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Large deviations for the first exit time on small random perturbations of dynamical systems *

by

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Summary

We consider small random perturbations of dynamical systems $\{X^\varepsilon(t)\}_{0\leq t}$ $(0 < \varepsilon)$ on a d-dimensional Euclidean space R^d when the origin $o \in R^d$ is a hyperbolic equilibrium point of unperturbed dynamical systems. The first exit time τ_D^ε of $\{X^\varepsilon(t)\}_{0\leq t}$ from a bounded domain $D(\ni o)$ of R^d obeys the large deviations phenominon with a variable decay rate.

1. Introduction.

Let $X^\varepsilon(t,x)$ $(t>0, x\in R^d, \varepsilon>0)$ be the solution of the following stochastic differential equation:

$$\text{(1.1).}\qquad \begin{aligned} dX^\varepsilon(t,x) &= b(X^\varepsilon(t,x))dt + \varepsilon^{1/2}\sigma(X^\varepsilon(t,x))dW(t),\\ X^\varepsilon(0,x) &= x, \end{aligned}$$

where $b(\cdot)=(b^i(\cdot))_{i=1}^d : R^d \mapsto R^d$ and $\sigma(\cdot)=(\sigma^{ij}(\cdot))_{i,j=1}^d : R^d \mapsto M_d(R)$ are twice continuously differentiable and have bounded derivatives up to the second order and $\sigma(\cdot)$ is uniformly nondegenerate, and where $W(\cdot)$ is a d-dimensional Wiener process (see Ikeda and Watanabe (1989)).

When $\varepsilon=0$, $X(t,x)\equiv X^0(t,x)$ is a solution of the following ordinary differential equation;

$$\text{(1.2).}\qquad \begin{aligned} dX(t,x)/dt &= b(X(t,x)),\\ X(0,x) &= x. \end{aligned}$$

The following is known; for any $T>0$, $\gamma>0$ and $x\in R^d$

$$\text{(1.3).}\qquad \lim_{\varepsilon\to 0} P\Big(\sup_{0\le t\le T}|X^\varepsilon(t,x)-X(t,x)|<\gamma\Big)=1$$

In this sense, $X^\varepsilon(t,x)_{\,0\le t,x\in R^d}$ can be considered as the small random perturbations of dynamical systems $\{X(t,x)\}_{0\le t,x\in R^d}$, for small $\varepsilon>0$ (see Freidlin and Wentzell (1984)).

Let $D\subset R^d$ be a bounded domain which contains o. In this paper we consider the asymptotic behavior of the first exit time of $X^\varepsilon(t,x)$ from D;

$$\text{(1.4).}\qquad \tau_D^\varepsilon(x)\equiv \inf\{t>0; X^\varepsilon(t,x)\notin D\}$$

when $b(x)=o$ if and only if $x=o$.

The weak low of large numbers for $\tau_D^\varepsilon(x)$ was obtained by Kifer (1981). Let us give notation to introduce his result. Put $A_1\equiv\{x\in\overline{D}$; There exists $s=s(x)\le 0$ such that $X(t,x)\notin\overline{D}$ for $t<s$ and such that $X(t,x)\in D$ for $t>s$. $X(t,x)\to o$ as $t\to\infty\}$; $A_2\equiv\{x\in\overline{D}$; There exists $s=s(x)\ge 0$ such that $X(t,x)\notin\overline{D}$ for $t>s$ and such that $X(t,x)\in D$ for $t<s$. $X(t,x)\to o$ as $t\to-\infty\}$; $A_3\equiv\{x\in\overline{D}$; There exist $s_1=s_1(x)\ge 0\ge s_2=s_2(x)$ such that $X(t,x)\notin\overline{D}$ for $s_2>t$ and $t>s_1$ and such that $X(t,x)\in D$ for $s_2<t<s_1\}$.

The following was the assumption in Kifer (1981).

(H.D). D has a C^2-boundary ∂D. $b(o)=o$. $\overline{D}=\{o\}\cup A_1\cup A_2\cup A_3$ and $A_2\cup A_3$ is not empty.

Remark1.1. From (H.D), the eigenvalues of $(\partial b^i(o)/\partial x_j)_{i,j=1}^d$ have non-zero real parts. That is, o is a hyperbolic equilibrium point of dynamical systems $\{X(t,x)\}_{0\le t,x\in R^d}$. Moreover, $\lambda\equiv$ the maximum of real parts of the eigenvalues of the matrix $(\partial b^i(o)/\partial x_j)_{i,j=1}^d$ is positive (see Hartman (1964), Chap. 9).

The following is proved in Kifer (1981).

Theorem 1.1.

Suppose that (H.D) holds. Then for any $\gamma > 0$ and $x \in A_1 \cup \{o\} \backslash \partial D$,

$$\lim_{\varepsilon \to 0} P(|\tau_D^\varepsilon(x)/\log(\varepsilon^{-1/(2\lambda)}) - 1| < \gamma) = 1, \tag{1.5}$$

and for any $\gamma > 0$ and $x \in A_2 \cup A_3 \backslash \partial D$,

$$\lim_{\varepsilon \to 0} P(|\tau_D^\varepsilon(x)/\tau_D^0(x) - 1| < \gamma) = 1. \tag{1.6}$$

In this paper we consider the large deviations for $\tau_D^\varepsilon(x)/\log(\varepsilon^{-1/(2\lambda)})$ for $x \in A_1 \cup \{o\} \backslash \partial D$. The large deviations for $\tau_D^\varepsilon(x)/\tau_D^0(x)$ for $x \in A_2 \cup A_3 \backslash \partial D$ can be obtained by the routine argument on large deviations (see Freidlin and Wentzell (1984), Chap. 3 and 5).

Let us introduce our assumption which is weaker than Kifer's.

(A.D). The eigenvalues $\{\lambda_i\}_{i=1}^d$ of $(\partial b^i(o)/\partial x_j)_{i,j=1}^d$ have non-zero real parts. λ is positive (see Remark 1.1 for notation). $b(x) = o$ if and only if $x = o$.

Remark 1.2. Under (A.D), $\{o\} \cup A_1 \cup A_2 \cup A_3$ is not always equl to $\overline{D}$. In 1-dimensional case $\{o\} \cup A_2 = \overline{D}$, and (A.D) is equivalent to (H.D).

Put

$$f(\varepsilon) = \log(\varepsilon^{-1/(2\lambda)}). \tag{1.7}$$

Then the following is our results.

Theorem 1.2.

Suppose that (A.D) holds. Then for any $x \in A_1 \cup \{o\} \backslash \partial D$ and $0 < T < 1$,

$$\lim_{\varepsilon \to 0} \log(-\log P(\tau_D^\varepsilon(x)/f(\varepsilon) < T))/\log \varepsilon = T - 1. \tag{1.8}$$

Theorem 1.3.

Suppose that (A.D) holds and that $d = 1$. Then for any $T \geq 1$,

$$\lim_{\varepsilon \to 0} \log P(\tau_D^\varepsilon(o)/f(\varepsilon) > T)/\log \varepsilon = T - 1. \tag{1.9}$$

For multi-dimensional case, we only have the following result.

Proposition 1.4.

Suppose that (A.D) holds. Then for any $x \in A_1 \cup \{o\} \backslash \partial D$ and $T \geq 1$,

$$\limsup_{\varepsilon \to 0} \log P(\tau_D^\varepsilon(x)/f(\varepsilon) > T)/\log \varepsilon < \infty. \tag{1.10}$$

Remark 1.3. Roughly speaking, (1.8)-(1.10) means the following; (1.8) implies, as $\varepsilon \to 0$,

$$P(\tau_D^\varepsilon(x)/f(\varepsilon) < T) \sim \exp(-\varepsilon^{T-1}), \tag{1.11}$$

and (1.9) implies, as $\varepsilon \to 0$,

$$P(\tau_D^\varepsilon(o)/f(\varepsilon) > T) \sim \varepsilon^{T-1}, \tag{1.12}$$

and (1.10) implies that there exists a positive constant C such that for sufficiently small $\varepsilon > 0$,

$$P(\tau_D^\varepsilon(x)/f(\varepsilon) > T) > \varepsilon^C. \tag{1.13}$$

From (4.37) in Kifer (1981), the following is known; for any $T > 1$ and $x \in A_1 \cup \{o\} \backslash \partial D$, there exists a positive constant C_1 such that for sufficiently small $\varepsilon > 0$,

$$P(\tau_D^\varepsilon(x)/f(\varepsilon) > T) < \varepsilon^{C_1}. \tag{1.14}$$

From Theorem 1.3, we have the following conjecture.

Conjecture 1.5.

Suppose that (A.D) holds. Then for any $x \in A_1 \cup \{o\} \backslash \partial D$ and $T \geq 1$,

$$\lim_{\varepsilon \to 0} \log P(\tau_D^\varepsilon(x)/f(\varepsilon) > T)/\log \varepsilon = \sum_{i=1}^{d} \max(0, (Re(\lambda_i)T/\lambda - 1)/2). \tag{1.15}$$

Put

$$u^\varepsilon(t,x) \equiv P(\tau_D^\varepsilon(x)/f(\varepsilon) > T). \tag{1.16}$$

Then $u^\varepsilon(t,x)$ satisfies the following PDE;

$$\partial u^\varepsilon(t,x)/\partial t = f(\varepsilon)[\varepsilon \sum_{i,j,k=1}^{d} \sigma^{ik}(x)\sigma^{jk}(x)[\partial^2 u^\varepsilon(t,x)/\partial x_i \partial x_j]/2$$

(1.17). $$+ \sum_{i=1}^{d} b^i(x)\partial u^\varepsilon(t,x)/\partial x_i] \quad \text{for } t > 0 \text{ and } x \in D,$$

$$u^\varepsilon(t,x) = 0 \quad \text{for } t \geq 0 \text{ and } x \in \partial D,$$

$$u^\varepsilon(t,x) = 1 \quad \text{for } t = 0 \text{ and } x \in D.$$

The PDE method such as the theory of viscosity solution might be useful to prove Conjecture 1.5.

In section 2, we give lemmas which are necessary for the proof of our results. In section 3, we prove Theorem 1.2. In section 4, we prove Theorem 1.3 and Proposition 1.4. In section 5, we show that Conjecture 1.5 is true in Gaussian case, that is, in case $b(x)$ is linear and $\sigma(x) = Identity$. In section 6, we correct the inequality (2.27) in Mikami (1993).

We introduce some notation; $Db(x) \equiv (\partial b^i(x)/\partial x_j)_{i,j=1}^d$; $C(\cdot)$ denotes a positive constant which depends on "$\cdot$"; $||A|| \equiv [\sum_{i,j=1}^d (A^{ij})^2]^{1/2}$ for $A = (A^{ij})_{i,j=1}^d$.

2. Lemmas.

In this section, we give lemmas which are necessary for the proof of our results.

Let $\Pi(t,x)$ $(t > 0,\ x \in D)$ be the solution of the following ordinary differential equation;

$$(2.1). \qquad \begin{aligned} d\Pi(t,x)/dt &= Db(X^0(t,x))\Pi(t,x), \\ \Pi(0,x) &= Id, \end{aligned}$$

where Id denotes the $d \times d$-identity matrix.

The following lemma can be found in Hartman (1964), Chap. 9.

Lemma 2.1.

Suppose that (A.D) holds. Then there exist $C_1 > 0$ and $\alpha > 0$ such that for any $x \in A_1 \cup \{o\}$ and all $t > 0$,

$$(2.2). \qquad |X^0(t,x)| \leq C_1|x|\exp(-\alpha t).$$

The following lemma can be obtained from the assumption.

Lemma 2.2.

Suppose that (A.D) holds. Then for any $\gamma_1 > 0$, there exist $C = C(\gamma_1) > 0$ such that

$$(2.3). \qquad ||\Pi(t,x)\Pi(s,x)^{-1}|| \leq C\exp(\lambda + \gamma_1)(t-s),$$

for all $t > s > 0$ and $x \in A_1 \cup \{o\}$.

(Proof). From (2.1), we get

$$(2.4). \qquad \begin{aligned} d[\Pi(t,x)\Pi(s,x)^{-1}]/dt &= Db(X^0(t,x))[\Pi(t,x)\Pi(s,x)^{-1}] \qquad \text{for } t > s, \\ \Pi(s,x)\Pi(s,x)^{-1} &= Id. \end{aligned}$$

For $\gamma_1 > 0$, take the $d \times d$ matrix $Q = Q(\gamma_1)$ so that

$$(2.5). \qquad < z, Q^{-1}Db(o)Qz > \leq (\lambda + \gamma_1)|z|^2$$

for all $z \in R^d$(see Hartman (1964), Chap. 9 and also Hirsh and Smale (1974), Chap. 7, section 1). From (2.4)-(2.5), we get

$$(2.6). \quad ||Q^{-1}\Pi(t,x)\Pi(s,x)^{-1}||^2 = ||Q^{-1}||^2 + 2\int_s^t \sum_{j=1}^{d} < (Q^{-1}\Pi(u,x)\Pi(s,x)^{-1})_j \\ , Q^{-1}Db(X^0(u,x))Q(Q^{-1}\Pi(u,x)\Pi(s,x))_j > du,$$

where we put $A_j \equiv (a^{ij})_{i=1}^d$ for $A \equiv (a^{ij})_{i,j=1}^d$.

From Lemma 2.1 and (2.6), we get

$$
\begin{aligned}
(2.7). \quad & ||Q^{-1}\Pi(t,x)\Pi(s,x)^{-1}||^2 \\
&= ||Q^{-1}||^2 + 2\int_s^t \sum_{j=1}^d < (Q^{-1}\Pi(u,x)\Pi(s,x)^{-1})_j, \\
&\qquad Q^{-1}Db(o)Q(Q^{-1}\Pi(u,x)\Pi(s,x))_j > du \\
&\quad + 2\int_s^t \sum_{j=1}^d < (Q^{-1}\Pi(u,x)\Pi(s,x)^{-1})_j, \\
&\qquad Q^{-1}[Db(X^0(u,x)) - Db(o)]Q(Q^{-1}\Pi(u,x)\Pi(s,x))_j > du \\
&\leq ||Q^{-1}||^2 + 2\int_s^t (\lambda + \gamma_1 \\
&\quad + ||Q||||Q^{-1}||C(D^2b)|X^0(u,x)|)||Q^{-1}\Pi(u,x)\Pi(s,x)^{-1}||^2 du \\
&\leq ||Q^{-1}||^2 + 2\int_s^t (\lambda + \gamma_1 \\
&\quad + ||Q||||Q^{-1}||C(D^2b)C_1|x|\exp(-\alpha u))||Q^{-1}\Pi(u,x)\Pi(s,x)^{-1}||^2 du,
\end{aligned}
$$

where $C(D^2b)$ is a constant which depends on the second derivatives of b.

From (2.7), we get, by the Gronwall's inequality

$$
\begin{aligned}
(2.8). \quad & ||Q^{-1}\Pi(t,x)\Pi(s,x)^{-1}||^2 \\
&\leq ||Q^{-1}||^2 \exp(2\int_s^t (\lambda + \gamma_1 + ||Q||||Q^{-1}||C(D^2b)C_1|x|\exp(-\alpha u))du)
\end{aligned}
$$

(see Hartman (1964), Chap. 3).

Q.E.D.

Lemma 2.3.

Suppose that (A.D) holds. Then for any x, y, $z \in R^d$ and $\gamma_1 > 0$,

$$
\begin{aligned}
(2.9). \quad & |X^0(t,z) - X^0(t,x) - X^0(t,y)| \\
&\leq C(\gamma_1)(|z-x-y| + C(D^2b)\int_0^t \exp(-(\lambda+\gamma_1)s)|X^0(s,x)||X^0(s,y)|ds) \\
&\quad \times \exp((\lambda+\gamma_1)t + C(\gamma_1)C(D^2b) \\
&\qquad \times \int_0^t (|X^0(s,z) - X^0(s,x) - X^0(s,y)| + |X^0(s,x) + X^0(s,y)|)ds),
\end{aligned}
$$

where $C(\gamma_1)$ is a positive constant in Lemma 2.2, and where $C(D^2b)$ is a positive constant which depends on the second derivatives of b.

(Proof). For any x, y, $z \in R^d$ and $t > 0$,

(2.10).

$$
\begin{aligned}
&X^0(t,z) - X^0(t,x) - X^0(t,y) \\
&= z - x - y + \int_0^t Db(o)(X^0(s,z) - X^0(s,x) - X^0(s,y))ds \\
&\quad + \int_0^t [b(X^0(s,z)) - b(X^0(s,x) + X^0(s,y)) \\
&\qquad - Db(X^0(s,x) + X^0(s,y))(X^0(s,z) - X^0(s,x) - X^0(s,y)) \\
&\qquad + (Db(X^0(s,x) + X^0(s,y)) - Db(o))(X^0(s,z) - X^0(s,x) - X^0(s,y)) \\
&\qquad + \int_0^1 d\theta_1 \int_0^1 d\theta_2 < D^2 b(\theta_1 X^0(s,x) + \theta_2 X^0(s,y))X^0(s,x), X^0(s,y) >]ds,
\end{aligned}
$$

where we put $< D^2 b(x)y, z > \equiv (\sum_{k,j=1}^d [\partial^2 b^i(x)/\partial x_k \partial x_j] y^k z^j)_{i=1}^d$ for $x \in R^d$, and $y = (y^k)_{k=1}^d$, $z = (z^j)_{j=1}^d \in R^d$. Here we considered as follows; for $0 \le s \le t$ and x, $y \in R^d$,

(2.11).
$$
\begin{aligned}
&b(X^0(s,x) + X^0(s,y)) - b(X^0(s,y)) - b(X^0(s,x)) \\
&= \int_0^1 Db(\theta_1 X^0(s,x) + X^0(s,y))X^0(s,x)d\theta_1 \\
&\quad - \int_0^1 d\theta_1 Db(\theta_1 X^0(s,x))X^0(s,x)d\theta_1 \quad \text{(since } b(o) = o \text{ from (A.D))} \\
&= \int_0^1 d\theta_1 \int_0^1 d\theta_2 < D^2 b(\theta_1 X^0(s,x) + \theta_2 X^0(s,y))X^0(s,x), X^0(s,y) > ds.
\end{aligned}
$$

From (2.10), we get

(2.12).

$$
\begin{aligned}
&X^0(t,z) - X^0(t,x) - X^0(t,y) \\
&= \exp(Db(o)t)(z - x - y) + \int_0^t \exp(Db(o)(t-s))[b(X^0(s,z)) - b(X^0(s,x) + X^0(s,y)) \\
&\quad - Db(X^0(s,x) + X^0(s,y))(X^0(s,z) - X^0(s,x) - X^0(s,y)) \\
&\quad + (Db(X^0(s,x) + X^0(s,y)) - Db(o))(X^0(s,z) - X^0(s,x) - X^0(s,y)) \\
&\quad + \int_0^1 d\theta_1 \int_0^1 d\theta_2 < D^2 b(\theta_1 X^0(s,x) + \theta_2 X^0(s,y))X^0(s,x), X^0(s,y) >]ds.
\end{aligned}
$$

From (2.12) and Lemma 2.2, we get, by the Taylor's theorem, the following; for any $\gamma_1 > 0$,

(2.13).

$$\exp(-(\lambda+\gamma_1)t)||X^0(t,z)-X^0(t,x)-X^0(t,y)|$$
$$\leq C(\gamma_1)|z-x-y|+C(\gamma_1)C(D^2b)\int_0^t \exp(-(\lambda+\gamma_1)s)|X^0(s,z)-X^0(s,x)-X^0(s,y)|$$
$$\times[|X^0(s,z)-X^0(s,x)-X^0(s,y)|+|X^0(s,x)+X^0(s,y)||ds$$
$$+C(\gamma_1)C(D^2b)\int_0^t \exp(-(\lambda+\gamma_1)s)|X^0(s,x)||X^0(s,y)|ds,$$

where $C(\gamma_1)$ is a positive constant in Lemma 2.2, and where $C(D^2b)$ is a positive constant which depends on the second derivatives of b.

From (2.13), by the Gronwall's inequality,

(2.14).

$$\exp(-(\lambda+\gamma_1)t)||X^0(t,z)-X^0(t,x)-X^0(t,y)|$$
$$\leq C(\gamma_1)(|z-x-y|+C(D^2b)\int_0^t \exp(-(\lambda+\gamma_1)s)|X^0(s,x)||X^0(s,y)|ds)$$
$$\times \exp(C(\gamma_1)C(D^2b)\int_0^t[|X^0(s,z)-X^0(s,x)-X^0(s,y)|+|X^0(s,x)+X^0(s,y)||ds)$$

(see Hartman (1964), Chap. 3).

Q. E. D.

Lemma 2.4.

Let $f(t)$ and $g(t)$ $(t \geq 0)$ be continuous functions such that for all $t \geq 0$

(2.15). $$f(t) \leq \exp(C\int_0^t f(s)ds)g(t)$$

for a positive constant C. Then

(2.16). $$f(t) \leq g(t)/(1-C\int_0^t g(s)ds)$$

as far as $1 > C\int_0^t g(s)ds$.

(Proof). From (2.15),

(2.17). $$-d[\exp(-C\int_0^t f(s)ds)]/dt \leq Cg(t).$$

Integrating both sides of (2.17), we get (2.16).

Q.E.D.

Let Γ_{max} denote the eigenspace of $Db(o)$ which corresponds to the eigenvalues whose real part is equal to λ. Then there exists a submanifold W_{max} which is tangent to Γ_{max} at o and which is invariant with respect to $X^0(t,\cdot)$ (see Kifer (1981), p. 77). The following lemma can be found in Kifer (1981), p. 90.

Lemma 2.5.

Suppose that (A.D) holds. Then there exists $\delta > 0$ so that for any $\gamma_1 > 0$, there exist $\tilde{C} = \tilde{C}(\gamma_1) > 0$ such that

$$(\tilde{C})^{-1}|x|\exp((\lambda-\gamma_1)t) \le |X^0(t,x)| \le \tilde{C}|x|\exp((\lambda+\gamma_1)t), \tag{2.18}$$

as far as $x \in W_{max}$ and $X^0(s,x) \in U_\delta(D)$ for all $0 < s < t$, where $U_\delta(D) \equiv \{y \in R^d; dist(y,D) < \delta\}$.

Put

$$T_a \equiv \inf\{t > 0; X^0(t,a) \notin U_\delta(D)\}, \tag{2.19}$$

The following lemma can be proved from Lemma 2.5.

Lemma 2.6.

Suppose that (A.D) holds. Then for any γ_4 and $\gamma_5 > 0$ for which $T > \gamma_4, \gamma_5$, there exists $\varepsilon_0 >$ such that

$$\begin{aligned}(2.20).\quad &\sup\{|X^0(t,y) - X^0(t,a) - X^0(t+f(\varepsilon)\gamma_5,x)|; |y-a-X^0(f(\varepsilon)\gamma_5,x)| \\ &\quad < \varepsilon^{(T+\gamma_4)/2}, a \in W_{max}(|a| = \varepsilon^{(T-\gamma_4)/2}), x \in A_1 \cup \{o\}, 0 \le t \le T_a\} \\ &< \varepsilon^{\min(\gamma_4, \alpha\gamma_5/(2\lambda))/2}\end{aligned}$$

for $\varepsilon < \varepsilon_0$.

(Proof). From Lemmas 2.3 and 2.4, for any $\gamma_1 > 0$, there exists $C(\gamma_1) > 0$ such that for any $t > 0$, and $y \in R^d$, $a \in W_{max}(|a| = \varepsilon^{(T-\gamma_4)/2})$ and $x \in A_1 \cup \{o\}$ for which $|y-a-X^0(f(\varepsilon)\gamma_5,x)| < \varepsilon^{(T+\gamma_4)/2}$,

(2.21).

$$\begin{aligned}&|X^0(t,y) - X^0(t,a) - X^0(t,X^0(f(\varepsilon)\gamma_5,x))| \\ &\le C(\gamma_1)(|y-a-X^0(f(\varepsilon)\gamma_5,x)| \\ &\qquad + C(D^2b)\int_0^t \exp(-(\lambda+\gamma_1)s)|X^0(s,a)||X^0(s,X^0(f(\varepsilon)\gamma_5,x))|ds) \\ &\quad \times \exp((\lambda+\gamma_1)t + C(\gamma_1)C(D^2b)\int_0^t |X^0(s,a) + X^0(s,X^0(f(\varepsilon)\gamma_5,x))|ds) \\ &\quad \times [1 - tC(\gamma_1)^2C(D^2b)(|y-a-X^0(f(\varepsilon)\gamma_5,x)| \\ &\qquad + C(D^2b)\int_0^t \exp(-(\lambda+\gamma_1)s)|X^0(s,a)||X^0(s,X^0(f(\varepsilon)\gamma_5,x))|ds) \\ &\qquad \times \exp((\lambda+\gamma_1)t + C(\gamma_1)C(D^2b)\int_0^t |X^0(s,a) + X^0(s,X^0(f(\varepsilon)\gamma_5,x))|ds)]^{-1}\end{aligned}$$

for a positive constant $C(D^2b)$, if

(2.22).

$$1 > tC(\gamma_1)^2C(D^2b)(|y-a-X^0(f(\varepsilon)\gamma_5,x)| + C(D^2b)\int_0^t \exp(-(\lambda+\gamma_1)s)|X^0(s,a)||X^0(s,X^0(f(\varepsilon)\gamma_5,x))|ds) \times \exp((\lambda+\gamma_1)t + C(\gamma_1)C(D^2b)\int_0^t |X^0(s,a)+X^0(s,X^0(f(\varepsilon)\gamma_5,x))|ds).$$

Let us show that (2.22) holds for $t = T_a$, if $|y-a-X^0(f(\varepsilon)\gamma_5,x)| < \varepsilon^{(T+\gamma_4)/2}$, $a \in W_{max}(|a| = \varepsilon^{(T-\gamma_4)/2})$ and $x \in A_1 \cup \{o\}$;

(2.23).
$$T_aC(\gamma_1)^2C(D^2b)(|y-a-X^0(f(\varepsilon)\gamma_5,x)| + C(D^2b)\int_0^{T_a} \exp(-(\lambda+\gamma_1)s)|X^0(s,a)||X^0(s+f(\varepsilon)\gamma_5,x)|ds) \times \exp((\lambda+\gamma_1)T_a + C(\gamma_1)C(D^2b)\int_0^{T_a} |X^0(s,a)+X^0(s+f(\varepsilon)\gamma_5,x)|ds)$$
$$< T_aC(\gamma_1)^2C(D^2b)(\varepsilon^{(T+\gamma_4)/2} + C(D^2b)\int_0^{T_a} \exp(-(\lambda+\gamma_1)s) \times \tilde{C}(\gamma_1)\exp((\lambda+\gamma_1)s)\varepsilon^{(T-\gamma_4)/2} \times C_1|x|\exp(-\alpha(s+f(\varepsilon)\gamma_5))ds) \times \exp((\lambda+\gamma_1)T_a + C(\gamma_1)C(D^2b)[\int_0^{T_a} |X^0(s,a)|ds + \int_0^{T_a} C_1|x|\exp(-\alpha(s+f(\varepsilon)\gamma_5))ds]).$$

From Lemma 2.5, for $\varepsilon > 0$ sufficiently small compared to $\gamma_1 > 0$,

(2.24). $$C(\gamma_1)C(D^2b)\int_0^{T_a} |X^0(s,a)|ds < T_a\gamma_1.$$

In fact, put for $\gamma_1 > 0$,

(2.25). $$S_{\gamma_1} \equiv \inf\{t > 0; |X^0(s,a)| = \gamma_1/[2C(\gamma_1)C(D^2b)]\}.$$

Then there exists $C(D) > 0$ such that

(2.26).
$$C(\gamma_1)C(D^2b)\int_0^{T_a} |X^0(s,a)|ds \le S_{\gamma_1}\gamma_1/2 + C(\gamma_1)C(D^2b)(T_a - S_{\gamma_1})C(D) \le S_{\gamma_1}\gamma_1/2 + 4C(\gamma_1)C(D^2b)C(D)\gamma_1/(\lambda-\gamma_1) < T_a\gamma_1,$$

for $\epsilon > 0$ sufficiently small compared to $\gamma_1 > 0$, since from Lemma 2.5,

(2.27). $$S_{\gamma_1} \geq [\log(\gamma_1/(2C(\gamma_1)C(D^2b)\tilde{C}(\gamma_1)|a|))]/(\lambda+\gamma_1),$$

(2.28). $$T_a < [\log(\tilde{C}(\gamma_1)/[|a| \max\{|y|; y \in \partial U_\delta(D)\}])]/(\lambda-\gamma_1),$$

and

(2.29). $$T_a > [\log(dist(o, \partial U_\delta(D))/[\tilde{C}(\gamma_1)|a|])]/(\lambda+\gamma_1).$$

From (2.23), (2.24), and (2.28),

(2.30).

$$\begin{aligned}
&T_a C(\gamma_1)^2 C(D^2b)(|y-a-X^0(f(\epsilon)\gamma_5,x)| \\
&\quad + C(D^2b)\int_0^{T_a} \exp(-(\lambda+\gamma_1)s)|X^0(s,a)||X^0(s+f(\epsilon)\gamma_5,x)|ds) \\
&\quad \times \exp((\lambda+\gamma_1)T_a + C(\gamma_1)C(D^2b)\int_0^{T_a} |X^0(s,a)+X^0(s+f(\epsilon)\gamma_5,x)|ds) \\
&< T_a C(\gamma_1)^2 C(D^2b)(\epsilon^{(T+\gamma_4)/2} + C(D^2b)\tilde{C}(\gamma_1)\epsilon^{(T-\gamma_4)/2}C_1|x|\epsilon^{(\alpha\gamma_5)/(2\lambda)}/\alpha) \\
&\quad \times \exp((\lambda+2\gamma_1)T_a + C(\gamma_1)C(D^2b)C_1|x|\epsilon^{(\alpha\gamma_5)/(2\lambda)}/\alpha) \\
&< \epsilon^{\min(\gamma_4,\alpha\gamma_5/(2\lambda))/2},
\end{aligned}$$

if γ_1 is sufficiently small compared to γ_4 and γ_5, and if $\epsilon > 0$ is sufficiently small compared to $\gamma_1 > 0$.

From (2.21)-(2.22), (2.29)-(2.30), we get (2.20).

Q.E.D.

3. Proof of Theorem 1.2.

In this section we prove Theorem 1.2. We devide the proof into the following two steps; for any $x \in A_1 \cup \{o\} \backslash \partial D$ and $0 < T < 1$,

$$\text{(3.1).} \qquad \limsup_{\varepsilon \to 0} \log(-\log P(\tau_D^\varepsilon(x)/f(\varepsilon) < T))/\log \varepsilon \leq T - 1,$$

$$\text{(3.2).} \qquad \liminf_{\varepsilon \to 0} \log(-\log P(\tau_D^\varepsilon(x)/f(\varepsilon) < T))/\log \varepsilon \geq T - 1.$$

Proof of (3.1).

Let $X_1(t,x)$ be the solution of the following:

$$\text{(3.3).} \qquad X_1(t,x) = \int_0^t Db(X^0(s,x))X_1(s,x)ds + \int_0^t \sigma(X^0(s,x))dW(s).$$

Put $C(D,x) \equiv \inf_{0 \leq t} dist(\partial D, X^0(t,x))$. Then for any $\gamma_1 > 0$, there exists a positive constant $C(\gamma_1)$ such that there exists $C(D^2b) > 0$ so that

$$\begin{aligned}
\text{(3.4).} \quad & P(\tau_D^\varepsilon(x) < f(\varepsilon)T) \\
& \leq P(\tilde{C}(D,x)/2 \leq \exp[(\lambda+\gamma_1)f(\varepsilon)T] \\
& \qquad \times \sup_{0 \leq s \leq f(\varepsilon)T} \exp[-(\lambda+\gamma_1)s]|\varepsilon^{1/2}X^1(s,x)|) \\
& \quad + P(\tilde{C}(D,x)/2 \leq \exp[(\lambda+\gamma_1)f(\varepsilon)T] \sup_{0 \leq s \leq f(\varepsilon)T} \exp[-(\lambda+\gamma_1)s] \\
& \qquad \times |\Pi(s,x)\varepsilon^{1/2} \int_0^s \Pi(u,x)^{-1}[\sigma(X^\varepsilon(u,x)) - \sigma(X^0(u,x))]dW(u)|),
\end{aligned}$$

for any $\tilde{C}(D,x) > 0$ for which

$$\text{(3.5).} \qquad \tilde{C}(D,x)/[1 - C(\gamma_1)C(D^2b)\tilde{C}(D,x)/(\lambda+\gamma_1)] < C(D,x).$$

(For any compact subset $K \subset D$, $C(D,x)$ and accordingly $\tilde{C}(D,x)^{-1}$ can be taken uniformly bounded in $x \in K$.)

The first probability on the right hand side of (3.4) can be considered as follows: there exists $C(\tilde{C}(D,x)) > 0$ such that

$$\begin{aligned}
\text{(3.6).} \quad & P(\tilde{C}(D,x)/2 \leq \exp[(\lambda+\gamma_1)f(\varepsilon)T] \sup_{0 \leq s \leq f(\varepsilon)T} \exp[-(\lambda+\gamma_1)s]|\varepsilon^{1/2}X_1(s,x)|) \\
& \leq P(\tilde{C}(D,x)/3 \leq \sup_{0 \leq s \leq 1} \varepsilon^{(1-T-2\gamma_1 T/\lambda)/2}|\tilde{W}(s)|) \\
& \qquad \text{(for some 1-dimensional Wiener process } \tilde{W}(\cdot)) \\
& \leq \exp(-C(\tilde{C}(D,x))/\varepsilon^{(1-T-2\gamma_1 T/\lambda)}),
\end{aligned}$$

for sufficiently small γ_1 and $\varepsilon > 0$ (see Freidlin and Wentzell (1984), Chap.3).

The second probability on the right hand side of (3.4) can be considered as follows: there exists $C_1(\tilde{C}(D,x)) > 0$ such that

$$
\begin{aligned}
(3.7). \quad & P(\tilde{C}(D,x)/2 \le \exp[(\lambda+\gamma_1)f(\varepsilon)T] \sup_{0\le s\le f(\varepsilon)T} \exp[-(\lambda+\gamma_1)s] \\
& \qquad \times |\Pi(s,x)\varepsilon^{1/2} \int_0^s \Pi(u,x)^{-1}[\sigma(X^\varepsilon(u,x)) - \sigma(X^0(u,x))]dW(u)|) \\
& \le P(\tilde{C}(D,x)/3 \le \sup_{0\le s\le 1} \varepsilon^{(1-T-2\gamma_1 T/\lambda)/2}|\tilde{W}(s)|) \\
& \qquad \text{(for some 1-dimensional Wiener process } \tilde{W}(\cdot)) \\
& \le \exp(-C_1(\tilde{C}(D,x))/\varepsilon^{(1-T-2\gamma_1 T/\lambda)}),
\end{aligned}
$$

for sufficiently small γ_1 and $\varepsilon > 0$ (see Freidlin and Wentzell (1984), Chap. 3).

Let us prove (3.4)-(3.7).

We first show that (3.4) is true.

$$
\begin{aligned}
(3.8). \quad & X^\varepsilon(t,x) - X^0(t,x) - \varepsilon^{1/2}X_1(t,x) \\
& = \int_0^t [b(X^\varepsilon(s,x)) - b(X^0(s,x)) - Db(X^0(s,x))(X^\varepsilon(s,x) - X^0(s,x))]ds \\
& \quad + \int_0^t Db(X^0(s,x))[X^\varepsilon(s,x) - X^0(s,x) - \varepsilon^{1/2}X_1(s,x)]ds \\
& \quad + \varepsilon^{1/2} \int_0^t [\sigma(X^\varepsilon(s,x)) - \sigma(X^0(s,x))]dW(s),
\end{aligned}
$$

and hence

$$
\begin{aligned}
(3.9). \quad & X^\varepsilon(t,x) - X^0(t,x) - \varepsilon^{1/2}X_1(t,x) \\
& = \Pi(t,x) \int_0^t \Pi(s,x)^{-1}\{[b(X^\varepsilon(s,x)) - b(X^0(s,x)) \\
& \qquad - Db(X^0(s,x))(X^\varepsilon(s,x) - X^0(s,x))]ds \\
& \qquad + \varepsilon^{1/2}[\sigma(X^\varepsilon(s,x)) - \sigma(X^0(s,x))]dW(s)\}
\end{aligned}
$$

(see section 2, (2.1) for notation). By way of Taylor expansion, for any $\gamma_1 > 0$, there exist $C(\gamma_1) > 0$ so that there exists a positive constant $C(D^2b)$ and for $x \in A_1 \cup \{o\}$,

$$
\begin{aligned}
(3.10). \quad & |X^\varepsilon(t,x) - X^0(t,x)| \\
& \le |\varepsilon^{1/2}X_1(t,x)| \\
& \quad + C(\gamma_1) \int_0^t \exp(\lambda+\gamma_1)(t-s)C(D^2b)|X^\varepsilon(s,x) - X^0(s,x)|^2 ds \\
& \quad + |\Pi(t,x)\varepsilon^{1/2} \int_0^t \Pi(s,x)^{-1}[\sigma(X^\varepsilon(s,x)) - \sigma(X^0(s,x))]dW(s),
\end{aligned}
$$

from (3.9) and Lemma 2.2.
From (3.10), we get

$$
\begin{aligned}
(3.11). \qquad & \exp[-(\lambda+\gamma_1)t]|X^\varepsilon(t,x)-X^0(t,x)| \\
& \le \exp[-(\lambda+\gamma_1)t]|\varepsilon^{1/2}X_1(t,x)| \\
& \quad + C(\gamma_1)\int_0^t \exp[-(\lambda+\gamma_1)s]C(D^2b)|X^\varepsilon(s,x)-X^0(s,x)|^2 ds \\
& \quad + \exp[-(\lambda+\gamma_1)t]|\Pi(t,x)\varepsilon^{1/2} \\
& \quad\quad \times \int_0^t \Pi(s,x)^{-1}[\sigma(X^\varepsilon(s,x))-\sigma(X^0(s,x))]dW(s).
\end{aligned}
$$

From (3.11), by the Gronwall's inequality,

$$
\begin{aligned}
(3.12). \quad & |X^\varepsilon(t,x)-X^0(t,x)| \\
& \le \exp[(\lambda+\gamma_1)t + C(\gamma_1)\int_0^t C(D^2b)|X^\varepsilon(s,x)-X^0(s,x)|ds] \\
& \quad \times \sup_{0\le s\le t}\{\exp[-(\lambda+\gamma_1)s]|\varepsilon^{1/2}X_1(s,x)| + \exp[-(\lambda+\gamma_1)s]|\Pi(s,x)\varepsilon^{1/2} \\
& \quad\quad \times \int_0^s \Pi(u,x)^{-1}[\sigma(X^\varepsilon(u,x))-\sigma(X^0(u,x))]dW(u)|\}
\end{aligned}
$$

(see Hartman (1964), Chap. 3). If

$$
(3.13) \qquad \tilde{C}(D,x)/2 > \exp[(\lambda+\gamma_1)f(\varepsilon)T] \sup_{0\le s\le f(\varepsilon)T} \exp[-(\lambda+\gamma_1)s]|\varepsilon^{1/2}X^1(s,x)|,
$$

and if

$$
\begin{aligned}
(3.14). \quad & \tilde{C}(D,x)/2 > \exp[(\lambda+\gamma_1)f(\varepsilon)T] \sup_{0\le s\le f(\varepsilon)T} \exp[-(\lambda+\gamma_1)s] \\
& \quad\quad \times |\Pi(s,x)\varepsilon^{1/2}\int_0^s \Pi(u,x)^{-1}[\sigma(X^\varepsilon(u,x))-\sigma(X^0(u,x))]dW(u)|,
\end{aligned}
$$

then from Lemma 2.4 and (3.12),

$$
(3.15). \quad \sup_{0\le s\le f(\varepsilon)T} |X^\varepsilon(t,x)-X^0(t,x)| \le \tilde{C}(D,x)/[1-C(\gamma_1)C(D^2b)\tilde{C}(D,x)/(\lambda+\gamma_1)] < C(D,x).
$$

This implies that $\tau_D^\varepsilon(x) > f(\varepsilon)T$.

Next we show that (3.6) is true. For γ_3, $\gamma_1 > 0$, and Q in Lemma 2.2, and

(3.16).

$$
\begin{aligned}
&(\exp[-2(\lambda+\gamma_1)s]|Q^{-1}X_1(s,x)|^2+\varepsilon^{2\gamma_3})^{1/2}\\
&=\varepsilon^{\gamma_3}+\int_0^s \exp[-2(\lambda+\gamma_1)u](\exp[-2(\lambda+\gamma_1)u]|Q^{-1}X_1(u,x)|^2+\varepsilon^{2\gamma_3})^{-1/2}\\
&\qquad\times(-(\lambda+\gamma_1)|Q^{-1}X_1(u,x)|^2du+<Q^{-1}X_1(u,x)\\
&\qquad\quad,Q^{-1}Db(X^0(u,x))QQ^{-1}X_1(u,x)du+Q^{-1}\sigma(X^0(u,x))dW(u)>)\\
&\quad+\int_0^s \exp[-2(\lambda+\gamma_1)u](||Q^{-1}\sigma(X^0(u,x))||^2-\exp[-2(\lambda+\gamma_1)u]\\
&\qquad\times|(Q^{-1}\sigma(X^0(u,x)))^*Q^{-1}X_1(u,x)|^2/(\exp[-2(\lambda+\gamma_1)u]|Q^{-1}X_1(u,x)|^2+\varepsilon^{2\gamma_3}))\\
&\qquad/(\exp[-2(\lambda+\gamma_1)u]|Q^{-1}X_1(u,x)|^2+\varepsilon^{2\gamma_3})^{1/2}du/2\\
&\le\varepsilon^{\gamma_3}+\int_0^s C(D^2b)||Q^{-1}||\cdot||Q||\,|X^0(u,x)|\\
&\qquad\times(\exp[-2(\lambda+\gamma_1)u]|Q^{-1}X_1(u,x)|^2+\varepsilon^{2\gamma_3})^{1/2}du\\
&\quad+\sup_{0\le u\le\infty}||Q^{-1}\sigma(X^0(u,x))||\sup_{0\le u\le 1}|\tilde{W}(u)|/[2(\lambda+\gamma_1)]^{1/2}\\
&\quad+\int_0^s \exp[-2(\lambda+\gamma_1)u]||Q^{-1}\sigma(X^0(u,x))||^2/(2\varepsilon^{\gamma_3})du,
\end{aligned}
$$

for some 1-dimensional Wiener process $\tilde{W}(\cdot)$ (see Ikeda and Watanabe (1989), Chap. 2, section 7), in the same way as in Lemma 2.2.

From (3.16), by the Gronwall's inequality (see Hartman (1964), Chap. 3),

(3.17).
$$
\begin{aligned}
&(\exp[-2(\lambda+\gamma_1)s]|Q^{-1}X_1(s,x)|^2+\varepsilon^{2\gamma_3})^{1/2}\\
&\le\exp(\int_0^s C(D^2b)||Q^{-1}||\cdot||Q||\cdot|X^0(u,x)|du)(\varepsilon^{\gamma_3}\\
&\qquad+\sup_{0\le u\le\infty}||Q^{-1}\sigma(X^0(u,x))||\sup_{0\le u\le 1}|\tilde{W}(u)|/[2(\lambda+\gamma_1)]^{1/2}\\
&\qquad+\int_0^s \exp[-2(\lambda+\gamma_1)u]||Q^{-1}\sigma(X^0(u,x))||^2du/(2\varepsilon^{\gamma_3}).
\end{aligned}
$$

Let γ_1, $\gamma_3 > 0$ sufficiently small depending on T and λ. Then we get (3.6), from Lemma 2.1.

Finally we prove (3.7) which can be proved in the same way as in (3.16)-(3.17). In fact, put

$$
(3.18).\quad Z^\varepsilon(s,x)\equiv\exp[-(\lambda+\gamma_1)s]\Pi(s,x)\int_0^s\Pi(u,x)^{-1}[\sigma(X^\varepsilon(u,x))-\sigma(X^0(u,x))]dW(u).
$$

Then there exist positive constants $C(D^2b)$ and $C(||Q^{-1}\sigma||)$ such that

(3.19). $(|Q^{-1}Z^\varepsilon(s,x)|^2+\varepsilon^{2\gamma_3})^{1/2}$

$$\le \varepsilon^{\gamma_3}+\int_0^s (|Q^{-1}Z^\varepsilon(u,x)|^2+\varepsilon^{2\gamma_3})^{-1/2}(-(\lambda+\gamma_1)|Q^{-1}Z^\varepsilon(u,x)|^2du$$
$$+<Q^{-1}Z^\varepsilon(u,x),Q^{-1}Db(X^0(u,x))QQ^{-1}Z^\varepsilon(u,x)du$$
$$+Q^{-1}\exp[-(\lambda+\gamma_1)u](\sigma(X^\varepsilon(u,x))-\sigma(X^0(u,x)))dW(u)>)$$
$$+\int_0^s \exp[-2(\lambda+\gamma_1)u]||Q^{-1}(\sigma(X^\varepsilon(u,x))-\sigma(X^0(u,x)))||^2$$
$$/(|Q^{-1}Z^\varepsilon(u,x)|^2+\varepsilon^{2\gamma_3})^{1/2}du/2$$
$$\le \varepsilon^{\gamma_3}+\int_0^s C(D^2b)||Q^{-1}||\cdot||Q||||X^0(u,x)|(|Q^{-1}Z^\varepsilon(u,x)|^2+\varepsilon^{2\gamma_3})^{1/2}du$$
$$+2C(||Q^{-1}\sigma||)\sup_{0\le u\le 1}|\tilde{W}(u)|/[2(\lambda+\gamma_1)]^{1/2}$$
$$+\int_0^s \exp[-2(\lambda+\gamma_1)u]C(||Q^{-1}\sigma||)^2/(2\varepsilon^{\gamma_3})du,$$

for some 1-dimensional Wiener process $\tilde{W}(\cdot)$ in the same way as in (3.16)-(3.17).

By the Gronwall's inequality, in the same way as in (3.17),

(3.20).

$$(|Q^{-1}Z^\varepsilon(s,x)|^2+\varepsilon^{2\gamma_3})^{1/2}$$
$$\le \exp(\int_0^s C(D^2b)||Q||\cdot||Q^{-1}||\cdot|X^0(u,x)|du)$$
$$\times[\varepsilon^{\gamma_3}+C(||Q^{-1}\sigma||)(2/(\lambda+\gamma_1))^{1/2}\sup_{0\le u\le 1}|\tilde{W}(u)|+C(||Q^{-1}\sigma||)^2/(4\varepsilon^{\gamma_3}(\lambda+\gamma_1))].$$

Let γ_1, $\gamma_3>0$ sufficiently small depending on T and λ. Then we get (3.7), from Lemma 2.1.

Next we prove (3.2).

Proof of (3.2).

For $\min(1-T,T)>\gamma_4>0$, take $a=a(\varepsilon,T,\gamma_4)\in W_{max}$ (see before Lemma 2.5 for notation) such that

(3.21). $$|a|=\varepsilon^{(T-\gamma_4)/2}.$$

For $T>\gamma_5>0$, put

(3.22). $$\varphi(t)\equiv\begin{cases} X^0(t,x) & \text{if } 0\le t\le f(\varepsilon)\gamma_5-1\ ,\\ a(t-f(\varepsilon)\gamma_5+1)+X^0(t,x) & \text{if } f(\varepsilon)\gamma_5-1\le t\le f(\varepsilon)\gamma_5.\end{cases}$$

Then

(3.23). $P(\tau_D^\varepsilon(x) < f(\varepsilon)T)$

$$\geq P(\sup_{0\leq t\leq f(\varepsilon)\gamma_5} |X^\varepsilon(t,x) - \varphi(t)| < \varepsilon^{(T+\gamma_4)/2},$$

$$\sup_{f(\varepsilon)\gamma_5\leq t\leq T_a+f(\varepsilon)\gamma_5} |X^\varepsilon(t,x) - X^0(t-f(\varepsilon)\gamma_5, X^\varepsilon(f(\varepsilon)\gamma_5,x))| < \gamma_6)$$

$$\geq P(\sup_{0\leq t\leq f(\varepsilon)\gamma_5} |X^\varepsilon(t,x) - \varphi(t)| < \varepsilon^{(T+\gamma_4)/2})$$

$$\times \inf_{|y-a-X^0(f(\varepsilon)\gamma_5,x)|<\varepsilon^{(T+\gamma_4)/2}} P(\sup_{0\leq t\leq T_a} |X^\varepsilon(t,y) - X^0(t,y)| < \gamma_6),$$

for sufficiently small $\gamma_6 > 0$ compared to δ (in Lemma 2.5), from Lemma 2.6 (see (2.19) for notation).

The following together with (3.23) completes the proof;

(3.24). $$\liminf_{\varepsilon\to 0} \log(-\log P(\sup_{0\leq t\leq f(\varepsilon)\gamma_5} |X^\varepsilon(t,x) - \varphi(t)| < \varepsilon^{(T+\gamma_4)/2}))/\log\varepsilon \geq T-1,$$

and

(3.25). $$\lim_{\varepsilon\to 0} \inf_{|y-a-X^0(f(\varepsilon)\gamma_5,x)|<\varepsilon^{(T+\gamma_4)/2}} P(\sup_{0\leq t\leq T_a} |X^\varepsilon(t,y) - X^0(t,y)| < \gamma_6) = 1.$$

Let us first show that (3.24) is true. Put

(3.26). $$X^{\varphi,\varepsilon}(t,x) \equiv \varphi(t) + \varepsilon^{1/2}\int_0^t \sigma(X^{\varphi,\varepsilon}(s,x))dW(s).$$

Then there exist positive constants $C(\sigma)$ and $C(Db) > 1$ such that

(3.27).

$$P(\sup_{0\leq t\leq f(\varepsilon)\gamma_5} |X^\varepsilon(t,x) - \varphi(t)| < \varepsilon^{(T+\gamma_4)/2})$$

$$= E[\exp([\int_0^{f(\varepsilon)\gamma_5} < \sigma(X^{\varphi,\varepsilon}(s,x))^{-1}(b(X^{\varphi,\varepsilon}(s,x)) - d\varphi(s)/ds), \varepsilon^{1/2}dW(s) >$$

$$- \int_0^{f(\varepsilon)\gamma_5} |\sigma(X^{\varphi,\varepsilon}(s,x))^{-1}(b(X^{\varphi,\varepsilon}(s,x)) - d\varphi(s)/ds)|^2 ds/2]/\varepsilon)$$

$$; \sup_{0\leq t\leq f(\varepsilon)\gamma_5} |X^{\varphi,\varepsilon}(t,x) - \varphi(t)| < \varepsilon^{(T+\gamma_4)/2}]$$

$$\geq \exp([-\varepsilon^T - 5C(\sigma)^2C(Db)^2 f(\varepsilon)\gamma_5\varepsilon^{(T-\gamma_4)}]/\varepsilon)$$

$$\times P(\int_0^{f(\varepsilon)\gamma_5} < \sigma(X^{\varphi,\varepsilon}(s,x))^{-1}(b(X^{\varphi,\varepsilon}(s,x)) - d\varphi(s)/ds), \varepsilon^{1/2}dW(s) >> -\varepsilon^T,$$

$$\sup_{0\leq t\leq f(\varepsilon)\gamma_5} |X^{\varphi,\varepsilon}(t,x) - \varphi(t)| < \varepsilon^{(T+\gamma_4)/2})$$

$$\geq 1/2\exp([-\varepsilon^T - 5C(\sigma)^2C(Db)^2 f(\varepsilon)\gamma_5\varepsilon^{(T-\gamma_4)}]/\varepsilon),$$

for sufficiently small $\varepsilon > 0$.

In fact, if $\sup_{0\le t\le f(\varepsilon)\gamma_5} |X^{\varphi,\varepsilon}(t,x) - \varphi(t)| < \varepsilon^{(T+\gamma_4)/2}$, then

(3.28).
$$\int_0^{f(\varepsilon)\gamma_5} |\sigma(X^{\varphi,\varepsilon}(s,x))^{-1}(b(X^{\varphi,\varepsilon}(s,x)) - d\varphi(s)/ds)|^2 ds$$
$$\le 9C(\sigma)^2 C(Db)^2 f(\varepsilon)\gamma_5 \varepsilon^{(T-\gamma_4)}$$

for some positive constants $C(\sigma)$ and $C(Db) > 1$, since

(3.29).
$$|b(X^{\varphi,\varepsilon}(s,x)) - d\varphi(s)/ds|$$
$$= \begin{cases} |b(X^{\varphi,\varepsilon}(s,x)) - b(\varphi(s))| & \text{if } 0 \le s \le f(\varepsilon)\gamma_5 - 1, \\ |b(X^{\varphi,\varepsilon}(s,x)) - b(\varphi(s)) + b(\varphi(s)) - b(X^0(s,x)) - a| & \text{if } f(\varepsilon)\gamma_5 - 1 \le s \le f(\varepsilon)\gamma_5, \end{cases}$$
$$\le C(Db)(\varepsilon^{(T+\gamma_4)/2} + |a|) + |a|$$
$$\le 3C(Db)\varepsilon^{(T-\gamma_4)/2}$$

for some constant $C(Db) > 1$ by the mean value theorem.

The last probability in (3.27) can be considered as follows;

(3.30).
$$P(\int_0^{f(\varepsilon)\gamma_5} < \sigma(X^{\varphi,\varepsilon}(s,x))^{-1}(b(X^{\varphi,\varepsilon}(s,x)) - d\varphi(s)/ds), \varepsilon^{1/2}W(s) >> -\varepsilon^T,$$
$$\sup_{0\le t\le f(\varepsilon)\gamma_5} |X^{\varphi,\varepsilon}(t,x) - \varphi(t)| < \varepsilon^{(T+\gamma_4)/2})$$
$$= P(\sup_{0\le t\le f(\varepsilon)\gamma_5} |X^{\varphi,\varepsilon}(t,x) - \varphi(t)| < \varepsilon^{(T+\gamma_4)/2})$$
$$- P(\int_0^{f(\varepsilon)\gamma_5} < \sigma(X^{\varphi,\varepsilon}(s,x))^{-1}(b(X^{\varphi,\varepsilon}(s,x)) - d\varphi(s)/ds), \varepsilon^{1/2}dW(s) >\le -\varepsilon^T,$$
$$\sup_{0\le t\le f(\varepsilon)\gamma_5} |X^{\varphi,\varepsilon}(t,x) - \varphi(t)| < \varepsilon^{(T+\gamma_4)/2}).$$

The first probability on the right hand side of (3.30) can be considered as follows;

(3.31).
$$P(\sup_{0\le t\le f(\varepsilon)\gamma_5} |X^{\varphi,\varepsilon}(t,x) - \varphi(t)| < \varepsilon^{(T+\gamma_4)/2})$$
$$= 1 - P(\sup_{0\le t\le f(\varepsilon)\gamma_5} |\varepsilon^{1/2} \int_0^t \sigma(X^{\varphi,\varepsilon}(s,x))dW(s)| \ge \varepsilon^{(T+\gamma_4)/2})$$
$$\ge 1 - \varepsilon^{1-(T+\gamma_4)} E[\int_0^{f(\varepsilon)\gamma_5} ||\sigma(X^{\varphi,\varepsilon}(s,x))||^2 ds]$$
$$\to 1 \quad (\text{as } \varepsilon \to 0),$$

by the martingale inequality (see Ikeda and Watanabe (1989), Chap. 1, section 6).

The second probability on the right hand side of (3.30) can be considered as follows;

$$
\begin{aligned}
(3.32).\ & P\Big(\int_0^{f(\varepsilon)\gamma_5} < \sigma(X^{\varphi,\varepsilon}(s,x))^{-1}(b(X^{\varphi,\varepsilon}(s,x)) - d\varphi(s)/ds), \varepsilon^{1/2}dW(s) > \leq -\varepsilon^T, \\
& \sup_{0\leq t\leq f(\varepsilon)\gamma_5} |X^{\varphi,\varepsilon}(t,x) - \varphi(t)| < \varepsilon^{(T+\gamma_4)/2}\Big) \\
& \leq P(3C(\sigma)C(Db)\varepsilon^{(1-\gamma_4)/2} f(\varepsilon)\gamma_5 \sup_{0\leq t\leq 1} |\tilde{W}(t)| \geq 1) \\
& \qquad \text{(for some 1-dimensional Wiener process } \tilde{W}(t), \text{ from (3.29))} \\
& \leq \exp(-\varepsilon^{(\gamma_4-1)/2})
\end{aligned}
$$

for sufficiently small $\varepsilon > 0$ (see Freidlin and Wentzell (1984), Chap.3).

Next we prove that (3.25) is true; for y for which $|y - a - X^0(f(\varepsilon)\gamma_5, x)| < \varepsilon^{(T+\gamma_4)/2}$,

$$
\begin{aligned}
(3.33).\ & P(\sup_{0\leq t\leq T_a} |X^\varepsilon(t,y) - X^0(t,y)| < \gamma_6) \\
& \geq P(\exp[(\lambda+\gamma_1)T_a] \sup_{0\leq s\leq T_a} \{\exp[-(\lambda+\gamma_1)s](|\varepsilon^{1/2}X_1(s,y)| \\
& \qquad + |\Pi(s,y)\varepsilon^{1/2}\int_0^s \Pi(u,y)^{-1}[\sigma(X^\varepsilon(u,y)) - \sigma(X^0(u,y))]dW(u)|\} < \gamma_7),
\end{aligned}
$$

for sufficiently small γ_7 comapred to γ_6 in the same way as in (3.12)-(3.15). The last probability in (3.33) converges to 1, as $\varepsilon \to 0$, uniformly in y for which $|y-a-X^0(f(\varepsilon)\gamma_5, x)| < \varepsilon^{(T+\gamma_4)/2}$, in the same way as in (3.6) and (3.7) from (2.24), since

$$
\begin{aligned}
(3.34).\quad & \int_0^{T_a} |X^0(s,y)|ds \\
& \leq \int_0^{T_a} |X^0(s,y) - X^0(s,a) - X^0(s+f(\varepsilon)\gamma_5, x)|ds \\
& \qquad \int_0^{T_a} |X^0(s,a)|ds + \int_0^{T_a} |X^0(s+f(\varepsilon)\gamma_5, x)|ds \\
& \leq T_a\varepsilon^{\min(\gamma_4, \alpha\gamma_5/(2\lambda))/2} + T_a\gamma_1 + \int_0^{T_a} C_1|x|\exp(-\alpha(s+f(\varepsilon)\gamma_5))ds,
\end{aligned}
$$

for sufficiently small $\varepsilon > 0$, and $\gamma_1 > 0$ from Lemma 2.6, (2.26) and Lemma 2.1, and since from (2.28)

$$
(3.35).\qquad T_a < \log(\tilde{C}(\gamma_1)/[|a|\max\{|y|; y \in \partial U_\delta(D)\}])/(\lambda - \gamma_1) < f(\varepsilon)T,
$$

for $\gamma_1 > 0$ sufficiently small compared to $\gamma_4 > 0$.

4. Proof of Theorem 1.3 and Proposition 1.4.

In this section we prove Theorem 1.3 and Proposition 1.4. We first prove Theorem 1.3, that is, 1-dimensional case.

Proof of Theorem 1.3.

We devide the proof into the following two step; for any $T \geq 1$,

$$\limsup_{\varepsilon \to 0} \log P(\tau_D^\varepsilon(o)/f(\varepsilon) > T)/\log \varepsilon \leq T - 1. \tag{4.1}$$

$$\liminf_{\varepsilon \to 0} \log P(\tau_D^\varepsilon(o)/f(\varepsilon) > T)/\log \varepsilon \geq T - 1. \tag{4.2}$$

(Since we consider 1-dimensional case, the set $A_1 \cup A_3$ is empty.)

We first prove (4.1).

Proof of (4.1). For any $1/2 < r < 1$ and $\gamma_2 > 0$, and sufficiently small $\gamma > 0$,

$$\begin{aligned}
&(4.3). P(\tau_D^\varepsilon(o)/f(\varepsilon) > T) \\
&\quad \geq P\Big(\sup_{0 \leq t \leq f(\varepsilon)(T-r)} |X^\varepsilon(t,o)| < \varepsilon^{(r+\gamma_2)/2}, \\
&\qquad \sup_{f(\varepsilon)(T-r) \leq t \leq f(\varepsilon)T} |X^\varepsilon(t,o) - X^0(t - f(\varepsilon)(T-r), X^\varepsilon(f(\varepsilon)(T-r),o))| < \gamma \Big) \\
&\quad \geq P\Big(\sup_{0 \leq t \leq f(\varepsilon)(T-r)} |X^\varepsilon(t,o)| < \varepsilon^{(r+\gamma_2)/2} \Big) \\
&\qquad \times \inf_{|y| < \varepsilon^{(r+\gamma_2)/2}} P\Big(\sup_{0 \leq t \leq f(\varepsilon)r} |X^\varepsilon(t,y) - X^0(t,y)| < \gamma \Big)
\end{aligned}$$

for sufficiently small $\varepsilon > 0$, since for $|y| < \varepsilon^{(r+\gamma_2)/2}$,

$$\sup_{0 \leq t \leq f(\varepsilon)r} |X^0(t,y)| < \varepsilon^{\gamma_2/4} \tag{4.4}$$

for sufficiently small $\varepsilon > 0$, from Lemma 2.5.

The following together with (4.3) completes the proof;

$$\limsup_{\varepsilon \to 0} \log P\Big(\sup_{0 \leq t \leq f(\varepsilon)(T-r)} |X^\varepsilon(t,o)| < \varepsilon^{(r+\gamma_2)/2} \Big)/\log \varepsilon \leq T - 1, \tag{4.5}$$

$$\lim_{\varepsilon \to 0} \inf_{|y| < \varepsilon^{(r+\gamma_2)/2}} P\Big(\sup_{0 \leq t \leq f(\varepsilon)r} |X^\varepsilon(t,y) - X^0(t,y)| < \gamma \Big) = 1. \tag{4.6}$$

Since (4.6) can be proved in the same way as in (3.13)-(3.20), we only prove (4.5).

Put

(4.7). $$\varphi^{\varepsilon}(t) \equiv \int_0^t \sigma(X^{\varepsilon}(s,o))^2 ds/\sigma(o)^2,$$

(4.8). $$Y^{\varepsilon}(t) \equiv X^{\varepsilon}((\varphi^{\varepsilon})^{-1}(t), o).$$

Then there exists a positive constant $C_1(\sigma)$ such that

(4.9). $$\begin{aligned} & P(\sup_{0\le t\le f(\varepsilon)(T-r)} |X^{\varepsilon}(t,o)| < \varepsilon^{(r+\gamma_2)/2}) \\ & = P(\sup_{0\le t\le \varphi^{\varepsilon}(f(\varepsilon)(T-r))} |Y^{\varepsilon}(t)| < \varepsilon^{(r+\gamma_2)/2}) \\ & \ge P(\sup_{0\le t\le f(\varepsilon)(T-r)(1+C_1(\sigma)\varepsilon^{(r+\gamma_2)/2})} |Y^{\varepsilon}(t)| < \varepsilon^{(r+\gamma_2)/2}). \end{aligned}$$

This is true, since in (4.9) we can assume the following; there exists a positive constant $C_2(\sigma)$ such that for any x,

(4.10). $$|\sigma(x) - \sigma(o)| < C_2(\sigma)\varepsilon^{(r+\gamma_2)/2},$$

and since there exists a positive constant $C_1(\sigma)$ such that

(4.11). $$\varphi^{\varepsilon}(t) \le t + C_1(\sigma)\varepsilon^{(r+\gamma_2)/2} t$$

from (4.7) and (4.10).

Put

(4.12). $$g(\varepsilon) \equiv f(\varepsilon)(1 + C_1(\sigma)\varepsilon^{(r+\gamma_2)/2}).$$

Then the last probability in (4.9) can be considered as follows;

(4.13).

$$\begin{aligned} & P(\sup_{0\le t\le f(\varepsilon)(T-r)(1+C_1(\sigma)\varepsilon^{(r+\gamma_2)/2})} |Y^{\varepsilon}(t)| < \varepsilon^{(r+\gamma_2)/2}) \\ & = E[\exp([\int_0^{g(\varepsilon)(T-r)} < \sigma(o)^{-1}(b(\varepsilon^{1/2}X_1(s,o))[\sigma(o)/\sigma(\varepsilon^{1/2}X_1(s,o))]^2 - \lambda\varepsilon^{1/2}X_1(s,o)) \\ & \quad , \varepsilon^{1/2}dW(s) > - \int_0^{g(\varepsilon)(T-r)} |\sigma(o)^{-1}(b(\varepsilon^{1/2}X_1(s,o))[\sigma(o)/\sigma(\varepsilon^{1/2}X_1(s,o))]^2 \\ & \quad - \lambda\varepsilon^{1/2}X_1(s,o))|^2 ds/2]/\varepsilon); \sup_{0\le t\le g(\varepsilon)(T-r)} |\varepsilon^{1/2}X_1(t,o)| < \varepsilon^{(r+\gamma_2)/2}] \\ & \ge 1/4P(|\int_0^{g(\varepsilon)(T-r)} < \sigma(o)^{-1}(b(\varepsilon^{1/2}X_1(s,o))[\sigma(o)/\sigma(\varepsilon^{1/2}X_1(s,o))]^2 \\ & \quad - \lambda\varepsilon^{1/2}X_1(s,o)), dW(s) > | < \varepsilon^{1/2}, \sup_{0\le t\le g(\varepsilon)(T-r)} |\varepsilon^{1/2}X_1(t,o)| < \varepsilon^{(r+\gamma_2)/2}) \end{aligned}$$

(see (3.3) for notation), since $1/2 < r$, and since

(4.14). $$dY^\varepsilon(t) = b(Y^\varepsilon(t))[\sigma(o)/\sigma(Y^\varepsilon(t))]^2 dt + \varepsilon^{1/2}\sigma(o)dW(t)$$

from (4.8).

The last probability in (4.13) can be considered as follows;

(4.15).

$$\begin{aligned}
&P(\Big|\int_0^{g(\varepsilon)(T-r)} < \sigma(o)^{-1}(b(\varepsilon^{1/2}X_1(s,o))[\sigma(o)/\sigma(\varepsilon^{1/2}X_1(s,o))]^2 \\
&\quad - \lambda\varepsilon^{1/2}X_1(s,o)), dW(s) > \Big| < \varepsilon^{1/2}, \sup_{0\le t\le g(\varepsilon)(T-r)} |\varepsilon^{1/2}X_1(t,o)| < \varepsilon^{(r+\gamma_2)/2}) \\
&= P(\sup_{0\le t\le g(\varepsilon)(T-r)} |\varepsilon^{1/2}X_1(t,o)| < \varepsilon^{(r+\gamma_2)/2}) \\
&\quad - P(\Big|\int_0^{g(\varepsilon)(T-r)} < \sigma(o)^{-1}(b(\varepsilon^{1/2}X_1(s,o))[\sigma(o)/\sigma(\varepsilon^{1/2}X_1(s,o))]^2 \\
&\quad - \lambda\varepsilon^{1/2}X_1(s,o)), dW(s) > \Big| \ge \varepsilon^{1/2}, \sup_{0\le t\le g(\varepsilon)(T-r)} |\varepsilon^{1/2}X_1(t,o)| < \varepsilon^{(r+\gamma_2)/2}).
\end{aligned}$$

The first probability on the right hand side of (4.15) can be considered as follows:

(4.16).

$$\begin{aligned}
&P(\sup_{0\le t\le g(\varepsilon)(T-r)} |\varepsilon^{1/2}X_1(t,o)| < \varepsilon^{(r+\gamma_2)/2}) \\
&\ge P(|\varepsilon^{1/2}X_1(g(\varepsilon)(T-r),o)| < \varepsilon^{(r+\gamma_2)/2}/2, \\
&\quad \max_{k=1}^{n(\varepsilon)} \varepsilon^{1/2}\exp(\lambda k) \sup_{k-1\le t\le g(\varepsilon)(T-r)} \Big|\int_{k-1}^t \exp(-\lambda s)\sigma(o)dW(s)\Big| < \varepsilon^{(r+\gamma_2)/2}/4) \\
&= P(|\varepsilon^{1/2}X_1(g(\varepsilon)(T-r),o)| < \varepsilon^{(r+\gamma_2)/2}/2) \\
&\quad - P(\max_{k=1}^{n(\varepsilon)} \varepsilon^{1/2}\exp(\lambda k) \sup_{k-1\le t\le g(\varepsilon)(T-r)} \Big|\int_{k-1}^t \exp(-\lambda s)\sigma(o)dW(s)\Big| \ge \varepsilon^{(r+\gamma_2)/2}/4),
\end{aligned}$$

where $n(\varepsilon) - 1$ denotes the integer part of $g(\varepsilon)(T-r)$. This is true, since for $k-1 \le t \le g(\varepsilon)(T-r)$,

(4.17). $$\begin{aligned}
|\varepsilon^{1/2}X_1(t,o)| &= \varepsilon^{1/2}\exp(\lambda t)\Big|\int_0^t \exp(-\lambda s)\sigma(o)dW(s)\Big| \\
&\le \varepsilon^{1/2}\exp(\lambda t)\Big|\int_0^{g(\varepsilon)(T-r)} \exp(-\lambda s)\sigma(o)dW(s)\Big|
\end{aligned}$$

$$+\varepsilon^{1/2}\exp(\lambda t)|\int_{k-1}^{g(\varepsilon)(T-r)}\exp(-\lambda s)\sigma(o)dW(s)|$$
$$+\varepsilon^{1/2}\exp(\lambda t)|\int_{k-1}^{t}\exp(-\lambda s)\sigma(o)dW(s)|$$
$$\leq \varepsilon^{1/2}\exp(\lambda g(\varepsilon)(T-r))|\int_{0}^{g(\varepsilon)(T-r)}\exp(-\lambda s)\sigma(o)dW(s)|$$
$$+\varepsilon^{1/2}\exp(\lambda k)|\int_{k-1}^{g(\varepsilon)(T-r)}\exp(-\lambda s)\sigma(o)dW(s)|$$
$$+\varepsilon^{1/2}\exp(\lambda k)|\int_{k-1}^{t}\exp(-\lambda s)\sigma(o)dW(s)|.$$

The first probability on the last part of (4.16) can be considered as follows; for sufficiently small $\varepsilon > 0$,

(4.18). $$P(|\varepsilon^{1/2}X_1(g(\varepsilon)(T-r),o)| < \varepsilon^{(r+\gamma_2)/2}/2) > \varepsilon^{(T-1)/2+\gamma_2}.$$

The second probability on the last part of (4.16) converges to 0 exponentially fast;

(4.19) $$P(\max_{k=1}^{n(\varepsilon)}\varepsilon^{1/2}\exp(\lambda k)\sup_{k-1\leq t\leq g(\varepsilon)(T-r)}|\int_{k-1}^{t}\exp(-\lambda s)\sigma(o)dW(s)| \geq \varepsilon^{(r+\gamma_2)/2}/4)$$
$$\leq n(\varepsilon)P(\varepsilon^{(1-r-2\gamma_2)/2}\sup_{0\leq t\leq 1}|\tilde{W}(t)| \geq 1)$$
(for some 1-dimensional Wiener process $\tilde{W}(t)$)
$$\leq \exp(-\varepsilon^{-(1-r-3\gamma_2)})$$

for sufficiently small $\varepsilon > 0$ (see Freidlin and Wentzell (1984), Chap. 3).

The second probability on the right hand side of (4.15) also converges to 0 exponentially fast;

(4.20).
$$P(|\int_{0}^{g(\varepsilon)(T-r)} < \sigma(o)^{-1}(b(\varepsilon^{1/2}X_1(s,o))[\sigma(o)/\sigma(\varepsilon^{1/2}X_1(s,o))]^2$$
$$-\lambda\varepsilon^{1/2}X_1(s,o)), dW(s) > | \geq \varepsilon^{1/2}, \sup_{0\leq t\leq g(\varepsilon)(T-r)}|\varepsilon^{1/2}X_1(t,o)| < \varepsilon^{(r+\gamma_2)/2})$$
$$\leq P(\varepsilon^{(r+\gamma_2/2-1/2)}\sup_{0\leq t\leq 1}|\tilde{W}(t)| > 1)$$
(for some 1-dimensional Wiener process $\tilde{W}(t)$)
$$\leq \exp(-\varepsilon^{-(2r-1)})$$

for sufficiently small $\varepsilon > 0$ (see Freidlin and Wentzell (1984), Chap. 3). Here we used the following;

(4.21). $$\sup\{|b(x)[\sigma(o)/\sigma(x)]^2 - \lambda x|/|x|^2; x \in R\} < \infty$$

from (A.D).

Q.E.D.

Next we prove (4.2).

Proof of (4.2). For $1/2 < r < 1$, $\gamma_2 > 0$ and sufficiently small $\varepsilon > 0$,

(4.22).

$$\begin{aligned}
&P(\tau_D^\varepsilon(o)/f(\varepsilon) > T) \\
&\le P(\sup_{0\le t\le f(\varepsilon)(T-r)} |X^\varepsilon(t,o)| < \varepsilon^{(r-\gamma_2)/2}) \\
&\quad + P(\sup_{0\le t\le f(\varepsilon)(T-r)} |X^\varepsilon(t,o)| \ge \varepsilon^{(r-\gamma_2)/2}, \\
&\qquad \sup_{\tau^\varepsilon_{(r-\gamma_2)/2}\le t\le \tau^\varepsilon_{(r-\gamma_2)/2}+T_{X^\varepsilon(\tau^\varepsilon_{(r-\gamma_2)/2})}} |X^\varepsilon(t) - X(t-\tau^\varepsilon_{(r-\gamma_2)/2}, X^\varepsilon(\tau^\varepsilon_{(r-\gamma_2)/2})| \ge \delta) \\
&\le P(\sup_{0\le t\le f(\varepsilon)(T-r)} |X^\varepsilon(t,o)| < \varepsilon^{(r-\gamma_2)/2}) \\
&\quad + \sup_{|y|=\varepsilon^{(r-\gamma_2)/2}} P(\sup_{0\le t\le T_y} |X^\varepsilon(t,y) - X(t,y)| \ge \delta)
\end{aligned}$$

(see (2.19) for natation), from Lemma 2.5. Here we put

(4.23). $$\tau^\varepsilon_{(r-\gamma_2)/2} \equiv \inf\{t > 0; |X^\varepsilon(t,o)| = \varepsilon^{(r-\gamma_2)/2}\}.$$

(4.22) together with the following completes the proof;

(4.24). $$\liminf_{\varepsilon\to 0} \log P(\sup_{0\le t\le f(\varepsilon)(T-r)} |X^\varepsilon(t,o)| < \varepsilon^{(r-\gamma_2)/2})/\log\varepsilon \ge T-1,$$

(4.25). $$\liminf_{\varepsilon\to 0} \log[\sup_{|y|=\varepsilon^{(r-\gamma_2)/2}} P(\sup_{0\le t\le T_y} |X^\varepsilon(t,y) - X(t,y)| \ge \gamma)]/\log\varepsilon = \infty.$$

We first show that (4.24) is true; there exists a positive constant $C_1(\sigma)$ such that

(4.26).

$$\begin{aligned}
&P(\sup_{0\le t\le f(\varepsilon)(T-r)} |X^\varepsilon(t,o)| < \varepsilon^{(r-\gamma_2)/2}) \\
&= P(\sup_{0\le t\le \varphi^\varepsilon(f(\varepsilon)(T-r))} |Y^\varepsilon(t)| < \varepsilon^{(r-\gamma_2)/2}) \\
&\le P(\sup_{0\le t\le f(\varepsilon)(T-r)(1-C_1(\sigma)\varepsilon^{(r-\gamma_2)/2})} |Y^\varepsilon(t)| < \varepsilon^{(r-\gamma_2)/2}),
\end{aligned}$$

in the same way as in (4.9).

Put

(4.27). $$g_-(\varepsilon) \equiv f(\varepsilon)(1 - C_1(\sigma)\varepsilon^{(r-\gamma_2)/2}).$$

Then the last probability in (4.26) can be considered as follows;

(4.28).

$$\begin{aligned}
&P(\sup_{0\le t\le f(\varepsilon)(T-r)(1-C_2(\sigma)\varepsilon^{(r-\gamma_2)/2})} |Y^\varepsilon(t)| < \varepsilon^{(r-\gamma_2)/2}) \\
&\le P(\sup_{0\le t\le g_-(\varepsilon)(T-r)} |Y^\varepsilon(t)| < \varepsilon^{(r-\gamma_2)/2}, \\
&\quad \int_0^{g_-(\varepsilon)(T-r)} < \sigma(o)^{-1}(b(Y^\varepsilon(s))[\sigma(o)/\sigma(Y^\varepsilon(s))]^2 - \lambda Y^\varepsilon(s)), \varepsilon^{1/2}dW(s) > \\
&\quad + \int_0^{g_-(\varepsilon)(T-r)} |\sigma(o)^{-1}(b(Y^\varepsilon(s))[\sigma(o)/\sigma(Y^\varepsilon(s))]^2 - \lambda Y^\varepsilon(s))|^2 ds/2 \ge \varepsilon) \\
&\quad + E[\exp([\int_0^{g_-(\varepsilon)(T-r)} < \sigma(o)^{-1}(b(\varepsilon^{1/2}X_1(s,o))[\sigma(o)/\sigma(\varepsilon^{1/2}X_1(s,o))]^2 \\
&\quad - \lambda\varepsilon^{1/2}X_1(s,o)), \varepsilon^{1/2}dW(s) > - \int_0^{g_-(\varepsilon)(T-r)} |\sigma(o)^{-1}(b(\varepsilon^{1/2}X_1(s,o)) \\
&\quad \times [\sigma(o)/\sigma(\varepsilon^{1/2}X_1(s,o))]^2 - \lambda\varepsilon^{1/2}X_1(s,o))|^2 ds/2]/\varepsilon) \\
&\quad ; \sup_{0\le t\le g_-(\varepsilon)(T-r)} |\varepsilon^{1/2}X_1(t,o)| < \varepsilon^{(r-\gamma_2)/2}, \int_0^{g_-(\varepsilon)(T-r)} < \sigma(o)^{-1}(b(\varepsilon^{1/2}X_1(s,o)) \\
&\quad \times [\sigma(o)/\sigma(\varepsilon^{1/2}X_1(s,o))]^2 - \lambda\varepsilon^{1/2}X_1(s,o)), \varepsilon^{1/2}dW(s) > \\
&\quad - \int_0^{g_-(\varepsilon)(T-r)} |\sigma(o)^{-1}(b(\varepsilon^{1/2}X_1(s,o))[\sigma(o)/\sigma(\varepsilon^{1/2}X_1(s,o))]^2 \\
&\quad - \lambda\varepsilon^{1/2}X_1(s,o))|^2 ds/2 \le \varepsilon] \\
&\le P(\varepsilon^{(r-2\gamma_2-1/2)} \sup_{0\le t\le 1} |\tilde{W}(t)| \ge 1) + 3P(\sup_{0\le t\le g_-(\varepsilon)(T-r)} |\varepsilon^{1/2}X_1(t,o)| < \varepsilon^{(r-\gamma_2)/2}) \\
&\qquad \text{(for a 1-dimensional Wiener process } \tilde{W}(t), \text{ from (4.21))} \\
&\le P(\varepsilon^{(r-2\gamma_2-1/2)} \sup_{0\le t\le 1} |\tilde{W}(t)| \ge 1) + 3P(|\varepsilon^{1/2}X_1(g_-(\varepsilon)(T-r),o)| < \varepsilon^{(r-\gamma_2)/2}) \\
&\le \exp(-\varepsilon^{-(2r-5\gamma_2-1)}) + \varepsilon^{(T-1-2\gamma_2)/2},
\end{aligned}$$

for sufficiently small $\gamma_2 > 0$ and $\varepsilon > 0$.

(4.25) can be proved as follows; the following holds uniformly in y for which $|y| = \varepsilon^{(r-\gamma_2)/2}$,

(4.29). $$P(\sup_{0\le t\le T_y} |X^\varepsilon(t,y) - X(t,y)| \ge \gamma) \le \exp(-\varepsilon^{-(1-r)/2})$$

for sufficiently small $\varepsilon > 0$, in the same way as in the proof of Theorem 1.2, since

(4.30). $$T_y < f(\varepsilon)r,$$

for sufficiently small $\gamma_2 > 0$, from (3.35).

Q.E.D.

Next we prove Proposition 1.4.

Proof of Proposition 1.4.

Put

(4.31). $$Z^{0,\varepsilon}(t,x) = X^{X^0(\cdot,x),\varepsilon}(t)$$

(see (3.26) for notation).

Then there exists a positive constant $C(\sigma, Db)$ such that for $x \in A_1 \cup \{o\} \backslash \partial D$, $T \geq 1$, and c_1, $c_2 > 0$,

(4.32).

$$\begin{aligned}
&P(\tau_D^\varepsilon(x)/f(\varepsilon) > T) \\
&\geq P(\sup_{0\leq t\leq f(\varepsilon)T} |X^\varepsilon(t,x) - X^0(t,x)| < c_1\varepsilon^{1/2}) \\
&= E[\exp([\int_0^{f(\varepsilon)T} < \sigma(Z^{0,\varepsilon}(s,x))^{-1}(b(Z^{0,\varepsilon}(s,x)) - b(X^0(s,x))), \varepsilon^{1/2}dW(s) > \\
&\quad - \int_0^{f(\varepsilon)T} |\sigma(Z^{0,\varepsilon}(s,x))^{-1}(b(Z^{0,\varepsilon}(s,x)) - b(X^0(s,x)))|^2 ds/2]/\varepsilon) \\
&\quad ; \sup_{0\leq t\leq f(\varepsilon)T} |Z^{0,\varepsilon}(t,x) - X^0(t,x)| < c_1\varepsilon^{1/2}] \\
&\geq \exp[-(C(\sigma,Db)c_1^2 + c_2)f(\varepsilon)T]P(\sup_{0\leq t\leq f(\varepsilon)T} |Z^{0,\varepsilon}(t,x) - X^0(t,x)| < c_1\varepsilon^{1/2}, \\
&\int_0^{f(\varepsilon)T} < \sigma(Z^{0,\varepsilon}(s,x))^{-1}(b(Z^{0,\varepsilon}(s,x)) - b(X^0(s,x))), dW(s) > \geq -c_2\varepsilon^{1/2}f(\varepsilon)T) \\
&= \varepsilon^{(C(\sigma,Db)c_1^2+c_2)T/(2\lambda)}[P(\sup_{0\leq t\leq f(\varepsilon)T} |Z^{0,\varepsilon}(t,x) - X^0(t,x)| < c_1\varepsilon^{1/2}) \\
&\quad - P(\sup_{0\leq t\leq f(\varepsilon)T} |Z^{0,\varepsilon}(t,x) - X^0(t,x)| < c_1\varepsilon^{1/2}, \\
&\int_0^{f(\varepsilon)T} < \sigma(Z^{0,\varepsilon}(s,x))^{-1}(b(Z^{0,\varepsilon}(s,x)) - b(X^0(s,x))), dW(s) >< -c_2\varepsilon^{1/2}f(\varepsilon)T)].
\end{aligned}$$

(4.32) together with the following completes the proof;

(4.33). $$\limsup_{\varepsilon\to 0} \log P(\sup_{0\leq t\leq f(\varepsilon)T} |Z^{0,\varepsilon}(t,x) - X^0(t,x)| < c_1\varepsilon^{1/2})/\log\varepsilon < \infty,$$

and for sufficiently small $\varepsilon > 0$,

(4.34). $$P(\sup_{0\le t\le f(\varepsilon)T}|Z^{0,\varepsilon}(t,x)-X^0(t,x)|<c_1\varepsilon^{1/2},\int_0^{f(\varepsilon)T}<\sigma(Z^{0,\varepsilon}(s,x))^{-1}$$
$$\times(b(Z^{0,\varepsilon}(s,x))-b(X^0(s,x))),dW(s)><-c_2\varepsilon^{1/2}f(\varepsilon)T)$$
$$<P(\sup_{0\le t\le f(\varepsilon)T}|Z^{0,\varepsilon}(t,x)-X^0(t,x)|<c_1\varepsilon^{1/2})/2$$

if c_2 is sufficiently large compared to c_1.

Let us first prove (4.33). There exists a positive constant $C_1(\sigma, Db)$ such that

(4.35).

$$P(\sup_{0\le t\le f(\varepsilon)T}|Z^{0,\varepsilon}(t,x)-X^0(t,x)|<c_1\varepsilon^{1/2})$$
$$=P(\sup_{0\le t\le f(\varepsilon)T}|\varepsilon^{1/2}\int_0^t\sigma(Z^{0,\varepsilon}(s,x))dW(s)|<c_1\varepsilon^{1/2})$$
$$\ge P(\sup_{0\le t\le f(\varepsilon)T}|\varepsilon^{1/2}W(t)|<C_1(\sigma,Db)c_1\varepsilon^{1/2},|(\int_0^t\sum_{j,k,\ell=1}^d\partial\sigma(Z^{0,\varepsilon}(s,x))^{ij}/\partial z_k$$
$$\times\varepsilon^{1/2}W^j(s)\varepsilon^{1/2}\sigma(Z^{0,\varepsilon}(s,x))^{k\ell}dW^\ell(s))_{i=1}^d|<c_1\varepsilon^{1/2}/3)$$
$$=P(\sup_{0\le t\le f(\varepsilon)T}|\varepsilon^{1/2}W(t)|<C_1(\sigma,Db)c_1\varepsilon^{1/2})$$
$$-P(\sup_{0\le t\le f(\varepsilon)T}|\varepsilon^{1/2}W(t)|<C_1(\sigma,Db)c_1\varepsilon^{1/2},|(\int_0^t\sum_{j,k,\ell=1}^d\partial\sigma(Z^{0,\varepsilon}(s,x))^{ij}/\partial z_k$$
$$\times\varepsilon^{1/2}W^j(s)\varepsilon^{1/2}\sigma(Z^{0,\varepsilon}(s,x))^{k\ell}dW^\ell(s))_{i=1}^d|\ge c_1\varepsilon^{1/2}/3)$$

for sufficiently small $\varepsilon > 0$ from Lemma 2.1, since for $i = 1, \cdots, d$,

(4.36).

$$\int_0^t\varepsilon^{1/2}\sum_{j=1}^d\sigma(Z^{0,\varepsilon}(s,x))^{ij}dW^j(s)$$
$$=\sum_{j=1}^d\sigma(Z^{0,\varepsilon}(t,x))^{ij}\varepsilon^{1/2}W^j(t)-\int_0^t\sum_{j,k=1}^d\partial\sigma(Z^{0,\varepsilon}(s,x))^{ij}/\partial z_k\varepsilon^{1/2}W^j(s)$$
$$\times(b^k(X^0(s,x))ds+\varepsilon^{1/2}\sum_{\ell=1}^d\sigma(Z^{0,\varepsilon}(s,x))^{k\ell}dW^\ell(s))$$
$$-\varepsilon\int_0^t[(\sum_{j,k,\ell,m=1}^d\partial^2\sigma(Z^{0,\varepsilon}(s,x))^{ij}/\partial z_k\partial z_m\sigma(Z^{0,\varepsilon}(s,x))^{k\ell}\sigma(Z^{0,\varepsilon}(s,x))^{m\ell})\varepsilon^{1/2}W^j(s)$$

$$+ (\sum_{k,m=1}^{d} \partial\sigma(Z^{0,\varepsilon}(s,x))^{ik}/\partial z_m \sigma(Z^{0,\varepsilon}(s,x))^{mk}]ds/2$$

by the Ito formula (see Ikeda and Watanabe (1989), Chap. 2, section 5).

With respect to (4.35), we have the following; there exists a positive constant C such that as $\varepsilon \to 0$,

$$(4.37).\ P(\sup_{0\le t\le f(\varepsilon)T} |\varepsilon^{1/2}W(t)| < C_1(\sigma, Db)c_1\varepsilon^{1/2}) \sim \exp(-Cf(\varepsilon)T/[C_1(\sigma, Db)c_1]^2)$$
$$= \varepsilon^{CT/(2\lambda[C_1(\sigma,Db)c_1]^2)}$$

(see Ikeda and Watanabe (1989), Chap. 6, section 8). We also have the following; for sufficiently small $\varepsilon > 0$,

$$(4.38) P(\sup_{0\le t\le f(\varepsilon)T} |\varepsilon^{1/2}W(t)| < C_1(\sigma, Db)c_1\varepsilon^{1/2}, |(\int_0^t \sum_{j,k,\ell=1}^{d} \partial\sigma(Z^{0,\varepsilon}(s,x))^{ij}/\partial z_k$$
$$\times \varepsilon^{1/2}W^j(s)\varepsilon^{1/2}\sigma(Z^{0,\varepsilon}(s,x))^{k\ell}dW^\ell(s))_{i=1}^d| \ge c_1\varepsilon^{1/2}/3)$$
$$\le P(\sup_{0\le t\le 1} \varepsilon^{3/8}|\tilde{W}(t)| \ge 1) \quad (\text{ for some 1-dimensional Wiener process } \tilde{W}(t))$$
$$\le \exp(-\varepsilon^{-1/2}).$$

(4.34) can be proved as follows; there exist positive constants C_1 and $C_2(\sigma, b)$ such that for sufficiently small $\varepsilon > 0$,

$$(4.39).$$
$$P(\sup_{0\le t\le f(\varepsilon)T} |Z^{0,\varepsilon}(t,x) - X^0(t,x)| < c_1\varepsilon^{1/2},$$
$$\int_0^{f(\varepsilon)T} < \sigma(Z^{0,\varepsilon}(s,x))^{-1}(b(Z^{0,\varepsilon}(s,x)) - b(X^0(s,x))), dW(s) >< -c_2\varepsilon^{1/2}f(\varepsilon)T)$$
$$\le P(\sup_{0\le t\le 1} (C_2(\sigma,b)c_1/c_2)(f(\varepsilon)T)^{-1/2}|\tilde{W}(t)| \ge 1)$$
$$(\text{for some 1-dimensional Wiener process } \tilde{W}(t))$$
$$\le \exp(-C_1(c_2/[C_2(\sigma,b)c_1])^2 f(\varepsilon)T)$$
$$= \varepsilon^{C_1(c_2/[C_2(\sigma,b)c_1])^2T/(2\lambda)}$$

(see Freidlin and Wentzell (1984), Chap. 3). Take c_2 sufficiently large compared to c_1. Then we get (4.34).

Q.E.D.

5. Example for Conjecture 1.5.

In this section we show that Conjecture 1.5 is true when $b(x) = Db(o)x$ and when $\sigma(x) = Identy$.

Example 5.1.

Suppose that (A.D) holds, that $b(x) = Db(o)x$ and that $\sigma(x) = Identy$. Then for any $x \in A_1 \cup \{o\} \backslash \partial D$ and $T \geq 1$,

$$\lim_{\varepsilon \to 0} \log P(\tau_D^\varepsilon(x)/f(\varepsilon) > T))/\log \varepsilon = \sum_{i=1}^{d} \max(0, (Re(\lambda_i)T/\lambda - 1)/2). \tag{5.1}$$

(Proof). We first show the lower bound. There exists a constant $C(D) > 0$ such that

$$P(\tau_D^\varepsilon(x)/f(\varepsilon) > T) < P(|X^\varepsilon(f(\varepsilon)T, o)| < C(D)), \tag{5.2}$$

since for $t > 0$ and $x \in R^d$,

$$X^\varepsilon(t, x) = X^\varepsilon(t, o) + X^0(t, x). \tag{5.3}$$

Let $\{\lambda_{\varphi(i)}\}_{i=1}^{\nu}$ be the eigenvalues of $Db(o)$ whose real parts are positive. Let Λ_i denotes the eigenspace which corresponds to $\lambda_{\varphi(i)}$ and put $d_i = dim\Lambda_i$ for $i = 1, \cdots, \nu$. For $\gamma_1 > 0$ take a $d \times d$-matrix Q so that

$$< Q^{-1}Db(o)Qz, z >> (Re(\lambda_{\varphi(i)}) - \gamma_1)|z|^2, \quad \text{for } z \in \Lambda_i \ (i = 1, \cdots, \nu) \tag{5.4}$$

(see Hirsch and Smale (1974), Chap. 7, section 1). Then

(5.5).

$$\begin{aligned}
&P(|X^\varepsilon(f(\varepsilon)T, o)| < C(D)) \\
&\leq P(|Q^{-1}X^\varepsilon(f(\varepsilon)T, o)| < ||Q^{-1}||C(D)) \\
&= \int_\varepsilon \int_0^{f(\epsilon)T} {}_{|\exp(Db(o)^* s)(Q^{-1})^* y|^2 ds < ||Q^{-1}||^2 C(D)^2} \exp(-|y|^2/2)/(2\pi)^{d/2} dy \\
&\leq \int_\varepsilon \int_0^{f(\epsilon)T} {}_{|Q^* \exp(Db(o)^* s)(Q^{-1})^* y|^2 ds < ||Q^*||^2 ||Q^{-1}||^2 C(D)^2} \exp(-|y|^2/2)/(2\pi)^{d/2} dy \\
&\leq \Pi_{i=1}^{\nu} \int_\varepsilon \int_0^{f(\epsilon)T} {}_{|Q^* \exp(Db(o)^* s)(Q^{-1})^* y_i|^2 ds < ||Q^*||^2 ||Q^{-1}||^2 C(D)^2, y_i \in \Lambda_i} \exp(-|y_i|^2/2)/(2\pi)^{d_i/2} dy_i \\
&\leq \Pi_{i=1}^{\nu} \int_\varepsilon \int_0^{f(\epsilon)T} {}_{\exp(2(Re(\lambda_{\varphi(i)}) - \gamma_1)s)|y_i|^2 ds < ||Q^*||^2 ||Q^{-1}||^2 C(D)^2, y_i \in \Lambda_i} \exp(-|y_i|^2/2)/(2\pi)^{d_i/2} dy_i \\
&\leq \Pi_{i=1}^{\nu} \int_{\varepsilon^{1-(Re(\lambda_{\varphi(i)}) - 2\gamma_1)T/\lambda}} {}_{|y_i|^2 < ||Q^*||^2 ||Q^{-1}||^2 C(D)^2, y_i \in \Lambda_i} \exp(-|y_i|^2/2)/(2\pi)^{d_i/2} dy_i \\
&\leq \Pi_{i=1}^{\nu} \varepsilon^{\max(d_i[(Re(\lambda_{\varphi(i)}) - 3\gamma_1)T/\lambda - 1]/2, 0)},
\end{aligned}$$

for sufficinetly small $\varepsilon > 0$.

Next we show the upper bound. There exists a constant $C_1(D) > 0$ such that

$$(5.6). \qquad P(\tau_D^\varepsilon(x)/f(\varepsilon) > T) > P(\sup_{0\le t\le f(\varepsilon)T} |X^\varepsilon(t,o)| < C_1(D)),$$

from (5.3), and

$$(5.7). \qquad P(\sup_{0\le t\le f(\varepsilon)T} |X^\varepsilon(t,o)| < C_1(D)) \ge P(\sup_{0\le t\le f(\varepsilon)T} |Q^{-1}X^\varepsilon(t,o)| < C_1(D)/||Q||)$$

for Q in (5.4).

Let $(Q^{-1}X^\varepsilon(t,o))_+$ and $(Q^{-1}X^\varepsilon(t,o))_-$ denote the components of $Q^{-1}X^\varepsilon(t,o)$ which belongs to the eigenspace corresponding to the positive and negative eigenvalues, respectively. Then the right hand side of (5.7) can be considered as follows;

$$\begin{aligned}(5.8). \qquad & P(\sup_{0\le t\le f(\varepsilon)T} |Q^{-1}X^\varepsilon(t,o)| < C_1(D)/||Q||) \\ & \ge P(\sup_{0\le t\le f(\varepsilon)T} |(Q^{-1}X^\varepsilon(t,o))_+| < C_1(D)/(2||Q||)) \\ & \quad - P(\sup_{0\le t\le f(\varepsilon)T} |(Q^{-1}X^\varepsilon(t,o))_-| \ge C_1(D)/(2||Q||)).\end{aligned}$$

The first probability on the right hand side of (5.8) can be considered as follows;

$$\begin{aligned}(5.9). \qquad & P(\sup_{0\le t\le f(\varepsilon)T} |(Q^{-1}X^\varepsilon(t,o))_+| < C_1(D)/(2||Q||)) \\ & \ge P(|(Q^{-1}X^\varepsilon(f(\varepsilon)T,o))_+| < C_1(D)/(4||Q||)) - \exp(-\varepsilon^{-1/2}),\end{aligned}$$

for sufficinetly small $\varepsilon > 0$, in the same way as in (4.16), (4.17) and (4.19). In the same way as in (5.5),

$$\begin{aligned}(5.10). \qquad & P(|(Q^{-1}X^\varepsilon(f(\varepsilon)T,o))_+| < C_1(D)/(4||Q||)) \\ & \ge \Pi_{i=1}^{\nu} \varepsilon^{\max(d_i[(Re(\lambda_{\varphi(i)})+3\gamma_1)T-1]/2,0)},\end{aligned}$$

for sufficinetly small $\varepsilon > 0$.

The second probability on the right hand side of (5.8) can be considered as follows;

$$\begin{aligned}(5.11). \qquad & P(\sup_{0\le t\le f(\varepsilon)T} |(Q^{-1}X^\varepsilon(t,o))_-| \ge C_1(D)/(2||Q||)) \\ & \le \exp(-\varepsilon^{-1/2})\end{aligned}$$

for sufficinetly small $\varepsilon > 0$ (see Freidlin and Wentzell (1984), Chap. 4).

Q.D.E.

6. Correction to Mikami (1993), Lemma 2.4.

Since the inequality (2.27) in the proof of Lemma 2.4 in Mikami (1993) is not correct if the set A_3 is not empty, we give the correction. In this section, all the notations including the numbers of displayed equations follow Mikami (1993).

(2.27) in Mikami (1993) should be changed as follows:

$$(2.27). \quad \limsup_{\varepsilon\to 0} \varepsilon \log[\sup_{x\in U_{\alpha_2}(o), 0\le h\le R/N_0} P_x(\tau_D^\varepsilon > R-h, \tau^\varepsilon_{U_{\alpha_1}(o)} \le R-h-R/N_0)] \le -2\alpha_3,$$

from Lemma 2.1, (2.23).

Accordingly the following change should be done; (2.23) in Mikami (1993) should be changed as follows:

$$(2.23). \quad \inf\{|X(t,x)|; |x|<\alpha_2, t\ge T_{\alpha_1,x}\} \ge 2\alpha_2,$$

where we put $T_{\alpha_1,x} \equiv \inf\{t>0 : |X(t,x)| = \alpha_1\}$; at (2.24) in Mikami (1993), $2/N_0$ should be $4/N_0$; at (2.25) in Mikami (1993), $\tau_D^\varepsilon > R$ should be $\tau_D^\varepsilon > R/N_0$; (2.26) in Mikami (1993) should be deleted; (2.29) in Mikami (1993) should be changed as follows:

$$(2.29). \quad Y_i^\varepsilon \equiv \begin{cases} 0 & \text{if } \inf_{iR\le t\le (i+1/N_0)R} |X^\varepsilon(t)| < \alpha_2 \\ & \quad \text{and if } \sup_{\tau_i^\varepsilon \le t\le (i+1-1/N_0)R} |X^\varepsilon(t)| < \alpha_1, \\ 1 & \text{otherwise,} \end{cases}$$

where we put $\tau_i^\varepsilon \equiv \inf\{\min(t, (i+1-1/N_0)R) > iR; |X^\varepsilon(t)| < \alpha_2\}$. We omit the technical change according to the change above.

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