HOKKAIDO UNIVERSITY

Title	Operator-theoretical analysis of representation of a supersymmetry algebra in Hilbert space
Author(s)	Arai, A.
Citation	Hokkaido University Preprint Series in Mathematics, 269, 2-12
Issue Date	1994-11-01
DOI	https://doi.org/10.14943/83416
Doc URL	https://hdl.handle.net/2115/69020
Type	departmental bulletin paper
File Information	re269.pdf

Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP

Operator-theoretical analysis of representation of a supersymmetry algebra in Hilbert space

Asao Arai

Series ♯269. November 1994

HOKKAIDO UNIVERSITY

PREPRINT SERIES IN MATHEMATICS

♯ 243 N. Hayashi, K. Kato and T. Ozawa, Dilation Method and Smoothing Effect of the Schrödinger Evolution Group, 10 pages. 1994.

♯ 244 D. Lehmann, T. Suwa, Residues of holomorphic vector fields relative to singular invariant subvarieties, 26 pages. 1994.

♯ 245 H. Kubo, Slowly decaying solutions for semilinear wave equations in odd space dimensions, 30 pages. 1994.

♯ 246 T. Nakazi, M. Yamada, (A_2)-Conditions and Carleson Inequalities, 27 pages. 1994.

♯ 247 N. Hayashi, K. Kato and T. Ozawa, Dilation Method and smoothing Effect of Solutions to the Benjamin-ono Equation, 17 pages. 1994.

♯ 248 H. Kikuchi, Sheaf cohomology theory for measurable spaces, 12 pages. 1994.

♯ 249 A. Inoue, Tauberian theorems for Fourier cosine transforms, 9 pages. 1994.

♯ 250 S. Izumiya, G. T. Kossioris, Singularities for viscosity solutions of Hamilton-Jacobi equations, 23 pages. 1994.

♯ 251 H. Kubo, K. Kubota, Asymptotic behaviors of radially symmetric solutions of $\Box u = |u|^p$ for super critical values p in odd space dimensions, 51 pages. 1994.

♯ 252 T. Mikami, Large Deviations and Central Limit Theorems for Eyraud-Farlie-Gumbel-Morgenstern Processes, 9 pages. 1994.

♯ 253 T. Nishimori, Some remarks in a qualitative theory of similarity pseudogroups, 19 pages. 1994.

♯ 254 T. Suwa, Residues of complex analytic foliations relative to singular invariant subvarieties, 15 pages. 1994.

♯ 255 T. Tsujishita, On Triple Mutual Information, 7 pages. 1994.

♯ 256 T. Tsujishita, Construction of Universal Modal World based on Hyperset Theory, 15 pages. 1994.

♯ 257 A. Arai, Trace Formulas, a Golden-Thompson Inequality and Classical Limit in Boson Fock Space, 35 pages. 1994.

♯ 258 Y-G. Chen, Y. Giga, T. Hitaka and M. Honma, A Stable Difference Scheme for Computing Motion of Level Surfaces by the Mean Curvature, 18 pages. 1994.

♯ 259 K. Iwata, J. Schäfer, Markov property and cokernels of local operators, 7 pages. 1994.

♯ 260 T. Mikami, Copula fields and its applications, 14 pages. 1994.

♯ 261 A. Inoue, An Abel-Tauber theorem for Fourier sine transforms, 6 pages. 1994.

♯ 262 N. Kawazumi, Homology of hyperelliptic mapping class groups for surfaces, 13 pages. 1994.

♯ 263 Y. Giga, M. E. Gurtin, A comparison theorem for crystalline evolution in the plane, 14 pages. 1994.

♯ 264 J. Wierzbicki, On Commutativity of Diagrams of Type II_1 Factors, 26 pages. 1994.

♯ 265 N. Hayashi, T. Ozawa, Schrödinger Equations with nonlinearity of integral type, 12 pages. 1994.

♯ 266 T. Ozawa, On the resonance equations of long and short waves, 8 pages. 1994.

♯ 267 T. Mikami, A sufficient condition for the uniqueness of solutions to a class of integro-differential equations, 9 pages. 1994.

♯ 268 Y. Giga, Evolving curves with boundary conditions, 10 pages. 1994.

Operator-theoretical analysis of representation of a supersymmetry algebra in Hilbert space*

Asao Arai
Department of Mathematics, Hokkaido University, Sapporo 060, Japan

Operator-theoretical analysis is made on (unbounded) representations, in Hilbert spaces, of a supersymmetry (SUSY) algebra coming from a supersymmetric quantum field theory in two-dimensional space-time. A basic idea for the analysis is to apply the theory of strongly anticommuting self-adjoint operators. A theorem on integrability of a representation of the SUSY algebra is established. Moreover, it is shown that strong anticommutativity of self-adjoint operators is a natural and suitable concept in analyzing representations of the SUSY algebra in Hilbert spaces.

PACS numbers: 02.30.Tb, 03.65.Db, 11.30.Pb

I. INTRODUCTION

In a relativistic supersymmetric quantum field theory (SSQFT) in two-dimensional space-time, the generators $\{Q_1, Q_2\}$ of supersymmetry (SUSY), called *supercharges*, play an important role, forming, together with the generators $\{H, P\}$ of the space-time translation, the SUSY algebra[1]

$$\begin{cases} Q_1^2 = H + P, \\ Q_2^2 = H - P, \\ Q_1 Q_2 + Q_2 Q_1 = 0. \end{cases} \tag{1.1}$$

In this paper we consider representations, in Hilbert spaces, of the SUSY algebra (1.1), concentrating our attention on their operator-theoretical contents. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{D}$ be a dense subspace in $\mathcal{H}$. We say that the set $\{\mathcal{H}, \mathcal{D}, H, P, Q_1, Q_2\}$ is a *symmetric representation* of (1.1) if H, P, Q_1 and Q_2 are symmetric operators in $\mathcal{H}$, leave $\mathcal{D}$ invariant and satisfy (1.1) on $\mathcal{D}$. An interesting problem is then to formulate integrability conditions for a symmetric representation $\{\mathcal{H}, \mathcal{D}, H, P, Q_1, Q_2\}$ of (1.1) in the sense that (i) H, P, Q_1 and Q_2 are essentially self-adjoint on $\mathcal{D}$ (denote their closures by $\bar{H}, \bar{P}, \bar{Q}_1$ and $\bar{Q}_2$, respectively); (ii) $\{\bar{H}, \bar{P}, \bar{Q}_1\}$ and $\{\bar{H}, \bar{P}, \bar{Q}_2\}$ are families of strongly commuting self-adjoint operators, respectively (see Definition 2.1(i) below); (iii) The pair $\{\bar{H}, \bar{P}\}$ satisfies

$$\pm\bar{P} \leq \bar{H}. \tag{1.2}$$

In the context of relativistic quantum field theory, where $\bar{H}$ and $\bar{P}$ denote the Hamiltonian and the momentum operator, respectively, the last condition together with the strong

*accepted for publication in *Journal of Mathematical Physics*

commutativity of $\bar{H}$ and $\bar{P}$ is called the *spectral condition*. Formulating such an integrability condition in an abstract form may be useful also in applications, in particular, in mathematically rigorous constructions of models of SSQFT[2-5].

A key to analyze symmetric representations of the SUSY algebra is to note that the supercharges Q_1 and Q_2 anticommute in a "naive" sense (see the last relation in (1.1)). In operator theory, there is a proper concept of anticommutativity for (unbounded) self-adjoint operators, which was introduced by Vasilescu[6] and has been developed by some authors[7-11]. In this paper we call it *strong anticommutativity* for clarity (see Definition 2.1(ii) below). Indeed, for *unbounded* self-adjoint operators, there is essential difference between strong anticommutativity and the ordinary "naive" anticommutativity (see Remark (2) after Definition 2.1)[12]. Thus we are naturally led to apply the theory of strongly anticommuting self-adjoint operators to operator-theoretical analysis of symmetric representations of the SUSY algebra. This is the basic idea of our method and this view-point is new[13]. We shall show that the concept of strong anticommutativity of self-adjoint operators is actually a natural concept to analyze represenations of the SUSY algebra.

As is well known in the physics literature[14], the SUSY algebra (1.1) may be regarded as the Lie superalgebra of a "supergroup" . In this connection, it may be desirable to formulate integrability conditions for representations of the SUSY algebra in such a way that H, P, Q_1 and Q_2 become the generators of a representation of the supergroup. However, the notion of supergroup representation cannot be realized within Hilbert space formalism (one would need a concept like a "Hilbert module over an infinite dimensional Grassmann algebra"). In this paper, we do not discuss this aspect; we stay within Hilbert space formalism and concentrate our attention on operator-theoretical contents of representations of the SUSY algebra in Hilbert spaces.

This paper is organized as follows. In Section II we prove a theorem on integrability of a representation of the SUSY algebra (1.1). In Section III, we show that, under a natural condition, strong anticommutativity for the supercharges is necessary for a symmetric representation of (1.1) in which $\{H, P\}$ has a strongly commuting self-adjoint extension. We also give an example of nonintegrable symmetric representations of (1.1) in which the supercharges do not strongly anticommute. In the last section, we present a method to construct an integrable symmetric representation of (1.1) from two strongly anticommuting self-adjoint supercharges.

II. INTEGRABILITY

We first give a definition of strong (anti-)commutativity of two self-adjoint operators.

Definition 2.1: Let A and B be self-adjoint operators in a Hilbert space.

(i) We say that A and B *strongly commute* if their spectral measures commute.
(ii) We say that A and B *strongly anticommute* if

$$e^{itA} B \subset B e^{-itA}$$

for all $t \in \mathbb{R}$.

Remark: (1) Definition 2.1(ii) is symmetric in A and B(see Ref.8).

(2) Let A and B be self-adjoint operators in a Hilbert space. If A and B are bounded, then the strong anticommutativity of A and B is equivalent to the ordinary anticommutativity of them:$AB + BA = 0$. However, in the case where A and B are unbounded, these two concepts of anticommutativity are not equivalent; It can be shown that the concept of strong anticommutativity is certainly stronger than that of the ordinary anticommutativity. In fact, the following holds (see Ref.6): *If A and B strongly anticommute, then $D(A) \cap D(B) \cap D(AB) = D(A) \cap D(B) \cap D(BA)$ and*

$$(AB + BA)f = 0, \quad f \in D(A) \cap D(B) \cap D(AB),$$

where $D(T)$ denotes the domain of the operator T. However, the converse is not true. See Example 3.3 in Ref.6 and Example in Section III below.

In what follows, the domain of the sum $A + B$ of two linear operators A and B in a Hilbert space is taken to be $D(A) \cap D(B)$, unless otherwise stated.

We shall prove the following.

Theorem 2.2: *Let $\{\mathcal{H}, \mathcal{D}, H, P, Q_1, Q_2\}$ be a symmetric representation of (1.1). Suppose that H is essentially self-adjoint on $\mathcal{D}$ (its closure is denoted $\bar{H}$). Then the following* (i)–(iii) *hold:*

(i) *The symmetric operators $Q_1 \pm Q_2, Q_1$ and Q_2 are essentially self-adjoint on $\mathcal{D}$ and the closures $\bar{Q}_1$ and $\bar{Q}_2$ strongly anticommute. Moreover, the following operator equalities hold:*

$$\begin{aligned} \bar{H} &= \frac{1}{2}\left(\overline{Q_1 \pm Q_2}\right)^2 \\ &= \frac{1}{2}\left(\bar{Q}_1 \pm \bar{Q}_2\right)^2 \\ &= \frac{1}{2}\left(\bar{Q}_1^2 + \bar{Q}_2^2\right) \end{aligned} \tag{2.1}$$

(ii) *(Spectral condition) P is essentially self-adjoint on $\mathcal{D}$ and $\bar{P}$ strongly commutes with $\bar{H}$. Moreover, $\{\bar{H}, \bar{P}\}$ satisfies (1.2).*

(iii) *Each $\bar{Q}_j$ $(j = 1, 2)$ strongly commutes with $\bar{P}$ and $\bar{H}$.*

Some remarks may be in order on this theorem:

(1) In constructing a model of SSQFT in a mathematically rigorous way, one usually starts with proving the essential self-adjointness of the Hamiltonian of the model on a suitable dense domain[15]. Thus the assumption of Theorem 2.2 is natural from such a view-point. Moreover, Theorem 2.2 may be remarkable in the sense that it shows that we need only to assume the essential self-adjointness of H on $\mathcal{D}$ to obtain all the desired properties for $\{H, P, Q_1, Q_2\}$

mentioned in the Introduction as well as the strong anticommutativity of $\bar{Q}_1$ and $\bar{Q}_2$.

(2) Theorem 2.2(i) implies the following: *if Q_1 and Q_2 are essentially self-adjoint on $\mathcal{D}$, but $\bar{Q}_1$ and $\bar{Q}_2$ do not strongly anticommute, then H cannot be essentially self-adjoint on $\mathcal{D}$.* This suggests that strong anticommutativity is a natural concept to characterize properties of the supercharges properly. In the next section, we shall discuss this aspect in some more detail.

(3) We can construct a symmetric representation $\{\mathcal{H}, \mathcal{D}, H, P, Q_1, Q_2\}$ of (1.1) in an abstract Boson-Fermion Fock space which satisfies the assumption of Theorem 2.2 and in which each Q_j is an infinite dimensional Dirac type operator and H and P are second qauntized operators[16].

To prove Theorem 2.2, we recall basic results in the theory of strongly anticommuting self-adjoint operators.

Lemma 2.3:(Refs.6, 8) *Let A and B be strongly anticommuting self-adjoint operators in a Hilbert space. Then the following* (i) and (ii) *hold:*

(i) *The operators $A+B$ and A^2+B^2 are self-adjoint and the operator equality*

$$(A+B)^2 = A^2 + B^2$$

holds.

(ii) *A (resp. B) and $|B|$ (resp. $|A|$) strongly commute.*

(iii) *$|A|$ and $|B|$ strongly commute.*

Theorem 2.4: *Let A and B be symmetric operators in a Hilbert space $\mathcal{H}$ and let $\mathcal{M}$ be a dense subspace of $\mathcal{H}$ which is left invariant by A and B. Assume that A^2+B^2 is essentially self-adjoint on $\mathcal{M}$ and $ABf + BAf = 0$ for all $f \in \mathcal{M}$. Then $A \pm B, A$ and B are essentially self-adjoint on $\mathcal{M}$ and the closures of A and B strongly anticommute. Moreover, the following operator equalities hold:*

$$\overline{A^2+B^2} = \left(\overline{A \pm B}\right)^2$$

Proof: Similar to the proof of Corollary 4.5 in Ref.8. ∎

Remark: The conlusion of Theorem 2.4 is slightly stronger than that of the corollary of the cited paper in that $\mathcal{M}$ is a common core for $A \pm B, A$ and B.

We also need the following lemma:

Lemma 2.5:[18] *Let A be a self-adjoint operator and B be a symmetric operator. Suppose that $D(A) \subset D(B)$ and $(Af, Bg) = (Bf, Ag)$ for all $f, g \in D(A)$. Then B is essentially self-adjoint and its closure strongly commutes with A.*

Proof of Theorem 2.2

(i) It follows from the first two equations in (1.1) that

$$H = \frac{1}{2}\left(Q_1^2 + Q_2^2\right), \tag{2.2}$$

$$P = \frac{1}{2}\left(Q_1^2 - Q_2^2\right), \tag{2.3}$$

on $\mathcal{D}$. Hence, by the assumption, $Q_1^2 + Q_2^2$ is essentially self-adjoint on $\mathcal{D}$. Therefore we can apply Theorem 2.4 with $A = Q_1, B = Q_2$ and $\mathcal{M} = \mathcal{D}$ to obtain the first half of part (i) and the first equality in (2.1). It is easy to see that $\overline{Q_1 \pm Q_2} = \bar{Q}_1 \pm \bar{Q}_2$. Hence the first equality in (2.1) and Lemma 2.3 imply the second and third equalities in (2.1).

(ii) By the third equation in (1.1), we have

$$(Q_1^2\phi, Q_2^2\phi) = (Q_1\phi, Q_2^2 Q_1\phi) \geq 0, \quad \phi \in \mathcal{D}.$$

By this fact and (2.2), we obtain

$$||Q_j^2\phi|| \leq 2||H\phi||, \quad \phi \in \mathcal{D}, j = 1,2. \tag{2.4}$$

By (2.3) and (2.4), we have for all $\phi \in \mathcal{D}$

$$\begin{aligned} ||P\phi|| &\leq \frac{1}{2}(||Q_1^2\phi|| + ||Q_2^2\phi|| \\ &\leq 2||H\phi||. \end{aligned} \tag{2.5}$$

A direct computation shows that

$$(P\phi, H\psi) = (H\phi, P\psi), \quad \phi, \psi \in \mathcal{D}.$$

By a limiting argument using (2.5) and the fact that $\mathcal{D}$ is a core for $\bar{H}$, we can show that $D(\bar{H}) \subset D(\bar{P})$ and

$$(\bar{P}\phi, \bar{H}\psi) = (\bar{H}\phi, \bar{P}\psi), \quad \phi, \psi \in D(\bar{H}),$$

where $\bar{P}$ is the closure of P restricted to $\mathcal{D}$. Thus, by applying Lemma 2.5 with $A = \bar{H}$ and $B = \bar{P}$, we conclude that $\bar{P}$ is self-adjoint and strongly commutes with $\bar{H}$.

By (2.2) and (2.3), we have

$$(\psi, \pm\bar{P}\psi) \leq (\psi, \bar{H}\psi), \quad \psi \in \mathcal{D}. \tag{2.6}$$

Then, by a limiting argument using (2.5), we can extend (2.6) to all $\psi \in D(\bar{H})$. Thus (1.2) follows.

(iii) By Lemma 2.3(ii), $\bar{Q}_j$ and $\bar{Q}_k^2$ strongly commute ($j, k = 1,2$). Hence, by the two variable functional calculus, we obtain the desired result. ∎

III. NECESSITY OF STRONG ANTICOMMUTATIVITY

In this section, we show that the strong anticommutativity of the supercharges is necesssary for a symmetric representation of the SUSY algebra (1.1) to be integrable. A basic fact is the following.

Theorem 3.1: *Let A and B be self-adjoint operators in a Hilbert space $\mathcal{H}$ and $\mathcal{M}$ be a dense subspace of $\mathcal{H}$ left invariant by A and B. Suppose that $\mathcal{M}$ is a common core for A^2 and B^2, $AB + BA = 0$ on $\mathcal{M}$, and A^2 and B^2 strongly commute. Then A and B strongly anticommute.*

The idea of proof of this theorem is to apply the following lemma.

Lemma 3.2:[19] *Let A and B be self-adjoint operators in a Hilbert space and $A = U_A|A|, B = U_B|B|$ be the polar decompositions of A and B, respectively. Then A and B strongly anticommute if and only if (i) $U_AU_B = -U_BU_A$; (ii) $U_A|B| \subset |B|U_A$; (iii) $U_B|A| \subset |A|U_B$; (iv) $|A|$ and $|B|$ strongly commute.*

The following lemma, which can be easily proven, is well known.

Lemma 3.3: *Let A and B be self-adjoint operators in a Hilbert space. Suppose that A is bounded. Then A and B strongly commute if and only if $AB \subset BA$.*

Proof of Theorem 3.1
It is shown[20] that

$$U_A f = \lim_{\varepsilon \downarrow 0} A(A^2 + \varepsilon)^{-1/2} f, \quad f \in \mathcal{H}.$$

The strong commutativity of A^2 and B^2 implies that of $(A^2+\varepsilon)^{-1/2}$ and B^2 for all $\varepsilon > 0$. Hence, by Lemma 3.3, we have

$$(A^2 + \varepsilon)^{-1/2} B^2 \subset B^2 (A^2 + \varepsilon)^{-1/2}$$

for all $\varepsilon > 0$. Note that

$$B^2 A f = A B^2 f, \quad f \in \mathcal{M}.$$

Using these facts, we have for all $f, g \in \mathcal{M}$

$$\begin{aligned}
(f, U_A B^2 g) &= \lim_{\varepsilon \downarrow 0} (f, AB^2 (A^2 + \varepsilon)^{-1/2} g) \\
&= \lim_{\varepsilon \downarrow 0} (B^2 A f, (A^2 + \varepsilon)^{-1/2} g) \\
&= \lim_{\varepsilon \downarrow 0} (AB^2 f, (A^2 + \varepsilon)^{-1/2} g) \\
&= \lim_{\varepsilon \downarrow 0} (B^2 f, A(A^2 + \varepsilon)^{-1/2} g) \\
&= (B^2 f, U_A g).
\end{aligned}$$

Since $\mathcal{M}$ is a core for B^2, it follows that $U_A g \in D(B^2)$ and $B^2 U_A g = U_A B^2 g$. By the condition that $\mathcal{M}$ is a core for B^2 again, this equality extends to all $g \in D(B^2)$, showing that $U_A B^2 \subset B^2 U_A$, which implies that U_A and B^2 strongly commute (Lemma 3.3). In particular, we have

$$(B^2+\varepsilon)^{-1/2} U_A = U_A (B^2+\varepsilon)^{-1/2}.$$

Since $|B| = \sqrt{B^2}$, it follows that U_A and $|B|$ strongly commute. Hence $U_A|B| \subset |B| U_A$. Similarly we can show that $U_B|A| \subset |A| U_B$. Moreover, we have for all $f, g \in \mathcal{M}$

$$\begin{aligned}
(U_A f, U_B g) &= \lim_{\varepsilon \downarrow 0}((A^2+\varepsilon)^{-1/2} A f, U_B g) \\
&= \lim_{\varepsilon \downarrow 0}(A f, U_B (A^2+\varepsilon)^{-1/2} g) \\
&= \lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0}((B^2+\delta)^{-1/2} B A f, (A^2+\varepsilon)^{-1/2} g) \\
&= \lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0}(B A f, (B^2+\delta)^{-1/2} (A^2+\varepsilon)^{-1/2} g) \\
&= -\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0}(A B f, (B^2+\delta)^{-1/2} (A^2+\varepsilon)^{-1/2} g) \\
&= -\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0}(B (B^2+\delta)^{-1/2} f, A (A^2+\varepsilon)^{-1/2} g) \\
&= -(U_B f, U_A g),
\end{aligned}$$

which implies that $U_A U_B + U_B U_A = 0$. Of course, $|A|$ and $|B|$ strongly commute. Thus we can apply Lemma 3.2 to obtain the desired result. ∎

The main result in this section is the following.

Theorem 3.4: *Let $\{\mathcal{H}, \mathcal{D}, H, P, Q_1, Q_2\}$ be a symmetric representation of (1.1) and let $\hat{H}$ (resp. $\hat{P}$) be any self-adjoint extension of H (resp. P). Suppose that, for $j = 1, 2$, Q_j^2 are essentially self-adjoint on $\mathcal{D}$ and $\hat{H}$ and $\hat{P}$ strongly commute. Then each Q_j is essentially self-adjoint on $\mathcal{D}$ and $\bar{Q}_1$ and $\bar{Q}_2$ strongly anticommute.*

To prove this theorem, we need a lemma.

Lemma 3.5: *Let A be a symmetric operator in a Hilbert space and $\mathcal{D}$ be a dense subspace of $\mathcal{H}$. Suppose that $\mathcal{D} \subset D(A^2)$ and A^2 is essentially self-adjoint on $\mathcal{D}$. Then A is essentially self-adjoint on $\mathcal{D}$ and the operator equality*

$$\overline{A^2} = \bar{A}^2 \tag{3.1}$$

holds.

Proof: This lemma is well known (e.g., Ref.21, Chapter X, p.341, Problem 28), but, to be self-contained, we give a proof of it. Let $T = \overline{A^2}$. For all $f \in \mathcal{D}$, we have $||Af|| \leq ||f||\,||Tf||$. Since $\mathcal{D}$ is a core for T, this inequality implies that $D(T) \subset D(\bar{A})$, where $\bar{A}$ is the closure of A restricted to $\mathcal{D}$, and

$$||\bar{A} f|| \leq ||f||\,||Tf||, \quad f \in D(T). \tag{3.2}$$

It is obvious that

$$(\bar{A}f, Tg) = (Tf, \bar{A}g), \quad f, g \in \mathcal{D}. \tag{3.3}$$

By (3.2) and the condition that $\mathcal{D}$ is a core for T, we can show that (3.3) extends to all $f, g \in D(T)$. Thus we can apply Lemma 2.5 to conclude that $\bar{A}$ is self-adjoint. We have $A^2 \subset \bar{A}^2$. Hence $T \subset \bar{A}^2$. Since the both sides are self-adjoint, (3.1) follows. ∎

Proof of Theorem 3.4

We have from (1.1)

$$Q_1^2 \restriction \mathcal{D} \subset \hat{H} + \hat{P}, \quad Q_2^2 \restriction \mathcal{D} \subset \hat{H} - \hat{P}.$$

By Lemma 3.5, Q_j is essentially self-adjoint on $\mathcal{D}$ and

$$\overline{Q_j^2} = \bar{Q}_j^2, j = 1, 2.$$

Hence we obtain

$$\bar{Q}_1^2 = \overline{\hat{H} + \hat{P}}, \quad \bar{Q}_2^2 = \overline{\hat{H} - \hat{P}}. \tag{3.4}$$

Since $\{\hat{H}, \hat{P}\}$ is a family of strongly commuting self-adjoint operators, (3.4) implies that $\bar{Q}_1^2$ strongly commutes with $\bar{Q}_2^2$. Hence, by Theorem 3.1, $\bar{Q}_1$ and $\bar{Q}_2$ strongly anticommute. ∎

Theorem 3.4 implies the following: *Let $\{\mathcal{H}, \mathcal{D}, H, P, Q_1, Q_2\}$ be a symmetric representation of (1.1). Suppose that, for $j = 1, 2$, Q_j^2 are essentially self-adjoint on $\mathcal{D}$ (then, by Lemma 3.5, Q_j is essentially self-adjoint on $\mathcal{D}$), but $\bar{Q}_1$ and $\bar{Q}_2$ do not strongly anticommute. Then $\{\hat{H}, \hat{P}\}$ cannot be a family of strongly commuting self-adjoint operators (in particular, $\{\hat{H}, \hat{P}\}$ does not satisfy the spectral condition).*

In Theorem 3.4, the condition that Q_j^2 ($j = 1, 2$) is essentially self-adjoint on $\mathcal{D}$ is not so restrictive for applications. On the other hand, it is natural to require, as a condition of (weaker) integrability of a symmetric representation $\{\mathcal{H}, \mathcal{D}, H, P, Q_1, Q_2\}$ of (1.1), that there exist self-adjoint extensions $\hat{H}$ and $\hat{P}$ of H and P repectively such that $\{\hat{H}, \hat{P}\}$ is a family of strongly commuting self-adjoint operators (otherwise the concept of the spectral condition could not be defined properly). In this sense, Theorem 3.4 tells us that the strong anticommutativity of the supercharges is necessary for integrability of the SUSY algebra $\{\mathcal{H}, \mathcal{D}, H, P, Q_1, Q_2\}$.

We conclude this section with a counter example of a symmetric representation of the SUSY algebra (1.1) in which each Q_j^2 is essentially self-adjoint on $\mathcal{D}$, but, $\bar{Q}_1$ and $\bar{Q}_2$ do not strongly anticommute. Although the example given below is taken from supersymmetric quantum mechanics , not from SSQFT, it is interesting both mathematically and physically[22,23].

Example: Let $\{\mathrm{a}_\nu\}_{\nu=1}^n$ be a finite set of isolated points in $\mathbf{R}^2$ and set $M := \mathbf{R}^2 \setminus$

$\{\mathbf{a}_\nu\}_{\nu=1}^n$. Let

$$A_1(\mathbf{r}) = -\sum_{\nu=1}^{n}\sum_{\alpha,\beta=0}^{m} C_{\alpha,\beta}^{(\nu)} D_x^\alpha D_y^\beta \left(\frac{y-a_{\nu 2}}{|\mathbf{r}-\mathbf{a}_\nu|^2}\right),$$

$$A_2(\mathbf{r}) = \sum_{\nu=1}^{n}\sum_{\alpha,\beta=0}^{m} C_{\alpha,\beta}^{(\nu)} D_x^\alpha D_y^\beta \left(\frac{x-a_{\nu 1}}{|\mathbf{r}-\mathbf{a}_\nu|^2}\right), \quad \mathbf{r}=(x,y)\in M,$$

where $C_{\alpha,\beta}^{(\nu)}$ are real constants, $m \geq 0$ is an integer, and D_x and D_y are the distributional partial differential operators in x and y, respectively. Note that

$$D_x A_2(\mathbf{r}) - D_y A_1(\mathbf{r}) = 2\pi \sum_{\nu=1}^{n}\sum_{\alpha,\beta=0}^{m} C_{\alpha,\beta}^{(\nu)} D_x^\alpha D_y^\beta \delta(\mathbf{r}-\mathbf{a}_\nu), \tag{3.5}$$

where $\delta(\mathbf{r})$ is the Dirac delta distribution on $\mathbf{R}^2$. Let

$$P_1 = -iD_x - A_1, \quad P_2 = -iD_y - A_2,$$

acting in $L^2(\mathbf{R}^2)$. Obviously each P_j leaves $C_0^\infty(M)$ invariant. It follows from (3.5) that

$$P_1 P_2 f = P_2 P_1 f, \quad f \in C_0^\infty(M). \tag{3.6}$$

In Ref.22 we proved that P_j is essentially self-adjoint on $C_0^\infty(M)$.

Let $\sigma_j, j=1,2$, be the first two of the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

In the Hilbert space

$$\begin{aligned}\mathcal{H} :&= L^2(\mathbf{R}^2)\otimes \mathbf{C}^2 \\ &= \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \middle| f,g \in L^2(\mathbf{R}^2) \right\},\end{aligned}$$

we define the self-adjoint operators

$$Q_j = \sigma_j \otimes \bar{P}_j, \quad j=1,2.$$

Let

$$\mathcal{D} = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \middle| f,g \in C_0^\infty(M) \right\}.$$

Then $\mathcal{D}$ is dense in $\mathcal{H}$ and *each* Q_j *is essentially self-adjoint on* $\mathcal{D}$. Moreover, Q_j leaves $\mathcal{D}$ invariant. By (3.6) and the anticommutativity of σ_1 and σ_2 , we have

$$Q_1 Q_2 \psi + Q_2 Q_1 \psi = 0, \quad \psi \in \mathcal{D}.$$

Let

$$H := \frac{1}{2}(I \otimes \bar{P}_1^2 + I \otimes \bar{P}_2^2), \quad P := \frac{1}{2}(I \otimes \bar{P}_1^2 - I \otimes \bar{P}_2^2).$$

Then we can easily see that the first two equations of (1.1) hold on $\mathcal{D}$. It follows that $\{\mathcal{H}, \mathcal{D}, H, P, Q_1, Q_2\}$ is a symmetric representation of the SUSY algebra (1.1).

In the same way as in Ref.22, we can show that *each* Q_j^2 *is essentially self-adjoint on* $\mathcal{D}$. By Theorem 3.4 in Ref.10, Q_1 *and* Q_2 *strongly anticommute if and only if* $\bar{P}_1$ *and* $\bar{P}_2$ *strongly commute.* On the other hand, it was shown in Ref.22 that $\bar{P}_1$ *and* $\bar{P}_2$ *strongly commute if and only if the constant* $C_{0,0}^{(\nu)}$ *is an integer for all* $\nu = 1, \cdots, n$ (in this case we say that the magnetic flux is *locally quantized*). Hence, if the magnetic flux is not locally quantized, then Q_1 and Q_2 do not strongly anticommute. Thus, by Theorem 3.4, any self-adjoint extension of $\{H, P\}$ is not a family of strongly commuting self-adjoint operators if the magnetic flux is not locally quantized. It also follows that, under the same condition, H *is not essentially self-adjoint on* $\mathcal{D}$ (by Theorem 2.2).

Remark: The operator $Q := Q_1 + Q_2$ is called a Dirac-Weyl operator. In the case where the magnetic flux is not locally quantized, it has nontrivial and interesting properties, see Ref.23 for the details.

IV. REPRESENTATION BASED ON TWO STRONGLY ANTICOMMUTING SELF-ADJOINT OPERATORS

In this section, we shall show that the strong anticommutativity of the self-adjoint supercharges is sufficient to construct an integrable symmetric representation of the SUSY algebra (1.1).

Let Q_1 and Q_2 be strongly anticommuting self-adjoint operators in a Hilbert space $\mathcal{H}$ and let E_1 and E_2 be the spectral measures of $|Q_1|$ and $|Q_2|$, respectively. Then, by Lemma 2.3(iii), we can define a two-dimensional spectral measure by

$$E = E_1 \otimes E_2.$$

By Lemma 2.3(i) (or the functional calculus),

$$H := \frac{1}{2}(Q_1^2 + Q_2^2) = \frac{1}{2}\int_{\mathbf{R}^2} (\lambda^2 + \mu^2) dE(\lambda, \mu) \tag{4.1}$$

is a nonnegative self-adjoint operator. It follows from Lemma 2.3(iii) and the functional calculus that $Q_1^2 - Q_2^2 \, (= |Q_1|^2 - |Q_2|^2)$ is essentially self-adjoint. Hence we can define the self-adjoint operator

$$P := \frac{1}{2}\overline{(Q_1^2 - Q_2^2)} = \int_{\mathbf{R}^2} (\lambda^2 - \mu^2) dE(\lambda, \mu). \tag{4.2}$$

Let

$$\mathcal{D} = \bigcup_{0 \le a < b < \infty, 0 \le c < d < \infty} \mathrm{Ran}\, E([a,b] \times [c,d]),$$

where $\mathrm{Ran}\, T$ denotes the range of the operator T. Then $\mathcal{D}$ is a dense subspace of $\mathcal{H}$ and left invariant by $|Q_j|, j = 1, 2$.

Lemma 4.1: *The set $\{\mathcal{H}, \mathcal{D}, H, P, Q_1, Q_2\}$ is a symmetric representation of (1.1).*

Proof: Let $Q_j = U_j|Q_j|$ $(j = 1, 2)$ be the polar decomposition of Q_j. Then, by Lemma 3.2, U_j commutes with E_k $(j, k = 1, 2)$. Hence it follows that H, P, Q_1 and Q_2 leaves $\mathcal{D}$ invariant, satisfying the first two equations in (1.1) on $\mathcal{D}$. The third equation in (1.1) follows from condition (i) in Lemma 3.2. ∎

Lemma 4.2: *The subspace $\mathcal{D}$ is a core for H.*

Proof. Let $\psi \in D(H)$ and $\psi_n = E([0,n] \times [0,n])\psi, n = 1, 2, \cdots$. Then $\psi_n \in \mathcal{D}$. By the functional calculus, one can easily show that $\psi_n \to \psi, H\psi_n \to H\psi$ as $n \to \infty$. Thus the desired result follows. ∎

By Lemmas 4.1 and 4.2, we can apply Theorem 2.2 to obtain the following result:

Theorem 4.3: *Let $\{\mathcal{H}, \mathcal{D}, H, P, Q_1, Q_2\}$ be as above. Then the conclusion of Theorem 2.2 holds with $\bar{H}, \bar{P}, \bar{Q}_1$ and $\bar{Q}_2$ replaced by H, P, Q_1 and Q_2, respectively.*

ACKNOLEDGMENT

This work is supported by the Grant-In-Aid 06640188 for science research from the Ministry of Education, Japan.

1. P.G.O.Freund, "Introduction to Supersymmetry," Cambridge University Press, Cambridge, 1986.
2. A.Jaffe and A.Lesniewski, *Supersymmetric quantum fields and infinite dimensional analysis*, in "Nonperturbative Quantum Field Theory," Ed. by G.'t Hoot, A.Jaffe, G.Mack, P.K.Mitter and R.Stora. Plenum Press, New York, 1988.
3. A. Jaffe, A. Lesniewski and C. Wieczerkowski, *Supersymmetry and the spectral condition on a cylinder*, Lett.Math.Phys. **16** (1988), 385-388.
4. A.Arai, *A general class of infinite dimensional Dirac operators and path integral representation of their index*, J.Funct.Anal. **105** (1992), 342-408.
5. A.Arai, *Dirac operators in Boson-Fermion Fock spaces and supersymmetric quantum field theory*, J.Geom.Phys. **11** (1993), 465–490.
6. F.-H.Vasilescu, *Anticommuting self-adjoint operators*, Rev.Roum.Math.Pures Appl. **28** (1983), 77-91.
7. Yu.S.Samoilenko, "Spectral Theory of Families of Self-Adjoint Operators," Kluwer Academic Publishers, Dordrecht, 1991.
8. S. Pedersen, *Anticommuting self-adjoint operators*, J.Funct.Anal. **89** (1990), 428-443.

9. A.Arai, *Commutation properties of anticommuting self-adjoint operators, spin representation and Dirac operators*, Integr. Equat. Oper. Th. **16** (1993), 38–63.
10. A.Arai, *Characterization of anticommutativity of self-adjoint operators in connection with Clifford algebra and applications*, Integr.Equat.Oper.Th. **17** (1993), 451–563.
11. A.Arai, *Analysis on anticommuting self-adjoint operators*, Adv.Stud.Pure Math. **23** (1994), 1–15.
12. The situation is similar to the case of commutativity for unbounded self-adjoint operators (see, e.g., §VIII.5 in Ref.17).
13. See also Ref.16.
14. See, e.g., Ref.1.
15. See Ref.2 and references therein.
16. A.Arai, *Fock space representations of the relativistic supersymmetry algebra in the two-dimensional space-time*, Hokkaido Univ. Preprint Series in Math. No.123, 1991.
17. M.Reed and B.Simon, "Methods of Modern Mathematical Physics Vol.I:Functional Analysis," Academic Press, New York, 1972.
18. A.E.Nussbaum, *A commutativity theorem for unbounded operators in Hilbert space*, Trans.Amer.Math.Soc. **140** (1969), 485–491. See also Proposition 4.1 in Ref.8.
19. Corollary 3.5 in Ref.8.
20. Lemma 3.1 in Ref.9.
21. M.Reed and B.Simon, "Methods of Modern Mathematical Physics Vol.II:Fourier Analysis, Self-Adjointness," Academic Press, New York, 1975.
22. A.Arai, *Momentum operators with gauge potentials, local quantization of magnetic flux, and representation of canonical commutation relations*, J.Math.Phys. **33** (1992), 3374–3378.
23. A.Arai, *Properties of the Dirac-Weyl operator with a strongly singular gauge potential*, J.Math.Phys. **34** (1993), 915–935.