Cohen-Macaulay partially ordered sets
with pure resolutions

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Cohen-Macaulay partially ordered sets with pure resolutions

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Abstract

We give a combinatorial classification of Cohen-Macaulay partially ordered sets P for which a minimal free resolution of the Stanley-Reisner ring $k[\Delta(P)]$ of the order complex $\Delta(P)$ of P is pure.

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1 Background

(1.1) Let $V = \{x_1, x_2, \ldots, x_v\}$ be a finite set, called the *vertex set*, and Δ a *simplicial complex* on V. Thus Δ is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for every $1 \leq i \leq v$ and (ii) $\sigma \in \Delta$, $\tau \subset \sigma \Rightarrow \tau \in \Delta$. Each element σ of Δ is called a *face* of Δ. Set $d = \max\{\#(\sigma); \sigma \in \Delta\}$. Here $\#(\sigma)$ is the cardinality of σ as a finite set. Then the *dimension* of Δ is defined by $\dim \Delta = d - 1$. A maximal face of Δ is also called a *facet* of Δ. We say that Δ is *pure* if $\#(\sigma) = d$ for every facet σ of Δ.

Given a subset W of V, we write Δ_W for the subcomplex of Δ defined by

$$\Delta_W = \{\sigma \in \Delta; \sigma \subset W\}.$$

Thus, in particular, $\Delta_V = \Delta$ and $\Delta_\emptyset = \{\emptyset\}$. On the other hand, if k is a field, then let $\tilde{H}_i(\Delta; k)$ denote the i-th *reduced simplicial homology group* of Δ with coefficients k. Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \{\emptyset\}$, and $\tilde{H}_{-1}(\{\emptyset\}; k) \cong k$, $\tilde{H}_i(\{\emptyset\}; k) = 0$ for each $i \geq 0$.

(1.2) Let $A = k[x_1, x_2, \ldots, x_v]$ be the polynomial ring in v variables over k. We identify each $x_i \in V$ with the indeterminate x_i of A. Define I_Δ to be the ideal of A which is generated by those square-free monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq v$, with $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \notin \Delta$. The quotient algebra $k[\Delta] := A/I_\Delta$ is called the *Stanley-Reisner ring* of Δ over k. In this paper, we consider A to be the graded ring $A = \bigoplus_{n \geq 0} A_n$ with the standard grading, i.e., each $\deg x_i = 1$, and may regard $k[\Delta] = \bigoplus_{n \geq 0} (k[\Delta])_n$ as a graded module over A with the quotient grading.

Let $A(j)$, $j \in \mathbf{Z}$, denote the graded module $A(j) = \bigoplus_{n \in \mathbf{Z}} [A(j)]_n$ over A with $[A(j)]_n := A_{n+j}$. Here $\mathbf{Z}$ is the set of integers.

(1.3) A graded *finite free resolution* of $k[\Delta]$ over A is an exact sequence

$$0 \longrightarrow \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{h_j}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{1_j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0 \quad (1)$$

of graded modules over A, where each $\bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{i_j}}$, $1 \leq i \leq h$, is a graded free module of rank $\sum_{j \in \mathbf{Z}} \beta_{i_j} < \infty$, and where each φ_i is degree-preserving. The *homological dimension* $\mathrm{hd}_A(k[\Delta])$ of $k[\Delta]$ over A is the minimal h possible in (1). It is known that $v - d \leq h \leq v$. The second inequality is Hilbert's syzygy theorem (see, e.g., [B–H, (2.2.14)]), and the first is a consequence of the Auslander-Buchsbaum formula (see, e.g., [B–H, (1.3.3)]). A finite free resolution (1) is called *minimal* if each $\sum_{j\in\mathbf{Z}}\beta_{i_j}$ is smallest possible. There exists a 'unique' minimal free resolution of $k[\Delta]$ over A.

Suppose that (1) is a minimal free resolution of $k[\Delta]$ over A; then in particular $h = \mathrm{hd}_A(k[\Delta])$. Hochster's formula [Hoc, Theorem (5.1)] guarantees that

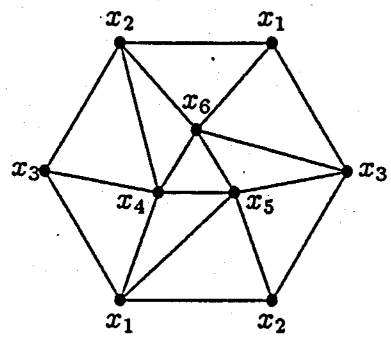
$$\beta_{i_j} = \sum_{W\subset V,\ \#(W)=j} \dim_k \tilde{H}_{j-i-1}(\Delta_W; k). \tag{2}$$

We say that $\beta_i^A = \beta_i^A(k[\Delta]) := \sum_{j\in\mathbf{Z}}\beta_{i_j}$ is the i-th *Betti number* of $k[\Delta]$ over A. A minimal free resolution (1) is called *pure* if, for each $1 \leq i \leq h$, there exists $c_i \in \mathbf{Z}$ such that $\beta_{i_j} = 0$ if $j \neq c_i$. Thus, (1) can be written as

$$0 \longrightarrow A(-c_h)^{\beta_h} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} A(-c_1)^{\beta_1} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0. \tag{3}$$

The *type* of a pure resolution (3) is the sequence $(c_1, c_2, \ldots, c_h) \in \mathbf{Z}^h$. Note that $2 \leq c_1 < c_2 < \cdots < c_h$. On the other hand, a pure resolution (3) is called *m-linear* if $c_i = (m-1)+i$ for every $1 \leq i \leq h$. We say that $k[\Delta]$ has a pure (resp. m-linear) resolution if a graded minimal free resolution of $k[\Delta]$ over A is pure (resp. m-linear). Thus, in particular, if $k[\Delta]$ has a pure resolution, then I_Δ is generated by square-free monomials of the same degree $(= c_1)$. Moreover, if $k[\Delta]$ is Cohen-Macaulay and has a pure resolution, then some relations (Herzog-Kuehl and Huneke-Miller formulas) between the type $(c_1, c_2, \ldots, c_h)$ and the Betti number sequence $(\beta_1, \beta_2, \ldots, \beta_h)$ are known, see [B–H, (4.1.17)].

(1.4) We say that a simplicial complex Δ has a pure resolution (resp. m-linear) resolution over k if $k[\Delta]$ has a pure (resp. m-linear) resolution. Note that this concept depends on the base field k. For example, if Δ is the simplicial complex (cf. [Rei]) drawn below (the facets of Δ are the triangles of the figure), then Δ has a 3-linear resolution if $\mathrm{char}(k) \neq 2$, while a minimal free resolution of $k[\Delta]$ is not pure if $\mathrm{char}(k) = 2$.



The final goal of our project is to find a combinatorial classification of simplicial complexes with pure resolutions. In the present paper, we give a characterization of Cohen-Macaulay partially ordered sets P for which the order complex $\Delta(P)$ has a pure resolution, see Theorem (3.6).

2 Simplicial complexes of dimension one

We first classify all the simplicial complexes of dimension one with pure resolutions. A simplicial complex Δ of dimension one on the vertex set V may be considered as a (simple) graph on V. We employ some standard terminologies (e.g., tree, forest, cycle) in graph theory.

(2.1) EXAMPLE. (a) Let Δ be a complete graph on the vertex set V. Then the subgraph Δ_W on W, $\emptyset \neq W \subset V$, is also complete. Hence $\tilde{H}_{\#(W)-i-1}(\Delta_W; k) = 0$ if $i \neq \#(W)-2$, and $\dim_k \tilde{H}_{\#(W)-(\#(W)-2)-1}(\Delta_W; k) = \binom{\#(W)-1}{2}$. Thus $k[\Delta]$ has a 3-linear resolution with $\mathrm{hd}_A(k[\Delta]) = \#(V)-2$ and $\beta_i^A(k[\Delta]) = \binom{i+1}{2}\binom{\#(V)}{i+2}$ for $1 \leq i \leq \#(V)-2$.

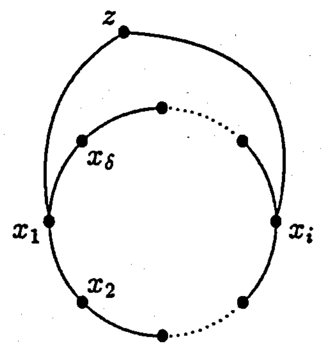
(b) Let Δ be a forest on V. If $\emptyset \neq W \subset V$, then Δ_W is a forest. Thus $\tilde{H}_{\#(W)-i-1}(\Delta_W; k) = 0$ if $i \neq \#(W)-1$, and $\dim_k \tilde{H}_{\#(W)-(\#(W)-1)-1}(\Delta_W; k)$ is one less than the number of connected components of Δ_W. Hence $k[\Delta]$ has a 2-linear resolution. If Δ is not connected, then $\mathrm{hd}_A(k[\Delta]) = \#(V)-1$. On the other hand, if Δ is connected, i.e., Δ is a tree on V, then $\mathrm{hd}_A(k[\Delta]) = \#(V)-2$ and $\beta_i^A(k[\Delta]) = i\binom{\#(V)-1}{i+1}$ for $1 \leq i \leq \#(V)-2$.

(c) Let Δ be a cycle on V. Then $\tilde{H}_{\#(V)-i-1}(\Delta; k) = 0$ if $i \neq \#(V) - 2$ and $\tilde{H}_{\#(V)-(\#(V)-2)-1}(\Delta; k) \cong k$. If $\emptyset \neq W \subset V$, $W \neq V$, then Δ_W is a forest. Thus $\mathrm{hd}_A(k[\Delta]) = \#(V) - 2$ and $k[\Delta]$ has a pure resolution of type $(2, 3, \ldots, \#(V) - 3, \#(V) - 2, \#(V))$. Note that $\beta_i^A(k[\Delta]) = \dfrac{i(\#(V) - i - 2)}{\#(V) - 1} \begin{pmatrix} \#(V) \\ i+1 \end{pmatrix}$ for $1 \leq i \leq \#(V) - 3$ and $\beta^A_{\#(V)-2}(k[\Delta]) = 1$.

We now show that a simplicial complex Δ of dimension one with a pure resolution is, in fact, one of the above (a), (b) and (c) in Example (2.1).

(2.2) PROPOSITION. *Let Δ be a simplicial complex of dimension one. Then the Stanley-Reisner ring $k[\Delta]$ has a pure resolution if and only if Δ is one of the the following*: (i) *complete graph*; (ii) *forest*; (iii) *cycle*.

Proof. Let Δ be a simplicial complex on the vertex set V with $\dim \Delta = 1$ and suppose that $k[\Delta]$ has a pure resolution. If we assume that Δ is neither a forest nor a cycle, then we can show that Δ is a complete graph in what follows. If $\{x, y\} \notin \Delta$ for some $x, y \in V$, then I_Δ is generated by square-free monomials of degree two. Hence Δ contains no cycle of length three. On the other hand, since Δ is not a forest, there exists a cycle in Δ. Thus a shortest cycle Γ in Δ is of length $\delta \geq 4$. Moreover, if $\delta = \#(V)$ then $\Delta = \Gamma$ by the minimality of the length δ of Γ, hence $\delta < \#(V)$ since Δ is not a cycle. Thus, there exists a vetex $z \in V$ which does not belong to Γ. If U is a vertex set of Γ, then $\tilde{H}_{\delta-(\delta-2)-1}(\Delta_U; k) \neq 0$. Hence, by Eq.(2), if $W \subset V$ and $\#(W) = \delta - 1$, then $\tilde{H}_{\#(W)-(\delta-2)-1}(\Delta_W; k) = 0$, i.e., Δ_W is connected. If $\delta = 4$ and $\Gamma = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_1\}\}$, then both $\Delta_{\{x_1,x_3,z\}}$ and $\Delta_{\{x_2,x_4,z\}}$ are connected, while $\{x_1, x_3\} \notin \Delta$ and $\{x_2, x_4\} \notin \Delta$, thus $\{x_3, z\} \in \Delta$ and $\{x_2, z\} \in \Delta$, hence $\Delta_{\{x_2,x_3,z\}}$ is a cycle of length three, a contradiction. On the other hand, if $\delta \geq 5$ and $\Gamma = \{\{x_1, x_2\}, \{x_2, x_3\}, \cdots, \{x_\delta, x_1\}\}$, then Δ_W with $W = \{z, x_1, \ldots, x_{\delta-2}\}$ is connected, say $\{z, x_1\} \in \Delta$. Moreover, since Δ_W with $W = \{z, x_2, \ldots, x_{\delta-1}\}$ is also connected, we have $\{z, x_i\} \in \Delta$ for some $2 \leq i \leq \delta - 1$:



This contradicts the minimality of the length δ of Γ. Q. E. D.

3 Cohen-Macaulay partially ordered sets

In this section we study a combinatorial classification of Cohen-Macaulay partially ordered sets with pure resolutions. Let P be a finite partially ordered set (*poset*, for short) and $\Delta(P)$ the *order complex* of P. Thus, $\Delta(P)$ is a simplicial complex on the vertex set P such that $C \subset P$ is a face of $\Delta(P)$ if and only if C is a *chain* (i.e., a totally ordered subset) of P. The *rank* of P is defined to be the dimension of $\Delta(P)$. We say that P is *pure* if the order complex $\Delta(P)$ is pure. If P is pure of rank $d-1$, then P is a disjoint union $P = P_0 \bigcup P_1 \bigcup \cdots \bigcup P_{d-1}$, called the *rank decomposition* of P, such that every maximal chain C of P is of the form $C : x_0 < x_1 < \cdots < x_{d-1}$ with each $x_i \in P_i$. On the other hand, we say that P has a pure resolution if the order complex $\Delta(P)$ has a pure resolution.

(3.1) LEMMA. *Let a poset P be pure of rank $d-1$ with $d \geq 3$ and $P = P_0 \bigcup P_1 \bigcup \cdots \bigcup P_{d-1}$ the rank decomposition of P. Suppose that*

(i) *the subposet $P_i \bigcup P_{i+1}$ of rank one is connected for every $0 \leq i \leq d-2$;*

(ii) *P does not contain the poset*  *as a subposet;*

(iii) *$k[\Delta]$ has a pure resolution.*

Then the number of maximal chains of P is equal to $\#(P) - d + 1$.

Proof. Set $Q_i = P_i \bigcup P_{i+1}$ for $0 \leq i \leq d-2$. First, we show that Q_i is a tree. Since $k[\Delta]$ has a pure resolution, thanks to Eq.(2), $k[\Delta(Q_i)]$ has a pure resolution. Thus, by Proposition (2.2), Q_i is either a tree or a cycle since Q_i is connected. Since $d \geq 3$, we have either $i > 0$ or $i+1 < d-1$. Let us assume $i > 0$. Since Q_{i-1} is connected, there exists $x \in P_i$ and $y, z \in P_{i-1}$ with $y \neq z$ such that $y < x$ and $z < x$. Hence, if Q_i is a cycle, then  is contained in $P_{i-1} \bigcup P_i \bigcup P_{i+1}$, thus  is contained in P as a subposet, a contradiction.

Now, let Q denote the subposet $P_0 \bigcup P_1 \bigcup \cdots \bigcup P_{d-2}$ of P. Then, by induction on d, the number of maximal chains of Q is $\#(Q) - d + 2$. (If $d = 3$, then $Q = P_0 \bigcup P_1$ is a tree, hence the number of edges of Q is $\#(Q) - 1$.) Let $P_{d-2} = \{y_1, y_2, \ldots, y_s\}$ and, for each $1 \leq j \leq s$, write p_j for the number of maximal chains C of Q with $y_j \in C$, and write q_j for the number of elements $x \in P_{d-1}$ with $y_j < x$. By (ii), if $p_j \geq 2$, then $q_j = 1$. Hence, the number of maximal chains of P is

$$\begin{aligned}
\sum_{j=1}^{s} p_j q_j &= \sum_{j=1}^{s} p_j + \sum_{j=1}^{s} (q_j - 1) \\
&= \sum_{j=1}^{s} p_j + \sum_{j=1}^{s} q_j - \#(P_{d-2}) \\
&= \sum_{j=1}^{s} p_j + \{\#(P_{d-2}) + \#(P_{d-1}) - 1\} - \#(P_{d-2}) \\
&\quad \text{(since } Q_{d-2} \text{ is a tree)} \\
&= \#(Q) - d + 2 + \#(P_{d-1}) - 1 \\
&= \#(P) - d + 1
\end{aligned}$$

as desired. Q. E. D.

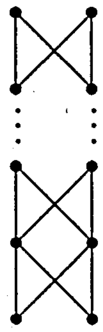
(3.2) LEMMA. *Let a poset P be pure of rank $d-1$ with $d \geq 2$ and $P = P_0 \bigcup P_1 \bigcup \cdots \bigcup P_{d-1}$ the rank decomposition of P. Suppose that*

(i) *each $\#(P_i) \geq 2$;*

(ii) *P contains the poset*  *as a subposet;*

(iii) *$k[\Delta]$ has a pure resolution.*

Then P is the complete intersection of rank $d-1$, i.e., P is the poset drawn below.



Proof. Since $\dim_k \tilde{H}_{4-2-1}(\Delta($$); k) = 1$, by (iii) and Eq.(2), every three-element subposet of P must be connected, i.e., P contains neither • • • nor • as a subposet. Hence each $\#(P_i) = 2$ and $P_i \bigcup P_{i+1} =$  as required.

Q. E. D.

We here recall some fundamental material on Cohen-Macaulay partially ordered sets. Let Δ be a simplicial complex of dimension $d-1$ and $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ the *f-vector* of Δ, i.e., f_i is the number of faces σ of Δ with $\#(\sigma) = i+1$. Define the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of Δ by

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}$$

with $f_{-1} := 1$. The *Hilbert series* of $k[\Delta] = \bigoplus_{n\geq 0}(k[\Delta])_n$ is the formal power series $F(k[\Delta], \lambda) = \sum_{n=0}^{\infty}(\dim_k(k[\Delta])_n)\lambda^n$. Then $F(k[\Delta], \lambda) = (h_0+h_1+\cdots+h_d\lambda^d)/(1-\lambda)^d$. We say that Δ is *Cohen-Macaulay* over k if $k[\Delta]$ is Cohen-Macaulay, i.e., $\mathrm{hd}_A(k[\Delta]) = v-d$ (with the notation in (1.3)). (See, e.g., [Sta], [H], [B–H] for the 'familiar' definition of Cohen-Macaulay rings with systems of parameters and regular sequences.) The h-vector $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of a Cohen-Macaulay complex Δ is non-negative, i.e., each $h_i \geq 0$. A one-dimensional simplicial complex Δ is Cohen-Macaulay if and only if Δ is connected. Every Cohen-Macaulay complex is pure. A finite poset P is called Cohen-Macaulay if the order complex $\Delta(P)$ is Cohen-Macaulay. Let P be a Cohen-Macaulay poset of rank $d-1$ with the rank decomposition $P = P_0 \bigcup P_1 \bigcup \cdots \bigcup P_{d-1}$. Then every rank-selected subposet $P_{i_0} \bigcup P_{i_1} \bigcup \cdots \bigcup P_{i_j}$,

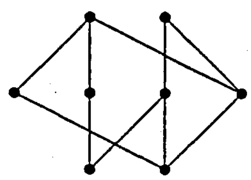
$0 \leq i_0 < i_1 < \cdots < i_j \leq d-1$, is Cohen-Macaulay. Consult [Hoc], [Sta], [B–H] and [H] for further information.

(3.3) PROPOSITION. *Suppose that P is a Cohen-Macaulay poset of rank $d-1$ such that the number of maximal chains of P is equal to $\#(P)-d+1$. Then $k[\Delta(P)]$ has a pure resolution.*

Proof. Let $f(\Delta(P)) = (f_0, f_1, \ldots, f_{d-1})$ be the f-vector of $\Delta(P)$ and $h(\Delta(P)) = (h_0, h_1, \ldots, h_d)$ the h-vector of $\Delta(P)$. Recall that $h_0 = 1$, $h_1 = f_0 - d$ and $f_{d-1} = h_0 + h_1 + \cdots + h_d$. Thus, it follows from $f_{d-1} = \#(P) - d + 1$ (and $f_0 = \#(P)$) that $h_2 + \cdots + h_d = 0$, while each $h_i \geq 0$ since P is Cohen-Macaulay. Hence $h_2 = \cdots = h_d = 0$. Thus, the Hilbert Series $F(k[\Delta], \lambda)$ $k[\Delta]$ is $(1 + (\#(P) - d)\lambda)/(1-\lambda)^d$.

It is known, see, e.g., [B–H, Exercise 4.1.7] (and, in fact, not difficult to prove), that if an ideal I of $A = k[x_1, x_2, \ldots, x_v]$ is generated by homogeneous polynomials of degree m (≥ 2) and if $R = A/I = \bigoplus_{n \geq 0} R_n$ is Cohen-Macaulay, then a minimal free resolution of R over A is m-linear if and only if the Hilbert Series $F(R, \lambda) = \sum_{n=0}^{\infty} (\dim_k R_n)\lambda^n$ is of the form $(h_0 + h_1\lambda + \cdots + h_{m-1}\lambda^{m-1})/(1-\lambda)^d$ with each $0 < h_i \in \mathbf{Z}$ for some $d \geq 0$. By virtue of this ring-theoretical result, a minimal free resolution of our $k[\Delta(P)]$ turns out to be 2-linear as desired. Q. E. D.

(3.4) EXAMPLE. Let P be the pure poset drawn below. Then the h-vector of $\Delta(P)$ is $(1, 5, -1, 1)$, thus P is not Cohen-Macaulay, while the number of maximal chains of P is $\#(P) - 3 + 1$. Since P contains , , and , as subposets, a minimal free resolution of $k[\Delta(P)]$ is not pure.



(3.5) EXAMPLE. Let P be the complete intersection of rank $d-1$ discussed in Lemma (3.2). Then P is Cohen-Macaulay and $k[\Delta(P)]$ has a pure resolution of type $(2,4,\ldots,2d)$ with Betti numbers $\beta_i^A(k[\Delta(P)]) = \binom{d}{i}$, $1 \le i \le d$ $(= \mathrm{hd}_A(k[\Delta]))$.

We now come to the main result of this paper.

(3.6) THEOREM. *Suppose that P is a Cohen-Macaulay partially ordered set of rank $d-1$ (with $d \ge 2$) which possesses the rank decomposition $P = P_0 \bigcup P_1 \bigcup \cdots \bigcup P_{d-1}$ with each $\#(P_i) \ge 2$. Then the Stanley-Reisner ring $k[\Delta(P)]$* of *P has a pure resolution if and only if we have one of the following:*

(i) *$d = 2$ and P is a cycle;*

(ii) *$d \ge 3$ and P is the complete intersection in* Lemma (3.2);

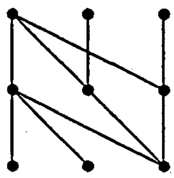
(iii) *the number of maximal chains of P is $\#(P) - d + 1$.*

Proof. First, by Example (2.1), Example (3.5) and Proposition (3.3), if P is one of the (i), (ii), and (iii), then $k[\Delta(P)]$ has a pure resolution. On the other hand, suppose that $k[\Delta(P)]$ has a pure resolution. If $d = 2$, then thanks to Proposition (2.2), P is either (i) a cycle or (iii) a tree, since P is connected. Let us assume $d > 2$. Note that every subposet $P_i \bigcup P_{i+1}$ of rank one is connected since $P_i \bigcup P_{i+1}$ is Cohen-Macaulay. Thus, by Lemma (3.1) together with Lemma (3.2), P is either (ii) or (iii) as required. Q. E. D.

REMARK. (a) Every simplicial complex of dimension zero has a 2-linear resolution.

(b) If $\#(P_i) = 1$ in the above Theorem (3.6), then P has a pure resolution if and only if the subposet $P_0 \bigcup \cdots \bigcup P_{i-1} \bigcup P_{i+1} \bigcup \cdots \bigcup P_{d-1}$ of P has a pure resolution.

(3.7) EXAMPLE. The Cohen-Macaulay poset P drawn below is an example of (iii) in Theorem (3.6).



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