



HOKKAIDO UNIVERSITY

Title	The 28th Sapporo Symposium on Partial Differential Equations
Author(s)	Jimbo, Shuichi
Description	The 28th Sapporo Symposium on Partial Differential Equations Organizers: T. Ozawa, Y. Giga, S. Jimbo, K. Tsutaya, Y. Tonegawa, G. Nakamura ² . Venues: Department of Mathematics, Faculty of Science, Hokkaido University July 23, 2003 (Wednesday) 9:30-10:30 Gregory SEREGIN (Steklov Institute/Keio Univ.) Interior regularity of L_3 -solutions to the Navier-Stokes equations 11:00-12:00 Takaaki NISHIDA (Kyoto Univ.) Heat convection problems and computer assisted proof 14:30-15:00 Akihiro SHIMOMURA (Gakushuin Univ.) Modified wave operators for Maxwell-Schrodinger equations 15:15-15:45 Hideaki SUNAGAWA (Osaka Univ.) Remarks on the large time asymptotics for nonlinear Klein-Gordon systems 16:00-16:30 Hirokazu NINOMIYA (Ryukoku Univ.) Curved traveling front of Allen-Cahn equations July 24, 2003 (Thursday) 9:30-10:30 Masahiro YAMAMOTO (Univ. Tokyo) Uniqueness in inverse scattering problems with a single incident wave 11:00-12:00 Shinya NISHIBATA (Tokyo Inst. Tech.) Asymptotic behavior of spherically symmetric solutions to the compressible Navier Stokes equation with external forces 14:30-15:00 Dening LI (West Virginia Univ.) Conical shock waves in supersonic flow 15:15-15:45 Yasushi TANIUCHI (Shinshu Univ.) Remarks on global solvability of 2-D Boussinesq equations with nondecaying initial data July 25, 2003 (Friday) 9:30-10:30 Ryuichi SUZUKI (Kokushikan Univ.) Blow-up of solutions of quasilinear parabolic equations with localized reactions 11:00-12:00 SAKAGUCHI (Ehime Univ.) Initial behavior of solutions of diffusion equations and symmetries of domains"
Citation	Hokkaido University technical report series in mathematics, 77, 1
Issue Date	2003-01-01
DOI	https://doi.org/10.14943/632
Doc URL	https://hdl.handle.net/2115/690
Type	departmental bulletin paper
File Information	2003pde001.pdf



Proceedings of the 28th Sapporo Symposium on
Partial Differential Equations

Edited by T. Ozawa, Y. Giga, S. Jimbo,

K. Tsutaya, Y. Tonegawa, G. Nakamura

Sapporo, 2003

Series #77. July, 2003

HOKKAIDO UNIVERSITY
TECHNICAL REPORT SERIES IN MATHEMATICS

- #50 H.-F. Yamada (Ed.), 1996 年度談話会・特別講演アブストラクト集 Colloquium Lectures, 99 pages. 1997.
- #51 N. Kawazumi (Ed.), リーマン面に関連する位相幾何学, 121 pages. 1997.
- #52 J. Inoue (Ed.), 第6回関数空間セミナー報告集, 89 pages. 1998.
- #53 Y. Giga (Ed.), Proceedings of the 23rd Sapporo Symposium on Partial Differential Equations, 77 pages. 1998.
- #54 N. Kawazumi (Ed.), リーマン面に関連する位相幾何学, 122 pages. 1998.
- #55 T. Ozawa and H.-F. Yamada (Eds.), 1997 年度談話会・特別講演アブストラクト集 Colloquium Lectures, 83 pages. 1998.
- #56 Y. Giga, 界面ダイナミクス-曲率の効果, 講義録, 48 pages. 1998.
- #57 J. Inoue (Ed.), 第7回関数空間セミナー報告集, 138 pages. 1999.
- #58 Y. Giga and R. Kobayashi (Eds.), Abstracts of Sapporo Symposium on Anisotropic Effects in a Crystal Growth Problem and its Mathematical Analysis (SAM), 51 pages. 1999.
- #59 Y. Giga and T. Ozawa (Eds.), Proceedings of the 24th Sapporo Symposium on Partial Differential Equations, 61 pages. 1999.
- #60 I. Tsuda and N. Kawazumi (Eds.), 1998 年度談話会・特別講演アブストラクト集, 55 pages. 1999.
- #61 T. Ozawa (Ed.), Proceedings of Sapporo Guest House Minisymposium on Nonlinear Wave Equations, 67 pages. 1999.
- #62 S. Miyajima, T. Takeo and J. Inoue (Eds.), 第8回関数空間セミナー報告集, 96 pages. 2000.
- #63 K. Ono and N. Honda (Eds.), 1999 年度談話会・特別講演アブストラクト集, 43 pages. 2000.
- #64 Y. Giga and T. Ozawa (Eds.), Proceedings of the 25th Sapporo Symposium on Partial Differential Equations, 55 pages. 2000.
- #65 H. Nakamura (Ed.), ガロア・タイヒミュラー群の LEGO 理論, 37 pages. 2000.
- #66 J. Inoue et al (Eds.), 関数空間セミナー報告集 2000, 134 pages. 2001.
- #67 Y. Giga and H. Yamashita (Eds.), 2000 年度談話会・特別講演アブストラクト集, 61 pages. 2001.
- #68 Y. Giga and T. Ozawa (Eds.), Proceedings of the 26th Sapporo Symposium on Partial Differential Equations, 67 pages. 2001.
- #69 M. Matsumoto, 基本群へのガロア作用, 50 pages. 2001.
- #70 T. Nakazi (Ed.), 第10回関数空間セミナー報告集, 97 pages. 2002.
- #71 Y. Giga (Ed.), Surface Evolution Equations - a level set method, 223 pages. 2002.
- #72 T. Suwa and T. Yamanouchi (Eds.), 2001 年度談話会・特別講演アブストラクト集, 44 pages. 2002.
- #73 T. Jimbo, T. Nakazi and M. Hayashi (Eds.), 第11回関数空間セミナー報告集, 135 pages. 2003.
- #74 T. Ozawa, Y. Giga, S. Jimbo and G. Nakamura (Eds.), Partial Differential Equations, 51 pages. 2002.
- #75 D. Matsushita (Ed.), Proceedings of the workshop "Hodge Theory and Algebraic Geometry", 191 pages. 2003.
- #76 M. Hayashi and G. Ishikawa (Eds.), 2002 年度談話会・特別講演アブストラクト集, 34 pages. 2003.

Proceedings of the 28th Sapporo Symposium on
Partial Differential Equations

Edited by T. Ozawa, Y. Giga, S. Jimbo, K. Tsutaya, Y. Tonegawa, G. Nakamura

Sapporo, 2003

Partially supported by Grant-in-Aid for Scientific Research, the Japan Society
for the Promotion of Science.

日本学術振興会科学研究費補助金 (基盤研究 A (2) 課題番号 14204011)

日本学術振興会科学研究費補助金 (基盤研究 A (1) 課題番号 13304011)

PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on July 23 through July 25 in 2003 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Professor Taira Shirota started the symposium more than 25 years ago. Professor Kôji Kubota and Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

T. Ozawa, Y. Giga, S. Jimbo, K. Tsutaya, Y. Tonegawa, G. Nakamura

CONTENTS

Programme

Gregory SEREGIN

Interior regularity of $L_{3,\infty}$ -solutions to the Navier-Stokes equations

Takaaki NISHIDA

Heat convection problems and computer assisted proof

Akihiro SHIMOMURA

Modified wave operators for Maxwell-Schrödinger equations

Hideaki SUNAGAWA

Remarks on the large time asymptotics for nonlinear Klein-Gordon systems

Hirokazu NINOMIYA

Curved traveling front of Allen-Cahn equations

Masahiro YAMAMOTO

Uniqueness in inverse scattering problems with a single incident wave

Shinya NISHIBATA

Asymptotic behavior of spherically symmetric solutions to the compressible Navier-Stokes equation with external forces

Dening LI

Conical shock waves in supersonic flow

Yasushi TANIUCHI

Remarks on global solvability of 2-D Boussinesq equations with nondecaying initial data

Norikazu YAMAGUCHI

L^q - L^r estimates of solution to the parabolic Maxwell equations and their application to the magnetohydrodynamic equations

Ryuichi SUZUKI

Blow-up of solutions of quasilinear parabolic equations with localized reactions

Shigeru SAKAGUCHI

Initial behavior of solutions of diffusion equations and symmetries of domains

The 28th Sapporo Symposium on Partial Differential Equations

(第28回偏微分方程式論札幌シンポジウム)

下記の要領でシンポジウムを行いますのでご案内申し上げます。

代表者: 小澤 徹, 儀我 美一, ☆神保秀一, 津田谷 公利, 利根川 吉廣, 中村 玄
Organizers: T. Ozawa, Y. Giga, S. Jimbo, K. Tsutaya, Y. Tonegawa, G. Nakamura

記

1. Period (期間) July 23, 2003 - July 25, 2003

2. Venues (場所)

Department of Mathematics, Faculty of Science, Hokkaido University

北海道大学大学院 理学研究科 数学教室

理学部 8号館 309号室 (23日 24日 午前)

理学部 5号館 大講義室 (23日 24日 午後, 25日)

Faculty of Science Building #8 ROOM 309 (Mornings of 23, 24)

Faculty of Science Building #5 Large Lecture Room (Afternoon of 23, 24 and July 25)

“注意: 理学部 8号館は数学事務室のある理学部 3号館から通りをはさんで
南西側 80メートルの新しい建物 (3階建)”

3. Programme (プログラム)

July 23, 2003 (Wednesday)

- | | |
|-------------|---|
| 9:30-10:30 | Gregory SEREGIN (Steklov Institute/Keio Univ.)
Interior regularity of $L_{3,\infty}$ -solutions to the Navier-Stokes equations |
| 11:00-12:00 | 西田孝明 (京大・理), Takaaki NISHIDA (Kyoto Univ.)
Heat convection problems and computer assisted proof |
| 14:00-14:30 | Free discussion time with speakers in the coffee-tea room* |
| 14:30-15:00 | 下村 明洋 (学習院大・理), Akihiro SHIMOMURA (Gakushuin Univ.)
Modified wave operators for Maxwell-Schrödinger equations |
| 15:15-15:45 | 砂川 秀明 (阪大・理), Hideaki SUNAGAWA (Osaka Univ.)
Remarks on the large time asymptotics for nonlinear Klein-Gordon systems |
| 16:00-16:30 | 二宮広和 (龍谷大・理工), Hirokazu NINOMIYA (Ryukoku Univ.)
Curved traveling front of Allen-Cahn equations |
| 16:30-17:00 | Free discussion time with speakers in the coffee-tea room* |

July 24, 2003 (Thursday)

- 9:30-10:30 山本昌宏 (東大・数理), Masahiro YAMAMOTO (Univ. Tokyo)
Uniqueness in inverse scattering problems with a single incident wave
Stability and uniqueness for inverse problems for non-stationary
- 11:00-12:00 西畑伸也 (東工大・情報), Shinya NISHIBATA (Tokyo Inst. Tech.) *Lamé system*
Asymptotic behavior of spherically symmetric solutions to
the compressible Navier-Stokes equation with external forces
- 14:00-14:30 Free discussion time with speakers in the coffee-tea room*
- 14:30-15:00 Dening LI (West Virginia Univ.)
Conical shock waves in supersonic flow
- 15:15-15:45 谷内 靖 (信州大・理), Yasushi TANIUCHI (Shinshu Univ.)
Remarks on global solvability of 2-D Boussinesq equations
with nondecaying initial data
- 16:00-16:30 山口範和 (早大・理工), Norikazu YAMAGUCHI (Waseda Univ.)
 L^q - L^r estimates of solution to the parabolic Maxwell equations
and their application to the magnetohydrodynamic equations
- 16:30-17:00 Free discussion time with speakers in the coffee-tea room*
- 18:00- Welcome Party at Trillium (エンレイソウ)

July 25, 2003 (Friday)

- 9:30-10:30 鈴木龍一 (国士舘大・工), Ryuichi SUZUKI (Kokushikan Univ.)
Blow-up of solutions of quasilinear parabolic equations with localized reactions
- 11:00-12:00 坂口茂 (愛媛大・理), Shigeru SAKAGUCHI (Ehime Univ.)
Initial behavior of solutions of diffusion equations and symmetries of domains
- 12:00-13:00 Free discussion time with speakers in the coffee-tea room*

* Lecturers in each session are invited to stay in the coffee-tea room during discussion time.

連絡先 〒060-0810 札幌市北区北10条西8丁目
北海道大学大学院理学研究科数学教室
☆ 電話兼 FAX (011)706-6926 (神保 秀一)
Correspondence to: S. Jimbo,
011-706-6926 (phone and Fax)
Secretariat: e-mail gjr@math.sci.hokudai.ac.jp
FAX 011-706-4672

... ..

... ..

Interior regularity of $L_{3,\infty}$ -solutions to the Navier-Stokes equations

G. Seregin

In this talk, we are going to discuss the following result.

Theorem 0.1 (*L. Escauriaza, G. Seregin, V. Šverák*) Consider two functions v and p defined in the space-time cylinder $Q = B \times]-1, 0[$, where $B(r) \subset \mathbb{R}^3$ stands for the ball of radius r with the center at the origin and $B = B(1)$. Assume that v and p have the following differentiability properties:

$$v \in L_{2,\infty}(Q) \cap L_2(-1, 0; W_2^1(B)), \quad p \in L_{\frac{3}{2}}(Q),$$

and satisfy the Navier-Stokes equations

$$\partial_t v + \operatorname{div} v \otimes v - \Delta v = -\nabla p, \quad \operatorname{div} v = 0$$

in Q in the sense of distributions.

Let, in addition,

$$\|v\|_{3,\infty,Q} < +\infty.$$

Then the function v is Hölder continuous in the closure of the set

$$Q(1/2) = B(1/2) \times]-(1/2)^2, 0[.$$

Remarks 0.2

1. The $L_{3,\infty}$ -case is the limit case of the so-called Ladyzhenskaya-Prodi-Serrin condition ($v \in L_{s,l}(\dots)$ with $\frac{3}{s} + \frac{2}{l} \leq 1$) which provides uniqueness and smoothness of solutions to the corresponding initial-boundary value problems. A local version of such kind of results was proved by Serrin for $\frac{3}{s} + \frac{2}{l} < 1$ and Struwe for $\frac{3}{s} + \frac{2}{l} = 1$.
2. The theorem stated above improves Struwe result proved under the condition that $L_{3,\infty}$ -norm is sufficiently small.
3. The theorem implies that, under "reasonable" conditions, $L_{3,\infty}$ -solutions to the Cauchy problem for the Navier-Stokes system are smooth.

4. Moreover, if, under “reasonable” conditions, $L_{3,\infty}$ -solutions to the initial-boundary value problems for the Navier-Stokes equations develop singularities, then the corresponding singular points must be on the boundary.

The main interest of the above result comes from the fact that they seem out to be of reach of “standard methods”. By those methods, we mean various conditions on (local) “smallness” of various norms of v which are invariant with respect to the natural scaling

$$u(x, t) \rightarrow \lambda u(\lambda x, \lambda^2 t), \quad p(x, t) \rightarrow \lambda^2 p(\lambda x, \lambda^2 t)$$

of the equations.

We note that finiteness of a norm $\|f\|_{s,l}$ with $s, l < \infty$ implies “local smallness” of f in this norm. This is not the case for $L_{3,\infty}$ -norm (which is still invariant under the scaling). This possible “concentration effect” was the main obstacle to proving regularity. To rule out concentration, we use a new method based on the reduction of the regularity problem to a backward uniqueness problem, which is then solved by finding suitable Carleman-type inequalities. The backward uniqueness result is new and seems to be of independent interest. It can be formulated as follows.

Proposition 0.3 *Assume that we are given a function $u : (\mathbb{R}^n \setminus \overline{B}) \times]0, 1[\rightarrow \mathbb{R}$ which meets the conditions:*

$$\begin{aligned} u &\in W_2^{2,1}((B(R) \setminus \overline{B}) \times]0, 1[), \quad \forall R > 1, \\ |u(x, t)| &\leq e^{M|x|^2}, \quad \forall (x, t) \in (\mathbb{R}^n \setminus \overline{B}) \times]0, 1[, \\ |\partial_t u - \Delta u| &\leq M(|u| + |\nabla u|) \quad \text{in } (\mathbb{R}^n \setminus \overline{B}) \times]0, 1[, \\ u(\cdot, 1) &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{B}. \end{aligned}$$

Then,

$$u = 0 \quad \text{in } (\mathbb{R}^n \setminus \overline{B}) \times]0, 1[.$$

In turn, the proof of this proposition is based on two Carleman-type inequalities. The first of them is of the form

$$\begin{aligned} &\int_{\mathbb{R}^n \times]0, 2[} h^{-2a}(t) e^{-\frac{|x|^2}{4t}} \left(\frac{a}{t} |u|^2 + |\nabla u|^2 \right) dx dt \\ &\leq c_0 \int_{\mathbb{R}^n \times]0, 2[} h^{-2a}(t) e^{-\frac{|x|^2}{4t}} |\partial_t u + \Delta u|^2 dx dt. \end{aligned}$$

It is valid for any function $u \in C_0^\infty(\mathbb{R}^n \times]0, 2[; \mathbb{R})$ and for any positive number a . Here, c_0 is an absolute positive constant and $h(t) = te^{\frac{1-t}{3}}$.

The second inequality is in a sense anisotropic:

$$\begin{aligned} & \int_{(\mathbb{R}_+^n + e_n) \times]0, 1[} t^2 e^{2\phi(x,t)} \left(a \frac{|u|^2}{t^2} + \frac{|\nabla u|^2}{t} \right) dx dt \\ & \leq c_* \int_{(\mathbb{R}_+^n + e_n) \times]0, 1[} t^2 e^{2\phi(x,t)} |\partial_t u + \Delta u|^2 dx dt. \end{aligned}$$

Here

$$\phi = \phi^{(1)} + \phi^{(2)},$$

where $\phi^{(1)}(x, t) = -\frac{|x'|^2}{8t}$ and $\phi^{(2)}(x, t) = a(1-t)\frac{x_n^2}{t^\alpha}$, $x' = (x_1, x_2, \dots, x_{n-1})$ so that $x = (x', x_n)$, and $e_n = (0, 0, \dots, 0, 1)$, $c_* = c_*(\alpha)$ is a positive constant and $\alpha \in]1/2, 1[$ is fixed. This Carleman-type inequality holds for any function $u \in C_0^\infty((\mathbb{R}_+^n + e_n) \times]0, 1[; \mathbb{R})$ and for any number $a > a_0(\alpha)$.

Our methods can be probably easily adopted to other parabolic problems with critical non-linearities. In fact, one could speculate that the general idea of the approach might be applicable to an even larger class of interesting equations with critical non-linearities, such as non-linear Schrödinger equations or non-linear wave equations. However, the local regularity issues arising in these cases would be slightly harder than in the parabolic case.

Heat Convection Problems and Computer Assisted Proofs

Takaaki NISHIDA
Kyoto University

We consider the Rayleigh-Bénard problem for the heat convection using the Boussinesq equations for the velocity, pressure and temperature in the dimensionless form :

$$\frac{1}{\mathcal{P}}\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}\right) + \nabla p = \Delta \vec{u} - \rho(T) \nabla z, \quad \nabla \cdot \vec{u} = 0, \quad \frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T = \Delta T$$

in the horizontal domain $\{x \in \mathbf{R}, y \in \mathbf{R}, 0 < z < 1\}$, where $\rho(T) = G - \mathcal{R}T$ is assumed for the density of the fluid, \mathcal{P} is the Prandtl number and \mathcal{R} is the Rayleigh number.

When the temperature $T = 1$ is given on the lower boundary and $T = 0$ on the upper boundary, the equilibrium state is the purely heat conduction solution, which exists for all parameter values $\mathcal{P} > 0, \mathcal{R} > 0$:

$$\vec{u} = 0, \quad T = 1 - z, \quad \rho = G - \mathcal{R}(1 - z), \quad p = G(1 - z) - \mathcal{R}\left(\frac{1}{2} - z + \frac{z^2}{2}\right) + p_a.$$

We first assume the stress free boundary condition for the velocity on the both boundaries ($z = 0, 1$), and the Dirichlet boundary condition for the temperature.

We will consider the bifurcation problems from this equilibrium state under the assumption that all perturbations are periodic in the horizontal direction, especially with the periodicity $0 \leq x \leq 2/a, 0 \leq y \leq 2/b$. The system for the perturbation to the equilibrium state is given by

$$\frac{1}{\mathcal{P}}\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}\right) + \nabla p = \Delta \vec{u} + \mathcal{R}\theta \nabla z, \quad \nabla \cdot \vec{u} = 0, \quad \frac{\partial \theta}{\partial t} + \vec{u} \cdot \nabla \theta = \Delta \theta + w.$$

Also we assume usual even- or odd-ness for the unknown functions to avoid the multiplicity of the eigenvalue. Then for example the temperature has the representation :

$$\theta(t, x, y, z) = \sum_{l, m, n} \theta_{lmn}(t) \cos al\pi x \cos bm\pi y \sin n\pi z.$$

The other unknowns have similar expressions.

The linearized system is selfadjoint and has real eigenvalues. Therefore when the biggest eigenvalue becomes $\lambda = 0$ at a critical Rayleigh number for fixed a and b and it is "simple", the usual stationary bifurcation theory can be applied and the stationary bifurcation occurs at the critical Rayleigh number.

However if we choose $a = 1/2\sqrt{2}$ and the aspect ratio $= b/a = \sqrt{3}$, which is one of the most interesting case, the critical Rayleigh number $\mathcal{R}_c = 6.75 \times \pi^4$ and the biggest eigenvalue $\lambda = 0$ has a two-dimensional eigenspace. One eigenfunction corresponds to the roll type solution, the other eigenfunction does to the rectangle type solution, and a linear combination of them does to the hexagonal type solution. Thus we can not apply the usual bifurcation theory. However if we restrict the function space for solutions to the subspace such that it corresponds to any one type of those eigenfunctions, we can apply the local bifurcation theory for the simple eigenvalue in the corresponding subspace and can obtain each type of solution respectively in a neighbourhood of the same bifurcation point.

Here we will obtain roll type solutions, rectangular cell and hexagonal cell solutions on the extended bifurcation curves numerically to have a better global bifurcation diagram for the full system. Then we consider the stability of them in the original function space as the time evolution problem, and propose an approach to prove the existence of those solutions and secondary bifurcation points as a computer assisted proof.

Then we consider similar problems with another boundary condition that the upper surface is stress free and the bottom surface is non-slip for the velocity.

T. Ikeda, H. Yoshihara and T. Nishida

" Pattern formation of heat convection problems ",

Proc. Internat. Symp. Math. Modeling and Numer. Simul. in Cont. Mech., Lecture Notes in Comput. Sci. Engin., Springer Verlag, ed. by T. Miyoshi, 2001, pp.209-218

Y. Watanabe, N. Yamamoto, M. T. Nakao and T. Nishida

" A numerical verification of bifurcated solutions for the heat convection problems ",

to appear in Journal of Mathematical Fluid Mechanics, 2003

MODIFIED WAVE OPERATORS FOR MAXWELL-SCHRÖDINGER EQUATIONS

AKIHIRO SHIMOMURA

Department of Mathematics,
Gakushuin University
1-5-1 Mejiro, Toshima, Tokyo 171-8588, Japan

1. INTRODUCTION

We study the scattering theory for the Maxwell-Schrödinger equations in three space dimensions ([7]). We consider those equations under two gauge conditions, that is, the Coulomb gauge and the Lorentz one. In this talk, we construct modified wave operators to the Maxwell-Schrödinger equations with above two gauge conditions for small scattered states without any restrictions on the support of the Fourier transform of them.

The Maxwell-Schrödinger equation in the Coulomb gauge is as follows:

$$\begin{cases} i\partial_t u = -\frac{1}{2}(\nabla - iA)^2 u + g(|u|^2)u, \\ \square A = \mathcal{P}(\text{Im}(\bar{u}(\nabla - iA)u)), \\ \nabla \cdot A = 0, \end{cases} \quad (\text{MS-C})$$

where

$$\begin{aligned} g(|u|^2) &\equiv \frac{1}{4\pi} \left(\frac{1}{|x|} * |u|^2 \right) u = (-\Delta)^{-1} |u|^2, \\ \square &\equiv \partial_t^2 - \Delta, \\ \mathcal{P} &\equiv 1 - \nabla \Delta^{-1} \nabla \cdot. \end{aligned}$$

Here u and A are complex and \mathbb{R}^3 -valued unknown functions of $(t, x) \in \mathbb{R} \times \mathbb{R}^3$, respectively. The last equation in the system (MS-C) is called the Coulomb gauge condition.

The Maxwell-Schrödinger equation in the Lorentz gauge is as follows:

$$\begin{cases} i\partial_t u = -\frac{1}{2}(\nabla - iA)^2 u + \phi u, \\ \square \phi = |u|^2, \\ \square A = \text{Im}(\bar{u}(\nabla - iA)u), \\ \partial_t \phi + \nabla \cdot A = 0. \end{cases} \quad (\text{MS-L})$$

Here u , ϕ and A are complex, real and \mathbb{R}^3 -valued unknown functions of $(t, x) \in \mathbb{R} \times \mathbb{R}^3$, respectively. The last equation in the system (MS-L) is called the Lorentz gauge condition.

According to linear scattering theory, it seems that the equations (MS-C) and (MS-L) in three space dimensions belongs to the borderline between the short range case and the long range one, because the solution of the three dimensional free wave equation decays like t^{-1} in L^∞ as $t \rightarrow \infty$. (The two dimensional Klein-Gordon-Schrödinger equation and the three dimensional Wave-Schrödinger equation also belong to the same case). There are some results of the long range scattering for the coupled systems of the Schrödinger equations and the second order hyperbolic

ones, that is, the Klein-Gordon-Schrödinger equation in two space dimensions in [4, 6], to the Wave-Schrödinger equation in three space dimensions in [1, 3, 5] and to the Maxwell-Schrödinger equation under the Coulomb gauge condition in three space dimensions in [2, 9]. In [1, 4, 9], the restriction on the support of the Fourier transform of the scattered state ϕ of the Schrödinger part is assumed, and in [2], the vanishing scattered states for the Maxwell part are considered. (Note that in Ginibre and Velo [1, 2], no size restriction on for data is assumed). Recently, the existence of wave operators for the two dimensional Klein-Gordon-Schrödinger equation and the three dimensional Wave-Schrödinger equation with the Yukawa type interaction have been proved in [6] and [5], respectively, for small scattered states without any restrictions on the support of the Fourier transform of them. Furthermore combining idea of [1] with that of [5], Ginibre and Velo [3] have proved the existence of modified wave operators for the three dimensional Wave-Schrödinger equation with restrictions on neither size of the scattered states nor the support of the Fourier transform of them.

In this talk, we construct modified wave operators to the equations (MS-C) and (MS-L) in three space dimensions for small scattered states with no restriction on the support of the Fourier transform of them. This talk is mainly based on [7].

Notations. For $s, m \in \mathbb{R}$, we introduce the weighted Sobolev spaces $H^{s,m}$ corresponding to the Lebesgue space L^2 :

$$H^{s,m} \equiv \{\psi \in \mathcal{S}' : \|\psi\|_{H^{s,m}} \equiv \|(1 + |x|^2)^{m/2} (1 - \Delta)^{s/2} \psi\|_{L^2} < \infty\}.$$

We also denote $H^{s,0}$ by H^s . For $1 \leq p \leq \infty$ and a positive integer k , we define the Sobolev space W_p^k corresponding to the Lebesgue space L^p by

$$W_p^k \equiv \left\{ \psi \in L^p : \|\psi\|_{W_p^k} \equiv \sum_{|\alpha| \leq k} \|\partial^\alpha \psi\|_{L^p} < \infty \right\}.$$

Note that for a positive integer k , $H^k = W_2^k$ and the norms $\|\cdot\|_{H^k}$ and $\|\cdot\|_{W_2^k}$ are equivalent. For $s > 0$, we define the homogeneous Sobolev spaces \dot{H}^s by the completion of \mathcal{S} with respect to the norm $\|w\|_{\dot{H}^s} \equiv \|(-\Delta)^{s/2} w\|_{L^2}$.

We set for $t \in \mathbb{R}$,

$$U(t) \equiv e^{\frac{it}{2}\Delta}, \quad \omega \equiv (-\Delta)^{1/2}.$$

2. MAIN RESULTS

We state our result for the case of the Coulomb gauge.

Let $(u_+^C, A_+^C, \dot{A}_+^C)$ be given scattered states, where u_+^C is complex valued, and A_+^C and \dot{A}_+^C are \mathbb{R}^3 -valued. Hereafter we assume that A_+^C and \dot{A}_+^C satisfy the divergence free condition, that is,

$$\nabla \cdot A_+^C = \nabla \cdot \dot{A}_+^C = 0.$$

Our result for the Coulomb gauge case is as follows.

Theorem 2.1. *Let $u_+^C \in H^{6,7}$, $A_+^C \in H^{5,3}$, $\dot{A}_+^C \in H^{4,3}$ and*

$$\delta \equiv \|u_+^C\|_{H^{6,7}} + \|A_+^C\|_{H^{5,3}} + \|\dot{A}_+^C\|_{H^{4,3}} \quad (2.1)$$

be sufficiently small. Assume that A_+^C and \dot{A}_+^C satisfy the condition (2). Then there exists a unique solution (u, A) of the equation (MS-C) satisfying

$$\begin{aligned} u &\in \bigcap_{k=0}^1 C^k([0, \infty); H^{3-2k}), \\ A &\in C([0, \infty); \dot{H}^1 \cap \dot{H}^3), \quad \partial_t A \in \bigcap_{k=0}^2 C^k([0, \infty); H^{2-k}), \\ \sup_{t \geq 2} &\left[\frac{t}{(\log t)^2} \left\{ \sum_{j=0}^1 \|\partial_t^j u(t) - \partial_t^j u_D^C(t)\|_{H^{3-2j}} + \left(\int_t^\infty \|u(s) - u_D^C(s)\|_{W_4^{8/3}}^{8/3} ds \right)^{3/8} \right\} \right] < \infty, \\ \sup_{t \geq 2} &\left[\frac{t^{3/2}}{(\log t)^2} \left(\|A(t) - A_0^C(t) - A_1^C(t)\|_{\dot{H}^1 \cap \dot{H}^2} \right. \right. \\ &\quad \left. \left. + \sum_{j=1}^2 \|\partial_t^j A(t) - \partial_t^j A_0^C(t) - \partial_t^j A_1^C(t)\|_{H^{2-j}} \right) \right. \\ &\quad \left. + \frac{t}{(\log t)^2} \left(\|A(t) - A_0^C(t) - A_1^C(t)\|_{\dot{H}^3} \right. \right. \\ &\quad \left. \left. + \sum_{j=1}^3 \|\partial_t^j A(t) - \partial_t^j A_0^C(t) - \partial_t^j A_1^C(t)\|_{H^{3-j}} \right) \right] < \infty. \end{aligned}$$

Here

$$A_0^C(t, x) \equiv ((\cos \omega t) A_+^C)(x) + ((\omega^{-1} \sin \omega t) \dot{A}_+^C)(x),$$

A_1^C is the solution of the following linear equation:

$$\begin{aligned} \partial_t^2 A_1^C - \Delta A_1^C &= \frac{1}{t^3} \mathcal{P} \left[\frac{x}{t} \left| \hat{u}_+^C \left(\frac{x}{t} \right) \right|^2 \right] z(t)^2, \\ \|A_1^C(t)\|_{\dot{H}^1 \cap \dot{H}^3} + \|\partial_t A_1^C(t)\|_{H^2} &\rightarrow 0, \end{aligned}$$

as $t \rightarrow +\infty$, where $z \in C^\infty(\mathbb{R}_t; \mathbb{R})$ such that $z(t) = 0$ for $|t| \leq 1/2$, $z(t) = 1$ for $|t| \geq 1$, and we define

$$\begin{aligned} u_D^C(t, x) &\equiv (U(t) e^{-\frac{i|\cdot|^2}{2t}} e^{-iS^C(t, -i\nabla)} u_+^C)(x) \\ &= \frac{1}{(it)^{3/2}} \hat{u}_+^C \left(\frac{x}{t} \right) e^{\frac{i|x|^2}{2t} - iS^C(t, \frac{x}{t})}, \end{aligned}$$

where

$$S^C(t, x) \equiv (g(|\hat{u}_+^C|^2))(x) \log t - x \cdot \int_1^t A_1^C(s, sx) ds,$$

for $t \geq 1$ and $x \in \mathbb{R}^3$.

A similar result holds for negative time.

Let

$$\begin{aligned} \mathcal{V}^C &\equiv \{(u_+^C, A_+^C, \dot{A}_+^C); \|u_+^C\|_{H^{6,7}} + \|A_+^C\|_{H^{5,3}} + \|\dot{A}_+^C\|_{H^{4,3}} \leq \delta, \\ &\quad \nabla \cdot A_+^C = \nabla \cdot \dot{A}_+^C = 0\}, \end{aligned}$$

where δ is defined by (2.1).

The following corollary is an immediate consequence of Theorem 2.1.

Corollary 2.1. For the equation (MS-C), the modified wave operator $W_+^C: (u_+^C, A_+^C, \dot{A}_+^C) \mapsto (u(0), A(0), \partial_t A(0))$ is well-defined on \mathcal{V}^C , where (u, A) is the solution to the equation (MS-C) obtained in Theorem 2.1. Similarly the modified wave operator W_-^C for negative time is also well-defined on \mathcal{V}^C .

Next we state our result for the case of the Lorentz gauge.

Let $(u_+^L, \phi_+^L, \dot{\phi}_+^L, A_+^L, \dot{A}_+^L)$ be given scattered states, where u_+^L is complex valued, ϕ_+^L and $\dot{\phi}_+^L$ are real valued, and A_+^L and \dot{A}_+^L are \mathbb{R}^3 -valued. Hereafter we assume that $\phi_+^L, \dot{\phi}_+^L, A_+^L$ and \dot{A}_+^L satisfy the condition

$$\begin{cases} \dot{\phi}_+^L + \nabla \cdot A_+^L = 0, \\ \Delta \phi_+^L + \nabla \cdot \dot{A}_+^L = 0. \end{cases}$$

Our result for the Lorentz gauge case is as follows.

Theorem 2.2. *Let $u_+^L \in H^{6,7}$, $\phi_+^L \in H^{5,2}$, $\dot{\phi}_+^L \in H^{4,2}$, $A_+^L \in H^{5,3}$, $\dot{A}_+^L \in H^{4,3}$, $\omega^{-1}A_+^L \in H^{0,2}$, $\omega^{-1}\dot{A}_+^L \in H^{0,2}$ and*

$$\begin{aligned} \eta \equiv & \|u_+^L\|_{H^{6,7}} + \|\phi_+^L\|_{H^{5,2}} + \|\dot{\phi}_+^L\|_{H^{4,2}} + \|A_+^L\|_{H^{5,3}} + \|\dot{A}_+^L\|_{H^{4,3}} \\ & + \|\omega^{-1}A_+^L\|_{H^{0,2}} + \|\omega^{-1}\dot{A}_+^L\|_{H^{0,2}} \end{aligned} \quad (2.2)$$

be sufficiently small. Assume that $\phi_+^L, \dot{\phi}_+^L, A_+^L$ and \dot{A}_+^L satisfy the condition (2). Then there exists a unique solution (u, ϕ, A) of the equation (MS-L) satisfying

$$u \in \bigcap_{k=0}^1 C^k([0, \infty); H^{3-2k}),$$

$$\phi \in C([0, \infty); \dot{H}^1 \cap \dot{H}^3), \quad \partial_t \phi \in \bigcap_{k=0}^2 C^k([0, \infty); H^{2-k}),$$

$$A \in C([0, \infty); \dot{H}^1 \cap \dot{H}^3), \quad \partial_t A \in \bigcap_{k=0}^2 C^k([0, \infty); H^{2-k}),$$

$$\sup_{t \geq 2} \left[\frac{t}{(\log t)^2} \left\{ \sum_{j=0}^1 \|\partial_t^j u(t) - \partial_t^j u_D^L(t)\|_{H^{3-2j}} + \left(\int_t^\infty \|u(s) - u_D^L(s)\|_{W_4^2}^{8/3} ds \right)^{3/8} \right\} \right] < \infty,$$

$$\begin{aligned} & \sup_{t \geq 2} \left[\frac{t^{3/2}}{(\log t)^2} \left(\|\phi(t) - \phi_0^L(t) - \phi_1^L(t)\|_{\dot{H}^1 \cap \dot{H}^2} \right. \right. \\ & \quad \left. \left. + \sum_{j=1}^2 \|\partial_t^j \phi(t) - \partial_t^j \phi_0^L(t) - \partial_t^j \phi_1^L(t)\|_{H^{2-j}} \right) \right. \\ & \quad \left. + \frac{t}{(\log t)^2} \left(\|\phi(t) - \phi_0^L(t) - \phi_1^L(t)\|_{\dot{H}^3} \right. \right. \\ & \quad \left. \left. + \sum_{j=1}^3 \|\partial_t^j \phi(t) - \partial_t^j \phi_0^L(t) - \partial_t^j \phi_1^L(t)\|_{H^{3-j}} \right) \right] < \infty. \end{aligned}$$

$$\begin{aligned} & \sup_{t \geq 2} \left[\frac{t^{3/2}}{(\log t)^2} \left(\|A(t) - A_0^L(t) - A_1^L(t)\|_{\dot{H}^1 \cap \dot{H}^2} \right. \right. \\ & \quad \left. \left. + \sum_{j=1}^2 \|\partial_t^j A(t) - \partial_t^j A_0^L(t) - \partial_t^j A_1^L(t)\|_{H^{2-j}} \right) \right. \\ & \quad \left. + \frac{t}{(\log t)^2} \left(\|A(t) - A_0^L(t) - A_1^L(t)\|_{\dot{H}^3} \right. \right. \\ & \quad \left. \left. + \sum_{j=1}^3 \|\partial_t^j A(t) - \partial_t^j A_0^L(t) - \partial_t^j A_1^L(t)\|_{H^{3-j}} \right) \right] < \infty. \end{aligned}$$

Here

$$\begin{aligned}\phi_0^L(t, x) &\equiv ((\cos \omega t) \phi_+^L)(x) + ((\omega^{-1} \sin \omega t) \dot{\phi}_+^L)(x), \\ A_0^L(t, x) &\equiv ((\cos \omega t) A_+^L)(x) + ((\omega^{-1} \sin \omega t) \dot{A}_+^L)(x),\end{aligned}$$

ϕ_1^L and A_1^L are the solutions of the following final value problems of the linear equations:

$$\begin{aligned}\partial_t^2 \phi_1^L - \Delta \phi_1^L &= \frac{1}{t^3} \left| \hat{u}_+^L \left(\frac{x}{t} \right) \right|^2 z(t)^2, \\ \|\phi_1^L(t)\|_{\dot{H}^1 \cap \dot{H}^3} + \|\partial_t \phi_1^L(t)\|_{H^2} &\rightarrow 0, \quad t \rightarrow +\infty,\end{aligned}$$

and

$$\begin{aligned}\partial_t^2 A_1^L - \Delta A_1^L &= \frac{1}{t^3} \frac{x}{t} \left| \hat{u}_+^L \left(\frac{x}{t} \right) \right|^2 z(t)^2, \\ \|A_1^L(t)\|_{\dot{H}^1 \cap \dot{H}^3} + \|\partial_t A_1^L(t)\|_{H^2} &\rightarrow 0, \quad t \rightarrow +\infty,\end{aligned}$$

respectively, where $z \in C^\infty(\mathbb{R}_t; \mathbb{R})$ such that $z(t) = 0$ for $|t| \leq 1/2$, $z(t) = 1$ for $|t| \geq 1$, and we define

$$\begin{aligned}u_D^L(t, x) &\equiv (U(t) e^{-\frac{i|x|^2}{2t}} e^{-iS^L(t, -i\nabla)} u_+^L)(x) \\ &= \frac{1}{(it)^{3/2}} \hat{u}_+^L \left(\frac{x}{t} \right) e^{\frac{i|x|^2}{2t} - iS^L(t, \frac{x}{t})},\end{aligned}$$

where

$$S^L(t, x) \equiv \int_1^t (\phi_1^L(s, sx) - x \cdot A_1^L(s, sx)) ds,$$

for $t \geq 1$ and $x \in \mathbb{R}^3$.

A similar result holds for negative time.

Let

$$\begin{aligned}\mathcal{V}^L &\equiv \{(u_+^L, \phi_+^L, \dot{\phi}_+^L, A_+^L, \dot{A}_+^L); \\ &\|u_+^L\|_{H^{6,7}} + \|\phi_+^L\|_{H^{5,2}} + \|\dot{\phi}_+^L\|_{H^{4,2}} + \|A_+^L\|_{H^{5,3}} + \|\dot{A}_+^L\|_{H^{4,3}} \\ &+ \|\omega^{-1} A_+^L\|_{H^{0,2}} + \|\omega^{-1} \dot{A}_+^L\|_{H^{0,2}} \leq \eta, \\ &\dot{\phi}_+^L + \nabla \cdot A_+^L = 0, \quad \Delta \phi_+^L + \nabla \cdot \dot{A}_+^L = 0\},\end{aligned}$$

where η is defined by (2.2).

The following corollary is an immediate consequence of Theorem 2.2.

Corollary 2.2. *For the equation (MS-L), the modified wave operator $W_+^L: (u_+^L, \phi_+^L, \dot{\phi}_+^L, A_+^L, \dot{A}_+^L) \mapsto (u(0), \phi(0), \partial_t \phi(0), A(0), \partial_t A(0))$ is well-defined on \mathcal{V}^L , where (u, ϕ, A) is the solution to the equation (MS-C) obtained in Theorem 2.2. Similarly the modified wave operator W_-^L for negative time is also well-defined on \mathcal{V}^L .*

3. IDEA OF PROOF

Our main idea of proof is as follows. We begin with the equation (MS-C), that is, the Coulomb gauge case. First we determine an asymptotic profile for the Maxwell part $A_0^C + A_1^C$, where A_0^C is the free wave solution and A_1^C is a suitable second correction term of the asymptotic profile. Secondly, we determine the asymptotic profile u_a^C for the Schrödinger component. Since the time decay estimate of A_1^C is not sufficient to prove the existence of ordinary wave operators, that is, the long range effect $(A_1^C \cdot (x/t) - g(|u_a^C|^2))u_a^C$ appears, we introduce the modified free dynamics u_a^C of the Dollard type for the Schrödinger equation such that $(i\partial_t + \frac{1}{2}\Delta)u_a^C + (A_1^C \cdot (x/t) - g(|u_a^C|^2))u_a^C$ decays faster than $(A_1^C \cdot (x/t) - g(|u_a^C|^2))u_a^C$ by

using the method of phase correction. On the other hand, another effect $A_0^C \cdot (x/t)u_a^C$ appears. In general, this decays slowly as $t \rightarrow \infty$ and brings difficulty. Furthermore, since all the derivatives with respect to x of the function $(x/t) \cdot A_0^C$ decay as fast as itself, we can not apply the method of phase correction to $(x/t) \cdot A_0^C u_a^C$. Fortunately in the Coulomb gauge case $A_0^C \cdot (x/t)u_a^C$ decays faster than in general case, since $\square A_0^C = 0$ and $\nabla \cdot A_0^C = 0$ imply $\square(x \cdot A_0^C) = 0$, that is, $x \cdot A_0^C$ is also a solution for the free wave equation under the Coulomb gauge condition.

We next explain idea of the proof for the equation (MS-L), that is, the Lorentz gauge case. As in the case of the Coulomb gauge, we first determine an asymptotic profile for the Maxwell part $(\phi_0^L + \phi_1^L, A_0^L + A_1^L)$, where (ϕ_0^L, A_0^L) are the free wave solution and (ϕ_1^L, A_1^L) is a suitable second correction term of the asymptotic profile. We construct a modified free profile u_p^L of the Dollard type by the method of phase correction in order to overcome difficulty from the long range effect $u_p^L(A_1^L \cdot (x/t) - \phi_1^L)$, which appears later, as in the case of Coulomb gauge. (u_p^L is a principal term of the asymptotic profile for the Schrödinger component). Finally, we note that since all the derivatives with respect to x of the function $\nabla \cdot A_0^L$ decay as fast as itself, we can not apply the method of phase correction to the slowly decaying term $(i/2)\nabla \cdot A_0^L u_p^L$ which appears later. To overcome this difficulty, we construct a suitable second correction term u_r^L of the asymptotic profile for the Schrödinger part such that $(i\partial_t + \frac{1}{2}\Delta)u_r^L - (i/2)\nabla \cdot A_0^L u_p^L$ decays faster than $(i/2)\nabla \cdot A_0 u_1$ so that the Cook-Kuroda method is applicable. This method is also used in [3, 5, 6, 8].

REFERENCES

- [1] J. Ginibre and G. Velo, *Long range scattering and modified wave operators for the Wave-Schrödinger system*, Ann. Henri Poincaré, **3** (2002), 537–612.
- [2] J. Ginibre and G. Velo, *Long range scattering and modified wave operators for the Maxwell-Schrödinger system I. The case of vanishing asymptotic magnetic field*, Preprint.
- [3] J. Ginibre and G. Velo, *Long range scattering and modified wave operators for the Wave-Schrödinger system II*, Preprint.
- [4] T. Ozawa and Y. Tsutsumi, *Asymptotic behavior of solutions for the coupled Klein-Gordon-Schrödinger equations*, preprint, RIMS-775 (1991); Adv. Stud. Pure Math., **23** (1994), 295–305.
- [5] A. Shimomura, *Modified wave operators for the coupled Wave-Schrödinger equations in three space dimensions*, To appear in Discrete Contin. Dyn. Syst.
- [6] A. Shimomura, *Wave operators for the coupled Klein-Gordon-Schrödinger equations in two space dimensions*, To appear in Funkcial. Ekvac.
- [7] A. Shimomura, *Modified wave operators for Maxwell-Schrödinger equations in three space dimensions*, To appear in Ann. Henri Poincaré.
- [8] A. Shimomura, *Scattering theory for Zakharov equations in three space dimensions with large data*, Preprint, UTMS 2003-1.
- [9] Y. Tsutsumi, *Global existence and asymptotic behavior of solutions for the Maxwell-Schrödinger equations in three space dimensions*, Comm. Math. Phys., **151** (1993), 543–576.

Remarks on the large time asymptotics for nonlinear Klein-Gordon systems

Hideaki Sunagawa

Department of Mathematics, Graduate School of Science, Osaka University
Toyonaka, Osaka 560-0043, Japan
sunagawa@math.sci.osaka-u.ac.jp

We consider the Cauchy problem for

$$\begin{cases} (\square + m^2)u = F(v), \\ (\square + \mu^2)v = G(u), \end{cases} \quad t > 0, x \in \mathbb{R} \quad (1)$$

with sufficiently small, smooth, compactly-supported initial data. Here $\square = \partial_t^2 - \partial_x^2$, m, μ are positive constants, F, G are smooth functions of unknowns and they are cubic nonlinear terms in the sense that

$$|F(w)| + |G(w)| \leq C|w|^3 \quad \text{if } |w| \leq \varsigma$$

for some constants C and $\varsigma > 0$. Though it is possible to consider much more general situations (including derivative nonlinear or quasi-linear cases), we do not go into such directions for the sake of simplicity.

Recently, much efforts are made for study of the large time behavior of solutions to the Cauchy problem for the systems of critical nonlinear Klein-Gordon equations with possibly different masses (see e.g. [14], [11], [12], [3], [4]). It is known that if $(m - \mu)(m - 3\mu)(3m - \mu) \neq 0$, the Cauchy problem (1) admits a unique global classical solution which tends to a free solution as $t \rightarrow \infty$. On the other hand, the case $(m - \mu)(m - 3\mu)(3m - \mu) = 0$ turns out to be more delicate and the previous works leave the problem open except a few partial results. (The scalar case has been extensively studied by Delort [2], after partial results [9], [7], [10], [5], [1] etc. See also the remark after Theorem 1.)

In this talk, we concentrate our attention to the following example:

$$\begin{cases} (\square + m^2)u = \alpha v^4, \\ (\square + \mu^2)v = \beta u^3, \\ (u, \partial_t u, v, \partial_t v) |_{t=0} = (\varepsilon u_0, \varepsilon u_1, \varepsilon v_0, \varepsilon v_1), \end{cases} \quad \begin{array}{l} t > 0, x \in \mathbb{R}, \\ x \in \mathbb{R}, \end{array} \quad (2)$$

where $\alpha, \beta \in \mathbb{R}$, $\varepsilon > 0$ is a small parameter, and $u_0, u_1, v_0, v_1 \in C_0^\infty(\mathbb{R})$. Our purpose is to show that the amplitude of v in (2) is modulated in the logarithmic

order when $\mu = m$ or $\mu = 3m$, whereas, when $\mu \neq m$, $\mu \neq 3m$, the influence of nonlinearity disappears eventually and v behaves like a free solution as $t \rightarrow \infty$. More precisely,

Theorem 1 *For any $u_0, u_1, v_0, v_1 \in C_0^\infty(\mathbb{R})$, there exists $\varepsilon_0 > 0$ such that (2) admits a unique global classical solution if $\varepsilon \in]0, \varepsilon_0]$. Moreover, the following asymptotics is valid as $t \rightarrow \infty$, uniformly with respect to $x \in \mathbb{R}$:*

$$\begin{aligned} u(t, x) &= \operatorname{Re} \left[\frac{e^{im(t^2 - |x|^2)_+^{1/2}}}{m\sqrt{t}} a(x/t) \right] + \mathcal{O}(t^{-1+\delta}), \\ v(t, x) &= \operatorname{Re} \left[\frac{e^{i\mu(t^2 - |x|^2)_+^{1/2}}}{\mu\sqrt{t}} \left\{ A(x/t) \log t + b(x/t) \right\} \right] + \mathcal{O}(t^{-1+\delta}). \end{aligned}$$

Here $(\cdot)_+$ stands for $\max\{\cdot, 0\}$, $i = \sqrt{-1}$, δ is an arbitrary small positive number, $a(y)$, $b(y)$ are \mathbb{C} -valued smooth functions which vanish when $|y| \geq 1$, and $A(y)$ is given by

$$A(y) = \begin{cases} \frac{\beta}{i8m^3} (1 - |y|^2)_+^{1/2} a(y)^3 & \text{if } \mu = 3m, \\ \frac{3\beta}{i8m^3} (1 - |y|^2)_+^{1/2} |a(y)|^2 a(y) & \text{if } \mu = m, \\ 0 & \text{if } \mu \neq 3m, \mu \neq m. \end{cases}$$

Remark It is interesting to compare this result with the corresponding one to the scalar case

$$(\square + 1)w = \beta w^3, \quad t > 0, x \in \mathbb{R}. \quad (3)$$

According to [2], w has the following asymptotics:

$$w(t, x) = \operatorname{Re} \left[\frac{1}{\sqrt{t}} e^{i\{(t^2 - |x|^2)_+^{1/2} + \varphi(x/t) \log t\}} a(x/t) \right] + \mathcal{O}(t^{-1+\delta}), \quad t \rightarrow \infty$$

with

$$\varphi(y) = -\frac{3\beta}{8} (1 - |y|^2)_+^{1/2} |a(y)|^2.$$

Roughly speaking, this shows that the long range character of nonlinearity appears at the level of the *phase* of oscillation of the solution for the scalar equation (3), while our main result claims that the long range character appears at the level of the *amplitude* of the solution for the system (2).

The same proof is available for a bit more general systems, such as

$$\begin{cases} (\square + m_1^2)u_1 = F_1(u, \partial u), \\ (\square + m_2^2)u_2 = F_2(u, \partial u), \\ (\square + m_3^2)u_3 = F_3(u, \partial u), \\ (\square + m_4^2)u_4 = \gamma u_1 u_2 u_3 + F_4(u, \partial u), \end{cases} \quad t > 0, x \in \mathbb{R}, \quad (4)$$

with the initial data

$$(u_j, \partial_t u_j) \Big|_{t=0} = (\varepsilon u_{0j}, \varepsilon u_{1j}), \quad j = 1, 2, 3, 4. \quad (5)$$

Here $u = (u_j)_{1 \leq j \leq 4}$, $\partial = (\partial_t, \partial_x)$, $\gamma \in \mathbb{R}$ and $F_j(u, \partial u) = O(|u|^4 + |\partial u|^4)$ ($1 \leq j \leq 4$). When we put

$$\Lambda := \left\{ (\lambda_1, \lambda_2, \lambda_3) \in \{\pm 1\}^3 \mid m_4 = \lambda_1 m_1 + \lambda_2 m_2 + \lambda_3 m_3 \right\},$$

the corresponding result to Theorem 1 is stated as follows:

Theorem 2 *For any $u_{0j}, u_{1j} \in C_0^\infty(\mathbb{R})$, there exists $\varepsilon_0 > 0$ such that (4)–(5) admits a unique global classical solution if $\varepsilon \in]0, \varepsilon_0]$. Moreover, the following asymptotics is valid as $t \rightarrow \infty$, uniformly with respect to $x \in \mathbb{R}$:*

$$\begin{aligned} u_j(t, x) &= \operatorname{Re} \left[\frac{e^{im_j(t^2 - |x|^2)_+^{1/2}}}{m_j \sqrt{t}} a_j(x/t) \right] + \mathcal{O}(t^{-1+\delta}), \quad j = 1, 2, 3, \\ u_4(t, x) &= \operatorname{Re} \left[\frac{e^{im_4(t^2 - |x|^2)_+^{1/2}}}{m_4 \sqrt{t}} \left\{ A(x/t) \log t + a_4(x/t) \right\} \right] + \mathcal{O}(t^{-1+\delta}). \end{aligned}$$

Here, δ is an arbitrary small positive number, a_j ($j = 1, 2, 3$) are \mathbb{C} -valued smooth functions which vanish when $|y| \geq 1$, and $A(y)$ is given by

$$A(y) = \begin{cases} \frac{\gamma}{i8m_1 m_2 m_3} (1 - |y|^2)_+^{1/2} \sum_{(\lambda_1, \lambda_2, \lambda_3) \in \Lambda} a_1^{(\lambda_1)}(y) a_2^{(\lambda_2)}(y) a_3^{(\lambda_3)}(y) & \text{if } \Lambda \neq \emptyset, \\ 0 & \text{if } \Lambda = \emptyset, \end{cases}$$

where $a_j^{(+1)}(y) = a_j(y)$, $a_j^{(-1)}(y) = \overline{a_j(y)}$.

Remark We can also show the analogous result for two-dimensional case, such as

$$\begin{cases} (\square + m^2)v_1 = \alpha v_2^3, \\ (\square + \mu^2)v_2 = \beta v_1^2, \end{cases} \quad t > 0, x \in \mathbb{R}^2, \quad (6)$$

where $\alpha, \beta \in \mathbb{R}$, or

$$\begin{cases} (\square + m_1^2)u_1 = F_1(u, \partial u), \\ (\square + m_2^2)u_2 = F_2(u, \partial u), \\ (\square + m_3^2)u_3 = \gamma u_1 u_2 + F_3(u, \partial u), \end{cases} \quad t > 0, \quad x \in \mathbb{R}^2, \quad (7)$$

where $u = (u_1, u_2, u_3)$, $\partial = (\partial_t, \partial_{x_1}, \partial_{x_2})$, $\gamma \in \mathbb{R}$ and $F_j(u, \partial u) = O(|u|^3 + |\partial u|^3)$. For the solution v_2 of (6) (resp. u_3 of (7)), the long range effect as in Theorem 1 (resp. Theorem 2) is observed if and only if $\mu = 2m$ (resp. $m_3 = \lambda_1 m_1 + \lambda_2 m_2$ for some $\lambda_1, \lambda_2 \in \{\pm 1\}$).

References

- [1] J.-M. Delort, Minoration du temps d'existence pour l'équation de Klein-Gordon non-linéaire en dimension 1 d'espace, *Ann. Inst. Henri Poincaré (Analyse non linéaire)*, **16** (1999), 563–591.
- [2] J.-M. Delort, *Existence globale et comportement asymptotique pour l'équation de Klein-Gordon quasi linéaire à données petites en dimension 1*. Ann. Sci. Ec. Norm. Sup. 4^e série **34** (2001), 1–61.
- [3] J.-M. Delort, D. Fang and R. Xue, *Global existence of small solutions for quadratic quasilinear Klein-Gordon systems in two space dimensions*. Prépublications Mathématiques de l'Université Paris 13, 2002-29.
- [4] D. Fang and R. Xue, *Global existence of small solutions for cubic quasilinear Klein-Gordon systems in one space dimension*. preprint, 2003.
- [5] V. Georgiev and B. Yordanov, *Asymptotic behaviour of the one-dimensional Klein-Gordon equation with a cubic nonlinearity*, preprint, 1996.
- [6] L. Hörmander, *Lectures on nonlinear hyperbolic differential equations*. Springer Verlag, Berlin, 1997.
- [7] S. Katayama, *A note on global existence of solutions to nonlinear Klein-Gordon equations in one space dimension*, J. Math. Kyoto Univ., **39** (1999), 203–213.
- [8] S. Klainerman, *Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions*. Comm. Pure. Appl. Math., **38** (1985), 631–641.
- [9] K. Moriyama, *Normal forms and global existence of solutions to a class of cubic nonlinear Klein-Gordon equations in one space dimension*, Diff. Integral Eqs., **10** (1997), 499–520.

- [10] K. Moriyama, S. Tonegawa and Y. Tsutsumi, Almost global existence of solutions for the quadratic semilinear Klein-Gordon equation in one space dimension, *Funkcialaj Ekvacioj*, **40** (1997), 313–333.
- [11] H. Sunagawa, *On global small amplitude solutions to systems of cubic nonlinear Klein-Gordon equations with different mass terms in one space dimension*. J. Diff. Eqs. **192** (2003), 308–325.
- [12] H. Sunagawa, *A note on the large time asymptotics for a system of Klein-Gordon equations*. Hokkaido Math. J. (to appear).
- [13] H. Sunagawa, *Large time asymptotics of solutions to nonlinear Klein-Gordon systems*. Osaka University Research Reports in Mathematics, 2003-5.
- [14] Y. Tsutsumi, *Stability of constant equilibrium for the Maxwell-Higgs equations*. Funkcialaj Ekvacioj, **46** (2003), 41-62.

The curved traveling front of the Allen-Cahn equation

Hirokazu Ninomiya (Ryukoku University)

and

Masaharu Taniguchi (Tokyo Institute of Technology)

In this talk we consider the Allen-Cahn equation:

$$u_t = \Delta u + f(u) \quad (x, y, t) \in \mathbf{R}^2 \times \mathbf{R}_+ \quad (1)$$

where f is of "bistable type". The typical example of the nonlinear term f is

$$f(u) = u(1-u)(u-a), \quad 0 < a < \frac{1}{2}. \quad (2)$$

The constant states 0 and 1 are stable under the diffusion-free system. By the assumption of a , the region of the state 1 is getting larger and larger and finally it covers the whole space. When the state 1 propagates, we can observe the characteristic profiles. In the one-dimensional space, one of the typical solutions is a *traveling wave* solution which never changes its shape without translation. Substituting $u(x, t) = \Phi(x - ct)$, we have

$$\Phi_{\xi\xi} - c\Phi_{\xi} + f(\Phi) = 0.$$

Actually for the nonlinearity (2), we have

$$\Phi(\xi) = \frac{1}{2} \left(1 - \tanh \frac{\xi}{2\sqrt{2}} \right), \quad c = \sqrt{2} \left(\frac{1}{2} - a \right).$$

As the appropriate singular limit, the interface between two states 1 and 0 becomes sharp and we can get the interface equation (see e.g. [4]):

$$V = H + k \quad (3)$$

where V is a normal velocity, H is the curvature, and k is a given constant. This equation is also observed in the filamentary vortex of the Ginzburg-Landau equation confined in a plane [3] and the BZ reaction [6].

The typical solutions of (3) are the circles and lines. In the case $k \neq 0$, unfortunately, some interfaces may possess some self-intersection points eventually, even if the initial interface has none. If the interface is represented by the graph $y = v(x, t)$, the equation (3) is reduced to

$$v_t = \frac{v_{xx}}{1 + v_x^2} + k\sqrt{1 + v_x^2} \quad x \in \mathbf{R}, t > 0, \quad (4)$$

Deckelnick et al in [3] proved the existence of the traveling curved front and studied the stability of the front under some restricted assumptions for u_0 . The authors relaxed the assumption for the initial data and classified all the traveling fronts in [7, 8]. They proved the following (see [7, Proposition 1.1, Theorem 1.2]).

Theorem 1 *Any traveling front of (3) with velocity $v(0, c)$ is one of the three, after appropriate translations,*

- (i) *lines $y = m_*x$, and $y = -m_*x$*
- (ii) *a traveling curved front $\Gamma_c(t)$ which possesses two asymptotes $y = \pm m_*x$,*
- (iii) *stationary circles with radius $1/|k|$ only in the case $c = 0$,*

where $m_* := \sqrt{c^2 - k^2}/k$. Moreover the explicit form of the traveling curved front $\Gamma_c(t) = \{y = \varphi(x) + ct\}$ with speed $c(\geq k)$ is given in

$$x(\theta; c) := \frac{\theta}{c} + \frac{k}{c\sqrt{c^2 - k^2}} \log \left| \frac{1 + \sqrt{\frac{c+k}{c-k}} \tan \frac{\theta}{2}}{1 - \sqrt{\frac{c+k}{c-k}} \tan \frac{\theta}{2}} \right|,$$

$$y(\theta; c) := -\frac{1}{c} \log \left(\frac{c \cos \theta - k}{c - k} \right),$$

for $\theta \in (-\arctan m_*, \arctan m_*)$.

The traveling curved front $\Gamma_c(t)$ is “V-shaped”, which connects two asymptotes. The existence of this traveling front is also reported in [2, 3] and in a liquid BZ reaction [6].

The asymptotic stability of the curved traveling front in (4) is discussed in [3, 8]. It is proved that the traveling curved front is asymptotically stable, if the initial perturbation is restricted to

$$BC_0^1 := \{v \in C^1(\mathbf{R}) \mid \sup_{-\infty < x < \infty} (|v(x)| + |v_x(x)|) < \infty, \lim_{|x| \rightarrow \infty} v(x) = 0\}.$$

and that if you take the perturbation space

$$BC^1 := \{v \in C^1(\mathbf{R}) \mid \sup_{-\infty < x < \infty} (|v(x)| + |v_x(x)|) < \infty\},$$

instead of BC_0^1 , the traveling curved front is not asymptotically stable (see [8, Theorem 1.1 and Theorem 4.1]).

By the above observation, we can expect that a “V-shaped” traveling wave solution of (1) exists. Actually we have the following theorem.

Theorem 2 *There exists a traveling wave solution $u(x, y, t) = U(x, y - ct)$ of (1) such that*

$$\lim_{R \rightarrow \infty} \sup_{(x,y) \in D_R} \left| U(x, y) - \Phi \left(\frac{k}{c}(y - m_*|x|) \right) \right| = 0$$

where

$$D_R := \{(x, y) \mid x^2 + y^2 \geq R^2\}.$$

Bonnet and Hamel [1] showed the existence of the “V-shaped” traveling wave solutions, if f is of the “ignition temperature” type (mono-stable type) instead of (2). Hamel and Monneau [5] shows the uniqueness of the traveling front of the corresponding singular limit problem.

References

- [1] A. BONNET AND F. HAMEL, Existence of nonplanar solutions of a simple model of premixed bunsen flames, *SIAM J. Math. Anal.*, **31**, (1999) 80–118.
- [2] P. K. BRAZHNİK, Exact solutions for the kinematic model of autowaves in two-dimensional excitable media, *Physica D*, **94**, (1996) 205–220.
- [3] K. Deckelnick, C. M. Elliott, and G. Richardson, Long time asymptotics for forced curvature flow with applications to the motion of a superconducting vortex, *Nonlinearity* **10** (1997), 655–678.
- [4] P. C. Fife, *Dynamics of Internal Layers and Diffusive Interfaces* (CBMS-NSF Reg. Conf. Ser. Appl. Math. 53), SIAM (1988).
- [5] F. HAMEL AND R. MONNEAU, Existence and uniqueness for a free boundary problem arising in combustion theory, *Interfaces and Free Boundaries*, **4**, (2002) 167–210.

- [6] V. Pérez-Muñuzuri, M. Gómez-Gesteira, A. P. Muñuzuri, V. A. Davydov, and V. Pérez-Villar, V-shaped stable nonspiral patterns, *Physical Review E* **51-2** (1995), 845–847.
- [7] H. Ninomiya and M. Taniguchi, Traveling curved fronts of a mean curvature flow with constant driving force, in " *Free boundary problems: theory and applications*, I' *GAKUTO Internat. Ser. Math. Sci. Appl.* **13** (2000), 206–221.
- [8] H. Ninomiya and M. Taniguchi, Stability of traveling curved fronts in a curvature flow with driving force, *Methods and Application of Analysis*, **8-2** (2001), 429–450.
- [9] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, (1984) Springer-Verlag.

UNIQUENESS IN INVERSE SCATTERING PROBLEMS WITH A SINGLE INCIDENT WAVE

J. CHENG AND M. YAMAMOTO

1. INTRODUCTION

Let $D \subset \mathbb{R}^2$ be a bounded domain and $k \in \mathbb{R}$. For $x \in \mathbb{R}^2$, we set $r = |x|$. We consider a scattering problem with sound-soft obstacle:

$$(1.1) \quad \Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus cl(D)$$

$$(1.2) \quad u = 0 \quad \text{on } \partial D$$

$$(1.3) \quad \lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial}{\partial r} u^S(x) - iku^S(x) \right) = 0.$$

Henceforth $cl(D)$ denotes the closure of a domain D , and we set $i = \sqrt{-1}$, $d \in S^1 \equiv \{x \in \mathbb{R}^2; |x| = 1\}$ and

$$u^S(x) = u(x) - e^{ikx \cdot d}.$$

Under suitable conditions on D , for $k \in \mathbb{R}$ and $d \in S^1$, there exists a unique H^1 -solution $u(x) = u(D)(x)$ to (1.1) - (1.3), and we can define the far field pattern $u_\infty(D) \left(\frac{x}{r} \right)$:

$$(1.4) \quad u^S(D)(x) = \frac{e^{ikr}}{\sqrt{r}} \left\{ u_\infty(D) \left(\frac{x}{r} \right) + O \left(\frac{1}{r} \right) \right\} \quad \text{as } r \rightarrow \infty.$$

Inverse scattering problem: Determine D from the far field pattern $u_\infty(D)$ for given k and d (possibly by changing them).

This inverse problem is also physically significant and has been studied by many authors. We refer for example to Colton and Kress [1].

The first basic topic for this inverse problem is the uniqueness: Does

$$(1.5) \quad u_\infty(D_1)(x) = u_\infty(D_2)(x), \quad |x| = 1$$

(for possible several d and k) imply $D_1 = D_2$?

There is a classical uniqueness result within smooth D_1, D_2 if (1.5) holds for an infinite number of $d \in S^1$, which is proved based on Schiffer's idea (see Theorem 5.1 in [1]). For the uniqueness by means of a finite number of $d \in S^1$, see Colton and Sleeman [2], Theorem 5.2 in [1]. Moreover the uniqueness is known with a *single* d , provided that D_1, D_2 are contained in a ball of radius ρ such that $k\rho < \pi$. See Corollary 5.3 in [1], [2].

An important open problem is the uniqueness in the inverse scattering problem with a *single* (d, k) . This problem is interesting from the theoretical point of view, because the far field patterns with many d are overdetermining data for determination of D and we can expect the uniqueness with a single far field pattern. Moreover the formulation with a single (d, k) is helpful for justification of numerical reconstruction of D , because one can usually use far field patterns observed by taking a single or a finite number of d .

2. MAIN RESULT

Let $k \in \mathbb{R}$ and $d \in S^1$ be arbitrarily fixed. Henceforth, for $P, Q \in \mathbb{R}^2$, we understand that \overline{PQ} is an open segment (not including the end points P and Q). Moreover for a polygonal domain D and $P \in \partial D$, $Q \notin cl(D)$ such that $\overline{PQ} \in \mathbb{R}^2 \setminus cl(D)$, by $\angle(\overline{PQ}, \partial D)$ we denote the least angle among the two angles in $\mathbb{R}^2 \setminus cl(D)$ formed by \overline{PQ} and ∂D . By a polygonal domain D , we mean that ∂D is composed of a finite number of segments.

Definition 2.1. Let $D \subset \mathbb{R}^2$ be a bounded polygonal domain. Let ℓ -points P_1, \dots, P_ℓ , $\ell \geq 2$, satisfy the following conditions (i) - (iv) :

(i) $P_1, \dots, P_\ell \in \partial D$.

For $1 \leq j \leq \ell$, we set

$$\theta_j = \begin{cases} \text{the exterior angle of } D \text{ at } P_j, & \text{if } P_j \text{ is a vertex of a polygon } D, \\ \pi, & \text{otherwise.} \end{cases}$$

(ii) $\overline{P_j P_{j+1}} \subset \mathbb{R}^2 \setminus cl(D)$ for $1 \leq j \leq \ell$.

(iii) $\angle(\overline{P_{j-1} P_j}, \partial D) = \angle(\overline{P_j P_{j+1}}, \partial D)$, $1 \leq j \leq \ell$, if $\overline{P_{j-1} P_j}$ does not bisect θ_j at P_j .

(iv) For $1 \leq j \leq \ell$, we have $\frac{\theta_j}{\angle(\overline{P_{j-1} P_j}, \partial D)} \in \mathbb{Q}$.

Here we set $P_0 = P_\ell$ and $P_{\ell+1} = P_1$ and

$$TR(D : P_1, \dots, P_\ell) = \begin{cases} \text{a closed broken line } P_1 \rightarrow P_2 \rightarrow \dots \rightarrow P_\ell \rightarrow P_1 \\ \text{if } \overline{P_1 P_\ell} \text{ does not bisect } \theta_1 \text{ at } P_1, \\ \text{a non-closed broken line } P_1 \rightarrow P_2 \rightarrow \dots \rightarrow P_\ell, \text{ otherwise.} \end{cases}$$

We call $TR(D : P_1, \dots, P_\ell)$ a *trapped ray of D with rational angles*.

By $TR(D)$, we denote the sum of all the trapped rays of D with rational angles. If $TR(D) \neq \emptyset$, then we call D *trapping with rational angles*.

In other words, if $TR(D) = \emptyset$, then there are no rays in $\mathbb{R}^2 \setminus cl(D)$ which go out to ∞ after finite times reflecting on ∂D subject to physical law (iii) with stricter constraint (iv) for angles of incidence.

We can state our main result:

Theorem 2.2. *Let $k \in \mathbb{R}$ and $d \in S^1$ be arbitrarily fixed and let*

$$(2.1) \quad \partial D_1 \cap TR(D_2) = \emptyset \quad \text{and} \quad \partial D_2 \cap TR(D_1) = \emptyset.$$

Then $u_\infty(D_1)(x) = u_\infty(D_2)(x)$, $|x| = 1$, implies $D_1 = D_2$.

Corollary 2.3. *Let D_1 and D_2 be star-shaped polygons. Then $u_\infty(D_1)(x) = u_\infty(D_2)(x)$, $|x| = 1$, implies $D_1 = D_2$.*

By the definition, the break of condition (2.1) happens rarely. However we do not know the uniqueness if (2.1) does not hold. In fact, we have the following trapping D_1, D_2 where our proof does not work.

Example 1. *Let us form D_1, D_2 as follows.*

(1) *We take a square $A_1 A_2 A_3 A_4$. For convenience, we set $A_1 = (0, 0)$, $A_2 = (1, 0)$, $A_3 = (1, 1)$, $A_4 = (0, 1)$.*

(2) *In the interior of the square $A_1 A_2 A_3 A_4$, we take a regular triangle $B_1 B_2 B_3$ (i.e., the lengths of the sides are equal). Here we choose vertices B_1, B_2, B_3 such that $B_1 \rightarrow B_2 \rightarrow B_3$ is counterclockwise and that $\overline{B_1 B_2} \parallel \overline{A_1 A_2}$.*

(3) *Take the midpoints P_1 and P_2 of the sides $\overline{B_1 B_3}$ and $\overline{B_2 B_3}$ respectively.*

(4) *Take a point Q_1 on the segment $\overline{B_3 P_2}$ arbitrarily.*

(5) *Take two points Q_2, Q_3 on the side $\overline{A_2 A_3}$ such that $\overline{B_3 Q_3} \parallel \overline{A_1 A_2}$ and $\overline{Q_1 Q_2} \parallel \overline{A_1 A_2}$.*

(6) By D_1 we denote the interior bounded by the closed broken line $A_1A_2Q_2Q_1B_2B_1B_3Q_3A_3A_4$ (which is a non-convex polygon with those vertices). By D_2 we denote the interior bounded by the closed broken line $A_1A_2Q_2Q_1P_2P_1B_3Q_3A_3A_4$ (Figure 1).

Then D_1 is trapping with rational angles. In fact, let P_3 be the midpoint of the side $\overline{B_1B_2}$. For D_1 , we can see that $P_1P_2P_3$ satisfies conditions (i) - (iv), and we have $TR(D_1) \cap \partial D_2 \supset \overline{P_1P_2} \neq \emptyset$, that is, condition (2.1) does not hold. In this example, we note that $TR(D_1 : P_1, P_2, P_3)$ is a closed broken line $P_1 \rightarrow P_2 \rightarrow P_3 \rightarrow P_1$. For these D_1 and D_2 , our proof does not work.

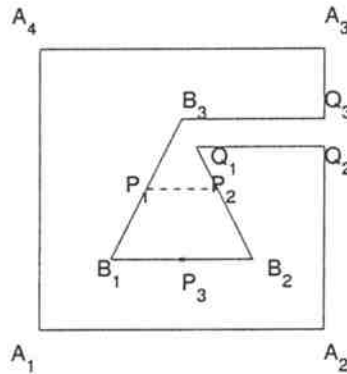


Figure 1

REFERENCES

1. Colton, D. and Kress, R., Inverse Acoustic and Electromagnetic Scattering Theory. the second edition, Springer-Verlag, Berlin, 1998.
2. Colton, D. and Sleeman, B.D., Uniqueness theorems for the inverse problem of acoustic scattering. IMA J. Appl. Math. 31 (1983), 253-259.

DEPARTMENT OF MATHEMATICS, FUDAN UNIVERSITY, SHANGHAI 200433, CHINA

E-mail address: jcheng@fudan.edu.cn

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, TOYKO 153-8914, JAPAN

E-mail address: myama@ms.u-tokyo.ac.jp

Asymptotic behavior of spherically symmetric solutions to the compressible Navier–Stokes equation with external forces

Shinya Nishibata*

Department of Mathematical and Computing Sciences
Tokyo Institute of Technology
Tokyo 152-8552, Japan

We study the large time behavior of an isentropic and spherically symmetric motion of compressible viscous gas in a field of external force over an unbounded exterior domain in \mathbb{R}^n ($n \geq 2$). The typical example of this problem appears in analysis of the behavior of atmosphere around the earth. First, we show that there exists a stationary solution satisfying an adhesion boundary condition and a positive spatial asymptotic condition. Then, it is shown that this stationary solution is a time asymptotic state to the initial boundary value problem with the same boundary and spatial asymptotic conditions. Here, the initial data can be chosen arbitrarily large if it belongs to the suitable Sobolev space. Moreover, if the external force is attractive, it also can be arbitrarily large. This condition includes the most typical external force, i.e., the gravitational force. In the proof of the stability theorem, it is the essential step to obtain the uniform positive lower bound for the density. It is derived through the energy method with the aid of a representation formula for the density.

The Navier–Stokes equation with external force for the isentropic motion of compressible viscous gas in the Eulerian coordinate is the system of equations given by

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{1a}$$

$$\rho \{u_t + (u \cdot \nabla)u\} = \mu_1 \Delta u + (\mu_1 + \mu_2) \nabla(\nabla \cdot u) - \nabla p(\rho) + \rho f. \tag{1b}$$

We study the asymptotic behavior of a solution (ρ, u) to (1) in an unbounded exterior domain $\Omega := \{\xi \in \mathbb{R}^n ; |\xi| > 1\}$, where n is a space dimension larger than or equal to 2. Here $\rho > 0$

*The author's work was supported in part by Grant-in-Aid for Scientific Research (C)(2) 14540200 of the Ministry of Education, Culture, Sports, Science and Technology Science and Technology.

is the mass density; $u = (u_1, \dots, u_n)$ is the velocity of gas; $p(\rho) = K\rho^\gamma$ ($K > 0, \gamma \geq 1$) is the pressure; f is the external force; μ_1 and μ_2 are constant viscosity-coefficients satisfying $\mu_1 > 0$ and $2\mu_1 + n\mu_2 > 0$.

It is assumed that the external force f is a spherically symmetric potential force and the initial data is also spherically symmetric. Namely, for $r := |\xi|$

$$(A1) \quad f := -\nabla U = \frac{\xi}{r} U_r(r), \quad U_r \in C^1[1, \infty), \quad \exists U_+ = \lim_{r \rightarrow \infty} U(r) = \lim_{r \rightarrow \infty} \int_1^r U_r(\eta) d\eta,$$

$$(A2) \quad \rho_0(x) = \hat{\rho}_0(r), \quad u_0(\xi) = \frac{\xi}{r} \hat{u}_0(r).$$

Under the assumptions (A1) and (A2), it is shown in [4] that the solution (ρ, u) is spherically symmetric. Here, the spherically symmetric solution is a solution to (1) in the form of

$$\rho(\xi, t) = \hat{\rho}(r, t), \quad u(\xi, t) = \frac{\xi}{r} \hat{u}(r, t). \quad (2)$$

Substituting (2) in (1), we reduce the system (1) to the equations for $(\hat{\rho}, \hat{u})(r, t)$. Hereafter, we omit the hat to express a spherically symmetric function without confusion. Then, the spherically symmetric solution $(\rho, u)(r, t)$ satisfies the system of equations

$$\rho_t + \frac{(r^{n-1} \rho u)_r}{r^{n-1}} = 0, \quad (3a)$$

$$\rho(u_t + uu_r) = \mu \left\{ \frac{(r^{n-1} u)_r}{r^{n-1}} \right\}_r - p(\rho)_r - \rho U_r, \quad (3b)$$

where $\mu := 2\mu_1 + \mu_2$ is supposed to be positive. The initial data to (3) is prescribed to be asymptotically constant in space:

$$\rho(r, 0) = \rho_0(r) > 0, \quad u(r, 0) = u_0(r), \quad \lim_{r \rightarrow \infty} (\rho(r, t), u(r, t)) = (\rho_+, u_+), \quad \rho_+ > 0. \quad (4)$$

As we interested in the behavior of gas around a solid sphere, an adhesion boundary condition is adopted:

$$u(1, t) = 0. \quad (5)$$

In addition, it is assumed that the initial data (4) is compatible with the boundary data (5). Since the characteristic speed of (3a) is zero on the boundary due to (5), one boundary condition is necessary and sufficient for the wellposedness of the initial boundary value problem (3), (4) and (5).

This problem is formulated to study the behavior of compressible viscous gas around the solid sphere in a field of external force. We show that the time asymptotic state of the

solution to the initial boundary value problem (3), (4), (5) is the stationary solution, which is a solution to (3) independent of time t , satisfying the same conditions (4) and (5). Hence, the stationary solution $(\bar{\rho}(r), \bar{u}(r))$ satisfies the system of equations

$$\frac{1}{r^{n-1}}(r^{n-1}\bar{\rho}\bar{u})_r = 0, \quad (6a)$$

$$\bar{\rho}\bar{u}\bar{u}_r = \mu \left\{ \frac{(r^{n-1}\bar{u})_r}{r^{n-1}} \right\} - p(\bar{\rho})_r - \bar{\rho}U_r \quad (6b)$$

and the boundary and the spatial asymptotic conditions

$$\bar{u}(1) = 0, \quad \lim_{r \rightarrow \infty} (\bar{\rho}(r), \bar{u}(r)) = (\rho_+, u_+). \quad (7)$$

Solving (6), we obtain an explicit formula of the stationary solution $(\bar{\rho}(r), \bar{u}(r))$:

$$\bar{u}(r) \equiv 0, \quad (8a)$$

$$\bar{\rho}(r) = \begin{cases} \left\{ \rho_+^{\gamma-1} + \frac{\gamma-1}{K\gamma}(U_+ - U(r)) \right\}^{\frac{1}{\gamma-1}} & \text{for } \gamma > 1, \\ \rho_+ \exp \left\{ \frac{1}{K}(U_+ - U(r)) \right\} & \text{for } \gamma = 1. \end{cases} \quad (8b)$$

Due to (8a), the spatial asymptotic data in (7) must satisfy $u_+ = 0$ for the existence of the stationary solution. The stability theorem on the stationary solution in (8) is summarized in the next theorem, which is the main result in the present research.

Theorem 1. *Suppose the initial data satisfies that for a certain $\sigma \in (0, 1)$*

$$\rho_0 \in \mathcal{B}_{\text{loc}}^{1+\sigma}[1, \infty), \quad u_0 \in \mathcal{B}_{\text{loc}}^{2+\sigma}[1, \infty), \quad (9a)$$

$$r^{\frac{n-1}{2}}(\rho_0 - \bar{\rho}), \quad r^{\frac{n-1}{2}}u_0, \quad r^{\frac{n-1}{2}}(\rho_0 - \bar{\rho})_r, \quad r^{\frac{n-1}{2}}u_{0r} \in L^2(1, \infty) \quad (9b)$$

and the compatibility condition holds. In addition, if there exists a positive constant δ , depending only on the initial data, such that $-\delta \leq U_r(r)$, then the initial boundary value problem (3), (4) and (5) has a unique solution (ρ, u) globally in time and the solution converges to the corresponding stationary solution. Precisely, it holds that

$$\lim_{t \rightarrow \infty} \sup_{r \in [1, \infty)} |(\rho(r, t) - \bar{\rho}(r), u(r, t))| = 0. \quad (10)$$

Notice that any smallness assumptions on the initial data is not necessary for the above stability theorem. Moreover, if $U_r \geq 0$, then U_r can be arbitrarily large. This is the case that the external force is attractive like the gravitational force. The Hölder continuity of the initial

data (9a) is necessary to ensure the validity of the transformation between the Eulerian and the Lagrangian coordinates. (See (11) below.) Actually, we show the asymptotic stability of the stationary solution in the Lagrangian without the Hölder continuity. In translating this result to that in the Eulerian coordinate, we need the differentiability of solutions. This is the reason we assume (9a), which gives the Hölder continuity of the solution with the aid of the Schauder theory for parabolic equations. The remainder of the present paper is devoted to a brief outline of the proof of Theorem 1. The readers are referred to the paper [11] for the detailed discussions.

In the proof of Theorem 1, we show the uniform a priori estimate by employing the energy method. For this purpose, it is convenient to adopt the Lagrangian coordinate rather than the Eulerian coordinate. The transformation from the Eulerian coordinate (r, t) to the Lagrangian coordinate (x, t) is executed by the relation:

$$x = \int_1^r \eta^{n-1} \rho(\eta, t) d\eta, \quad r_t = u, \quad r_x = \frac{v}{r^{n-1}}, \quad (11)$$

where $v = 1/\rho$ is the specific volume. Using (11), we deduce the system (3) to

$$v_t = (r^{n-1}u)_x, \quad (12a)$$

$$u_t = \mu r^{n-1} \left(\frac{(r^{n-1}u)_x}{v} \right)_x - r^{n-1} p_x - U_r. \quad (12b)$$

The initial and the boundary conditions for (v, u) are derived from (4) and (5) as

$$v(x, 0) = v_0(x) := 1/\rho_0(x), \quad u(x, 0) = u_0(x), \quad \lim_{x \rightarrow \infty} v_0(x) = v_+ := 1/\rho_+, \quad (13)$$

$$u(0, t) = 0. \quad (14)$$

The spatial variable r in the Eulerian coordinate is regarded as a function of (x, t) in the Lagrangian coordinate. Thus, the density $\bar{\rho}$ in the stationary solution also depends on (x, t) , that is, $\bar{\rho}(x, t) := \bar{\rho}(r(x, t))$. Also, let $\bar{\rho}_0(x) := \bar{\rho}(r(x, 0))$.

Define the energy form \mathcal{E} by

$$\mathcal{E} := \frac{1}{2}u^2 + \Psi(v, \bar{v}),$$

$$\Psi(v, \bar{v}) := p(\bar{v})(v - \bar{v}) - \varphi, \quad \varphi := \int_{\bar{v}}^v p(\eta) d\eta, \quad p(v) := Kv^{-\gamma}, \quad \bar{v} := 1/\bar{\rho}.$$

If $c \leq v(x, t) \leq C$ for positive constants c and C , then $\Psi(v, \bar{v})$ is equivalent to $|v - \bar{v}|^2$. Namely, $c|v - \bar{v}|^2 \leq \Psi(v, \bar{v}) \leq C|v - \bar{v}|^2$ for positive constants c and C . Then the energy form \mathcal{E} is equivalent to $|u|^2 + |v - \bar{v}|^2$.

We state several a priori estimates for the solution (v, u) without detailed proofs.

Proposition 2. (*Basic estimate*) Suppose that $v_0 - \bar{v}_0, u_0 \in L^2(0, \infty)$. Then the solution satisfies

$$\int_0^\infty \mathcal{E}(x, t) dx + \mu \int_0^t \int_0^\infty (n-1) \frac{v}{r^2} u^2 + \frac{r^{2n-2}}{v} u_x^2 dx d\tau \leq \int_0^\infty \mathcal{E}(x, 0) dx. \quad (15)$$

Applying the Sobolev inequality on (15), we have

Corollary 3.

$$\int_0^t \|(r^{n-2} u^2)(\tau)\|_\infty d\tau \leq C, \quad (16)$$

where C is a positive constant depending only on the initial data.

In order to obtain the pointwise bound for the specific volume $v(x, t)$ uniformly in time, we employ a “cut-off-function” defined by

$$\eta(x) = \begin{cases} 1, & x \leq k, \\ k+1-x, & k \leq x \leq k+1, \\ 0, & k+1 \leq x. \end{cases} \quad \text{for } k = 1, 2, \dots$$

By using (12) with the cut-off-function $\eta(x)$, we have a representation formula of the density.

Lemma 4. $v(x, t)$ is represented by

$$v(x, t)^\gamma = \frac{v_0(x)^\gamma + \frac{K\gamma}{\mu} \int_0^t A(x, \tau) B(x, \tau) d\tau}{A(x, t) B(x, t)},$$

for $x \in [k-1, k)$ and $t \geq 0$, where

$$A(x, t) := \exp\left(\frac{K\gamma}{\mu} \int_0^t \int_k^{k+1} v^{-\gamma} dx d\tau + \frac{\gamma}{\mu} \int_0^t \int_x^\infty \frac{U_r}{r^{n-1}} \eta dx d\tau\right),$$

$$B(x, t) := \exp\left(\frac{\gamma}{\mu} \int_x^\infty \left(\frac{u}{r^{n-1}} - \frac{u_0}{r_0^{n-1}}\right) \eta dx + \frac{\gamma}{\mu} \int_0^t \int_x^\infty (n-1) \frac{u^2}{r^n} \eta dx d\tau - \gamma \int_k^{k+1} \log \frac{v}{v_0} dx\right).$$

Proposition 2 and Lemma 4 yield the upper and the lower bounds of $v(x, t)$.

Proposition 5. There exist positive constants c and C , depending only on the initial data, such that

$$c \leq v(x, t) \leq C \quad (17)$$

for $x \geq 0$ and $t \geq 0$.

The estimate (17) immediately gives the pointwise bounds for the density, $0 < c \leq \rho \leq C$. To obtain the a priori estimate for the derivatives of the solution, it is convenient to use the function

$$\varphi(x, t) := \int_{\tilde{v}}^v p(\eta) d\eta.$$

Proposition 6. *Suppose that $v_0 - \tilde{v}_0$, u_0 , $r_0^{n-2}(v_0 - \tilde{v}_0)_x$, $r_0^{n-2}u_{0x} \in L^2(0, \infty)$. Then we have*

$$\int_0^\infty (r^{2n-4}\varphi_x^2)(x, t) dx + c \int_0^t \int_0^\infty (r^{2n-4}\varphi_x^2)(x, \tau) dx d\tau \leq C, \quad (18)$$

$$\int_0^\infty (r^{2n-4}u_x^2)(x, t) dx + c \int_0^t \int_0^\infty (r^{4n-6}u_{xx}^2)(x, \tau) dx d\tau \leq C, \quad (19)$$

where c and C are positive constants depending only on the initial data.

The estimate for $(v - \tilde{v})_x$ follows from (18);

$$\int_0^\infty (r^{2n-4}(v - \tilde{v})_x^2)(x, t) dx \leq C. \quad (20)$$

Using the estimates (15), (18), (19) and (20), we show the global existence of the solution (v, u) in the Lagrangian coordinate. Moreover, these estimates give the asymptotic stability of the solution in (10). In these discussions, the Hölder continuity of the solution is not necessary. It is used to ensure the validity of the translation of these results to those in Eulerian coordinate. Actually, we show that, by applying the Schauder theory for the parabolic equations (see [2]), the solution (ρ, u) is also belonging to the Hölder space if the initial data satisfies (9a). It immediately gives the corresponding stability theorem in the Eulerian coordinate.

Related results. The Navier–Stokes equation has been attracting interests of a lot of researchers in the fields of not only physics but also mathematics for these decays. Thus, we have so many preceding researches and have to restrict ourselves to a certain problem. Here we mainly state several results on the spherically symmetric motion of compressible viscous fluid in an exterior domain.

The first of all, we need to mention the book [1] written by S. N. Antontsev, A. V. Kazhikhov, and V. N. Monakhov, which gives comprehensive introduction to the mathematical theory of the compressible and viscous fluid. The first notable research in the equations on the exterior domain is given by A. Matsumura and T. Nishida in [9], where the stability of the stationary solution is first proved under the smallness assumptions. Note that this research covers more general solutions on more general domain than the present research, studying the spherically symmetric solution on the exterior domain.

Another pioneering work is given by N. Itaya [4], which establishes the existence of the spherically symmetric solution globally in time on a bounded annulus domain without smallness assumptions on the initial data. This paper has drawn attention of the researchers to the spherically symmetric solution. Then, A. Matsumura in [8] shows that the spherically symmetric solution to the isothermal model with external force on the annulus domain exists globally in time and it converges to the corresponding stationary solution as time tends to infinity. Moreover, it shows that the convergence is exponentially fast. The research by K. Higuchi in [3] extends this result to the isentropic model. In addition, it considers the equations of heat-conductive ideal gas on the annulus domain. The present research aims to extend the results in [8] and [3] to those on an unbounded exterior domain.

The study of the spherically symmetric solution over an unbounded exterior domain is started by S. Jiang in [5], where the global existence of the solution is established for the model of heat-conductive ideal gas. Moreover, the partial result on the asymptotic state is obtained. Precisely, it shows that, for the space dimension $n = 3$, $\|u(t)\|_{2j} \rightarrow 0$ as $t \rightarrow \infty$, where $j (\geq 2)$ is an arbitrarily fixed integer.

In the case of one dimensional space $n = 1$, the problem on the unbounded exterior domain is coincide with the half-space problem. A. Matsumura and K. Nishihara in [10] start to investigate this problem for the compressible Navier–Stokes equation. In [10], several kinds of boundary conditions are proposed. Namely, inflow, outflow and no flow boundary conditions. Then, it classifies the asymptotic behaviors of the solution into the several cases subject to the relation between the boundary data and the spatial asymptotic data. Moreover, it proves the stability theorem for some cases by using the Lagrangian coordinate. The research [6] by S. Kawashima, S. Nishibata and P. Zhu also studies the same one dimensional half space problem. It obtains the a priori estimates directly in the Eulerian coordinate and proves the stability of the stationary solution. The Hölder continuity of the solution is also discussed in [6].

Notation. For a region Ω , an integer l and $0 < \sigma < 1$, $\mathcal{B}^{l+\sigma}(\Omega)$ denotes the space of Hölder continuous functions over Ω which have the l -th order derivatives of Hölder continuity with exponent σ . $\mathcal{B}_{loc}^{l+\sigma}(\Omega)$ is the space of functions belonging to $\mathcal{B}^{l+\sigma}(\omega)$ for an arbitrarily compact set $\omega \subset \Omega$. For $1 \leq p \leq \infty$, $L^p(\Omega)$ denotes the standard Lebesgue space over Ω equipped with the norm $\|\cdot\|_p$. c and C denote several generic positive constants.

Acknowledgment. The present result is obtained through the joint research with Dr. Tohru Nakamura at Tokyo institute of Technology and Prof. Shigenori Yanagi at Ehime University. The details are published in the paper [11].

References

- [1] S. N. ANTONTSEV, A. V. KAZHIKHOV, AND V. N. MONAKHOV, *Boundary value problems in mechanics of nonhomogeneous fluids*, North Holland, 1990.
- [2] A. FRIEDMAN, *Partial differential equations of parabolic type*, Prentice Hall, 1964.
- [3] K. HIGUCHI, *Global existence of the spherically symmetric solution and the stability of the stationary solution to compressible Navier-Stokes equation*, Master thesis of Kanazawa Univ., (1992). (Japanese).
- [4] N. ITAYA, *On a certain temporally global solution, spherically symmetric, for the compressible NS equations*, The Jinbun ronshu of Kobe Univ. Comm., 21 (1985), pp. 1–10. (Japanese).
- [5] S. JIANG, *Global spherically symmetric solutions to the equations of a viscous polytropic ideal gas in an exterior domain*, Commun. Math. Phys., 178 (1996), pp. 339–374.
- [6] S. KAWASHIMA, S. NISHIBATA AND P. ZHU, *Asymptotic stability of the stationary solution to the compressible Navier-Stokes equations in the half space*, to appear in Comm. Math. Phys..
- [7] A. V. KAZHIKHOV AND V. V. SHELUKHIN, *Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas*, J. Appl. Math. Mech., 41 (1977), pp. 273–282.
- [8] A. MATSUMURA, *Large-time behavior of the spherically symmetric solutions of an isothermal model of compressible viscous gas*, Trans. theorem and statist. phys., 21 (1992), pp. 579–592.
- [9] A. MATSUMURA AND T. NISHIDA, *Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids*, Commun. Math. Phys., 89 (1983), pp. 445–464.
- [10] A. MATSUMURA AND K. NISHIHARA, *Large-time behaviors of solutions to an inflow problem in the half space for a one-dimensional system of compressible viscous gas*, Comm. Math. Phys., 222 (2001), pp. 449–474.
- [11] T. NAKAMURA, S. NISHIBATA AND S. YANAGI, *Large-time behaviors of spherically symmetric solutions to an isentropic model of compressible viscous fluids in a field of external forces*, to appear.
- [12] S. YANAGI, *Asymptotic stability of the spherically symmetric solutions for a viscous polytropic gas in a field of external forces*, Trans. theorem and statist. phys., 29 (2000), pp. 333–353.

Conical Shock Waves in Supersonic Flow

Dening Li

Department of Mathematics, West Virginia University, Morgantown, USA

Abstract

1 Physical background of shock waves

1. A steady shock front is produced as supersonic flow passes a solid projectile. The most important two typical configurations of the flying projectiles:
 - Long wing: two dimensional object;
 - Conical head: three dimensional object.
2. Depending upon the shape of the leading front of solid object, the shock front will be
 - detached from the projectile if the projectile has a blunt head;
 - attached to to the head of the projectile if the projectile has a narrow and sharp pointed head.
3. It is of practical importance because the sharp jump of pressure across shock front produces great resistance to the flying object and therefore is to be avoided.
4. We study the case of steady shock wave attached to sharp pointed front of projectile.

2 Mathematical models

1. Euler system of equations: quasi-linear, hyperbolic,

$$\begin{cases} \partial_t \rho + \sum_{j=1}^3 \partial_{x_j} (\rho v_j) = 0, \\ \partial_t (\rho v_i) + \sum_{j=1}^3 \partial_{x_j} (\rho v_i v_j + \delta_{ij} p) = 0, \quad i = 1, 2, 3 \\ \partial_t (\rho E) + \sum_{j=1}^3 \partial_{x_j} (\rho v_j E + p v_j) = 0. \end{cases} \quad (1)$$

ρ - density, \mathbf{v} - velocity, $E = e + \frac{1}{2}|\mathbf{v}|^2$ - total energy, $p = p(\rho, E)$ - pressure.

2. Various simplified models to (1) can be introduced.

(a) Linearization: small perturbation.

(b) Geometrical simplification:

- One dimensional model: shock transition relations and Lax shock inequality.
- Geometrically symmetric model: cylindrical and spherical model.

(c) Steady flow model: time-independent flow.

(d) Thermodynamical simplification:

- Polytropic gas model: $p = A\rho^\gamma$;
- Isentropic model: $p = p(\rho)$;
- Irrotational model: $\nabla \times \mathbf{v} = 0$.

3 Conical shock wave for steady irrotational and isentropic flow

1. Irrotational flow: $\nabla \times \mathbf{v} = 0$ implies $\mathbf{v} = \nabla\phi$, ϕ - velocity potential.

2. Second order scalar equation for ϕ

$$\begin{aligned} \left(\frac{v_1^2}{a^2} - 1\right) \phi_{x_1 x_1} + \left(\frac{v_2^2}{a^2} - 1\right) \phi_{x_2 x_2} + \left(\frac{v_3^2}{a^2} - 1\right) \phi_{x_3 x_3} \\ + \frac{2v_1 v_2}{a^2} \phi_{x_1 x_2} + \frac{2v_1 v_3}{a^2} \phi_{x_1 x_3} + \frac{2v_2 v_3}{a^2} \phi_{x_2 x_3} = 0. \end{aligned} \quad (2)$$

3. (2) is hyperbolic in the region of supersonic flow ($|\mathbf{v}| > a$), and is elliptic in the region of subsonic flow ($|\mathbf{v}| < a$).

4. Symmetric conical shock wave:

- shock polar and apple curve;
- weak and strong shock.

5. Theorems: Linear stability of conical shock wave and existence.

6. Mathematical tools: conical coordinates, energy estimate for linearized problem, linear iteration

7. Generalized hodograph transformation: to transform the domain with free boundary into a fixed annular region.

4 Stability for oblique shock wave

My current work on the stability of oblique shock waves for isentropic system of Euler equations.

1. Boundary value problem for $m \times m$ hyperbolic system:

$$\begin{cases} \partial_t u + \sum_{j=1}^n A_j \partial_{x_j} u + C u = f, & \text{in } x_1 > 0; \\ P u = g & \text{on } x_1 = 0. \end{cases} \quad (3)$$

2. Well-posedness of (3): if there is energy estimate

$$\eta \|u\|_\eta^2 + |u|_\eta^2 \leq C_0 \left(\frac{1}{\eta} \|f\|_\eta^2 + |g|_\eta^2 \right) \quad (4)$$

$$\|u\|_\eta = \left(\int_{R^1} \int_{R^{n-1}} \int_0^\infty e^{2\eta t} |u(t, x_1, x')|^2 dx_1 dx' dt \right)^{\frac{1}{2}},$$

$$|u|_\eta = \left(\int_{R^1} \int_{R^{n-1}} e^{2\eta t} |u(t, 0, x')|^2 dx' dt \right)^{\frac{1}{2}}.$$

3. Kreiss' condition for well-posedness:

- $$M(s, i\omega) = -A_1^{-1} \left(sI + i \sum_{j=2}^n \omega_j A_j \right), \quad (5)$$

with $s = \eta + i\tau$ and $\omega \in R^{n-1}$.

- Eigenvectors and generalized eigenvectors with negative eigenvalues: u_j .
 - Kreiss' condition: $P u_j$ are uniformly linearly independent.
4. Analysis of stability of oblique shock front.
 - Oblique shock front is stable for weak shock.
 - Mathematical condition and its physical implication.

Remarks on global solvability of 2-D Boussinesq equations with nondecaying initial data

Yasushi TANIUCHI Shinshu Uni. Matsumoto 390-8621, taniuchi@math.shinshu-u.ac.jp

1. INTRODUCTION

We consider the natural convection in a viscous incompressible fluid described by the Boussinesq equations:

$$(B) \quad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = g\theta & \text{in } t > 0 \text{ and } x \in \mathbf{R}^n, \\ \partial_t \theta - \Delta \theta + u \cdot \nabla \theta = 0 & \text{in } t > 0 \text{ and } x \in \mathbf{R}^n, \\ \operatorname{div} u = 0 & \text{in } t > 0 \text{ and } x \in \mathbf{R}^n, \\ u|_{t=0} = u_0, \quad \theta|_{t=0} = \theta_0 & \text{in } x \in \mathbf{R}^n, \end{cases}$$

where $u = (u^1(x, t), u^2(x, t), \dots, u^n(x, t))$, $\theta = \theta(x, t)$ and $p = p(x, t)$ denote the unknown velocity vector field, the unknown temperature and the unknown pressure of the fluid at the point $(x, t) \in \mathbf{R}^n \times (0, \infty)$, respectively. (u_0, θ_0) is a given initial data and g is the given constant vector which denotes acceleration of gravity.

Many researchers have investigated the Boussinesq equations. See e.g. Cannon–DiBenedetto[1], Fife–Joseph[5], Foias–Mainley–Tennam[4], Morimoto[14][15], Oeda[16][17], Hishida[6], Hishida–Yamada[7], Kagei[10], Kagei–von Wahl[11] and papers cited there.

These results, however, are imposed the integrability condition on the initial data $u_0, \theta_0 \in L^q$ for some $q < \infty$. This condition implies that $u_0(x)$ and $\theta_0(x)$ decay at infinity in some sense. On the other hand, Giga–Inui–Matsui[8], Cannon–Knightly[2] and Cannone[3] showed the time-local existence of solution to the Navier–Stokes equations:

$$(N-S) \quad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0 & \text{in } t > 0 \text{ and } x \in \mathbf{R}^n, \\ u|_{t=0} = u_0 & \text{in } x \in \mathbf{R}^n \end{cases}$$

with nondecaying initial data $u_0 \in L^\infty$. It is notable that in the two dimensional case Giga–matsui–Sawada[9] proved the unique global solvability of (N-S) with $u_0 \in L^\infty(\mathbf{R}^2)$.

In [18], we proved the unique local existence theorem for the Boussinesq equations with nondecaying initial data. To be precise, we showed that if $u_0 \in L^\infty$ with $\operatorname{div} u = 0$ in \mathcal{D}' and $\theta_0 \in \dot{B}_{\infty,1}^0$, then there exists the strong solution

$$(1.1) \quad (u, \theta) \in C_w([0, T]; L^\infty(\mathbf{R}^n)) \times C([0, T]; \dot{B}_{\infty,1}^0(\mathbf{R}^n))$$

to the Boussinesq equations with $\nabla p \in C([0, T]; \dot{B}_{\infty,1}^0(\mathbf{R}^n))$. We note that the space $\dot{B}_{\infty,1}^0$ includes some nondecaying functions, for example, $\sin(a \cdot x) + (1 + |x_1|^2)^{-1}$. Moreover, in [18] it was shown that there exists a unique global solutions to the 2-D Boussinesq equations when the initial data $(u_0, \theta_0) \in L^\infty(\mathbf{R}^2) \times L^p(\mathbf{R}^2)$ for some $p \in (2, \infty)$. Although the initial velocity need not to be decay at infinity, the initial temperature θ_0 was restricted to decay at infinity.

The purpose of this paper is to remove this decay condition on θ_0 . Now we state our main Theorem.

Theorem 1 (global existence). *Let the initial data $(u_0, \theta_0) \in L^\infty(\mathbf{R}^2) \times \dot{B}_{\infty,1}^0(\mathbf{R}^2)$ with $\operatorname{div} u_0 = 0$. Then there exists a unique solution $(u, \theta) \in C_w([0, \infty); L^\infty(\mathbf{R}^2)) \times C([0, \infty); \dot{B}_{\infty,1}^0(\mathbf{R}^2))$ to (B) with $\nabla p \in C((0, \infty); \dot{B}_{\infty,1}^0)$.*

Remark. $\dot{B}_{\infty,1}^0$ contains some nondecaying functions as well as L^∞ . For example, $\sin(a \cdot x)$ and $\cos(b \cdot x) \in \dot{B}_{\infty,1}^0$, where a and b are constant vectors and $b \neq 0$. Furthermore, periodic functions $f \in W^{1,\infty}(T)$ with $\int_T f dx = 0$ belongs to $\dot{B}_{\infty,1}^0(\mathbf{R}^n)$. We observe this fact by the Fourier series. We also see that if $f_1(x_1, x_2, \dots, x_{n-1}) \in W^{1,\infty}(\mathbf{R}^{n-1})$ and $f_2(x_n) \in W^{1,\infty}(\mathbf{R}) \cap L^p(\mathbf{R})$ ($p < \infty$), then $f_1 f_2 \in \dot{B}_{\infty,1}^0(\mathbf{R}^n)$.

In two dimension, taking rotation to first equation of (B), we get the rotation equation

$$(Rot) \quad \omega_t - \Delta \omega + u \cdot \nabla \omega = \operatorname{rot}(g\theta),$$

where $\omega = \operatorname{rot} u$ is a scalar function. If $\theta \equiv 0$, the maximum principle yields $\|\omega(t)\|_\infty \leq \|\omega(0)\|_\infty$. In [9], this estimate plays an important role in proving the global solution to the 2-D Navier-Stokes equations. We, however, can not apply the maximum principle for the rotation equation directly, nevertheless, $\partial \theta$ is bounded. To overcome this difficulty, we establish the following uniformly local L^p estimate for the vorticity equation.

Lemma 1.1. *Let $u \in L^\infty(0, T; W^{1,\infty}(\mathbf{R}^2))$ with $\nabla \cdot u = 0$ and let $v \in L^\infty(0, T; L^\infty(\mathbf{R}^2))$ be a solution to the 2 dimensional vorticity equation*

$$(1.2) \quad \frac{\partial}{\partial t} v - \Delta v + u \cdot \nabla v = f + \partial_j g, \quad \text{in } \mathbf{R}^2 \times (0, T), \quad v|_{t=0} = v_0.$$

Then there holds for all $t \in [0, T]$ and all $p \geq 2$

$$(1.3) \quad \|v(t)\|_{L_{ul}^p} \leq C \left(1 + t + \int_0^t \|u(\tau)\|_\infty d\tau \right)^{2/p} \left\{ \|v_0\|_{L_{ul}^p} + \int_0^t \|f\|_{L_{ul}^p} d\tau + p^{1/2} \left(\int_0^t \|g(\tau)\|_{L_{ul}^p}^2 d\tau \right)^{1/2} \right\}$$

where C is an absolute constant and $\|f\|_{L_{ul}^p} \equiv \sup_x \left(\int_{|x-y|<1} |f(y)|^p dy \right)^{1/p}$.

2. FUNCTION SPACES

Before proving the main results, we first recall definition of the homogeneous Besov space. For integer j , let φ_j be the Littlewood–Paley decomposition satisfying $\hat{\varphi}_j(\xi) = \hat{\varphi}_0(2^{-j}\xi) \in C_0^\infty(\mathbf{R}^n)$, $\operatorname{supp} \hat{\varphi}_0 \subset \{1/2 < |\xi| < 2\}$ and $\sum_{j=-\infty}^\infty \hat{\varphi}_j(\xi) = 1$ excepting $\xi = 0$.

Definition 1 *The homogeneous Besov space $\dot{B}_{p,q}^s = \dot{B}_{p,q}^s(\mathbf{R}^n)$ is introduced by*

$$\dot{B}_{p,q}^s \equiv \{f \in \mathcal{Z}'; \|f\|_{\dot{B}_{p,q}^s} < \infty\}$$

for $s \in \mathbf{R}$, $1 \leq p, q \leq \infty$, where

$$\|f\|_{\dot{B}_{p,q}^s} \equiv \begin{cases} \left[\sum_{j=-\infty}^{\infty} (2^{js} \|\varphi_j * f\|_p)^q \right]^{1/q} & \text{if } q < \infty, \\ \sup_{-\infty < j < \infty} 2^{js} \|\varphi_j * f\|_p & \text{if } q = \infty. \end{cases}$$

Here $Z \equiv \{f \in \mathcal{S}; D^\alpha \hat{f}(0) = 0, \forall \alpha \text{ is multi-index}\}$.

If the exponents satisfy the following condition:

$$(2.1) \quad \text{either } s < n/p \quad \text{or} \quad s = n/p \quad \text{and} \quad q = 1,$$

then $\dot{B}_{p,q}^s$ can be regarded as subspace of \mathcal{S}' . To be precise, there holds

$$(2.2) \quad \dot{B}_{p,q}^s \cong \{f \in \mathcal{S}'; \|f\|_{\dot{B}_{p,q}^s} < \infty \text{ and } f = \sum_{j=-\infty}^{\infty} \varphi_j * f \text{ in } \mathcal{S}'\},$$

if s, p and q satisfy (2.1). For the details one can see e.g. Kozono-Yamazaki[13, Proposition 2.10].

In what follows, we deal with (2.2) as the definition of $\dot{B}_{p,q}^s$, when s, p and q satisfy (2.1). We note that any polynomial excepting 0 no longer belongs to $\dot{B}_{p,q}^s$, when s, p and q satisfy (2.1).

Finally, we give some definitions. Let $B(x, r)$ denote the ball centered at x of radius r and let

$$\|f\|_{p;\Omega} \equiv \left(\int_{y \in \Omega} |f(y)|^p dy \right)^{1/p}, \quad \|f\|_{p,\lambda} \equiv \sup_{x \in \mathbf{R}^2} (\|f\|_{p;B(x,\lambda)}) = \sup_x \left(\int_{|x-y| < \lambda} |f(y)|^p dy \right)^{1/p},$$

$$\|f\|_{L_{ul}^p} \equiv \|f\|_{p,1} = \sup_x \left(\int_{|x-y| < 1} |f(y)|^p dy \right)^{1/p}, \quad L_{ul}^p \equiv \{f \in L_{loc}^1; \|f\|_{p,1} < \infty\}.$$

3. PRELIMINARY

To prove Theorem 1 and Lemma 1.1, we need the followings.

Lemma 3.1 (Sawada-Taniuchi[18]). *Assume that the initial data $(u_0, \theta_0) \in L^\infty(\mathbf{R}^n) \times \dot{B}_{\infty,1}^0(\mathbf{R}^n)$ with $\text{div } u_0 = 0$ in the sense of distribution. Then there exists $T > 0$ and a unique solution $(u, \theta, \nabla p)$ to (B) with*

$$(3.1) \quad u \in C_w([0, T]; L^\infty(\mathbf{R}^n)) \cap C^1((0, T); L^\infty(\mathbf{R}^n)) \cap C((0, T); W^{2,\infty}(\mathbf{R}^n)),$$

$$(3.2) \quad \theta \in C_w([0, T]; \dot{B}_{\infty,1}^0) \cap C^1((0, T); L^\infty(\mathbf{R}^n)) \cap C((0, T); W^{2,\infty}(\mathbf{R}^n)),$$

$$(3.3) \quad \nabla p \in C((0, T); \dot{B}_{\infty,1}^0)$$

$$(3.4) \quad \partial_t u - \Delta u + P(u \cdot \nabla u) = P(g\theta) \quad \text{in } L^\infty,$$

$$(3.5) \quad \partial_t \theta - \Delta \theta + u \cdot \nabla \theta = 0 \quad \text{in } L^\infty$$

$$(3.6) \quad \nabla p = (1 - P)(u \cdot \nabla u + g\theta).$$

Here $P = \{P_{kl}\}_{1 \leq l, k \leq n} = \{\delta_{kl} + R_l R_k\}_{1 \leq l, k \leq n}$. Moreover there exists $C = C(n)$ such that

$$(3.7) \quad T > \frac{C}{(\|u_0\|_\infty + \|\theta_0\|_{\dot{B}_{\infty,1}^0})^2 + |g| + 1}.$$

Remark. To solve (B) we reduce to construct a solution (u, θ) to the integral equations:

$$(I.E.B)_1 \quad u(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-\tau)\Delta} P(u \cdot \nabla u)(\tau) d\tau + \int_0^t e^{(t-\tau)\Delta} P(g\theta)(\tau) d\tau$$

$$(I.E.B)_2 \quad \theta(t) = e^{t\Delta} \theta_0 - \int_0^t e^{(t-\tau)\Delta} (u \cdot \nabla \theta)(\tau) d\tau.$$

Proposition 3.1. *There holds for all $f \in L^\infty$*

$$(3.8) \quad \|e^{t\Delta} f\|_{\dot{B}_{\infty,1}^s} \leq C(n, s) t^{-s/2} \|f\|_\infty \quad \text{for } t > 0 \text{ and } s > 0.$$

Proposition 3.2. *If $m \geq 1$, then*

$$\|f\|_{q, m\lambda} \leq (2m^2)^{1/q} \|f\|_{q, \lambda} \quad \text{for all } f \in L_{ul}^q(\mathbf{R}^2), \lambda > 0.$$

Proposition 3.3. *Let $1 \leq q \leq \infty$, $j = 0, \pm 1, \pm 2, \dots$, $\phi \in \mathcal{S}$ and let $f \in L_{ul}^q(\mathbf{R}^2)$. Then there holds*

$$(3.9) \quad \|\phi_j * f\|_\infty \leq \begin{cases} C 2^{2j/q} \|f\|_{q,1} & \text{for all } j \geq 0, \\ C \|f\|_{q,1} & \text{for all } j \leq 0, \end{cases}$$

where C is independent of q, j and f . Here $\phi_j(\cdot) = 2^{2j} \phi(2^j \cdot)$.

Proposition 3.4 (Gronwall's inequality). *Let A_1, A_2 and A_3 be nonnegative function on $(0, T)$ with $\int_0^T A_j(t) dt < \infty$ for $j = 1, 2, 3$, and let A_0, γ_1, γ_2 be positive constants with $0 < \gamma_2 < \gamma_1 < 1$. Assume that $z \in C([0, T])$ and*

$$0 \leq z(t) \leq A_0 + \int_0^t A_1(\tau) z(\tau)^{1-\gamma_1} d\tau + \int_0^t A_2(\tau) z(\tau)^{1-\gamma_2} d\tau + \int_0^t A_3(\tau) z(\tau) d\tau \quad \text{for all } 0 \leq t < T.$$

Then

$$z(t) \leq \left\{ \left(A_0^{\gamma_1} + \gamma_1 \int_0^t A_1(\tau) d\tau \right)^{\frac{\gamma_2}{\gamma_1}} + \gamma_2 \int_0^t A_2(\tau) d\tau \right\}^{\frac{1}{\gamma_2}} \exp \int_0^t A_3(\tau) d\tau \quad \text{for all } 0 \leq t < T.$$

4. OUTLINE OF THE PROOF OF L_{ul}^p ESTIMATE FOR THE VORTICITY EQUATION

In this section we sketch the proof of Lemma 1.1. We first fix $x \in \mathbf{R}^2$ and $\lambda \geq 1$. Let $\rho \in C_0^\infty(\mathbf{R}^2)$ with $\rho(y) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2 \end{cases}$, $\|\nabla \rho\|_\infty \leq 2$, $\|\Delta \rho\|_\infty \leq 4$ and let $\rho_{x,\lambda}(y) = \rho(\frac{y-x}{\lambda})$. We easily see that

$$(4.10) \quad \begin{aligned} & \frac{\partial}{\partial t}(\rho_{x,\lambda} v) - \Delta(\rho_{x,\lambda} v) + u \cdot \nabla(\rho_{x,\lambda} v) \\ &= \rho_{x,\lambda} f + \rho_{x,\lambda} \partial_j g - 2 \nabla \rho_{x,\lambda} \cdot \nabla v - (\Delta \rho_{x,\lambda}) v + u \cdot (\nabla \rho_{x,\lambda}) v. \end{aligned}$$

Taking inner product in $L^2(\mathbf{R}^2)$ between (4.10) and $|\rho_{x,\lambda}v|^{p-2}\rho_{x,\lambda}v$ and taking supremum of it over $x \in \mathbf{R}^2$, we obtain

$$(4.11) \quad \begin{aligned} \|v(t)\|_{p,\lambda}^p &\leq 8 \left\{ \|v_0\|_{p,\lambda}^p + p \int_0^t \|f\|_{p,\lambda} \|v\|_{p,\lambda}^{p-1} d\tau + \frac{p(p+1)}{2} \int_0^t \|g\|_{p,\lambda}^2 \|v\|_{p,\lambda}^{p-2} d\tau \right. \\ &\quad \left. + p \int_0^t \left(\frac{264}{\lambda^2} + \frac{2\|u(\tau)\|_\infty}{\lambda} \right) \|v(\tau)\|_{p,\lambda}^p d\tau \right\}, \end{aligned}$$

where we used to

$$\|f\|_{p,2\lambda}^p \leq 8 \|f\|_{p,\lambda}^p, \quad \text{and} \quad \sup_{x \in \mathbf{R}^2} \left(\int_0^t \int_{B(x,2\lambda)} |f| dy d\tau \right) \leq 8 \sup_{x \in \mathbf{R}^2} \left(\int_0^t \int_{B(x,\lambda)} |f| dy d\tau \right).$$

Then Proposition 3.4 with $z(t) = \|v(t)\|_{p,\lambda}^p$ and Proposition 3.2 yield

$$(4.12) \quad \begin{aligned} \|v(t)\|_{p,1} &\leq \|v(t)\|_{p,\lambda} \\ &\leq (2\lambda^2)^{1/p} \left\{ 8^{1/p} \|v_0\|_{p,1} + 8^{1/2}(p+1)^{1/2} \left(\int_0^t \|g\|_{p,1}^2 d\tau \right)^{1/2} + \int_0^t 8 \|f\|_{p,1} d\tau \right\} \\ &\quad \times \exp \left\{ \frac{8 \cdot 264}{\lambda^2} t + \frac{16}{\lambda} \int_0^t \|u(\tau)\|_\infty d\tau \right\}. \end{aligned}$$

Now let $s \in (0, T]$ and set $\lambda = 1 + s + \int_0^s \|u(\tau)\|_\infty d\tau$. Then there holds

$$\|v(t)\|_{p,1} \leq C \left(1 + s + \int_0^s \|u(\tau)\|_\infty d\tau \right)^{2/p} \left\{ \|v_0\|_{p,1} + p^{1/2} \left(\int_0^t \|g\|_{p,1}^2 d\tau \right)^{1/2} + \int_0^t \|f\|_{p,1} d\tau \right\}$$

for all $0 < t \leq s \leq T$. Since $s \in (0, T]$ is arbitrary, we get the desired estimate (1.3).

5. OUTLINE OF THE PROOF OF THEOREM 1

In this section we shall sketch the proof of Theorem 1. We use the similar method as in [20]. (See also Serfati[19].) We show that the time-local solution given in Lemma 3.1 can be extended to a global one. Thanks to (3.1) and (3.2), we may assume $u_0 \in W^{2,\infty}$, $\theta_0 \in \dot{B}_{\infty,1}^0 \cap W^{2,\infty}$ and assume $(u, \theta) \in C([0, T]; W^{2,\infty}) \times C([0, T]; W^{2,\infty} \cap B_{\infty,1}^0)$. It suffices to show

$$(5.1) \quad \sup_{0 < t < T} (\|u(t)\|_\infty + \|\theta(t)\|_{\dot{B}_{\infty,1}^0}) < \infty \quad \text{if } T < \infty.$$

By the equations (3.5) and $(I.E.B)_2$ we have

$$(5.2) \quad \|\theta(t)\|_\infty \leq \|\theta_0\|_\infty,$$

$$(5.3) \quad \|\theta(t)\|_{\dot{B}_{\infty,1}^0} \leq \|\theta_0\|_{\dot{B}_{\infty,1}^0} + Ct^{1/2} \left(\sup_{0 \leq \tau \leq t} \|u(\tau)\|_\infty \right) \|\theta_0\|_\infty.$$

Therefore for (5.1) we only need to show

$$(5.4) \quad \sup_{0 \leq \tau < T} \|u(\tau)\|_\infty < \infty, \quad \text{if } T < \infty.$$

Here we recall that the Littlewood-Paley decomposition:

$$1 = \hat{\psi}_N(\xi) + \sum_{j \geq N} \hat{\varphi}_j(\xi) \quad (\xi \in \mathbf{R}^2, N = 0, \pm 1, \pm 2, \dots),$$

where $\psi_N = 2^{2N} \psi(2^N \cdot) = \mathcal{F}^{-1}(1 - \sum_{j=N} \hat{\varphi}_j)$. We easily see that

$$(5.5) \quad \|u(t)\|_\infty \leq \|\psi_N * u(t)\|_\infty + \sum_{j=N}^{\infty} \|\phi_j * u(t)\|_\infty \equiv J_1 + J_2$$

By $(I.B.E)_1$ and $(I.E.B)_2$ we have for all $0 \leq s \leq t < T$

$$(5.6) \quad \begin{aligned} J_1 &= \left\| \psi_N * e^{(t-s)\Delta} u(s) + \int_s^t P \nabla \cdot \psi_N * e^{(t-\tau)\Delta} (u \otimes u)(\tau) d\tau \right. \\ &\quad \left. + \int_s^t \psi_N * e^{(t-\tau)\Delta} P \left(g e^{\tau\Delta} \theta_0 + g \int_0^\tau \nabla \cdot e^{(\tau-\tau')\Delta} (u\theta)(\tau') d\tau' \right) d\tau \right\|_\infty \\ &\leq \|u(s)\|_\infty + C_1 2^N \left(\sup_{0 \leq \tau \leq t} \|u(\tau)\|_\infty \right)^2 (t-s) \\ &\quad + C_2 |g| \|\theta_0\|_{\dot{B}_{\infty,1}^0} (t-s) + C_3 |g| 2^N \left(\sup_{0 \leq \tau \leq t} \|u(\tau)\|_\infty \right) \|\theta_0\|_\infty (t^2 - s^2), \end{aligned}$$

since $\|\nabla P \psi_N\|_{L^1} \leq \|\nabla P \psi_N\|_{\mathcal{H}^1} \leq C \|\nabla \psi_N\|_{\mathcal{H}^1} \leq C 2^N$. Here \mathcal{H}^1 denotes Hardy space.

To estimate J_2 , we use the Biot-Savart Law $(\frac{\partial}{\partial x_2} \omega, -\frac{\partial}{\partial x_1} \omega) = -\Delta u$, which yields $\|\phi_j * u\|_\infty \leq C 2^{-j} \|\phi_j * \omega\|_\infty$. Let $p \geq 4$. Then by this inequality and Proposition 3.3 we have

$$(5.7) \quad \sum_{j=N}^{\infty} \|\phi_j * u(t)\|_\infty \leq C \sum_{j=N}^{\infty} 2^{-j} \|\phi_j * \omega(t)\|_\infty \leq C 2^{-N} \max\{2^{2N/p}, 1\} \|\omega(t)\|_{L_{ul}^p}.$$

Let $s \leq t < \min\{T, s+1\}$, (i.e. $t-s < 1$). Then Lemma 1.1 and (5.7) yield

$$(5.8) \quad J_2 \leq C 2^{-N} \max\{2^{2N/p}, 1\} \left(1 + \sup_{s \leq \tau \leq t} \|u(\tau)\|_\infty \right)^{2/p} (\|\omega(s)\|_{L_{ul}^p} + p^{1/2} (t-s)^{1/2} |g| \|\theta_0\|_\infty)$$

for all $p \geq 4$ and $s \leq t < \min\{T, s+1\}$.

Let $h(t) \equiv 1 + t + \sup_{0 \leq \tau \leq t} \|u(\tau)\|_\infty$. Then Gathering the estimates (5.6) and (5.8) with (5.5), we obtain

$$(5.9) \quad \begin{aligned} h(t) &\leq h(s) + (t-s) + C_1 2^N h(t)^2 (t-s) \\ &\quad + C_2 |g| \|\theta_0\|_{\dot{B}_{\infty,1}^0} (t-s) + 2C_3 |g| 2^N h(t) \|\theta_0\|_\infty h(s) (t-s) \\ &\quad + C_4 2^{-N} \max\{2^{2N/p}, 1\} h(t)^{2/p} (\|\omega(s)\|_{L_{ul}^p} + p^{1/2} (t-s)^{1/2} |g| \|\theta_0\|_\infty) \end{aligned}$$

for all $p \geq 4$ and all $0 \leq s \leq t < \min\{T, s+1\}$.

Now we fix $s \in [0, T)$. Since $h(t)$ is a continuous function, there holds $h(t) \leq 2h(s)$ for t sufficiently close to s . Let

$$T_1(s) \equiv \sup\{\tau \in (0, T-s) ; h(s+\tau) \leq 2h(s)\}, \quad (T_1(s) \leq T-s).$$

(Since h is a nondecreasing function, $h(t) \leq 2h(s)$ for all $s \leq t < s + T_1(s)$.) Then (5.9) yields

$$(5.10) \quad h(t) < h(s) + \frac{1}{2} + C_5 2^N h(s)^2 (t-s) + C_6 2^{-N} \max\{2^{2N/p}, 1\} h(s)^{2/p} (\|\omega(s)\|_{L_{ul}^p} + p^{1/2})$$

for all $s \leq t < s + \min\{\frac{1}{4}, \frac{1}{4C_2|g|\|\theta_0\|_{\dot{B}_{\infty,1}^0}}, T_1(s)\}$ and $p \geq 4$. Here $C_5 \equiv 4C_1 + 4C_3|g|\|\theta_0\|_{\infty}$ and $C_6 = \sqrt{2}C_4 \cdot (1 + |g|\|\theta_0\|_{\infty})$. Now we choose N suitably as follows.

When $\frac{h(s)^{1-\frac{2}{p}}}{8C_6(\|\omega(s)\|_{L_{ut}^p} + p^{1/2})} \leq 1$, we choose $N \geq 0$ such as $2^{-(1-2/p)N} \sim \frac{h(s)^{1-\frac{2}{p}}}{8C_6(\|\omega(s)\|_{L_{ut}^p} + p^{1/2})}$. When $\frac{h(s)^{1-\frac{2}{p}}}{8C_6(\|\omega(s)\|_{L_{ut}^p} + p^{1/2})} > 1$, we choose $N \leq -1$ such as $2^{-N} \sim \frac{h(s)^{1-\frac{2}{p}}}{8C_6(\|\omega(s)\|_{L_{ut}^p} + p^{1/2})}$. Then the last term of (5.10) is smaller than $1/(8h(s))$. Therefore, since $\frac{1}{2} \leq \frac{4}{8}h(s)$, we obtain

$$h(t) < \frac{13}{8}h(s) + C_7 \max\left\{(\|\omega(s)\|_{L_{ut}^p} + p^{1/2})^{\frac{p}{p-2}}, (\|\omega(s)\|_{L_{ut}^p} + p^{1/2})h(s)^{\frac{2}{p}}\right\} h(s)(t-s)$$

for all $s \leq t < s + \min\{\frac{1}{4}, \frac{1}{4C_2|g|\|\theta_0\|_{\dot{B}_{\infty,1}^0}}, T_1(s)\}$ and $p \geq 4$. Here $C_7 = 2C_5(8C_6 + 1)^2$. Hence we have

$$(5.11) \quad h(t) < \frac{15}{8}h(s) \text{ for all } s \leq t < s + \min\{\delta(s, p), T_1(s)\},$$

where

$$\delta(s, p) \equiv \min\left\{\frac{1/(4C_7)}{(\|\omega(s)\|_{L_{ut}^p} + p^{1/2})^{\frac{p}{p-2}}}, \frac{1/(4C_7)}{h(s)^{2/p}(\|\omega(s)\|_{L_{ut}^p} + p^{1/2})}, \frac{1}{4}, \frac{1}{4C_2|g|\|\theta_0\|_{\dot{B}_{\infty,1}^0}}\right\}.$$

Obviously from the definition of $T_1(s)$ we obtain $T_1(s) \geq \min\{\delta(s, p), T - s\}$ and hence

$$(5.12) \quad h(t) \leq 2h(s) \text{ for all } s \leq t \leq s + \min\{\delta(s, p), T - s\}.$$

We note that this estimate holds for all $s \in (0, T)$ and $p \geq 4$.

Let $p_k = k + 4$ and $\{t_k\}_{k=0}^{\infty}$ be the increasing sequence defined by

$$t_0 \equiv 0, \quad t_{k+1} - t_k \equiv \delta(t_k, p_k).$$

Now we assume that

$$(5.13) \quad \delta(t_k, p_k) < T - t_k \text{ for all } k = 0, 1, 2, \dots, \text{ (i.e., } t_{k+1} < T \text{ for all } k = 0, 1, 2, \dots).$$

Then by (5.12) we have

$$(5.14) \quad h(t_k) \leq 2h(t_{k-1}) \leq \dots \leq 2^k h(0)$$

and hence

$$(5.15) \quad h(t_k)^{2/p_k} \leq (2^k h(0))^{2/(k+4)} \leq Ch(0)^{2/(k+4)} \leq Ch(0)^{1/2}.$$

Since $t_k = \sum_{j=0}^{k-1} \delta(t_j, p_j) \leq \sum_{j=0}^{k-1} \frac{1/(8C_7)}{p_j^{1/2}} \leq Ck^{1/2}$ and since $\|\omega(0)\|_{L_{ut}^p} \leq \pi^{2/p}\|\omega_0\|_\infty$, by Lemma 1.1 and (5.15) we see

$$\begin{aligned} \|\omega(t_k)\|_{L_{ut}^{p_k}} &\leq C \left(1 + t_k \left(1 + \sup_{0 \leq \tau \leq t_k} \|u(\tau)\|_\infty\right)\right)^{2/p_k} (\|\omega_0\|_{L_{ut}^{p_k}} + p_k^{1/2} t_k^{1/2} |g| \|\theta_0\|_\infty) \\ &\leq C [2k^{1/2} h(t_k)]^{2/p_k} (\|\omega_0\|_{L_{ut}^{p_k}} + k^{3/4} |g| \|\theta_0\|_\infty) \\ &\leq C (2k^{1/2})^{2/(k+4)} h(t_k)^{2/p_k} (\|\omega_0\|_{L^\infty} + k^{3/4} |g| \|\theta_0\|_\infty) \\ &\leq Ch(0)^{1/2} k^{3/4} (\|\omega_0\|_{L^\infty} + |g| \|\theta_0\|_\infty) \end{aligned}$$

Therefore we observe that for all large k

$$\delta(t_k, p_k) \geq C \min \left\{ \left(\frac{1}{k^{3/4}} \right)^{(k+4)/(k+2)}, \frac{1}{k^{3/4}} \right\} \geq \frac{C}{2} \left(\frac{1}{k} \right)^{\frac{3}{4}},$$

$$(5.16) \quad t_k = \sum_{j=0}^{k-1} \delta(t_j, p_j) \geq Ck^{1/4},$$

which contradicts the assumption $T < \infty$.

Hence we see that if $T < \infty$, there exists some integer m such that

$$\delta(t_m, p_m) \geq T - t_m, \quad (\text{i.e., } t_{m+1} \geq T),$$

which and (5.12) yield

$$(5.17) \quad h(t) \leq 2h(t_k) \text{ for all } t \in [t_k, T].$$

This implies the desired estimate (5.4) and proves Theorem 1.

Remark. Moreover, letting $T = \infty$, we have by (5.14) and (5.16)

$$h(Ck^{1/4}) \leq h(t_k) \leq 2^k h(0) \text{ for all } k = 1, 2, \dots,$$

which implies

$$(5.18) \quad \|u(t)\|_\infty \leq C (\|u_0\|_\infty + 1) \exp(Ct^4)$$

for all $t > 0$.

REFERENCES

- [1] Cannon, J.R., DiBenedetto, E., *The initial value problem for Boussinesq equations with data in L^p* , Approximation Methods for Navier-Stokes problems, Edited by Rautmann, R., Lect. Notes in Math., **771** Springer-Verlag, Berlin (1980) 129-144.
- [2] Cannon, J.R., Knightly, G.H., *A note on the Cauchy problem for the Navier-Stokes equations*, SIAM J. Appl. Math., **18** (1970) 641-644.
- [3] Cannone, M., *Ondelettes, Praproducts et Navier-Stokes*, Diderot Editeur, Arts et Sciences Paris-New York-Amsterdam (1995).

- [4] Foias, C., Mainley, O., Temmam, R., *Attractors for the Bérard problem: existence and physical bounds of their fractal dimension.*, *Nonlinear Anal. T. M. A.*, **11** (1987), 939-967.
- [5] Fife, P.C., Joseph, D.D., *Existence convective solutions of the generalized Bénard problem which are analytic in their norm.*, *Arch. Rational Mech. Anal.*, **33** (1969), 116-138.
- [6] Hishida, T., *Existence and Regularizing properties of solutions for the nonstationary convection problem*, *Funkcial. Ekvac.*, **34** (1991), 449-474.
- [7] Hishida, T., Yamada Y., *Global solutions for the heat convection equations in an exterior domain*, *Tokyo J. Math.*, **15** (1992), 135-151.
- [8] Giga, Y., Inui, K., Matsui, S., *On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data*, *Quaderni di Matematica*, **4** (1999), 28-68.
- [9] Giga, Y., Matsui, S., Sawada, O., *Global Existence of Two-Dimensional Navier-Stokes Flow with nondecaying initial velocity*, *J. Math. fluid Mech.*, **3** (2001), 302-315.
- [10] Kagei, Y., *On weak solutions of nonstationary Boussinesq equations*, *Differential Integral Equations*, **6** (1993), 587-611.
- [11] Kagei, Y., von Wahl, W., *The Eckhaus criterion for convection roll solutions of the Oberbeck-Boussinesq equations*, *Int. J. Non-Linear Mechanics*, **32** (1997), 563-620.
- [12] Koch, H., Tataru, D., *Well-posedness for the Navier-Stokes equations*, *Adv. Math.*, **157** (2001), 22-35.
- [13] Kozono, H., Yamazaki, M., *Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data*, *Comm. P. D. E.*, **19** (1994), 191-200
- [14] Morimoto, H., *On the existence of weak solutions of equation of natural convection*, *J. Fac. Sci. Univ. Tokyo, Sect. IA*, **36** (1989), 87-102.
- [15] Morimoto, H., *Non-stationary Boussinesq equations*, *J. Fac. Sci. Univ. Tokyo, Sect. IA*, **39** (1992), 61-75.
- [16] Oeda, K., *On the initial Value problem for the heat convection equation of Boussinesq approximation in a time-dependent domain.*, *Proc. Japan Acad., Ser A*, **64** (1988), 143-146.
- [17] Oeda, K., *Weak and strong solutions of the heat convection equations in regions with moving boundaries*, *J. Fac. Sci. Univ. Tokyo, Sect. IA*, **36** (1989), 491-536.
- [18] Sawada, O., Taniuchi, T. *On Boussinesq flow with nondecaying initial data*, preprint.
- [19] Serfati, P., *Solutions C^∞ en temps, $n - \log$ lipshitz bornées en espace et équation d'Euler*, *C.R.Acad. Sci. Paris, t. 302 Série I*, (1995), 555-558.
- [20] Taniuchi, Y., *Uniformly local L^p estimate for 2-D vorticity equation and its application to Euler equations with initial vorticity in \mathbf{bmo}* , preprint

L^q - L^r estimates of solution to the parabolic Maxwell equations and their application to the magnetohydrodynamic equations*

Norikazu Yamaguchi

Department of Mathematical Sciences, Graduate School of Science and Engineering, Waseda University, Ohkubo 3-4-1, Shinjuku-ku, Tokyo 169-8555, Japan.

1. Introduction and main results

Let Ω be a simply connected and unbounded domain in the three dimensional Euclidean space \mathbf{R}^3 whose boundary $\partial\Omega$ is a compact and sufficiently smooth hypersurface. Suppose that there is some $R_0 > 0$ such that $\partial\Omega \subset B_{R_0}(0) = \{x \in \mathbf{R}^3 \mid |x| < R_0\}$. In this paper we are concerned with the initial boundary value problem of the magnetohydrodynamic equations in $\Omega \times (0, \infty)$ concerning the velocity vector field $v = (v_1(x, t), v_2(x, t), v_3(x, t))$, the magnetic vector potential $H = (H_1(x, t), H_2(x, t), H_3(x, t))$ and the scalar pressure $p = p(x, t)$:

$$\left\{ \begin{array}{ll} v_t - \Delta v + (v \cdot \nabla)v + \nabla p + H \times \text{curl } H = 0 & \text{in } \Omega \times (0, \infty), \\ H_t + \text{curl } \text{curl } H + (v \cdot \nabla)H - (H \cdot \nabla)v = 0 & \text{in } \Omega \times (0, \infty), \\ \text{div } v = 0, \quad \text{div } H = 0 & \text{in } \Omega \times (0, \infty), \\ v = 0, \quad \text{curl } H \times \nu = 0, \quad \nu \cdot H = 0 & \text{on } \partial\Omega \times (0, \infty), \\ v(x, 0) = a, \quad H(x, 0) = b & \text{in } \Omega. \end{array} \right. \quad (\text{MHD})$$

Here $a = (a_1(x), a_2(x), a_3(x))$ and $b = (b_1(x), b_2(x), b_3(x))$ are prescribed initial data and ν is the unit outward normal on $\partial\Omega$. We impose the perfectly conducting wall condition on the magnetic vector potential on the boundary. The perfectly conducting wall condition means that the obstacle, $\mathbf{R}^3 \setminus \overline{\Omega}$, is the perfect conductor body. The magnetohydrodynamic equations were proposed by Cowling [3] or Landau and Lifshitz

*This is joint work with Prof. Y. Shibata of Waseda University.

[10]. They are known to be one of the mathematical model describing the motion of viscous incompressible resistive fluid.

The main purpose of this paper is to show the global solvability of (MHD). The initial boundary value problem of the magnetohydrodynamic equations was treated in a bounded domain by the Galerkin method. However, in general the Galerkin method does not work well in the unbounded domain case. Thus we shall take another approach. On the other hand, there are some works in exterior domain. Zhao [17] considered (MHD) with nonperfect conductor body, that is the boundary condition of the magnetic vector potential is replaced by the homogeneous Dirichlet condition. However from a physical viewpoint, the case of the perfectly conducting wall is also important. The author knows only the result by Kozono [9] concerning the perfectly conducting wall case, where the weak solution was dealt with. There has been no work on the global strong solvability to the (MHD) in the exterior domain.

Our approach is based on the argument of T. Kato [8]. Kato proved the global solvability of the Cauchy problem of the Navier-Stokes equations in \mathbf{R}^N ($N \geq 2$) with small initial velocity with respect to L^N -norm. The argument of Kato is based on the estimates of various L^q -norm of the Stokes semigroup. In particular L^q - L^r type estimates play crucial role in it. The argument of Kato was extended to the case of exterior domain by Iwashita [7]. Our aim of this talk is to show the global solvability of (MHD), by use of the argument of Kato and Iwashita which is known to work well in exterior domain. In order to do this, one of the main points is to study the linearized problems corresponding to (MHD). They are consisted of two systems of equations. First is the system of well known nonstationary Stokes equations and second is the system of the Maxwell equations of parabolic type with perfectly conducting wall condition:

$$\left\{ \begin{array}{ll} u_t + \text{curl curl } u = 0 & \text{in } \Omega \times (0, \infty), \\ \text{div } u = 0 & \text{in } \Omega \times (0, \infty), \\ \text{curl } u \times \nu = 0 & \text{on } \partial\Omega \times (0, \infty), \\ \nu \cdot u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = b & \text{in } \Omega. \end{array} \right. \quad (\text{PM})$$

Here $u = (u_1(x, t), u_2(x, t), u_3(x, t))$ is unknown vector valued function and b is a prescribed initial function. In order to derive good linear estimates, we have to study the parabolic Maxwell equations (PM). The parabolic Maxwell equations are important not only for (MHD) but also for another equations involving the Maxwell equations. For example, the time dependent Ginzburg-Landau-Maxwell superconductivity model and the magneto-micropolar fluid equations.

Before stating our main results we shall introduce some notations. Let $1 < q < \infty$.

It is well known that the Banach space $L^q(\Omega)^3$ admits the Helmholtz decomposition:

$$L^q(\Omega)^3 = L_\sigma^q(\Omega) \oplus G^q(\Omega), \quad \oplus : \text{direct sum.}$$

Here

$$\begin{aligned} L_\sigma^q(\Omega) &= \overline{\{f \in C_0^\infty(\Omega)^3 \mid \operatorname{div} f = 0 \text{ in } \Omega\}}^{\|\cdot\|_{L^q(\Omega)^3}}, \\ G^q(\Omega) &= \{f \in L^q(\Omega)^3 \mid f = \nabla p \text{ for some } p \in L_{\text{loc}}^q(\overline{\Omega})\}. \end{aligned}$$

By the assumption that $\partial\Omega$ is sufficiently smooth, the space $L_\sigma^q(\Omega)$ is characterized as (see e.g., Galdi [6, Chapter 3])

$$L_\sigma^q(\Omega) = \{f \in L^q(\Omega)^3 \mid \operatorname{div} f = 0 \text{ in } \Omega, \nu \cdot f = 0 \text{ on } \partial\Omega\}.$$

Let $P = P_{q,\Omega}$ be a continuous projection from $L^q(\Omega)^3$ onto $L_\sigma^q(\Omega)$ and let us define the operators A and B as follows:

$$\begin{aligned} \mathcal{D}(A) &= L_\sigma^q(\Omega) \cap W^{2,q}(\Omega)^3 \cap W_0^{1,q}(\Omega)^3, \\ Av &= -P\Delta v \quad \text{for } v \in \mathcal{D}(A), \\ \mathcal{D}(B) &= L_\sigma^q(\Omega) \cap \{H \in W^{2,q}(\Omega)^3 \mid \operatorname{curl} H \times \nu = 0 \text{ on } \partial\Omega\}, \\ BH &= P(\operatorname{curl} \operatorname{curl} H) = \operatorname{curl} \operatorname{curl} H \quad \text{for } H \in \mathcal{D}(B). \end{aligned}$$

From Akiyama, Kasai, Shibata and M. Tsutsumi [1], Borchers and Sohr [2] and Miyakawa [12, 13] both $-A$ and $-B$ generate bounded analytic semigroups $\{e^{-tA}\}$ and $\{e^{-tB}\}$ in $L_\sigma^q(\Omega)$, respectively. By use of operators A and B , (MHD) is converted into the following system of integral equations:

$$\begin{aligned} v(t) &= e^{-tA}a - \int_0^t e^{-(t-\tau)A} P[(v(\tau) \cdot \nabla)v(\tau) + H(\tau) \times \operatorname{curl} H(\tau)] d\tau, \\ H(t) &= e^{-tB}b - \int_0^t e^{-(t-\tau)B} P[(v(\tau) \cdot \nabla)H(\tau) - (H(\tau) \cdot \nabla)v(\tau)] d\tau. \end{aligned} \tag{INT}$$

In analyzing (INT) we require L^q - L^r estimates for the semigroup $\{e^{-tA}\}$ and $\{e^{-tB}\}$. The first was already proved by Iwashita (see also Maremonti and Solonnikov [11] and Enomoto and Shibata [5]). Therefore we have to do is to derive L^q - L^r estimates for the semigroup $\{e^{-tB}\}$.

We are now in a position to state our main results. The first result is concerned with the local energy decay property for the semigroup $\{e^{-tB}\}$.

Theorem 1.1 (Local energy decay). *Let $1 < q < \infty$. For any $R > R_0$ and any integer $m \geq 0$, there exists $C = C(q, R, m) > 0$ such that*

$$\|\partial_t^m e^{-tB} f\|_{W^{2,q}(\Omega_R)} \leq C t^{-(3/2+m)} \|f\|_{L^q(\Omega)}, \quad t \geq 1,$$

for any $f \in L_\sigma^q(\Omega) \cap L_R^q(\Omega)$, where $\Omega_R = \Omega \cap B_R$ and $L_R^q(\Omega) = \{u \in L^q(\Omega)^3 \mid u = 0 \text{ for } |x| \geq R\}$.

By use of Theorem 1.1, one can obtain the following L^q - L^r estimates for $\{e^{-tB}\}$.

Theorem 1.2 (L^q - L^r estimates).

(i) Let $1 \leq q \leq r \leq \infty$ and $(q, r) \neq (1, 1), (\infty, \infty)$. Then there exists $C = C(q, r) > 0$ such that

$$\|e^{-tB} f\|_{L^r(\Omega)} \leq C t^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{r})} \|f\|_{L^q(\Omega)}, \quad t > 0$$

for any $f \in L^q_\sigma(\Omega)$.

(ii) Let $1 < q \leq r \leq 3$. Then there exists $C = C(q, r) > 0$ such that

$$\|\nabla e^{-tB} f\|_{L^r(\Omega)} \leq C t^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{r}) - \frac{1}{2}} \|f\|_{L^q(\Omega)}, \quad t > 0$$

for any $f \in L^q_\sigma(\Omega)$.

The basic idea to prove Theorem 1.1 and Theorem 1.2 is similar to that of Iwashita [7] which deals with the nonstationary Stokes equations. However the boundary condition of (PM), perfectly conducting wall condition, is quite different from the boundary condition of the Stokes equations that is homogeneous Dirichlet condition, nonslip boundary condition. Therefore in constructing the parametrix of the resolvent problem corresponding to (PM) (see (2.1)), we have to introduce a new idea which is based on a theorem due to von Wahl [16, Theorem 3.2].

The following result by Iwashita is concerning the L^q - L^r estimates for the Stokes semigroup, which is refined by Maremonti and Solonnikov and Enomoto and Shibata.

Theorem 1.3 ([7, 11, 5]).

(i) Let $1 \leq q \leq r \leq \infty$ and $(q, r) \neq (1, 1), (\infty, \infty)$. Then there exists $C = C(q, r) > 0$ such that

$$\|e^{-tA} f\|_{L^r(\Omega)} \leq C t^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{r})} \|f\|_{L^q(\Omega)}, \quad t > 0$$

for any $f \in L^q_\sigma(\Omega)$.

(ii) Let $1 < q \leq r \leq 3$. Then there exists $C = C(q, r) > 0$ such that

$$\|\nabla e^{-tA} f\|_{L^r(\Omega)} \leq C t^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{r}) - \frac{1}{2}} \|f\|_{L^q(\Omega)}, \quad t > 0$$

for any $f \in L^q_\sigma(\Omega)$.

Combining Theorem 1.2 and Theorem 1.3 we obtain the global solvability of (MHD) with small initial data.

Theorem 1.4. There exists a constant $\epsilon > 0$ such that if $(a, b) \in L^3_\sigma(\Omega) \times L^3_\sigma(\Omega)$ and $\|(a, b)\|_3 < \epsilon$, then a unique strong solution (u, H) to (MHD) exists and satisfies the following properties:

$$t^{(1-3/q)/2}(u, H) \in BC([0, \infty); L^q_\sigma(\Omega) \times L^q_\sigma(\Omega)) \quad \text{for any } q, 3 \leq q \leq \infty, \quad (1.1)$$

$$t^{1/2} \nabla(u, H) \in BC([0, \infty); L^3(\Omega) \times L^3(\Omega)), \quad (1.2)$$

where $BC(\cdot)$ denotes the class of bounded continuous functions. All the values in (1.1)–(1.2) vanish at $t = 0$ except for $q = 3$ in (1.1), and in case $q = 3$, $(u(0), H(0)) = (a, b)$.

2. Sketch of the proof of Theorem 1.1

As stated in the previous section, in view of argument of Iwashita we know that once obtaining the local energy decay, by cut-off technique we have L^q - L^r estimates. Therefore, in this section we will give a sketch of our proof of Theorem 1.1.

To prove Theorem 1.1 we have to study the resolvent problem corresponding to (PM). In view of Miyakawa [12], to do this it suffices to study the following Laplace resolvent system with perfectly conducting wall condition:

$$\begin{cases} \lambda u - \Delta u = f & \text{in } \Omega, \\ \operatorname{curl} u \times \nu = 0 & \text{on } \partial\Omega, \\ \nu \cdot u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

Here $\lambda \in \Sigma_\epsilon = \{\lambda \in \mathbf{C} \setminus \{0\} \mid |\arg \lambda| < \pi - \epsilon, 0 < \epsilon < \pi/2\}$ and $f = (f_1(x), f_2(x), f_3(x))$ is given function. Our aim of this section is to prove the following theorem.

Theorem 2.1. *Let $1 < q < \infty$ and m be a nonnegative integer. There exists solution operator $R(\lambda) \in B(W_R^{m,p}(\Omega), W^{m+2,p}(\Omega_{R+2}))$ such that $R(\lambda)$ depends on $\lambda \in \Sigma_\epsilon$ meromorphically and has the following properties:*

- (i) *The set Λ of the poles is discrete.*
- (ii) *$u = R(\lambda)f$ is a solution of (2.1) for $\lambda \in \Sigma_\epsilon \setminus \Lambda$ and $f \in W_R^{m,p}(\Omega)$.*
- (iii) *$R(\lambda) \in B(W_R^{m,p}(\Omega), W^{m+2,p}(\Omega))$ for each $\lambda \in \Sigma_\epsilon \setminus \Lambda$.*
- (iv) *Let $\Sigma_\epsilon(\delta) = \{\lambda \in \Sigma_\epsilon \mid |\lambda| < \delta\}$. There exists $\delta_0 > 0$ such that $\Sigma_\epsilon(\delta_0) \cap \Lambda = \emptyset$ and $R(\lambda)$ has the following expansion of $\lambda \in \Sigma_\epsilon(\delta_0)$ in $B(W_R^{m,p}(\Omega), W^{m+2,p}(\Omega_R))$:*

$$R(\lambda) = \lambda^{1/2}G_1 + G_2(\lambda) + \lambda^{1/2}G_3(\lambda). \quad (2.2)$$

Here $G_1 \in B(W_R^{m,p}(\Omega), W^{m+2,p}(\Omega_R))$, $G_2(\lambda)$ is $B(W_R^{m,p}(\Omega), W^{m+2,p}(\Omega_R))$ -valued holomorphic function of $\lambda \in \Sigma_\epsilon(\delta_0)$ and $G_3(\lambda)$ is bounded.

Here $W_R^{m,p}(\Omega) = \{f \in W^{m,p}(\Omega) \mid f = 0 \text{ for } |x| > R\}$.

In order to prove Theorem 2.1, first of all we construct the parametrix to (2.1). Choose a positive number $R > 0$ such that $R > R_0 + 3$. Here R_0 is introduced in the previous

section. Let Φ be a mapping of $f \in L^q(\Omega_{R+3})$ to unique solution $u \in W^{2,p}(\Omega_{R+3})$ of the following problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega_{R+3}, \\ \text{curl } u \times \nu = 0 & \text{on } \partial\Omega_{R+3}, \\ \nu \cdot u = 0 & \text{on } \partial\Omega_{R+3}. \end{cases}$$

Then $\Phi \in B(L^q(\Omega_{R+3}), W^{2,q}(\Omega_{R+3}))$. Put $\varphi \in C_0^\infty(\mathbf{R}^3)$ such that $\varphi = 1$ for $|x| \leq R+1$ and $= 0$ for $|x| \geq R+2$. For $f \in L^q(\Omega)$, let f_{R+3} be the restriction of f to Ω_{R+3} and let f_0 be the zero extension of f to \mathbf{R}^3 , that is, $f_0 = f$ in Ω and $f_0 = 0$ in $\mathbf{R}^3 \setminus \Omega$. Let us define an operator $A(\lambda)$ by

$$A(\lambda)f = (1 - \varphi)R_0(\lambda)f_0 + \varphi\Phi f_{R+3}. \quad (2.3)$$

Here $R_0(\lambda)$ denotes the solution operator to $\lambda u - \Delta u = f$ in \mathbf{R}^3 (The properties of $R_0(\lambda)$ is stated in Murata [14]). For $A(\lambda)f$ we have

$$\begin{cases} (\lambda - \Delta)A(\lambda)f = f + S(\lambda)f & \text{in } \Omega, \\ \text{curl } (A(\lambda)f) \times \nu = 0 & \text{on } \partial\Omega, \\ \nu \cdot (A(\lambda)f) = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$S(\lambda)f = 2\nabla\varphi \cdot \nabla R_0(\lambda)f_0 + (\Delta\varphi)R_0(\lambda)f_0 + \lambda\varphi\Phi f_{R+3} - 2\nabla\varphi \cdot \nabla\Phi f_{R+3} - (\Delta\varphi)\Phi f_{R+3}$$

From the definition of the cut-off function φ , $\text{supp } S(\lambda)f \subset \Omega_{R+3}$. If $f \in L^p_{R+2}(\Omega)$, then by the Fourier multiplier theorem and the property of $R_0(\lambda)$, we obtain

$$\|R_0(\lambda)f_0\|_{W^{2,p}(B_{R+3})} \leq C\|f\|_{L^p(\Omega)}.$$

Lemma 2.2. *The inverse $(I + S(\lambda))^{-1}$ of $I + S(\lambda)$ exists as a $B(L^p_{R+2}(\Omega), L^p_{R+2}(\Omega))$ -valued meromorphic function of $\lambda \in \Sigma_\epsilon$. The set Λ of poles is discrete and has no intersection with $\Sigma_\epsilon(\delta_0)$ for some $\delta_0 > 0$. Furthermore, $(I + S(\lambda))^{-1}$ has the same type of expansion as (2.2).*

This lemma will follow from the following lemma.

Lemma 2.3. *$I + S(0)$ has the bounded inverse $(I + S(0))^{-1}$.*

Before stating the proof of Lemma 2.3, we introduce the following uniqueness result which will be required in the proof of Lemma 2.3.

Proposition 2.4. Let $1 < q < \infty$. Suppose that $u \in W_{\text{loc}}^{2,q}(\Omega)$ satisfies

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \text{curl } u \times \nu = 0 & \text{on } \partial\Omega, \\ \nu \cdot u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.4)$$

and $u(x) = O(|x|^{-1})$, $\nabla u(x) = O(|x|^{-2})$. Then $u = 0$ in Ω .

Remark 2.5. The assumption that Ω is simply connected is essentially required to prove Proposition 2.4 only.

Proof of Proposition 2.4. By virtue of the local theory for the elliptic equations, one can take $u \in W_{\text{loc}}^{2,r}(\Omega)$ for any $r \in (1, \infty)$. In particular, now we take $u \in W_{\text{loc}}^{2,2}(\Omega)$. We consider a function $\psi \in C_0^\infty(\mathbf{R}^3)$ with the properties $0 \leq \psi(x) \leq 1$, $\psi(x) = 1$ for $|x| \leq 1/2$ and $= 0$ for $|x| \geq 1$ and define $\psi_R(x) := \psi(x/R)$. According to the well known formula $\Delta u = \nabla \text{div } u - \text{curl } \text{curl } u$, the divergence theorem and the assumption we get

$$\begin{aligned} 0 &= \int_{\Omega} -\Delta u \cdot \psi_R u \, dx = \int_{\Omega_R} \text{curl } u \cdot (\nabla \psi_R \times u) \, dx + \int_{\Omega_R} (\text{div } u)(\nabla \psi_R \cdot u) \, dx \\ &\quad + \int_{\Omega_R} \psi_R [(\text{div } u)^2 + \text{curl } u \cdot \text{curl } u] \, dx. \end{aligned}$$

Since $\text{supp } \psi_R \subset \{x \in \mathbf{R}^3 \mid R/2 < |x| < R\}$, we have

$$\left| \int_{\Omega_R} \text{curl } u \cdot (\nabla \psi_R \times u) \, dx + \int_{\Omega_R} (\text{div } u)(\nabla \psi_R \cdot u) \, dx \right| \leq \frac{C}{R}.$$

Therefore letting $R \rightarrow \infty$, we have $\|\text{curl } u\|_{L^2(\Omega)}^2 + \|\text{div } u\|_{L^2(\Omega)}^2 = 0$. This implies that $\text{curl } u = 0$ and $\text{div } u = 0$ in Ω and moreover by virtue of theorem due to von Wahl [16], we obtain $\nabla u = 0$. Hence $u = \text{const}$ in Ω . From the assumption that u satisfies $\nu \cdot u = 0$ on $\partial\Omega$, we have $u = 0$ in Ω in Ω . This completes the proof. \blacksquare

Now we shall show Lemma 2.3.

Proof of Lemma 2.3. Since the operator $S(0)$ is compact, by the Fredholm alternative theorem it suffices to show injectivity of $I + S(0)$. Let us pick up $f \in L_{R+2}^p(\Omega)$ so that $(I + S(0))f = 0$. Then it follows from (2.3), $A(0)f$ satisfies (2.4) and moreover $A(0)f$ has the properties that $A(0)f = O(|x|^{-1})$ and $\nabla(A(0)f) = O(|x|^{-2})$. Therefore from Proposition 2.4, $A(0)f = 0$. Namely we have

$$(1 - \varphi)R_0(0)f_0 + \varphi\Phi f_{R+3} = 0 \quad \text{in } \Omega.$$

By the definition of the cut-off function φ we have $\Phi f_{R+3} = 0$ for $|x| \leq R+1$ and $R_0(0)f_0 = 0$ for $|x| \geq R+2$. Put $w = \Phi f_{R+3}$ for $x \in \Omega_{R+3}$ and $= 0$ for $x \notin \Omega$. Then w satisfies

$$\begin{cases} -\Delta w = f_0 & \text{in } B_{R+3}, \\ \text{curl } w \times \nu = 0 & \text{on } \partial B_{R+3}, \\ \nu \cdot w = 0 & \text{on } \partial B_{R+3}. \end{cases}$$

On the other hand, from $R_0(0)f_0 = 0$ for $|x| \geq R+2$, we also have

$$\begin{cases} -\Delta R_0(0)f_0 = f_0 & \text{in } B_{R+3}, \\ \text{curl } (R_0(0)f_0) \times \nu = 0 & \text{on } \partial B_{R+3}, \\ \nu \cdot (R_0(0)f_0) = 0 & \text{on } \partial B_{R+3}. \end{cases}$$

Hence we obtain $w = R_0(0)f_0$ in Ω_{R+3} . Therefore

$$0 = A(0)f = R_0(0)f_0 + \varphi(\Phi f_{R+3} - R_0(0)f_0) = R_0(0)f_0.$$

This implies $f_0 = 0$ in Ω . ■

Proof of Lemma 2.2. Let $M = \|(I + S(\lambda))^{-1}\|$, where $\|\cdot\|$ denotes the operator norm. From the fact that $S(\lambda)$ is continuous in $\Sigma_\epsilon \cup \{0\}$, there is some $\delta_0 > 0$ such that $\|S(\lambda) - S(0)\| < 1/2M$ for any $\lambda \in \Sigma_\epsilon(\delta_0)$. Hence, for $\lambda \in \Sigma_\epsilon(\delta_0)$,

$$(I + S(\lambda))^{-1} = \sum_{j=1}^{\infty} [(I + S(0))^{-1}(S(0) - S(\lambda))]^j (I + S(0))^{-1}. \quad (2.5)$$

Since $S(\lambda)$ is holomorphic in $\lambda \in \Sigma_\epsilon$, by analytic Fredholm's alternative theorem (see e.g., Dunford and Schwartz [4, p. 592, Lemma 13]) we obtain $(I + S(\lambda))^{-1}$ for any $\lambda \in \Sigma_\epsilon$ as a meromorphic function and we see that the set Λ of poles is discrete in Σ_ϵ . The expansion of $(I + S(\lambda))^{-1}$ follows from that of the expansion of $R_0(\lambda)$, Lemma 2.3 and (2.5). ■

With help of above results, we prove Theorem 2.1.

Proof of Theorem 2.1. Define $R(\lambda)$ by

$$R(\lambda) = A(\lambda)(I + S(\lambda))^{-1}. \quad (2.6)$$

Then the assertions are immediately derived from the expansion of $R_0(\lambda)$, Lemma 2.2 and (2.6). ■

In the rest of this paper, we shall state the strategy of the proof of Theorem 1.1. Let $0 < \epsilon < \epsilon_1 < \pi/2$ and let γ be a contour as follows: $\gamma = \gamma_1 \cup \gamma_2$, where

$$\begin{aligned}\gamma_1 &= \{\lambda \in \mathbf{C} \mid 0 < |\lambda| \leq \delta_0/2, |\arg \lambda| = \pi - \epsilon_1\}, \\ \gamma_2 &= \{\lambda \in \mathbf{C} \mid |\lambda| > \delta_0/2, |\arg \lambda| = \pi - \epsilon_1\}.\end{aligned}$$

According to Theorem 2.1, the semigroup e^{-tB} is represented as

$$e^{-tB} = \frac{1}{2\pi i} \int_{\gamma_1} e^{\lambda t} R(\lambda) d\lambda + \frac{1}{2\pi i} \int_{\gamma_2} e^{\lambda t} (\lambda + B)^{-1} d\lambda,$$

in $L^q_\sigma(\Omega) \cap L^q_{R+2}(\Omega)$. To prove Theorem 1.1, we introduce the following well known lemma concerning the gamma function $\Gamma(\sigma)$.

Lemma 2.6. *For $\sigma > 0$ and $t > 0$, it holds that*

$$\left| \frac{1}{2\pi i} \int_{\gamma_1} e^{\lambda t} \lambda^{\sigma-1} d\lambda - \frac{\sin \sigma \pi}{\pi} \Gamma(\sigma) t^{-\sigma} \right| \leq C e^{-ct}.$$

Finally combining Theorem 2.1 and Lemma 2.6, we have the local energy decay property of the semigroup e^{-tB} , Theorem 1.1.

References

- [1] T. Akiyama, H. Kasai, Y. Shibata, and M. Tsutsumi. On a resolvent estimate of a system of Laplace operators with perfect wall condition. Preprint, 2003.
- [2] W. Borchers and H. Sohr. On the semigroup of the Stokes operator for exterior domains in L^q -spaces. *Math. Z.*, 196(3):415–425, 1987.
- [3] T. G. Cowling. *Magnetohydrodynamics*. Interscience Tracts on Physics and Astronomy, No. 4. Interscience Publishers, Inc., New York, 1957.
- [4] N. Dunford and J. T. Schwartz. *Linear operators. Part I*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. General theory, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original, A Wiley-Interscience Publication.
- [5] Y. Enomoto and Y. Shibata. On the rate of decay of the Oseen semigroup in exterior domains and its application to Navier-Stokes equations. Preprint, 2003.
- [6] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*. Springer-Verlag, New York, 1994. Linearized steady problems.
- [7] H. Iwashita. L_q - L_r estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in L_q spaces. *Math. Ann.*, 285(2):265–288, 1989.

- [8] T. Kato. Strong L^p -solutions of the Navier-Stokes equation in \mathbf{R}^m , with applications to weak solutions. *Math. Z.*, 187(4):471–480, 1984.
- [9] H. Kozono. On the energy decay of a weak solution of the MHD equations in a three-dimensional exterior domain. *Hokkaido Math. J.*, 16(2):151–166, 1987.
- [10] L. D. Landau and E. M. Lifshitz. *Electrodynamics of continuous media*. Course of Theoretical Physics, Vol. 8. Translated from the Russian by J. B. Sykes and J. S. Bell. Pergamon Press, Oxford, 1960.
- [11] P. Maremonti and V. A. Solonnikov. On nonstationary Stokes problem in exterior domains. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 24(3):395–449, 1997.
- [12] T. Miyakawa. The L^p approach to the Navier-Stokes equations with the Neumann boundary condition. *Hiroshima Math. J.*, 10(3):517–537, 1980.
- [13] T. Miyakawa. On nonstationary solutions of the Navier-Stokes equations in an exterior domain. *Hiroshima Math. J.*, 12(1):115–140, 1982.
- [14] M. Murata. Large time asymptotics for fundamental solutions of diffusion equations. *Tohoku Math. J. (2)*, 37(2):151–195, 1985.
- [15] M. Sermange and R. Temam. Some mathematical questions related to the MHD equations. *Comm. Pure Appl. Math.*, 36(5):635–664, 1983.
- [16] W. von Wahl. Estimating ∇u by $\operatorname{div} u$ and $\operatorname{curl} u$. *Math. Methods Appl. Sci.*, 15(2):123–143, 1992.
- [17] C. Zhao. Initial boundary value problem for the evolution system of MHD type describing geophysical flow in three-dimensional domains. *Math. Methods Appl. Sci.*, 26(9):759–781, 2003.

BLOW-UP OF SOLUTIONS OF QUASILINEAR PARABOLIC EQUATIONS WITH LOCALIZED REACTIONS

RYUICHI SUZUKI AND ISAMU FUKUDA

Department of Mathematics
Faculty of Engineering
Kokushikan University
4-28-1 Setagaya Setagaya-ku Tokyo, 154 Japan

Let $R > 0$ and

$$(1) \quad B = \{x \in \mathbf{R}^N \mid |x| < R\}.$$

We shall consider the initial boundary value problem for the quasilinear parabolic equation with both a nonlocal source (localized reaction) $bu^q(x_0, t)$ and a local source au^p ,

$$(2) \quad u_t = \Delta u^m + au^p + bu^q(x_0, t), \quad (x, t) \in B \times (0, T),$$

$$(3) \quad u(x, t) = 0, \quad (x, t) \in \partial B \times (0, T),$$

$$(4) \quad u(x, 0) = u_0(x), \quad x \in B,$$

where $u_t = \partial u / \partial t$, Δ is the N -dimensional Laplacian, $p, q > m \geq 1$, $a, b > 0$, $x_0 \in B$ and $u_0(x) \geq 0$ in B . We assume that $u_0(x) = u_0(r) \in C(\bar{B})$ ($r = |x|$) is a radially symmetric function in $x \in B$ and a non-increasing function in $r \geq 0$, and $u_0(R) = 0$.

We shall only consider non-negative weak solutions $u = u(x, t)$.

Under these conditions, a non-negative weak solution of (2)-(4) exists locally in time (See [4] and [10] for $m = 1$ and [12] for $m \geq 1$). When $m = 1$, non-negative solutions of (2)-(4) are unique (See [3] and [10]). When $m > 1$, we do not know whether or not non-negative solutions of (2)-(4) are unique (But, when $b = 0$, the uniqueness of solutions can be proved [1]).

Moreover, we see that the solution $u(x, t)$ blows up in finite time for large initial data (see [3], [10] ($m = 1$) and [12] ($m > 1$) for $a = 0$ and see e.g. [7] for $b = 0$).

Namely, for some $T \in (0, \infty)$,

$$(5) \quad \lim_{t \uparrow T} \|u(\cdot, t)\|_{L^\infty(B)} = \infty.$$

We say this time T the blow-up time.

We are interested in the shape of the blow-up set S of a blow-up solution of (2)-(4):

$$(6) \quad S = \{x \in \bar{B} \mid \text{there exists a sequence } (x_m, t_m) \in B \times (0, T) \\ \text{satisfying } x_m \rightarrow x, t_m \uparrow T \text{ and } u(x_m, t_m) \rightarrow \infty \text{ as } m \rightarrow \infty\}.$$

Each x in S is called "a blow-up point" of u .

When $a = 0$, we see that the blow-up solution of (2)-(4) blows up in whole domain B (See [2, 3, 10] for $m = 1$ and [12] for $m > 1$). Namely,

$$(7) \quad S = \bar{B}.$$

We call this phenomena "total blow-up".

On the other hand, when $b = 0$, it is well known that if $u(x, t)$ is non-decreasing in $t \geq 0$ (When $m = 1$, this assumption is not required), the blow-up solution goes to infinity only at the origin (see [13] [5] [6] for $m = 1$ and [8] for $m > 1$). Namely,

$$(8) \quad S = \{0\}.$$

We call this blow-up phenomena "single point blow-up".

So, we consider the initial boundary value problem (2)-(4) with $a, b > 0$. Then, we are interested in the problem whether total blow-up or single point blow-up occurs according to the relations of the values of p and q .

On this problem, when $m = 1$ Okada-Fukuda [9] obtain the interesting results :
When $x_0 = 0$,

(I) Let $p > q+1$. Let $\Phi(x)$ be an eigenfunction corresponding to the first eigenvalue of $-\Delta$ with zero Dirichlet condition. If for some $\mu > 0$

$$(9) \quad \Delta u_0 + u_0^p + u^q(x_0) \geq \mu \Phi$$

then the blow-up solution of (2)-(4) blows up only at the origin, namely, $S = \{0\}$;

(II) Let $p \leq q+1$. then the blow-up solution blows up in whole domain B , namely, $S = \bar{B}$;

When $x_0 \neq 0$,

(I) Let $p > q + 1$. If (9) holds then the blow-up solution of (2)-(4) blows up only at the origin, namely, $S = \{0\}$;

(II) Let $p < q$. Let $|x_0| < r_1 < R$ and $0 < \varepsilon_0 < 1$. If

$$(10) \quad \frac{\partial u}{\partial t} \geq 0 \quad \text{in } B \times (0, T)$$

and

$$(11) \quad u_0(r_1) \geq \varepsilon_0 u_0(0) \text{ and } u_0(0) \text{ is large enough,}$$

then the solution of (2)-(4) blows up in whole domain B , namely, $S = \bar{B}$.

Thus, when $m = 1$, in the case $x_0 = 0$, they showed that $p = q + 1$ is the cut off “straight line” between the cases where single point blow-up occurs and single point blow-up does not occurs. But, in the case $x_0 \neq 0$, they can not see clearly which blow-up phenomena occurs or not, especially, when $p - 1 \leq q \leq p$.

So, the main purpose of this paper is to solve these problems completely in the case $x_0 \neq 0$ and to extend these results to the case $m \geq 1$. We summarize our results as follows :

Theorem 1. Assume $a, b > 0$ and $p, q > m \geq 1$.

(I) Assume $p > q$. If $u(x, t)$ is non-decreasing in $t \geq 0$ for each $x \in B$ then the blow-up solution of (2)-(4) blows up only at the origin, that is, $S = \{0\}$;

(II) Assume $p < q$.

(i) If (11) holds then the solution of (2)-(4) blows up in whole domain B , that is, $S = \bar{B}$.

(ii) There exists a solution of (2)-(4) such that it blows up only at the origin, that is, $S = \{0\}$.

Remark 2. When $m = 1$, in (I) of Theorem 1 we do not need the assumption that $u(x, t)$ is non-decreasing in $t \geq 0$ for each $x \in B$.

Remark 3. There exists a solution $u(x, t)$ of (2)-(4) such that it is non-decreasing in $t \geq 0$ for each $x \in B$, if the inequality

$$(12) \quad \Delta u_0^m + u_0^p + u_0^q(x_0) \geq 0 \quad \text{in } B$$

holds in the next sense : For any bounded domain $D \subset B$ with smooth boundary ∂D , and nonnegative $\varphi(x) \in C^2(\bar{D})$ which vanishes on the boundary ∂D ,

$$(13) \quad \int_D \{u_0^m \Delta \varphi + u_0^p \varphi + u_0^q(x_0) \varphi\} dx - \int_{\partial D} u_0 \partial_n \varphi dS \geq 0$$

where n denotes the outer unit normal to the boundary.

Namely, in the case where $m = 1$ and $x_0 \neq 0$, we see that $p = q$ is the cut off “straight line” between the cases where single pint blow-up occurs and single point blow-up does not occurs. This result seems to be very interesting, since the cut off “straight lines” are different between the cases $x_0 \neq 0$ and $x_0 = 0$, when $m = 1$.

Remark 4. We can not get any result in the case where $p = q$.

Remark 5. If $p, q < m$ then the solution of (2)-(4) exists globally in time [12].

Thus, we completely solve the open problem when $x_0 \neq 0$ in Okada-Fukuda [9], except for the case $p = q$. We note that even if $m = 1$ (semilinear case), the results of the existence of single point blow-up solutions where $p < q$ are new. It seems to be interesting that the problem has both a single point blow-up solution and a total blow-up solution by only choosing initial data u_0 as in (II), since the other problems do not have such phenomena.

Our methods which are based on the maximum principle, are essentially due to Fujita-Chen [6] for the proof of (I) of Theorem 1 and due to Okada-Fukuda [9] for the proof of (i) of (I) in Theorem 1. But, we can not apply their methods directly, since our equation is degenerate at $u = 0$ and some of our results are refinements of their results.

Finally, we mention the existence, uniqueness and (total) blow-up of local solutions of (2)-(4) more in detail.

The existence of local solutions of (2)-(4) was shown by Cannon-Yin [4] when $m = 1$. Souplet [10] extended their results to more general form which contain various nonlocal problems. But, in quasilinear case $m > 1$ there are few papers studying these problems and we show the existence of local solutions of (2)-(4) in the quasilinear case in [12].

The uniqueness of local solutions was studied by Chadam-Peirce-Yin [3] and Souplet [10] (for more general equations) when $m = 1$. When $m > 1$ and $b = 0$, the

uniqueness of local solutions was studied by Aronson-Crandall-Peletier [1]. When $m > 1$ and $b > 0$, however, there are few papers studying the uniqueness of solutions and we can not show it. So, our discussions are derived without using the uniqueness of local solutions of (2) when $m > 1$.

The existence of blow-up solutions of (2)-(4) was also shown in [3] and [10] in the case $m = 1$. It was shown in [3] that if initial data $u_0(x)$ is large enough in the neighborhood of fixed x_0 , then the solution of (2)-(4) blows up in finite time. The results from [3] was extended in [10] to moving $x_0 = x_0(t)$. We note that in [10] the assumptions that initial data $u_0(x)$ is large enough in the neighborhood of x_0 is not required. In [12] we also obtain the blow-up results which are the extensions of [10] to $m > 1$.

Further, when $m = 1$ and $a = 0$, it was also shown in [3] (see also [11]) that the blow-up solution of (2)-(4) blows up in whole domain B . In [12] we extend the results from [3] and [11] to quasilinear case $m > 1$.

REFERENCES

- [1] D. G. Aronson, M. G. Crandall and L. A. Peletier, *Stabilization of solutions of a degenerate nonlinear diffusion problem*, Nonlinear Anal. **6** (1982), 1001–1022.
- [2] J. Bebernes, A. Bressan and A. Lacey, *Total blow-up versus single point blow-up* J. Differ. Equations **73** (1988), 30–44.
- [3] J. M. Chadam, A. Peirce and H. -M. Yin, *The blowup property of solutions to some diffusion equations with localized nonlinear reactions*, J. Math. Anal. Appl. **169** (1992), 313–328.
- [4] J. R. Cannon and H. M. Yin, *A class of non-linear non-classical parabolic equations*, J. Differ. Equations **79** (1989), 266–288 .
- [5] A. Friedman and B. Mcleod, *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ. Math. J. **34** (1985), 425–447.
- [6] H. Fujita and Y.-G. Chen, *On the set of blow-up points and asymptotic behaviours of blow-up solutions to a semilinear parabolic equation* Analyse Mathématique et Applications, Contrib. Honneur Jaques-Louis Lions, (1988) ,181–201
- [7] T. Imai and K. Mochizuki, *On blow-up of solutions for quasilinear degenerate parabolic equations*, Publ. RIMS, Kyoto Univ. **27** (1991), 695–709.
- [8] K. Mochizuki and R. Suzuki, *Blow-up sets and asymptotic behavior of interfaces for quasilinear degenerate parabolic equations in \mathbf{R}^N* , J. Math. Soc. Japan **44** (1992), 485–504.

- [9] A. Okada and I. Fukuda, *Total versus single point blow-up of solutions of a semilinear parabolic equation with localized reaction*, J. Math. Anal. Appl. **281** (2003), 485–500.
- [10] P. Souplet, *Blow-up in nonlocal reaction-diffusion equations*, SIAM J. Math. Anal. **29** (1998), 1301–1334 .
- [11] P. Souplet, *Uniform blow-up profiles and boundary behavior for diffusion equations with nonlocal nonlinear source*, J. Differ. Equations **153** (1999), 374–406.
- [12] R. Suzuki and I. Fukuda, *Quasilinear parabolic equations with localized reaction*, preprint .
- [13] F. B. Weissler, *Single point blow-up for a semilinear initial value problem*, J. Differ. Equations **55** (1984), 204–224.

Initial Behavior of Solutions of Diffusion Equations and Symmetries of Domains

Shigeru Sakaguchi

Department of Mathematical Sciences, Faculty of Science

Ehime University, Matsuyama 790-8577 Japan

e-mail: sakaguch@dpc.ehime-u.ac.jp

1 Introduction

This is based on the author's recent work with R. Magnanini. We study the relation between the initial behavior of solutions of diffusion equations and symmetries of domains. We deal with the porous medium equation and the heat equation.

Concerning the porous medium equation, we consider the flow of a gas into a bounded porous container Ω with smooth boundary $\partial\Omega$. Initially Ω is empty and at all times the density of the gas is kept constant on $\partial\Omega$. Choose a number $R > 0$ sufficiently small to have that, for any point x in Ω having distance R from $\partial\Omega$, the closed ball B with radius R centered at x intersects $\partial\Omega$ only at one point. We show that if the gas content of all such balls B is constant at each given time, then Ω must be a ball.

Concerning the heat equation, we consider a convex and polygonal heat conductor Ω whose inscribed circle touches every side of $\partial\Omega$. Initially Ω has zero temperature and at all times the temperature is kept constant on $\partial\Omega$. The cold spot is the point at which the temperature attains its minimum at each given time. It is proved that, if the cold spot is stationary, then Ω must satisfy two geometric conditions. In particular, we show that these geometric conditions yield some symmetries provided Ω is either pentagonal or hexagonal.

First, we consider the porous medium equation. Let Ω be a bounded domain in \mathbb{R}^N , $N \geq 2$, with smooth boundary $\partial\Omega$. The physical situation can be modeled as the following initial-boundary value problem for the porous medium equation:

$$u_t = \Delta u^m \quad \text{in} \quad \Omega \times (0, \infty), \quad (1.1)$$

$$u = 1 \quad \text{on} \quad \partial\Omega \times (0, \infty), \quad (1.2)$$

$$u = 0 \quad \text{on} \quad \Omega \times \{0\}, \quad (1.3)$$

where $u = u(x, t)$ denotes the normalized density of the gas at a point $x \in \Omega$ and at a time $t > 0$, and $m > 1$ is a constant. Existence and uniqueness of the weak solution are guaranteed by the standard theory for the porous medium equation. Since the diffusivity mu^{m-1} vanishes with u , an important feature of the solution is that the flow propagates disturbance from rest with finite speed. See [Ar]. Therefore, it is known that for any point $x \in \Omega$ there exists a time $T(x) > 0$ satisfying

$$u(x, t) \begin{cases} = 0 & \text{for all } t \in [0, T(x)], \\ > 0 & \text{for all } t \in (T(x), \infty). \end{cases} \quad (1.4)$$

For any point $x \in \bar{\Omega}$, define the function $d = d(x)$ by

$$d(x) = \inf\{|x - y| : y \in \partial\Omega\}. \quad (1.5)$$

This function $d(x)$ denotes the distance between x and $\partial\Omega$. For each $\rho > 0$, set

$$\Omega_\rho = \{x \in \Omega : d(x) < \rho\}. \quad (1.6)$$

Since $\partial\Omega$ is smooth and compact, there exists a number $\delta > 0$ such that

$$\forall x \in \Omega_\delta \exists_1 y = y(x) \in \partial\Omega \text{ with } d(x) = |x - y(x)|; \quad (1.7)$$

$$\max_{1 \leq j \leq N-1} \kappa_j(y) < \frac{1}{\delta} \text{ for any } y \in \partial\Omega, \quad (1.8)$$

where $\kappa_1(y), \dots, \kappa_{N-1}(y)$ denote the principal curvatures of $\partial\Omega$ at $y \in \partial\Omega$ with respect to the interior normal direction to Ω . Let us quote a theorem from C. Cortázar, M. Del Pino, and M. Elgueta [CDE] (see Theorem 1.1, p. 134).

Theorem 1.1 ([CDE]) *There exists a number $\varepsilon_0 > 0$ such that for any $x \in \Omega_{\varepsilon_0}$*

$$T(x) = d^2\{T_0 - H(y(x))T_1d + o(d)\}, \quad d = d(x), \quad (1.9)$$

where T_0, T_1 are positive constants depending only on m , and $H(y(x)) = \sum_{j=1}^{N-1} \kappa_j(y(x))$.

Combining this theorem with Aleksandrov's sphere theorem (see Theorem 3.1) yield

Corollary 1.2 *Suppose that there exists a number $\rho > 0$ such that, for any pair of points $x, z \in \Omega_\rho$, $d(x) = d(z)$ implies $T(x) = T(z)$. Then Ω must be a ball.*

One purpose of this talk is to report another sphere theorem. Take a number $R > 0$ with $0 < R < \delta$, where δ is the number given in (1.7) and (1.8). Set

$$\Gamma_R = \{x \in \Omega : d(x) = R\}. \quad (1.10)$$

Then we have from (1.7)

$$\overline{B_R(x)} \cap \partial\Omega = \{y(x)\} \quad \text{for any } x \in \Gamma_R, \quad (1.11)$$

where $B_R(x)$ denotes an open ball in \mathbb{R}^N centered at x and with radius R . Our sphere theorem is the following.

Theorem 1.3 *Suppose that there exists a function $a : (0, \infty) \rightarrow (0, \infty)$ satisfying*

$$\int_{B_R(x)} u(z, t) dz = a(t) \quad \text{for any } x \in \Gamma_R \text{ and for any } t > 0. \quad (1.12)$$

Then Ω must be a ball.

This theorem is shown by combining the following lemma with Aleksandrov's sphere theorem.

Lemma 1.4 *For any $x \in \Gamma_R$, as $t \rightarrow +0$*

$$t^{-\frac{N+1}{4}} \int_{B_R(x)} u(z, t) dz \longrightarrow c(m, N) \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(y(x)) \right) \right\}^{-\frac{1}{2}}, \quad (1.13)$$

where $c(m, N)$ is a positive constant depending only on m and N , and $y(x) \in \partial\Omega$ is a point given in (1.7).

Next, let us consider the heat equation. Let Ω be a bounded convex domain in \mathbb{R}^N . The physical situation can be modeled as the following initial-boundary value problem for the heat equation:

$$u_t = \Delta u \quad \text{in } \Omega \times (0, \infty), \quad (1.14)$$

$$u = 1 \quad \text{on } \partial\Omega \times (0, \infty), \quad (1.15)$$

$$u = 0 \quad \text{on } \Omega \times \{0\}, \quad (1.16)$$

where $u = u(x, t)$ denotes the normalized temperature at a point $x \in \Omega$ and at a time $t > 0$. Since Ω is convex, a result of [BL] shows that $\log(1 - u(x, t))$ is concave in x , which, together with the analyticity of u in x , implies that for each time $t > 0$ there exists a unique point $x(t) \in \Omega$ satisfying

$$\{x \in \Omega : \nabla u(x, t) = 0\} = \{x(t)\}. \quad (1.17)$$

The point $x(t)$ is the unique cold spot for each time $t > 0$. Put $I = \{x \in \Omega : d(x) = \max_{y \in \Omega} d(y)\}$, where d is the function defined by (1.5). Then, we have

$$\text{dist}(x(t), I) \rightarrow 0 \text{ as } t \rightarrow +0, \quad (1.18)$$

since the function $-4t \log u(x, t)$ attains its maximum at $x = x(t)$ for each $t > 0$ and a result of Varadhan [V] shows that

$$-4t \log u(x, t) \rightarrow d(x)^2 \text{ as } t \rightarrow +0 \text{ uniformly on } \bar{\Omega}. \quad (1.19)$$

In conclusion, the cold spot $x(t)$ starts from I . Also, as $t \rightarrow \infty$, $x(t)$ tends to the point at which a positive first eigenfunction of $-\Delta$ with the homogeneous Dirichlet boundary condition attains its maximum. There was a conjecture of Klamkin concerning the behavior of the cold spot. For simplicity, let Ω contain the origin. The conjecture of Klamkin [Kl] stated that if the origin is a stationary cold spot, that is, if $x(t) \equiv 0$, then Ω must be centro-symmetric with respect to the origin. This was denied by Gulliver-Willms [GW] and Kawohl [Ka]. A typical counterexample is an equilateral triangle in the plane. After that Chamberland-Siegel [CS] posed the following conjecture.

Conjecture 1.5 (Chamberland-Siegel) *If the origin is a stationary cold spot in a bounded convex domain Ω , then Ω is invariant under the action of an essential subgroup G of orthogonal transformations.*

A subgroup G is said to be *essential* if, for every $x \neq 0$, there exists an element $g \in G$ such that $gx \neq x$. Of course, the condition that Ω is invariant under the action of an essential subgroup G of orthogonal transformations is a sufficient condition for the origin being a stationary cold spot. Indeed, if Ω is invariant under the action of an essential subgroup G of orthogonal transformations, then by the unique solvability of the initial-Dirichlet problem (1.14)-(1.16) the solution u itself is invariant under the action of G . Namely, we have $u(x, t) \equiv u(gx, t)$ ($x \in \Omega$, $t > 0$, $g \in G$). This, together with the assumption that G is essential, implies that $\nabla u(0, t) = 0$ ($t > 0$), and then it follows from (1.17) that the origin is a stationary cold spot.

Our previous results to Conjecture 1.5 is

Theorem 1.6 ([MS 4]) *When $N = 2$, the following hold true.*

- (1) *If Ω is a triangle and the origin is a stationary cold spot, then Ω must be an equilateral triangle centered at the origin.*
- (2) *If Ω is a convex quadrangle and the origin is a stationary cold spot, then Ω must be a parallelogram centered at the origin.*

- (3) If Ω is a non-convex quadrangle, then there is no stationary critical point of u in Ω . In particular, there is no stationary cold spot.

In (1) of Theorem 1.6, G is the cyclic group generated by the rotation of the angle $\frac{2\pi}{3}$, and in (2) $G = \{I, -I\}$ where I is the identity mapping. The proof is based on two ingredients; one is the balance law around stationary critical points of the heat flow (see Theorem 3.2 (ii)) and the other is making use of the asymptotic behavior as $t \rightarrow +0$ of solutions of the heat equation due to Varadhan [V].

Another purpose of this talk is to report the following results.

Theorem 1.7 *Let $N = 2$ and suppose that Ω is a convex polygon whose inscribed circle touches every side of $\partial\Omega$. Then the following hold true.*

- (1) *If Ω is a pentagon and the origin is a stationary cold spot, then Ω must be a regular pentagon centered at the origin.*
- (2) *If Ω is a hexagon and the origin is a stationary cold spot, then Ω is invariant under the action of the rotation of one of angles $\frac{\pi}{3}, \pi, \frac{2\pi}{3}$.*

This theorem is shown by the following general result.

Theorem 1.8 *Let $N = 2$ and let Ω be a convex m -polygon with $m \geq 5$. Suppose that the origin is a stationary cold spot and the sphere $\partial B_R(0)$ touches every side of $\partial\Omega$, say $\partial\Omega \cap \partial B_R(0) = \{p_1, \dots, p_m\}$. Let q_1, \dots, q_k ($1 \leq k \leq m$) be the nearest k corners of $\partial\Omega$ to the origin. Then the following hold.*

$$\sum_{j=1}^m p_j = 0 \quad \text{and} \quad \sum_{j=1}^k q_j = 0. \quad (1.20)$$

The first equation of (1.20) follows directly from Theorem 3.2 (ii), and the second equation of (1.20) is new.

2 Outlines of proofs

Proof of Lemma 1.4 (an outline) : In [CDE], for sufficiently short time interval $(0, t_0)$ a subsolution and a supersolution for problem (1.1)-(1.3) were constructed. We use these to know that there exist a number $\xi_0 > 0$ and a nonnegative function $F = F(\xi)$ ($\xi \geq 0$) satisfying $F(0) = 1$ and $F(\xi) = 0$ for $\xi \geq \xi_0$ such that for sufficiently small $t > 0$ an approximate formula

$$u(x, t) \sim F\left(\frac{d(x)}{\sqrt{t}}\right) \quad (2.1)$$

holds true, roughly speaking. For any $x_0 \in \Gamma_R$ and for any $t \in (0, t_0)$ we have from the coarea formula

$$\int_{B_R(x_0)} F\left(\frac{d(x)}{\sqrt{t}}\right) dx = \int_0^{2R} F\left(\frac{s}{\sqrt{t}}\right) \left\{ \int_{\Gamma_s \cap B_R(x_0)} dS_x \right\} ds, \quad (2.2)$$

where Γ_s is given as in (1.10) and dS_x denotes the area element of Γ_s . We calculate

$$\lim_{s \rightarrow +0} s^{-\frac{N-1}{2}} \int_{\Gamma_s \cap B_R(x_0)} dS_x = 2^{\frac{N-1}{2}} \omega_{N-1} \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(y(x_0)) \right) \right\}^{-\frac{1}{2}}, \quad (2.3)$$

where ω_{N-1} denotes the volume of unit ball in \mathbb{R}^{N-1} . Hence

$$\begin{aligned} \int_{B_R(x_0)} F\left(\frac{d(x)}{\sqrt{t}}\right) dx &= \int_0^{2R} F\left(\frac{s}{\sqrt{t}}\right) s^{\frac{N-1}{2}} \left\{ s^{-\frac{N-1}{2}} \int_{\Gamma_s \cap B_R(x_0)} dS_x \right\} ds \\ &= t^{\frac{N+1}{4}} \int_0^{\frac{2R}{\sqrt{t}}} F(\xi) \xi^{\frac{N-1}{2}} \left\{ (\sqrt{t}\xi)^{-\frac{N-1}{2}} \int_{\Gamma_{\sqrt{t}\xi} \cap B_R(x_0)} dS_x \right\} d\xi. \end{aligned}$$

Therefore, with the aid of the Lebesgue's dominated convergence theorem, from (2.3) we see that as $t \rightarrow +0$

$$t^{-\frac{N+1}{4}} \int_{B_R(x_0)} F\left(\frac{d(x)}{\sqrt{t}}\right) dx \longrightarrow c(m, N) \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(y(x_0)) \right) \right\}^{-\frac{1}{2}},$$

where $c(m, N)$ is a positive constant depending only on m and N . This, together with (2.1), gives the formula (1.13).

Proof of Theorem 1.8 (an outline): We introduce a subsolution $u^- = u^-(x, t)$ and a supersolution $u^+ = u^+(x, t)$ for problem (1.14)-(1.16). Denote by ν_1, \dots, ν_m the interior normal unit vectors to $\partial\Omega$ at the points p_1, \dots, p_m , respectively. Define the function $f = f(\xi)$ by

$$f(\xi) = \frac{1}{\sqrt{\pi}} \int_{\xi}^{\infty} e^{-\frac{1}{4}s^2} ds \quad \text{for all } \xi \in \mathbb{R}. \quad (2.4)$$

Hence u^- and u^+ are given by

$$u^-(x, t) = \max_{1 \leq j \leq m} f\left(\frac{(x - p_j) \cdot \nu_j}{\sqrt{t}}\right), \quad (2.5)$$

$$u^+(x, t) = \sum_{j=1}^m f\left(\frac{(x - p_j) \cdot \nu_j}{\sqrt{t}}\right). \quad (2.6)$$

Then it follows from the comparison principle that

$$u^- \leq u \leq u^+ \text{ in } \Omega \times (0, \infty). \quad (2.7)$$

Let $R^* = |q_1|$ ($= |q_2| = \dots = |q_k|$). Then $R^* > R$. By using the fact that Ω is a convex polygon, with a reflection argument we can extend the solution u to a solution $u^* = u^*(x, t)$ of the heat equation in a larger domain $\Omega^* \times (0, \infty) \supset \Omega \times (0, \infty)$, where $B_{R^*}(0) \subset \Omega^*$. Since the origin is a stationary cold spot, we see that the origin is a stationary critical point of u^* . Therefore, by using Theorem 3.2 (ii), we get

$$\int_{B_{R^*}(0)} xu^*(x, t) dx = 0 \text{ for any } t > 0. \quad (2.8)$$

Letting $t \rightarrow +0$ yields

$$2 \int_{B_{R^*}(0) \setminus \Omega} x dx = 0, \quad (2.9)$$

which implies the first equation of (1.20).

Denote by \mathcal{R} the region obtained by reflecting $B_{R^*}(0) \setminus \bar{\Omega}$ with respect to each side of $\partial\Omega$, and put $E = (B_{R^*}(0) \cap \Omega) \setminus \bar{\mathcal{R}}$. Note that $\mathcal{R} \subset \Omega$ and $E \subset \Omega$. Denote by x^* the reflection of the point x with respect to each side of $\partial\Omega$. Then $u^*(x, t) \equiv 2 - u^*(x^*, t)$ because of (1.15). Therefore it follows from (2.9) that for any $t > 0$

$$0 = \int_{B_{R^*}(0)} xu^*(x, t) dx = \int_E xu(x, t) dx + \int_{\mathcal{R}} (x - x^*)u(x, t) dx. \quad (2.10)$$

Then, with the help of (2.7), we investigate the limit

$$\lim_{t \rightarrow +0} \frac{1}{t} \{ \text{the right-hand side of (2.10)} \},$$

and by using the first equation of (1.20), we see that the second equation of (1.20) holds true.

3 Appendices and Remarks

1. We quote Aleksandrov's sphere theorem from [Alek].

Theorem 3.1 (Aleksandrov) *Let $N \geq 3$, and let $\Phi = \Phi(\kappa_1, \dots, \kappa_{N-1})$ be a C^1 function defined for $\kappa_1 \geq \dots \geq \kappa_{N-1}$ and satisfying $\frac{\partial \Phi}{\partial \kappa_i} > 0$ ($i = 1, \dots, N-1$). Suppose that we have a closed C^2 hypersurface S embedded in \mathbb{R}^N . If on S the function Φ of its principal curvatures $\kappa_1, \dots, \kappa_{N-1}$ is constant, then S must be a sphere.*

2. We quote the balance law of the temperature around stationary zeros and stationary critical points of the heat flow from [MS 3].

Theorem 3.2 (Balance law) *Let G be a domain in \mathbb{R}^N ($N \geq 2$). Fix a point $x_0 \in G$, and set $d_* = \text{dist}(x_0, \partial G)$. Suppose that $v = v(x, t)$ satisfies the heat equation in $G \times (0, \infty)$. Then the following hold:*

(i) $v(x_0, t) = 0$ ($t > 0$) if and only if

$$\int_{\partial B_r(x_0)} v(x, t) dS_x = 0 \quad ((r, t) \in (0, d_*) \times (0, \infty)), \quad (3.1)$$

where dS_x denotes the area element of the sphere $\partial B_r(x_0)$.

(ii) $\nabla v(x_0, t) = 0$ ($t > 0$) if and only if

$$\int_{\partial B_r(x_0)} (x - x_0)v(x, t) dS_x = 0 \quad ((r, t) \in (0, d_*) \times (0, \infty)). \quad (3.2)$$

(ii) was proved in [MS 1], and (i) was stated in [MS 2]. In [MS 3], another proof of (i) and a shorter proof of (ii) were given.

3. For the heat equation, the sphere theorem corresponding to Theorem 1.3 is the following.

Theorem 3.3 ([MS 3]) *Let Ω be a bounded domain in \mathbb{R}^N ($N \geq 2$) satisfying the exterior sphere condition and suppose that D is a domain satisfying the interior cone condition, and such that $\bar{D} \subset \Omega$. Assume that the boundary ∂D is a stationary isothermic surface of the solution u of problem (1.14)-(1.16), that is,*

$$u(x, t) = a(t) \quad ((x, t) \in \partial D \times (0, \infty)) \quad (3.3)$$

for some function $a : (0, \infty) \rightarrow (0, \infty)$. Then Ω must be a ball.

The proof in [MS 3] is different from that of Theorem 1.3. The outline is as follows. The asymptotic behavior (1.19), together with assumption (3.3), implies that

$$\partial D = \{x \in \Omega : d(x) = R\} (= \Gamma_R) \quad (3.4)$$

for some $R > 0$. Then Theorem 3.2 (i) yields that

$$\int_{\partial B_R(x)} u(z, t) dS_z = b(t) \quad \text{for any } x \in \Gamma_R \text{ and for any } t > 0 \quad (3.5)$$

for some function $b : (0, \infty) \rightarrow (0, \infty)$ (see [MS 3] for details).

For the unique solution u of the initial-Dirichlet problem (1.14)-(1.16), define the function $W = W(x, s)$ by

$$W(x, s) = s \int_0^{\infty} u(x, t) e^{-st} dt \quad ((x, s) \in \bar{\Omega} \times (0, \infty)). \quad (3.6)$$

Then, W satisfies the following.

$$\Delta W - s W = 0 \quad \text{in } \Omega \quad \text{and} \quad W = 1 \quad \text{on } \partial\Omega. \quad (3.7)$$

By a result of Varadhan [V], we have

$$-\frac{1}{\sqrt{s}} \log W(x, s) \rightarrow d(x) \quad \text{as } s \rightarrow \infty \quad \text{uniformly on } \bar{\Omega}. \quad (3.8)$$

In [V], Varadhan first showed (3.8) and then, by using it, he showed (1.19). Roughly speaking, (3.8) means that for sufficiently large $s > 0$ we may have an approximate formula

$$W(x, s) \sim e^{-\sqrt{sd(x)}}.$$

This formula, together with Laplace's method, yields

Theorem 3.4 ([MS 3]) *Let Ω be a bounded domain in \mathbb{R}^N with piecewise C^2 boundary $\partial\Omega$. Assume that, for an open ball $B_r(x_0) \subset \Omega$, the set $\partial\Omega \cap \partial B_r(x_0)$ consists of a finite number of points p_1, \dots, p_K , and $\kappa_j(p_k) < \frac{1}{r}$ ($j = 1, \dots, N-1$) for any $k = 1, \dots, K$. Then, for any continuous function φ on \mathbb{R}^N , the following formula holds:*

$$\lim_{s \rightarrow \infty} s^{\frac{N-1}{4}} \int_{\partial B_r(x_0)} \varphi(x) W(x, s) dS_x = (2\pi)^{\frac{N-1}{2}} \sum_{k=1}^K \varphi(p_k) \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{r} - \kappa_j(p_k) \right] \right\}^{-\frac{1}{2}}. \quad (3.9)$$

By setting $r = R$, $x_0 = x \in \Gamma_R$, $K = 1$, and $\varphi \equiv 1$, we use Theorem 3.4. Hence, in view of (3.5) and (3.6), we use Aleksandrov's sphere theorem to obtain the conclusion of Theorem 3.3. Remark that we can not use the function W to prove Theorem 1.3.

Acknowledgement.

This research was partially supported by Grants-in-Aid for Scientific Research (B) (# 12440042) and (B) (# 15340047) of Japan Society for the Promotion of Science.

References

- [Alek] A. D. Aleksandrov, Uniqueness theorems for surfaces in the large V, *Vestnik Leningrad Univ.* 13, no. 19 (1958), 5–8. (English translation: *Amer. Math. Soc. Transl., Ser. 2*, 21 (1962), 412–415.)
- [Ar] D. G. Aronson, The porous medium equation, in "Nonlinear Diffusion Problems" (A. Fasano & M. Primicerio, eds.), Springer Lecture Notes in Math. 1224 (1986), 1–46.
- [BL] H. L. Brascamp and E. H. Lieb, On extensions of Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log-concave functions, and with an application to the diffusion equation. *J. Funct. Anal.* 22 (1976), 366–389.
- [CS] M. Chamberland and D. Siegel, Convex domains with stationary hot spots. *Math. Methods Appl. Sci.* 20 (1997), 1163–1169.
- [CDE] C. Cortázar, M. Del Pino, and M. Elgueta, On the short-time behavior of the free boundary of a porous medium equation, *Duke Math. J.* 87 (1997), 133–149.
- [GW] R. Gulliver and N. B. Willms, A conjectured heat flow problem, in *Solutions*, *SIAM Rev.* 37 (1995), 100–104.
- [Ka] B. Kawohl, A conjectured heat flow problem, in *Solutions*, *SIAM Rev.* 37 (1995), 104–105.
- [Kl] M. S. Klamkin, A conjectured heat flow problem, in *Problems*, *SIAM Rev.* 36 (1994), 107.
- [MS 1] R. Magnanini and S. Sakaguchi, The spatial critical points not moving along the heat flow, *J. Anal. Math.* 71 (1997), 237–261.
- [MS 2] R. Magnanini and S. Sakaguchi, Spatial critical points not moving along the heat flow II : The centrosymmetric case, *Math. Z.* 230 (1999), 695–712, Corrigendum, 232 (1999), 389.
- [MS 3] R. Magnanini and S. Sakaguchi, Matzoh ball soup: Heat conductors with a stationary isothermic surface, *Ann. of Math. (2)* 156 (2002), 931–946.
- [MS 4] R. Magnanini and S. Sakaguchi, On heat conductors with a stationary hot spot, *Ann. Mat. Pura Appl.*, in press.
- [V] S. R. S. Varadhan, On the behavior of the fundamental solution of the heat equation with variable coefficients, *Comm. Pure Appl. Math.* 20 (1967), 431–455.

