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On time periodic solutions of the Dirichlet problem for degenerate parabolic equations of nondivergence type

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Abstract. In this paper, we are concerned with the existence of periodic solutions of a quasilinear parabolic equation

$$u_t = u^\gamma(\Delta u + u + f) \quad \text{in } \Omega \times \mathbf{R}$$

with the Dirichlet boundary condition, where Ω is a smoothly bounded domain in $\mathbf{R}^N$ and f is a given function periodic in time defined on $\bar{\Omega} \times \mathbf{R}$. Our results depend on the first eigenvalue λ_1 of $-\Delta$ in Ω with the Dirichlet boundary condition. If $\lambda_1 > 1$, then there exists a unique positive periodic solution for a positive f ($\gamma \in \mathbf{R}$). In the case of $\lambda_1 < 1$, we construct a nonnegative periodic solution for a negative f ($1 \leq \gamma < 3$).

1 Introduction.

We are concerned with the following quasilinear parabolic equation

$$\begin{cases} u_t = u^\gamma(\Delta u + u + f) & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t+T) = u(t) & \text{in } \Omega \times \mathbf{R}, \end{cases} \quad (1)$$

where Ω is a smoothly bounded domain in $\mathbf{R}^N$ and f is a given function on $\bar{\Omega} \times \mathbf{R}$ which is T -periodic in time.

Degenerate parabolic equations of the form (1) are often studied in the literature. When $N = 1$, the equation (1) with $\gamma = 2, f = 0$ describes the evolution of curvature of convex curves moved by curvature under the Gauss map parametrization. Since the space variable x is actually the argument of normals of curves, 2π -periodic boundary condition (instead of the Dirichlet condition) is imposed to describe closed curves. If initial data is positive, solution blows up in a finite time and its profile is studied in Gage [6] and Gage and Hamilton [8]. Geometrically speaking, they proved that a convex embedded curve moved by curvature shrinks to a "round point" in a finite time. Blow up profiles of convex immersed curves were classified by Angenent [2] based on results of Abresch and Langer [1] and Epstein and Weinstein [4], where $2\pi m$ -periodic boundary condition is imposed ($m = 1, 2, \dots$). The Dirichlet condition arises when the curve is complete but non-closed and its curvature equals to zero at the end. The initial value problem of general dimension N for (1) with $\gamma = 2$ and $f = 0$ was investigated in Friedman and McLeod [5] and Gage [7] for positive initial data. These results contain the one dimensional case. For an open set $D \subset \mathbf{R}^N$, let $\lambda_1(D)$ denote the first eigenvalue of $-\Delta$ in D with the Dirichlet boundary condition. In particular, we write $\lambda_1 = \lambda_1(\Omega)$. It was proved in [5] that

- i) If $\lambda_1 > 1$, then there exists a unique global solution which decays to zero as $t \rightarrow \infty$
- ii) If $\lambda_1 < 1$, then there exists a unique solution which blows up in a finite time.

The estimate of the blow up set in [5] was improved in [7]. Recently, results of type i) and ii) are extended by Wiegner [12] for general γ . There are nonuniqueness results for $\gamma = 1$ and $f = 0$ by Ughi [11] (for one dimensional case) and Dal Passo and Luckhaus [3]. They showed that weak solutions may not be unique if initial data takes zero in some open set of the domain because of degeneracy of the equation and discussed a class of solutions so that uniqueness holds.

It seems that there is only one result for the periodic problem for (1) in [9], where the one-dimensional case of 2π -periodic boundary condition is considered. Since they treat motions of closed curves, their solution satisfies the certain constraint. The constraint yields a positive lower bound for the class of all positive solutions, so the equation in [9] is actually nondegenerate. However the equation (1) in this paper is degenerate. When the Dirichlet boundary condition is imposed, the relation between λ_1 and 1 also gives a serious effect on results. The purpose of this paper is to prove the following two theorems.

Theorem 1. Assume that $\gamma \in \mathbb{R}$ and that f is a continuous positive function on $\bar{\Omega} \times \mathbb{R}$ which is T -periodic in time. If $\lambda_1 > 1$, then there exists a unique positive solution u of (1) in $\bigcap_{p>1} W_{p,loc}^{2,1}(\Omega \times \mathbb{R}) \cap C(\bar{\Omega} \times \mathbb{R})$. Furthermore, if f is Hölder continuous, then the solution u belongs to $C^{2+\alpha,1+\alpha}(\Omega \times \mathbb{R}) \cap C(\bar{\Omega} \times \mathbb{R})$ with some $0 < \alpha < 1$.

Theorem 2. Assume that $1 \leq \gamma < 3$ and that f is a continuous positive function on $\bar{\Omega} \times \mathbb{R}$ with $f_t \in C(\bar{\Omega} \times \mathbb{R})$ which is T -periodic in time. Then there is a nonnegative solution $u \in C(\bar{\Omega} \times \mathbb{R})$ of (1) such that $u \in \bigcap_{p>1} W_{p,loc}^{2,1}(\Omega_+ \times \mathbb{R})$, $u > 0$ in $\Omega_+ \times \mathbb{R}$, $u \equiv 0$ on $(\Omega \setminus \Omega_+) \times \mathbb{R}$ and $\nabla u = 0$ on $(\partial\Omega_+ \setminus \partial\Omega) \times \mathbb{R}$, where $\Omega_+ = \bigcup_{i=1}^k \Omega_i$ with a class $\{\Omega_i : 1 \leq i \leq k\}$ of connected open subsets in Ω with $\lambda_1(\Omega_i) < 1$ for $1 \leq i \leq k$.

We prove Theorem 1 in Section 2 and Theorem 2 in Section 3. We seek for a solution of (1) as a limit of its approximate solutions. The Leray-Schauder degree

theory is adapted to solve the approximate equations. We need an a priori upper bound for positive approximate solutions. To do that, inequalities of Harnack type play an important part. We refer to the reference in [9] for literatures on applications of the Leray-Schauder degree theory and Harnack's inequality.

2 Proof of Theorem 1.

We first assume that f is smooth and solve an approximate equation which is nondegenerate. Take positive constants a_1, a_2 such that $a_1 \leq f \leq a_2$ on $\bar{\Omega} \times \mathbf{R}$. Let $\varphi_i \in C^\infty(\bar{\Omega})$ be the solution of

$$\begin{cases} \Delta \varphi_i + \varphi_i + a_i = 0 & \text{in } \Omega \\ \varphi_i = 0 & \text{on } \partial\Omega. \end{cases} \quad (2)$$

Since $\lambda_1 > 1$, there is no point such that $\varphi_i(x) < -a_i$; otherwise there is a domain $D \subset \Omega$ such that $\lambda_1(\Omega) \leq \lambda_1(D) \leq 1$ which leads a contradiction. The maximum principle implies that φ_i cannot take a minimum of the value $\geq -a_i$ in Ω so $\varphi_i > 0$ in Ω . For $\varepsilon > 0$ and $\tau \in [0, 1]$, we consider

$$\begin{cases} u_t = a(u)(\Delta u + u + \tau f) & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t+T) = u(t) & \text{in } \Omega \times \mathbf{R}, \end{cases} \quad (3)$$

where $a : \mathbf{R} \rightarrow \mathbf{R}$ is a smooth function such that

$$\frac{\varepsilon^\gamma}{2} \leq a(r) \leq (2m + \varepsilon)^\gamma \quad \text{for } r \in \mathbf{R},$$

$$a(r) = (r + \varepsilon)^\gamma \quad \text{for } 0 \leq r \leq m$$

with $m = \max \varphi_2$. Then each solution of (3) is smooth and positive by standard regularity theory and maximum principle (see [10]). Letting u be a solution of (3) and $v = \varphi_2 - u$, we have

$$v_t = a(u)(\Delta v + v + a_2 - \tau f) \quad \text{in } \Omega \times \mathbf{R}.$$

Since $a_2 - \tau f \geq 0$, we observe that $v \geq 0$ in $\Omega \times \mathbf{R}$. Indeed, solution of the linear equation

$$w_t = a(u)(\Delta w + w + g) \quad (4)$$

is unique for given g . To see this we may assume $g = 0$. Multiplying $w_t/a(u)$ and integrating over Q by parts yields $w_t = 0$. Since $\lambda_1 > 1$ and $\Delta w + w = 0$, we see $w \equiv 0$. By the maximum principle solution v of

$$v_t = a(u)(\Delta v + v + g)$$

is positive provided that $g \geq 0$. The existence of solution is not difficult to show so this implies $v \geq 0$ for

$$v_t = a(u)(\Delta v + v + a_2 - \tau f) \quad \text{in } \Omega \times \mathbf{R}.$$

In other words, $u \leq \varphi_2$ in $\Omega \times \mathbf{R}$. Therefore it follows from [10] that there is $C = C(\varepsilon, \|f\|_\infty, p) > 0$ such that $\|u\|_{W_p^{2,1}(Q)} \leq C$ for each solution u of (3) with $p > N + 1$, where $Q = \Omega \times (0, T)$. Choose $\alpha > 0$ such that $W_p^{2,1}(Q)$ is compactly embedded into $C^{\alpha, \alpha/2}(\bar{Q})$. Put $X = C^{\alpha, \alpha/2}(\bar{Q})$ and define $S(v, \tau)$ by the unique solution of

$$\begin{cases} u_t = a(v)(\Delta u + u + \tau f) & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t + T) = u(t) & \text{in } \Omega \times \mathbf{R}. \end{cases} \quad (5)$$

It is easily seen that S is a compact continuous operator from $X \times [0, 1]$ into X and $S(v, 0) = 0$ for all $v \in X$. According to the Leray-Schauder fixed point theorem, there exists $u \in X$ with $S(u, 1) = u$. By $u \leq \varphi_2$, u is a smooth positive solution of

$$\begin{cases} u_t = (u + \varepsilon)^\gamma(\Delta u + u + f) & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t + T) = u(t) & \text{in } \Omega \times \mathbf{R}. \end{cases} \quad (6)$$

It is immediate that $u \geq \varphi_1$ in $\bar{\Omega} \times \mathbf{R}$ in the same way as the proof of $u \leq \varphi_2$.

For continuous function f not necessarily smooth, take $\tilde{a}_1, \tilde{a}_2 > 0$ such that $\tilde{a}_1 < f < \tilde{a}_2$ in $\bar{\Omega} \times \mathbf{R}$ and a positive smooth solution $\tilde{\varphi}_i$ ($i = 1, 2$) of (2) with $a_i = \tilde{a}_i$. Choose a sequence $\{f_\varepsilon\} \subset C^\infty(\bar{\Omega} \times \mathbf{R})$ of T -periodic-in-time functions such that

$$f_\varepsilon \rightarrow f \quad \text{in } C(\bar{\Omega} \times \mathbf{R}) \quad \text{as } \varepsilon \rightarrow 0,$$

$$\tilde{a}_1 \leq f_\varepsilon \leq \tilde{a}_2 \quad \text{in } \bar{\Omega} \times \mathbf{R} \quad \text{for each } \varepsilon > 0.$$

By the first paragraph of this proof, for each $\varepsilon > 0$, there exists a positive smooth solution u_ε of (6) with $f = f_\varepsilon$ such that

$$\tilde{\varphi}_1 \leq u_\varepsilon \leq \tilde{\varphi}_2 \quad \text{in } \bar{\Omega} \times \mathbf{R}.$$

Then we may assume that $\{u_\varepsilon\}$ converges weakly to some u in $\bigcap_{p>1} W_{p,loc}^{2,1}(\Omega \times \mathbf{R})$ as $\varepsilon \rightarrow 0$. It is clear that u satisfies

$$u_t = u^\gamma(\Delta u + u + f) \quad \text{in } \Omega \times \mathbf{R},$$

$$u(t+T) = u(t) \quad \text{in } \Omega \times \mathbf{R}.$$

Furthermore, we see that u is positive in $\Omega \times \mathbf{R}$, zero on $\partial\Omega \times \mathbf{R}$ and continuous in $\bar{\Omega} \times \mathbf{R}$ since $\tilde{\varphi}_1 \leq u \leq \tilde{\varphi}_2$.

It remains to show the uniqueness of positive solution of (1) in $\bigcap_{p>1} W_{p,loc}^{2,1}(\Omega \times \mathbf{R}) \cap C(\bar{\Omega} \times \mathbf{R})$. Put

$$\Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}$$

for $\varepsilon > 0$. Let $\varphi_\varepsilon > 0$ be an eigenfunction corresponding to $\lambda(\Omega_\varepsilon)$. For $\delta > 0$, denote by sgn_δ the piecewise linear function on $\mathbf{R}$ with $\text{sgn}_\delta(s) = -1$ for $s \leq -\delta$ and $\text{sgn}_\delta(s) = 1$ for $s \geq \delta$ and define $\varphi_\delta : \mathbf{R} \rightarrow \mathbf{R}$ by

$$\varphi_\delta(s) = \int_0^s \text{sgn}_\delta(r) dr \quad \text{for } s \in \mathbf{R}.$$

Suppose that u, v are positive solutions of (1) in $\bigcap_{p>1} W_{p,loc}^{2,1}(\Omega \times \mathbf{R}) \cap C(\bar{\Omega} \times \mathbf{R})$. Then it holds that

$$(F(u))_t = \Delta u + u + f \tag{7}$$

and

$$(F(v))_t = \Delta v + v + f, \quad (8)$$

where $F(u)$ is a primitive of $u^{-\gamma}$ such that

$$F(u) = \begin{cases} \frac{u^{1-\gamma}}{1-\gamma} & \text{for } \gamma \neq 1 \\ \log u & \text{for } \gamma = 1. \end{cases}$$

Subtracting (8) from (7), multiplying $\text{sgn}_\delta(u-v) \cdot \phi_\varepsilon$ and integrating over Q yields

$$\int_0^T \int_\Omega \{(F(u))_t - (F(v))_t\} \text{sgn}_\delta(u-v) \cdot \phi_\varepsilon dx dt = \int_0^T \int_\Omega \{\Delta(u-v) + (u-v)\} \text{sgn}_\delta(u-v) \cdot \phi_\varepsilon dx \quad (9)$$

where we define $\phi_\varepsilon = 0$ outside Ω_ε . Now integrating by parts, it follows that

$$\begin{aligned} \int_\Omega \Delta(u-v) \cdot \text{sgn}_\delta(u-v) \cdot \phi_\varepsilon dx &\leq - \int_\Omega \text{sgn}_\delta(u-v) \nabla(u-v) \nabla \phi_\varepsilon dx \\ &= \int_\Omega \varphi_\delta(u-v) \Delta \phi_\varepsilon dx \\ &= -\lambda_1(\Omega_\varepsilon) \int_\Omega \varphi_\delta(u-v) \phi_\varepsilon dx. \end{aligned}$$

Since $\varphi_\delta(u-v) \rightarrow |u-v|$ in Ω as $\delta \rightarrow 0$,

$$\int_\Omega \varphi_\delta(u-v) \phi_\varepsilon dx \rightarrow \int_\Omega |u-v| \phi_\varepsilon dx \quad \text{as } \delta \rightarrow 0.$$

On the other hand, since u and v are T -periodic, we have

$$\int_0^T \int_\Omega \{(F(u))_t - (F(v))_t\} \text{sgn}_\delta(u-v) \phi_\varepsilon dx dt \rightarrow \int_0^T \int_\Omega |F(u) - F(v)|_t \phi_\varepsilon dt dx = 0$$

as $\delta \rightarrow 0$. Therefore letting $\delta \rightarrow 0$ in (9), we get

$$0 \leq -\lambda_1(\Omega_\varepsilon) \int_0^T \int_\Omega |u-v| \phi_\varepsilon dx dt + \int_0^T \int_\Omega |u-v| \phi_\varepsilon dx dt.$$

From $\lambda_1(\Omega_\varepsilon) \geq \lambda_1 > 1$, it follows that $u = v$ in $\Omega_\varepsilon \times \mathbb{R}$. This implies $u = v$ in $\Omega \times \mathbb{R}$ since $\varepsilon > 0$ is arbitrary. $\square$

Remark 1. If f is nonpositive, then the problem (1) admits no positive solutions in $\bigcap_{p>1} W_{p,loc}^{2,1}(\Omega \times \mathbb{R}) \cap C(\bar{\Omega} \times \mathbb{R})$. In fact, for any $\varepsilon > 0$ there is $\rho_\varepsilon > 0$ such that $u \leq \varepsilon$ on $\partial\Omega_\rho \times \mathbb{R}$ for each $0 < \rho \leq \rho_\varepsilon$, where $\Omega_\rho = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \rho\}$. Let $\phi_1, \phi_\rho > 0$ be the eigenfunctions corresponding to λ_1 and $\lambda_1(\Omega_\rho)$ normalized in $L^1(\Omega)$, respectively. Then $\{\phi_\rho\}$ converges to ϕ_1 in $L^1(\Omega)$ as $\rho \rightarrow 0$. Multiplying $\frac{u_t}{u^\gamma}$ with (1) and integrating over $\Omega_\rho \times \mathbb{R}$ yields

$$\int_0^T \int_{\Omega_\rho} \frac{u_t}{u^\gamma} \phi_\rho dx dt = \int_0^T \int_{\Omega_\rho} \{(1 - \lambda_1(\Omega_\rho))u\phi_\rho + f\phi_\rho\} dx dt - \int_0^T \int_{\partial\Omega_\rho} u \frac{\partial\phi_\rho}{\partial\nu} ds dt.$$

Since there is $C > 0$ satisfying

$$\int_{\partial\Omega_\rho} \left| \frac{\partial\phi_\rho}{\partial\nu} \right| ds \leq C$$

for sufficiently small ρ , we have

$$\int_0^T \int_{\Omega_\rho} \{(1 - \lambda_1(\Omega_\rho))u\phi_\rho + f\phi_\rho\} dx dt \geq -C\varepsilon$$

for sufficiently small ρ . Letting $\rho \rightarrow 0$, it follows that

$$\int_0^T \int_{\Omega} \{(1 - \lambda_1)u\phi_1 + f\phi_1\} dx dt \geq -C\varepsilon$$

and hence

$$\int_0^T \int_{\Omega} \{(1 - \lambda_1)u\phi_1 + f\phi_1\} dx dt \geq 0.$$

This implies that (1) cannot possess positive solutions in the above class.

3 Proof of Theorem 2.

This section is devoted to proof of Theorem 2, so we assume that $\lambda_1 < 1$ throughout this section. We suppose that $f \in C^\infty(\bar{\Omega} \times \mathbb{R})$ except for the proof of Theorem 2. We start with inequalities of Harnack type in time and in space direction for

$$\begin{cases} v_t = v^\gamma(\Delta v + g(v, x, t)) & \text{in } \Omega \times \mathbb{R}, \\ v = \varepsilon & \text{on } \partial\Omega \times \mathbb{R}, \\ v(t+T) = v(t) & \text{in } \Omega \times \mathbb{R}, \end{cases} \quad (10)$$

where g is a smooth function on $(0, \infty) \times \bar{\Omega} \times \mathbf{R}$ and $\gamma \geq 1$. Let $v \in C^\infty(\bar{\Omega} \times \mathbf{R})$ be a solution of (10) with $v > \varepsilon$ in $\Omega \times \mathbf{R}$. Putting $z = (\log v)_t = v_t/v$, we observe, in the same way as in [9], that

$$z \geq -\frac{(g_v v - g)_+}{\gamma} v^{\gamma-1} - \left(\frac{|g_t|}{\gamma}\right)^{1/2} v^{(\gamma-1)/2} \quad (11)$$

at a minimizer of z over $\Omega \times \mathbf{R}$, where f_+ means the positive part of f . Since $z = 0$ on $\partial\Omega$, it turns out that (11) holds on $\bar{\Omega} \times \mathbf{R}$. In the same way as in [9] we have Harnack type inequalities.

Lemma 1. Assume that there are positive constants c_i ($i = 0, 1, 2$) such that

$$g_\rho \rho - g \leq c_0 \quad \text{and} \quad \|g_t\|_\infty^{1/2} \leq c_1 \quad (12)$$

for all $(\rho, x, t) \in (0, \infty) \times \bar{\Omega} \times \mathbf{R}$ and $\max_{\bar{\Omega} \times \mathbf{R}} v \geq c_2$ for each solution v of (10) with $v > \varepsilon$ in $\Omega \times \mathbf{R}$. Then there exists $C_0 = C_0(c_0, c_1, c_2) > 0$ such that if v solves (10) with $v > \varepsilon$ in $\Omega \times \mathbf{R}$, then

$$v(x, t) \leq v(x, s) \exp(-C_0 M^{\gamma-1}(t-s)) \quad (13)$$

for all $(x, t), (x, s) \in \Omega \times \mathbf{R}$ with $s - T \leq t \leq s$, where $M = \max_{\bar{\Omega} \times \mathbf{R}} v$.

In a high dimensional domain, an inequality of Harnack type in space direction is described in terms of means in balls centered at a maximizer of solution instead of Lemma 2.4 in [9]. The difficulty of the proof in this case is that maximizers may approach to the boundary of Ω , while this difficulty does not happen in the case periodic boundary condition. To overcome that, we need the following proposition.

Proposition 1. Assume that $w \in C^2(\bar{\Omega})$ equals some constant α on the boundary $\partial\Omega$, where $\partial\Omega$ is at least C^2 . If $\tilde{w}$ is the extension of w to $\mathbf{R}^N$ such that $w = \alpha$ outside Ω , then $g(\tilde{w}) \in C^2(\mathbf{R}^N)$ provided that $g \in C^2(\mathbf{R})$ and that $g'(0) = g''(0) = 0$.

Proof. We may assume $\alpha = 0$ and $g(0) = 0$. Let x_0 be a point in $\partial\Omega$. It suffices to prove $g(\tilde{w})$ is C^2 at $x_0 \in \mathbb{R}^N$. We may assume that Ω is a half space

$$\Omega = \{(x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_N > 0\}$$

and that $x_0 = 0$ without the loss of generality. It now suffices to prove that

$$\lim_{h \downarrow 0} \frac{1}{h} v(h e_N) = 0, \quad \lim_{h \downarrow 0} \frac{1}{h} \frac{\partial v}{\partial x_N}(h e_N) = 0$$

for $v = g(\tilde{w})$, where e_N is the unit vector $(0, \dots, 0, 1) \in \mathbb{R}^N$.

By the mean value theorem

$$v(h e_N) = g(w(h e_N)) - g(0) = g'(\eta) w(h e_N)$$

for some η between 0 and $w(h e_N)$, so we see

$$\frac{1}{h} v(h e_N) \rightarrow g'(0) \frac{\partial w}{\partial x_N}(0) \quad \text{as } h \downarrow 0.$$

Since $g'(0) = 0$, this yields

$$\frac{1}{h} v(h e_N) \rightarrow 0 \quad \text{as } h \downarrow 0 \quad \text{and} \quad \frac{\partial v}{\partial x_N}(0) = 0.$$

Similarly, since $g'(0) = 0$, we observe that

$$\begin{aligned} \frac{\partial v}{\partial x_N}(h e_N) &= g'(w(h e_N)) \frac{\partial w}{\partial x_N}(h e_N) \\ &= g''(\zeta) w(h e_N) \frac{\partial w}{\partial x_N}(h e_N) \end{aligned}$$

for some ζ between 0 and $w(h e_N)$. Thus, since $g''(0) = 0$, we see

$$\frac{1}{h} \frac{\partial v}{\partial x_N}(h e_N) \rightarrow g''(0) \left(\frac{\partial w}{\partial x_N}(0) \right)^2 = 0 \quad \text{as } h \downarrow 0. \quad \square$$

For any $w \in C^2(\bar{\Omega})$ with a constant value on $\partial\Omega$, we suppose that $\tilde{w}$ means the extension of w defined in Proposition 1 in the rest of this section.

Lemma 2. Assume $\gamma \geq 1$ and (12) on g . Then if $v \in C^\infty(\bar{\Omega} \times \mathbf{R})$ solves (10) with $v > \varepsilon$ in $\Omega \times \mathbf{R}$, then

$$(M^\gamma - \varepsilon^\gamma)^3 \leq \frac{1}{\omega_N \rho^N} \int_{B_\rho(x_0)} \{\tilde{v}(x, t_0)^\gamma - \varepsilon^\gamma\}^3 dx + \frac{3\gamma C_M (M^\gamma - \varepsilon^\gamma)^2}{2(N+2)} \rho^2, \quad (14)$$

where

$$C_M = \frac{c_0}{\gamma} M^{\gamma-1} + \frac{c_1}{\gamma^{1/2}} M^{(\gamma-1)/2} + M^{\gamma-1} g_M,$$

$$g_M = \max\{(g(v, x, t))_+ : \varepsilon < v \leq M, (x, t) \in \bar{\Omega} \times \mathbf{R}\},$$

$$\max_{\bar{\Omega} \times \mathbf{R}} v = v(x_0, t_0)$$

and ω_N means the volume of unit balls in $\mathbf{R}^N$.

Proof. Let $v \in C^\infty(\bar{\Omega} \times \mathbf{R})$ be a solution of (10) with $v > \varepsilon$ in $\Omega \times \mathbf{R}$. Since (11) implies

$$v^{\gamma-1} \Delta v = z - v^{\gamma-1} g \geq -C_M \quad \text{in } \Omega \times \mathbf{R}, \quad (15)$$

we have

$$\Delta(v^\gamma) = \gamma(v^{\gamma-1} \Delta v + (\gamma-1)v^{\gamma-2} |\nabla v|^2) \geq -\gamma C_M \quad \text{in } \Omega \times \mathbf{R}.$$

Puttting $w = v^\gamma - \varepsilon^\gamma$, it follows that $\tilde{w}^3 \equiv (\tilde{v}^\gamma - \varepsilon^\gamma)^3 \in C^2(\mathbf{R}^N \times \mathbf{R})$ and

$$\Delta(\tilde{w}^3) = 0 \quad \text{on } (\mathbf{R}^N \setminus \Omega) \times \mathbf{R}$$

from Proposition 1. We also see

$$\begin{aligned} \Delta(\tilde{w}^3) &= 3(2\tilde{w}|\nabla \tilde{w}|^2 + \tilde{w}^2 \Delta \tilde{w}) \\ &\geq 3\tilde{w}^2 \Delta \tilde{w} \\ &\geq -3\gamma C_M (M^\gamma - \varepsilon^\gamma)^2 \quad \text{in } \Omega \times \mathbf{R}. \end{aligned}$$

Therefore we get

$$\Delta(\tilde{w}^3) \geq -3\gamma C_M (M^\gamma - \varepsilon^\gamma)^2 \quad \text{in } \mathbf{R}^N \times \mathbf{R},$$

that is,

$$\Delta\{\tilde{w}^3 + \frac{3\gamma C_M(M^\gamma - \varepsilon^\gamma)^2}{2N}|x - x_0|^2\} \geq 0 \quad \text{in } \mathbf{R}^N \times \mathbf{R}.$$

Then for each $\rho > 0$, the mean value theorem yields

$$\begin{aligned} & (M^\gamma - \varepsilon^\gamma)^3 \\ & \leq \frac{1}{\omega_N \rho^N} \int_{B_\rho(x_0)} \left\{ \tilde{w}^3(x, t_0) + \frac{3\gamma C_M(M^\gamma - \varepsilon^\gamma)^2}{2N}|x - x_0|^2 \right\} dx \\ & = \frac{1}{\omega_N \rho^N} \int_{B_\rho(x_0)} (v(x, t_0)^\gamma - \varepsilon^\gamma)^3 dx + \frac{3\gamma C_M(M^\gamma - \varepsilon^\gamma)^2}{2(N+2)} \rho^2. \quad \square \end{aligned}$$

For $\varepsilon > 0$, we consider the following approximate equation of (1)

$$\begin{cases} u_t = (u + \varepsilon)^\gamma \left\{ \Delta u + \frac{u}{u + \varepsilon}(u + f) \right\} & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t + T) = u(t) & \text{in } \Omega \times \mathbf{R}. \end{cases} \quad (16)$$

We adapt the Leray-Schauder degree theory to get a positive solution of (16). To do that, an a priori bound of positive solutions for

$$\begin{cases} u_t = (u + \varepsilon)^\gamma \left\{ \Delta u + \frac{u}{u + \varepsilon}(u + \tau f) + (1 - \tau)(u + 1) \right\} & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t + T) = u(t) & \text{in } \Omega \times \mathbf{R} \end{cases} \quad (17)$$

with $\tau \in [0, 1]$. Putting $v = u + \varepsilon$, v satisfies

$$v_t = v^\gamma \left\{ \Delta v + \frac{v - \varepsilon}{v}(v - \varepsilon + \tau f) + (1 - \tau)(v + 1 - \varepsilon) \right\} \quad \text{in } \Omega \times \mathbf{R}. \quad (18)$$

Lemma 3. There are positive constants C_1, C_2 depending only on $\|f\|_\infty$ and $\|f_t\|_\infty$ such that

$$\int_0^T \int_\Omega v dx dt \leq C_1 \quad \text{and} \quad \int_0^T \int_\Omega \frac{v_t^2}{v^\gamma} dx dt \leq C_2$$

for $0 < \varepsilon < 1, 0 \leq \tau \leq 1$ and solution v of (18) with $v > \varepsilon$ in $\Omega \times \mathbf{R}$.

Proof. Let v be a solution of (18) with $v > \varepsilon$ in $\Omega \times \mathbf{R}$ and put

$$V(x) = \int_0^T v(x, t) dt \quad \text{for } x \in \bar{\Omega}.$$

Multiplying $\frac{1}{v^\gamma}$ with (18) and integrating over $(0, T)$ yields

$$\Delta V + V = F \quad \text{in } \Omega, \tag{19}$$

where

$$F(x) = \int_0^T \left\{ \left(1 - \frac{v - \varepsilon}{v}\right)v + \frac{v - \varepsilon}{v}(-\tau f + \varepsilon) - (1 - \tau)(v + 1 - \varepsilon) \right\} \quad \text{for } x \in \bar{\Omega}.$$

Multiplying a positive first eigenfunction ϕ_1 of the Laplacian with (19) and integrating over Ω , it follows that

$$(1 - \lambda_1) \int_{\Omega} V \phi_1 dx = \int_{\Omega} F \phi_1 dx + \varepsilon T \int_{\partial\Omega} \frac{\partial \phi_1}{\partial \nu} ds$$

and hence

$$\int_{\Omega} V dx \leq \left(\int_{\Omega} V \phi_1 dx \right)^{1/3} \left(\int_{\Omega} V \phi_1^{-1/2} dx \right)^{2/3} \leq KM^{2/3} \tag{20}$$

for some $K > 0$ independent of V , where $M = \max V = V(x_0)$. On the other hand, it follows that

$$\Delta V \geq -M - 1 \quad \text{in } \Omega$$

from (19) and $F \geq -1$. We proceed as in the proof of Lemma 2. Setting $W = V - \varepsilon T$, Proposition 1 implies $\tilde{W}^3 \equiv (\tilde{V} - \varepsilon T)^3 \in C^2(\mathbf{R}^N)$ and

$$\Delta(\tilde{W}^3) = 0 \quad \text{on } \mathbf{R}^N \setminus \Omega.$$

We also see

$$\Delta(\tilde{W}^3) \geq -3(M + 1)(M - \varepsilon T)^2 \quad \text{in } \Omega,$$

so

$$\Delta(\tilde{W}^3) + \frac{3(M + 1)(M - \varepsilon T)^2}{2N} |x - x_0|^2 \geq 0 \quad \text{in } \mathbf{R}^N.$$

Therefore for each $\rho > 0$, the mean value theorem yields

$$\begin{aligned} & (M - \varepsilon T)^3 \\ & \leq \frac{1}{\omega_N \rho^N} \int_{B_\rho(x_0)} \left\{ \tilde{W}^3(x) + \frac{3(M+1)(M - \varepsilon T)^2}{2N} \|x - x_0\|^2 \right\} dx \\ & = \frac{1}{\omega_N \rho^N} \int_{B_\rho(x_0)} (\tilde{V}(x) - \varepsilon T)^3 dx + \frac{3(M+1)(M - \varepsilon T)^2}{2(N+2)} \rho^2. \end{aligned}$$

From this inequality and (20), it follows that

$$(M - \varepsilon T)^3 \leq \frac{K}{\omega_N \rho^N} M^{8/3} + \frac{3(M+1)(M - \varepsilon T)^2}{2(N+2)} \rho^2.$$

Choosing $0 < \rho < (\frac{N+2}{3})^{1/2}$, we obtain

$$(M - \varepsilon T)^3 \leq \frac{K}{\omega_N \rho^N} M^{8/3} + \frac{1}{2} (M+1)(M - \varepsilon T)^2.$$

This yields some $M_0 > 0$ such that

$$\int_0^T v(x, t) dt \leq M_0 \quad \text{for } x \in \bar{\Omega}.$$

Therefore the first inequality in this lemma holds with $C_1 = M_0 |\Omega|$, where $|\Omega|$ denotes the volume of Ω . Define

$$\Phi(s) = \int_\varepsilon^s \frac{r - \varepsilon}{r} dr \quad \text{for } s > \varepsilon.$$

Multiplying $\frac{v_t}{v^\gamma}$ with (18) and integrating over Q yields

$$\int_0^T \int_\Omega \frac{v_t^2}{v^\gamma} dx dt = -\tau \int_0^T \int_\Omega \frac{v - \varepsilon}{v} v_t f dx dt = \tau \int_0^T \int_\Omega \Phi(v) f_t dx dt \leq C_1 \|f_t\|_\infty. \quad \square$$

Combining Lemmas 2-3, we derive an a priori upper bound for solutions v of (18) with $v > \varepsilon$ in $\Omega \times \mathbb{R}$.

Theorem 3. If $1 \leq \gamma < 3$ and $f \in C^\infty(\bar{\Omega} \times \mathbb{R})$ is negative on $\bar{\Omega} \times \mathbb{R}$ and T -periodic in time, then there exists $M_0 = M_0(\|f\|_\infty, \|f_t\|_\infty) > 0$ such that $\max v \leq M_0$ for each $\varepsilon \in (0, 1)$, $\tau \in [0, 1]$ and each solution $v \in C^\infty(\bar{\Omega} \times \mathbb{R})$ of (18) with $v > \varepsilon$ in $\Omega \times \mathbb{R}$.

Proof. For each solution $v \in C^\infty(\bar{\Omega} \times \mathbf{R})$ with $v > \varepsilon$ in $\Omega \times \mathbf{R}$, let $M = \max v = v(x_0, t_0)$. Putting $g(v, x, t) = \frac{(v - \varepsilon)}{v}(v - \varepsilon + \tau f) + (1 - \tau)(v + 1 - \varepsilon)$, we easily see

$$g_v v - g \leq 2 + \|f\|_\infty \equiv c_0 \quad \text{and} \quad \|g_t\|_\infty^{1/2} \leq \|f_t\|_\infty^{1/2} \equiv c_1$$

and hence

$$C_M \leq \frac{c_0}{\gamma} M^{\gamma-1} + \frac{c_1}{\gamma^{1/2}} M^{(\gamma-1)/2} + M^{\gamma-1}(M + 1) \leq K M^\gamma$$

for some $K = K(c_0, c_1) > 0$. Taking $0 < \rho < (\frac{N+2}{3\gamma K})^{1/2}$, it follows that

$$\int_{\Omega} \{v(x, t_0)^\gamma - \varepsilon^\gamma\}^3 dx \geq \omega_N \rho^N (M^\gamma - \varepsilon^\gamma)^2 (\frac{1}{2} M^\gamma - \varepsilon^\gamma) \quad (21)$$

from Lemma 2. On the other hand, choose $s_0 \in [0, T)$ such that

$$\int_{\Omega} v(x, s_0) dx \leq \frac{1}{T} \int_0^T \int_{\Omega} v(x, t) dx dt.$$

Lemma 3 and Hölder's inequality yield

$$\begin{aligned} \int_{\Omega} v(x, t_0) dx &= \int_{\Omega} v(x, s_0) dx + \int_{\Omega} \int_{s_0}^{t_0} v_t dx dt \\ &\leq \int_{\Omega} v(x, s_0) dx + \left(\int_0^T \int_{\Omega} v^\gamma dx dt \right)^{1/2} \left(\int_0^T \int_{\Omega} \frac{v_t^2}{v^\gamma} dx dt \right)^{1/2} \\ &\leq \frac{C_1}{T} + (C_1 C_2)^{1/2} M^{(\gamma-1)/2} \end{aligned}$$

and hence

$$\int_{\Omega} v(x, t_0)^{3\gamma} dx \leq M^{3\gamma-1} \left(\frac{C_1}{T} + (C_1 C_2)^{1/2} M^{(\gamma-1)/2} \right). \quad (22)$$

By (21) and (22), we obtain

$$\omega_N \rho^N (M^\gamma - \varepsilon^\gamma)^2 (\frac{1}{2} M^\gamma - \varepsilon^\gamma) \leq M^{3\gamma-1} \left(\frac{C_1}{T} + (C_1 C_2)^{1/2} M^{(\gamma-1)/2} \right).$$

This yields our desired upper bound. $\square$

Lemma 4. Assume that a is a positive continuous function on $\mathbf{R}$. Assume that there are constants $\lambda, \Lambda > 0$ such that

$$\lambda \leq a(r) \leq \Lambda \quad \text{on } \mathbf{R}.$$

Let $u \in W_p^{2,1}(Q)$, $p > n + 1$ be a solution of

$$\begin{cases} u_t = a(u)(\Delta u + c(x)u + h(x)) & \text{in } \Omega \times (0, T) = Q, \Omega \subset \mathbf{R}^N \\ u = 0 & \text{on } \partial\Omega \times (0, T) \end{cases} \quad (23)$$

with continuous functions c and h . Then there are constants $0 < \alpha < 1$ (depending only on $N, \lambda, \Lambda, \max |c|, \max |h|, M$) and $C = C(t_0, N, \lambda, \Lambda, \max |c|, \max |h|, M)$ such that

$$\|u\|_{C^{\alpha, \alpha/2}(Q')} \leq C, \quad Q' = \Omega \times (t_0, T)$$

provided that $\max_Q |u| \leq M$.

Remark 2. One can prove that this lemma along the same line of the proof of the Hölder estimate for quasilinear parabolic equation of divergence type [22, p.419, Chap.V, Theorem 1.1]. The main ingredient is an energy type identity obtained by multiplying $\zeta^2 u/a(u)$ with the equation (23) and integrating by parts on $\Omega \times (t_1, t)$ with use of $u = 0$ on $\partial\Omega$; here $\zeta \in C_0^\infty(\bar{\Omega} \times (0, T])$. The resulting identity is

$$\begin{aligned} & \int_{\Omega} \zeta^2 B(u(x, t)) dx + \int_{t_1}^t \int_{\Omega} |\nabla u(x, \tau)|^2 \zeta^2 dx d\tau \\ &= \int_{\Omega} \zeta^2 B(u(x, t_1)) dx + \int_{t_1}^t \int_{\Omega} 2\zeta \zeta_t B(u) dx d\tau - \int_{t_1}^t \int_{\Omega} 2\zeta u \nabla \zeta \cdot \nabla u dx d\tau \\ & \quad + \int_{t_1}^t \int_{\Omega} c \zeta^2 u^2 dx d\tau + \int_{t_1}^t \int_{\Omega} \zeta^2 h u dx d\tau. \end{aligned}$$

Here $B(v) = \int_0^v \frac{\sigma}{a(\sigma)} d\sigma$. Since $\lambda \leq a \leq \Lambda$, we see

$$\frac{v^2}{2\Lambda} \leq B(v) \leq \frac{v^2}{2\lambda}.$$

If $B(v) = v^2$, from this identity we apply the embedding lemma [22, Chap.II, §7] to get the Hölder estimate. Since $B(v)$ is comparable with v^2 , one can modify the proof to get the desired Hölder estimate.

Lemma 5. Assume that a is a positive continuous function satisfying

$$\lambda \leq a(r) \leq \Lambda \quad \text{for all } r \in \mathbf{R}$$

with some constant $\lambda, \Lambda > 0$. Let k be a positive number. Then for each $h \in C(\bar{Q})$ there is a unique solution $u \in \bigcap_{p>1} W_p^{2,1}(Q)$ of

$$\begin{cases} u_t = a(u)(\Delta u - ku + h) & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ u(t+T) = u(t) & \text{in } \Omega \times \mathbb{R}. \end{cases}$$

Moreover for each $p > 1$ there is $C = C(\lambda, \Lambda, p)$ such that

$$\|u\|_{W_p^{2,1}(Q)} \leq C \|h\|_\infty.$$

Sketch of the proof of Lemma 5. We appeal to the continuity method by considering

$$\begin{cases} u_t = a_\tau(v)(\Delta u - ku + h) & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ u(t+T) = u(t) & \text{in } \Omega \times \mathbb{R} \end{cases} \quad (24)$$

with $a_\tau(v) = \tau a(v) + (1-\tau)\lambda$. Let $X = W_q^{2,1}(Q)$ for some $q > n+1$ so that $X \subset C(\bar{Q})$ for $Q = \Omega \times (0, T)$. Let S be a mapping from $X \times [0, 1]$ to X by setting a solution $u = S(v, \tau)$ of (24) for given $v \in X$. It is rather standard to see that S is compact and continuous by L^p -theory [10].

To apply the Leray-Schauder fixed point theory it suffices to get a estimate for $u = S(u, \tau)$, i.e., there is $K > 0$ such that

$$\|u\|_X \leq K \quad \text{for all } u = S(u, \tau), 0 \leq \tau \leq 1.$$

By the maximum principle we have

$$\|u\|_\infty \leq \frac{\|h\|_\infty}{k}.$$

By a priori Hölder estimate we have

$$\|u\|_{C^{\alpha, \alpha/2}} \leq C \quad \text{for } u = S(u, \tau), 0 \leq \tau \leq 1$$

with $C > 0, 0 < \alpha < 1$ depending only on $N, \lambda, \Lambda, k, \|h\|_\infty$. By L^p -theory for linear equations we have

$$\|u\|_{W_q^{2,1}(Q)} \leq C'$$

with C' independent of u and τ . Thus we find a solution u of the original problem is given by $u = S(u, 1)$.

The uniqueness is the same as in [9]. $\square$

Now take $k_\varepsilon > 0$ such that

$$\Phi(s) = k_\varepsilon s + \frac{s}{s + \varepsilon}(s + f) > 0 \quad \text{for all } s > 0.$$

We suppose that $\Phi(s) = 0$ for $s \leq 0$.

By Lemma 5, for $h \in C(\bar{Q})$, we can define $S(h)$ by the unique solution of

$$\begin{cases} u_t = (u + \varepsilon)^\gamma(\Delta u - k_\varepsilon u + h) & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t + T) = u(t) & \text{in } \Omega \times \mathbf{R}. \end{cases} \quad (25)$$

Then S is a continuous compact operator from $C(\bar{Q})$ into itself by Lemma 5. Thus, the Leray-Schauder degree for $I - S \circ \Phi$ in $C(\bar{Q})$ is well-defined.

Lemma 6. If $f \in C^\infty(\bar{\Omega} \times \mathbf{R})$ is negative on $\bar{\Omega} \times \mathbf{R}$, then there is a sufficiently small $r > 0$ such that the Leray-Schauder degree of $I - S \circ \Phi$ of the value zero in $B_r(0)$ equals one, that is,

$$\deg(I - S \circ \Phi, B_r(0), 0) = 1,$$

where $B_r(0)$ denotes the ball with radius r centered at 0 in $C(\bar{Q})$.

Proof. For $\tau \in [0, 1]$, we consider

$$\begin{cases} u_t = (u + \varepsilon)^\gamma(\Delta u - k_\varepsilon u + \tau\Phi(u)) & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t + T) = u(t) & \text{in } \Omega \times \mathbf{R}. \end{cases} \quad (26)$$

Multiplying $\frac{\phi_1}{(u+\varepsilon)^\gamma}$ with (26) and integrating over Q yields

$$0 = \int_0^T \int_\Omega \left\{ -\lambda_1 u + (\tau - 1)k_\varepsilon u + \tau \frac{u}{u+\varepsilon} (u+f) \right\} \phi_1 dx dt.$$

This implies $\max u \geq \min(-f)$ for any $\varepsilon > 0, \tau \in [0, 1]$ and each positive solution u of (26). Choose $r < \min(-f)$ so that $S \circ (\tau\Phi)$ has no fixed points on $\partial B_r(0)$ for all $\tau \in [0, 1]$. From the homotopy invariance of degree, it follows that

$$\deg(I - S \circ \Phi, B_r(0), 0) = \deg(I, B_r(0), 0) = 1. \quad \square$$

Lemma 7. If $1 \leq \gamma < 3$ and $f \in C^\infty(\bar{\Omega} \times \mathbf{R})$ is negative on $\bar{\Omega} \times \mathbf{R}$, then there is $R > r$ such that

$$\deg(I - S \circ \Phi, B_R(0), 0) = 0$$

for $0 < \varepsilon < 1$.

Proof. Define $\tilde{\Phi} : \mathbf{R} \rightarrow \mathbf{R}$ by

$$\tilde{\Phi}(s) = k_\varepsilon s_+ + \frac{s^2}{s_+ + \varepsilon} + s_+ + 1 \quad \text{for } s \in \mathbf{R}.$$

For $\tau \in [0, 1]$, we consider

$$\begin{cases} u_t = (u + \varepsilon)^\gamma \{ \Delta u - k_\varepsilon u + \tau \Phi(u) + (1 - \tau) \tilde{\Phi}(u) \} & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t + T) = u(t) & \text{in } \Omega \times \mathbf{R}. \end{cases} \quad (27)$$

By Theorem 3, there is $M_0 > 0$ such that $\max u \leq M_0$ for any $\varepsilon \in (0, 1), \tau \in [0, 1]$ and each positive solution of (27). Take $R > \max(M_0, r)$ so that $S \circ (\tau\Phi + (1 - \tau)\tilde{\Phi})$ has no fixed points on $\partial B_R(0)$ for all $\tau \in [0, 1]$. It now follows that

$$\deg(I - S \circ \Phi, B_R(0), 0) = \deg(I - S \circ \tilde{\Phi}, B_R(0), 0).$$

Multiplying $\frac{\phi_1}{(u+\varepsilon)^\gamma}$ with (27) with $\tau = 0$ and integrating over Q yields

$$0 = \int_0^T \int_\Omega \left\{ (1 - \lambda_1)u + \frac{u^2}{u + \varepsilon} + 1 \right\} \phi_1 dx dt > 0.$$

This implies that there are no solutions for (27) with $\tau = 0$. We thus obtain

$$\deg(I - S \circ \tilde{\Phi}, B_R(0), 0) = 0.$$

This completes the proof. $\square$

Lemma 8. If $1 \leq \gamma < 3$ and $f \in C^\infty(\bar{\Omega} \times \mathbb{R})$ is negative on $\bar{\Omega} \times \mathbb{R}$, then for each $\varepsilon \in (0, 1)$, there is a positive smooth solution of (16).

Proof. Let $0 < \varepsilon < 1$. From Lemmas 6, 7, it follows that

$$\deg(I - S \circ \Phi, B_R(0) \setminus B_r(0), 0) = -1,$$

which implies the existence of positive smooth solution of (16). $\square$

We are now in a position to prove Theorem 2.

Proof of Theorem 2. Take a sequence $\{f_\varepsilon\} \subset C^\infty(\bar{\Omega} \times \mathbb{R})$ of negative functions such that each f_ε is T -periodic in t and

$$f_\varepsilon \rightarrow f \quad \text{and} \quad f_{\varepsilon t} \rightarrow f_t \quad \text{in} \quad C(\bar{Q}) \quad \text{as} \quad \varepsilon \rightarrow 0.$$

Then it follows that $\|f_\varepsilon - f\|_\infty \leq 1$, $\|f_{\varepsilon t} - f_t\|_\infty \leq 1$ for sufficiently small ε . Let u_ε be a positive smooth solution of (16) with $f = f_\varepsilon$ for $\varepsilon \in (0, 1)$ obtained in Lemma 8. Define $U_\varepsilon, F_\varepsilon : \bar{\Omega} \rightarrow \mathbb{R}$ by

$$U_\varepsilon(x) = \int_0^T u_\varepsilon(x, t) dt$$

and

$$F_\varepsilon(x) = \int_0^T \left\{ \left(1 - \frac{u_\varepsilon}{u_\varepsilon + \varepsilon}\right) u_\varepsilon - \frac{u_\varepsilon}{u_\varepsilon + \varepsilon} f_\varepsilon \right\} dt.$$

Multiplying $\frac{1}{(u_\varepsilon + \varepsilon)^\gamma}$ with (16) and integrating over $(0, T)$ yields

$$\Delta U_\varepsilon + U_\varepsilon = F_\varepsilon \quad \text{in } \Omega. \tag{28}$$

Let $M_0 > 0$ be an upper bound for $\{u_\varepsilon\}$ by Theorem 3. Since $0 \leq F_\varepsilon \leq (2 + \|f\|_\infty)T$ and $\|U_\varepsilon\|_\infty \leq M_0 T$, there is $C_p > 0$ such that $\|U_\varepsilon\|_{W^{2,p}(\Omega)} \leq C_p$ for sufficiently small ε

with $p > \frac{N}{2}$. Then we may assume that $\{U_\varepsilon\}$ converges to some U weakly in $W^{2,p}(\Omega)$ and strongly in $C(\bar{\Omega})$ as $\varepsilon \rightarrow 0$. Integrating (13) in Lemma 1 over $(s-T, s)$ and $(t, t+T)$ respectively yields some positive constants λ, Λ independent of ε such that

$$\lambda u_\varepsilon(x, t) \leq U_\varepsilon(x) \leq \Lambda u_\varepsilon(x, t) \quad \text{in } Q. \quad (29)$$

Multiplying $\frac{u_\varepsilon}{(u_\varepsilon + \varepsilon)^\gamma}$ with (16) and integrating by parts over Q , we get

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 dx dt \leq (M_0 + \|f\|_\infty + 1) M_0 T |\Omega|.$$

Multiplying $\frac{u_{\varepsilon t}}{(u_\varepsilon + \varepsilon)^\gamma}$ with (16) and integrating by parts over Q yields

$$\int_0^T \int_\Omega \frac{u_{\varepsilon t}^2}{(u_\varepsilon + \varepsilon)^\gamma} dx dt \leq (\|f_t\|_\infty + 1) M_0 T |\Omega|$$

and hence

$$\int_0^T \int_\Omega u_{\varepsilon t}^2 \leq (M_0 + 1)^\gamma (\|f_t\|_\infty + 1) M_0 T |\Omega|.$$

Therefore we may suppose that $\{u_\varepsilon\}$ converges to some u a.e. in Q . From (29), it follows that

$$\lambda u(x, t) \leq U(x) \leq \Lambda u(x, t) \quad \text{a.e. in } Q. \quad (30)$$

Since there is $\delta_0 > 0$ such that $\max u_\varepsilon \geq \delta_0$ by the same argument in the proof of Lemma 6, we see $\max U \geq \lambda \delta_0 > 0$ from (29). Put $\Omega_+ = \{x \in \Omega : U(x) > 0\}$ and decompose $\Omega_+ = \bigcup_{i=1}^k \Omega_i$ into its connected components $\Omega_i, i = 1, \dots, k$ so that $U = 0$ on $\partial\Omega_i$. Setting

$$F(x) = - \int_0^T f(x, t) dt \quad \text{for } x \in \bar{\Omega}$$

and letting $\varepsilon \rightarrow 0$ in (28) yields

$$\Delta U + U = F > 0 \quad \text{in } \Omega_+. \quad (31)$$

Multiplying U with (31) and integratig over Ω_i , we have

$$\int_{\Omega_i} (-|\nabla U|^2 dx + U^2) dx = \int_{\Omega_i} F U dx$$

and hence

$$\lambda_1(\Omega_i) \leq \frac{\int_{\Omega_i} |\nabla U|^2 dx}{\int_{\Omega_i} U^2 dx} < 1 \quad \text{for } i = 1, \dots, k.$$

Since $U_\varepsilon \rightarrow U > 0$ uniformly in Ω_+ and $U_\varepsilon \leq \Lambda u_\varepsilon$ in $\Omega \times \mathbf{R}$, we may assume that $u_\varepsilon \rightarrow u$ weakly in $W_{p,loc}^{2,1}(\Omega_+ \times \mathbf{R})$ and strongly in $C_{loc}(\Omega_+ \times \mathbf{R})$. Therefore $u \in C(\Omega_+ \times \mathbf{R})$,

$$u(t+T) = u(t) \quad \text{in } \Omega \times \mathbf{R}$$

and

$$u_t = u^\gamma(\Delta u + u + f) \quad \text{in } \Omega_+ \times \mathbf{R}.$$

By (30), we get $u \equiv 0$ on $(\Omega \setminus \Omega_+) \times \mathbf{R}$ and hence $u \in C(\bar{\Omega} \times \mathbf{R})$. We finally derive $\nabla u = 0$ on $(\partial\Omega_+ \setminus \partial\Omega) \times \mathbf{R}$ by (30) since $\nabla U = 0$ on $\partial\Omega_+ \setminus \partial\Omega$. This completes the proof. $\square$

Remark 3. If f is nonnegative, there are no nonnegative solutions of (1) with properties stated in Theorem 2 by the same method as in Remark 1.

Remark 4. In the case of $\lambda_1 > 1$, (1) possesses the positive solution (Theorem 1), while in general there are no positive solutions when $\lambda_1 < 1$ as the following example

$$\begin{cases} u_{xx} + u - 1 = 0 & \text{in } (0, 3\pi), \\ u(0) = u(3\pi) = 0, \end{cases}$$

shows.

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