# REALIZATION THEOREMS OF GEOMETRIC SINGULARITIES FOR HAMILTON-JACOBI EQUATIONS

S. Izumiya and G. T. Kossioris

Series ♯218. November 1993

## HOKKAIDO UNIVERSITY
## PREPRINT SERIES IN MATHEMATICS

♯ 189: T. Nakazi, Multipliers of invariant subspaces in the bidisc, 12 pages. 1993.

♯ 190: S. Izumiya, G.T. Kossioris, Semi-local classification of geometric singularities for Hamilton-Jacobi equations, 24 pages. 1993.

♯ 191: N. Hayashi, T. Ozawa, Finite energy solutions of nonlinear Schrödinger equations of derivative type, 21 pages. 1993.

♯ 192: J. Seade, T. Suwa, A residue formula for the index of a holomorphic flow, 22 pages. 1993.

♯ 193: H. Kubo, Blow-up of solutions to semilinear wave equations with initial data of slow decay in low space dimensions, 8 pages. 1993.

♯ 194: F. Hiroshima, Scaling limit of a model of quantum electrodynamics, 52 pages. 1993.

♯ 195: T. Ozawa, Y. Tsutsumi, Global existence and asymptotic behavior of solutions for the Zakharov equations in three space dimensions, 34 pages. 1993.

♯ 196: H. Kubo, Asymptotic behaviors of solutions to semilinear wave equations with initial data of slow decay, 25 pages. 1993.

♯ 197: Y. Giga, Motion of a graph by convexified energy, 32 pages. 1993.

♯ 198: T. Ozawa, Local decay estimates for Schrödinger operators with long range potentials, 17 pages. 1993.

♯ 199: A. Arai, N. Tominaga, Quantization of angle-variables, 31 pages. 1993.

♯ 200: S. Izumiya, Y. Kurokawa, Holonomic systems of Clairaut type, 17 pages. 1993.

♯ 201: K.-S. Saito, Y. Watatani, Subdiagonal algebras for subfactors, 7 pages. 1993.

♯ 202: K. Iwata, On Markov properties of Gaussian generalized random fields, 7 pages. 1993.

♯ 203: A. Arai, Characterization of anticommutativity of self-adjoint operators in connection with Clifford algebra and applications, 13 pages. 1993.

♯ 204: J. Wierzbicki, An estimation of the depth from an intermediate subfactor, 7 pages. 1993.

♯ 205: N. Honda, Vanishing theorem for the tempered distributions, 11 pages. 1993.

♯ 206: T. Hibi, Betti number sequences of simplicial complexes, Cohen-Macaulay types and Möbius functions of partially ordered sets, and related topics, 25 pages. 1993.

♯ 207: A. Inoue, Regularly varying correlations, 23 pages. 1993.

♯ 208: S. Izumiya, B. Li, Overdetermined systems of first order partial differential equations with singular solution, 9 pages. 1993.

♯ 209: T. Hibi, Hochster's formula on Betti numbers and Buchsbaum complexes, 7 pages. 1993.

♯ 210: T. Hibi, Star-shaped complexes and Ehrhart polynomials, 5 pages. 1993.

♯ 211: S. Izumiya, G. T. Kossioris, Geometric singularities for solutions of single conservation laws, 28 pages. 1993.

♯ 212: A. Arai, On self-adjointness of Dirac operators in Boson-Fermion Fock spaces, 43 pages. 1993.

♯ 213: K. Sugano, Note on non-commutative local field, 3 pages. 1993.

♯ 214: A. Hoshiga, Blow-up of the radial solitions to the equations of vibrating membrane, 28 pages. 1993.

♯ 215: A. Arai, Scaling limit of anticommuting self-adjoint operators and nonrelativistic limit of Dirac operators, 35 pages. 1993.

♯ 216: Y. Giga, N. Mizoguchi, Existence of periodic solutions for equations of evolving curves, 45 pages. 1993.

♯ 217: T. Suwa, Indices holomorphic vector fields relative to invariant curves, 10 pages. 1993.

# REALIZATION THEOREMS OF GEOMETRIC SINGULARITIES FOR HAMILTON-JACOBI EQUATIONS

SHYUICHI IZUMIYA[1] AND GEORGIOS T. KOSSIORIS[2]

## 0. INTRODUCTION

In ([6],[7]) we described the geometric framework for the study of generation and propagation of shock waves in $\mathbb{R}^n$ appearing in viscosity solutions of Hamilton-Jacobi equations

$$
\text{(P)} \qquad \begin{cases} \dfrac{\partial y}{\partial t} + H(t, x_1, \cdots, x_n, \dfrac{\partial y}{\partial x_1}, \cdots, \dfrac{\partial y}{\partial x_n}) = 0 \\ y(0, x_1, \cdots, x_n) = \phi(x_1, \cdots, x_n), \end{cases}
$$

where $H$ and $\phi$ are $C^\infty$-functions.

The *geometric solution* $y$ of (P) has been defined in ([6],[7]) in the framework of *one-parameter Legendrian unfoldings* and it is constructed by the method of characteristics. Although $y$ is initially smooth there is in general a critical time beyond which characteristics cross. The geometric solution past the critical time is multi-valued, that is singularities appear. The classification of singularities of $y$ has been studied in ([6],[7]). The purpose of the papers ([6],[7]) is to give a correct geometrical setting for describing the shock waves of viscosity solutions (cf.,[3],[4]) Existence and uniqueness of the solution of (P) in the viscosity sense have been established in [3]. The viscosity solution is continuous and coincides with the smooth geometric solution until the first critical time. After the characteristic cross, the viscosity solution developes *shock waves* i.e., surfaces across which the gradient of the viscosity solution is discontinuous. The viscosity solution of (P) in a neighbourhood of the first critical time has been constructed in [8] by selecting a continuous single-valued branch of the graph of the geometric solution. Our project is to continue this process and construct the viscosity solutions around each singularity of the geometric solution. So we need to classify the singularities of geometric solutions. For the purpose we have to prove that Legendrian unfoldings is the correct class of geometric solutions. In ([6],[7]) we presented the realization theorems for Legendrian unfoldings as geometric solutions of a given Hamilton-Jacobi equation. The purpose of this paper is to give more detailed results (cf., Theorems 3.2, 3.3). We can also specify the point at where the generic bifurcations of geometric singularities occur.

In [2] Bogaevskii constructed different geometric framework of the Cauchy problem in the case for the special kind of Hamiltonians. Here, our framework is so fundamental that we can apply the our geometric framework to general Hamiltonians. For example, the notion of nondegeneracy for the Hamiltonian in this paper is much weaker than the condtion that the Hamiltonian is convex or concave (cf., §3).

[1]Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060, Japan
[2]Department of Mathematics, University of Crete, Iraklion, Greece

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

We consider the case when the Hamiltonian depends only on momenta, $H(\frac{\partial y}{\partial x_1}, \cdots, \frac{\partial y}{\partial x_n})$ in §4. In this case we can state much stronger assertion for realizations of bifircations of singularities (cf., Proposition 4.1).

All maps considered here are differentiable of class $C^\infty$, unless stated otherwise.

## 1. Geometric framework for Hamilton-Jacobi equations

In the present paper we treat Hamilton-Jacobi equations in the framework of the geometric theory of first order partial differential equations described e.g., in [6]. In this section we briefly describe the geometric framework and present the necessary notation.

Let $J^1(\mathbb{R}^n, \mathbb{R})$ be the 1-jet bundle of functions of $n$-variables which may be considered as $\mathbb{R}^{2n+1}$ with a natural coordinate system $(x_1, \dots, x_n, y, p_1, \dots, p_n)$, where $(x_1, \dots, x_n)$ is a coordinate system of $\mathbb{R}^n$. We also have a natural projection $\pi : J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ given by $\pi(x, y, p) = (x, y)$.

An immersion germ $i : (L_0, u_0) \to J^1(\mathbb{R}^n, \mathbb{R})$ is said to be *a Legendrian immersion germ* (i.e., Legendrian submanifold germ) if $\dim L = n$ and $i^*\theta = 0$, where $\theta = dy - \sum_{i=1}^n p_i \cdot dx_i$. The image of $\pi \circ i$ is called *the wave front set* of $i$ and it is denoted by $W(i)$.

We also consider the 1-jet bundle $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and the canonical 1-form $\Theta$ on that space. Let $(t, x_1, \dots, x_n)$ be a canonical coordinate system on $\mathbb{R} \times \mathbb{R}^n$ and

$$(t, x_1, \dots, x_n, y, s, p_1, \dots, p_n)$$

the corresponding coordinate system on $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. Then, the canonical 1-form is given by

$$\Theta = dy - \sum_{i=1}^{n} p_i \cdot dx_i - s \cdot dt = \theta - s \cdot dt.$$

We define the natural projection $\Pi : J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \to (\mathbb{R} \times \mathbb{R}^n) \times \mathbb{R}$ by $\Pi(t, x, y, s, p) = (t, x, y)$. We call the above 1-jet bundle *an unfolded 1-jet bundle.*

*A Hamilton-Jacobi equation* is defined to be a hypersurface

$$E(H) = \left\{(t, x, y, s, p) \in J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) | s + H(t, x, p) = 0\right\}$$

in $J^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$. *A geometric solution* of $E(H)$ is a Legendrian submanifold $L$ in $J^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$ lying in $E(H)$.

We say that *a generalized Cauchy problem (GCP) (with initial condition $L'$) is given for an equation $E(H)$* if there is given an $n$-dimensional submanifold $i : L' \subset E(H)$ such that $i^*\Theta = 0$ and $X_H \notin T(L')$ at any point of $L'$ where $X_H$ is the characteristic vector field given by

$$X_H = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x_i} + (\sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i} - H) \frac{\partial}{\partial y} - \frac{\partial H}{\partial t} \frac{\partial}{\partial s} - \sum_{i=1}^{n} \frac{\partial H}{\partial x_i} \frac{\partial}{\partial p_i}.$$

We have the following existence theorem:

**Theorem 1.1. (Classical existence theorem [9]).** *A GCP $i : L' \subset E(H)$ has a unique solution, that is, there is a Legendrian submanifold $L \subset E(H)$, $L' \subset L$ and any two such Legendrian submanifolds coincide in a neighbourhood of $L'$.*

In order to study (P) we need a more restricted framework. For any $c \in (\mathbb{R}, 0)$, we define

$$E(H)_c = \left\{(c, x, y, -H(c, x, p), p) | (x, y, p) \in J^1(\mathbb{R}^n, \mathbb{R})\right\}.$$

Then, $E(H)_c$ is a $(2n+1)$-dimensional submanifold of $J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and $\Theta_c = \Theta|E(H)_c = dz - \sum_{i=1}^n p_i dx_i$ gives a contact structure on $E(H)_c$. We define a mapping $\iota_c : J^1(\mathbb{R}^n,\mathbb{R}) \to E(H)_c$ by $\iota_c(x,y,p) = (c,x,y,-H(c,x,p),p)$. The mapping $i_c$ is a contact diffeomorphism and the following diagram is commutative:

$$\begin{array}{ccc} J^1(\mathbb{R}^n,\mathbb{R}) & \xrightarrow{\iota_c} & E(H)_c \\ \pi\big\downarrow & & \big\downarrow\pi_c \\ \mathbb{R}^n\times\mathbb{R} & = & \mathbb{R}^n\times\mathbb{R}. \end{array}$$

We say that *a generalized Cauchy problem (with initial condition $L'$) associated with the time parameter(GCPT) is given for an equation* $E(H)$ if a GCP $i : L' \subset E(H)$ with $i(L') \subset E(H)_c$ for some $c \in (\mathbb{R},0)$ is given.

**Remark.** The Cauchy problem (P) is a GCPT. The initial submanifold is given by

$$L_{\phi,0} = \left\{(0,x,\phi(x),-H(0,x,\frac{\partial\phi}{\partial x}),\frac{\partial\phi}{\partial x})|x\in\mathbb{R}^n\right\} \subset E(H)_0.$$

The problem of studying the singularities of the graph of the geometric solution is formulated as follows:

**Geometric problem.** *Classify the generic bifurcations of wave fronts of*

$$\pi_t| : L\cap E(H)_t \to \mathbb{R}^n\times\mathbb{R}$$

*with respect to the parameter $t$ (i.e., the generic bifurcations of wave fronts of geometric solutions along the time parameter).*

Following [6], in order to study the singularities of the geometric solution we identify geometric solutions with one-parameter Legendrian unfoldings. Such a characterization, which is given in Section 3 permits the use of the available singularity theory of one-parameter Legendrian unfoldings. In the next section we present the necessary background material that we use in Section 3.

## 2. One parameter Legendrian unfoldings

We now describe the notion of one-parameter Legendrian unfoldings. Let $R$ be an $(n+1)$-dimensional smooth manifold, $\mu : (R,u_0) \to (\mathbb{R},t_0)$ be a submersion germ and $\ell : (R,u_0) \to J^1(\mathbb{R}^n,\mathbb{R})$ be a smooth map germ. We say that the pair $(\mu,\ell)$ is *a Legendrian family* if $\ell_t = \ell|\mu^{-1}(t)$ is a Legendrian immersion germ for any $t\in(\mathbb{R},t_0)$. Then we have the following simple but very important lemma.

**Lemma 2.1.** *Let $(\mu,\ell)$ be a Legendrian family. Then there exist a unique element $h \in C^\infty_{u_0}(R)$ such that $\ell^*\theta = h\cdot d\mu$, where $C^\infty_{u_0}(R)$ is the ring of smooth function germs at $u_0$.*

Define a map germ $\mathcal{L} : (R,u_0) \to J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ by

$$\mathcal{L}(u) = (\mu(u), x\circ\ell(u), y\circ\ell(u), h(u), p\circ\ell(u)).$$

We can easily show that $\mathcal{L}$ is a Legendrian immersion germ. If we fix 1-forms $\Theta$ and $\theta$, the Legendrian immersion germ $\mathcal{L}$ is uniquely determined by the Legendrian family $(\mu,\ell)$. We call $\mathcal{L}$ *a Legendrian unfolding associated with the Legendrian family* $(\mu, \ell)$. In order to study bifurcations of wave fronts of Legendrian unfoldings, we introduce the following equivalence relation. Let $\mathcal{L}_i : (R, u_i) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ $(i = 0, 1)$ be Legendrian unfoldings. We say that $\mathcal{L}_0$ and $\mathcal{L}_1$ are *P-Legendrian equivalent* if there exist a contact diffeomorphism germ

$$K : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_1)$$

of the form

$$K(t, x, y, s, p) = (\phi_1(t), \phi_2(t, x, y), \phi_3(t, x, y), \phi_4(t, x, y, s, p), \phi_5(t, x, y, s, p))$$

and a diffeomorphism germ $\Psi : (R, u_o) \to (R, u_1)$ such that $K \circ \mathcal{L}_0 = \mathcal{L}_1 \circ \Psi$.

In order to understand the meaning of P-Legendrian equivalence, we introduce the following equivalence: We say that *two wave front sets* $W(\mathcal{L}_0)$ *and* $W(\mathcal{L}_1)$ *have diffeomorphic bifurcations* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), \Pi(z_0)) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), \Pi(z_1))$$

of the form $\Phi(t, x, y) = (\phi_1(t), \phi_2(t, x, y), \phi_3(t, x, y))$ such that $\Phi(W(\mathcal{L})) = W(\mathcal{L}')$. If $\mathcal{L}_0$ and $\mathcal{L}_1$ are P-Legendrian equivalent then these wavefronts have diffeomorphic bifurcations. By the theorem of Zakalyukin [11; Assertion , Section 1.1], the converse is also true for generic Legendrian unfoldings. We can define the notion of stability with respect to the P-Legendrian equivalence in the same way as for the ordinary Legendrian stability (see [1],[11]).

Motivated by Arnol'd-Zakalyukin's theory ([1],[11]), we can construct generating families of Legendrian unfoldings. A function germ $F : ((\mathbb{R} \times \mathbb{R}^n) \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ is called a *a generalized phase function germ* if $d_2F|0 \times \mathbb{R}^n \times \mathbb{R}^k$ is non-singular, where $d_2F(t, x, q) = (\frac{\partial F}{\partial q_1}(t, x, q), \dots, \frac{\partial F}{\partial q_k}(t, x, q))$. Then $C(F) = d_2F^{-1}(0)$ is a smooth $(n+1)$-manifold germ and $\pi_F : (C(F), 0) \to \mathbb{R}$ is a submersion germ, where $\pi_F(t, x, q) = t$.

Define map germs $\tilde{\Phi}_F : (C(F), 0) \to J^1(\mathbb{R}^n, \mathbb{R})$ by

$$\tilde{\Phi}_F(t, x, q) = (x, F(t, x, q), \frac{\partial F}{\partial x}(t, x, q))$$

and $\Phi_F : (C(F), 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ by

$$\Phi_F(t, x, q) = (t, x, F(t, x, q), \frac{\partial F}{\partial t}(t, x, q), \frac{\partial F}{\partial x}(t, x, q)).$$

Since $\frac{\partial F}{\partial q_i} = 0$ on $C(F)$, we can easily show that $(\tilde{\Phi}_F)^*\theta = \frac{\partial F}{\partial t}|C(F) \cdot dt|C(F)$. By definition, $\Phi_F$ is a Legendrian unfolding associated with the Legendrian family $(\pi_F, \tilde{\Phi}_F)$. Following the lines of Arnol'd-Zakalyukin ([1],[11]), we can show the following proposition.

**Proposition 2.2.** *All Legendrian unfolding germs are constructed by the above method.*

We define a function germ $\tilde{F} : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ by $\tilde{F}(t, x, y, q) = F(t, x, q) - y$. We call $\tilde{F}$ *a generating family of* $\Phi_F$. We also consider an equivalence relation among generating families of Legendrian unfoldings. Let

$$\tilde{F}_i : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0) \ (i = 0, 1)$$

be generating families of $\Phi_{F_i}$. We say that $\tilde{F}_0$ and $\tilde{F}_1$ are *t-P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k, 0) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k, 0)$$

of the form

$$\Phi(t, x, y, q) = (\phi_1(t), \phi_2(t, x, y), \phi_3(t, x, y), \phi_4(t, x, y, q))$$

such that

$$\langle \tilde{F}_1 \circ \Phi \rangle_{\mathcal{E}_{(t,x,y,q)}} = \langle \tilde{F}_0 \rangle_{\mathcal{E}_{(t,x,y,q)}},$$

where $\langle \tilde{F}_0 \rangle_{\mathcal{E}_{(t,x,y,q)}}$ denotes the ideal generated by $\tilde{F}_0$ in the ring $\mathcal{E}_{(t,x,y,q)}$ of function germs of $(t, x, y, q)$ -variables at the origin. The definition of *the stable t-P-$\mathcal{K}$-equivalence* is given in the usual way (see [1],[11]).

For a generating family $\tilde{F}$ of $\Phi_F$, we define

$$T_e(\text{P-}\mathcal{K})(\tilde{f}) = \langle \frac{\partial \tilde{f}}{\partial q_1}, \ldots, \frac{\partial \tilde{f}}{\partial q_k}, \tilde{f} \rangle_{\mathcal{E}_{(x,y,q)}} + \langle \frac{\partial \tilde{f}}{\partial x_1}, \ldots, \frac{\partial \tilde{f}}{\partial x_n}, 1 \rangle_{\mathcal{E}_{(x,y)}}$$

and P-$\mathcal{K}$-cod $\tilde{f} = \dim_{\mathbb{R}} \mathcal{E}_{(x,y,q)}/T_e(\text{P-}\mathcal{K})(\tilde{f})$, where $\tilde{f} = \tilde{F}|0 \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k$. We also say that $\tilde{F}$ is *a P-$\mathcal{K}$-versal deformation of $\tilde{f}$* if

$$\mathcal{E}_{(x,y,q)} = \langle \frac{\partial \tilde{F}}{\partial t}|0 \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k \rangle_{\mathbb{R}} + T_e(\text{P-}\mathcal{K})(\tilde{f}).$$

Then we have the following proposition whose proof is like that of the ordinary theory of Legendrian singularities ([1],[11]).

**Proposition 2.3.** (1) *Let $\tilde{F}_i$ $(i = 0, 1)$ be generating families of $\Phi_{F_i}$. Then $\Phi_{F_0}$ and $\Phi_{F_1}$ are P-Legendrian equivalent if and only if $\tilde{F}_0$ and $\tilde{F}_1$ are stably t-P-$\mathcal{K}$-equivalent.*
(2) *Let $\tilde{F}$ be a generating family of $\Phi_F$, then $\Phi_F$ is stable with respect to the P-Legendrian equivalence if and only if $\tilde{F}$ is a P-$\mathcal{K}$-versal deformation of $\tilde{f}$.*

We consider the Legendrian unfolding which is given by the following generating family:

$({}^0A_r)$
$$q_1^{r+1} \pm q_2^2 + \sum_{i=1}^{r-1} x_i q^i - y \ (1 \le r \le n)$$

$({}^0D_r)$
$$q_1^2 q_2 \pm q_2^{r-1} + \sum_{i=2}^{r-1} x_i q_2^{i-1} + x_1 q_1 - y \ (4 \le r \le n)$$

$({}^1A_r)$
$$q_1^{r+1} \pm q_2^2 + q_1^{r-1}(t \pm x_{r-1}^2 \pm \cdots \pm x_n^2) + \sum_{i=1}^{r-2} x_i q_1^i - y \ (3 \le r \le n+2)$$

$({}^1D_r)$
$$q_1^2 q_2 \pm q_2^{r-1} + q_2^{r-2}(t \pm x_{r-1}^2 \pm \cdots \pm x_n^2) + \sum_{i=2}^{r-2} x_i q_2^{i-1} + x_1 q_1 - y \ (4 \le r \le n+2)$$

$({}^1E_6)$
$$q_1^3 + q_2^4 + q_1 q_2^2 t + x_4 q_1 q_2 + x_3 q_2^2 + x_1 q_1 + x_2 q_2 - y.$$

Each germ in the above list is a generating family of P-Legendrian stable Legendrian unfoldings. Furthermore, it is known that this gives a generic classification of Legendrian unfoldings for $n \le 5$ (cf. [7],[11]).

## 3. Realization theorems

In this section we identify the geometric solution of a (GCPT) introduced in Section 2 with the notion of one-parameter Legendrian unfoldings. Let $i : L' \subset E(H)_0 \subset E(H)$ be the initial condition of a (GCPT) and let $L$ be the unique solution. Since $X_H \notin TE(H)_c$, then $L$ is transverse to $E(H)_c$ in $E(H)$ for any $c \in (\mathbb{R}, 0)$. It follows that $L_c = L \cap E(H)_c$ is an $n$-dimensional submanifold of $E(H)_c$ and it satisfies $\Theta_c|L_c = 0$ (i.e., $L_c$ is a Legendrian submanifold of $E(H)_c$). If we consider the local parametrization of $L$, we may assume that $L$ is the image of an immersion germ $\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \to E(H)$ such that $\mathcal{L}|(c \times \mathbb{R}^n)$ is a Legendrian immersion germ of $E(H)_c$. Hence the coordinate representation of $\mathcal{L}$ is given by $\mathcal{L}(t, u) = (t, x(t, u), y(t, u), -H(t, x(t, u), p(t, u)), p(t, u))$.

Let $\tilde{\pi} : J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \to J^1(\mathbb{R}^n, \mathbb{R})$ be the canonical projection defined by $\tilde{\pi}(t, x, y, s, p) = (x, y, p)$. Then the map germ $\ell = \tilde{\pi} \circ \mathcal{L}$ satisfies that $\ell_t = \ell|\pi_1^{-1}(t)$ is a Legendrian immersion germ for any $t \in (\mathbb{R}, 0)$. Hence $\mathcal{L}$ is a Legendrian unfolding associated with $(\pi_1, \tilde{\pi} \circ \mathcal{L})$, where $\pi_1$ is the canonical projection $\pi_1 : (\mathbb{R} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. This completes the proof of the first part of Theorem 3.1. The proof of the second part is given in [6].

**Theorem 3.1.** (1) *The local solution of the generalized Cauchy problem associated with the time parameter for the Hamilton-Jacobi equation*

$$s + H(t, x, p) = 0$$

*is a Legendrian unfolding*

$$\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}).$$

(2) *Let*

$$\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$$

*be a Legendrian unfolding associated with $(\pi_1, \ell)$. Then there exists a $C^\infty$-function germ $H(t, x_1, \dots, x_n, p_1, \dots, p_n)$ such that $\mathcal{L}$ is a local solution of the generalized Cauchy problem associated with the time parameter for Hamilton-Jacobi equation*

$$s + H(t, x, p) = 0,$$

*where the initial condition is given by $\ell(0, u)$.*

The above theorem guarantees that the class of Legendrian unfoldings supplies the correct class to describe the geometric solutions of (GCPT) for Hamilton-Jacobi equations. Thus, generic results for the singularities of Legendrian unfoldings can be translated to generic results in the class of all Hamiltonians and all initial conditions. However, we must also concern ourselves with what are the types of singularities that the geometric solution to a given Hamilton-Jacobi equation might exhibit. For the purpose, we need a kind of non-degeneracy condition on the Hamiltonian function. We say that a Hamilitonian function $H(t, x, p)$ is *$s$-non-degenerated at* $(t_0, x_0, p_0)$ if $\frac{\partial^s H}{\partial p_1^{i_1} \cdots \partial p_n^{i_n}}(t_0, x_0, p_0) \neq 0$ for some $(i_1, \dots, i_n) \in (\mathbb{N} \cup \{0\})^n$ with $i_1 + \cdots + i_n = s \geq 2$. We simply say that $H(t, x, p)$ is *non-degenerated at* $(t_0, x_0, p_0)$ if it is 2-non-degenerated at $(t_0, x_0, p_0)$. The following theorem is a realization theorem for generic singularities for a given Hamilton-Jacobi equation.

**Theorem 3.2.** *Let $H(t,x,p)$ be a non-degenerate Hamiltonian function germ at $(t_0,x_0,p_0)$ and $\mathcal{L} : (R,u_0) \to (J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),(t_0,x_0,y_0,s_0,p_0))$ be a P-Legendrian stable Legendrian unfolding associated with $(\mu,\ell)$. Then there exists a Legendrian unfolding $\mathcal{L}'$ which is a geometric solution of the Hamilton-Jacobi equation $s+H(t,x,p)=0$ such that $\mathcal{L}$ and $\mathcal{L}'$ are P-Legendrian equivalent.*

In [7] we have shown the above result (c.f., Theorem 3.5 in [7]) under a much stronger assumption (i.e., the matrix $(\frac{\partial^2 H}{\partial p_i \partial p_j}(t_0,x_0,p_0))$ is positive (or negative) definite). We can state the conclusion under the weaker assumption as the above, the proof is given by the similar step as that of Theorem 3.5 in [7]. However, it is not so popular and the methods will be used in the later, then we give the proof here.

*Proof.* Without loss of generality, we assume that $(t_0,x_0,y_0)=(0,0,0)$. Let $\tilde{G} : (\mathbb{R}\times(\mathbb{R}^n\times\mathbb{R})\times\mathbb{R}^k,0) \to (\mathbb{R},0)$ be a generating family of the Legendrian unfolding $\mathcal{L}$. Since $d_2G|0\times\mathbb{R}^n\times\mathbb{R}^k$ is non-singular, the set

$$\phi_g = \{(0,x,-H(0,x,\frac{\partial g}{\partial x}(x,q)),\frac{\partial g}{\partial x}(x,q)) | \frac{\partial g}{\partial q_i}(x,q)=0\ i=1,\dots,k\}$$

is the initial condition for the corresponding (GCPT), where $g=G|0\times\mathbb{R}^n\times\mathbb{R}^k$. By the arguments of the last part of §1, we can construct a Legendrian unfolding $\mathcal{L}'$ which is the local unique geometric solution around $\phi_g$.

We now choose a generating family $\tilde{F} : (\mathbb{R}\times(\mathbb{R}^n\times\mathbb{R})\times\mathbb{R}^{k'},0) \to (\mathbb{R},0)$ of $\mathcal{L}'$. By definition, Image $\Phi_F|t=0$ is equal to $\phi_g$, so that we may assume that $k=k'$ and $\tilde{f},\tilde{g}$ are $P$-$\mathcal{K}$-equivalent, where $f=F|0\times\mathbb{R}^n\times\mathbb{R}^k$, $\tilde{f}(x,y,q)=f(x,q)-y$ and $\tilde{g}(x,y,q)=g(x,q)-y$.

If $P$-$\mathcal{K}$-cod $\tilde{g}=0$, then $P$-$\mathcal{K}$-cod $\tilde{f}=0$, so that $\tilde{f}$ is already a $P$-$\mathcal{K}$-versal deformation of itself. Thus, $\tilde{F}$ and $\tilde{G}$ are $P$-$\mathcal{K}$-versal deformations of $\tilde{f}$ and $\tilde{g}$ respectively. By the uniqueness theorem of $P$-$\mathcal{K}$-versal deformations (see [5]), $\tilde{F}$ is t-$P$-$\mathcal{K}$-equivalent to $\tilde{G}$.

We now assume that $P$-$\mathcal{K}$-cod $\tilde{g}=1$, so that $P$-$\mathcal{K}$-cod $\tilde{f}=1$. If $\frac{\partial F}{\partial t}|_{t=0} \notin T_e(P\text{-}\mathcal{K})(\tilde{f})$, then we can get the required assertion by the uniqueness of the $P$-$\mathcal{K}$-versal deformation as in the previous case.

Suppose that $\frac{\partial F}{\partial t}|_{t=0} \in T_e(P\text{-}\mathcal{K})(\tilde{f})$ for any generating family $\tilde{F}$ of $\mathcal{L}'$. Since $\Phi_F$ is a geometric solution of $s+H(t,x,p)=0$, we have a relation

$$-\frac{\partial F}{\partial t} \equiv H(t,x,\frac{\partial F}{\partial x}) \bmod \langle\frac{\partial F}{\partial q_1},\dots,\frac{\partial F}{\partial q_k}\rangle_{\mathcal{E}_{(t,x,q)}},$$

so that

$$-\frac{\partial F}{\partial t}|_{t=0} \equiv H(0,x,\frac{\partial f}{\partial x}) \bmod \langle\frac{\partial f}{\partial q_1},\dots,\frac{\partial f}{\partial q_k}\rangle_{\mathcal{E}_{(x,q)}}.$$

Therefore

$$-\frac{\partial F}{\partial t}|_{t=x=0} \equiv H(0,0,\frac{\partial f}{\partial x}(0,q)) \bmod \langle\frac{\partial f_0}{\partial q_1},\dots,\frac{\partial f_0}{\partial q_k}\rangle_{\mathcal{E}_q},$$

where $f_0(q)=f(0,q)$. We may assume that $f_0 \in \mathfrak{M}_q^3$, where $\mathfrak{M}_q$ is the unique maximal ideal of $\mathcal{E}_q$.

We now consider the Taylor polynomial of $H(t,x,p)$ of degree 2 at $(t,x,p_0)$ with respect to $p=(p_1,\dots,p_n)$-variables as follows :

$$H(t,x,p) = H(t,x,p_0)+\sum_{i=1}^{n}\frac{\partial H}{\partial p_i}(t,x,p_0)(p_i-p_{0,i})$$
$$+\frac{1}{2}\sum_{i,j}\frac{\partial^2 H}{\partial p_i\partial p_j}(t,x,p_0)(p_i-p_{0,i})(p_j-p_{0,j}) + \text{higher term}.$$

If we set $t=0$, $x=0$, and $p_i=\frac{\partial f}{\partial x_i}(0,q)$, then

$$H(0,0,\frac{\partial f}{\partial x_i}(0,q)) = H(0,0,p_0)+\sum_{i=1}^{n}\frac{\partial H}{\partial p_i}(x,p_0)(\frac{\partial f}{\partial x_i}(0,q)-p_{0,i})$$
$$+\frac{1}{2}\sum_{i,j}\frac{\partial^2 H}{\partial p_i\partial p_j}(x,p_0)(\frac{\partial f}{\partial x_i}(0,q)-p_{0,i})(\frac{\partial f}{\partial x_j}(0,q)-p_{0,j}) + \text{higher term}.$$

Since $F(t,x,q)$ is a generalized phase function germ, rank$(\frac{\partial^2 F}{\partial q_i\partial x_j}|_{t=0})=k$. Then we may assume that $\frac{\partial^2 F}{\partial q_i \partial x_j}(0)=\delta_{ij}$ for $i,j=1,\dots,k$ and $\frac{\partial F}{\partial x_\ell}|_{(t,x)=(0,0)}\in\mathfrak{M}_q^2$ for $\ell=k+1,\dots,n$. It follows that $\frac{\partial F}{\partial x_i}(0,0,q)-p_{0i}=q_i+\psi(q)$, where $\psi(q)\in\mathfrak{M}_q^2$. Since $H$ is non-degenerate, the quadratic form $\sum_{i=1}^{k}\frac{\partial^2 F}{\partial p_i\partial p_j}(0,0,p_0)q_iq_j$ does not vanish.

On the other hand

$$-\frac{\partial F}{\partial t}|_{t=0}\in\langle f(x,q)-y,\frac{\partial f}{\partial q}(x,q)\rangle_{\mathcal{E}_{(x,y,q)}}+\langle 1,\frac{\partial f}{\partial x}(x,q)\rangle_{\mathcal{E}_{(x,y)}}.$$

It follows that

$$H(0,0,\frac{\partial f}{\partial x}(0,q))\in\langle f(x,q)-y,\frac{\partial f_0}{\partial q}(q)\rangle_{\mathcal{E}_q}+\langle 1,\frac{\partial f_0}{\partial x}(q)\rangle_{\mathbb{R}},$$

and

$$\frac{1}{2}\sum_{1\le i,j\le n}\frac{\partial^2 H}{\partial p_i\partial p_j}(0,0,p_0)(\frac{\partial f}{\partial x_i}(0,q)-p_{0,i})(\frac{\partial f}{\partial x_j}(0,q)-p_{0,j})$$
$$\equiv H(0,0,\frac{\partial f}{\partial x}(0,q))-H(0,0,p_0)-\sum_{i=1}^{n}\frac{\partial H}{\partial p_i}(0,0,p_0)(\frac{\partial f}{\partial x_i}(0,q)-p_{0,i}) \bmod \mathfrak{M}_q^3$$
$$=H(0,\frac{\partial f}{\partial x}(0,q))-H(0,0,p_0)-\sum_{i=1}^{n}\frac{\partial H}{\partial p_i}(0,0,p_0)\frac{\partial f}{\partial x_i}(0,q)+\sum_{i=1}^{n}p_{0,i}\frac{\partial H}{\partial p_i}(0,0,p_0)$$
$$\in\langle 1,\frac{\partial f}{\partial x_1}(0,q),\dots,\frac{\partial f}{\partial x_n}(0,q)\rangle_{\mathbb{R}}+\langle f_0,\frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_q^3.$$

For any linear isomorphism $A:\mathbb{R}^n\to\mathbb{R}^n$, we have a relation

$$\frac{\partial f(Ax,q)}{\partial x_i}=\sum_{j=1}^{n}A_{ij}\frac{\partial f}{\partial x_j}(Ax,q).$$

Since

$$\sum_{1\le i,j\le n} \frac{\partial^2 H}{\partial p_i \partial p_j}(0,0,p_0)(\frac{\partial f}{\partial x_i}(0,q)-p_{0,i})(\frac{\partial f}{\partial x_j}(0,q)-p_{0,j})$$
$$\equiv \sum_{1\le i,j\le k} \frac{\partial^2 H}{\partial p_i \partial p_j}(0,0,p_0)q_iq_j \bmod \mathfrak{M}_q^3$$

and the vector space

$$\langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q} + \langle 1, \frac{\partial f}{\partial x_1}(0,q),\ldots,\frac{\partial f}{\partial x_1}(0,q)\rangle_{\mathbb{R}} \bmod \mathfrak{M}_q^3$$

is an invariant under the action of the linear isomorphism $A$, then any quadratic form of $q = (q_1,\ldots,q_k)$-variables is contained in the above vector space. If there exists a quadratic form of $q$-variables that it is contained in $\langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_q^3$, then all quadratic forms are contained in it for the same reason as above. In this case, since the vector space $\langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_q^3$ has at most dimension $k$, then $k$ should be 1 and $f_0$ is an $A_2$-type function germ. It follows from the classification (cf. the list after Proposition 2.3) that $F(t,x,q)$ is of ${}^0A_2$-type, so that this case is contained in the case of $P$-$\mathcal{K}$-cod$(\tilde{f}) = 0$.

We may assume that any quadratic form of $q$-variables is contained in

$$\langle 1, \frac{\partial f}{\partial x_1}(0,q),\ldots,\frac{\partial f}{\partial x_n}(0,q)\rangle_{\mathbb{R}} \bmod \mathfrak{M}_q^3.$$

Since $\mathcal{K}$-cod$(f_0)$ is finite (for the definition of $\mathcal{K}$-finiteness, see [10]), then there exists $r \in \mathbb{N}$ such that $\mathfrak{M}_q^r \subset \langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q}$. By the same arguments as those of the previous paragraph, we can assert that every monomial of $q$-variables of degree 3 is contained in the vector space

$$\langle 1, \frac{\partial f}{\partial x_1}(0,q),\ldots,\frac{\partial f}{\partial x_n}(0,q)\rangle_{\mathbb{R}} + \langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_q^4.$$

If there exists a monomial of degree 3 which is contained in $\langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_q^4$, then $k$ should be 1 and $f_0$ is an $A_3$-type function germ. It follows that

$$\dim_{\mathbb{R}}\langle 1, \frac{\partial f}{\partial x_1}(0,q),\ldots,\frac{\partial f}{\partial x_n}(0,q)\rangle_{\mathbb{R}} \ge 3 \bmod \mathfrak{M}_q^4.$$

By the clasification (cf. the list after Proposition 2.3, $F(t,x,q)$ should be of ${}^0A_3$-type, then this case is contained in the case of $P$-$\mathcal{K}$-cod$(\tilde{f}) = 0$.

For ${}^0A_\ell$ or ${}^1A_\ell$-type germs, we get same normal forms as in the list after Proposition 2.3 without the assumption $n \le 5$ (cf. Theorem 2.2 in [11]). We can continue this procedure up to degree $r-1$. Eventually, it remains the case that every polynomial of degree $r-1$ is contained in the vector space

$$\langle 1, \frac{\partial f}{\partial x_1}(0,q),\ldots,\frac{\partial f}{\partial x_n}(0,q)\rangle_{\mathbb{R}} + \langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_q^r.$$

Since $\mathfrak{M}_q^r \subset \langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q}$, then

$$\mathcal{E}_q = \langle 1, \frac{\partial f}{\partial x_1}(0,q), \dots, \frac{\partial f}{\partial x_n}(0,q)\rangle_{\mathbb{R}} + \langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q}.$$

It follows that

$$\mathcal{E}_{(x,q)} = T_e(P\text{-}\mathcal{K})(\tilde{f}) + \mathfrak{M}_q \mathcal{E}_{(x,q)},$$

so that we have $\mathcal{E}_{(x,q)} = T_e(P\text{-}\mathcal{K})(f)$ by the Malgrange preparation theorem. This contradicts the fact that $P$-$\mathcal{K}$-cod$(\tilde{f}) = 1$. This completes the proof.

In the list of generating families for Legendrian unfoldings in Section 2, the germs of ${}^1A_r$, ${}^1D_r$ and ${}^1E_6$ describe the situation how geometric singularities bifurcate. Especially, ${}^1A_3$ singularity describes how the singularity appears from a smooth solution. All of these are P-Legendrian stable Legendrian unfoldings, so that these can be realized as geometric solutions at the non-degenerated point for a given Hamilton-Jacobi equation. We consider that at where such bifurcations should be appeared.

**Theorem 3.3.** *Let $s+H(t,x,p)=0$ be a Hamilton-Jacobi equation. If one of ${}^1A_r$, ${}^1D_r$ or ${}^1E_6$ appears at a point $(t_0,x_0,p_0)$ of a geometric solution of $E(H)$, then $H$ is $s$-non-degenerated at $(t_0,x_0,p_0)$ for some $s=2,\dots,r-1$ (for ${}^1E_6$ case, $2 \le s \le 5$).*

*Proof.* For the proof, we may assume that $(t_0,x_0,p_0)=(0,0,0)$.

We remark that for any function germ $f : (\mathbb{R}^n \times \mathbb{R}^k, 0) \to (\mathbb{R},0)$ with a property that $\frac{\partial f}{\partial q}(0) = (\frac{\partial f}{\partial q_1}(0), \dots, \frac{\partial f}{\partial q_k}(0))$ is non zero vector, we can construct a Legendrian submanifold

$$\phi_f = \{x, f(x,q), \frac{\partial f}{\partial x}(x,q)) | \frac{\partial f}{\partial q}(x,q) = 0\}$$

of the 1-jet space $J^1(\mathbb{R}^n,\mathbb{R})$.

Firstly we prove the assertion for the germ ${}^1A_1$. Since our consern is invariant under the stable t-P-$\mathcal{K}$-equivalence, we may assume that $k=1$. We consider the function germ

$$f(x,q) = q^{r+1} + q^{r-1}(\pm x_{r-1}^2 \pm \cdots \pm x_n^2) + \sum_{i=1}^{r-2} x_i q^i - y,$$

then $\frac{\partial f}{\partial q} = (r+1)q^r + (r-1)q^{r-2}(\pm x_{r-1}^2 \pm \cdots \pm x_n^2) + \sum_{i=1}^{r-1} i x_i q^{i-1}$, so that $\phi_f$ is a Legendrian submanifold of $J^1(\mathbb{R}^n,\mathbb{R})$ whose generating family is given by

$$\tilde{f}(x,y,q) = f(x,q) - y = q^{r+1} + q^{r-1}(\pm x_{r-1}^2 \pm \cdots \pm x_n^2) + \sum_{i=1}^{r-2} x_i q^i - y.$$

Since $\ell_0 = i_0(\phi_f)$ gives the initial condition for the (GCPT), we can get a Legendrian unfolding $(\mathcal{L},0) \subset E(H)$ by the method of characteristics. Let $\tilde{G}(t,x,y,q) = G(t,x,q) - y$ be a generating family of $(\mathcal{L},0)$, then $\tilde{g}(x,y,q) = g(x,q) - y = \tilde{G}(0,x,y,q)$ is a generating family of $\ell_0$. By the proof of Theorems 19.4 and 20.8 in [1], there exists a diffeomorphism germ $\Psi : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ of the form $\Psi(x,q) = (x, \psi(x,q))$ such that $g \circ \Psi(x,q) = f(x,q)$. Define a function germ $F : (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R},0)$ by $F(t,x,q) = G(t,x,\psi(x,q))$, then the

Legendrian unfolding Image $\Phi_F$ is equal to Image $\Phi_G = \mathcal{L}$ and $F|_{t=0} = f$. It follows that $\Phi_F$ is the geometric solution of $E(H)$ and $\frac{\partial F}{\partial t} + H(t,x,\frac{\partial F}{\partial x}) \equiv 0 \bmod \langle \frac{\partial F}{\partial q}|_{t=0}\rangle_{\mathcal{E}_{(t,x,y,q)}}$. If we put $t=0$, then we have

$$\frac{\partial F}{\partial t}|_{t=0} + H(0,x,q) \equiv 0$$

$$\bmod \langle (r+1)q^r + (r-1)q^{r-2}(\pm x_{r-1}^2 \pm \cdots \pm x_n^2) + \sum_{i=1}^{r-2} i x_i q^{i-1}\rangle_{\mathcal{E}_{(x,y,q)}}.$$

We now consider the Taylor polynomial of $H(t,x,p)$ for a sufficiently higher degree at $(t,x,p_0)$ with respect to $p=(p_1,\dots,p_n)$-variables as follows :

$$H(t,x,p) = H(t,x,0) + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}(t,x,0)p_i + \frac{1}{2}\sum_{1\le i,j\le n}\frac{\partial^2 H}{\partial p_i \partial p_j}(t,x,0)p_ip_j + \text{higher}.$$

On the other hand, we have

$$\frac{\partial f}{\partial x_i} = \begin{cases} q^i & 1 \le i \le r-2 \\ \pm 2x_i q^{r-1} & r-1 \le i \le n. \end{cases}$$

If we set $t=0$ and $p_i = \frac{\partial f}{\partial x_i}(0,q)$, then

$$\begin{aligned} H(0,x,\frac{\partial f}{\partial x}(0,q)) =& H(0,x,0) + \sum_{i=1}^{r-2}\frac{\partial H}{\partial p_i}(0,x,0)q^i + \sum_{i=r-1}^{n}\frac{\partial H}{\partial p_i}(0,x,0)(\pm 2x_iq^{r-1}) \\ &+\Psi(0,x,0))q^{r-1} + \frac{1}{2}\sum_{i+j\neq r-1}\frac{\partial^2 H}{\partial p_i\partial p_j}(0,x,0)q^{i+j} \\ &+\frac{1}{2}\sum_{i,j}\frac{\partial^2 H}{\partial p_i \partial p_j}(0,x,0)q^{i-1}(\pm 2q^{r-1}x_j) + \text{higher term}, \end{aligned}$$

where

$$\begin{aligned} \Psi(0,x,0) =& \frac{1}{2}\sum_{i+j=r-1}\frac{\partial^2 H}{\partial p_i\partial p_j}(0,x,0) + \frac{1}{3!}\sum_{i+j+k=r-1}\frac{\partial^3 H}{\partial p_i \partial p_j \partial p_k}(0,x,0) \\ &+\cdots+\frac{1}{s!}\sum_{i_1+\cdots i_s=r-1}\frac{\partial^s H}{\partial p_1^{i_1}\cdots\partial p_s^{i_s}}(0,x,0) + \cdots + \frac{1}{(r-1)!}\frac{\partial^{r-1}H}{\partial p_1^{(r-1)}}(0,x,0). \end{aligned}$$

So we have

$$\begin{aligned} \frac{\partial F}{\partial t}|_{t=0} + H(0,x,\frac{\partial f}{\partial x}(0,q)) =& \frac{\partial F}{\partial t}|_{t=0} + H(0,x,0) + \sum_{i=1}^{r-1}\frac{\partial H}{\partial p_i}(0,x,0)q^{i-1} \\ &+\sum_{i=1}^{r-1}\frac{\partial H}{\partial p_i}(0,x,0)(\pm 2x_iq^{r-1}) + \Psi(0,x,0)q^{r-1} \\ &+\frac{1}{2}\sum_{i+j\neq r-1}\frac{\partial^2 H}{\partial p_i\partial p_j}(0,x,0)q^{i+j} + \frac{1}{2}\sum_{i,j}\frac{\partial^2 H}{\partial p_i\partial p_j}(0,x,0)q^{i-1}(\pm 2q^{r-1}x_j) \\ &+\text{higher term.} \equiv 0 \end{aligned}$$

$$\bmod \langle (r+1)q^r + (r-1)q^{r-2}(\pm x_r^2 \pm \cdots \pm x_n^2) + \sum_{i=1}^{r-1} i x_i q^{i-2}\rangle_{\mathcal{E}_{(x,y,q)}}.$$

Since $T_e(\text{P-}\mathcal{K})(\tilde{f})$ is equal to

$$\langle q^{r+1} + q^{r-1}(\pm x_{r-1}^2 \pm \cdots \pm x_n^2) + \sum_{i=1}^{r-2} x_i q^i - y,$$
$$(r+1)q^r + (r-1)q^{r-2}(\pm x_{r-1}^2 \pm \cdots \pm x_n^2) + \sum_{i=1}^{r-2} i x_i q^{i-1}\rangle_{\mathcal{E}_{(x,y,q)}}$$
$$+\langle 1, q, \ldots, q^{r-2}, 2q^{r-1}x_{r-1}, \ldots, 2q^{r-1}x_n\rangle_{\mathcal{E}_{(x,y)}},$$

we can show that every element of $\mathcal{E}_{(x,y,q)}$ except $q^{r-1}$ is contained in $T_e(P\text{-}\mathcal{K})(f)$. It follows that

$$\frac{\partial F}{\partial t}|_{t=0} + \Psi(0,x,0)q^{r-1} \in T_e(\text{P-}\mathcal{K})(\tilde{f}).$$

Suppose that $\frac{\partial^s H}{\partial p_1^{i_1}\cdots\partial p_n^{i_n}}(0,0,0) = 0$ for $s = 2, \ldots, r-1$, then $\Psi(0,x,0) \in \mathfrak{M}_x$, so that

$$\Psi(0,x,0)q^{r-1} \in T_e(\text{P-}\mathcal{K})(\tilde{f}).$$

It follows that $\frac{\partial F}{\partial t}|_{t=0} \in T_e(\text{P-}\mathcal{K})(\tilde{f})$. This formula shows that $\tilde{F}$ cannot be a $P$-$\mathcal{K}$-versal deformation of $\tilde{f}$. However the generating family of type ${}^1A_r$ is a $P$-$\mathcal{K}$-versal deformation of $\tilde{f}$, so that $\tilde{F}$ is not t-$P$-$\mathcal{K}$-equivalent to the germ of type ${}^1A_r$.

On the other hand, let $\tilde{F}'$ be a generating family of a geometric solution of $s+H(t,x,p)=0$. Suppose that $\tilde{F}'$ is $t$-P-$\mathcal{K}$-equivalent to the germ of ${}^1A_r$, then there exist a diffeomorphism germ

$$\Phi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}, 0) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}, 0)$$

of the form

$$\Phi(t,x,y) = (\phi_1(t), \phi_2(t,x,y), \phi_3(t,x,y), \phi_4(t,x,y,q))$$

and an element $\lambda(t,x,y,q) \in \mathcal{E}_{(t,x,y,q)}$ such that $\lambda(0) \neq 0$ and $\lambda(t,x,y,q)\tilde{F}' \circ \Phi(t,x,y,q) = \tilde{F}(t,x,y,q)$, where

$$F(t,x,y,q) = q^{r+1} + q^{r-1}(t \pm x_{r-1}^2 \pm \cdots \pm x_n^2) + \sum_{i=1}^{r-2} x_i q^i \ (3 \le r \le n+2).$$

Thus we have $\lambda(0,x,y,q)\tilde{F}'(0,\phi_2(0,x,y),\phi_3(0,x,y),\phi_4(0,x,y,q)) = \tilde{f}(x,y,q)$. Let $\tilde{G}$ be a generating family of the geometric solution which is constructed by the characteristic method from the initial data $\phi_f$. By the uniqueness of the geometric solution, we can assert that $\tilde{F}'$ is $t$-P-$\mathcal{K}$-equivalent to $\tilde{G}$. However, by the previous arguments, $\tilde{G}$ cannot be $t$-P-$\mathcal{K}$-equivalent to $\tilde{F}$. This contradicts to the assumption that $\tilde{F}'$ is $t$-P-$\mathcal{K}$-equivalent to $\tilde{F}$. Thus the Legendrian unfolding of type ${}^1A_r$ cannot be realized as a geometric solution of $s + H(t,x,p) = 0$, where $H(t,x,p)$ is not non-degenerated at $(t_0, x_0, p_0)$.

Since we can prove the assertion for the germs ${}^1D_r$ and ${}^1E_6$ by exactly the same way as the above arguments, we omit the proof of these cases.

**Corollary 3.4.** *If an $^1A_3$ singularity appears at $(t_0, x_0, p_0)$, then $H(t,x,p)$ is non-degenerated at $(t_0, x_0, p_0)$.*

The corollary describes that the first singularities appears at non-degenerated point of the Hamiltonian function if the initial data is smooth.

**Example 3.5.** Consider the equation : $s + p^3 = 0$ (i.e. $n = 1$ and $H(t,x,p) = p^3$).

In this case the bifurcation of geometric singularities of type $^1A_3$ : $q^4 + xq + tq^2 - y$ describe the situation how the first singularity appears from a smooth initial data. By the above theorem the generating family of a Legendrian unfolding which is a geometric solution of $s + p^3 = 0$ cannot be t-$P$-$\mathcal{K}$-equivalent to the germ of type $^1A_3$. Thus the Legendrian unfolding of type $^1A_3$ cannot be realized as a geometric solution of $s + p^3 = 0$ at the origin.

## 4. The case when the Hamiltonian depends only on momenta

We can asserts the much stronger statement for the case that the Hamiltonian function depends only on $(p_1, \ldots, p_n)$-variables. In this case the Cauchy problem is given by

$$(\mathrm{P}') \qquad \begin{cases} \dfrac{\partial y}{\partial t} + H(\dfrac{\partial y}{\partial x_1}, \cdots, \dfrac{\partial y}{\partial x_n}) = 0 \\ y(0, x_1, \cdots, x_n) = \phi(x_1, \cdots, x_n), \end{cases}$$

where $H$ and $\phi$ are $C^\infty$-functiions. It follows that the characteristic equation is given by

$$(\mathrm{C}') \qquad \begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial H}{\partial p_i}(p) \ (i = 1, \ldots, n), \\ \dfrac{dp_i}{dt} = 0 \ (i = 1, \ldots, n), \\ \dfrac{dy}{dt} = -H(p) + \displaystyle\sum_{i=1}^{n} p_i \cdot \dfrac{\partial H}{\partial p_i}(p), \ x, p \in \mathbb{R}^n, \\ x(0) = u, \ p(0) = \dfrac{\partial \phi}{\partial u}(u), \ y(0) = \phi(u), \ u \in \mathbb{R}^n. \end{cases}$$

We can explicity solve the charcteristic equation as follows:

$$(\mathrm{S}') \qquad \begin{cases} x_i(t,u) = u_i + t\dfrac{\partial H}{\partial p_i}(\dfrac{\partial \phi}{\partial u}(u)) \ (i = 1, \ldots, n), \\ p_i(t,u) = \dfrac{\partial \phi}{\partial u_i}(u) \ (i = 1, \ldots, n), \\ y(t,u) = t\{-H(\dfrac{\partial \phi}{\partial u}(u)) + \displaystyle\sum_{i=1}^{n} \dfrac{\partial \phi}{\partial u_i}(u) \cdot \dfrac{\partial H}{\partial p_i}(\dfrac{\partial \phi}{\partial u}(u))\} + \phi(u). \end{cases}$$

Then we have the following proposition.

**Proposition 4.1.** *Let $s + H(p) = 0$ be a Hamilton-Jacobi equation. If a singularity of geometric solution for the Cauchy problem $(P')$ appears at a point $(t_0, x_0, p_0)$, then $H$ is non-degenrated at $(t_0, x_0, p_0)$.*

*Proof.* The Jacobian matrix of the mapping $x : \mathbb{R}^n \to \mathbb{R}^n$ with respect to $u$ is

$$\begin{pmatrix} 1+tc_{11} & tc_{12} & \dots & tc_{1n} \\ tc_{21} & 1+tc_{22} & \dots & tc_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ tc_{n1} & tc_{n2} & \dots & 1+tc_{nn} \end{pmatrix},$$

where $c_{ij} = \sum_{k=1}^{n} \frac{\partial^2 H}{\partial p_i \partial p_k}(\phi(u))\frac{\partial^2 \phi}{\partial u_k \partial u_j}(u)$. By the definiton of singularity of geometric solutions, the point $(t_0, x_0, p_0) = (t_0, x(t_0, u_0), p(t_0, u_0))$ is a singularitiy of the geometric solution if and only if the rank of the Jacobian matrix of $x : \mathbb{R}^n \to \mathbb{R}^n$ is less than $n$. However, if the Hamiltonian function $H(p)$ is degenerated at the point $(t_0, x_0, p_0)$, then the Jacobian matrix is equal to the identity matrix, so that the geometric solution is non-singular at the point.

## References


1. V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps*, Birkhauser, 1986.
2. I. A. Bogaevskii, *Modifications of singularities of minimum functions and bifurcations of shock waves of the Burgers eqaution with vanishing viscosity*, Leningrad Math. J. **1** (1990), 807–823.
3. M. G. Crandall and P. -Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277** (1983), 1–42.
4. M. G. Crandall, H. Ishii and P.-Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. **27** (1992), 1–67.
5. J. Damon, *The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$*, Memoirs of Amer. Math. Soc. **50-306** (1984).
6. S. Izumiya, *Geometric singularities for Hamilton-Jacobi equation*, Advanced Studies in Pure Math. **22** (1993), 89–100.
7. S. Izumiya and G. T. Kossioris, *Semi-local classification of geometric singularities for Hamilton-Jacobi equaitons*, to appear in Journal of Differential Equations.
8. G. T. Kossioris, *Formation of singularities for viscosity solutions of Hamilton-Jacobi equations in higher dimensions*, Comm. in Partial Differential Equations **18** (1993), 1085–1108.
9. V. V. Lychagin, *Local classification of non-linear first order partial differential equations*, Russian Math. Surveys **30** (1975), 105–175.
10. C. T. C. Wall, *Finite determinacy of smooth map germs*, Bull. London Math. Soc. **14** (1981), 481–539.
11. V. M. Zakalyukin, *Reconstructions of fronts and caustics depending on a parameter and versality of mappings*, Journal of Soviet Math. **27** (1983), 2713–2735.