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**THE ASYMPTOTIC BEHAVIOUR
OF RADIAL SOLUTIONS NEAR THE
BLOW-UP POINT TO QUASI-LINEAR
WAVE EQUATIONS IN TWO SPACE
DIMENSIONS**

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THE ASYMPTOTIC BEHAVIOUR OF RADIAL SOLUTIONS NEAR THE BLOW-UP POINT TO QUASI-LINEAR WAVE EQUATIONS IN TWO SPACE DIMENSIONS

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1. Introduction.

Consider the Cauchy problem:

$$u_{tt} - c^2(u_t, u_r)(u_{rr} + \frac{1}{r}u_r) = \frac{1}{r}u_r G(u_t, u_r), \quad (r, t) \in (0, \infty) \times (0, T_\epsilon), \quad (1.1)$$

$$u(r, 0) = \epsilon f(r), \quad u_t(r, 0) = \epsilon g(r), \quad r \in (0, \infty), \quad (1.2)$$

where

$$c(u_t, u_r) = 1 + \frac{a_1}{2}u_t^2 + \frac{a_2}{2}u_t u_r + \frac{a_3}{2}u_r^2 + O(|u_t|^3 + |u_r|^3),$$

$$G(u_t, u_r) = O(u_r^2 + u_t^2),$$

near $u_t = u_r = 0$. Equation (1.1) is a radially symmetric form of quasi-linear wave equation in two space dimensions which involves the equation of vibrating membrane. In [4], we obtained the following blow up result:

$$\limsup_{\epsilon \rightarrow 0} \epsilon^2 \log(1 + T_\epsilon) \leq \frac{1}{H},$$

where T_ϵ is the lifespan of the radial solution of the Cauchy problem (1.1), (1.2) and H is a constant depending only on f, g and $\partial^2 c(0, 0)$. More precisely, the blow up occurs as follows. If we set

$$w(r, t) = \frac{c(u_t, u_r)v_{rr} - v_{rt}}{2c(u_t, u_r)} \quad \text{with} \quad v(r, t) = r^{\frac{1}{2}}u(r, t),$$

then we find that

$$|w(r, t)| \longrightarrow \infty \quad \text{as} \quad \epsilon^2 \log(1 + t) \rightarrow \frac{1}{H}$$

along a pseudo-characteristic curve for sufficiently small ϵ .

In this paper, we investigate the asymptotic behaviour of $w(r, t)$ when $\varepsilon^2 \log(1+t)$ tends to $\frac{1}{H}$.

2. Statement of Results.

As we did in [3], we assume $f, g \in C_0^\infty(\mathbb{R}^2)$, $|f| + |g| \not\equiv 0$ and $f(r) = g(r) = 0$ for $r \geq M$. Moreover we assume $a_1 - a_2 + a_3 = a \neq 0$ which means (1.1) does not satisfy the *null-condition*. Then we can define a positive constant H by

$$\begin{aligned} H &= \max_{\rho \in \mathbb{R}} (-a\mathcal{F}'(\rho)\mathcal{F}''(\rho)) \\ &= -a\mathcal{F}'(\rho_0)\mathcal{F}''(\rho_0), \end{aligned}$$

where $\mathcal{F}(\rho)$ is the Friedlander radiation field which is constructed by f and g (see [4]). We introduce a variable $s = \varepsilon^2 \log(1+t)$ and we write $t = t_X$ when $s = X$, i.e.,

$$X = \varepsilon^2 \log(1 + t_X).$$

To state our results, we have to recall the facts which are obtained in [4].

Firstly, for any $B > H$ we consider the Burgers equation:

$$U_{\rho s} + \frac{a}{2}(U_{\rho})^2 U_{\rho\rho} = 0, \quad (\rho, s) \in \mathbb{R} \times [0, \frac{1}{B}],$$

$$U_{\rho}(\rho, 0) = \mathcal{F}'(\rho), \quad \rho \in \mathbb{R},$$

then, there exists an $\varepsilon(B) > 0$ such that the Cauchy problem (1.1), (1.2) has a smooth solution in $0 \leq t \leq t_{\frac{1}{B}}$ and the following holds.

$$\begin{aligned} |\partial_r^l \partial_t^m u(r, t_{\frac{1}{B}}) - \varepsilon r^{-\frac{1}{2}} (-1)^m \partial_{\rho}^{l+m} U(r - t_{\frac{1}{B}}, \frac{1}{B})| &\leq C_{l,m,B} \varepsilon^{\frac{5}{4}} r^{-\frac{1}{2}} \\ \text{for } r - t_{\frac{1}{B}} &> -\frac{1}{3\varepsilon} \quad \text{and } l+m \neq 0 \end{aligned} \quad (2.1)$$

for $\varepsilon < \varepsilon(B)$. Moreover, U satisfies

$$\begin{aligned} U(\rho(s), s) &= \mathcal{F}'(\rho_0), \\ U_{\rho\rho}(\rho(s), s) &= \frac{\mathcal{F}''(\rho_0)}{1 + a\mathcal{F}'(\rho_0)\mathcal{F}''(\rho_0)s} = \frac{\mathcal{F}''(\rho)}{1 - Hs}, \end{aligned} \quad (2.2)$$

for $0 \leq s \leq \frac{1}{B}$ along the curve Λ_{ρ_0} defined by

$$\frac{d\rho}{ds} = \frac{a}{2}(U_{\rho})^2 \quad \text{for } s \geq 0, \quad \rho = \rho_0 \quad \text{for } s = 0.$$

These facts are proved in section 3 of [4] by using the energy inequality and the Klainerman inequality.

Secondly, we define a pseudo-characteristic curve Z by

$$\frac{dr}{dt} = c(u_t, u_r) \quad \text{for } t \geq t_{\frac{1}{B}}, \quad r = \rho\left(\frac{1}{B}\right) + t_{\frac{1}{B}} \quad \text{for } t = t_{\frac{1}{B}}$$

and a function w by

$$w(r, t) = \frac{cv_{rr} - v_{rt}}{2c} \quad \text{with} \quad v(r, t) = r^{\frac{1}{2}}u(r, t).$$

Then, for any $A < H$ there exists an $\bar{\varepsilon}(A) > 0$ such that if $\varepsilon < \bar{\varepsilon}(A)$, then w should satisfy

$$w'(t) = \alpha_0(t)w(t)^2 + \alpha_1(t)w(t) + \alpha_2(t) \quad \text{for} \quad t_{\frac{1}{B}} \leq t \leq t_{\frac{1}{A}}, \quad (2.3)$$

$$w(t_{\frac{1}{B}}) = \varepsilon U_{\rho\rho}(\rho(\frac{1}{B}), \frac{1}{B}) + O(\varepsilon^{\frac{5}{4}}), \quad (2.4)$$

where

$$w(t) = w(r(t), t) \quad \text{for} \quad (r(t), t) \in Z$$

and

$$\begin{aligned} \alpha_0(t) &= -a\varepsilon \mathcal{F}'(\rho_0)(1+t)^{-1} + O(\varepsilon^{\frac{5}{4}}(1+t)^{-1}), \\ \alpha_1(t) &= O(\varepsilon^4(1+t)^{-1} + \varepsilon^2(1+t)^{-2}), \\ \alpha_2(t) &= O(\varepsilon(1+t)^{-2}), \end{aligned} \quad (2.5)$$

as long as u exists. Here $X = O(Y)$ means $|X| \leq CY$ with constant C depending only on B, f, g, ρ_0, a and M . This fact is proved in section 4 and 5 of [4] by using (2.1), (2.2) and *a priori* estimates of u .

Now we state our results.

Theorem. For any $\delta > 0$ there exists an $\varepsilon_\delta > 0$ such that $w(t)$ is well-defined in $t_{\frac{1}{B}} \leq t \leq t_{\frac{1}{H}-\delta}$ for $\varepsilon < \varepsilon_\delta$ and at the point $t = t_{\frac{1}{H}-\delta}$,

$$\lim_{\varepsilon \rightarrow 0} \left(\frac{1}{H} - \varepsilon^2 \log(1+t) \right) \frac{w(t)}{\varepsilon} = \frac{1}{H} \mathcal{F}''(\rho_0)$$

holds.

However, since we are interested in the behaviour of w when $\varepsilon^2 \log(1+t)$ tends to $\frac{1}{H}$, we reduce the above result into

Corollary.

$$\lim_{\varepsilon \rightarrow 0, \varepsilon^2 \log(1+t) \rightarrow \frac{1}{H}} \left(\frac{1}{H} - \varepsilon^2 \log(1+t) \right) \frac{w(t)}{\varepsilon} = \frac{1}{H} \mathcal{F}''(\rho_0).$$

In three space dimensions, for the radial solution of the Cauchy problem:

$$\begin{aligned} u_{tt} - c^2(u_t)(u_{rr} + \frac{2}{r}u_r) &= 0, \\ u(r, 0) &= \varepsilon f(r), \quad u_t(r, 0) = \varepsilon g(r), \end{aligned}$$

with $c(u_t) = 1 + au_t + O(u_t^2)$ and $a \neq 0$, F. John [5] and L. Hörmander [2] have shown a blow up result

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log(1 + T_\varepsilon) \leq \frac{1}{\max(a\mathcal{F}''(\rho))}.$$

In this case, if we set $H = \max(a\mathcal{F}''(\rho)) = a\mathcal{F}''(\rho_0)$, we also expect

$$\lim_{\varepsilon \rightarrow 0, \varepsilon \log(1+t) \rightarrow \frac{1}{H}} \left(\frac{1}{H} - \varepsilon \log(1+t) \right) \frac{w(t)}{\varepsilon} = \frac{1}{H} \mathcal{F}''(\rho_0),$$

which would be obtained in parallel.

For the non radially symmetric case, S. Alinhac [1] studies the Cauchy problem

$$\partial_t^2 u - \Delta u = \sum_{i,j,k=0}^2 g_{ij}^k \partial_k u \partial_{ij}^2 u, \quad (x, t) \in \mathbb{R}^2 \times (0, T_\varepsilon),$$

$$u(x, 0) = u^0(x; \varepsilon), \quad u_t(x, 0) = u^1(x; \varepsilon), \quad x \in \mathbb{R}^2,$$

where $\partial_0 = \partial_t$ and g_{ij}^k are constants. Note that this problem differs from ours in the power of $\partial_k u$. If u^0, u^1 and g_{ij}^k satisfy the *non degenerate* condition (ND), he finds the *asymptotic lifespan* T_ε^a which satisfies the following: For any $N \in \mathbb{N}$, there exists an $\varepsilon_N > 0$ such that if $\varepsilon < \varepsilon_N$, then

$$T_\varepsilon > T_\varepsilon^a - \varepsilon^N$$

and

$$\frac{1}{C} \leq (T_\varepsilon^a - t) \|\partial^2 u(t)\|_{L_x^\infty} \leq C \quad \text{for} \quad \frac{C}{\varepsilon^2} \leq t \leq T_\varepsilon^a - \varepsilon^N$$

holds for some constant C . Since he estimates $\partial^2 u$ not along a pseud-characteristic curve but in whole space $\mathbb{R}^2$, it seems difficult to determine the constant C .

In the rest of this paper, we concentrate on the proof of Theorem.

3. Proof of Theorem.

In [3], we have proved that there exists an $\varepsilon_1(\delta) > 0$ such that for $\varepsilon < \varepsilon_1$ the Cauchy problem (1.1), (1.2) has a smooth solution u in $0 \leq t \leq t_{\frac{1}{H}-\delta}$ and therefore $w(t)$ is well-defined in $t_{\frac{1}{B}} \leq t \leq t_{\frac{1}{H}-\delta}$. Thus we have only to prove that for any $\eta > 0$ there exists an $\varepsilon_0(\delta, \eta) > 0$ such that

$$\left| \left(\frac{1}{H} - s \right) \frac{w(t)}{\varepsilon} - \frac{1}{H} \mathcal{F}''(\rho_0) \right| < \eta$$

for $\varepsilon < \varepsilon_0$ and $s = \frac{1}{H} - \delta$. If we take $\frac{1}{A} = \frac{1}{H} + \delta$ in the argument in section 2, there exist an $\varepsilon_2(\delta) > 0$ such that if $\varepsilon < \varepsilon_2$, $w(t)$ should satisfy the ordinary differential equation (2.3), (2.4) in $t_{\frac{1}{B}} \leq t \leq t_{\frac{1}{H}+\delta}$ as long as u exists. Thus we find that for $\varepsilon < \min(\varepsilon_1, \varepsilon_2)$ the ordinary differential equation (2.3), (2.4) make sense in $t_{\frac{1}{B}} \leq t \leq t_{\frac{1}{H}-\delta}$.

Now the following lemma is useful.

Lemma. Let $w(t)$ be a solution of the ordinary differential equation

$$w'(t) = \alpha_0(t)w(t)^2 + \alpha_1(t)w(t) + \alpha_2(t) \quad \text{for } t_0 \leq t \leq T$$

and assume

$$\begin{aligned} \alpha_0(t) &\geq 0 \quad \text{for } t_0 \leq t \leq T, \\ w(t_0) &> K \end{aligned}$$

where

$$K = \int_{t_0}^T |\alpha_2(t)| \exp\left(-\int_{t_0}^t \alpha_1(\tau) d\tau\right) dt.$$

Then $w(t)$ satisfies

$$w(t) \exp\left(-\int_{t_0}^t \alpha_1(\tau) d\tau\right) \geq \frac{w(t_0) - K}{1 - (w(t_0) - K) \int_{t_0}^t \alpha_0(\tau) \exp\left(\int_{t_0}^\tau \alpha_1(\xi) d\xi\right) d\tau}$$

and

$$w(t) \exp\left(-\int_{t_0}^t \alpha_1(\tau) d\tau\right) \leq \frac{w(t_0) + K}{1 - (w(t_0) + K) \int_{t_0}^t \alpha_0(\tau) \exp\left(\int_{t_0}^\tau \alpha_1(\xi) d\xi\right) d\tau}.$$

Proof of Lemma. At first we consider the case $\alpha_1(t) \equiv 0$. Let $w_1(t)$ be a solution of

$$w_1'(t) = \alpha_0(t)(w_1(t) - K)^2, \quad (3.1)$$

$$w_1(t_0) = w(t_0) \quad (3.2)$$

and set

$$w_2(t) = \int_{t_0}^t |\alpha_2(\tau)| d\tau.$$

Since $\alpha_0(t) \geq 0$, we find that

$$w_1(t) \geq w(t_0) > K = w_2(T) \geq w_2(t)$$

and that

$$\begin{aligned} (w_1(t) - w_2(t))' &= \alpha_0(t)(w_1(t) - K)^2 - |\alpha_2(t)| \\ &\leq \alpha_0(t)(w_1(t) - w_2(t))^2 + \alpha_2(t), \\ w_1(t_0) - w_2(t_0) &= w(t_0). \end{aligned}$$

Thus the usual comparison theorem leads

$$w_1(t) - w_2(t) \leq w(t). \quad (3.3)$$

By solving the ordinary differential equation (3.1), (3.2), $w_1(t)$ is represented by

$$w_1(t) = K + \frac{w(t_0) - K}{1 - (w(t_0) - K) \int_{t_0}^t \alpha_0(\tau) d\tau}.$$

Substituting this equality into (3.3), we find

$$\begin{aligned} w(t) &\geq K + \frac{w(t_0) - K}{1 - (w(t_0) - K) \int_{t_0}^t \alpha_0(\tau) d\tau} - w_2(t) \\ &\geq \frac{w(t_0) - K}{1 - (w(t_0) - K) \int_{t_0}^t \alpha_0(\tau) d\tau}. \end{aligned}$$

On the other hand, if we let $w_3(t)$ be a solution of

$$w_3'(t) = \alpha_0(t)(w_3(t) + K)^2,$$

$$w_3(t_0) = w(t_0),$$

then we find

$$\begin{aligned} (w_3(t) + w_2(t))' &= \alpha_0(t)(w_3(t) + K)^2 + |\alpha_2(t)| \\ &\geq \alpha_0(t)(w_3(t) + w_2(t))^2 + \alpha_2(t), \\ w_3(t_0) + w_2(t_0) &= w(t_0). \end{aligned}$$

Thus we obtain

$$w_3(t) + w_2(t) \geq w(t).$$

Since $w_3(t)$ is represented by

$$w_3(t) = -K + \frac{w(t_0) + K}{1 - (w(t_0) + K) \int_{t_0}^t \alpha_0(\tau) d\tau},$$

we obtain

$$\begin{aligned} w(t) &\leq -K + \frac{w(t_0) + K}{1 - (w(t_0) + K) \int_{t_0}^t \alpha_0(\tau) d\tau} + w_2(t) \\ &\leq \frac{w(t_0) + K}{1 - (w(t_0) + K) \int_{t_0}^t \alpha_0(\tau) d\tau}. \end{aligned}$$

For the general case, setting

$$W(t) = w(t) \exp\left(-\int_{t_0}^t \alpha_1(\tau) d\tau\right)$$

and applying the result we have just proved to $W(t)$, we obtain the inequalities we wanted.

Now we want to apply Lemma to (2.3), (2.4) as $t_0 = t_{\frac{1}{B}}$ and $T = t_{\frac{1}{H}-\delta}$. By (2.5), we have

$$\begin{aligned} \exp\left(\pm \int_{t_{\frac{1}{B}}}^t \alpha_1(\tau) d\tau\right) &= \exp\left(O\left(\int_{t_{\frac{1}{B}}}^t \varepsilon^4(1+\tau)^{-1} d\tau\right)\right) \\ &= \exp\left(O(\varepsilon^4 \log(1+t)) + O(\varepsilon^4 \log(1+t_{\frac{1}{B}}))\right) \\ &= \exp(O(\varepsilon^2)) = 1 + O(\varepsilon^2) \quad \text{for } t_{\frac{1}{B}} \leq t \leq t_{\frac{1}{H}-\delta}, \end{aligned}$$

$$\begin{aligned}
K &= \int_{t_{\frac{1}{B}}}^{t_{\frac{1}{H}-\delta}} |\alpha_2(t)| \exp\left(-\int_{t_{\frac{1}{B}}}^t \alpha_1(\tau) d\tau\right) dt \\
&= O((1+\varepsilon^2)\varepsilon \int_{t_{\frac{1}{B}}}^{t_{\frac{1}{H}-\delta}} (1+t)^{-2} dt) \\
&= O(\varepsilon(1+t_{\frac{1}{B}})^{-1}) + O(\varepsilon(1+t_{\frac{1}{H}-\delta})^{-1}) \\
&= O(\varepsilon^3), \\
&\int_{t_{\frac{1}{B}}}^t \alpha_0(\tau) \exp\left(\int_{t_{\frac{1}{B}}}^{\tau} \alpha_1(\xi) d\xi\right) d\tau \\
&= (1+O(\varepsilon^2))(-a\varepsilon\mathcal{F}'(\rho_0) + O(\varepsilon^{\frac{5}{4}})) \int_{t_{\frac{1}{B}}}^t (1+\tau)^{-1} d\tau \\
&= (-a\varepsilon\mathcal{F}'(\rho_0) + O(\varepsilon^{\frac{5}{4}}))(\log(1+t) - \log(1+t_{\frac{1}{B}})) \\
&\quad \text{for } t_{\frac{1}{B}} \leq t \leq t_{\frac{1}{H}-\delta}.
\end{aligned} \tag{3.4}$$

Since $H > 0$, $-a\mathcal{F}'(\rho_0)$ and $\mathcal{F}''(\rho_0)$ have the same sign. Without loss of generality, we can assume that both are positive and then it follows from (2.4) and (3.4) that there exists an $\varepsilon_3 > 0$ such that

$$w(t_{\frac{1}{B}}) > K$$

and

$$\alpha_0(t) \geq 0$$

hold for $\varepsilon < \varepsilon_3$. Thus we can apply Lemma and obtain

$$\begin{aligned}
&(1 + C\varepsilon^2)w(t) \\
&\geq \frac{w(t_{\frac{1}{B}}) - C\varepsilon^3}{1 - (w(t_{\frac{1}{B}}) - C\varepsilon^3)(-a\varepsilon\mathcal{F}'(\rho_0)U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{5}{4}})(\log(1+t) - \log(1+t_{\frac{1}{B}}))} \\
&= \frac{\varepsilon U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{5}{4}}}{1 - (-a\mathcal{F}'(\rho_0)U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{1}{4}})(s - \frac{1}{B})} \quad \text{for } \frac{1}{B} \leq s \leq \frac{1}{H} - \delta,
\end{aligned}$$

where $U_{\rho\rho}(\frac{1}{B}) = U_{\rho\rho}(\rho(\frac{1}{B}), \frac{1}{B})$ and C is a constant depending only on B, f, g, ρ_0, a and M and it varies from line to line. By (2.4), we get

$$\begin{aligned}
\frac{w(t)}{\varepsilon} &\geq (1 - C\varepsilon^2) \frac{U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{1}{4}}}{1 - (-a\mathcal{F}'(\rho_0)U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{1}{4}})(s - \frac{1}{B})} \\
&= \frac{U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{1}{4}}}{1 - \frac{s - \frac{1}{B}}{\frac{1}{H} - \frac{1}{B}} + C\varepsilon^{\frac{1}{4}}} \\
&= \frac{\frac{1}{H}\mathcal{F}''(\rho_0) - C\varepsilon^{\frac{1}{4}}}{\frac{1}{H} - s + C\varepsilon^{\frac{1}{4}}} \quad \text{for } \frac{1}{B} \leq s \leq \frac{1}{H} - \delta.
\end{aligned}$$

If we set $s = \frac{1}{H} - \delta$, we have

$$\begin{aligned} \left(\frac{1}{H} - s\right) \frac{w(t)}{\varepsilon} &\geq \left(\frac{1}{H} \mathcal{F}''(\rho_0) - C\varepsilon^{\frac{1}{4}}\right) \frac{\frac{1}{H} - s}{\frac{1}{H} - s + C\varepsilon^{\frac{1}{4}}} \\ &= \frac{1}{H} \mathcal{F}''(\rho_0) \frac{\delta}{\delta + C\varepsilon^{\frac{1}{4}}} - \frac{C\delta\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}} \\ &= \frac{1}{H} \mathcal{F}''(\rho_0) - \frac{C\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}} - \frac{C\delta\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}}. \end{aligned}$$

There exists an $\varepsilon_4(\delta, \eta) > 0$ such that if $\varepsilon < \varepsilon_4$, then

$$\frac{C\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}} + \frac{C\delta\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}} < \eta,$$

i.e.,

$$\left(\frac{1}{H} - s\right) \frac{w(t)}{\varepsilon} - \frac{1}{H} \mathcal{F}''(\rho_0) > -\eta$$

holds. Similarly, using the other inequality in Lemma we find that there exists an $\varepsilon_5(\delta, \eta) > 0$ such that if $\varepsilon < \varepsilon_5$, then

$$\left(\frac{1}{H} - s\right) \frac{w(t)}{\varepsilon} - \frac{1}{H} \mathcal{F}''(\rho_0) < \eta$$

holds. Thus if we take $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5)$, we find that

$$\left| \left(\frac{1}{H} - s\right) \frac{w(t)}{\varepsilon} - \frac{1}{H} \mathcal{F}''(\rho_0) \right| < \eta$$

holds for $\varepsilon < \varepsilon_0$ and this completes the proof of Theorem.

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