On Commutativity of Diagrams of Type II_1 Factors

Jerzy Wierzbicki

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On Commutativity of Diagrams of Type II_1 Factors

By

Jerzy Wierzbicki*

Abstract

We show that a diagram $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ of type II_1 factors, with the relative commutant $S' \cap K = \mathbb{C}$, must be a commuting square, if we only impose some conditions on the inclusion $S \subset Q$.

§1. Introduction

The object of our study are diagrams of von Neumann algebras $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$. This kind of situation was first considered by S.Popa ([P1],[P2],[P3]) and appeared later in other papers, for example [GHJ],[P4],[P5],[SW],[K],[S],[WW]. When the von Neumann algebra K is equipped with a finite tracial weight τ and E_Q^K, E_R^K and E_S^K are τ - preserving conditional expectations of K onto Q, R and S, then a special situation may occur:

$$E_Q^K E_R^K = E_R^K E_Q^K = E_S^K.$$

Then the diagram $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ is called a commuting square. At present, this concept plays important role in the theory of subfactors. In the special case, when $S = \mathbb{C}$, the subalgebras Q and R were called by S.Popa orthogonal ([P1]). This case, in the classical probability theory, corresponds to the condition of independence of two σ - fields.

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*Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060,Japan.

Everywhere in this paper, the von Neumann algebras, which form such a diagram, are type II_1 factors. This situation was considered by T.Sano and Y.Watatani in [SW], where the authors introduced the notion of angles between two subfactors Q and R. Our paper is, in a sense, an extension of the study presented in [SW]. It may seem a little surprising that for certain inclusions, say $S \subset Q$, a diagram $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$, with (as always in this work) $[K : S] < \infty$ and $S' \cap K = \mathbb{C}$, must be a commuting square. We will show several results of this kind. The notion of commuting square of type II_1 factors is strictly connected to the concept of so called co-commuting square of type II_1 factors, which will be described in the next section. There are plenty of diagrams $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ which are completely irregular in the sense that $Q \vee R = K$, $Q \cap R = S$ and they are neither commuting nor co-commuting squares. We can construct them, like in [SW], by "tensoring" degenerated (in the sense of S.Popa's [P5]) diagrams. However, when the index $[Q : S]$ is "small" or the second relative commutant of the inclusion "$S \subset Q$" is only 2 - dimensional, then the situation is different. For example, we can prove such property:

Let $\mathcal{D} = \begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ *be a diagram of type* II_1 *factors with* $S' \cap K = \mathbb{C}$ *and* $[K : S] < \infty$. *If* $[Q : S] = 4\cos^2 \frac{\pi}{n}$ *for a prime number* n, *then the diagram* $\mathcal{D}$ *is a commuting square.*

The crucial observation is that Ocneanu's convolution $E_Q^K * E_R^K$ is a scalar multiple of a projection. Then, we can use results from [SW] or [P3] to show that the "right angle" between two subfactors is not so special in certain situations.

T.Sano and Y.Watatani ([SW]) showed that commutativity of a diagram is equivalent to co-commutativity of some other related diagram. Therefore, sufficient conditions for a diagram to be, for example, a co-commuting square can immediately be reformulated in terms of a commuting square. We will try then not to repeat ourselves and we will keep rather to the co-commuting square terminology.

§2. Fourier Transform and convolution.

We recall here the Ocneanu's Fourier transform and convolution in the third relative commutant. We prove also a few properties which will be useful later. We believe that most of them are known to other persons working in this area. It is convenient to keep in mind a lemma which is an immediate consequence of [P2, Lemma 1.2.2.] and [PP1, Corollary 4.5]. If τ is a faithful, normal and finite trace on a von Neumann algebra B and A is its von Neumann subalgebra $K \subset L$ then by ${}^{\tau}E_A^B$ we denote τ preserving conditional expectation of B onto A. When there is no risk of confusion we will write it simply E_A^B or just E_A.

Lemma 2.1. *Let $K \subset A \subset B \subset L$ be all type II_1 factors and let the inclusion $K \subset L$ be extremal with $[L:K] < \infty$. If τ and τ' are the traces on L and on K', then for any $x \in K' \cap L$,*

$$ {}^{\tau}E_A^B(x) = {}^{\tau}E_{K'\cap A}^{K'\cap B}(x) = {}^{\tau'}E_{K'\cap A}^{K'\cap B}(x) \text{ and } {}^{\tau'}E_{B'}^{A'}(x) = {}^{\tau'}E_{A'\cap L}^{B'\cap L}(x) = {}^{\tau}E_{A'\cap L}^{B'\cap L}(x). $$

We will often use the lemma without referring to it.

Let $S \subset K$ be an irreducible inclusion of type II$_1$ factors with $[K:S] = \gamma^{-1} < \infty$. Consider corresponding Jones' tower

$$ S \subset K \subset^{e_S} L \subset^{e_K} L_1. $$

Note that by [PP2] the inclusion $S \subset L_1$ is extremal, and so Lemma 2.1 can be applied.

Definition 2.2. The mapping $\mathcal{F} : K' \cap L_1 \to S' \cap L$ given by

$$ \mathcal{F}(x) = \gamma^{-\frac{3}{2}} E_L(x e_S e_K) $$

will be called the Fourier transform of $K' \cap L_1$ into $S' \cap L$. By the inverse Fourier transform we mean the mapping $\mathcal{F}^* : S' \cap L \to K' \cap L_1$ defined by

$$ \mathcal{F}^*(y) = \gamma^{-\frac{3}{2}} E_{K'\cap L_1}(y e_K e_S). $$

We show that the name "inverse Fourier transform" is justified. The algebra $S' \cap L_1$ has canonical Hilbert space structure determined by the trace $\tau_{L_1} = \tau_{S' \cap L_1}$. Let us consider $S' \cap L$ and $K' \cap L$ as its Hilbert subspaces.

Proposition 2.3. *The mappings $\mathcal{F}$ and $\mathcal{F}^*$ are Hilbert space isomorphisms between $S' \cap L$ and $K' \cap L_1$. Moreover,*

$$\mathcal{F}\mathcal{F}^* = Id_{|S' \cap L} \quad \text{and} \quad \mathcal{F}^*\mathcal{F} = Id_{|K' \cap L_1}.$$

In [W] was shown a property of the Pimsner-Popa basis, which is very useful in proof of the above proposition.

Lemma 2.4. *Let $N \subset M$ be a finite index inclusion of type II_1 factors. If $\{m_i\}_i$ is a Pimsner-Popa basis of N' over M', then the trace preserving conditional expectation E_N^M of M onto N is expressed as follows:*

$$E_N^M(x) = [M:N]^{-1} \sum_i m_i x m_i^*.$$

Proof of Proposition 2.3. We show first that $\mathcal{F}\mathcal{F}^* = Id_{|S' \cap L}$. Let $\{m_i\}_{i=0}^n$ be Pimsner-Popa basis of K over S such that $m_0 = 1$. The preceding lemma gives: $E_{K'}^{S'}(y) = \gamma \sum_i m_i y m_i^*$ for trace preserving conditional expectation $E_{K'}^{S'}$. From Lemma 2.1,

$$^{\tau'}E_{K' \cap L_1}^{S' \cap L_1}(y e_K e_S) = E_{K'}^{S'}(y e_K e_S) = \gamma \sum_i m_i y e_K e_S m_i^*,$$

for $y \in S' \cap L$. Hence

$$\gamma^3 \mathcal{F}\mathcal{F}^*(y) = \gamma E_L(\sum_i m_i y e_K e_S m_i^* e_S e_K) =$$

$$= \gamma E_L(\sum_i m_i y e_K E_S(m_i^* m_0) e_S e_K) = \gamma E_L(y e_K e_S e_K) = \gamma^2 E_L(y e_K) = \gamma^3 y.$$

We show now that $\mathcal{F}^*$ is a contraction. For $y \in S' \cap L$

$$\|\mathcal{F}(y)\|_2^2 = \gamma^{-3} \tau(E_{K' \cap L_1}(y e_K e_S) E_{K' \cap L_1}(y e_K e_S)^*) =$$

$$\gamma^{-1}\sum_{i,j}\tau(m_i y e_K e_S m_i^* m_j e_S e_K y^* m_j^*) =$$

$$= \gamma^{-1}\sum_{i<n}\tau(m_i y e_K e_S e_K y^* m_i^*) + \gamma^{-1}\tau(m_n y e_K p e_S e_K y^* m_n^*),$$

where $p = E_S(m_n^* m_n)$ is a projection, cf.[PP1]. Since $[p, e_K] = 0$,

$$\|\mathcal{F}(y)\|_2^2 = \sum_{i<n}\tau(m_i y e_K y^* m_i^*) + \tau(m_n y e_K p e_K y^* m_n^*) \leq \sum_{i\leq n}\tau(m_i y e_K y^* m_i^*) =$$

$$= \sum_{i\leq n}\tau(E_L(m_i y e_K y^* m_i^*)) = \gamma\tau(\sum_{i\leq n} m_i y y^* m_i^*) = \gamma\tau(y y^* \sum_{i\leq n} m_i^* m_i) =$$

$$\gamma\|y\|_2^2\tau(\sum_{i\leq n} m_i m_i^*) = \|y\|_2^2.$$

Similarly, using a Pimsner-Popa basis of L' over L_1' we obtain that $\mathcal{F}$ is a contraction too. So $\mathcal{F}$ and $\mathcal{F}^*$ are isometries and the standard polarization argument completes the proof.

Q.E.D.

We will always use notation $\langle K, S, e\rangle$, for an algebraic basic construction ([PP2]) of a finite index inclusion $S \subset K$ of type II_1 factors, where e is the corresponding Jones projection, i.e. $exe = E_S^K(x)e$, for $x \in K$ and $\langle K, S, e\rangle$ is generated, as a von Neumann algebra, by K and e. If $\langle K, S, e\rangle$ is represented on $L^2(K,\tau)$ and e is just an extension of E_S^K, then, as in [J], we will write simply $\langle K, e\rangle$. Let us consider an intermediate subfactor Q between S and K, with indices $[Q : S] = \alpha^{-1}$ and $[K : Q] = \lambda^{-1}$. By [J] we know that $\alpha\lambda = \gamma$. Let $M = \langle K, e_Q\rangle$ and $M_1 = \langle L, e_M\rangle$ be Jones' basic constructions. We have

$$S \subset Q \subset K \subset^{e_Q} M \subset L \subset^{e_M} M_1 \subset L_1,$$

where, as before, $L = \langle K, e_S\rangle$ and $L_1 = \langle L, e_K\rangle$. Obviously, $e_Q \in S' \cap L$ and $e_M \in K' \cap L_1$. Also $e_Q e_S = e_S$ and $e_M e_K = e_K$.

Proposition 2.5.

$$\mathcal{F}(\alpha^{-\frac{1}{2}}e_M) = \lambda^{-\frac{1}{2}}e_Q \ \textit{and} \ \mathcal{F}^*(\lambda^{-\frac{1}{2}}e_Q) = \alpha^{-\frac{1}{2}}e_M.$$

Proof. We will show only the second equality. First, we remark that $e_Q e_K e_Q = \lambda e_M e_Q$. Indeed, let us represent all these operators on $L^2(L,\tau)$ and let the vector sign denote the canonical embedding of L in $L^2(L,\tau)$. Then for any $\overrightarrow{ae_Sb} \in L^2(L,\tau)$, $a, b \in K$ we have:

$$e_Q e_K e_Q \overrightarrow{ae_Sb} = \overrightarrow{e_Q E_K(e_Q a e_S b)} = \overrightarrow{e_Q E_K(e_Q a e_Q e_S b)} =$$

$$= \overrightarrow{e_Q E_Q(a) E_K(e_S) b} = \gamma \overrightarrow{e_Q E_Q(a) b} =$$

$$= \lambda \overrightarrow{E_Q(a) \alpha e_Q b} = \lambda \overrightarrow{E_M(E_Q(a) e_S b)} = \lambda e_M e_Q \overrightarrow{ae_Sb}.$$

In the last line we used the property $E_M(e_S) = \alpha e_Q$, cf. [SW, Lemma 7.1].

Now,

$$\mathcal{F}^*(e_Q) = \gamma^{-\frac{3}{2}} E_{K'\cap L_1}(e_Q e_K e_S) = \gamma^{-\frac{3}{2}} E_{K'\cap L_1}(\lambda e_M e_S) =$$

$$\gamma^{-\frac{3}{2}} \lambda e_M E_{K'\cap L_1}(e_S) = \sqrt{\frac{\lambda}{\alpha}} e_M,$$

where we used [PP1, Corollary 4.5].

Q.E.D.

Definition 2.6. For $x, y \in S' \cap L$ we set

$$x * y = \mathcal{F}(\mathcal{F}^*(x)\mathcal{F}^*(y)).$$

Similarly, for $x, y \in K' \cap L_1$

$$x \hat{*} y = \mathcal{F}^*(\mathcal{F}(x)\mathcal{F}(y)).$$

Following [O], we call the operation "$*$" or "$\hat{*}$" convolution. The convolution "$\hat{*}$" in $K' \cap L_1$ should not be confused with the convolution "$*$", which we define by building up another basic construction, say $L_2 = \langle L_1, e_L \rangle$, and putting $x * y = \mathcal{F}_1(\mathcal{F}_1^*(x)\mathcal{F}_1^*(y))$, where $\mathcal{F}_1$ and $\mathcal{F}_1^*$ are the shifted Fourier and inverse Fourier transforms: $\mathcal{F}_1^* : K' \cap L_1 \to L' \cap L_2$, $\mathcal{F}_1^*(x) = \gamma^{-\frac{3}{2}} E_{L'}(x e_L e_K)$ and $\mathcal{F}_1 : L' \cap L_2 \to K' \cap L_1$, $\mathcal{F}_1(y) = \gamma^{-\frac{3}{2}} E_{L_1}(y e_K e_L)$. Similarly, we may define "$\hat{*}$" in $S' \cap K$.

Proposition 2.7.

$$\textit{For } x \in S' \cap L,\ \sqrt{\frac{\alpha}{\lambda}} e_Q * x = E_M(x).$$

$$\textit{For } y \in K' \cap L_1,\ \sqrt{\frac{\lambda}{\alpha}} e_M \hat{*} y = E_{M'}(y) = E_{M' \cap L_1}(y).$$

Proof. Let $\tilde{x} = \mathcal{F}^*(x) \in K' \cap L_1$. By [PP1, Lemma 1.2] and Proposition 2.5,

$$E_L(e_M \tilde{x} e_S e_K) = \gamma^{-1} E_L(e_M E_L(\tilde{x} e_S e_K) e_K) = \gamma^{-1} E_L(e_M E_L(\tilde{x} e_S e_K) e_M e_K) =$$

$$= \gamma^{-1} E_L(E_M(\tilde{x} e_S e_K) e_K) = E_M(\tilde{x} e_S e_K) = \gamma^{\frac{3}{2}} E_M(\mathcal{F}(\tilde{x})) = \gamma^{\frac{3}{2}} E_M(x).$$

Since $S' = \langle K', L', e_S \rangle$ and $Q' = \langle K', M', e_Q \rangle$, the proof of the other equality is almost the same.

Q.E.D.

Lemma 2.8. *Let $S \subset Q \subset K$ be type II_1 factors, $[K : S] < \infty$, $M = \langle K, Q, e_Q \rangle$ and $S' \cap Q = \mathbb{C}$. Then*

(1) *e_Q is minimal projection in $S' \cap M$.*

(2) *e_Q is central in $S' \cap M$, iff $S' \cap K = \mathbb{C}$.*

Proof. (1) Suppose $e \leq e_Q$ for some non-zero projection $e \in P(S' \cap M)$. Exactly as in [P3, Proposition 7.3] we obtain a projection $f \in S' \cap K$ such that $E_K(e) = \lambda f$ ($\lambda = [K : Q]^{-1}$). Now,

$$e = e e_Q = \lambda^{-1} E_K(e e_Q) e_Q = \lambda^{-1} E_K(e) e_Q = f e_Q.$$

Hence $[f, e_Q] = 0$; therefore $f \in S' \cap Q = \mathbb{C}$, which implies $e = e_Q$.

(2) "$\Rightarrow$" Fix $q_1, q_2 \in P(S' \cap K)$ and a number $\theta \in (0, 2\pi]$ such that $q_1 + q_2 = 1$ and $|\tau(q_1) + \exp(i\theta)\tau(q_2)| \neq 1$. $u = q_1 + \exp(i\theta) q_2$ is a unitary in $S' \cap K$ and since e_Q is central, $e_Q = e_Q u e_Q u^* e_Q$. Hence $|\tau(u)| = 1$, which gives contradiction.

"$\Leftarrow$"Let $c = C(e_Q, S' \cap M)$ be the central support of e_Q in $S' \cap M$. For any projection $f \in (S' \cap M)_c$ satisfying $\tau(f) = \lambda$, there is a unitary $u \in S' \cap M$ such that $f = ue_Qu^*$. Hence, by [PP1, Lemma 1.2] and Lemma 2.1

$$ue_Q = \lambda^{-1} E_K(ue_Q)e_Q = \lambda^{-1}\tau(ue_Q)e_Q.$$

Thus $f = \lambda^{-2}|\tau(ue_Q)|^2 e_Q$. This implies $c = e_Q$.

Q.E.D.

From now on, we consider two intermediate subfactors Q, R between S and K: $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$. We always consider non-degenerated diagrams, in the sense that $Q \not\subset R$ and $R \not\subset Q$, but we do not assume, like in [SW], that always $K = Q \vee R$ and $S = Q \cap R$. Throughout the rest of the paper we keep to the notation preceding Proposition 2.5 and we add the following one: $N = \langle K, e_R \rangle$ and $N_1 = \langle L, e_N \rangle$ will be the Jones basic constructions. For brevity we assume that $\eta = [K : R]^{-1}$ and $\beta = [R : S]^{-1}$. The inclusions

$$\begin{matrix} N & \subset & L \\ \cup & & \cup \\ K & \subset & M \end{matrix} \quad \text{and} \quad \begin{matrix} M_1 & \subset & L_1 \\ \cup & & \cup \\ L & \subset & N_1 \end{matrix}$$

hold true. J_K (or J_L) will be modular conjugation in $L^2(K, \tau)$ (or $L^2(L, \tau)$). We always assume that the inclusion $S \subset K$ is irreducible.

Theorem 2.9. *There exists a projection p in $R' \cap M$ such that:*

(1)

$$e_Q * e_R = \tau(e_Q e_R)\gamma^{-\frac{1}{2}} p = \sqrt{\frac{\lambda}{\alpha}} E_M(e_R) = \sqrt{\frac{\eta}{\beta}} E_{R'}(e_Q),$$

(2)

$$e_R * e_Q = J_K e_Q * e_R J_K = \tau(e_Q e_R)\gamma^{-\frac{1}{2}} J_K p J_K = \sqrt{\frac{\lambda}{\alpha}} E_{Q'}(e_R) = \sqrt{\frac{\eta}{\beta}} E_N(e_Q),$$

(3) *p is minimal and central projection in $R' \cap M$ and $p \geq e_R \vee e_Q$,*

(4) *$\hat{p} = J_K p J_K$ is minimal and central projection in $Q' \cap N$ and $\hat{p} \geq e_R \vee e_Q$.*

Proof. Let $p = \bigwedge\{f | f \geq e_Q, f \in P(R' \cap M)\}$. Suppose p_1, $0 \neq p_1 \leq p$, is another projection in $R' \cap M$. By Lemma 2.8 $e_Q p_1 = 0$ or $e_Q p_1 = e_Q$. This implies $e_Q \leq p - p_1 \leq p$ or $e_Q \leq p_1 \leq p$. So the minimality of p gives us $p_1 = 0$ or $p_1 = p$. Therefore p is minimal in $R' \cap M$.

Let $E_{R'}(e_Q) = \sum_{1 \leq i \leq n} \alpha_i p_i$, $\alpha_i > 0$ be the spectral decomposition. Then, by $e_Q \leq p$, it follows that $\sum \alpha_i p_i \leq p$ and, in consequence $\sum p_i \leq p$. Therefore, by minimality of p, we see that $n = 1$ and $E_{R'}(e_Q) = \theta p$, for some scalar θ. From this and Lemma 2.8 we see that p is also central in $R' \cap M$.

Identically we obtain $E_{Q'}(e_R) = \kappa g$ for a minimal and central projection g in $Q' \cap N$ and a scalar κ. It is easy to check that

$$\forall x \in S' \cap L, \quad E_{N \cap Q'}(x) = J_K E_{R' \cap M}(J_K x J_K) J_K.$$

Using this we see that for $\hat{g} = J_K g J_K$,

$$\kappa \hat{g} = J_K E_{Q' \cap N}(e_R) J_K = E_{R' \cap M}(J_K e_R J_K) = E_M(e_R) = \sqrt{\frac{\alpha}{\lambda}} e_Q * e_R.$$

Also, $\hat{g}$ is minimal and central projection in $R' \cap M$. Since $e_R \leq g$, we have $e_R \leq \hat{g}$. Hence

$$0 \neq e_S \leq e_Q \wedge e_R \leq p \wedge \hat{g},$$

which implies $p = \hat{g}$. Computation of coefficients θ and κ ends the proof. For example, let us multiply the equality $\kappa p = E_M(e_R)$ by e_Q and take the trace: $\kappa e_Q = E_M(e_R e_Q)$, and so $\kappa = \lambda^{-1} \tau(e_R e_Q)$.

Q.E.D.

Let us write $[x]$ for the range projection of an operator x. Similarly as above, we obtain the following corollary.

Corollary 2.10. *$[e_M \hat{*} e_N]$ is a minimal and central projection in $M' \cap N_1$, $[e_M \hat{*} e_N] \geq e_M \vee e_N$ and*

$$e_M \hat{*} e_N = J_L e_N \hat{*} e_M J_L = \tau(e_M e_N) \gamma^{-\frac{1}{2}} [e_M \hat{*} e_N] = \sqrt{\frac{\alpha}{\lambda}} E_{M'}(e_N) = \sqrt{\frac{\beta}{\eta}} E_{N_1}(e_M).$$

Remark. From the above theorem and corollary, it is easy to see the following property of the "shifted" convolution "$*$" in $K' \cap L_1$: $e_N * e_M = e_M \hat{*} e_N$. Then it is natural to ask, wether this relation holds true for arbitrary operators in $K' \cap L_1$. Right now, it is not clear for us.

For later convenience, we carry out some computations. From [J] we know, that $\frac{\lambda}{\beta} = \frac{\eta}{\alpha}$. We denote this quotient by δ.

Proposition 2.11.

$$\frac{\tau([e_Q * e_R])}{\tau([e_M \hat{*} e_N]} = \frac{\tau(e_Q e_R)}{\tau(e_M e_N)} = \delta.$$

Proof. From Propositions 2.5 and 2.3 and Theorem 2.9 we have:

$$\tau(e_M e_N) = \|e_M e_N\|_2^2 = \|\mathcal{F}(e_M e_N)\|_2^2 = \frac{\alpha\beta}{\lambda\eta}\|e_Q * e_R\|_2^2 =$$

$$= \frac{\alpha\beta}{\lambda\eta}\tau(e_Q e_R)^2 \gamma^{-1}\frac{\lambda\eta}{\tau(e_Q e_R)} = \delta^{-1}\tau(e_Q e_R).$$

Q.E.D.

T.Sano and Y.Watatani, cf. [SW], introduced notion of angles between subfactors Q and R. The diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$, with $K = Q \vee R$ and $\text{Op-ang}_K(Q, R) = \{\frac{\pi}{2}\}$ will be called co-commuting square. (Cf. [K],[WW],[Wi].) The following characterization can be found in [SW].

Proposition 2.12. *The following conditions are equivalent:*

(1) *Diagram* $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ *of finite factors is a co-commuting square.*

(2) *Diagram of their commutants* $\begin{matrix} R' & \subset & S' \\ \cup & & \cup \\ K' & \subset & Q' \end{matrix}$ *forms a commuting square.*

(3) *Diagram* $\begin{matrix} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{matrix}$ *is a commuting square.*

(4) *Diagram* $\begin{matrix} M_1 & \subset & L_1 \\ \cup & & \cup \\ L & \subset & N_1 \end{matrix}$ *is a co-commuting square.*

(5) $E_M(e_R) \in \mathbb{C}$.

As an application of Theorem 2.9 we give a few other conditions on commutativity. (Remember that $K' \cap L = \mathbb{C}$ and $[L : K] = [K : S] = \gamma^{-1} < \infty$.)

Corollary 2.13. *The following conditions are equivalent:*

(1) $\begin{array}{ccc} N & \subset & L \\ \cup & & \cup \\ K & \subset & M \end{array}$ *is a co-commuting square.*

(2) $N' \cap M_1 = \mathbb{C}$.

(3) $\tau(e_N e_M) = \alpha\beta$.

(4) $e_N \hat{*} e_M$ *is a scalar.*

(5) $e_N \hat{*} e_M = e_M \hat{*} e_N$ *and* $N \vee M = L$.

Proof. "(2) $\Rightarrow$ (1)" follows from Proposition 2.12.

"(1) $\Rightarrow$ (2)" If (1) is satisfied then, from Proposition 2.12, $E_{M_1}(e_N) \in \mathbb{C}$. Hence and by Theorem 2.9 the identity is minimal in $N' \cap M_1$.

"(1) $\Leftrightarrow$ (3)" Since, by computation in 2.11, (3) is equivalent to $e_Q e_R = e_S$, this is immediate consequence of Proposition 2.12.

"(2) $\Leftrightarrow$ (4)" is obvious because, by Theorem 2.9, $e_M \hat{*} e_N$ is a scalar multiple of a minimal projection in $M' \cap N_1$.

"(1) $\Leftrightarrow$ (5)" is direct consequence of Proposition 2.5.

Q.E.D.

By the remark following Corollary 2.10, we can replace the symbol "$\hat{*}$" in (4) and (5) of the above corollary, by "$*$".

Example. Let μ be an outer action of a finite group G on type II_1 factor S. If A and B are subgroups of G with trivial intersection $A \cap B = \{e\}$, then we can consider diagram of crossed products $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$, where $Q = S \rtimes_\mu A$, $R = S \rtimes_\mu B$ and $K = S \rtimes_\mu G$. The diagram of corresponding basic constructions, cf. [SW, Lemma 5.1], can be identified with the following one $\begin{array}{ccc} N & \subset & L \\ \cup & & \cup \\ K & \subset & M \end{array}$, where

$M = (S \otimes L^\infty(G/A)) \rtimes_\mu G$, $N = (S \otimes L^\infty(G/B)) \rtimes_\mu G$, $L = (S \otimes L^\infty(G)) \rtimes_\mu G$. $L^\infty(G/A)$ ($L^\infty(G/B)$) contains functions constant on left cosets of A (of B) and the action μ is extended in obvious way, cf. [SW]. If χ_D denotes the characteristic function of a subset $D \subset G$, then

$$e_Q \approx 1 \otimes \chi_A, \quad e_R \approx 1 \otimes \chi_B, \quad \text{and} \quad e_S \approx 1 \otimes \chi_{\{e\}}.$$

One can easily verify that $R' \cap M$ is isomorphic to the subspace of $L^\infty(G)$ which is composed of functions constant on double cosets BgA, $g \in G$. Similarly $Q' \cap N \approx L^\infty(A \setminus G/B)$ and we have

$$[e_Q * e_R] \approx \chi_{BA} \quad \text{and} \quad [e_R * e_Q] \approx \chi_{AB}.$$

Also, since $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ is a commuting square, $[e_N \hat{*} e_M] = [e_M \hat{*} e_N] = 1$.

§3. Angles for subfactors with "small" second relative commutant.

The following lemma may be considered as a generalization of Lemma 5.3 in [SW].

Lemma 3.1. *If the diagram* $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ *is not a co-commuting square and* $\dim(Q' \cap M) = 2$, *then the only non-zero element of spectrum* $Sp(e_M e_N e_M - e_K)$ *is* $\frac{1}{1-\lambda}(\frac{\tau(e_N e_M)}{\alpha} - \lambda)$.

Proof. Note that our assumption $\dim(Q' \cap M) = 2$ implies $Q \vee R = N \cap M = K$. Let us denote $E_0 = E_M E_N E_M$ and $E = E_0 - E_K$. It is sufficient to show (see [SW, Corollary 3.1]) that $E^2 = \frac{1}{1-\eta}(\frac{\tau(e_N e_M)}{\beta} - \eta)E$. Since E has the bimodule property, by [PP1, Lemma 1.2], we need only to make sure that $E^2(e_S) = \frac{1}{1-\lambda}(\frac{\tau(e_N e_M)}{\alpha} - \lambda)E(e_S)$. By [SW, Lemma 7.1] and Theorem 2.9, for the projection $\hat{p} = [e_R * e_Q] \in Q' \cap N$ we have

$$E_0(e_S) = E_M E_N(\alpha e_Q) = \frac{\tau(e_Q e_R)}{\delta} E_M(\hat{p}).$$

Since $\dim(Q' \cap M) = 2$,

$$E_M(\hat{p}) = \frac{\tau(\hat{p}e_Q)}{\tau(e_Q)}e_Q + \frac{\tau(\hat{p}(1-e_Q))}{\tau(1-e_Q)}(1-e_Q) = e_Q + \frac{\tau(\hat{p}) - \lambda}{1-\lambda}(1-e_Q).$$

Therefore, setting

$$\theta = \tau(e_Q e_R)/\eta \quad \text{and} \quad \kappa = \frac{\lambda}{\eta}\frac{\eta - \tau(e_Q e_R)}{1-\lambda}, \quad \text{we get}$$

$$E_0(e_S) = \alpha E_0(e_Q) = \alpha\theta e_Q + \alpha\kappa(1-e_Q).$$

Hence,

$$E_0^2(e_S) = E_0(\alpha\theta e_Q + \alpha\kappa(1-e_Q)) = (\theta-\kappa)E_0(e_S) + \alpha\kappa \quad \text{which gives}$$

$$E^2(e_S) = (E_0 - E_K)(E_0 - E_K)(e_S) = (E_0^2 - E_K)(e_S) = (\theta-\kappa)E_0(e_S) + \alpha\kappa - \gamma$$

$$= (\theta-\kappa)E(e_S) + (\theta-\kappa)\gamma + \alpha\kappa - \gamma = (\theta-\kappa)E(e_S)$$

and

$$\theta - \kappa = \frac{1}{1-\lambda}(\frac{\tau(e_Q e_R)}{\eta} - \lambda) = \frac{1}{1-\lambda}(\frac{\tau(e_N e_M)}{\alpha} - \lambda).$$

Q.E.D.

Theorem 3.2. *Let* $\mathcal{D} = \begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ *be a diagram of type type* II_1 *factors* $([K:S] < \infty,\ S' \cap K = \mathbb{C})$ *and let* $[K:R] < [K:Q]$. *If* $\dim Q' \cap M = 2$ $(M = \langle K, Q, e_Q \rangle)$, *then* $\mathcal{D}$ *is a co-commuting square.*

Proof. Suppose that $\mathcal{D}$ is not a co-commuting square. From the preceding lemma, the spectrum $Sp(e_M e_N e_M - e_K) - \{0\} = \{\sigma\}$, where $\sigma = \frac{1}{1-\lambda}(\frac{\tau(e_N e_M)}{\alpha} - \lambda)$. Since

$$Sp(e_N e_M e_N - e_K) - \{0\} = Sp(e_N e_M - e_K)(e_N e_M - e_K)^* - \{0\}$$

$$= Sp(e_N e_M - e_K)^*(e_N e_M - e_K) - \{0\} = \{\sigma\},$$

it follows that

$$e_M e_N e_M = e_K + \sigma g, \quad e_N e_M e_N = e_K + \sigma f,$$

for some projections $g, f \in S' \cap L$. Then, by Proposition 2.12, $\tau(e_M e_N) \neq \gamma$ $(= [K:S]^{-1})$, which implies $\tau(g) = (\tau(e_M e_N) - \gamma)\frac{1}{\sigma} = \alpha - \gamma$, but

$$f \leq e_N - e_K \text{ and } \tau(f) = \tau(g),$$

and so $\alpha - \gamma \leq \beta - \gamma$, which gives contradiction with $\lambda < \eta$.

Q.E.D.

Corollary 3.3. *If* $\mathcal{D} = \begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$, $\dim Q' \cap M = \dim R' \cap M = 2$ *and* $[K:Q] \neq [K:R]$, *then* $\mathcal{D}$ *is a co-commuting square.*

§4. Angles for subfactors with "small" indices.

We use here results of S.Popa, cf. [P3], to consider another sufficient conditions which lead to the same effect. Let us outline some of those results. We assume the following notations: $P_n(x)$ will be Jones' polynomials,

$$P_{-1} \equiv P_0 \equiv 1 \quad P_{n+1}(x) = P_n(x) - xP_{n-1},$$

if $x > \frac{1}{4}$ then

$$\Lambda_0(x) = \{\frac{P_{k-1}(x)}{P_{k-2}(x)} \mid 2 \leq k \leq n-2\} = \{x\frac{P_{k-1}(x)}{P_k(x)} \mid 0 \leq k \leq n-4\} \text{ and } \Lambda_1 = \emptyset,$$

if $x \leq \frac{1}{4}$ then

$$\Lambda_0(x) = \{x\frac{P_{k-1}(x)}{P_k(x)} \mid 0 \leq k\} \cup \{\frac{P_{k+1}(x)}{P_k(x)} \mid 0 \leq k\}$$

$$\text{and} \quad \Lambda_1 = [\frac{1-\sqrt{1-4x}}{2}, \frac{1+\sqrt{1-4x}}{2}],$$

$$\Lambda(x) = \Lambda_0(x) \cup \Lambda_1(x) \quad \text{and} \quad \tilde{\Lambda}(x) = \Lambda(x)) \cap (x + \Lambda(x)).$$

If $B \subset A$ is a finite index inclusion of type II_1 factors, then

$$P(A,B) = \{e \in P(A) | E_B(e) \text{ is a scalar }\}$$

and as in [P3] we denote: $\Lambda(A,B)=\{\tau(e)|e\in P(A,B)\}$.

Theorem 4.1. *For a finite index inclusion of type II_1 factors $B\subset A$, $\Lambda(A,B)\subset\Lambda([A:B]^{-1})$. Moreover, if $[A:B]\leq 4$, then $\Lambda(A,B)=\Lambda([A:B]^{-1})$.*

Next proposition is a slight modification of [P3, Proposition 4.5]. We will set

$$\mathcal{L}^s(B\subset A)=\{F|\ F \text{ is factor, } B\subset F\subset A \text{ and } [A:F]^{-1}=s\}.$$

Proposition 4.2. *Let $D\subset B\subset A$ be a triple of type II_1 factors with $[A:D]<\infty$ and $D'\cap B=\mathbb{C}$. If $f\in P(D'\cap A)$ and $\tau(f)\in\Lambda_0(t)$ ($t=[A:B]^{-1}$), then there exists $F\in\mathcal{L}^s(D\subset B)$, $s=\frac{t^{k+1}}{P_k(t)^2}$, where $k\geq 0$ is determined by $\tau(f)=t\frac{P_{k-1}(t)}{P_k(t)}$ or by $\tau(f)=\frac{P_{k+1}(t)}{P_k(t)}$.*

Proof. We may assume that $\tau(f)=t\frac{P_{k-1}(t)}{P_k(t)}$, for some $k\geq 0$. Since evidently $f\in P(A,B)$, exactly as in [P3, Proposition 4.5] we obtain a subfactor $F\subset B$ with $[B:F]^{-1}=s$. It is easy to verify that the additional assumption $f\in D'$ yields $D\subset F$.

Q.E.D.

Let us now come back to diagrams of type II_1 factors. We keep to all the notations from the preceding sections as well as to the assumptions $[K:S]<\infty$ and $S'\cap K=\mathbb{C}$.

Lemma 4.3.

$$\tau([e_Q*e_R])\in[[\tilde{\Lambda}(\lambda)\cap\tilde{\Lambda}(\eta)]\cup\{1\}]\cap\delta[[\tilde{\Lambda}(\alpha)\cap\tilde{\Lambda}(\beta))]\cup\{1\}].$$

Proof. Suppose that $p=[e_Q*e_R]\neq 1$. Since $p\in R'\cap M$ and $R'\cap K=\mathbb{C}$, by Lemma 2.1, $p\in P(M,K)$, and so $\tau(p)\in\Lambda(M,K)\subset\Lambda(\lambda)$. The same reasoning applied to the projection $p-e_Q\in S'\cap M$ also gives $\tau(p)-\lambda\in\Lambda(\lambda)$. Hence $\tau(p)\in\tilde{\Lambda}(\lambda)$. Identically, we obtain $\tau(\hat{p})=\tau(J_KpJ_K)\in\tilde{\Lambda}(\eta)$, but $\tau(p)=\tau(\hat{p})$; therefore,

$$\tau(p)\in\tilde{\Lambda}(\lambda)\cap\tilde{\Lambda}(\eta).$$

Again, the projection $q = [e_M \hat{*} e_N] \in P(N_1, L)$ and analogically we obtain:

$$\tau(q) \in \tilde{\Lambda}(\alpha) \cap \tilde{\Lambda}(\beta)$$

or $\tau(q) = 1$. Now, by Proposition 2.11, $\tau(p) = \delta\tau(q)$, which completes the proof.

Q.E.D.

Remark. Note that the condition $\tau(p) \in \Lambda(\lambda)$, which is weaker than the above lemma, is a kind of generalization (at least in $S' \cap K = \mathbb{C}$ case) of Popa's [P3, Theorem 6.1]. Indeed, if diagram $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ is a commuting square then $\tau(p) = \delta$. Hence

$$\delta = \frac{\lambda}{\beta} \in \Lambda(\lambda) \Leftrightarrow \beta \in \lambda\Lambda(\lambda)^{-1} = \Lambda(\lambda),$$

where the last equality comes directly from the definition of the set $\Lambda(\lambda)$.

Remark. Incidentally, with a little extra effort we can make the Popa's condition a bit stronger. If diagram $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ is a commuting square ($S' \cap K = \mathbb{C}$) then $\beta \in \{\lambda, \frac{\lambda}{1-\lambda}\} \cup \Lambda_1(\lambda)$ and $\alpha \in \{\eta, \frac{\eta}{1-\eta}\} \cup \Lambda_1(\eta)$. In the proof we use Proposition 4.2 to obtain a factor in $\mathcal{L}^s(N \subset M_1)$ with "too big" index s^{-1}.

Example. It is easy to see that for $\lambda^{-1} < 2 + \sqrt{5}$, the set $\tilde{\Lambda}(\lambda)$ is finite. Let λ be a transcendental and η algebraic numbers, $1 < \lambda^{-1}, \eta^{-1} < 2 + \sqrt{5}$. Then, by the above lemma, the diagram $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ is a co-commuting square.

Example. It is interesting to see, how some of the above results reflect in the group theory. Let μ be an outer action of a finite group G on a type II_1 factor S. Let, for example, H and F be non-trivial subgroups of G such that $H \cap F = \{1\}$. We can consider diagram $\begin{array}{ccc} S \rtimes_\mu H & \subset & S \rtimes_\mu G \\ \cup & & \cup \\ S & \subset & S \rtimes_\mu F \end{array}$ and apply [NT], [SW], Proposition 4.2 and Theorem 2.9 to get a property of groups which does not seem quite trivial. If $[G : H] = 3$ and $G \neq HF$, then $|F| = 2$ and there is an intermediate subgroup D such that $F \subset D \subset G$, with $[G : D] = 3$ or $[G : D] = 4$.

Theorem 4.4. *Let* $\mathcal{D} = \begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ *be a diagram of type II_1 factors with $S' \cap K = \mathbb{C}$ and $[K : S] < \infty$. If $[K : Q] = 4\cos^2 \frac{\pi}{n}$ for a prime number n, then the diagram $\mathcal{D}$ is a co-commuting square.*

Proof. Let $n \geq 5$ be a prime number and $[K : Q] = \lambda^{-1} = 4\cos^2 \frac{\pi}{n}$. By Lemma 4.3, it is sufficient to prove that the set $\tilde{\Lambda}(\lambda)$ is empty. From its definition, it is not empty, iff there exist integers k, s, such that

$$(*) \qquad n-2 \geq s > k \geq 2 \text{ and } \frac{P_{k-1}(\lambda)}{P_{k-2}(\lambda)} = \frac{P_{s-1}(\lambda)}{P_{s-2}(\lambda)} + \lambda.$$

From [J, Lemma 4.2.4], the equation is equivalent to the following one:

$$\frac{1}{\sin \frac{2\pi}{n}} = \frac{1}{\tan \frac{k\pi}{n}} - \frac{1}{\tan \frac{s\pi}{n}}.$$

From this we see that if k, s satisfy $(*)$ then $k' = n-s, s' = n-k$ also satisfy $(*)$. Therefore, we are allowed to assume that $k + s \leq n$. Directly from the definition of Jones' polynomials

$$P_{k-1}(\lambda)P_{s-2}(\lambda) - P_{s-1}(\lambda)P_{k-2}(\lambda) = \lambda(P_{k-2}(\lambda)P_{s-3}(\lambda) - P_{s-2}(\lambda)P_{k-3}(\lambda)).$$

Then $(*)$ is equivalent to the following equation

$$(**) \qquad P_{k-2}(\lambda)P_{s-2}(\lambda) - P_{k-2}(\lambda)P_{s-3}(\lambda) + P_{s-2}(\lambda)P_{k-3}(\lambda) = 0.$$

By [J, Lemma 4.2.4], the degree of the above polynomial does not exceed $[\frac{k-1}{2}] + [\frac{s-1}{2}]$. We show that the degree of minimum polynomial of λ is $\frac{n-1}{2}$. We use the terminology and results presented in [ST]. If we denote $\zeta = \exp(\frac{2\pi i}{n})$, then $\lambda = \frac{1}{4} - \frac{1}{4}(\frac{1-\zeta}{1+\zeta})^2$ is an element of the cyclotomic field $Q(\zeta)$. By Theorem 2.5 and Lemma 3.4 in [ST], degree of the minimum polynomial of λ is $\frac{n-1}{m}$, where m is the number of solutions in l, $1 \leq l \leq n-1$ of the following equation:

$$\lambda = \frac{1}{4} - \frac{1}{4}(\frac{1-\zeta^l}{1+\zeta^l})^2.$$

It is easy matter to check that $m = 2$. Now, the inequality $[\frac{k-1}{2}] + [\frac{s-1}{2}] \geq \frac{n-1}{2}$ clearly contradicts our assumption $k + s \leq n$.

Q.E.D.

Corollary 4.5. *Let* $\mathcal{D} = \begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ *be a diagram of type* II_1 *factors with* $S' \cap K = \mathbb{C}$ *and* $[K : S] < \infty$. *If* $[Q : S] = 4\cos^2 \frac{\pi}{n}$ *for a prime number* n, *then the diagram* $\mathcal{D}$ *is a commuting square.*

Proof. This is an immediate consequence from Proposition 2.12 and Theorem 4.4.

Q.E.D.

Example. T.Teruya, cf. [Te], considered characteristic intermediate subfactors. If $S \subset Q \subset K$, $S' \cap K = \mathbb{C}$ is a triple of type II_1 factors then, analogically to group theory, he called Q a characteristic intermediate subfactor, if for any automorphism $\sigma \in \mathrm{Aut}(K)$ such that $\sigma(S) = S$ we have also $\sigma(Q) = Q$. Also, he showed that, if K is an extension of S by a finite group G, then this notion coincides with characteristic subgroups. By Theorem 4.4 we can easily obtain examples of characteristic intermediate subfactors. Let n and m, $n \neq m$, $n, m > 4$ be any prime numbers and let K be a hyperfinite type II_1 factor. Let $Q \subset K$, and $S \subset Q$ be II_1 subfactors with $[K : Q] = 4\cos^2 \frac{\pi}{n}$ and $[Q : S] = 4\cos^2 \frac{\pi}{m}$. Then, no matter how we pick up S, the subfactor Q is characteristic.

Indeed, from Proposition 4.2 and [J], we get $S' \cap K = \mathbb{C}$. If for some $\sigma \in \mathrm{Aut}(K, S)$, $Q \neq \sigma(Q) = R$ then, by Theorem 4.4, the diagram $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ is commuting and co-commuting square, which, by [SW, Theorem 7.1], contradicts the assumption $n \neq m$.

Example. Let μ be an outer action of the symmetric group S_3 on a type II_1 factor S. Consider the following diagram:

$$\begin{array}{ccc} S \rtimes_\mu \langle(1,2)\rangle & \subset & S \rtimes_\mu S_3 \\ \cup & & \cup \\ S & \subset & S \rtimes_\mu \langle(2,3)\rangle \end{array}.$$

Clearly, it is not co-commuting square, and so for $n = 6$ the statement of Theorem 4.4 is not satisfied.

The prime numbers n are not the only ones for which the set $\tilde{\Lambda}(\lambda)$, $\lambda^{-1} = 4\cos^2\frac{\pi}{n}$ is empty. We can easily check that for $n = 4$ or $n = 9$ it is empty too. With a little digital help, we conjecture that it is empty for all odd numbers (≥ 5). For n even $\tilde{\Lambda}(\lambda)$ is not empty, because it contains $\{1-\lambda, 2\lambda\}$. We were only able to prove the following

Proposition 4.6. *Let $[K:Q] = \lambda^{-1} = 4\cos^2\frac{\pi}{n}$ and let the diagram* $\mathcal{D} = \begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ *be not co-commuting square. Then n is not a prime number and the following statements hold true*

(1) *There exists a factor*

$$P \in \bigcup_{0\leq k\leq n-4} \mathcal{L}^{s_k}(R\subset K), \text{ where } \quad s_k = \frac{\lambda^{k+1}}{P_k(\lambda)^2}.$$

(2) *If $2\lambda^{-1} \neq [K:R] = \eta^{-1} < (\lambda^{-1}-1)^2$, then n must be even, the inclusion "$R\subset K$" is isomorphic to "$Q\subset K$" and*

$$\cos(\text{Op-ang}_K(Q,R)) = \{(\frac{\lambda}{1-\lambda})^2\}.$$

(3) *If $n = 6$, then the principal graph of the inclusion "$Q\subset K$" is A_5.*

Proof. (1) If the diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ is not a co-commuting square, then the projection $p = [e_Q * e_R] \in P(M,K)$ so, by Proposition 4.2, we obtain a factor $P \in \mathcal{L}^{s_k}(R\subset K)$, for some integer $k\in[0,n-4]$.

(2) Since $(0,(\lambda^{-1}-1)^2)\cap\{s_k^{-1} \mid 0\leq k\leq n-4\} = \{\lambda^{-1}\}$, the only possibility for the factor P in (1) is $[K:P] = \lambda^{-1}$. Since $[P:R] = \frac{\lambda}{\eta} < \lambda(\lambda^{-1}-1)^2 < \frac{9}{4}$, by [J], $[P:R] = 1$ or $[P:R] = 2$, with the second possibility eliminated by the assumption $\frac{\lambda}{\eta} \neq 2$. From the construction of the factor P (see [P3, Proposition

4.5]), in the case of $k = 0$ or $k = n-4$, we see that it is a downward basic construction for the pair $K \subset M$. Thus $M = \langle K, R, \tilde{e}_R \rangle$ where $\tilde{e}_R$ is minimal and central projection in $R' \cap M$. By Theorem 2.9 and Lemma 4.3, $p\tilde{e}_R = 0$; therefore, $R' \cap M = \mathbb{C}p \oplus \tilde{e}_R$. Otherwise, there would be another projection, say $f \in R' \cap M$, $f + p + \tilde{e}_R = 1$, $\tau(f) \geq \lambda$, which gives contradiction with $\lambda > 1/4$. By [PP1, Lemma 1.8], there is a unitary $u \in K$ such that $\tilde{e}_R = ue_Qu^*$ and $R = uQu^*$, which gives desired isomorphism.

Since $e_Q + \tilde{e}_R$ is a projection and $E_K(e_Q + \tilde{e}_R) = 2\lambda < 1$, by Theorem 4.1, there exists an integer k, $2 \leq k \leq n-3$ such that

$$\frac{P_{k-1}(\lambda)}{P_{k-2}(\lambda)} = 2\lambda, \quad \text{hence} \quad P_{k-1}(\lambda) - \lambda P_{k-2}(\lambda) = \lambda P_{k-2}, \text{ and so}$$

$$P_k(\lambda) = -P_k(\lambda) + P_{k-1}(\lambda) \text{ and consequently } \frac{P_k(\lambda)}{P_{k-1}(\lambda)} = \frac{1}{2}.$$

Therefore n can not be odd. Value of the angle is computed in Lemma 3.1.

(3) If the principal graph of "$Q \subset K$" is D_4, then $Q' \cap M = \mathbb{C}e_Q \oplus \mathbb{C}f \oplus \mathbb{C}g$ for some projections f and g such that $\tau(f) = \tau(g) = \lambda (= \frac{1}{3})$. If f_1 is a projection in $S' \cap M$, then, since $E_K(f_1)$ is a scalar, $\tau(f_1) \geq \lambda$. Therefore, if $S' \cap M \neq Q' \cap M$ then $S' \cap M = \mathbb{C}e_Q \oplus M_2(\mathbb{C})$. Since f and g are Jones projections $(E_K(f) = \lambda, E_K(g) = \lambda)$ they are central (see [PP1] or Lemma 2.8). Thus we have $S' \cap M = Q' \cap M$ and, in consequence,

$$R' \cap M = (R' \cap M) \cap (Q' \cap M) = R' \cap Q' \cap M = K' \cap M = \mathbb{C}.$$

Then, by Corollary 2.13, $\mathcal{D}$ must be a co-commuting square.

Q.E.D.

Almost identically as in (3) above we can prove a little more.

Corollary 4.7. *If* $\mathcal{D} = \begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ *is a diagram of type II_1 factors, $Q \vee R = K$ ($[K:S] < \infty$, $S' \cap K = \mathbb{C}$) and K is a crossed product of Q by an action μ of a finite group G, $K = Q \rtimes_\mu G$, then $\mathcal{D}$ is a co-commuting square.*

Example. Let $S \subset Q \subset K$ be a triple of type II_1 factors with $S' \cap K = \mathbb{C}$. Let Q be a crossed product of S by a finite abelian group and let K be a crossed product of Q by any finite group. From the above corollary and by [Wi, Theorem 6.], we see that if there is another intermediate subfactor R between S and K such that $R \cap Q = S$ and $R \vee Q = K$, then the inclusion "$S \subset K$" has finite depth.

Corollary 4.8. *If* $\mathcal{D} = \begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ *is like above,* $[K:S] \neq \frac{P_k(\lambda)^2}{t^{k+1}}$ *for* $0 < k < n-4$, $[K:R] \neq [K:Q] < 2+\sqrt{5}$, *$Q$ and R are the only intermediate subfactors between S and K, then $\mathcal{D}$ is co-commuting square.*

Proof. If $p = [e_Q * e_R]$ is like in Theorem 2.9, then $p \in P(M,K)$ and $p - e_Q \in P(M,K)$ and hence $\tau(e_Q)$ is an element of the algebraic difference $\Lambda(\lambda) - \Lambda(\lambda)$. Since $\lambda^{-1} < 2+\sqrt{5}$ and (in case $\lambda^{-1} \geq 4$) $\Lambda_1(\lambda) - \Lambda_1(\lambda) = [-\sqrt{1-4\lambda}, \sqrt{1-4\lambda}]$, we can write a little more:

$$\lambda \in (\Lambda_0(\lambda) - \Lambda_0(\lambda)) \cup (\Lambda_0(\lambda) - \Lambda_1(\lambda)) \cup (\Lambda_1(\lambda) - \Lambda_0(\lambda)).$$

In any case $\tau(p) \in \Lambda_0(\lambda)$ or $\tau(p - e_Q) \in \Lambda_0(\lambda)$. If $p \neq 1$, then by Proposition 4.2, we obtain an intermediate subfactor F between S and K. By our assumption, F can not be S. If $F = R$ then, identically as in Proposition 4.6(2), R is downward basic construction for pair $K \subset M$, which gives contradiction with $[K:R] \neq [K:Q]$. If $F = Q$ then $\tau(p - e_Q) = \lambda$, which implies non-singularity of "$Q \subset K$". This means that either there is an intermediate (of index 2) between Q and K or K is an extension of Q by cyclic group of 3 elements. First possibility is eliminated by our assumptions and in the second case we may apply Corollary 4.7.

Q.E.D.

§5. Some consequences of the commutativity of a diagram.

We assume in this section that diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ is a commuting square, or equivalently, that diagram $\begin{matrix} N & \subset & L \\ \cup & & \cup \\ K & \subset & M \end{matrix}$ is a co-commuting square.

Proposition 5.1. *There exists Jones' projection f for the inclusion $M \subset L$ (i.e. $f \in L$ and $E_M(f) = [L : M]^{-1} = \alpha$) such that*

$$e_R = e_R f = [e_Q * e_R] f.$$

Proof. Let $p = [e_Q * e_R]$. Take any Jones' projection $e_0 \in L$ and the corresponding downward basic construction D so that $L = \langle M, D, e_0 \rangle$. Take such unitary $u \in M$ that $upu^* \in D$. If we set $e_1 = u^* e_0 u$ and $D_1 = u^* D u$, then, by [PP1, Corollary 1.8], $L = \langle M, D_1, e_1 \rangle$ and $p \in D_1$. Since $\tau(e_1 p) = \alpha \tau(p) = \eta = \tau(e_R)$, we can take such unitary $v \in M_p$ that $e_R = v^* e_1 p v$. It is easy to check that $f = (v^* + 1 - p) e_1 (v + 1 - p)$ will do the job.

Q.E.D.

It is natural to ask what the von Neumann algebras $A = M \cap \{e_R\}$ and $B = R \vee \{e_Q\}$ look like. Obviously, $R \subset B \subset A \subset M$ and if, in addition, the diagram $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ is also a co-commuting square then $A = B$ is a factor and $\begin{array}{ccc} K & \subset & M \\ \cup & & \cup \\ R & \subset & A \end{array}$ is a commuting and co-commuting square too. (Cf. [K],[P5],[Wi].)

Let f be like in the above proposition and let M_0 be the corresponding downward basic construction, $M_0 = M \cap \{f\}$ and $L = \langle M, M_0, f \rangle$. For a projection e and a von Neumann algebra F we will denote

$$V(e, F) = \bigvee \{ueu^* \mid u \text{ is a unitary in } F\ \}.$$

Lemma 5.2. *Let $p = [e_Q * e_R]$. Then*

$$A_p = B_p = (M_0)_p = Ae_Q A = Be_Q B = Re_Q R$$

$$\textit{and} \quad p = V(e_Q, R) = V(e_Q, B) = V(e_Q, A).$$

Proof. First, we show that $Ae_Q A = Re_Q R$. Let $x \in A$. By [PP1, Lemma 1.2], $xe_Q = ye_Q$, for $y \in K$. Multiplying the equality $ye_Q e_R = e_R y e_Q$ by e_S we obtain

$ye_S = e_R ye_S = E_R(y)e_S$ which implies $y \in R$. Therefore $Ae_QA = Re_QR = Be_QB$.

Obviously $V(e_Q, R) \in R' \cap M$, hence (by Theorem 2.9) $V(e_Q, R) \geq p$. The equality follows from $ue_Qu^* \leq p$ which hold for all unitaries $u \in R$. By the first part of the proof, it follows that also $p = V(e_Q, B) = V(e_Q, A)$. Now, it is easy to see that $A_p = B_p = Re_QR$.

Proposition 5.1 implies $(M_0)_p \subset A_p$. From Theorem 2.9, A_p is a factor. Now, computation of indices completes the proof.

Q.E.D.

Remark. Let $T = Re_QR$ be the set characterized above. We remark that T is a basic construction for pair $S_p \subset R_p$ and a downward basic construction for pair $M_p \subset L_p$. We can write

$$T = \langle R_p, S_p, e_Q \rangle \quad \text{and} \quad L_p = \langle M_p, T, e_R \rangle.$$

Finally, we obtain the following decomposition for von Neumann algebras A and B.

Corollary 5.3. *Let as before* $p = [e_Q * e_R]$. *Then*

$$A = T \oplus M_{1-p} \quad \textit{and} \quad B = T \oplus R_{1-p}.$$

Proof. The elements $x = r + \sum_i a_i e_Q b_i$, with $r, a_i, b_i \in R$ form a dense $*$-subalgebra of B. By Theorem 2.9 we see that $(1-p)x(1-p) = (1-p)r$. Since p is central in B,

$$B = B_p \oplus B_{1-p} = T \oplus R_{1-p}.$$

If we apply this equality to the algebra $Q \vee \{e_R\}$, then, using the modular conjugation J_K, we obtain the remaining one.

Q.E.D.

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