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RESIDUES OF COMPLEX ANALYTIC FOLIATIONS RELATIVE TO SINGULAR INVARIANT SUBVARIETIES

Tatsuo Suwa

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RESIDUES OF COMPLEX ANALYTIC FOLIATIONS RELATIVE TO SINGULAR INVARIANT SUBVARIETIES*)

TATSUO SUWA

In this article, we summarize the results in [LS] and, at the same time, supplement relevant material that are omitted there.

In section 1, we review the index and the index formula of C. Camacho and P. Sad [CS], which is the starting point of our research. If S is a complex analytic surface and if C is a compact non-singular curve in S, there is such a global invariant as the first Chern class $c_1(N_C)$ of the normal bundle N_C of C in S. It is determined by the way C is embedded in S and is equal to the self-intersection number C^2 of C in this case. In [CS] it is found that, if there is a one dimensional complex analytic foliation (i.e., a foliation determined locally by a holomorphic vector field) on S leaving C invariant, at each of the points on C which are singular points of the foliation, one can define an index and the sum of the indices is equal to $c_1(N_C)$. The formula was effectively used to prove the existence of a local integral curve (separatrix) for arbitrary holomorphic vector field with an isolated singularity at the origin in $\mathbb{C}^2$. In section 2, after we review some basic definitions and facts about complex analytic singular foliations and their invariant subvarieties, we state the main results. Let $\mathcal{F}$ be an s-dimensional singular foliation on a complex manifold W and V a p-dimensional subvariety of W which is left invariant by $\mathcal{F}$. We denote by $S(\mathcal{F}, V)$ the set of singular points of V and singular points of $\mathcal{F}$ on V. We assume that V is a locally complete intersection and that its normal sheaf N_V, which is locally free on V, admits an extension N (as a C^∞ vector bundle) to a neighborhood of V. For each compact connected component Z of $S(\mathcal{F}, V)$ and a symmetric and homogeneous polynomial φ of degree $d > p-s$, we have the residue in the homology $H_{2p-2d}(Z; \mathbb{C})$ of Z. Moreover, if V is compact, then the sum of residues (taken in $H_{2p-2d}(V; \mathbb{C})$) is equal to $\varphi(N) \cap [V]$ (Theorem (2.4)). In Theorem (2.7), we give an explict integral formula for the residue when $s = 1$ and the singularity is isolated. In section 3, we recall some basic material used to obtain these results. The key fact here is that the foliation induced on the invariant subvariety "acts" on the normal bundle of the subvariety away from the singularity. From this we have a Bott type vanishing theorem, which is the essential ingredient in defining the residues, i.e. in localizing the characteristic classes of N_V to the singular set $S(\mathcal{F}, V)$. The actual

*) Report based on a joint work with D. Lehmann

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proofs and computations are conveniently done using the integration theory on the Čech-de Rham cohomology. The assumption that there is an extension N is needed to construct forms to represent the characteristic classes of N_V in the neighborhood of the singularity. These are explained in section 4.

1. The Camacho-Sad index

We briefly recall the Camacho-Sad index of a holomorphic vector field on a complex surface relative to a (non-singular) invariant curve and their index theorem ([CS]). Let $X = A_1\frac{\partial}{\partial z_1} + A_2\frac{\partial}{\partial z_2}$ be a holomorphic vector field near the origin 0 in $\mathbb{C}^2 = \{(z_1, z_2)\}$ with possibly an isolated singularity at 0. Suppose that X leaves the z_1-axis, denoted by C, invariant, which means that $X(z_2)$ is a multiple of z_2. Thus we may write $A_2 = cz_2$, for some holomorphic function c near 0. Then the index of X at 0 relative to C is defined to be the residue of the meromoprphic 1-form $\frac{c}{A_1}|_{z_2=0}dz_1$ on the z_1-plane. In fact it depends only on the foliation $\mathcal{F}$ generated by X and not on X itself. Thus we denote it by $\mathrm{Ind}_0(\mathcal{F}, C)$;

$$\mathrm{Ind}_0(\mathcal{F}, C) = \frac{1}{2\pi\sqrt{-1}}\int_\gamma \frac{cdz_1}{A_1},$$

where γ denotes a closed path in the z_1-plane C going around 0 once counterclockwise.

(1.1) Example. If $X = \lambda_1 z_1\frac{\partial}{\partial z_1} + \lambda_2 z_2\frac{\partial}{\partial z_2}$ with λ_1 and λ_2 complex numbers, it leaves the z_1 axis invariant, and we compute

$$\mathrm{Ind}_0(\mathcal{F}, C) = \frac{1}{2\pi\sqrt{-1}}\int_\gamma \frac{\lambda_2 dz_1}{\lambda_1 z_1} = \frac{\lambda_2}{\lambda_1}.$$

Note that X also leaves the z_2-axis invariant and the index relative to that is $\frac{\lambda_1}{\lambda_2}$.

Thus the above index is not an integer, in general. Geometrically it is determined by the first order term in the holonomy.

Now let S be a complex surface, i.e. a complex manifold of dimension 2, and C a compact non-singular curve in S. Suppose we have a one dimensional complex analytic foliation (foliation locally defined a single vector field) $\mathcal{F}$ on S leaving C invariant. Then the Camacho-Sad index theorem says that

$$\sum_{p\in S(\mathcal{F})} \mathrm{Ind}_p(\mathcal{F}, C) = c_1(N_C),$$

where $S(\mathcal{F})$ denotes the singular set of $\mathcal{F}$ and $c_1(N_C)$ the first Chern class of the normal bundle N_C of C in S, which is equal to the self-intersection number C^2 of C.

(1.2) *Remarks.* 1°. In the above formula, the right hand side is an integer which depends only on the way C is embedded in S and is independent of the foliation.

2°. The intersection number of two distinct curves is always localized to the points of intersection. On the other hand the self-intersection number of a curve may not be (complex analytically) localized. However, in the presence of a foliation leaving the curve invariant, it is localized to the singular points.

3°. We are naturally led to consider generalizations in two directions. One is to consider the case of singular invariant curves. Generalizations in this direction are done by A. Lins Neto [LN1], M Soares [So] and the author [Su2]. It turns out that the natural global invariant is again the self-intersection number C^2 of the invariant curve C. The other is to consider the case of a higher dimensional singular foliation on a higher dimensional manifold leaving a (higher dimensional) submanifold invariant. This was worked out by J.-P. Brasselet (unpublished), B. Gmira [G] and Lins Neto [LN2] in special cases and by D. Lehmann [L3] in general. In the following sections we describe how it is generalized to the case of higher dimensional singular invariant subvarieties so that it includes the cases considered by the above authors.

2. The residues

For generalities on complex analytic singular foliations on complex manifolds, we refer to [BB] and [Su1]. Let W be a (connected) complex manifold of dimension n and TW the holomorphic tangent bundle of W. We denote by $\mathcal{O}_W$ the sheaf of germs of holomorphic functions on W and by $\Theta_W = \mathcal{O}_W(TW)$ the tangent sheaf on W. In general, for a coherent sheaf $\mathcal{S}$ on W, we set

$$Sing(\mathcal{S}) = \{\, w \in W \mid \mathcal{S}_x \text{ is not } \mathcal{O}_{W,w} - \text{free} \,\},$$

where $\mathcal{S}_w$ and $\mathcal{O}_{W,w}$ denote the stalks at w of the corresponding sheaves. The rank of $\mathcal{S}$ is defined to be the rank of the locally free sheaf $\mathcal{S}|_{W-Sing(\mathcal{S})}$.

We define (the tangent sheaf of) a singular foliation on W to be an integrable coherent subsheaf of Θ_W. Thus let $\mathcal{F}$ be a coherent subsheaf of Θ_W. We set

$$S(\mathcal{F}) = Sing(\Theta_W/\mathcal{F}).$$

Note that $S(\mathcal{F})$ is an analytic set containing $Sing(\mathcal{F})$. We may describe $S(\mathcal{F})$ explicitly as follows. Let U be a sufficiently small coordinate neighborhood with complex analytic coordinates $(z_1, z_2, \ldots, z_n)$ and let $X_1, X_2, \ldots, X_r$ be generators of $\mathcal{F}$ on U. We write $X_i = \sum_{j=1}^{n} A_{ij}(z)\frac{\partial}{\partial z_j}$. Then we have

$$S(\mathcal{F}) \cap U = \{\, z \in U \mid \operatorname{rank}(A_{ij}(z)) < s \,\},$$

where s is the rank of $\mathcal{F}$.

(2.1) Definition. (The tangent sheaf of) a singular foliation on W is a coherent subsheaf $\mathcal{F}$ of Θ_W which satisfies the "integrability condition", i.e.,

$$(*) \qquad [\mathcal{F}_w, \mathcal{F}_w] \subset \mathcal{F}_w \quad \text{for } w \in W - S(\mathcal{F}).$$

We say that $\mathcal{F}$ is reduced if it is full in Θ_X, i.e., for any open set U in W,

$$\Gamma(U, \Theta_X) \cap \Gamma(U - S(\mathcal{F}), \mathcal{F}) = \Gamma(U, \mathcal{F}),$$

where $\Gamma(U, \mathcal{S})$ denotes, for a sheaf $\mathcal{S}$, the space of sections of $\mathcal{S}$ on U.

(2.2) *Remarks.* 1°. If s is the rank of $\mathcal{F}$, then $\mathcal{F}$ defines a non-singular foliation of dimension s on $X - S(\mathcal{F})$. We call $\mathcal{F}$ a singular foliation of dimension s.

2°. Given a singular foliation $\mathcal{F}$ on W. There is a canonical way of obtaining a reduced one from $\mathcal{F}$ ([Su1]).

3°. If $\mathcal{F}$ is reduced, then the condition (*) holds for all w in W.

4°. If $\mathcal{F}$ is locally free, then $\mathcal{F}$ is reduced if and only if $\operatorname{codim} S(\mathcal{F}) \geq 2$.

Now let V be a reduced and irreducible analytic subvariety of W of dimension p. We denote by $\mathcal{I}_V$ the ideal sheaf of $\mathcal{O}_W$ defining V. Thus the quotient sheaf $\mathcal{O}_V = \mathcal{O}_W/\mathcal{I}_V$ is the sheaf of germs of holomorphic functions on V. We also denote by $Sing(V)$ the singular set of V. We consider the sheaf $\Theta_W(\log V)$ of logarithmic vector fields of V ([Sa], [BR]), i.e., $\Theta_W(\log V) = \{ X \in \Theta_W \mid X(\mathcal{I}_V) \subset \mathcal{I}_V \}$. Note that a germ X in Θ_W is in $\Theta_W(\log V)$ if and only if it is tangential to $V - Sing(V)$. We define the tangent sheaf Θ_V of V to be the images of the sheaf homomorphisms $\Theta_W(\log V) \otimes \mathcal{O}_V \to \Theta_W \otimes \mathcal{O}_V$. Note that there is a natural action of Θ_V on $\mathcal{O}_V$. The sheaf Θ_V is also defined as the dual of the sheaf of holomorphic 1-forms Ω_V on V, which is defined by the exact sequence

$$\mathcal{I}_V/\mathcal{I}_V^2 \to \Omega_W \otimes \mathcal{O}_V \to \Omega_V \to 0.$$

We have $\Theta_V = \mathcal{H}om_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$ so that we have the exact sequence

$$(2.3) \qquad 0 \to \Theta_V \to \Theta_W \otimes \mathcal{O}_V \xrightarrow{\pi} N_V,$$

where $N_V = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{I}_V/\mathcal{I}_V^2, \mathcal{O}_V)$. The restriction of N_V to $V - Sing(V)$ is the sheaf of germs of holomorphic sections of the normal bundle of $V - Sing(V)$ in W. Let $\mathcal{F}$ be a reduced dim s singular foliation and assume that

(1) $\mathcal{F}$ leaves V invariant, i.e., $\mathcal{F} \subset \Theta_W(\log V)$, and that
(2) $V \not\subset S(\mathcal{F})$.

Thus $\mathcal{F}$ induces a dim s singular foliation on $V - Sing(V)$, which may not be reduced. The singular foliation $\mathcal{F}_V$ on V induced from $\mathcal{F}$ is defined to be the image of the sheaf homomorphism $\mathcal{F} \otimes \mathcal{O}_V \to \Theta_W \otimes \mathcal{O}_V$. Note that from the above condition

(1) above, $\mathcal{F}_V$ is a subsheaf of Θ_V. We set $S(\mathcal{F},V) = Sing(V) \cup (V \cap S(\mathcal{F}))$. In what follows, we assume that V is a locally complete intersection of codimension $q = n-p$. Thus the sheaf N_V is locally free on V. We further assume that there is a C^∞ vector bundle N on a neighborhood U of V which extend the holomorphic vector bundle, again denoted by N_V, defining the sheaf N_V. Note that every hypersurface satisfies the above assumptions. More generally, a subvariety defined by a regular section of a holomorphic vector bundle on W satisfies the assumptions.

(2.4) Theorem. *Let W, $\mathcal{F}$ and V be as above and let φ be a symmetric and homogeneous polynomial of degree $d > p - s$.*
(1) If Z is a connected component of $S(\mathcal{F},V)$ which is compact, there is an element

$$\mathrm{Res}_\varphi(\mathcal{F},V;Z) \in H_{2p-2d}(Z;\mathbb{C}),$$

which depends only on the local behavior of $\mathcal{F}$ and V near Z.
(2) If V is compact, we have

$$\sum_Z i_* \mathrm{Res}_\varphi(\mathcal{F},V;Z) = i^*\varphi(N) \cap [V] \quad \text{in} \quad H_{2p-2d}(V;\mathbb{C}),$$

where the first i denotes the imbedding $Z \hookrightarrow V$ and the second i the imbedding $V \hookrightarrow U$.

(2.5) *Remarks.* 1°. A symmetric polynomial φ may be written as a polynomial $P(\sigma_1,\sigma_2,\dots)$ of elementary symmetric functions $\sigma_1,\sigma_2,\dots$. Then $\varphi(N)$ is defined to be the class in $H^{2d}(U;\mathbb{C})$ given by $\varphi(N) = P(c_1(N),c_2(N),\dots)$.

2°. If V is non-singular, we have the Poincaré duality $H_{2p-2d}(V;\mathbb{C}) \simeq H^{2d}(V;\mathbb{C})$ and the class $i^*\varphi(N)\cap[V]$ corresponds to $i^*\varphi(N) = \varphi(N_V)$. Thus the above formula reduces to the one in [L3].

3°. If V is a hypersurface in W, there is always a holomorphic line bundle N on W such that V is defined by a regular section of N. Namely, if $\{U_\alpha\}_{\alpha\in A}$ is a covering of W such that V is defined by $f_\alpha = 0$ on each U_α, then N is the line bundle on W represented by the cocycle $\{f_\alpha/f_\beta\}$.

Next we give an explicit form of the residue in the case $s = 1$ and Z is an isolated point. Thus we consider the following situation. Let X be a holomorphic vector field near the origin 0 in $\mathbb{C}^n$ and V a complete intersection variety through 0 defined by reduced equations $f_1 = \cdots = f_q = 0$ with possibly isolated singularity at 0. Assume that the set of zeros of X intersects with V only at 0, thus $Z = \{0\}$.

(2.6) Lemma. *There is a coordinate system $(z_1,\dots,z_n)$ near the origin 0 in $\mathbb{C}^n$ such that if we write $X = \sum_{i=1}^n A_i(z)\frac{\partial}{\partial z_i}$, then $A_1,\dots,A_p,f_1,\dots,f_q$, $p = n-q$ form a regular sequence in $\mathcal{O}_n$.*

For the proof of the above lemma, we refer to [LS].

From the condition that V is invariant by $\mathcal{F} = (X)$, we may write $X(f_i) = \sum_{j=1}^{r} c_{ij} f_j$ with c_{ij} holomorphic functions near 0. Let C denote the matrix $(c_{ij}$. For each $i = 1, 2, \dots$, we define $\sigma_i(C)$ by

$$\det(I + tC) = 1 + t\sigma_1(C) + t^2\sigma_2(C) + \cdots$$

and for $\varphi = P(\sigma_1, \sigma_2, \dots)$, we set $\varphi(C) = P(\sigma_1(C), \sigma_2(C), \dots)$. In our case, since $s = 1$, we have residues for polynomials φ of degree $p = n - q$. Hence $\mathrm{Res}_\varphi(\mathcal{F}, V; \{0\})$ is in $H_0(\{0\}; \mathbb{C}) = \mathbb{C}$ and is a number.

(2.7) Theorem. *In terms of the coordinate system of Lemma (2.6), we have*

$$\mathrm{Res}_\varphi(\mathcal{F}, V; \{0\}) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^p \int_\Gamma \frac{\varphi(C) dz_1 \wedge dz_2 \wedge \cdots \wedge dz_p}{A_1 A_2 \dots A_p},$$

where Γ is the p-cycle in V defined by

$$\Gamma = \{ z \mid |A_1(z)| = \cdots = |A_p(z)| = \varepsilon,\ f_1(z) = \cdots = f_q(z) = 0 \}$$

with orientation so that $d(\arg A_1) \wedge \cdots \wedge d(\arg A_p)$ is positive.

(2.8) *Remark.* In [LS], $\mathrm{Res}_\varphi(\mathcal{F}, V; \{0\})$ is denoted by $I(\mathcal{F}, V, \varphi, \nu)$, $\nu = N_V$ and $\varphi(C)$ already includes the constant $\left(\frac{\sqrt{-1}}{2\pi}\right)^p$.

Now we consider some special cases of Theorem (2.7):

(I) The case V is non-singular. In this case, it is obvious that the residue in Theorem (2.7) is identical with the one in [L3] Théorème 1. Especially if $p = q = 1$ and $\varphi = \sigma_1$, we recover the index of [CS] (in section 1, we denoted $\mathrm{Res}_{\sigma_1}(\mathcal{F}, C; \{0\})$ by $\mathrm{Ind}_0(\mathcal{F}, C)$).

(II) The case $q = 1$. In this case, since the rank of N is one, for the polynomial φ, we have essentially $\varphi = \sigma_1^p$, $p = n - 1$. Since $c_1(N)$ is the dual of the homology class $[V]$ in this case, the corresponding global invariant $i^*\varphi(N) \cap [V]$ in $H_0(V; \mathbb{C}) = \mathbb{C}$ is $c_1(N)^{n-1} \cap [V] = c_1(N)^n = V^n$, the self-intersection number of V in W. If we write $X(f) = cf$, for some holomorphic function c near 0, we have

$$\mathrm{Res}_\varphi(\mathcal{F}, V; \{0\}) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^{n-1} \int_\Gamma \frac{c^{n-1} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{n-1}}{A_1 A_2 \dots A_{n-1}},$$

where Γ is the $(n-1)$-cycle in V defined by

$$\Gamma = \{ z \mid |A_1(z)| = \cdots = |A_{n-1}(z)| = \varepsilon,\ f(z) = 0 \}$$

with orientation so that $d(\arg A_1) \wedge \cdots \wedge d(\arg A_{n-1})$ is positive. Especially, if $n = 2$, we recover the index defined in [Su2] (which coincides with the one in [LN1], if V is locally irreducible).

(2.9) Example. Let $W = \mathbb{C}^3 = \{(z_1, z_2, z_3)\}$ and V the subvariety defined by

$$f(z_1, z_2, z_3) = z_1^\ell + z_2^\ell + z_3^\ell = 0.$$

If $X = \sum_{i=1}^3 z_i \frac{\partial}{\partial z_i}$, then $X(f) = \ell f$. Since z_1, z_2 and f form a regular sequence in this case, we have

$$\mathrm{Res}_\varphi(\mathcal{F}, V; \{0\}) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_\Gamma \frac{\ell^2 dz_1 \wedge dz_2}{z_1 z_2},$$

where $\varphi = \sigma_1^2$ and Γ is the 2-cycle in V defined by

$$\Gamma = \{\, z \mid |z_1| = |z_2| = \varepsilon,\ f(z) = 0 \,\}$$

with orientation so that $d(\arg z_1) \wedge d(\arg z_2)$ is positive. On Γ, we have $z_3^\ell = -(z_1^\ell + z_2^\ell)$. Thus if we set $z_1 = \varepsilon e^{\sqrt{-1}\theta_i}$, $i = 1, 2$, Γ is an ℓ-fold covering of the set of (θ_1, θ_2) in the square $[0, 2\pi]^2$. Since $\frac{dz_1 \wedge dz_2}{z_1 z_2} = -d\theta_1 \wedge d\theta_2$, we get

$$\int_\Gamma \frac{dz_1 \wedge dz_2}{z_1 z_2} = -4\pi^2 \ell.$$

Hence we have

$$\mathrm{Res}_{\sigma_1^2}(F, V; \{0\}) = \ell^3.$$

The natural compactification V_ℓ of V in $\mathbb{CP}^3$ does not acquire singularities at infinity. The foliation $\mathcal{F}$ generated by X can also extended to a foliation on $\mathbb{CP}^3$ which does not have singularities on V_ℓ at infinity ([LS]). Thus we recover:

$$V_\ell^3 = c_1^2(N) \cap [V_\ell] = \mathrm{Res}_{\sigma_1^2}(F, V; \{0\}) = \ell^3$$

for the self-intersection number of V_ℓ.

For more examples, see [LS].

3. Relevant material

(I) The Chern-Weil theory: For details, see, for example, [B], [BB], [GH] and [MS]. Let W be a differentiable manifold (of class C^∞) and E a C^∞ complex vector bundle of rank r. We denote by $C^\infty(W)$ the complex vector space of C^∞ functions on W and by $C^\infty(E)$ the $C^\infty(W)$-module of C^∞ sections of E on W. Denoting by $T_{\mathbb{R}}W$ the real tangent bundle of W, we set

$$\tau = T_{\mathbb{R}}^c W = T_{\mathbb{R}} W \otimes \mathbb{C}.$$

Note that if W is a complex manifold we have $\tau = TW \oplus \bar{T}W$. A connection ω for E is determined by a $\mathbb{C}$-linear map

$$\nabla : C^\infty(E) \longrightarrow C^\infty(\tau^* \otimes E) \qquad \text{with}$$

$$\nabla(fs) = df \otimes s + f\nabla(s) \quad \text{for} \quad f \in C^\infty(W) \text{ and } s \in C^\infty(E).$$

For connections $\omega_0, \omega_1, \ldots, \omega_k$ on E and for a symmetric and homogeneous polynomial φ of degree d, we have a $2d-k$ form $\Delta_{\omega_0\omega_1\cdots\omega_k}(\varphi)$ on W such that

$$d\Delta_{\omega_0\omega_1\cdots\omega_k}(\varphi) = \sum_{i=0}^{k}(-1)^i \Delta_{\omega_0\cdots\hat{\omega}_i\cdots\omega_k}(\varphi).$$

Thus, for a single connection ω, $\Delta_\omega(\varphi)$ defines a class in the de Rham cohomology $H^{2d}_{DR}(W;\mathbb{C})$ depending only on E and not on the connection chosen. We set

$$\varphi(E) = \left(\frac{\sqrt{-1}}{2\pi}\right)^d [\Delta_\omega(\varphi)].$$

Especially, when $\varphi = \sigma_i$, the i-th elementary symmetric polynomial, $\sigma_i(E) = c_i(E)$, the i-th Chern class of E.

(II) The vanishing theorem: Let V be a p-dimensional (complex) submanifold of a complex manifold W and $\mathcal{F}$ an s-dimensional foliation on W. Assume that $\mathcal{F}$ leaves V invariant and that $S(\mathcal{F}) \cap V = \phi$. Thus $\mathcal{F}$ induces a non-singular s-dimensional foliation $\mathcal{F}_V$ on V. Note that π in (2.3) is surjective in this case. The following lemma shows that there is a natural action of $\mathcal{F}_V$ on N_V (Cf. [L3] Lemme Fondamental 2.1).

(3.1) Lemma. *There is a $\mathbb{C}$-bilinear map*

$$\alpha : \mathcal{F}_V \times N_V \to N_V$$

such that, if we denote $\alpha(\xi,\nu)$, for ξ and ν in the stalks of $\mathcal{F}_V$ and N_V, respectively, over the same point, by $[\xi,\nu]$,

$$[f\xi,\nu] = f[\xi,\nu] \quad \text{and} \quad [\xi,f\nu] = \xi(f)\nu + f[\xi,\nu] \quad \text{for} \quad f \in \mathcal{O}_V.$$

Proof. Denoting by 1 the germ in $\mathcal{O}_V$ represented by the function with the constant value 1, a germ ξ in $\mathcal{F}_V$ is written as $\xi = X \otimes 1$ (tensor product in $\Theta_W \otimes \mathcal{O}_V$) with X in $\mathcal{F}$ and a germ ν in N_V as $\pi(Y \otimes 1)$. If we set

$$\alpha(\xi,\nu) = \pi([X,Y] \otimes 1),$$

from the integrability of $\mathcal{F}$ and the fact that the sheaf homomorphism $\mathcal{F} \otimes \mathcal{O}_V \to \Theta_W \otimes \mathcal{O}_V$ is injective away from $S(\mathcal{F})$, we see readily that it is well-defined. The rest of the lemma follows from the property of the Poisson bracket of vector fields. □

On the complex manifold V, $\mathcal{F}_V$ and N_V are locally free and $\mathcal{F}_V$ may be thought of as a subbundle of TV. We define

$$\delta : C^\infty(N_V) \longrightarrow C^\infty((\mathcal{F}_V \oplus \bar{T}V)^* \otimes N_V) = C^\infty(\mathrm{Hom}(\mathcal{F}_V, N_V) \oplus (\bar{T}^*V \otimes N_V)),$$

by

$$\delta(\nu) = (\xi \mapsto [\xi, \nu], \bar{\partial}\nu).$$

We call a connection ω for N_V *special* if it is determined by ∇ which extends δ, i.e., if the following diagram is commutative:

$$\begin{array}{ccc} C^\infty(N_V) & \xrightarrow{\nabla} & C^\infty(\tau^* \otimes N_V) \\ {\scriptstyle id}\downarrow & & {\scriptstyle \rho\otimes 1}\downarrow \\ C^\infty(N_V) & \xrightarrow{\delta} & C^\infty((\mathcal{F}_V \oplus \bar{T}V)^* \otimes N_V), \end{array}$$

where ρ denotes the canonical projection $\tau^* \to (\mathcal{F}_V \oplus \bar{T}V)^*$ (transposed of the natural inclusion $\mathcal{F}_V \oplus \bar{T}V \to \tau$). Note that, in general, a connection for a holomorphic vector bundle extends $\bar{\partial}$ if and only if the entries of the connection matrix with respect to a holomorphic frame are all of type $(1,0)$ ([BB] p.293). Note also that a special connection for N_V always exists ([BB] Lemma (2.5)).

We quote the vanishing theorem ([L3] Théorème 2, [KT]), which can also be proved in a similar way as [BB] Proposition (3.27) and Lemma (5.3).

(3.2) Lemma. *Let $\omega_0, \omega_1, \ldots, \omega_k$ be special connections for N_V. If φ is a symmetric and homogeneous polynomial of degree $d > p - s$, then*

$$\Delta_{\omega_0\omega_1\cdots\omega_k}(\varphi) = 0.$$

(III) Integration on the Čech-de Rham cohomology: For details on this subject, we refer to [BT], [L1] - [L3]. For an open set U in a differentiable manifold W, we denote by $A^q(U)$ the space of (complex valued) differentiable q-forms on U. Let $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ be an open covering of X whose index set $\mathcal{I}$ is a countable ordered set. We set

$$\mathcal{I}^p = \{\, I = (i_0, \ldots, i_p) \mid i_0 < \cdots < i_p,\ i_\nu \in \mathcal{I} \,\}$$

and $\mathcal{I}^* = \bigcup_{p\geq 0} \mathcal{I}^p$. Also, for $I = (i_0, \ldots, i_p)$ in $\mathcal{I}^p$, we set $|I| = p$ and $U_I = U_{i_0\cdots i_p}$ $= \bigcap_{\nu=0}^p U_{i_\nu}$ and define $C^p(\mathcal{U}, A^q)$ to be the direct product

$$C^p(\mathcal{U}, A^q) = \prod_{I\in\mathcal{I}^p} A^q(U_I).$$

Thus an element σ in $C^p(\mathcal{U}, A^q)$ assigns to each I in $\mathcal{I}^p$ an element σ_I in $A^q(U_I)$. The coboundary operator $\delta : C^p(\mathcal{U}, A^q) \to C^{p+1}(\mathcal{U}, A^q)$ is defined by

$$(\delta\sigma)_{i_0 \cdots i_{p+1}} = \sum_{\nu=0}^{p+1} (-1)^\nu \sigma_{i_0 \cdots \hat{i}_\nu \cdots i_{p+1}}.$$

This together with the exterior derivative d makes $C^*(\mathcal{U}, A^*)$ a doule complex. The (simple) complex obtained from this is denoted by $(A^*(\mathcal{U}), D)$. Thus $A^r(\mathcal{U}) = \bigoplus_{p+q=r} C^p(\mathcal{U}, A^q)$ and the differential $D : A^r(\mathcal{U}) \to A^{r+1}(\mathcal{U})$ is given by

$$(D\sigma)_I = \sum_{\nu=0}^{p} (-1)^\nu \sigma_{I-i_\nu} + (-1)^p \, d\sigma_I,$$

for I with $|I| = p$. We call $(A^*(\mathcal{U}), D)$ the Čech-de Rham complex and its cohomology $H^*(A^*(\mathcal{U}))$ the Čech-de Rham cohomology with respect to the covering $\mathcal{U}$. Since one of the spectral sequences associated to the double complex $C^*(\mathcal{U}, A^*)$ degenerates, we see that the restriction map $A^r(X) \to C^0(\mathcal{U}, A^r) \subset A^r(\mathcal{U})$ induces an isomorphism

$$H^r(W; \mathbb{C}) \simeq H^r(A^*(\mathcal{U})) \tag{3.3}$$

of the de Rham cohomology onto the Čech-de Rham cohomology. The pairing

$$A^r(\mathcal{U}) \times A^s(\mathcal{U}) \to A^{r+s}(\mathcal{U})$$

defined, for $\sigma \in A^r(\mathcal{U})$ and $\tau \in A^s(\mathcal{U})$, by

$$(\sigma \cup \tau)_{i_0 \cdots i_p} = \sum_{\nu=0}^{p} (-1)^{(r-\nu)(p-\nu)} \sigma_{i_0 \cdots i_\nu} \wedge \tau_{i_\nu \cdots i_p} \tag{3.4}$$

induces the cup product

$$H^r(A^*(\mathcal{U})) \times H^s(A^*(\mathcal{U})) \to H^{r+s}(A^*(\mathcal{U}))$$

with the usual properties.

Next, we recall the integration theory on stratified spaces ([L1] - [L3]). Let W be an oriented differentiable manifold of real dimension N and V a closed subset of W. Given a (locally finite) stratification $V = \coprod_{i \in \mathcal{I}'} V_i$ of V by submanifolds V_i of W. Contrary to [L1], we order the index set $\mathcal{I}'$ by $i \leq j$ if and only if $V_j \subset \bar{V}_i$. Let U_i be a tubular neighborhood of V_i in W. We set $U_0 = W - V$ and $\mathcal{I} = \mathcal{I}' \cup \{0\}$ and we put an order relation in $\mathcal{I}$ by $0 < i$ for any i in $\mathcal{I}'$. We denote by $\mathcal{U}$ the

covering $\{U_i\}_{i\in I}$ of W. We set $T_0 = U_0$ and, for each i in $\mathcal{I}'$, we let T_i be a closed disk bundle in U_i. Moreover, for each i in $\mathcal{I}$, we set

$$R_i = T_i \cap (\bigcap_{j>i}(X - \operatorname{Int} T_j))$$

and, for $I = (i_0, \ldots, i_p)$ in $\mathcal{I}^p$, we set $R_I = R_{i_0} \cap \cdots \cap R_{i_p} = \partial R_{i_0} \cap \cdots \cap \partial R_{i_p}$. We orient R_I as in [L1]. If W is compact, the integration

$$\int_W : A^N(\mathcal{U}) \to \mathbb{C}$$

defined by assigning to σ in $A^N(\mathcal{U})$ the sum $\sum_{I\in\mathcal{I}^*} \int_{R_I} \sigma_I$ induces integration on the cohomology $H^N(A^*(\mathcal{U}))$ which is compatible with the usual integration on the de Rham cohomology $H^N(W;\mathbb{C})$ ((3.3)). Also the pairing

$$B : A^r(\mathcal{U}) \times A^{N-r}(\mathcal{U}) \to A^N(\mathcal{U}) \to \mathbb{C}$$

defined as the composition of the cup product and the integration induces the Poincaré duality

$$H^r(A^*(\mathcal{U})) \simeq H_{N-r}(W;\mathbb{C}).$$

Suppose now that V is compact (W may not be). We denote by $A^r(\mathcal{U}, U_0)$ the kernel of the canonical projection $A^r(\mathcal{U}) \to A^r(U_0)$ and by $A^r(\mathcal{U},\mathcal{U}')$ the kernel of the canonical projection $A^r(\mathcal{U}) \to A^r(\mathcal{U}')$, where $\mathcal{U}' = \{U_i\}_{i\in\mathcal{I}'}$, a covering of V. Note that the cohomology of the complex $A^*(\mathcal{U}, U_0)$ is $H^*(W, W - V;\mathbb{C})$ and that of $A^*(\mathcal{U}')$ is $H^*(V;\mathbb{C})$. Since the restriction of the cup product to $A^r(\mathcal{U}, U_0) \times A^{N-r}(\mathcal{U},\mathcal{U}')$ is zero, we have the paring

$$B : A^r(\mathcal{U}, U_0) \times A^{N-r}(\mathcal{U}') \to \mathbb{C},$$

which induces the Alexander-Lefschetz duality

$$H^r(W, W - V;\mathbb{C}) \simeq H_{N-r}(V;\mathbb{C}).$$

4. Proofs

(I) Proof of Theorem (2.4): Let $S = \coprod_\alpha Z_\alpha$ be a decomposition of S into connected components and assume that each Z_α is compact. For each α, we take a stratification $\coprod_{i_\alpha\in\mathcal{I}_\alpha} V_{\alpha,i_\alpha}$ of Z_α by submanifolds V_{α,i_α} of W. Also, let U_{α,i_α} and U_1, respectively, be sufficiently small tubular neighborhoods of V_{α,i_α} and of $V_1 = V - S$. We denote by $\mathcal{U}_\alpha$ the covering $\{U_{\alpha,i_\alpha}\}_{i_\alpha\in\mathcal{I}_\alpha}$ of $U_\alpha = \bigcup_{i_\alpha\in\mathcal{I}_\alpha} U_{\alpha,i_\alpha}$ and by $\mathcal{U}$ the covering $\{U_1\} \cup (\bigcup_\alpha \mathcal{U}_\alpha)$ of $U = U_1 \cup (\bigcup_\alpha U_\alpha)$. Note that we may assume that U_α and U are regular neighborhoods of Z_α and of V, respectively. If

we further set $\mathcal{I} = \{1\} \cup (\bigcup_\alpha \mathcal{I}_\alpha)$, then we have a decomposition $\mathcal{I}^* = \mathcal{I}_1^* \coprod (\coprod_\alpha \mathcal{I}_\alpha^*)$, where $\mathcal{I}_1^*$ denotes the set of sequences I of indices in increasing order starting with 1. We take $R_1 \subset U_1$ and $R_{\alpha,i_\alpha} \subset U_{\alpha,i_\alpha}$ as in section 3 and we use a variant of the integration theory described there. Thus if V is compact, we have the evaluation

$$\int_V : A^{2p}(\mathcal{U}) \to \mathbb{C},$$

which is defined by $\int_V \sigma = \sum_{I \in \mathcal{I}^*} \int_{R_I \cap V} \sigma_I$ for σ in $A^{2p}(\mathcal{U})$. This defines a pairing

$$B : A^k(\mathcal{U}) \times A^{2p-k}(\mathcal{U}) \to \mathbb{C}$$

defined by

$$B(\sigma, \tau) = \int_V \sigma \cup \tau$$

for σ in $A^k(\mathcal{U})$ and τ in $A^{2p-k}(\mathcal{U})$, which induces a map

$$\kappa_V : H^k(A^*(\mathcal{U})) \simeq H^k(U; \mathbb{C}) \to H_{2p-k}(U; \mathbb{C}) \simeq H_{2p-k}(V; \mathbb{C}),$$

which coinsides with the cap product with the homology class $[V]$. If we take σ with $\sigma_1 = 0$, then, from the definition (3.4), we see that the cup product $\sigma \cup \tau$ depends only on τ_I for I in $\coprod_\alpha \mathcal{I}_\alpha^*$. So if we define $A^k(\mathcal{U}, U_1)$ by the exact sequence

$$0 \to A^k(\mathcal{U}, U_1) \to A^k(\mathcal{U}) \to A^k(U_1) \to 0,$$

even if V is not compact, we have the pairing

$$B : A^k(\mathcal{U}, U_1) \times A^{2p-k}(\bigcup_\alpha \mathcal{U}_\alpha) \to \mathbb{C},$$

which induces a map

$$\kappa_V : H^k(U, U_1; \mathbb{C}) \to H_{2p-k}(\bigcup_\alpha U_\alpha; \mathbb{C}) \simeq H_{2p-k}(S; \mathbb{C}) = \bigoplus_\alpha H_{2p-k}(Z_\alpha; \mathbb{C}).$$

Now, if we denote by $\rho : U_1 \to V_1$ the projection map, the restriction $N|_{U_1}$ of N to U_1 is diffeomorphic to the pull-back $\rho^* N_{V_1}$ of the normal bundle N_{V_1} of V_1. Let ω be a special connection for N_{V_1} and ω_1 the connection for N on U_1 defined by $\omega_1 = \rho^* \omega$. On each U_{α,i_α} we choose an arbitrary connection ω_{α,i_α} for N. The class $\varphi(N)$ in $H^{2d}(A^*(\mathcal{U}))$ is then represented by a cocycle σ with

$$\sigma_I = \left(\frac{\sqrt{-1}}{2\pi}\right)^d \Delta_{\omega_I}(\varphi),$$

where $\Delta_{\omega_I}(\varphi) = \Delta_{\omega_{i_1}\cdots\omega_{i_k}}(\varphi)$ for $I = (i_1, \ldots, i_k)$. Since ω is special, we have $\Delta_{\omega_1}(\varphi) = 0$. Hence, from the above remark, the class $\varphi(N)$ is in $H^{2d}(A^*(\mathcal{U}, U_0))$. We define the residue of $\mathcal{F}$ on Z_α relative to V with respect to φ, denoted by $\mathrm{Res}_\varphi(\mathcal{F}, V; Z_\alpha)$, to be the α component of $\kappa_V(\varphi(N))$ so that we may write

$$\kappa_V(\varphi(N)) = \sum_\alpha \mathrm{Res}_\varphi(\mathcal{F}, V; Z_\alpha).$$

It is now clear that we have the residue formula (2) above if V is compact. □

(4.1) *Remarks.* 1°. The above procedure can be summarized in the following diagram:

$$\begin{array}{ccc} H^{2d}(U;\mathbb{C}) & \xleftarrow[j^*]{} & H^{2d}(U, U_1;\mathbb{C}) \\ \kappa_V \downarrow & & \kappa_V \downarrow \\ H_{2p-2d}(V;\mathbb{C}) & \xleftarrow[i_*]{} & H_{2p-2d}(S;\mathbb{C}) \simeq \oplus_\alpha H_{2p-2d}(Z_\alpha;\mathbb{C}) \end{array}$$

where j^* denotes the canonical homomorphism. If we compute the class $\varphi(N)$ in $H^{2d}(U;\mathbb{C})$, which corresponds to $i^*\varphi(N) \cap [V]$ in $H_{2p-2d}(V;\mathbb{C})$ by κ_V, using a special connection, we see that it has a natural reduction in $H^{2d}(U, U_1;\mathbb{C})$ whose image by κ_V is the sum $\sum_\alpha \mathrm{Res}_\varphi(E, V; Z_\alpha)$ of the residues.

2°. In fact, it is not necessary to take a stratification of Z_α. Take a regular neighborhood U_α for each Z_α and work directly on the Čech-de Rham cohomology with respect to the covering $\{U_1\} \cup \{U_\alpha\}_\alpha$.

(II) Outline of the proof of Theorem (2.7): Let U_1 be a tubular neighborhood of $V_1 = V - \{0\}$ with the projection $\rho : U_1 \to V_1$ and U_α a small open disk about 0. We consider the Čech-de Rham cohomology $H^r(A^*(\mathcal{U}))$ associated with the covering $\mathcal{U} = \{U_1, U_\alpha\}$ of $U = U_1 \cup U_\alpha$. Here we recall the exact sequence of vector bundles

$$0 \to TV_1 \to TU|_{V_1} \xrightarrow{\pi} N_{V_1} \to 0.$$

We denote by $\nu_1, \ldots, \nu_q$ the frame $\pi(\frac{\partial}{\partial f_1}), \ldots, \pi(\frac{\partial}{\partial f_q})$ for the normal bundle N_{V_1}. Let ω_1 be the connection for N on U_1 given by $\omega_1 = \rho^*\omega$, where ω is a special connection for N_{V_1} on V_1. We take as ω_α an arbitrary connection for N on U_α for the moment. Since $\Delta_{\omega_1}(\varphi) = 0$, by definition the residue is given by

$$\mathrm{Res}_\varphi(\mathcal{F}, V; \{0\}) = \left(\frac{\sqrt{-1}}{2\pi}\right)^p \left(\int_{R_\alpha \cap V} \Delta_{\omega_\alpha}(\varphi) + \int_{R_{1\alpha} \cap V} \Delta_{\omega_1\omega_\alpha}(\varphi)\right),$$

where R_α and $R_{1\alpha}$ are defined as in section 3. In our case, denoting by T_1 a closed disk bundle over V_1 in U_1 and by T_α a closed disk in U_α, we have $R_1 = T_1 \cap (U - \mathrm{Int}\, T_\alpha)$ and $R_\alpha = T_\alpha$. Hence, $R_{1\alpha}$ is the restriction of the bundle

$\rho : T_1 \to V_1$ to $L = \partial T_\alpha \cap V$, which is the link of the singularity of V at 0, and we have $R_{1\alpha} \cap V = L$. In order to further reduce the computation, we take as ω_α the connection for N on U_α flat with respect to the frame for N obtained by extending the frame $\rho^*\nu_1, \ldots, \rho^*\nu_q$ throughout U_α. Then we have

$$\text{Res}_\varphi(E, V; \{0\}) = \left(\frac{\sqrt{-1}}{2\pi}\right)^p \int_L \Delta_{\omega_1 \omega_\alpha}(\varphi). \tag{4.2}$$

The rest of the proof is done in three steps:
1) Let $V_i = \{ m \in L \mid A_i(m) \neq 0 \}$ for $i = 1, \ldots, p$ and consider the Čech-de Rham cohomology with respect to the covering $\mathcal{V} = \{V_i\}$ of L. We study the properties of the connection ω_i' of N on V_i whose connection matrix with respect to the given trivialization is $C\frac{dz_i}{A_i}$.
2) Then we prove that $\Delta_{\omega_1 \omega_\alpha}(\varphi)$, which is a cocycle on L, is cohomologous, when imbedded in the total Čech-de Rham complex $A^*(\mathcal{V})$, to the element τ in $A^{2p-1}(\mathcal{V})$ defined by:

$$\begin{cases} \tau_{I_0} = \Delta_{\omega_1 \omega_1' \omega_2' \ldots \omega_p'}(\varphi), & \text{for } I_0 = (1, \ldots, p) \\ \tau_I = 0, & \text{for any simplex } I \text{ of dim} \neq p-1 \text{ in the nerve of } \mathcal{V}. \end{cases}$$

3) Finally, we prove that

$$\tau_{I_0} = \frac{\varphi(C) dz_1 \wedge dz_2 \wedge \cdots \wedge dz_p}{A_1 A_2 \cdots A_p}.$$

Using integration on $A^*(\mathcal{U})$ as recalled in section 3, this will achieve the proof. For details we refer to [LS].

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