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Frobenius extensions**

Kozo Sugano

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Note on H-separable Frobenius extensions

Kozo Sugano

Throughout this paper A will be a ring with the identity element 1 , B a subring of A containing 1 , C the center of A and D will be the centralizer of B in A . We will use the same notation as the author's previous paper [8]. For any subset X of A and any A - A -module M we will write

$$V_A(X) = \{a \in A \mid ax = xa \text{ for any } x \in X\}, \text{ and}$$

$$M^A = \{m \in M \mid am = ma \text{ for any } a \in A\}$$

respectively. Thus we have $D = A^B = V_A(B)$ and $(A \otimes_B A)^A = \{\sum a_i \otimes b_i \in A \otimes_B A \mid \sum x a_i \otimes b_i = \sum a_i \otimes b_i x \text{ for any } x \in A\}$.

A is said to be a Frobenius extension of B in the case where A is left B -f.g. (finitely generated) projective and there exists a left A and right B -isomorphism of A to $\text{Hom}({}_B A, {}_B B)$. This is the case if and only if there exist finite $x_k, y_k \in A$ and $h \in \text{Hom}({}_B A, {}_B B)$ such that $x = \sum h(x x_k) y_k = \sum x_k h(y_k x)$ hold for each $x \in A$. In this case we call the set $\{x_k, y_k, h\}$ a Frobenius system, and the map h a Frobenius homomorphism, of $A|B$ respectively.

Now for any $h \in \text{Hom}({}_B A, {}_B B)$ we can define a multiplication among the elements of $A \otimes_B A$ by $(a \otimes b)(c \otimes d) = ah(bc) \otimes d$ for any $a, b, c, d \in A$ (See Proposition 4.1 [2]). This multiplication is well defined, and by this definition we can make $A \otimes_B A$ an associative ring which does not always have the identity element. On the other hand we can define the following maps

$$\phi_r: A \otimes_B A \longrightarrow \text{Hom}(A_B, A_B)$$

$$\phi_l: A \otimes_B A \longrightarrow \text{Hom}({}_B A, {}_B A)$$

by $\phi_r(a \otimes b)(x) = ah(bx)$ and $\phi_l(a \otimes b)(x) = h(xa)b$ for any $a, b, x \in A$. Direct calculation shows that ϕ_r and ϕ_l are ring, and opposite ring, homomorphisms respectively. Now we have

Lemma 1. For an $h \in \text{Hom}({}_B A, {}_B B)$ the following conditions are equivalent;

- (i) A is a Frobenius extension of B with h a Frobenius homomorphism
- (ii) ϕ_r defined as above is an isomorphism
- (iii) ϕ_l defined as above is an isomorphism
- (vi) $A \otimes_B A$ has the identity element as a ring defined as above

If $\sum x_k \otimes y_k$ is the identity of $A \otimes_B A$, then $\{x_k, y_k, h\}$ is a Frobenius system of $A|B$.

Proof. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are well known (See e.g. the proof of Theorem 1 on page 94 [4]), and (ii) $\Rightarrow$ (vi) and (iii) $\Rightarrow$ (vi) are obvious. Let $\sum x_k \otimes y_k$ be the identity of $A \otimes_B A$. Then for any $x \in A$ we have $1 \otimes x = (1 \otimes x)(\sum x_k \otimes y_k) = \sum h(xx_k) \otimes y_k$ and $x \otimes 1 = (\sum x_k \otimes y_k)(x \otimes 1) = \sum x_k h(y_k x) \otimes 1$. Then we have $x = \sum h(xx_k) y_k = \sum x_k h(y_k x)$. Thus we have (vi) $\Rightarrow$ (i) and the last assertion.

There are some special maps as follows

$$\eta : A \otimes_B A \rightarrow \text{Hom}(C D, C A) \quad \eta(a \otimes b)(x) = axb$$

$$\eta_l : D \otimes C A \rightarrow \text{Hom}(B A, B A) \quad \eta_l(d \otimes a)(x) = dxa$$

$$\eta_t : D \otimes C D \rightarrow \text{Hom}(B A B, B A B) \quad \eta_t(d \otimes e)(x) = dxe$$

for $a, b, x \in A$ and $d, e \in D$. $\eta_r : A \otimes C D \rightarrow \text{Hom}(A B, A B)$ is defined similarly. η , η_l and η_r are A - A -maps and η_t is a D - D -map. A is an H -separable extension of B if and only if η is an isomorphism and D is C -f.g. projective. This is the case if and only if $1 \otimes 1 = \sum d_i \sum x_{ij} \otimes y_{ij}$ for some $d_i \in D$ and $\sum x_{ij} \otimes y_{ij} \in (A \otimes_B A)^A$. We call such set $\{d_i, \sum x_{ij} \otimes y_{ij}\}$ an H -system of $A|B$. In the case where A is an H -separable extension of B all the above maps are isomorphisms. The next lemma is an immediate consequence of Corollary 3 [6].

Lemma 2. In the case where A is left B -f.g. projective the following conditions are equivalent:

- (i) A is an H -separable extension of B
- (ii) η_l is an isomorphism and D is C -f.g. projective.
- (iii) There exists an A - A -split epimorphism of finite direct sum of copies of A to $\text{Hom}(B A, B A)$.

Let $\{x_k, y_k, h\}$ be a Frobenius system of $A|B$ and $\sum a_j \otimes b_j$ an arbitrary in $(A \otimes_B A)^A$. For any $x \in A$ we have $\sum x a_j h(b_j) = \sum a_j h(b_j x)$ and for any $b \in B$ $\sum b a_j h(b_j) = \sum a_j h(b_j b) = \sum a_j h(b_j) b$. Thus $\sum a_j h(b_j) \in D$, and $\sum a_j \otimes b_j = \sum x_k h(y_k a_j) \otimes h(b_j x_1) y_1 = \sum x_k h(y_k a_j) h(b_j x_1) \otimes y_1 = \sum x_k h(y_k a_j h(b_j x_1)) \otimes y_1 = \sum a_j h(b_j x_1) \otimes y_1 = \sum x_1 a_j h(b_j) \otimes y_1 \in \sum x_1 D \otimes y_1$. Thus $(A \otimes_B A)^A \subset \sum x_k D \otimes y_k$. Since $\sum x_k \otimes y_k \in (A \otimes_B A)^A$, the converse inclusion is clear. Thus we have $(A \otimes_B A)^A = \sum x_k D \otimes y_k$ (See page 370 [1]). By this equality we have

Theorem 1. Let A be a Frobenius extension of B with a Frobenius system $\{x_k, y_k, \tilde{h}\}$. Then the following conditions are equivalent;

- (i) A is an H -separable extension of B
- (ii) There exist finite $d_i, e_i \in D$ such that $1 \otimes 1 = \sum d_i x_k e_i \otimes y_k$ in $A \otimes_B A$

If there exist $d_i, e_i \in D$ which satisfy the condition of (ii), then we have the following assertions

- (1) $\sum d_i \otimes e_i \in (D \otimes_C D)^D$, and we have $\tilde{h}(x) = \sum d_i x e_i$ for each $x \in A$
- (2) We can obtain a Frobenius system $\{d_i, e_i, h\}$ of $D|C$, where h is defined by $h(d) = \sum x_k d y_k$ for each d in D .

Proof. Since A is H -separable extension of B if and only if $1 \otimes 1 \in D(A \otimes_B A)^A$, we have the equivalence of (i) and (ii) immediately by $(A \otimes_B A)^A = \sum x_k D \otimes y_k$. Assume (ii). Then $\sum x_k e_i \otimes y_k \in (A \otimes_B A)^A$, and for each x in A we have $1 \otimes x = \sum d_i x_k e_i \otimes y_k x = \sum d_i x x_k e_i \otimes y_k$, and $1 \otimes \tilde{h}(x) = \sum d_i x x_k e_i \otimes \tilde{h}(y_k)$. But $\tilde{h}(y_k) \in B$ and $d_i \in D$. Hence we have $\tilde{h}(x) = \sum d_i x x_k e_i \tilde{h}(y_k) = \sum d_i x x_k \tilde{h}(y_k) e_i = \sum d_i x e_i \in B$. Now for each $d \in D$ we have $\sum d d_i x e_i = \sum d_i x e_i d$. Then since η is an isomorphism by (i), we have $\sum d d_i \otimes e_i = \sum d_i \otimes e_i d$ in $D \otimes_C D$. Thus we have proved (1). Next we will prove (2). The map h defined in (2) is in $\text{Hom}({}_C D, {}_C C)$, since $\sum x_k D y_k \subset C$. Then since $\sum d_i x e_i \in V_A(D)$ for each $x \in A$, we have $\sum d_i h(e_i d) = \sum d_i x_k e_i d y_k = \sum d d_i x_k e_i y_k = d \sum h(x_k) y_k = d$. Similarly we have $\sum h(d d_i) e_i = d$. Thus $\{d_i, e_i, h\}$ is a Frobenius system of $D|C$.

Theorem 2. Assume $B = V_A(D)$, and let D be a Frobenius C -algebra with a Frobenius system $\{d_i, e_i, h\}$. Then the following three conditions are equivalent

- (i) A is an H -separable extension of B .
- (ii) There exist finite $x_k, y_k \in A$ such that $1 \otimes 1 = \sum d_i \otimes x_k e_i y_k$ in $D \otimes C A$.
- (iii) There exists $\sum x_k \otimes y_k \in (A \otimes_B A)^A$ such that $1 \otimes 1 = \sum d_i x_k e_i \otimes y_k$ in $A \otimes_B A$.

In the case where there exist $x_k, y_k \in A$ which satisfy the condition of (ii), we have the following assertions:

- (1) $\sum x_k \otimes y_k \in (A \otimes_B A)^A$ and $h(d) = \sum x_k d y_k$ holds for any $d \in D$.
- (2) A is a Frobenius extension of B with a Frobenius system $\{x_k, y_k, \tilde{h}\}$, where $\tilde{h}$ is defined by $\tilde{h}(x) = \sum d_i x e_i$ for each $x \in A$.

Proof. (ii) $\Rightarrow$ (i) is obvious, since $\sum x_k e_i \otimes y_k \in (A \otimes_B A)^A$. Assume A is an H -separable extension of B , and consider the isomorphism η introduced above. η induces the isomorphism $(A \otimes_B A)^A \cong \text{Hom}(C D, C C)$. Therefore there exists $\sum x_k \otimes y_k \in (A \otimes_B A)^A$ such that $\eta(\sum x_k \otimes y_k) = h$. Then since $1 = \sum d_i h(e_i) = \sum d_i x_k e_i y_k$ and $\sum d_i x_k e_i \in V_A(D) = B$, we have in $A \otimes_B A$ that $1 \otimes 1 = 1 \otimes \sum d_i x_k e_i y_k = \sum d_i x_k e_i \otimes y_k$, while in $D \otimes C A$ we have $1 \otimes 1 = \sum d_i x_k e_i y_k \otimes 1 = \sum d_i \otimes x_k e_i y_k$, since $\sum x_k e_i y_k \in C$. Thus we have (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). Now assume (ii). Since $\sum d_i \otimes e_i \in (D \otimes C D)^D$, we have also $\sum d_i a x_k e_i \in B$ for each $a \in A$ and k . Hence we can obtain left B -homomorphisms f_k of A to B defined by $f_k(a) = \sum d_i a x_k e_i$ for each $a \in A$, which satisfy $a = \sum d_i a x_k e_i y_k = \sum f_k(a) y_k$ for each $a \in A$. Thus $\{y_k, f_k\}$ forms a dual basis for ${}_B A$. Now consider the map η_1 of $D \otimes C A$ to $\text{Hom}({}_B A, {}_B A)$ introduced above. For any $f \in \text{Hom}({}_B A, {}_B A)$ and $a \in A$ we have $\eta_1(\sum d_i \otimes x_k e_i f(y_k))(a) = \sum d_i a x_k e_i f(y_k) = \sum f_k(a) f(y_k) = f(\sum f_k(a) y_k) = f(a)$. Hence we have $\eta_1(\sum d_i \otimes x_k e_i f(y_k)) = f$, which means that η_1 is an epimorphism. Next suppose $\sum c_j \otimes a_j \in \text{Ker } \eta_1$. Then $\sum c_j y_k a_j = 0$ and $\sum d_i \otimes x_k e_i \in (D \otimes C D)^D$ for each k , and we have $\sum c_j \otimes a_j = \sum c_j d_i \otimes x_k e_i y_k a_j = \sum d_i \otimes x_k e_i c_j y_k a_j = 0$. Thus η_1 is a monomorphism, and we see that η_1 is an isomorphism. But D is C -f.g. projective, and A is left B -f.g. projective. Therefore A is an H -separable extension of B by Lemma 2. Thus we have proved (ii) $\Rightarrow$ (i). Now we will prove (1) and (2) of the second assertion under the condition of (ii). For any $d \in D$ we have $d \otimes 1 = \sum d d_i \otimes x_k e_i y_k$, and $h(d) = \sum h(d d_i) x_k e_i y_k = \sum x_k h(d d_i) e_i y_k = \sum x_k d y_k \in$

C. Then since $\sum x_k D y_k \subset C$, we have $h = \eta(\sum x_k \otimes y_k) \in \text{Hom}(cD, cC) \cong (A \otimes_B A)^A$. Therefore we have $\sum x_k \otimes y_k \in (A \otimes_B A)^A$. Since $\sum d_i \otimes e_i \in (D \otimes_C D)^D$, we can define the map $\tilde{h}$ of A to $B (=V_A(D))$ by $\tilde{h}(x) = \sum d_i x e_i$ for each $x \in A$. Then $\sum \tilde{h}(a x_k) y_k = \sum d_i a x_k e_i y_k = \sum d_i x_k e_i y_k a = a$. Similarly we have $\sum x_k \tilde{h}(y_k a) = a$. Thus $\{x_k, y_k, \tilde{h}\}$ is a Frobenius system of $A|B$.

Proposition 1. Let A be a Frobenius extension of B with a Frobenius system $\{x_k, y_k, h\}$. Consider the following two conditions;

(i) A is an H -separable extension of B .

(ii) There exists $\sum d_i \otimes e_i \in (D \otimes_C D)^D$ with $1 \otimes 1 = \sum d_i \otimes x_k e_i y_k$ in $D \otimes_C A$

(i) always implies (ii). If $V_A(D) = B$, (i) and (ii) are equivalent.

Proof. Assume (i). By Theorem 1 there exists $\sum d_i \otimes e_i \in (D \otimes_C D)^D$ such that $1 \otimes 1 = \sum d_i x_k e_i \otimes y_k$ holds in $A \otimes_B A$. Then since $\sum x_k e_i y_k \in C$, we have $\sum d_i \otimes x_k e_i y_k = \sum d_i x_k e_i y_k \otimes 1 = 1 \otimes 1$ in $D \otimes_C A$. Thus we have (i) $\Rightarrow$ (ii). Next let $B = V_A(D)$, and assume (ii). Then for the completely same reason as the proof of (ii) $\Rightarrow$ (i) of Theorem 2 we see that the map η_1 of $D \otimes_C A$ to $\text{End}({}_B A)$ is an isomorphism. On the other hand we have $d = \sum d_i d_i x_k e_i y_k = \sum d_i x_k e_i d y_k$ for each $d \in D$, since $1 = \sum d_i x_k e_i y_k$ and $\sum d_i x_k e_i \in V_A(D)$. But $\sum x_k e_i d y_k \in C$, since $\sum x_k \otimes y_k \in (A \otimes_B A)^A$. Therefore if we define maps g_i by $g_i(d) = \sum x_k e_i d y_k$ for each $d \in D$ and each i , we have $g_i \in \text{Hom}(cD, cC)$ and $d = \sum d_i g_i(d)$ for each $d \in D$. Hence D is C -f.g. projective. Then by Lemma 2 we have (i).

As is introduced in [8] when we write $\{A/B, S/T\}$, we mean that S is a ring containing A as subring with the common identity, and T is a subring of S containing B . In this case we will always write $\tilde{D} = V_S(T)$ and $\tilde{C} = V_S(S)$, the center of S . There exists also the canonical homomorphism $\tilde{\eta}$ of $S \otimes_T S$ to $\text{Hom}(\tilde{C}\tilde{D}, \tilde{C}S)$ defined by $\tilde{\eta}(s \otimes t)(\tilde{d}) = s \tilde{d} t$ for $s, t \in S$ and $\tilde{d} \in \tilde{D}$. $\{A/B, S/T\}$ is said to have the centralizer property in the case where $V_S(A) = \tilde{C}$, $V_S(B) = \tilde{D}$ and $T = V_S(\tilde{D})$ hold. By the same argument as is stated on page 602 [7] we have the next lemma

Lemma 3. Let $\{A/B, S/T\}$ have the centralizer property, and assume

that A is an H -separable extension of B . Then we have

(1) The canonical map $i \otimes i$ of $A \otimes_B A$ to $S \otimes_T S$ is a monomorphism, where i is the inclusion map of A to S .

(2) If $\tilde{\eta}$ is a monomorphism, then S is an H -separable extension of T , and we have $(i \otimes i)[(A \otimes_B A)^A] \subset (S \otimes_T S)^S$.

Proof. We will give the proof very briefly following the same lines as page 602 [7]. Since $D \otimes_C \tilde{C} = \tilde{D}$ via $d \otimes \tilde{c} \rightarrow d\tilde{c}$ for $d \in D$ and $\tilde{c} \in \tilde{C}$ we have a natural isomorphism ϕ of $\text{Hom}(\tilde{c}\tilde{D}, \tilde{c}S)$ to $\text{Hom}(cD, cS)$ such that $\phi(f)(d) = f(d)$ for $f \in \text{Hom}(\tilde{c}\tilde{D}, \tilde{c}S)$ and $d \in D$ and the following commutative diagram;

$$\begin{array}{ccccc} A \otimes_B A & \xrightarrow{\quad \eta \quad} & \text{Hom}(cD, cA) & & \\ \downarrow i \otimes i & \searrow \tilde{\eta} & \downarrow i \cdot = \text{Hom}(D, i) & & \\ S \otimes_T S & \xrightarrow{\quad \tilde{\eta} \quad} & \text{Hom}(\tilde{c}\tilde{D}, \tilde{c}S) & \xrightarrow{\quad \phi \quad} & \text{Hom}(cD, cS) \end{array}$$

Since $i \cdot \eta$ is a monomorphism, so is $i \otimes i$. On the other hand we have

$$\begin{aligned} i \cdot \eta [(A \otimes_B A)^A] &= i \cdot [(\text{Hom}(cD, cA))^A] \subset [\text{Hom}(cD, cS)]^A = \text{Hom}(cD, cV_S(A)) \\ &= \text{Hom}(cD, c\tilde{C}) = \phi [\text{Hom}(\tilde{c}\tilde{D}, \tilde{c}\tilde{C})] \end{aligned}$$

Therefore if $\tilde{\eta}$ is a monomorphism, we have $(i \otimes i)[(A \otimes_B A)^A] \subset (S \otimes_T S)^S$. Then since $D \subset \tilde{D}$ each H -system of $A|B$ is an H -system of $S|T$. Thus we have (2).

Theorem 3. Let $\{A/B, S/T\}$ have the centralizer property, and assume that A is an H -separable Frobenius extension of B with a Frobenius system $\{x_k, y_k, \tilde{h}\}$. Then S is also an H -separable Frobenius extension of T with a Frobenius system $\{x_k, y_k, \tilde{h}^*\}$ such that $\tilde{h}^*|_A = \tilde{h}$.

Proof. Since A is left (and right) B -f.g. projective, S is an H -separable extension of T by Theorem 1.1 [8]. By Theorem 1 there exists $\sum d_i \otimes e_i \in (D \otimes_C D)^D$ such that $1 \otimes 1 = \sum d_i x_k e_i \otimes y_k$ in $A \otimes_B A$, and $\{d_i, e_i, h\}$ is a Frobenius system of $D|C$, where h is defined by $h(d) = \sum x_k d y_k$ for $d \in D$. Then since $\tilde{D} = D\tilde{C}$ and $\sum x_k \otimes y_k \in (A \otimes_B A)^A \subset (S \otimes_T S)^S$ by Lemma 3, if we define a map h^* by $h^*(\tilde{d}) = \sum x_k \tilde{d} y_k$ for $\tilde{d} \in \tilde{D}$, we have $h^* \in \text{Hom}(\tilde{c}\tilde{D}, \tilde{c}\tilde{C})$ with $h^*|_D = h$, and $\{d_i, e_i, h^*\}$ forms a Frobenius system of $\tilde{D}|\tilde{C}$. On the other hand we have $1 \otimes 1 = \sum d_i \otimes x_k e_i y_k$ in $A \otimes_B A$ by the proof of (i) $\Rightarrow$ (ii) Proposition 3. The same equality holds also in $S \otimes_T S$. Then by Theorem 2 we see that

$\{x_k, y_k, \tilde{h}^*\}$ is a Frobenius system of $S|T$, where $\tilde{h}^*(x) = \sum d_i x e_i$ for any $x \in S$. We have $\tilde{h}^*|_A = \tilde{h}$, since $\tilde{h}(x) = \sum d_i x e_i$ for $x \in A$ by Theorem 1 (2).

The next proposition which is the improvement of Theorems 4 and 5 [5] is a modification of Theorems 1 and 2 [3] which were proved by using H-system.

Proposition 2. Let A be an H-separable extension of B . Then we have

(1) Assume that A is a Frobenius extension of B with a Frobenius system $\{x_k, y_k, \tilde{h}\}$, and define a map h by $h(d) = \sum x_k d y_k$ for any $d \in D$. Then for any $d_i, e_i \in D$ such that $\eta(\sum d_i \otimes e_i) = \tilde{h}(d_i, e_i, h)$ is a Frobenius system of $D|C$.

(2) Assume that D is a Frobenius C-algebra with a Frobenius system $\{d_i, e_i, h\}$, and define a map $\tilde{h}$ by $\tilde{h}(x) = \sum d_i x e_i$ for each $x \in A$. Then for any $x_k, y_k \in A$ such that $\eta(\sum x_k \otimes y_k) = h(x_k, y_k, \tilde{h})$ is a Frobenius system of $A|B'$, where $B' = V_A(V_A(B))$.

Proof. (1). Let $d \in D$. We have $\sum d_i h(e_i d) = \sum d_i x_k e_i d y_k = \sum d d_i x_k e_i y_k = \sum d h(x_k) y_k = d$. Similarly we have $d = \sum h(d d_i) e_i$. (2). Let $x \in A$. Then we have $\sum x_k h(y_k x) = \sum x_k d_i y_k x e_i = \sum x x_k d_i y_k e_i = \sum x h(d_i) e_i = x$. Similarly we have $x = \sum h(x x_k) y_k$. Obviously h is a $B'-B'$ -map of A to B' .

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