx &\geq \int_{|x-x_0|<\sigma} u^\gamma(x, t_0) dx \geq \int_0^\sigma r^{n-1} (M^\gamma - \gamma c' M^\gamma r^2/n) dr \\ &\geq M^\gamma \sigma^n / 2n \quad \text{for small } \sigma. \end{aligned}$$

The remaining argument is the same as in the proof of Theorem 3.2.

4 Constraints and lower bounds.

We consider a positive (periodic) solution $u \in C^\infty(K)$ of (1.1) with the constraint (1.2). Our goal in this section is to establish an *a priori* lower bound for u when $f > 0$. Since an upper bound is obtained in Theorem 3.2, this will complete the proof of Theorem 1.4. As noticed in the Introduction, the constraint (1.2) plays an important role for a lower bound. We first study the stationary problem of (1.1):

$$(4.1) \quad U_{xx} + U = F \quad \text{in } T = \mathbb{R}/2\pi\mathbb{Z}.$$

4.1. Lemma on distance of zeros. For $a \in \mathbb{R}$ and $d > 0$ suppose that V solves

$$V_{xx} + V \geq 0 \quad \text{on } (a, a+d)$$

with $V(a) = V_x(a) = 0$ and $V(a+d) = 0$. Suppose that

$$V \geq 0 \quad \text{on } (a, a+d) \quad \text{and} \quad V \not\equiv 0.$$

Then $d > \pi$. Here V_x is assumed to be Lipschitz on $[a, a+d]$.

Proof. We may assume $a = 0$. Since $V(0) = V(d) = 0$, integration by parts yields

$$\int_0^d \sin\left(\frac{\pi x}{d}\right) \{V_{xx} + \left(\frac{\pi}{d}\right)^2 V\} dx = 0.$$

Suppose that $\pi/d \geq 1$. Since $\sin(\pi x/d) \geq 0$, $V \geq 0$ on $(0, d)$, we now observe that

$$\int_0^d \sin\left(\frac{\pi x}{d}\right) (V_{xx} + V) dx \leq \int_0^d \sin\left(\frac{\pi x}{d}\right) \{V_{xx} + \left(\frac{\pi}{d}\right)^2 V\} dx = 0.$$

Since $V_{xx} + V \geq 0$, this implies

$$V_{xx} + V = 0 \quad \text{on } (0, d).$$

By $V(0) = V(d) = 0$ we see

$$V(x) = R \sin(\pi x/d)$$

with some constant R . Since $V_x(0) = 0$, this implies $V \equiv 0$, which contradicts $V \not\equiv 0$. We thus proved $\pi/d < 1$. $\square$

We next discuss the way to get a lower bound of solutions of (4.1) from constraints. For later convenience we state here a general version. Let $\mu^\pm = \{\mu_\epsilon^\pm\}_{\epsilon>0}$ be a sequence of nonincreasing, continuous, positive functions on $(0, \infty)$ such that $\mu_\epsilon^\pm$ converges to some (nonincreasing, positive,) function $\mu_0^\pm$ on $(0, \infty)$. We assume that μ_0^- is continuous in $(0, \infty)$ and that the convergence $\mu_\epsilon^- \rightarrow \mu_0^-$ is uniform in every compact subset of $(0, \infty)$ as $\epsilon \rightarrow 0$. Let $h^- = \{h_\epsilon^-\}_{\epsilon>0}$ be a bounded sequence in $L^\infty(0, \infty)$ such that $0 \leq h_\epsilon^- \leq 1$. Assume that h_ϵ^- converges to $h_0^- \equiv 1$ uniformly in every compact subset of $(0, \infty)$ as $\epsilon \rightarrow 0$. For $U > 0$ in (4.1) and $\epsilon \geq 0$ we set

$$(4.2) \quad A_\epsilon^\pm(\zeta, U) = \int_0^{2\pi} \sin_\pm(x - \zeta) \mu_\epsilon^\pm(U) h_\epsilon^\pm(U) dx, \quad \zeta \in \mathbb{R},$$

where $\sin_+ z = \max(\sin z, 0)$, $\sin_- z = -\min(\sin z, 0)$ and $h_\epsilon^+ \equiv 1$ for $\epsilon \geq 0$.

4.2. Lemma on a lower bound for stationary solutions. Let k_j ($0 \leq j \leq 4$) be a positive constant. Suppose that

$$(4.3) \quad \int_0^1 \mu_0^-(x^2) dx = \infty.$$

Then there are positive constants ϵ_0 , δ_0 depending only on k_j 's and $\mu^\pm$, h^- such that a positive solution $U \in C^2(\mathbb{T})$ of (4.1) satisfies $\min_{\mathbb{T}} U \geq \delta_0$ if U and ϵ ($0 \leq \epsilon < \epsilon_0$) fulfill following properties.

$$(U1) \quad 0 \leq F \leq k_0, \quad \text{where } F = U_{xx} + U,$$

$$(U2) \quad k_1 \leq \max_T U \leq k_2$$

$$(U3) \quad A_\varepsilon^-(\zeta, U) \leq k_3 A_\varepsilon^+(\zeta, U) + k_4 \quad \text{for all } \zeta \in \mathbb{R}.$$

Proof. We argue by a contradiction. Suppose that for some k_j ($0 \leq j \leq 4$) there would exist sequences $\{U_n\}$, $\{\varepsilon_n\}$ ($\varepsilon_n \geq 0$) satisfying (U1)-(U3) with $U = U_n$, $\varepsilon = \varepsilon_n$, $F = F_n$ such that

$$(4.4) \quad 0 < \min_T U_n \rightarrow 0 \quad \text{and} \quad \varepsilon_n \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

Let $x_n \in T$ be a point such that $\min_T U_n = U_n(x_n)$. We may assume $x_n \rightarrow x_0$ for some $x_0 \in T$ by taking a subsequence if necessary. Since F_n and U_n are bounded by (U1) and (U2), the equation (4.1) implies that $\{U_{nxx}\}$ is bounded in $C(T)$. By Ascoli-Arzelà's theorem U_n converges to some function U in $C^1(T)$, by taking a subsequence if necessary. Since $\{U_{nxx}\}$ is bounded, U_x is Lipschitz in T and solves

$$U_{xx} + U \geq 0 \quad (\text{by } F_n \geq 0).$$

By (4.4) we would obtain

$$U(x_0) = \lim_{n \rightarrow \infty} U_n(x_n) = 0$$

since the convergence $U_n \rightarrow U$ is uniform. By (U2) we see $U \not\equiv 0$ i.e. there is $x_* \in \mathbb{R}$ with $U(x_*) > 0$. We set

$$a = \sup\{x \in \mathbb{R}; U(x) = 0, x < x_*\} \in \mathbb{R}.$$

Since U takes zero at x_0 this is well-defined. By continuity of U_x we see $U(a) = U_x(a) = 0$ and $U(x) > 0$ for $a < x \leq x_*$. We now apply Lemma 4.1 to observe that the distance of another zero (bigger than a) of U and a is more than π . Thus there is a small $0 < \rho_0 < \pi/2$ such that if $0 < \rho < \rho_0$ then

$$\lambda_\rho = \inf\{U(x); a + \rho \leq x \leq a + \rho + \pi\} > 0.$$

We fix $0 < \rho < \rho_0$ and set $\zeta = a + \rho$. Since $U_n \rightarrow U$ is uniform, we have a uniform lower bound for U_n :

$$(4.5) \quad U_n \geq \kappa = \lambda_\rho/2 \quad \text{on} \quad [\zeta, \zeta + \pi]$$

for sufficiently large n .

Since U_{xx} is bounded, $U_x(a) = U(a) = 0$ would imply that

$$(4.6) \quad 0 \leq U(x) \leq R(x-a)^2 \quad \text{for all } x \in \mathbb{R}$$

with some $R > 0$. From a bound (4.5) it follows that

$$\overline{\lim}_{n \rightarrow \infty} A_n^+ = \overline{\lim}_{n \rightarrow \infty} \int_{\zeta}^{\zeta+\pi} \sin(x-\zeta) \mu_{\varepsilon_n}^+(U_n) dx \leq \int_{\zeta}^{\zeta+\pi} \sin(x-\zeta) dx \mu_0^+(\kappa) = 2\mu_0^+(\kappa),$$

where $A_n^\pm = A_\varepsilon^\pm(\zeta, U_n)$ with $\varepsilon = \varepsilon_n$. By (U3) this implies

$$\overline{\lim}_{n \rightarrow \infty} A_n^- \leq 2k_3\mu_0^+(\kappa) + k_4 \equiv L < \infty.$$

Using Fatou's lemma, we observe that

$$\begin{aligned} \int_a^\zeta (-\sin(x-\zeta)) \mu_0^-(U) dx &\leq \underline{\lim}_{n \rightarrow \infty} \int_a^\zeta (-\sin(x-\zeta)) \mu_{\varepsilon_n}^-(U_n) h_{\varepsilon_n}^-(U_n) dx \\ &\leq \underline{\lim}_{n \rightarrow \infty} A_n^- \leq L, \end{aligned}$$

since the uniform convergence $\mu_{\varepsilon_n}^- \rightarrow \mu_0^-$, $h_{\varepsilon_n}^- \rightarrow h_0^-$ on every compact set in $(0, \infty)$ implies that

$$\lim_{n \rightarrow \infty} \mu_{\varepsilon_n}^-(U_n(x)) = \mu_0^-(U(x)), \quad \lim_{n \rightarrow \infty} h_{\varepsilon_n}^-(U_n(x)) = 1$$

for $a < x < \zeta$. From (4.6) it would follow that

$$\int_a^\zeta (-\sin(x-\zeta)) \mu_0^-(R(x-a)^2) dx \leq L.$$

Since $-\sin(x-\zeta) > 0$ on $(\zeta-\pi, \zeta) \ni a$, this contradicts (4.3). We have thus obtain a lower bound δ_0 for U for small $\varepsilon \geq 0$.

4.3. Corollary. *Let $f \in C(\mathbb{T})$ be a positive function and $\sigma > 0$. There are positive constants δ_0, M_0 (depending only on $\sigma, \|f\|_\infty$ and $\min_{\mathbb{T}} f$) such that if a positive function $u \in C^2(\mathbb{T})$ solves*

$$u_{xx} + u = f \quad \text{in } \mathbb{T} \quad \text{with} \quad \int_0^{2\pi} e^{ix} u^{-\sigma} dx = 0,$$

then $\delta_0 \leq u \leq M_0$ provided that $1/2 \leq \sigma$.

Proof. Although an upper bound M_0 is obtained by a general Theorem 3.2, we sketch the proof in this context which reflects an essential step of the proof of Theorem 3.2. Clearly $uu_{xx} = fu - u^2 \geq -M^2$ with $M = \max_{\mathbb{T}} u$. This implies

$$(u^2)_{xx} = 2u_x^2 + 2uu_{xx} \geq -2M^2,$$

which yields

$$u^2(x) \geq M^2 - M^2(x - x_0)^2, \quad M = \max_{\mathbf{T}} u = u(x_0).$$

We now integrate to get

$$M^2 \int_{x_0}^{x_0+1} (1 - (x - x_0)^2) dx \leq \int_0^{2\pi} u^2 dx \leq M \int_0^{2\pi} u dx.$$

This yields a bound M_0 of M since we observe

$$\int_0^{2\pi} u dx = \int_0^{2\pi} f dx \leq 2\pi \|f\|_{\infty}$$

by integrtrng $u_{xx} + u = f$ on $\mathbf{T}$.

By the maximum principle we have

$$\max u \geq \min f.$$

Since the constraint $\int e^{ix} u^{-\sigma} dx = 0$ is equivalent to

$$\int_0^{2\pi} \sin(x - \zeta) u^{-\sigma} dx = 0 \quad \text{for all } \zeta \in \mathbf{R},$$

(U3) is fulfilled with $k_3 = 1$, $k_4 = 0$, $\mu_{\varepsilon}^{\pm}(v) = v^{-\sigma}$, $h_{\varepsilon}^{\pm}(v) \equiv 1$ for all $\varepsilon \geq 0$. We observe that (U2) is fulfilled with $k_1 = \min_{\mathbf{T}} f > 0$, $k_2 = M_0$. Since (4.3) is fulfilled for $\mu_0^{-}(v) = v^{-\sigma}$, we now apply Lemma 4.2 with $k_0 = \|f\|_{\infty}$ to get a desired lower bound δ_0 of u . $\square$

We now study time dependent problem (1.1) or its general form

$$(4.7) \quad u_t = u^{\gamma}(u_{xx} + u - f) \quad \text{in } K.$$

For $u \in C(K)$ it is convenient to define

$$U(x) = \int_0^T u(x, t) dt.$$

Divide (4.7) by u^{γ} and integrate over $(0, T)$ to get

$$(4.8) \quad U_{xx} + U = F \quad \text{in } \mathbf{T} \quad \text{with} \quad F = \int_0^T f(x, t) dt,$$

since u is T -periodic in time. Harnack's inequality (2.4) allows us to compare u and U .

4.4. Lemma. *Suppose that $u \in C(K)$ satisfies (2.4). Then there are $\lambda, \Lambda > 0$ (depending only on C, γ, M, T) such that $\lambda u(x, t) \leq U(x) \leq \Lambda u(x, t)$ for $(x, t) \in K$.*

Proof. Since u is T -periodic, integrating (2.4) on $(t_0 - T, t_0)$ yields

$$U(x) = \int_{t_0-T}^{t_0} u(x, t) dt \leq u(x, t_0) \Lambda, \quad \Lambda = \int_{t_0-T}^{t_0} \exp(-CM^{\gamma-1}(t - t_0)) dt;$$

Λ is independent of t_0 . The inequality $\lambda u \leq U$ is similarly obtained by integrating

$$u(x, t_0) \exp(CM^{\gamma-1}(t_0 - t)) \leq u(x, t) \quad \text{on} \quad t_0 \leq t \leq t_0 + T.$$

4.5. Proof of Theorem 1.4. Since an upper bound of a positive smooth solution of (1.1) or (4.7) with $1 \leq \gamma < 3$ is obtained by Theorem 3.2, it suffices to establish a lower bound assuming the constraint (1.2).

By the maximum principle we see

$$(4.9) \quad \max_K u \geq \min_K f \equiv c_2.$$

Since Harnack's inequality (2.4) is available for (4.7) with $c_0 = \|f\|_\infty$, $c_1 = \|f_t\|_\infty$, $\gamma = 2$, Lemma 4.4 now yields

$$(4.10) \quad \max_T U \geq U(x_0) \geq \lambda u(x_0, t_0) \geq \lambda c_2 \equiv k_1$$

by taking (x_0, t_0) as a maximizer of u over K . From an upper bound M_0 of u , it follows that U fulfills (U2) with $k_2 = M_0 T$. Since $0 \leq f \leq \|f\|_\infty$, we see that (U1) is fulfilled with $k_0 = T\|f\|_\infty$. If $U \geq \delta_0$ on K , Lemma 4.4 gives a lower bound $\delta = \Lambda^{-1}\delta_0$ for u .

It remains to check (U3) with (4.3) by choosing $\mu_\varepsilon^\pm(v) = v^{-1}$, $h_\varepsilon^\pm(v) \equiv 1$ for $\varepsilon \geq 0$, so that (4.3) is fulfilled. We shall drop subscript ε since all quantities we handle are independent of ε . We estimate A^- by Lemma 4.4 to get

$$A^-(\zeta, U) \leq \lambda^{-1} \int_0^{2\pi} \frac{\sin_-(x - \zeta)}{u(x, t)} dx.$$

By the constraint (1.2) we have

$$\int_0^{2\pi} \frac{\sin_-(x - \zeta)}{u(x, t)} dx = \int_0^{2\pi} \frac{\sin_+(x - \zeta)}{u(x, t)} dx, \quad \zeta \in \mathbb{R}$$

since $\int_0^{2\pi} \sin(x - \zeta) u^{-1} dx = 0$. Using again Lemma 4.4, we get

$$A^-(\zeta, U) \leq \lambda^{-1} \Lambda A^+(\zeta, U)$$

which is (U3) with $k_3 = \lambda^{-1}\Lambda$ and $k_4 = 0$. This enables us to apply Lemma 4.2 to get a lower bound δ_0 for U . If we examine properties of all constants, we conclude that the constant δ_0 has the desired property.

4.6. Generalization. If we examine the proof, it turns out that the assumptions in Theorem 1.4 are weakened. First of all we may replace (1.1) by (4.7) with $1 \leq \gamma < 3$. Second, (1.2) may be replaced by

$$\int_0^{2\pi} \frac{e^{ix}}{u^\sigma(x, t)} dx = 0$$

for some $\sigma \geq 1/2$ and *some* t . Here $\mu_\varepsilon^\pm(v)$ should be $v^{-\sigma}$ in the proof.

5 Approximate penalized equations.

To solve (1.1) with (1.2) for positive $f \in C^\infty(K)$ satisfying (1.3) we consider an approximate penalized equation. Let $m > 1$ be a large number such that

$$(5.1) \quad \min_K f - \frac{1}{m} \geq \frac{1}{2} \min_K f.$$

For $0 < \varepsilon, \varepsilon' < 1$ we take a smooth nondecreasing function $\xi = \xi_\varepsilon$ on $[0, \infty)$ such that

$$(5.2) \quad \xi(v) = v \quad \text{for } v \geq m\varepsilon + \varepsilon'$$

$$(5.3) \quad v \vee m\varepsilon \leq \xi(v) \leq \ell(v \vee m\varepsilon) \quad \text{for } v \geq 0$$

with some $1 < \ell < 2$ independent of $\varepsilon, \varepsilon'$ and v .

Our approximate penalized equation is of the form

$$(5.4) \quad w_t = (w + \varepsilon')^2(w_{xx} + \frac{w^2}{(w + \varepsilon')^2}(w + \frac{\varepsilon}{\xi(w + \varepsilon')} - f)) \quad \text{in } K.$$

The term $\varepsilon/\xi(w + \varepsilon')$ plays the role of a penalty term. (If we do not need to have smoothness of this term, we may take $\varepsilon/(w + \varepsilon') \vee m\varepsilon$.) The cutting off by $m\varepsilon$ of ξ is introduced so that $f - \varepsilon/\xi(w + \varepsilon')$ is always positive, which will be important to get a lower bound. The parameter ε' is taken so that the equation (5.4) is uniformly parabolic while ε is for penalization. To get a lower bound for w , the penalization effect must dominate over other approximation so ε' should tend to zero faster than ε when $\varepsilon \rightarrow 0$. For simplicity we shall take $\varepsilon' = \varepsilon^2$ for the rest of our argument. We set $w + \varepsilon^2 = u$ in (5.4) to get

$$(5.5) \quad u_t = u^2(u_{xx} + \varphi_\varepsilon(u)(u + \psi_\varepsilon(u) - f - \varepsilon^2)) \quad \text{in } K$$

with

$$(5.6) \quad \varphi_\varepsilon(v) = \frac{(v - \varepsilon^2)^2}{v^2}, \quad \psi_\varepsilon(v) = \frac{\varepsilon}{\xi(v)} \quad \text{for } v > 0.$$

A direct calculation shows

$$(5.7) \quad \varphi_\varepsilon(v) = 1 - \frac{2\varepsilon^2}{v} + \frac{\varepsilon^4}{v^2}$$

$$(5.8) \quad \varphi'_\varepsilon(v) = \frac{2\varepsilon^2}{v^2} \left(1 - \frac{\varepsilon^2}{v}\right).$$

Since φ_ε and ψ_ε satisfy

$$(5.9)$$

$$0 \leq \varphi_\varepsilon(v) \leq 1, \quad 0 \leq \varphi'_\varepsilon(v)v^2 \leq 2\varepsilon^2 \leq 2, \quad v(1 - \varphi_\varepsilon(v)) \leq 2\varepsilon^2 \leq 2, \quad \text{for } v > \alpha = \varepsilon^2$$

$$(5.10)$$

$$\min_K f - \psi_\varepsilon(v) > 0, \quad \psi'_\varepsilon(v) \leq 0 \quad \text{for } v > \alpha,$$

all assumptions in Theorem 3.2 are fulfilled with $c_3 = 1$, $c_4 = 2$, $c_5 = 2$, $c_6 = 0$; to get

(5.10)₁ we use (5.1)-(5.3).

5.1. Upper bound Proposition. *There is a positive constant $M_0 = M_0(T, \|f\|_\infty, \|f_t\|_\infty)$ such that if $u \in C^\infty(K)$ solves (5.5) with $\varphi_\varepsilon, \psi_\varepsilon$ satisfying (5.9)-(5.10), and $u > \varepsilon^2$ on K , then $u \leq M_0$ on K . Here and hereafter $\varphi_\varepsilon, \psi_\varepsilon$ are assumed to be smooth on $(0, \infty)$.*

As in §4 we set

$$U(x) = \int_0^T u(x, t) dt.$$

Applying the maximum principle for (5.5) we see

$$\max_K u \geq \min_K f - \psi_\varepsilon(u) \geq \min_K f - \frac{1}{m} \geq \frac{1}{2} \min_K f \equiv c_2$$

by (5.1)-(5.3). Since other assumptions for Harnack's inequality (2.4) hold by (5.9)-(5.10), U and u are compared by Lemma 4.4.

5.2. Comparison Lemma. *Assume that φ_ε and ψ_ε fulfills (5.9), (5.10) and that*

$$\min_K f - \psi_\varepsilon(v) \geq c_2 > 0 \quad \text{for } v > \varepsilon^2$$

for some constant c_2 . Then there are positive constants λ, Λ (depending only on $T, \|f\|_\infty, \|f_t\|_\infty, c_2$) such that if $u \in C^\infty(K)$ solves (5.5) with $u > \varepsilon^2$ on K , then

$$(5.11) \quad \lambda u(x, t) \leq U(x) \leq \Lambda u(x, t) \quad \text{for } (x, t) \in K.$$

We shall derive a uniform lower bound of u independent of ε by assuming (1.3) for f , i.e.,

$$(5.12) \quad \int_0^{2\pi} f(x, t) \sin(x - \zeta) dx = 0 \quad \text{for all } \zeta, t \in \mathbb{R}.$$

5.3. Lemma on approximate constraints. Assume that $f \in C^\infty(K)$ satisfies (5.12). Assume that (5.9)₁ and

$$(5.13) \quad 1 - \varphi_\varepsilon(v) \leq c_7 \varepsilon^2 v^{-1} \quad \text{for } v > \varepsilon^2$$

with some $c_7 > 0$. Then if $u \in C^\infty(K)$ solves (5.5) with $u > \varepsilon^2$ on K , we have

$$(5.14) \quad \left| \int_0^T \int_0^{2\pi} \{ \varphi_\varepsilon(u) \psi_\varepsilon(u) + (1 - \varphi_\varepsilon(u))(f + \varepsilon^2) \} \sin(x - \zeta) dx dt \right| \leq 4Tc_7 \varepsilon^2.$$

for all $\zeta \in \mathbb{R}$.

Proof. Since u is T -periodic in t we have

$$\int_0^T \frac{u_t}{u^2} dt = \int_0^T \left(-\frac{1}{u} \right)_t dt = 0.$$

Since u is 2π -periodic in x , integrating by parts yields

$$\int_0^{2\pi} (u_{xx} + u) \sin(x - \zeta) dx = 0 \quad \text{for all } \zeta \in \mathbb{R}.$$

Multiply (5.5) with u^{-2} to get

$$u_t u^{-2} = u_{xx} + u - (f + \varepsilon^2) + \varphi_\varepsilon(u) \psi_\varepsilon(u) + (1 - \varphi_\varepsilon(u))(f + \varepsilon^2) + (\varphi_\varepsilon(u) - 1)u.$$

Multiplying $\sin(x - \zeta)$ with the both sides and integrating over K now yields

$$\int_0^T \int_0^{2\pi} \{ \varphi_\varepsilon(u) \psi_\varepsilon(u) + (1 - \varphi_\varepsilon(u))(f + \varepsilon^2) \} \sin(x - \zeta) dx dt = \int_0^T \int_0^{2\pi} (1 - \varphi_\varepsilon(u)) u \sin(x - \zeta) dx dt.$$

We use (5.13) to get (5.14). $\square$

Since φ_ε in (5.6) satisfies (5.7), the property (5.13) with $c_7 = 2$ holds for φ_ε defined by (5.6). By its form of φ_ε and of ψ_ε in (5.6), it is clear that

$$(5.15) \quad \varphi_\varepsilon(v) \rightarrow 1, \quad \psi_\varepsilon(v)/\varepsilon \rightarrow v^{-1} \quad \text{uniformly on every compact subset of } (0, \infty)$$

as $\varepsilon \rightarrow 0$.

5.4. Lemma on constraints for the limit of approximate solutions. Assume that φ_ϵ and ψ_ϵ fulfill (5.9)₁, (5.13), (5.15). Assume that $u_\epsilon \in C^\infty(K)$ ($u_\epsilon > \epsilon^2$) solves (5.5) with $f = f_\epsilon \in C^\infty(K)$ satisfying (5.12). Assume that $\|f_\epsilon\|_\infty$ is bounded for $0 < \epsilon < 1$. Suppose that u_ϵ converges to a positive $u \in C(K)$ uniformly in K . Then

$$(5.16) \quad \int_0^T \int_0^{2\pi} \sin(x - \zeta) u^{-1}(x, t) dx dt = 0 \quad \text{for all } \zeta \in \mathbb{R}.$$

Proof. Dividing (5.14) ($f = f_\epsilon$, $u = u_\epsilon$) with $0 < \epsilon < 1$ and using (5.13), we observe that

$$(5.17) \quad \left| \int_0^T \int_0^{2\pi} \sin(x - \zeta) \varphi_\epsilon(u_\epsilon) \psi_\epsilon(u_\epsilon) \epsilon^{-1} dx dt \right| \leq c_7 \epsilon + c_7 \|f_\epsilon + 1\|_\infty \int_0^T \int_0^{2\pi} \frac{\epsilon}{u_\epsilon} dx dt.$$

Since $u > 0$, u_ϵ is bounded away from zero, say $u_\epsilon \geq \delta_0 > 0$ uniformly for sufficiently small $\epsilon > 0$. The right hand side of (5.17) now converges to zero as $\epsilon \rightarrow 0$. By the convergence (5.15), sending ϵ to zero in (5.17) yields (5.16), since $u_\epsilon \geq \delta_0$. $\square$

The basic strategy to get a lower bound for u in (5.5) is similar to the proof of Theorem 1.4. This time we should check assumptions of Lemma 4.2 by using our approximate constraint (5.14). For this purpose we note a property of φ_ϵ of (5.6)₁. The assumptions on φ_ϵ in the following Lemma is automatically satisfied for φ_ϵ of (5.6)₁.

5.5. Lemma. Assume that φ_ϵ fulfills (5.9) and (5.15). Let Λ_0 be a positive number. There exists a sequence of continuous functions h_ϵ , $0 \leq h_\epsilon \leq 1$ on $(0, \infty)$ such that h_ϵ converges to 1 uniformly in every compact subset of $(0, \infty)$ as $\epsilon \rightarrow 0$ and that

$$\int_0^T \varphi_\epsilon(v(t)) dt \geq T h_\epsilon(V), \quad V = \int_0^T v(t) dt$$

for all $v \in C[0, T]$ with $v > \epsilon^2$ and $V \leq \Lambda_0 v(t)$, $0 \leq t \leq T$.

Proof. We set a continuous function

$$h_\epsilon(\rho) = \varphi_\epsilon(\rho \Lambda_0^{-1} \vee \epsilon^2), \quad \rho > 0$$

and observe that $h_\epsilon \rightarrow 1$ uniformly in every compact subset of $(0, \infty)$ and that $0 \leq h_\epsilon \leq 1$.

Since φ_ϵ is nondecreasing on (ϵ^2, ∞) by (5.9)₂, we thus conclude

$$\int_0^T \varphi_\epsilon(v(t)) dt \geq \int_0^T h_\epsilon(V) dt = T h_\epsilon(V). \square$$

We set

$$I_\epsilon^\pm = I_\epsilon^\pm(\zeta, u) = \epsilon^{-1} \int_0^T \int_0^{2\pi} \{\varphi_\epsilon(u) \psi_\epsilon(u) + (1 - \varphi_\epsilon(u))(f + \epsilon^2)\} \sin_\pm(x - \zeta) dx dt.$$

The approximate constraint (5.14) is rewritten as

$$(5.18) \quad |I_\varepsilon^+ - I_\varepsilon^-| \leq 4Tc_7\varepsilon.$$

5.6. Proposition on estimate of $A_\pm$. Assume that $u \in C(K)$ satisfies the comparison (5.11) with U and $u > \varepsilon^2$ on K and that $f \geq 0$.

(i) Suppose that φ_ε fulfills (5.9) and (5.15). Suppose that $\psi_\varepsilon(v) \geq \varepsilon\nu_\varepsilon(v)/\ell$ for $v > 0$, where $\nu_\varepsilon(v) = 1/(v \vee m\varepsilon)$ with some $m > 0$, $\ell > 0$. Let h_ε be as in Lemma 5.5 and set $h_\varepsilon^- = h_\varepsilon$ and $\mu_\varepsilon^-(v) = \nu_\varepsilon(\lambda^{-1}v)$. Let A_ε^- be defined by (4.2). Then

$$TA_\varepsilon^-(\zeta, U) \leq \ell I_\varepsilon^-(\zeta, u), \quad \zeta \in \mathbb{R}.$$

(ii) Suppose that φ_ε satisfies (5.9)₁, (5.13). Suppose that ψ_ε satisfies $\psi_\varepsilon(v) \leq \varepsilon v^{-1}$ for $v > 0$. Let A_ε^+ be defined by (4.2) with $\mu_\varepsilon^+(v) = v^{-1}$. Then

$$I_\varepsilon^+(\zeta, u) \leq T\Lambda(1 + \varepsilon c_7 \|f + \varepsilon^2\|_\infty) A_\varepsilon^+(\zeta, U), \quad \zeta \in \mathbb{R}.$$

Proof. (i) Since $\ell\varepsilon^{-1}\psi_\varepsilon(u) \geq \nu_\varepsilon(u) \geq \mu_\varepsilon^-(U)$ we see

$$\begin{aligned} I_\varepsilon^- &\geq \varepsilon^{-1} \int_0^T \int_0^{2\pi} \varphi_\varepsilon(u) \psi_\varepsilon(u) \sin_-(x - \zeta) dx dt \quad (\text{by } f \geq 0, \varphi_\varepsilon \leq 1) \\ &\geq \ell^{-1} \int_0^T \left(\int_0^{2\pi} \varphi_\varepsilon(u) dt \right) \mu_\varepsilon^-(U) \sin_-(x - \zeta) dx dt. \end{aligned}$$

Apply Lemma 5.5 to get $\ell I_\varepsilon^- \geq TA_\varepsilon^-$.

(ii) We estimate φ_ε , $\psi_\varepsilon\varepsilon^{-1}$, $1 - \varphi_\varepsilon$ from above to get

$$\begin{aligned} I_\varepsilon^+ &\leq \int_0^T \int_0^{2\pi} \left(\frac{1}{u} + \frac{\varepsilon c_7}{u} \|f + \varepsilon^2\|_\infty \right) \sin_+(x - \zeta) dx dt \\ &\leq T\Lambda(1 + \varepsilon c_7 \|f + \varepsilon^2\|_\infty) \int_0^{2\pi} \frac{1}{U} \sin_+(x - \zeta) dx \end{aligned}$$

which completes the proof.

5.7. Lower bound Theorem. Assume that $f \in C^\infty(K)$ fulfills (5.12) with $f > 0$. Let φ_ε and ψ_ε be defined by (5.6) with (5.2), (5.3), $m > 1$. Then there are positive constants ε_0 , δ depending only on T , $\|f\|_\infty$, $\|f_t\|_\infty$, $\min_K f$ such that if $u \in C^\infty(K)$ solves (5.5) _{ε} with $\bar{m}$ satisfying (5.1), then $u \geq \delta$ on K for $0 < \varepsilon < \varepsilon_0$ provided that $u > \varepsilon^2$ on K .

Proof. Since φ_ε and ψ_ε in (5.6) fulfill all assumptions on φ_ε , ψ_ε in §5.1-§5.6 including Propositions 5.6, we may apply all these results. By the comparison Lemma 5.2 it suffices to apply Lemma 4.2 to get a lower bound for U for sufficiently small ε .

We shall prove (U1)-(U3) in Lemma 4.2. Dividing (5.5) with u^2 and integrating over $(0, T)$ yields

$$0 = U_{xx} + U - F \quad \text{on } T$$

$$F = \int_0^T \{(f + \varepsilon^2 - \psi_\varepsilon(u))\varphi_\varepsilon(u) + (1 - \varphi_\varepsilon(u))u\} dt.$$

Since $f - \psi_\varepsilon(u) > 0$ by (5.10) and $\varphi_\varepsilon(u) \leq 1$ we see $F \geq 0$. Moreover, by (5.13) with $c_7 = 2$, we observe that

$$F \leq \int_0^T (\|f + \varepsilon^2\|_\infty + \frac{2\varepsilon^2}{u}) dt = T(\|f\|_\infty + 3) \equiv k_0 \quad (0 < \varepsilon < 1),$$

which proves (U1). By (5.1)-(5.3) applying the maximum principle yields

$$\max_K u (= u(x_0, t_0)) \geq \frac{1}{2} \min_K f \equiv c_2$$

as in §5.1. By Lemma 5.2 we see

$$\max_T U \geq U(x_0) \geq \lambda u(x_0, t_0) \geq \frac{\lambda}{2} \min_K f \equiv k_1 > 0;$$

(cf. the proof of (4.10).) Since $u \leq M_0$ by Proposition 5.1, (U2) is now fulfilled with $k_2 = TM_0$.

It remains to show (U3) since $\mu_\varepsilon^\pm, h_\varepsilon^-$ satisfy desired properties including (4.3). Applying the approximate constraint (5.18) to Proposition 5.6 we observe that

$$\begin{aligned} A_\varepsilon^-(\zeta, U) &\leq T^{-1} \ell I_\varepsilon^-(\zeta, u) \leq 2T^{-1} I_\varepsilon^-(\zeta, u) \\ &\leq 2T^{-1} (I_\varepsilon^+(\zeta, u) + 8T\varepsilon) \leq 2T^{-1} I_\varepsilon^+(\zeta, u) + 2 \cdot 8 \\ &\leq 2\Lambda(1 + 2(\|f\|_\infty + 1))A_\varepsilon^+(\zeta, U) + 16 \end{aligned}$$

where we have used $0 < \varepsilon < 1$, $\ell < 2$, $c_7 = 2$. We have thus proved (U3). The proof is now complete.

6 Existence of periodic solutions.

We shall construct a positive solution $w \in C^\infty(K)$ of the approximate penalized equation (5.4) (with $\varepsilon' = \varepsilon^2$) for arbitrary positive $f \in C^\infty(K)$. We always assume (5.1)-(5.3) and positivity of $f \in C^\infty(K)$ in this section. Also we fix $\varepsilon > 0$ except in the proof of the

Main Theorem given in §6.8. We begin with a solvability result for a uniformly parabolic equation since (5.4) is uniformly parabolic for $w > 0$. For $\omega > 0$, $T > 0$ we set

$$K_\omega = (\mathbb{R}/\omega\mathbb{Z}) \times (\mathbb{R}/T\mathbb{Z}).$$

We consider a uniform parabolic equation

$$(6.1) \quad v_t = a(v)(v_{xx} - bv + h) \quad \text{in } K_\omega.$$

6.1. Unique solvability Lemma. *Assume that b is a positive constant and that a is a continuous function on $\mathbb{R}$ such that $a(\sigma) \geq a_0 > 0$ for all $\sigma \in \mathbb{R}$ with some constant a_0 . For each $h \in C(K_\omega)$ there is a unique solution $v \in \bigcap_{q \geq 1} W_q^{2,1}(K_\omega) \subset C(K_\omega)$ of (6.1). Moreover the solution operator $h \mapsto v$ is a continuous, compact operator from $C(K_\omega)$ into itself. There exist $\theta_0 = \theta_0(a_0, \|h\|_\infty, b, \omega, T) > 0$ and $C_0 = C_0(a_0, \|h\|_\infty, b, \omega, T) > 0$ such that estimate*

$$\|v\|_{W_q^{2,1}} \leq C_0 \|h\|_\infty$$

holds for all $2 \leq p \leq 2 + \theta_0$ and $h \in C(K_\omega)$.

We postpone the proof in the next section.

6.2. Mappings. We next interpret a solution w of (5.4) as a fixed point of a mapping. Let $b > 0$ be taken so that

$$\phi(w, x, t) \equiv bw_+ + \frac{(w_+)^2}{(w_+ + \varepsilon^2)^2} \left(w_+ + \frac{\varepsilon}{\xi(w_+ + \varepsilon^2)} - f(x, t) \right) \geq 0, \quad w_+ = w \vee 0,$$

for all $w \in \mathbb{R}$, $(x, t) \in K$ and $\phi > 0$ on K if $w > 0$. It is possible to choose $b > 0$ because $(w_+)^2$ vanishes faster than w_+ . Indeed, $w \leq 0$ implies $\phi \equiv 0$. If $w \geq \max_K f > 0$, then $\phi > 0$ for any $b > 0$ since $\xi > 0$. If otherwise (but $w > 0$), we observe that

$$\begin{aligned} \phi(w, x, t) &= \frac{w}{(w + \varepsilon^2)^2} \{ b(w + \varepsilon^2)^2 + w^2 + \left(\frac{\varepsilon}{\xi(w + \varepsilon^2)} - f \right) w \} \\ &\geq \frac{w}{(w + \varepsilon^2)^2} (2b\varepsilon^2 w - fw) > 0 \end{aligned}$$

provided that $b > \|f\|_\infty / 2\varepsilon^2$.

For this choice of b let S denote the solution operator of

$$v_t = (v_+ + \varepsilon^2)^2 (v_{xx} - bv + h) \quad \text{in } K$$

i.e. $S(h) = v$. By Lemma 6.1 S is well-defined as an operator from $C(K)$ into itself.

Lemma 6.1 also yields;

(i) $S : C(K) \rightarrow C(K)$ is compact and continuous,

(ii) $S(h)$ is Hölder continuous on K for $h \in C(K)$.

The second assertion follows from the Sobolev embedding since $S(h) \in W_p^{2,1}(K)$ for all $p > 1$. Using (ii) and the Schauder theory of linear parabolic equations [LSU] we observe that

(iii) $v = S(h) \in C^{2,1}(K)$ (i.e., $v, v_{xx}, v_t \in C(K)$) provided that h is Hölder continuous in K .

By the strong maximum principle for classical solutions we see

(iv) $v = S(h) > 0$ in K if $h \geq 0$ and $h \not\equiv 0$, where h is Hölder continuous.

For $u \in C(K)$ we define a continuous operator $H : C(K) \rightarrow C(K)$ by

$$H(u)(x, t) \equiv \phi(u(x, t), x, t).$$

Since ϕ is locally Lipschitz in w , we observe that

(v) if u is Hölder continuous in K so is $H(u)$.

Suppose that $w \in C(K)$ is a fixed point of $S \circ H$ i.e., $S \circ H(w) = w$. Then by (ii) w is Hölder continuous. This implies $S(H(w)) \in C^{2,1}(K)$ by (iii), (v). Since $\phi > 0$ for $w > 0$,

(vi) $H(u) \geq 0$ and $H(u) \not\equiv 0$ unless $u \equiv 0$, so $w = S(H(w)) > 0$ in K if $w \not\equiv 0$ by (iv).

The equivalence of (5.4) and $S \circ H(w) = w$ is formally easy. We now obtain a positive solution $w \in C^{2,1}(K)$ of (5.4). Of course, this solution is in $C^\infty(K)$ since $f \in C^\infty(K)$ by the higher regularity theory [LSU].

6.3. Lemma on the local degree near zero. *There is a constant $r_0 > 0$ such that the (Leray-Schauder) degree of $I - S \circ H$ of the value zero in $B_r(0)$ equals one for $0 < r < r_0$, i.e.,*

$$\deg(I - S \circ H, B_r(0), 0) = 1,$$

where I denotes the identity operator and

$$B_r(0) = \{h \in C(K); \|h\|_\infty \leq r\}.$$

Proof. Since S is compact and S, H are continuous, the degree of $I - S \circ \tau H$ is well-defined for $0 \leq \tau \leq 1$. Since $S(0) = 0$ by $b > 0$, the homotopy invariance of degree implies

$$\deg(I - S \circ H, B_r(0), 0) = \deg(I, B_r(0), 0) = 1$$

provided that

$$(6.2) \quad (S \circ \tau H)(u) \neq u$$

for all $u \in \partial B_r(0)$, $0 \leq \tau \leq 1$ if r is sufficiently small.

It remains to prove (6.2). We argue by a contradiction. We may assume that there were $\{\tau_n\}$ ($0 < \tau_n \leq 1$) and $\{u_n\}$ such that $u_n \neq 0$ and

$$(S \circ \tau_n H)(u_n) = u_n \quad \text{with} \quad \|u_n\|_\infty \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

By §6.2 (iv)-(vi) we observe that $u_n > 0$ and solves

$$(6.3) \quad u_t = (u + \varepsilon^2)^2(u_{xx} - bu + \tau H(u)) \quad \text{in} \quad K$$

with $\tau = \tau_n$ in a classical sense. By (5.1)-(5.3) we see

$$f - \frac{\varepsilon}{\xi(u_n + \varepsilon^2)} \geq \frac{1}{2} \min_K f > 0.$$

Since $\|u_n\|_\infty \rightarrow 0$, this would imply

$$G_n \equiv f - \frac{\varepsilon}{\xi(u_n + \varepsilon^2)} - u_n > 0 \quad \text{for large } n.$$

Multiplying (6.3) _{τ} with $(u + \varepsilon^2)^{-2}$ and integrating over K with $u = u_n$, $\tau = \tau_n$ yields

$$0 = \int_0^T \int_0^{2\pi} u_n \left\{ b(1 - \tau_n) + \frac{\tau_n u_n}{(u_n + \varepsilon^2)^2} G_n \right\} dx dt$$

as in the proof of Lemma 3.1. Since $0 < \tau_n \leq 1$, $b > 0$ and $u_n > 0$, this implies $G_n = 0$.

This contradicts the positivity of G_n for large n . We have thus proved (6.2).

6.4. Lemma on the degree in a large ball. *There is $R_0 > 0$ such that*

$$\deg(I - S \circ H, B_R(0), 0) = 0 \quad \text{for} \quad R > R_0.$$

Proof. We set

$$\Phi(w) = bw_+ + \frac{w^2}{(w_+ + \varepsilon^2)^2} w_+ + 1 \quad \text{for} \quad w \in \mathbb{R}$$

and its affine homotopy to H :

$$\Phi_\tau(w) = \tau H(w) + (1 - \tau)\Phi(w), \quad 0 \leq \tau \leq 1$$

for $w \in C(K)$. We first observe that

$$\deg(I - S \circ \Phi, B_R(0), 0) = 0 \quad \text{for} \quad R > 0.$$

Indeed, since Φ_τ also satisfies (v)(vi) of §6.2 with $H = \Phi_\tau$, if $S \circ \Phi(u) = u$, then $u > 0$ and it is a $C^{2,1}$ solution of

$$u_t = (u + \varepsilon^2)^2(u_{xx} + \frac{u^2}{(u + \varepsilon^2)^2}u + 1) \quad \text{in } K.$$

At the minimum point of u we see $u_t = 0$ and $u_{xx} \geq 0$ but this is impossible by the equation for u . So there is no fixed point of $S \circ \Phi$.

By the homotopy invariance of degree the proof will be complete if we prove *a priori* bound (independent of τ) for w satisfying $S \circ \Phi_\tau(w) = w$ so that we set

$$R_0 = \sup\{\|w\|_\infty; S \circ \Phi_\tau(w) = w, 0 \leq \tau \leq 1\}.$$

As in §6.2, $S \circ \Phi_\tau(w) = w$ implies that w is positive and it is a $C^{2,1}$ solution of

$$(6.4) \quad w_t = (w + \varepsilon^2)^2\{w_{xx} - bw + \Phi_\tau(w)\} \quad \text{in } K.$$

Since f is smooth, so is w . The following *a priori* bound is sufficient for our purpose.

6.5. Upper bound Lemma. *There is a constant $R_1 > 0$ such that if $w \in C^\infty(K)$ with $w > 0$ solves (6.4) then $w \leq R_1$ in K .*

Proof. In (6.4) we set $w + \varepsilon^2 = u$ to get

$$u_t = u^2[u_{xx} + \varphi_\varepsilon(u)\{u + \tau\psi_\varepsilon(u) + (1 - \tau)/\varphi_\varepsilon(u) - \tau f - \varepsilon^2\}]$$

with $\varphi_\varepsilon, \psi_\varepsilon$ satisfying (5.6). If we denote

$$\varphi = \varphi_\varepsilon, \quad \psi = \tau\psi_\varepsilon + (1 - \tau)/\varphi_\varepsilon,$$

then $u > \varepsilon^2$ solves

$$u_t = u^2(u_{xx} + \varphi(u)\{u + \psi(u) - \tau f - \varepsilon^2\}).$$

It suffices to check assumptions of Theorem 3.2 by setting $\alpha = \varepsilon^2$. By (5.9) all we have to prove are

$$(6.5) \quad \varphi'(v)(\psi(v) - \tau f - \varepsilon^2) + \varphi(v)\psi'(v) \leq 0, \quad \text{for } v > \varepsilon^2,$$

$$(6.6) \quad \varphi(v)(\psi(v) - \min_K(\tau f + \varepsilon^2)) \leq c_6(v + 1), \quad \text{for } v > \varepsilon^2.$$

The left hand side of (6.5) is

$$\begin{aligned} & \varphi'(\tau(\psi_\varepsilon - f) - \varepsilon^2 + (1 - \tau)\varphi^{-1}) + \varphi(\tau\psi'_\varepsilon + (1 - \tau)(\varphi^{-1})') \\ & \leq \varphi'\varphi^{-1}(1 - \tau) + \varphi(\varphi^{-1})'(1 - \tau) = (1 - \tau)(\varphi'\varphi^{-1} - \varphi\varphi'\varphi^{-2}) = 0 \end{aligned}$$

since $\varphi' \geq 0$, $\psi_\varepsilon - f < 0$, $\psi'_\varepsilon \leq 0$ on $[\varepsilon^2, \infty)$ by (5.1)-(5.3); here $\varphi^{-1} = 1/\varphi$. The left hand side of (6.6) is

$$\varphi(\tau(\psi_\varepsilon - \min_K f) + (1 - \tau)\varphi^{-1} - \varepsilon^2) \leq \varphi\varphi^{-1}(1 - \tau) \leq 1.$$

We have thus proved (6.5) and (6.6) (with $c_6 = 1$).

6.6. Remark. For semilinear equation

$$\begin{aligned} u_t &= \Delta u + m(t)u^p & \text{in } (0, T) \times \Omega \\ u &= 0 & \text{on } (0, T) \times \partial\Omega \end{aligned}$$

with T -periodic m Esteban [E1] [E2] obtained a positive T -periodic solution u provided that $1 < p < n/(n-2)$ where Ω is a smoothly bounded domain in $\mathbb{R}^n$. The key argument is to prove similar statement to Lemmas 6.4 and 6.5. The method to get upper bound in [E1], [E2] is based on a blow up argument. This is used in [Gi] to get a bound for positive solutions of the initial-boundary value problem

$$u_t = \Delta u + u^p \quad \text{in } (0, \infty) \times \Omega$$

with $u = 0$ on $(0, \infty) \times \partial\Omega$ for $1 < p < (n+2)/(n-2)$. However, it seems to be difficult to obtain an upper bound by this method in our problem. Application of Harnack's inequality to get a bound seems to be new.

6.7. Existence Theorem for approximate penalized equation. *Assume that $f \in C^\infty(K)$ and $f > 0$. Then for every $\varepsilon > 0$ there is a positive solution $w \in C^\infty(K)$ of (5.4) with $\varepsilon' = \varepsilon^2$ and (5.1)-(5.3).*

Proof. By Lemmas 6.3 and 6.4 we have

$$\deg(I - S \circ H, B_R(0) \setminus B_r(0), 0) = -1$$

for $r < R$ with $r < r_0$, $R > R_0$. It now follows that there exists $w \neq 0$ such that $S \circ H(w) = w$. This yields a desired solution w of (5.4) as remarked at the end of §6.2.

6.8. Proof of Theorem 1.1. We approximate f by $f_\varepsilon \in C^\infty(K)$ satisfying (1.3) such that

$$f_\varepsilon \rightarrow f \text{ in } C(K) \text{ as } \varepsilon \rightarrow 0 \quad \text{with} \quad \|f_{\varepsilon t}\|_\infty \leq 2\|f_t\|_\infty, \quad \|f_\varepsilon\|_\infty \leq 2\|f\|_\infty, \quad 0 < \varepsilon < 1.$$

Of course this is possible by convoluting f with a mollifier in space and a mollifier in time so that (1.3) is inherited to f_ϵ .

Let w_ϵ be a solution of the approximate penalized equation (5.4) (with $\epsilon' = \epsilon^2 > 0$ and $f = f_\epsilon$) obtained in Theorem 6.7. By Proposition 5.1 and Theorem 5.7 we have $\delta \leq w_\epsilon + \epsilon^2 \leq M_0$ with $M_0, \delta > 0$ independent of sufficiently small $\epsilon > 0$, say, $\epsilon < \epsilon_0$. This implies that w_ϵ solves

$$w_t = a_\epsilon(w)(w_{xx} - w + F_\epsilon) \quad \text{in } K$$

with $a_\epsilon(\sigma) = ((\sigma + \epsilon^2)^2 \vee \delta^2) \wedge M_0^2$ and $\|F_\epsilon\|_\infty \leq N_0$ for $0 < \epsilon < \epsilon_0$, where N_j is a constant depending only on $T, \|f\|_\infty, \|f_t\|_\infty, \min_K f$. Since $a_\epsilon(\sigma) \geq \delta^2$, from Lemma 6.1 it follows that

$$\|w_\epsilon\|_{W_p^{2,1}} \leq C_0 \|F_\epsilon\|_\infty \leq N_1, \quad 0 < \epsilon < \epsilon_0$$

at least for some $p > 2$. Applying the Sobolev inequality yields a bound on ν -Hölder norm of w_ϵ for some $0 < \nu < 1$:

$$\|w_\epsilon\|_{C^\nu} \leq N_2, \quad 0 < \epsilon < \epsilon_0.$$

By the Banach-Alaoglu theorem and the Ascoli-Arzelà theorem there is a subsequence (still denoted w_ϵ) and $u \in W_p^{2,1} \cap C^\nu$ such that

$$w_\epsilon \rightarrow u \quad \text{weakly in } W_p^{2,1}, w_\epsilon \rightarrow u \quad \text{strongly in } C(K).$$

Letting $\epsilon \rightarrow 0$ in (5.4) now yields (1.1) for u since $\epsilon/\xi(w_\epsilon + \epsilon^2) \rightarrow 0$ by $w_\epsilon + \epsilon^2 \geq \delta$. The higher regularity

$$u \in W_p^{2,1}(K) \quad \text{for all } p > 1 \quad \text{or}$$

$$u \in C^\infty(K) \quad \text{for } f \in C^\infty(K)$$

follows from the standard linear L^p and Schauder regularity theory [LSU] since the equation (1.1) is uniformly parabolic if $\delta^2 \leq u \leq M^2$.

It remains to prove the constraint (1.2). As in the Introduction, if u solves (1.1) and f satisfies (1.3), then

$$-\frac{d}{dt} \int_0^{2\pi} \frac{\sin(x - \zeta)}{u(x, t)} dx = - \int_0^{2\pi} f \sin(x - \zeta) dx = 0, \quad t, \zeta \in \mathbb{R}.$$

Applying Lemma 5.4 yields

$$\int_0^T \int_0^{2\pi} \frac{\sin(x - \zeta)}{u(x, t)} dx dt = 0.$$

We thus conclude that

$$\int_0^{2\pi} \frac{\sin(x - \zeta)}{u(x, t)} dx = 0 \quad \text{for all } t, \zeta \in \mathbb{R}$$

which is the same as (1.2).

7 Unique solvability of a class of quasilinear equations.

Our goal in this section is to prove Lemma 6.1. We first prove uniqueness of solution by reflecting Lipschitz continuity of solutions w.r.t. data in L^1 . In the proof we use smoothed signature functions which was often used to prove L^1 -contractivity for scalar conservation law; see Crandall [Cr]. We next establish a priori bounds for linear parabolic equation by a perturbation method. Such a method is often used in a different context; see Campanato [Ca] and [GYo].

7.1. Uniqueness Lemma. *Suppose that $v_i \in C(K_\omega)$ solves (6.1) with $h = h_i \in C(K_\omega)$ with $i = 1, 2$, where $b > 0$. Suppose that $v_i \in W_q^{2,1}(K_\omega)$ for some $1 < q < \infty$ for $i = 1, 2$. Then*

$$b \int_0^T \int_0^\omega |v_1 - v_2| dx dt \leq \int_0^T \int_0^\omega |h_1 - h_2| dx dt.$$

In particular solution of (6.1) is unique in $C(K_\omega) \cap W_q^{2,1}(K_\omega)$.

Proof. Divide (6.1)_i with $a(v_i)$ and subtract one equation from the other to get

$$(7.1) \quad A(v_1)_t - A(v_2)_t - w_{xx} + bw + k = 0 \quad \text{in } K = K_\omega$$

with $w = v_1 - v_2$, $k = h_2 - h_1$ and $A(s) = \int_0^s a^{-1}(r) dr$. Let $\text{sgn}_\rho(s)$, $\rho > 0$ denote the piecewise linear continuous function such that $\text{sgn}_\rho(s) = 1$ for $s \geq \rho$, $\text{sgn}_\rho(s) = -1$ for $s \leq -\rho$ and sgn_ρ is linear for $|s| \leq \rho$. Multiplying (7.1) with $\text{sgn}_\rho(w)$ and integrating it on K yields

$$(7.2) \quad 0 = \int_0^T \int_0^\omega \{(A(v_1) - A(v_2))_t \text{sgn}_\rho(w) - w_{xx} \text{sgn}_\rho w + bw \text{sgn}_\rho w + k \text{sgn}_\rho w\} dx dt.$$

Note that

$$- \int_0^\omega w_{xx} \text{sgn}_\rho w dx = \int_0^\omega w_x \text{sgn}'_\rho(w) w_x dx \geq 0$$

since $\text{sgn}'_\rho \geq 0$. Sending ρ to 0 in (7.2) yields

$$\int_0^T \int_0^\omega \{(A(v_1) - A(v_2))_t \text{sgn}(w) + b|w|\} dx dt \leq \int_0^T \int_0^\omega |k| dx dt.$$

It remains to prove

$$(7.3) \quad \int_0^T \int_0^\omega \{(A(v_1) - A(v_2))_t \operatorname{sgn}(w)\} dx dt = 0.$$

Since $a(r) > 0$ we observe that $\operatorname{sgn} w = \operatorname{sgn}(W)$ with

$$W = A(v_1) - A(v_2).$$

Thus

$$\int_0^T \int_0^\omega W_t \operatorname{sgn} W dx dt = \int_0^T \int_0^\omega |W|_t dx dt = 0,$$

since W is T -periodic in time. This is the same as (7.3). $\square$

We next derive *a priori* estimates for linear parabolic equation of form

$$(7.4) \quad u_t = a(x, t)(u_{xx} - bu + h) \quad \text{in } K_\omega$$

with a constant $b > 0$. We use a perturbation method.

7.2. Linear estimate Lemma. *Let a_i be a positive constant ($i = 0, 1$). There exist positive constants $\theta_1 = \theta_1(a_i, b, \omega, T)$ and $C_* = C_*(a_i, b, \omega, T)$ such that the following estimate holds. For $a, h \in C(K_\omega)$ with $a_0 \leq a \leq a_1$ there is a unique solution $u \in W_p^{2,1}(K_\omega)$ satisfying*

$$(7.5) \quad \|u\|_{W_p^{2,1}} \leq C_* \|h\|_\infty, \quad \text{for } 2 \leq p \leq \theta_1 + 2.$$

The solution u always satisfies

$$(7.6) \quad \|u\|_\infty \leq b^{-1} \|h\|_\infty.$$

Proof. We may assume $a_1 = 1$ by dilation of the time variable. We may assume that $a, h \in C^\infty(K)$ with $K = K_\omega$. Indeed, for $a, h \in C(K)$ there are $a_\varepsilon, h_\varepsilon \in C^\infty(K)$ such that $a_\varepsilon \rightarrow a, h_\varepsilon \rightarrow h$ in $C(K)$ as $\varepsilon \rightarrow 0$ and that $a_0 \leq a_\varepsilon \leq a_1 = 1$. There is a unique solution u_ε of (7.4) with $a = a_\varepsilon, h = h_\varepsilon$. By our results for u_ε we have

$$(7.7) \quad \|u_\varepsilon\|_{W_p^{2,1}} \leq C_* \|h_\varepsilon\|_\infty, \quad \|u_\varepsilon\|_\infty \leq b^{-1} \|h_\varepsilon\|_\infty$$

for $2 \leq p \leq 2 + \theta$. We thus observe that

$$u_\varepsilon \rightarrow u_0 \text{ weakly in } W_p^{2,1} \text{ and } u_\varepsilon \rightarrow u_0 \text{ strongly in } C(K)$$

as $\varepsilon \rightarrow 0$ by taking a subsequence if necessary. If $w \in W_2^{2,1}(K)$ solves

$$w_t = a(w_{xx} - bw) \quad \text{in } K$$

then we see $w = 0$. Indeed, multiplying the equation with w_t/a and integrating on K by parts yields $w_t = 0$ so that $w = 0$. Thus solution $u \in W_2^{2,1}$ for (7.4) is unique. Since u_0 solves (7.4) it follows that $u = u_0$. Sending $\varepsilon \rightarrow 0$ in (7.7) yields (7.5)-(7.6).

We first admit that for (smooth) a there is a unique solution $u \in W_2^{2,1}(K)$ of (7.4) for $h \in L^2(K)$. This solution $u \in C^\infty(K)$ if h is smooth by the standard regularity theory.

By the maximum principle for $u \in C^\infty(K)$ we see

$$\min_K h \leq bu \leq \max_K h.$$

This yields (7.6).

We next consider L^2 estimate of the linear equation with constant coefficients:

$$(7.8) \quad u_t = u_{xx} - bu + g \quad \text{on } K.$$

Multiplying (7.8) with u_{xx} and integrating over K by parts yields

$$\int_0^T \int_0^\omega (u_{xx}^2 + bu_x^2) dx dt = - \int_0^T \int_0^\omega g u_{xx} dx dt.$$

Thus by the Cauchy inequality we have

$$\iint (u_{xx}^2 + bu_x^2) dx dt \leq \frac{1}{2} \iint (u_{xx}^2 + g^2) dx dt,$$

which in particular yields estimates of $L^2(K)$ norm:

$$(7.9) \quad \|u_{xx}\|_2 \leq 1 \cdot \|g\|_2$$

Let L_3 denote the operator norm of $g \mapsto u_{xx}$ from $L^3(K)$ into itself. By L^p theory [LSU] L_3 is finite and depending only on ω, T, b . Applying Riesz' interpolation between $L^2(K)$ and $L^3(K)$ to (7.9) yields

$$(7.10) \quad \|u_{xx}\|_p \leq L_p \|g\|_p$$

with $L_p \rightarrow 1$ as $p \rightarrow 2$, $p > 2$, where $L_p = L_p(\omega, T, b)$.

We shall use a perturbation argument. Rewrite (7.4) as $u_t = u_{xx} - bu + g$ with $g = (a-1)(u_{xx} - bu + h) + h$ and apply (7.10) to get

$$\|u_{xx}\|_p \leq L_p [\|h\|_p + (1-a_0)(\|u_{xx}\|_p + b\|u\|_p + \|h\|_p)].$$

We take θ_1 such that $2 \leq p \leq 2 + \theta_1$ implies

$$c = \sup\{L_p; 2 \leq p \leq 2 + \theta_1\} < \frac{1}{1 - a_0}$$

since $L_p \rightarrow 1$ as $p \rightarrow 2$. We thus have the estimate

$$\|u_{xx}\|_p \leq c_* L_p(2\|h\|_p + b\|u\|_p) \quad \text{with} \quad c_* = (1 - c(1 - a_0))^{-1}$$

Since (7.6) yields

$$\|u\|_p \leq S\|u\|_\infty \leq Sb^{-1}\|h\|_\infty \quad \text{with} \quad S = S(T, \omega, p),$$

we now observe that

$$\|u_{xx}\|_p \leq c_* L_p(2 + 1)S\|h\|_\infty \quad \text{for} \quad 2 \leq p \leq 2 + \theta_1.$$

Interpolating (7.6) with this inequality, we conclude that

$$\|u\|_{W^{2,1}} \leq C_* \|h\|_\infty$$

for $2 \leq p \leq 2 + \theta_1$ by estimating u_t by the equation (7.4).

It remains to prove the unique existence of solution in $W^{2,1}_2$ of (7.4) for $h \in L^2$. As in the first paragraph of the proof, we have the uniqueness of solution of

$$-\varphi_t = (a\varphi)_{xx} - b(a\varphi) \quad \text{in} \quad K.$$

By the regularity theory only L^2 solution is zero. This implies that the dual operator L^* of the bounded linear operator $L : u \mapsto h$ (defined by (7.4)) from $W^{2,1}_2$ to L^2 is injective. The closed range theorem now implies that L is surjective provided that the range of L is closed. For (7.4) in the same way to obtain (7.9) we get

$$\|u_{xx}\|_2 + \|u_x\|_2 \leq c\|Lu\|_2.$$

Multiplying u_t with (7.4) and integrating now yields

$$\|u_t\|_2 \leq C'\|Lu\|_2.$$

Since $\int uu_t dt = 0$. Use (7.4) to get

$$\|u\|_{W^{2,1}_2} \leq C''\|Lu\|_2$$

which implies the range of L is closed, so the proof is now complete.

7.3. A priori estimate. We consider

$$(7.11) \quad v_t = a_\tau(v)(v_{xx} - bv + h) \quad \text{in } K = K_\omega$$

with $a_\tau(p) = \tau a(p) + (1 - \tau)a_0$, $0 \leq \tau \leq 1$, where $a(p) \geq a_0$ is given in Lemma 6.1 and $b > 0$. The estimate (7.6) yields

$$\|v\|_\infty \leq b^{-1}\|h\|_\infty$$

for a $W_2^{2,1}$ solution v of (7.11) with $v \in C(K)$. We set

$$(7.12) \quad a_1 = \max\{a(p); |p| \leq b^{-1}\|h\|_\infty\} < \infty$$

to get

$$a_0 \leq a_\tau(v) \leq a_1$$

for a solution $v \in C(K)$ of (7.11). By Lemma 7.2 there are positive constants θ_1 and C_* such that for $2 \leq p \leq 2 + \theta_1$ a solution $v \in C(K) \cap W_2^{2,1}(K)$ fulfills

$$(7.13) \quad \|v\|_{W_2^{2,1}} \leq C_* \|h\|_\infty.$$

Here θ_1, C_* only depends on $\|h\|_\infty, a_0, b, \omega, T$.

7.4. A homotopy of mappings. We fix p such that $2 < p < 2 + \theta_1$. Since $K = K_\omega$ is two-dimensional, there is $0 < \nu < 1$ such that the inclusion $W_p^{2,1}(K)$ into $C^\nu(K)$ is continuous by the Sobolev inequality. We fix ν and set $X = C^\nu(K)$. For $(v, \tau) \in X \times [0, 1]$ let $w = R(v, \tau)$ be the unique solution of

$$(7.14) \quad w_t = a_\tau(v)(w_{xx} - bw + \tau h) \quad \text{in } K.$$

7.5. Proposition. *The mapping R gives a compact, continuous mapping from $X \times [0, 1]$ into X . The image of R is contained in $\bigcap_{q>1} W_q^{2,1}(K)$. Moreover $R(v, 0) = 0$ for all $v \in X$.*

Proof. For $v \in X$ (7.6) implies

$$a_0 \leq a_\tau(v) \leq a_1$$

where a_1 is defined by (7.12). Suppose that $\|v\|_X \leq M$. Then $a(v)$ has a modulus of continuity m depending on v only through M . Since m is a modulus of continuity of a_τ for $0 \leq \tau \leq 1$, L^q -theory of linear parabolic equation provides

$$\|R(v, \tau)\|_{W_q^{2,1}} \leq Z \|h\|_q \quad \text{for } \|v\|_X \leq M$$

for $1 < q < \infty$ with $Z = Z(a_0, a_1, q, M)$ where a_1 is as in (7.12). We observe that the image of R is contained in $W_q^{2,1}(K)$ for all q . Take q large so that $W_q^{2,1} \subset X$ is compact. Thus

$$\{R(v, \tau); \|v\|_X \leq M, 0 \leq \tau \leq 1\}$$

is compact in X , so $R : X \times [0, 1] \rightarrow X$ is compact.

From (7.14) it is clear that $R(v, 0) = 0$.

It remains to prove the continuity of R . Suppose that $v_j \rightarrow v$ in X , $\tau_j \rightarrow \tau$ as $j \rightarrow \infty$. We apply L^q theory to estimate (for $1 < q < \infty$)

$$\|u_j\|_{W_q^{2,1}} \leq B, \quad u_j = R(v_j, \tau_j)$$

with $B = B(a_0, a_1, q, \|h\|_\infty)$ since $\{v_j\}$ is bounded so that $a_{\tau_j}(v_j)$ has a modulus of continuity independent of j .

For sufficiently large q we observe that

$$(7.15) \quad u_j \rightarrow u \text{ weakly in } W_q^{2,1}(K) \text{ and } u_j \rightarrow u \text{ in } C^\nu(K)$$

for some $u \in X \cap W_q^{2,1}(K)$ by taking a subsequence if necessary. Sending $j \rightarrow \infty$ in

$$u_{jt} = a_{\tau_j}(v_j)(u_{jxx} - bu_j + \tau_j h)$$

now yields (7.14) with $w = u$. Since (7.14) admits a unique solution for given v , we see $u = R(v, \tau)$. We have thus proved that $R : X \times [0, 1] \rightarrow X$ is continuous.

7.6. Proof of Lemma 6.1. Since ν is chosen so that $W_p^{2,1}(K) \subset C^\nu(K) = X$ is continuous, a priori estimate (7.13) yields

$$\|v\|_X \leq SC_* \|h\|_\infty$$

with $S = S(p, \nu, \omega, T)$. By Proposition 7.5 and this estimate the Leray-Schauder fixed point theorem yields $v \in X$ such that $R(v, 1) = v$. By (7.15) we see $v \in \bigcap_{q>1} W_q^{2,1}(K) \subset C(K)$. $R(v, 1) = v$ means v solves (6.1). A priori estimate of Lemma 6.1 follows from (7.13). Since solution of (6.1) is unique, it remains to prove that $h \mapsto v$ is compact, continuous from $C(K)$ into itself.

The compactness follows from a priori estimates in Lemma 6.1 since K is two-dimensional. Suppose that $h_j \rightarrow h$ in $C(K)$ and that v_j is a solution of (6.1) with $h = h_j$. By a priori estimates there is v such that

$$v_j \rightarrow v \text{ weakly in } W_p^{2,1} \text{ and } v_j \rightarrow v \text{ in } C(K) (j \rightarrow \infty)$$

by taking some subsequence if necessary, since K is two-dimensional. Letting $j \rightarrow \infty$ in (6.1) with $v = v_j$, $h = h_j$ yields (6.1) for v and h . This implies the continuity of $h \mapsto v$.

8 Appendix.

We shall prove that (1.1) with constraint (1.2) admits at most one positive solution if f is time independent. We shall also give a geometric interpretation of the assumption (1.8).

8.1. Proof of Remark 1.3. Multiplying (1.4) with u_t and integrating by parts over K yields

$$\int_0^T \int_0^{2\pi} \frac{u_t^2}{u^2} dx dt = \int_0^T \int_0^{2\pi} \left\{ -\frac{1}{2}(u_x^2)_t + \frac{1}{2}(u^2)_t - f u_t \right\} dx dt = 0$$

if f is time independent. This implies $u_t = 0$ on K .

It remains to prove that

$$(8.1) \quad u_{xx} + u = f(x) \quad \text{on} \quad T = \mathbb{R}/2\pi\mathbb{Z}$$

admits at most one positive solution satisfying (1.2). Suppose that $u > 0$ solves (8.1) with (1.2). Then all positive solution U of (8.1) is of the form

$$U(x; \xi, \eta) = u(x) + \xi \cos x + \eta \sin x$$

for some $\xi, \eta \in \mathbb{R}$. It is easy to see that

$$D = \{(\xi, \eta); U(x; \xi, \eta) > 0 \quad \text{for all} \quad x \in T\} \subset \mathbb{R}^2$$

is open and convex. Since $-\log$ is strictly convex, so is the function

$$E(\xi, \eta) = \int_0^{2\pi} -\log U(x, \xi, \eta) dx \quad \text{on} \quad D.$$

The strictly convexity of E on D implies that E has at most one critical point. Since

$$\frac{\partial E}{\partial \xi} = \int_0^{2\pi} -\frac{\cos x}{U} dx, \quad \frac{\partial E}{\partial \eta} = \int_0^{2\pi} -\frac{\sin x}{U} dx,$$

$U(x; \xi, \eta)$ satisfies (8.1) with (1.2) if and only if (ξ, η) is a critical point of E in D . By the choice of u , E takes its critical point at the origin, so there is no other critical point of E on D . In other words there is no $U(x, \xi, \eta)$ satisfying (1.2) unless (ξ, η) is the origin. We have thus proved the uniqueness of positive solution u of (8.1) fulfilling the constraint (1.2).

8.2. Curvature of Frank diagram. Let $\mathcal{F}$ be the Frank diagram of $Q > 0$ as in §1.3. Let $\kappa(\theta)$ be the inward curvature of $\mathcal{F}$ at $p = (Q(\theta))^{-1}(\cos \theta, \sin \theta)$. Then

$$(8.2) \quad \kappa = Q^3(Q + Q'')/(Q'^2 + Q^2)^{3/2}.$$

In particular, (1.8) is equivalent to say that the curvature of $\mathcal{F}$ is positive.

Proof. For $p \in \mathbb{R}^2$ we define $q(p)$ by

$$p = q(p)(Q(\theta))^{-1}(\cos \theta, \sin \theta),$$

where θ is the argument of p . This q is a well defined positive function (except the origin) and is positively homogeneous of degree one. Moreover, $\mathcal{F}$ is 1-level set of q . We observe that

$$(8.3) \quad \kappa = \operatorname{div}\left(\frac{\nabla q}{|\nabla q|}\right) = \frac{1}{(q_1^2 + q_2^2)^{3/2}}(q_2^2 q_{11} - 2q_1 q_2 q_{12} + q_1^2 q_{22}),$$

where $q_i = \partial q / \partial p_i$, $i = 1, 2$.

We may assume that $\theta = 0$ and $Q(0) = 1$ by rotation and dilation of coordinates. Since q is homogeneous of degree one so that q_i is of degree zero, we see

$$(8.4) \quad q = q_1 \cos \theta + q_2 \sin \theta, \quad 0 = q_{11} \cos \theta + q_{12} \sin \theta, \quad 0 = q_{21} \cos \theta + q_{22} \sin \theta,$$

where functions are evaluated at $(\cos \theta, \sin \theta)$. From these identities it follows that $q_{21}(p_*) = 0$, $q_{11}(p_*) = 0$, and $q_1(p_*) = q(p_*) = 1$ for $p_* = (1, 0)$ since $Q(\theta) = q(\cos \theta, \sin \theta)$. Differentiate Q in θ and use (8.4)₁ to get

$$Q'(\theta) = -q_1 \sin \theta + q_2 \cos \theta$$

$$Q''(\theta) + Q(\theta) = q_{11} \sin^2 \theta - 2q_{12} \cos \theta \sin \theta + q_{22} \cos^2 \theta$$

which yields

$$q_2(p_*) = Q'(0) \quad \text{and} \quad q_{22}(p_*) = Q''(0) + Q(0) = Q''(0) + 1.$$

Plugging these values in (8.3) we obtain

$$\kappa(0) = \frac{1}{(1 + Q'(0)^2)^{3/2}}(Q''(0) + 1).$$

This completes the proof.

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