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Existence of periodically evolving convex curves moved by anisotropic curvature

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1 Introduction.

This note reports our recent results [2] on the existence of time-periodic solution of curvature flow equations in the plane. The present paper includes a natural extension of results in [2].

Let $\{\Gamma_t\}$ be a smooth one parameter family of closed embedded curves bounding a domain in the plane. Let θ be the argument of the inward unit normal $\mathbf{n}$ of Γ_t. The normal velocity of Γ_t in the direction of $\mathbf{n}$ denotes V. We consider an equation of Γ_t of the form

$$V = a(\theta)k - Q(\theta, t), \tag{1}$$

where k is the inward curvature of Γ_t and a and Q are given functions. Since θ is argument, a and Q are assumed to be 2π -periodic in θ, i.e., $a(\theta + 2\pi) = a(\theta)$ and $Q(\theta + 2\pi, t) = Q(\theta, t)$. We assume that a is strictly positive so that our problem is parabolic.

Existence Theorem. Assume that Q is T-periodic in time, i.e., $Q(\theta, T+t) = Q(\theta, t)$ for all $0 \leq \theta < 2\pi, t \in \mathbf{R}$. Assume that $a > 0$ and $Q > 0$ is continuous with partial derivatives $Q_{\theta\theta}, Q_t, Q_{\theta\theta t}$. Assume that

$$Q_{\theta\theta} + Q > 0 \quad \text{for all } \theta \text{ and } t. \tag{2}$$

Then there are a constant vector $\mathbf{c} \in \mathbf{R}^2$ and a closed evolving curve Γ_t solving (1) and

$$\Gamma_{t+T} = \Gamma_t + \mathbf{c} \quad \text{for all } t \in \mathbf{R}. \tag{3}$$

The curvature of Γ_t is always positive and the quantities in (1) is continuous. If Q is smooth, so is Γ_t.

This shows the existence of time-periodic solution of (1) when Q is time-pereiodic. Several exmples of (1) are provided in [3] where a standard form of (1) for thermodynamics is derived. A general motion by anisotropic curvature is described as

$$V = \frac{1}{\beta(\theta)}((\sigma''(\theta) + \sigma(\theta))k - c(t)) \tag{4}$$

where $\beta > 0$ is called a kinetic coefficient and $\sigma > 0$ is called the surface energy density of material; $c(t)$ is the temperature difference. Since the condition (2) is equivalent to say that the Frank diagram of $Q(\cdot, t)$ has a positive curvature everywhere for $Q > 0$, our Existence Theorem yields:

Corollary. Assume that $\beta > 0, \sigma > 0$ and c are continuous with the second derivative σ''. Assume that c is T-periodic and that the Frank diagram of σ and $1/\beta$ has a positive curvature everywhere. Then there is a closed evolving curve Γ_t solving (4) which is T-periodic in the sense of (3). The curvature of Γ_t is always positives.

In our previous paper [2] these existence results are proved only for (1) with $a \equiv 1$. It turns out the method applies to general (1) with a small modification.

Curvature evolution equations. Since a solution Γ_t we seek is convex, we may use θ as a coordinate to represent Γ_t. In θ -coordinate evolution of curvature is described by

$$k_t = k^2(V_{\theta\theta} + V)$$

as in [3]. If Γ_t evolves by (1), k fulfills

$$k_t = k^2((ak)_{\theta\theta} + ak - (Q_{\theta\theta} + Q)) \tag{5}$$

Since Γ_t is closed, k fulfills the constraint

$$\int_0^{2\pi} \frac{e^{i\theta}}{k(\theta, t)} d\theta = 0. \tag{6}$$

Of course, since θ is the argument of a normal, k is 2π -periodic in θ. As in [2] to show Existence Theorem it suffices to find a positive T-periodic solution k of (5), (6) with 2π -periodicity in θ. To simplify the notation we set

$$\mathrm{T} = \mathrm{R}/2\pi\mathrm{Z} \quad \text{and} \quad K = \mathrm{T} \times (\mathrm{R}/T\mathrm{Z}).$$

By $h \in C(K)$ we mean that h is continuous in R^2 and that $h(x,t)$ is 2π -periodic in x and T-periodic in time t. As in [2], the following existence result implies the existence of k satisfying (5), (6) by setting $Q_{\theta\theta} + Q = f, \theta = x, k = w$ and it yields our Existence Theorem.

Theorem 1. Assume that $a \in C(\mathrm{T})$ is positive. Assume that $f \in C(K)$ with $f > 0$ and $f_t \in C(K)$ satisfies

$$\int_0^{2\pi} f(x,t) e^{ix} dx = 0 \quad \text{for all } t \in \mathrm{R}. \tag{7}$$

Then there is a positive solution $w \in C(K)$ (with $aw \in \bigcap_{p>1} W_p^{2,1}(K)$)

$$w_t = w^2((aw)_{xx} + aw - f) \quad \text{in } K \tag{8}$$

satisfying

$$\int_0^{2\pi} \frac{e^{ix}}{w(x,t)} dx = 0 \quad \text{for all } t \in \mathbf{R}. \tag{9}$$

Here $W_p^{2,1}$ denotes L^p - Sobolev space of order 2 in x, 1 in t.

If we set $u = aw$, u solves

$$au_t = u^2(u_{xx} + u - f).$$

The outline of the proof of Theorem 1 is the same as that of the Main Existence Theorem in [2] where a is assumed to equal one. Instead of presenting a whole proof, we point out necessary alternations when a depends on space variable x.

The main idea is to get a priori lower and upper bounds for approximate penalized equations admitting a solution. The penalty method applies to recover constraint (9).

The biography of [2] includes many references to recent work on equations of form

$$u_t = u^\gamma(u_{xx} + g(u,x,t)), \quad \gamma \geq 1$$

where g is a given function. We do not repeat it again here.

2 Harnack type inequalities.

In this section, we consider the equation

$$au_t = u^\gamma\{u_{xx} + g(u,x,t)\} \quad \text{in } K, \tag{10}$$

where $\gamma \in \mathrm{R}$ and a is a continuous positive function on K with $a_t \in C(K)$.

Putting $z = u_t/u$, we have

$$z_x = \frac{u_{xt}}{u} - \frac{u_x u_t}{u^2},$$

$$z_{xx} = \frac{u_{xxt}}{u} - \frac{2u_x z_x}{u} - \frac{u_{xx} u_t}{u^2}.$$

Differentiating $az = u^{\gamma-1}(u_{xx} + g)$ in t yields

$$az_t = u^\gamma z_{xx} + 2u^{\gamma-1} u_x z_x + \gamma a z^2 + \{u^{\gamma-1}(g_u u - g) - a_t\} z + u^{\gamma-1} g_t.$$

Let (x_0, t_0) be a minimizer of z over K. Then we have

$$\gamma a z^2 + \{u^{\gamma-1}(g_u u - g) - a_t\} z + u^{\gamma-1} g_t \leq 0 \quad \text{at } (x_0, t_0)$$

and hence

$$z \geq -u^{\gamma-1} \frac{(g_u u - g)_+}{\gamma a} - \frac{(a_t)_+}{\gamma a} - u^{\frac{\gamma-1}{2}} \frac{|g_t|^{1/2}}{(\gamma a)^{1/2}} \quad \text{at } (x_0, t_0).$$

Such differential identity is obtained for (8) with $a = 1, f = 0$ by Gage [1]. Inequalities of Harnack type in time direction (Lemma1) and in space direction (Lemma2) follow from this estimate of $\min\limits_K z$ as in [2,§2].

Lemma 1. Assume that $\gamma \geq 1$ and $\alpha \geq 0$. Suppose that there are positive constants c_0, c_1, c_2 such that

$$v g_v(v, x, t) - g(v, x, t) \leq c_0, \ |g_t(v, x, t)|^{1/2} \leq c_1 \tag{11}$$

for all $(v, x, t) \in (\alpha, \infty) \times K$ and $\max\limits_K u \geq c_2$ for each positive solution u of (10). Then there exists $C = C(c_0, c_1, c_2, \max\limits_K a, \min\limits_K a, \max\limits_K |a_t|, \gamma) > 0$ such that for each solution u of (10) with $u > \alpha$

$$u(x, t) \leq u(x, s) \exp(-CM^{\gamma-1}(t - s)) \tag{12}$$

for all $(x, t), (x, s) \in K$ with $s - T \leq t \leq s$, where $M = \max u$.

Lemma 2. Assume that $\gamma \geq 1, \alpha \geq 0$ and (11) for g. If u is a solution of (10) with $u > \alpha$, then

$$u(x, t_0)^\gamma \geq M^\gamma - \frac{\gamma C_M}{2}(x - x_0)^2 \quad \text{in } K,$$

where

$$C_M = \frac{c_0}{\gamma}M^{\gamma-1} + \frac{\max(a_t)_+}{\gamma} + \frac{c_1(\max a)^{1/2}}{\gamma^{1/2}}M^{\frac{\gamma-1}{2}} + M^{\gamma-1}g_M,$$
$$g_M = \max\{(g(v,x,t))_+; \alpha < v < M, (x,t) \in K\}.$$
$$M = \max_K u = u(x_0,t_0),$$

3 Upper bounds.

We shall obtain an a priori upper bound for positive smooth solutions of

$$au_t = u^\gamma\{u_{xx} + \varphi(u)(u + \psi(x,u) - f(x,t))\} \quad \text{in } K, \tag{13}$$

where φ, ψ are smooth functions on $(0,\infty), \mathbf{T}\times(0,\infty)$, respectively. Hear and hearafter, $a \in C(K)$ is assumed to be time independent. This equation corresponds to the equation (3.1) in [2], in which ψ is independent of x. The dependence of ψ on x has no effect on proofs in the rest of this paper.

Lemma 3. Suppose that $\psi \geq 0, f \geq 0$ and $0 \leq \varphi \leq c_3, v - \varphi(v)v \leq c_4$ on (α,∞) with $\alpha > 0$ for some positive constants c_3 and c_4. Then for each solution $u \in C^\infty(K)$ of (13) with $u > \alpha$

$$\int\int_K u\,dxdt \leq 2\pi T(c_3\|f\|_\infty + c_4) \equiv C_1$$
$$\int\int_K \frac{u_t^2}{u^\gamma}dxdt \leq \frac{c_3C_1\|f_t\|_\infty}{\min a} \equiv C_2.$$

Proof. The first inequality is obtained in the same way as the proof of Lemma 3.1 in [2]. Multiplying u_t/u^γ with (13) and integrating over K yields

$$\int\int_K a\frac{u_t^2}{u^\gamma}dxdt = -\int\int_K \varphi(u)u_tf\,dxdt = \int\int_K \Phi(u)f_t\,dxdt,$$

where

$$\Phi(s) = \int_0^s \varphi(r)dr \quad \text{for } s \in \mathrm{R}.$$

We thus have

$$\int\int_K a\frac{u_t^2}{u^\gamma}dxdt \leq c_3C_1\|f_t\|_\infty.$$

This implies the second inequality. □

Lemmas 2 - 3 yield the following theorem.

Theorem 2. Suppose that $1 \leq \gamma < 3$ and $\alpha > 0$. In addition to the hypotheses in Lemma 3, assume that

$$\varphi'(v)(\psi(x,v) - f) + \varphi(v)\psi'(v) \leq 0,$$

$$0 \leq \varphi'(v)v^2 \leq c_5, \varphi(v)(\psi(x,v) - \min_K f) \leq c_6(v+1)$$

on $\mathbf{T} \times (\alpha, \infty)$ for some constants $c_5, c_6 > 0$. Then there is a positive constant M_0 depending only on $c_j (3 \leq j \leq 6), T, \|f\|_\infty, \|f_t\|_\infty, \gamma, \min_{\mathbf{T}} a$ such that $\max_K u \leq M_0$ for each solution $u \in C^\infty(K)$ with $u > \alpha$.

4 Lower bounds.

We consider the equation

$$au_t = u^2\{u_{xx} + \varphi_\varepsilon(u)(u + \psi_\varepsilon(x,u) - f_\varepsilon)\} \quad \text{in } K \tag{14}$$

in this section. To get a positive lower bound for positive smooth solutions of (14), we investigate the stationary problem

$$U_{xx} + U = F \quad \text{in } \mathbf{T}. \tag{15}$$

The coefficient a clearly gives no effect when we treat the stationary problem.

The following lemma is a key as in [2].

Lemma 4. Let $b \in \mathrm{R}$ and $d > 0$. Suppose that $V \geq 0$ on $(b, b+d)$, $V \not\equiv 0$ and V_x is Lipschitz continuous on $[b, b+d]$. If $V_{xx} + V \geq 0$ on $(b, b+d)$ with $V(b) = V_x(b) = 0$ and $V(b+d) = 0$, then $d > \pi$.

Let $\{\mu_\varepsilon^\pm\}_{\varepsilon\geq 0}$ be a sequence of positive functions on $\mathbf{T}\times(0,\infty)$ such that $\mu_\varepsilon^\pm(x,\cdot)$ is nonincreasing for each $x\in\mathbf{T}$ and $\mu_\varepsilon^\pm\to\mu_0^\pm$ in $\mathbf{T}\times(0,\infty)$ as $\varepsilon\to 0$. Suppose that $\mu_\varepsilon^-\to\mu_0^-$ uniformly in every conpact subset of $\mathbf{T}\times(0,\infty)$ as $\varepsilon\to 0$. Let $\{h_\varepsilon^-\}_{\varepsilon\geq 0}$ be a sequence in $L^\infty(0,\infty)$ with $0\leq h_\varepsilon^-\leq 1$ such that $h_\varepsilon^-\to h_0^-\equiv 1$ uniformly in every compact subset in $\mathbf{T}\times(0,\infty)$ as $\varepsilon\to 0$. Put $h_\varepsilon^+\equiv 1$ for all $\varepsilon\geq 0$. For a positive function U on $\mathbf{T}$ and $\varepsilon\geq 0$, we set

$$A_\varepsilon^\pm(\zeta,U)=\int_0^{2\pi}\sin_\pm(x-\zeta)\mu_\varepsilon^\pm(x,U)h_\varepsilon^\pm(U)dx\quad\text{for }\zeta\in\mathbf{R},$$

where $\sin_+ z=\max(\sin z,0)$ and $\sin_- z=-\min(\sin z,0)$.

The following lemma is the same as Lemma 4.2 in [2] except for the dependence of $\mu_\varepsilon^\pm$ on $x\in\mathbf{T}$, which does not affect the proof.

Lemma 5. Assume that there are positive constants $k_j(0\leq j\leq 4)$ such that for each positive solution $U\in C^2(\mathrm{T})$

i) $0\leq F\leq k_0$, where $F=U_{xx}+U$

ii) $k_1\leq\max U\leq k_2$,

iii) $A_\varepsilon^-(\zeta,U)\leq k_3A_\varepsilon^+(\zeta,U)+k_4\quad$ for all $\zeta\in\mathbf{R}$.

Suppose that

$$\int_0^1\mu_0^-(x^2)dx=\infty.$$

Then there are positive constants δ_0,ε_0 depending only on k_j 's and $\{\mu_\varepsilon^\pm\},\{h_\varepsilon^-\}$ such that $\min_{\mathbf{T}}U\geq\delta_0$ for each positive solution $U\in C^2(\mathrm{T})$ of (15) and $0\leq\varepsilon\leq\varepsilon_0$.

The following is the same as Lemma 4.4 in [2].

Lemma 6. If $u\in C(K)$ satisfies (12), then there are $\lambda,\Lambda>0$ depending only on C,γ,M,T such that

$$\lambda u(x,t)\leq U(x)\leq\Lambda u(x,t)\quad\text{for }(x,t)\in K,$$

where $U(x) = \int_0^T u(x,t)dt$.

Lemma 5.3 in [2] remains valid even if $\psi_\varepsilon(u)$ is replaced by $\psi_\varepsilon(x,u)$ as stated below.

Lemma 7. Assume that $f_\varepsilon \in C^\infty(K)$ satisfies (7). If $0 \leq \varphi_\varepsilon \leq 1$ and $1 - \varphi_\varepsilon(v) \leq c_7\varepsilon^2 v^{-1}$ for $v > \varepsilon^2$ with some positive constant c_7, then

$$|\int\int_K \{\varphi_\varepsilon(u)\psi_\varepsilon(x,u) + (1-\varphi_\varepsilon(u))f_\varepsilon\}\sin(x-\zeta)dxdt| \leq 4Tc_7\varepsilon^2 \tag{16}$$

for each solution $u \in C^\infty(K)$ of (14) with $u > \varepsilon^2$ and $\zeta \in \mathbf{R}$.

Using Lemmas 5-7, we can prove our lower bound theorem in the same way as the proof of Theorem 5.7 in [2].

Theorem 3. Assume that $f_\varepsilon \in C^\infty(K)$ satisfies (7) with $f_\varepsilon > 0$ and that $\varphi_\varepsilon, \psi_\varepsilon$ fulfill

$$0 \leq \varphi_\varepsilon(v) \leq 1, 0 \leq \varphi_{\varepsilon v}(x,v) \leq 2, \varepsilon^2 \leq 2v(1-\varphi_\varepsilon(v)) \leq 2\varepsilon^2 \leq 2 \quad \text{for } v > \varepsilon^2$$

$$\min_{\varepsilon>0}\min_K(f_\varepsilon - \psi_\varepsilon(x,v)) > 0, \psi_{\varepsilon v}(x,v) \leq 0, \quad \text{for } v > \varepsilon^2, x \in \mathbf{T}.$$

Then there are positive constants ε_0, δ_0 depending only on $T, \|f\|_\infty, \|f_t\|_\infty, \min_K f_\varepsilon, \min_{\mathbf{T}} a, \max_{\mathbf{T}} a$ such that $\min_K u \geq \delta_0$ for each solution $u \in C^\infty(K)$ of (14) with $u > \varepsilon^2$ and $0 < \varepsilon \leq \varepsilon_0$.

5 Existence of periodic solutions.

We start with approximate equations

$$aw_t = (w+\varepsilon^2)^2\{w_{xx} + \frac{w^2}{(w+\varepsilon^2)^2}(w + \frac{\varepsilon a}{\xi_\varepsilon(x, aw+\varepsilon^2)} - f)\} \quad \text{in } K, \tag{17}$$

where $a \in C^\infty(\mathbf{T}), f \in C^\infty(K)$, $\xi_\varepsilon : \mathbf{T} \times (0,\infty) \to (0,\infty)$ is a smooth function such that $\xi_\varepsilon(x,\cdot)$ is nondecreasing for every $x \in \mathbf{T}$,

$$\xi_\varepsilon(x,v) = v \quad \text{for } v \geq m\varepsilon a, x \in \mathbf{T},$$

$$v \vee (m\varepsilon a) \le \xi_\varepsilon(x,v) \le l(v \vee (m\varepsilon a)) \quad \text{for } v \ge 0, x \in \mathbf{T}$$

with some $1 < l < 2$ and

$$\min_K f - \frac{1}{m} \ge \frac{1}{2} \min_K f.$$

To solve (17), we need the following fact, in which the coefficient $a(v)$ of v_{xx} in Lemma 6.1 in [2] is replaced by $a(v,x,t)$ and we can prove in the same way as the proof of Lemma 6.1

Lemma 8. Assume that b is a positive constant and that a is a continuous function on $\mathbf{R} \times K$ such that $a(\sigma,x,t) \ge a_0$ for all $\sigma \in \mathbf{R}$ on K with some positive constant a_0. Then for each $h \in C(K)$ there exists a unique solution $v \in \bigcap_{q>1} W_q^{2,1}(K) \subset C(K)$ of

$$v_t = a(v,x,t)(v_{xx} - bv + h) \quad \text{in } K.$$

Moreover the solution operator $h \mapsto v$ is a continuous, compact operator from $C(K)$ into itself. There are positive constants θ_0, C_0 depending only on $a_0, \|h\|_\infty, b, T, \sup_K a$ such that

$$\|v\|_{W_p^{2,1}} \le C_0 \|h\|_\infty \quad \text{for } 2 \le p \le 2 + \theta_0, h \in C(K).$$

Take $b > 0$ such that

$$\phi(w,x,t) = bw_+ + \frac{(w_+)^2}{(w_+ + \varepsilon^2)^2}(w_+ + \frac{\varepsilon a}{\xi_\varepsilon(x, aw + \varepsilon^2)} - f) \ge 0$$

for all $w \in \mathbf{R}, (x,t) \in K$ and $\phi > 0$ if $w > 0$. For this b let S be the solution operator of

$$av_t = (v_+ + \varepsilon^2)^2(v_{xx} - bv + h) \quad \text{in } K,$$

which is well-defined by Lemma 8. Lemma 8 also yields;

i) S is a continuous compact operator from $C(K)$ into itself,

ii) $S(h)$ is Hölder continuous on K for $h \in C(K)$.

By standard regularity theory and maximum principle, we see that each fixed point of $S \circ \phi$ in $C(K)$ is a positive smooth solution of (17).

We can calculate values of the Leray-Schauder degree in a large and a small ball in $C(K)$ in the same way as in Lemmas 6.3, 6.4 in [2].

Lemma 9. There is $r_0 > 0$ such that the degree of $I - S \circ \phi$ of the value zero in $B_r(0)$ equals one , i.e.,

$$\deg\,(I - S \circ \phi, B_r(0), 0) = 1$$

for $0 < r < r_0$.

Lemma 10. There is $R_0 > 0$ such that

$$\deg\,(I - S \circ \phi, B_R(0), 0) = 0 \quad \text{for } R > R_0.$$

We sketch proof of Theorem 1.

Proof of Theorem 1. Choose an approximate sequence $\{a_\varepsilon\} \in C^\infty(\mathrm{T})$ and $\{f_\varepsilon\} \in C^\infty(K)$ satisfying (7) such that

$$a_\varepsilon \to a \text{ in } C(\mathrm{T}), f_\varepsilon \to f, f_{\varepsilon t} \to f_t \text{ in } C(K) \text{ as } \varepsilon \to 0.$$

From Lemmas 9, 10, for each $\varepsilon > 0$ there exists a positive solution $v_\varepsilon \in C^\infty(K)$ of (17) with $a = a_\varepsilon$ and $f = f_\varepsilon$ for each $\varepsilon > 0$. Putting $u_\varepsilon = v_\varepsilon + \varepsilon^2$, u_ε satisfies

$$a_\varepsilon u_t = u^2\{u_{xx} + \frac{(u-\varepsilon^2)^2}{u^2}(u + \frac{\varepsilon a_\varepsilon}{\xi_\varepsilon(x, u+\varepsilon^2)} - f_\varepsilon - \varepsilon^2)\} \quad \text{in } K. \tag{18}$$

Setting

$$\varphi_\varepsilon(v) = \frac{(v-\varepsilon^2)^2}{v^2}, \psi_\varepsilon(x,v) = \frac{\varepsilon a_\varepsilon}{\xi_\varepsilon(x, v+\varepsilon^2)},$$

$\varphi_\varepsilon, \psi_\varepsilon$ satisfy the assumptions of Theorems 2, 3, so there are positive constants $M_0, \delta_0, \varepsilon_0$ such that $\delta_0 \le u_\varepsilon \le M_0$ on K for $0 < \varepsilon < \varepsilon_0$. Then we obtain a positive solution u of

$$a u_t = u^2(u_{xx} + u - f) \tag{19}$$

as the limit of a subsequence of $\{v_\varepsilon\}$ in $W_p^{2,1}(K)$ with $p > 2$. It remains to prove the constraint (9) for $w = u/a$. Multiplying $\sin(x-\zeta)/u^2$ with (19) and integrating over $(0, 2\pi)$ yields

$$-\frac{d}{dt}\int_0^{2\pi} \frac{a}{u(x,t)} \sin(x-\zeta)dx = -\int_0^{2\pi} f \sin(x-\zeta)dx = 0$$

for all $t, \zeta \in \mathbf{R}$. Letting $\varepsilon \to 0$ in (16), it follows that

$$\int\int_K \frac{a}{u} \sin(x-\zeta)dxdt = 0 \quad \text{for all } \zeta \in \mathbf{R}.$$

These imply that $w = u/a$ satisfies the constraint (9). Therefore u is our desired solution of (19). □

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