Analysis of a Family of Strongly Commuting Self-adjoint Operators with Applications to Perturbed D'Alembertians and the External Field Problem in Quantum Field Theory

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Analysis of a Family of Strongly Commuting Self-adjoint Operators with Applications to Perturbed D'Alembertians and the External Field Problem in Quantum Field Theory

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Abstract: A family of strongly commuting self-adjoint operators associated with some objects in the d-dimensional Minkowski space is introduced and operator calculi concerning these self-adjoint operators and the canonical momentum operator $p = (p_0, p_1, \cdots, p_{d-1})$ are developed. It is shown that a class of unitary transformations of p_μ is given by a class of operator-valued Lorentz transformations of perturbed p_μ's. Moreover, the integral kernels of the unitary groups of perturbed d'Alembertians are explicitly computed. As an application, a detailed analysis of the quantum theory of a charged spinless relativistic particle in an external electromagnetic field is given. The present analysis clarifies a general mathematical structure behind Schwinger's *proper-time method* for the external field problem in quantum field theory.

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I. INTRODUCTION

In the external field problem in quantum field theory, which concerns a quantized scalar field or a quantized Dirac field interacting with a classical (unquantized) electromagnetic field, it is important to investigate the properties of the Green functions ("propagators") of the Dirac or the Klein-Gordon operators with vector potentials. Schwinger [Sch] presented a beautiful method, called the *proper-time method*, to obtain explicit formulae of the Green functions for some classes of electromagnetic fields (cf. also [I-Z, §2-5-4]). Recently Vaidya et al [V-S-H] have reconsidered the proper-time method from an algebraic point of view and algebraically computed the Green function for a spinless charged particle in an external plane-wave electromagnetic field. The discussions in [V-S-H] are heuristic, but suggest some ideas that may lead one to a complete and mathematically rigorous understanding for the proper-time method from an operator-theoretical point of view. With this motivation, we develop in this paper an operator theory associated with some objects in the d-dimensional Minkowski space $\mathbf{M}^d$. The operator theory presented here does not only clarifies algebraic-analytic structures of the proper-time method for a class of vector potentials, but also is interesting in its own right. In particular, it has applications to perturbed d'Alembertians or Klein-Gordon operators.

Before describing the outline of the present paper, we introduce some basic symbols. We denote a vector in $\mathbf{M}^d$ or the Euclidean space $\mathbf{R}^d$ as $x = (x^0, \cdots, x^{d-1}) = (x^\mu)_{\mu=0}^{d-1}$ and by $g = (g_{\mu\nu})_{\mu,\nu=0,\cdots,d-1}$ the metric tensor of $\mathbf{M}^d$ with

$$g_{00} = -g_{jj} = 1, \quad j = 1, \cdots, d-1,$$
$$g_{\mu\nu} = 0, \quad \mu \neq \nu,$$

so that the indefinite inner product of $\mathbf{M}^d$ is given by

$$xy = g_{\mu\nu}x^\mu y^\nu = x^0 y^0 - \sum_{j=1}^{d-1} x^j y^j. \tag{1.1}$$

Here, and in what follows, *summation over repeated Greek indices with respect to upper and lower ones is understood* unless otherwise stated. We write $xx = x^2$.

We define

$$x_\mu = g_{\mu\nu}x^\nu, \quad \mu = 0, \cdots, d-1.$$

Then we can write $xy = x^\mu y_\mu$. The inverse g^{-1} of the matrix g is given by $g^{-1} = (g^{\mu\nu})_{\mu,\nu=0,\cdots,d-1}$ with $g^{\mu\nu} = g_{\mu\nu}, \mu, \nu = 0, \cdots, d-1$, so that one can write

$$x^\mu = g^{\mu\nu}x_\nu, \quad \mu = 0, \cdots, d-1.$$

For each $a \in \mathbf{M}^d$, the function ax defines a self-adjoint multiplication operator on $L^2(\mathbf{R}^d)$ with domain $D(ax) = \{\psi \in L^2(\mathbf{R}^d) | ax\psi \in L^2(\mathbf{R}^d)\}$ (we denote by $D(\,\cdot\,)$ operator domain).

Let ∂_μ be the generalized partial differential operator in x^μ acting in $L^2(\mathbf{R}^d)$, so that the μ-th component

$$p_\mu = i\partial_\mu \tag{1.2}$$

of the canonical momentum operator $p = (p_0, \cdots, p_{d-1})$ is self-adjoint. For each $b \in \mathbf{M}^d$, we can define a self-adjoint operator on $L^2(\mathbf{R}^d)$, denoted bp, as follows:

$$D(bp) = \left\{ \psi \in L^2(\mathbf{R}^d) \middle| \int_{\mathbf{R}^d} \left| b\xi \widetilde{\psi}(\xi) \right|^2 d\xi < \infty \right\}, \tag{1.3a}$$

$$(\widetilde{bp\psi})(\xi) = b\xi\widetilde{\psi}(\xi), \quad \psi \in D(bp), \tag{1.3b}$$

where

$$\widetilde{\psi}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} e^{i\xi x}\psi(x)dx, \quad \xi \in \mathbf{R}^d,$$

is the Fourier transform of ψ with ξx being the *Minkowski inner product* of ξ and x as in (1.1).

We introduce a subset of $\mathbf{M}^d \times \mathbf{M}^d$:

$$\mathbf{M}_0 = \{(a, b) \in \mathbf{M}^d \times \mathbf{M}^d | \ a \neq 0, b \neq 0, ab = 0\}. \tag{1.4}$$

In Section II we first show that, for each $(a, b) \in \mathbf{M}_0$, ax and bp strongly commute. In this case, via the two-variable functional calculus, one can define, a self-adjoint operator $u(ax, bp)$ for a real-valued Borel measurable, almost everywhere (a.e.) finite function u on $\mathbf{R}^2$ with respect to (w.r.t.) the Lebesgue measure on $\mathbf{R}^2$. We compute the unitary transformation of p_μ and of the free d-dimensional d'Alembertian by the unitary operator $e^{iu(ax,bp)}$. As an application, we prove essential self-adjointness of a perturbed d'Alembertian.

Section III concerns a self-adjoint operator strongly commuting with ax and bp. For a $d \times d$ real antisymmetric constant matrix $f = (f_{\mu\nu})_{\mu,\nu=0,\cdots,d-1}$, we introduce a self-adjoint operator L_f, which is a linear combination of the components of the angular momentum operator in $\mathbf{M}^d$. It is shown that, if f is in a subset of $d \times d$ real antisymmetric matrices, then L_f strongly commutes with ax and bp.

In Section IV we introduce operator-valued Lorentz transformations in an abstract framework and present some elementary facts on them.

Section V is devoted to operational calculus of the self-adjoint operators ax, bp, L_f and p_μ. The main result of this section is the unitary transformation property of p_μ by a unitary operator defined in terms of ax, bp and L_f. We show that an operator-valued Lorentz transformation of a perturbed momentum operator is unitarily equivalent to p on a dense domain. As a corollary, we obtain a self-adjoint extension of a perturbed d'Alembertian.

In Section VI, we explicitly compute the integral kernels of the unitary groups generated by perturbed d'Alembertians. This section should clarifies a general mathematical structure behind Schwinger's proper-time method.

Section VII is an application of the results in Sections V and VI to the external field problem of a charged spinless particle for a class of vector potentials. The method employed, which includes a regularization procedure or a deformation of the vector potential with a parameter $\varepsilon > 0$ and limting arguments with respect to the limit $\varepsilon \to 0$, does not only give a mathematically rigorous basis to the discussions in [V-S-H], but also present more general results than those in [V-S-H].

In the last section, we give a remark on the mathematical meaning of what is done in Section VII from a view-point of representation theory of "partially broken canonical commutation relations".

II. Operational calculus associated with two vectors in the Minkowski space and essential self-adjointness of a perturbed d'Alembertian

We say that two self-adjoint operators on a Hilbert space *strongly commute* if their spectral measures commute. The following fact is well known (see, e.g., [R-S1, Theorem VIII.13]).

PROPOSITION 2.1. *Let A and B be self-adjoint operators on a Hilbert space. Then the following (i)–(iii) are equivalent:*

(i) *A and B strongly commute.*
(ii) *For all $s, t \in \mathbf{R}$, $e^{itA}e^{isB} = e^{isB}e^{itA}$.*
(iii) *For all $t \in \mathbf{R}$, $e^{itA}B \subset Be^{itA}$.*

Let $(a, b) \in \mathrm{M}_0$ (see (1.4)). We first show that the operators ax and bp defined in the Introduction strongly commute. Then we consider a unitary transformation defined by the functional calculus of ax and bp.

LEMMA 2.2. *The self-adjoint operators ax and bp strongly commute.*

PROOF: It is easy to see that

$$(e^{itbp}\psi)(x) = \psi(x - tb). \tag{2.1}$$

Using this formula and the condition $ab = 0$, we have for all $s \in \mathbf{R}$

$$\left(e^{itbp}e^{isax}\psi\right)(x) = e^{isa(x-tb)}\psi(x - tb) = e^{isax}\psi(x - tb) = \left(e^{isax}e^{itbp}\psi\right)(x).$$

Hence $e^{itbp}e^{isax} = e^{isax}e^{itbp}$ for all $s, t \in \mathbf{R}$. Thus, by Proposition 2.1, ax and bp strongly commute. ∎

We denote by $\mathrm{B}_{\mathrm{real}}(\mathbf{R}^d)$ the set of real-valued, Borel measurable fucntions on $\mathbf{R}^d$ which are a.e. finite w.r.t. the d-dimensional Lebesgue measure. Let $E_{ax}(\cdot)$ and $E_{bp}(\cdot)$ be the spectral measures of ax and bp, respectively. Then, by Lemma 2.2, there exists a unique two-dimensional spectral measure $E_{ax,bp}(\cdot)$ such that, for all Borel sets B_1, B_2 in $\mathbf{R}$, $E_{ax,bp}(B_1 \times B_2) = E_{ax}(B_1)E_{bp}(B_2)$. As is well known, the spectral measures of p_μ and

x_μ, respectively, are absolutely continuous w.r.t. the Lebesgue measure on $\mathbf{R}$. Hence so do $E_{ax}(\cdot)$ and $E_{bp}(\cdot)$, respectively. Thus, for each $u \in \mathrm{B}_{\mathrm{real}}(\mathbf{R}^2)$, the operator

$$u(ax, bp) := \int_{\mathbf{R}^2} u(\lambda_1, \lambda_2) dE_{ax,bp}(\lambda_1, \lambda_2)$$

is self-adjoint.

We denote by $L^\infty(\mathbf{R}^d)$ the set of essentially bounded Borel measurable functions on $\mathbf{R}^d$ and set $||\psi||_\infty := \text{ess.sup}_{x\in\mathbf{R}^d}|\psi(x)|$, the essential supremum of $\psi \in L^\infty(\mathbf{R}^d)$. In what follows, for simplicity, we mean by " a bounded function on $\mathbf{R}^d$" an element of $L^\infty(\mathbf{R}^d)$. The subset of real-valued functions in $L^\infty(\mathbf{R}^d)$ is denoted $L^\infty_{\mathrm{real}}(\mathbf{R}^d)$.

Let $\mathrm{N}_0 = \{0, 1, 2, \cdots\}$. For $r \in \mathrm{N}_0$, we denote by $C^r_{\mathrm{real}}(\mathbf{R}^d)$ the set of r times continuously differentiable, real-valued fucntions on $\mathbf{R}^d$ and by $\mathfrak{B}^r(\mathbf{R}^d)$ the set of bounded functions u in $C^r_{\mathrm{real}}(\mathbf{R}^d)$ such that, for all $j = 1, \cdots, r$, the partial derivatives of u of order j is bounded on $\mathbf{R}^d$.

We say that a real-valued function $u = u(x_1, x_2)$ on $\mathbf{R}^2$ is in the set $\mathfrak{B}^{r,\infty}$ $(r \in \mathrm{N}_0)$ if, for a.e. x_2, $u(\cdot, x_2) \in \mathfrak{B}^r(\mathbf{R})$ and, for all $j = 0, \cdots, r$, the function $\partial_1^j u(x_1, x_2) := \partial^j u(x_1, x_2)/\partial x_1^j$ is bounded on $\mathbf{R}^2$. We say that a function u is in $\mathfrak{B}^{\infty,r}$ if the function $\tilde{u}(x_1, x_2) := u(x_2, x_1)$ is in $\mathfrak{B}^{r,\infty}$. In this case, we write $\partial_2^j u(x_1, x_2) := \partial^j u(x_1, x_2)/\partial x_2^j$.

We denote by $C_0^\infty(\mathbf{R}^d)$ the set of infinitely differentiable functions on $\mathbf{R}^d$ with compact support.

LEMMA 2.3. *For all $u \in \mathfrak{B}^{r,\infty}(\mathbf{R}^2)$, there exists a sequence $\{u_k\}_k$ of real-valued functions in $C_0^\infty(\mathbf{R}^2)$ such that, for all $j = 0, \cdots, r$,*

$$\sup_{k\geq 1} |\partial_1^j u_k(x_1, x_2)| \leq C,$$
$$\lim_{k\to\infty} \partial_1^j u_k(x_1, x_2) = \partial_1^j u(x_1, x_2), \qquad \text{a.e. } (x_1, x_2),$$

where C is a constant.

PROOF: This follows from a standard limting argument using Friedrichs' mollifier. ∎

LEMMA 2.4. *(i) Let $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. Then, for each $\mu = 0, 1, \cdots, d-1$, $u(ax, bp)$ leaves $D(p_\mu)$ invariant and*

$$[p_\mu, u(ax, bp)] = ia_\mu \partial_1 u(ax, bp) \quad \text{on } D(p_\mu), \tag{2.2}$$

where $[A, B] := AB - BA$.

(ii) Let $u \in \mathfrak{B}^{\infty,1}(\mathbf{R}^2)$. Then, for each $\mu = 0, 1, \cdots, d-1$, $u(ax, bp)$ leaves $D(x_\mu)$ invariant and

$$[x_\mu, u(ax, bp)] = -ib_\mu \partial_2 u(ax, bp) \quad \text{on } D(x_\mu).$$

PROOF: (i) Let $\psi \in D(p_\mu)$. We first consider the case where $u \in C_0^\infty(\mathbf{R}^2)$. Then we have

$$u(ax, bp)\psi = \frac{1}{2\pi}\int_{\mathbf{R}^2} \hat{u}(\xi_1, \xi_2)e^{i\xi_1 ax}e^{i\xi_2 bp}\psi d\xi_1 d\xi_2, \tag{2.3}$$

where

$$\hat{u}(\xi_1, \xi_2) = \frac{1}{2\pi}\int_{\mathbf{R}^2} e^{-i(\xi_1 x_1 + \xi_2 x_2)}u(x_1, x_2)dx_1 dx_2$$

is the standard Fourier transform of u and the integral on the right hand side (RHS) of (2.3) is taken in the strong topology. By (2.1), we have

$$(e^{i\xi_1 ax}e^{i\xi_2 bp}\psi)(x) = e^{i\xi_1 ax}\psi(x - b\xi_2).$$

Hence the function $e^{i\xi_1 ax}e^{i\xi_2 bp}\psi$ is differentiable and

$$\partial_\mu(e^{i\xi_1 ax}e^{ixi_2 bp}\psi)(x) = ia_\mu\xi_1 e^{i\xi_1 ax}\psi(x - b\xi_2) + e^{i\xi_1 ax}(\partial_\mu\psi)(x - b\xi_2),$$

which implies that $\partial_\mu(e^{i\xi_1 ax}e^{i\xi_2 bp}\psi) \in L^2(\mathbf{R}^d)$. Hence $e^{i\xi_1 ax}e^{i\xi_2 bp}\psi \in D(p_\mu)$ and

$$p_\mu e^{i\xi_1 ax}e^{i\xi_2 bp}\psi = -a_\mu\xi_1 e^{i\xi_1 ax}e^{i\xi_2 bp}\psi + e^{i\xi_1 ax}e^{i\xi_2 bp}p_\mu\psi,$$

which implies that

$$\int_{\mathbf{R}^2} ||\hat{u}(\xi_1, \xi_2)p_\mu e^{i\xi_1 ax}e^{i\xi_2 bp}\psi|| d\xi_1 d\xi_2 < \infty.$$

It follows from the closedness of p_μ that $u(ax, bp)\psi \in D(p_\mu)$ and

$$\begin{aligned}
&p_\mu u(ax, bp)\psi \\
&= \frac{1}{2\pi}\left\{-a_\mu\int_{\mathbf{R}^2} \hat{u}(\xi_1, \xi_2)\xi_1 e^{i\xi_1 ax}e^{i\xi_2 bp}\psi d\xi_1 d\xi_2 + \int_{\mathbf{R}^2} \hat{u}(\xi_1, \xi_2)e^{i\xi_1 ax}e^{i\xi_2 bp}p_\mu\psi d\xi_1 d\xi_2\right\} \\
&= ia_\mu\partial_1 u(ax, bp)\psi + u(ax, bp)p_\mu\psi.
\end{aligned}$$

Thus (2.2) follows.

We next consider the case where $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. Then we can take a sequence $\{u_k\}_k$ as given in Lemma 2.3. By the preceding result, we have

$$p_\mu u_k(ax, bp)\psi = ia_\mu\partial_1 u_k(ax, bp)\psi + u_k(ax, bp)p_\mu\psi.$$

By the functional calculus, $u_k(ax, bp) \to u(ax, bp), \partial_1 u_k(ax, bp) \to \partial_1 u_k(ax, bp)$ strongly as $k \to \infty$. Hence $p_\mu u_k(ax, bp)\psi \to ia_\mu\partial_1 u(ax, bp)\psi + u(ax, bp)p_\mu\psi$ as $k \to \infty$. By the closedness of p_μ, $u(ax, bp)\psi \in D(p_\mu)$ and (2.2) holds.

It is easy to see that, for all $\psi \in D(x_\mu)$,

$$x_\mu e^{i\xi_1 ax}e^{i\xi_2 bp}\psi = b_\mu\xi_2 e^{i\xi_1 ax}e^{i\xi_2 bp}\psi + e^{i\xi_1 ax}e^{i\xi_2 bp}x_\mu\psi..$$

Hence, in the same way as in the case of p_μ, we can prove part (ii). ∎

We say that a function u in $\mathbf{B}_{\text{real}}(\mathbf{R}^2)$ is in the set $\mathfrak{C}^r(\mathbf{R}^2)$ if, for a.e. $x_2 \in \mathbf{R}$, $u(\cdot, x_2) \in C^r_{\text{real}}(\mathbf{R})$ and there exists a sequence $\{u_k\}_k$ in $\mathfrak{B}^{r,\infty}(\mathbf{R}^2)$ such that, for $j = 0, \cdots, r$,

$$\sup_{k\geq 1} |\partial_1^j u_k(x_1, x_2)| \leq C|\partial_1^j u(x_1, x_2)|, \tag{2.4}$$

$$\lim_{k\to\infty} \partial_1^j u_k(x_1, x_2) = \partial_1^j u(x_1, x_2), \qquad \text{a.e.}(x_1, x_2), \tag{2.5}$$

where C is a constant.

THEOREM 2.5. *Let $u \in \mathfrak{C}^1(\mathbf{R}^2)$ and $\psi \in D(p_\mu) \cap D(\partial_1 u(ax, bp))$. Then, for $\mu = 0, \cdots, d-1$, $e^{-iu(ax,bp)}\psi \in D(p_\mu)$ and*

$$e^{iu(ax,bp)} p_\mu e^{-iu(ax,bp)}\psi = [p_\mu + a_\mu \partial_1 u(ax, bp)]\psi. \tag{2.6}$$

PROOF: We first consider the case where $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. Then $u(ax, bp)$ is bounded. Hence we have

$$e^{-iu(ax,bp)}\phi = \sum_{k=0}^{\infty} \frac{(-i)^k u(ax, bp)^k \phi}{k!}, \quad \phi \in L^2(\mathbf{R}^d).$$

Let $\psi \in D(p_\mu)$ and $\psi_N = \sum_{k=0}^N (-i)^k u(ax, bp)^k \psi/k!$. Then, $\psi_N \to e^{-iu(ax,bp)}\psi$ $(N \to \infty)$. Moreover, by Lemma 2.4, $\psi_N \in D(p_\mu)$ and

$$p_\mu \psi_N = \sum_{k=1}^{N} \frac{(-i)^k}{(k-1)!} i a_\mu \partial_1 u(ax, bp) \cdot u(ax, bp)^{k-1}\psi + \sum_{k=0}^{\infty} \frac{(-i)^k u(ax, bp)^k}{k!} p_\mu \psi.$$

Hence

$$p_\mu \psi_N \to a_\mu \partial_1 u(ax, bp) e^{-iu(ax,bp)}\psi + e^{-iu(ax,bp)} p_\mu \psi$$

as $N \to \infty$. By the closedness of p_μ, $e^{-iu(ax,bp)}\psi \in D(p_\mu)$ and

$$p_\mu e^{-iu(ax,bp)}\psi = a_\mu \partial_1 u(ax, bp) e^{-iu(ax,bp)}\psi + e^{-iu(ax,bp)} p_\mu \psi.$$

Thus (2.6) holds.

We next consider the case where $u \in \mathfrak{C}^1(\mathbf{R}^2)$. Then there exists a sequence $\{u_k\}_k$ in $\mathfrak{B}^{1,\infty}(\mathbf{R}^2)$ satisfying (2.4) and (2.5) with $j = 0, 1$. Let $\psi \in D(p_\mu) \cap D(\partial_1 u(ax, bp))$. By the preceding result, we have

$$p_\mu e^{-iu_k(ax,bp)}\psi = e^{-iu_k(ax,bp)}[p_\mu + a_\mu \partial_1 u_k(ax, bp)]\psi.$$

By the functional calculus, one can easily show that

$$\lim_{k\to\infty} [p_\mu + a_\mu \partial_1 u_k(ax, bp)]\psi = [p_\mu + a_\mu \partial_1 u(ax, bp)]\psi.$$

Similarly, for all $\eta \in D(u(ax, bp))$,

$$\lim_{k\to\infty} u_k(ax, bp)\eta = u(ax, bp)\eta.$$

Hence, by standard convergence theoreoms ([R-S1, Theorem VIII.25, Theorem VIII.21]),

$$\mathrm{s} - \lim_{k\to\infty} e^{-iu_k(ax,bp)} = e^{-iu(ax,bp)},$$

where s − lim denotes strong limit. Thus we obtain

$$\lim_{k\to\infty} p_\mu e^{-iu_k(ax,bp)}\psi = e^{-iu(ax,bp)}[p_\mu + a_\mu \partial_1 u(ax, bp)]\psi.$$

By the closedness of p_μ, $e^{-iu(ax,bp)}\psi \in D(p_\mu)$ and

$$p_\mu e^{-iu(ax,bp)}\psi = e^{-iu(ax,bp)}[p_\mu + a_\mu \partial_1 u(ax, bp)]\psi,$$

which imply the desired result. ∎

REMARK: By virtue of Lemma 2.4(ii), we can also obtain formulae for x_μ in the same way as in the case of p_μ, e.g., the formula corresponding to (2.6) is given

$$e^{iu(ax,bp)} x_\mu e^{-iu(ax,bp)}\psi = [x_\mu - b_\mu \partial_2 u(ax, bp)]\psi$$

under a condition parallel to the one in Theorem 2.5. This kind of translations from formulas of p_μ into the ones of x_μ is easy. Thus, in this paper, we concentrate our attention only on formulae of p_μ.

As a corollary of Theorem 2.5, we obtain the following.

THEOREM 2.6. *Let $u \in \mathfrak{B}^{2,\infty}(\mathbf{R}^2)$. Then, for each $\mu = 0, \cdots, d-1$, $e^{-iu(ax,bp)}$ leaves $D(p_\mu^2)$ invariant and*

$$e^{iu(ax,bp)} p_\mu^2 e^{-iu(ax,bp)}\psi = (p_\mu + a_\mu \partial_1 u(ax, bp))^2\psi, \quad \psi \in D(p_\mu^2),\ \mu = 0, 1, \cdots, d-1. \quad (2.7)$$

PROOF: Let $\phi, \psi \in D(p_\mu^2)$. Then, by Theorem 2.5, we have

$$(p_\mu^2\phi, e^{-iu(ax,bp)}\psi) = (p_\mu\phi, p_\mu e^{-iu(ax,bp)}\psi) = (p_\mu\psi, \eta),$$

where

$$\eta = e^{-iu(ax,bp)} p_\mu\psi + e^{-iu(ax,bp)} a_\mu \partial_1 u(ax, bp)\psi.$$

Since $p_\mu\psi \in D(p_\mu)$, it follows from Theorem 2.5 again that $e^{-iu(ax,bp)}p_\mu\psi \in D(p_\mu)$. Since $u(ax, bp)$ and $\partial_1 u(ax, bp)$ commute, we have

$$e^{-iu(ax,bp)}\partial_1 u(ax, bp)\psi = \partial_1 u(ax, bp)e^{-iu(ax,bp)}\psi.$$

Since $\partial_1 u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$, it follows from Lemma 2.4 that $\partial_1 u(ax, bp)e^{-iu(ax,bp)}\psi$ is in $D(p_\mu)$. Thus $\eta \in D(p_\mu)$. Hence

$$(p_\mu^2\phi, e^{-iu(ax,bp)}\psi) = (\phi, p_\mu\eta),$$

which implies that $e^{-iu(ax,bp)}\psi \in D(p_\mu^2)$ and $p_\mu^2 e^{-iu(ax,bp)}\psi = p_\mu\eta$. The last equation combined with (2.6) gives (2.7). ∎

Let

$$\square = p^2 = \partial_0^2 - \sum_{j=1}^{d-1}\partial_j^2 \tag{2.8}$$

be the free d'Alembertian with $D(\square) := \cap_{\mu=0}^{d-1}D(p_\mu^2)$. It is easy to see that $\square$ is essentially self-adjoint on $\mathcal{S}(\mathbf{R}^d)$ (the set of rapidly decreasing C^∞ functions on $\mathbf{R}^d$). We denote the closure of $\square$ by H_0:

$$H_0 = \overline{\square}. \tag{2.9}$$

For $u \in \mathfrak{C}^1(\mathbf{R}^2)$, we can define

$$\begin{aligned}\square_u &= (p + a\partial_1 u(ax, bp))^2 \\ &= (p_0 + a_0\partial_1 u(ax, bp))^2 - \sum_{j=1}^{d-1}(p_j + a_j\partial_1 u(ax, bp))^2\end{aligned} \tag{2.10}$$

with $D(\square_u) = \cap_{\mu=0}^{d-1}D([p_\mu + a_\mu\partial u_1(ax, bp)]^2)$, which gives a perturbed d'Alembertian.

THEOREM 2.7. *Let $u \in \mathfrak{B}^{2,\infty}(\mathbf{R}^2)$. Then $\square_u$ is essentially self-adjoint on $D(\square)$ and the following operator equality holds:*

$$e^{iu(ax,bp)}H_0e^{-iu(ax,bp)} = \overline{\square}_u.$$

PROOF: By Theorem 2.6, we have

$$e^{iu(ax,bp)}\square e^{-iu(ax,bp)}\psi = (p + a\partial_1 u(ax, bp))^2\psi, \quad \psi \in D(\square).$$

Theorem 2.6 also implies that $e^{-iu(ax,bp)}$ maps $D(\square)$ onto itself. Hence the essential self-adjointness of $\square$ on $D(\square)$ gives the desired result. ∎

REMARK: As is easily seen, analysis similar to that in this section can be made also in the Euclidean space $\mathbf{R}^d$ in quite a parallel way.

III. A linear combination of the components of the angular momentum operator in the Minkowski space

We denote by $M_d^{\rm as}(\mathbf{R})$ the set of $d \times d$ real anti-symmetric matrices:

$$M_d^{\rm as}(\mathbf{R}) = \{f = (f_{\mu\nu}) \mid f_{\mu\nu} \in \mathbf{R},\ f_{\mu\nu} = -f_{\nu\mu},\ \mu, \nu = 0, 1, \cdots, d-1\}. \tag{3.1}$$

For each $f \in M_d^{\rm as}(\mathbf{R})$ and a constant vector $q \in \mathrm{M}^d$, we define

$$Q_\mu(x) = f_{\mu\nu}(x^\nu - q^\nu), \quad \mu = 0, 1, \cdots, d-1. \tag{3.2}$$

It is easy to show that the multiplication operator Q_μ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$. Moreover, Q_μ leaves $\mathcal{S}(\mathbf{R}^d)$ invariant and satisfies the commutation relations

$$[p_\nu, Q_\mu]\psi = i f_{\mu\nu}\psi, \quad \psi \in \mathcal{S}(\mathbf{R}^d),\ \mu, \nu = 0, \cdots, d-1. \tag{3.3}$$

We define an operator L_f by

$$L_f = Q_\mu p^\mu, \tag{3.4}$$

with $D(L_f) = \cap_{\mu=0}^{d-1} D(Q_\mu p^\mu)$, where, for notational simplicity, we suppress the dependence of L_f on q. Note that L_f can be written

$$L_f = \sum_{\nu<\mu} f_{\nu\mu}\{(x^\mu - q^\mu)p^\nu - (x^\nu - q^\nu)p^\mu\},$$

i.e., L_f is is a linear combination of $(x^\mu - q^\mu)p^\nu - (x^\nu - q^\nu)p^\mu$ $(\nu < \mu)$, the components of the angular momentum operator in the coordinate system translated by q.

It follows from the antisymmetry of f and (3.3) that, for all $\psi \in D(L_f)$, $Q_\mu\psi \in D(p^\mu)$ and

$$Q_\mu p^\mu \psi = p^\mu Q_\mu \psi, \quad \psi \in D(L_f), \tag{3.5}$$

which implies that L_f is a symmetric operator on $L^2(\mathbf{R}^d)$. Moreover, we can prove the following fact.

Proposition 3.1. *L_f is esentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$.*

Proof: The idea of proof is to apply Nelson's commutator theorem ([R-S2, Theorem X.37]). Let

$$|x| = \sqrt{(x^0)^2 + (x^1)^2 + \cdots + (x^{d-1})^2},$$

the Euclidean norm of $x = (x^0, \cdots, x^{d-1})$, and Δ be the d-dimensional Laplacian:

$$\Delta = \sum_{\mu=0}^{d-1} p_\mu^2.$$

It is well known that the operator

$$N := -\Delta + |x|^2, \tag{3.6}$$

the Hamiltonian of the harmonic oscillator on $\mathbf{R}^d$, is self-adjoint with $D(N) = D(\Delta) \cap D(|x|^2)$ and essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$. We have

$$||-\Delta\psi||^2 + ||\,|x|^2\psi||^2 \le ||N\psi||^2 + 2d||\psi||^2, \quad \psi \in D(N). \tag{3.7}$$

Let $\psi \in C_0^\infty(\mathbf{R}^d)$. Then, by (3.3), we have

$$\begin{aligned}||L_f\psi||^2 &= |-if_{\mu\nu}(p^\mu\psi, Q^\nu\psi) + (p_\nu p_\mu\psi, Q^\nu Q^\mu\psi)| \\ &\le C(f)\sqrt{\sum_{\mu=0}^{d-1}||p_\mu\psi||^2}\sqrt{\sum_{\nu=0}^{d-1}||Q_\nu\psi||^2} + \sqrt{\sum_{\mu,\nu=0}^{d-1}||p_\mu p_\nu\psi||^2}\sqrt{\sum_{\mu,\nu=0}^{d-1}||Q^\mu Q^\nu\psi||^2},\end{aligned}$$

where

$$C(f) = \sqrt{\sum_{\mu,\nu=0}^{d-1}|f_{\mu\nu}|^2}. \tag{3.8}$$

By (3.7), we have the following estimates:

$$\sqrt{\sum_{\mu=0}^{d-1}||p_\mu\psi||^2} = ||(-\Delta)^{1/2}\psi|| \le ||N^{1/2}\psi||, \tag{3.9}$$

$$\sqrt{\sum_{\mu,\nu=0}^{d-1}||p_\mu p_\nu\psi||^2} = ||-\Delta\psi|| \le ||N\psi|| + \sqrt{2d}||\psi||,$$

$$\begin{aligned}\sqrt{\sum_{\nu=0}^{d-1}||Q_\nu\psi||^2} &\le C(f)||\,|x-q|\psi|| \\ &\le C(f)(||\,|x|\psi|| + |q|\,||\psi||) \\ &\le C(f)(||N^{1/2}\psi|| + |q|\,||\psi||),\end{aligned}$$

$$\begin{aligned}\sqrt{\sum_{\mu,\nu=0}^{d-1}||Q^\mu Q^\nu\psi||^2} &\le C(f)^2||\,|x-q|^2\psi|| \\ &\le C(f)^2(||\,|x|^2\psi|| + 2|q|\,||\,|x|\psi|| + |q|^2||\psi||) \\ &\le C(f)^2(||N\psi|| + \sqrt{2d}||\psi|| + 2|q|||N^{1/2}\psi|| + |q|^2||\psi||).\end{aligned}$$

Hence we obtain

$$||L_f\psi||^2 \le C||(N+1)\psi||^2, \tag{3.10}$$

where $C > 0$ is a constant. Moreover, by using (3.3), we have

$$[L_f, N]\psi = -2i \sum_{\nu=0}^{d-1} f_{\mu\nu} p^\mu p_\nu \psi - 2i \sum_{\mu=0}^{d-1} Q_\mu x_\mu \psi,$$

which gives

$$|(L_f\psi, N\psi) - (N\psi, L_f\psi)| \leq C' \|(N+1)^{1/2}\psi\|^2,$$

where $C' > 0$ is a constant. Hence we can apply Nelson's commutator theorem to obtain the desired result. ∎

For each $a \in \mathrm{M}^d$, we introduce a class of $d \times d$ real antisymmetric matrices:

$$\mathcal{F}_a = \{f \in M_d^{\mathrm{as}}(\mathbf{R}) \mid a^\mu f_{\mu\nu} = 0, \nu = 0, 1, \cdots, d-1\}. \tag{3.11}$$

PROPOSITION 3.2. *Let $a \in \mathrm{M}^d$.*

(i) *If $f \in \mathcal{F}_a$, then each Q_μ strongly commutes with ap.*
(ii) *If $f \in \mathcal{F}_a$, then $\overline{L}_f$ strongly commutes with ax and ap.*

PROOF: (i) Let $\psi \in C_0^\infty(\mathbf{R}^d)$. Then, by (2.1), $e^{itap}\psi$ is in $C_0^\infty(\mathbf{R}^d)$ and

$$\begin{aligned}(Q_\mu e^{itap}\psi)(x) &= f_{\mu\nu}(x^\nu - q^\nu)\psi(x - at) \\ &= f_{\mu\nu}(x^\nu - a^\nu t - q^\nu)\psi(x - at) \\ &= (e^{itap} Q_\mu \psi)(x),\end{aligned}$$

where, in the second equality, we have used the property $f_{\mu\nu}a^\nu = 0$. Hence

$$Q_\mu e^{itap}\psi = e^{itap} Q_\mu \psi.$$

Since $C_0^\infty(\mathbf{R}^d)$ is a core of Q_μ as already mentioned, it follows that $e^{itap}Q_\mu \subset Q_\mu e^{itap}$. Hence, by Proposition 2.1, Q_μ strongly commutes with ap.

(ii) Similar to the proof of part (i). ∎

REMARK: One can easily show that, if each Q_μ strongly commutes with ap, then $f \in \mathcal{F}_a$.

IV. OPERATOR-VALUED LORENTZ TRANSFORMATIONS

In this section we take an interlude to introduce operator-valued Lorentz transformations and to prove some basic facts on them.

Let $\mathcal{H}_\mu, \mu = 0, 1, \cdots, d-1$, be complex Hilbert spaces and

$$\mathcal{H} = \bigoplus_{\mu=0}^{d-1} \mathcal{H}_\mu = \{\psi = \{\psi^\mu\}_{\mu=0}^{d-1} \mid \psi^\mu \in \mathcal{H}_\mu,\ \mu = 0, 1, \cdots, d-1\}$$

be the direct sum of $\mathcal{H}_\mu, \mu = 0, 1, \cdots, d-1$. The metric tensor g of M^d naturally defines a bounded linear operator on $\mathcal{H}$, denoted also g, by

$$(g\psi)^\mu = g_{\mu\nu}\psi^\nu, \quad \mu = 0, 1, \cdots, d-1,\ \psi = \{\psi^\mu\}_{\mu=0}^{d-1} \in \mathcal{H}. \tag{4.1}$$

DEFINITION 4.1: A densely defined linear operator L on $\mathcal{H}$ is called an operator-valued Lorentz transformation on $\mathcal{H}$ if $L^*gL \subset g$.

Every linear operator T on $\mathcal{H}$ is represented as a matrix operator $T = (T^\mu{}_\nu)$ with $T^\mu{}_\nu$ a linear operator from $\mathcal{H}_\nu$ to $\mathcal{H}_\mu$. We call $T^\mu{}_\nu$'s the components of T. In terms of components, a bounded linear operator L on $\mathcal{H}$ is a Lorentz transformation if and only if

$$\left(L^\lambda{}_\mu\right)^* g_{\lambda\rho} L^\rho{}_\nu = g_{\mu\nu}.$$

PROPOSITION 4.2. *Let T be a bounded linear operator on $\mathcal{H}$ such that $T^*g = -gT$. Then e^T is a bounded operator-valued Lorentz transformation on $\mathcal{H}$.*

PROOF: Putting $L = e^T (= \sum_{k=0}^\infty T^k/k!)$, we have

$$L^*g = \sum_{k=0}^\infty \frac{(T^*)^k g}{k!} = \sum_{k=0}^\infty g\frac{(-1)^k T^k}{k!} = ge^{-T} = gL^{-1},$$

which implies the desired result. ∎

We denote by $\mathrm{N} = \{1, 2, \cdots\}$ the set of natural numbers. For a densely defined linear operator T on $\mathcal{H}$ such that $T^{N+1} = 0$ for some $N \in \mathrm{N}$, we deifine e^T by

$$e^T = \sum_{k=0}^N \frac{T^k}{k!} \tag{4.2}$$

with $D(e^T) = D(T^{N+1})$.

PROPOSITION 4.3. *Let T be a densely defined linear operator T on $\mathcal{H}$ such that, for some $N \in \mathrm{N}$, $T^{N+1} = 0$ with $D(T^{N+1})$ dense and $T^*g \supset -gT$. Then e^T is an operator-valued Lorentz transformation on $\mathcal{H}$.*

PROOF: We have $(e^T)^* \supset \sum_{k=0}^N (T^*)^k/k!$. By the assumption $T^*g \supset -gT$, g maps $D(T^k)$ into $D((T^*)^k)$ for all $k \in \mathrm{N}$ and $(T^*)^k g\psi = g(-1)^k T^k \psi$ for all $\psi \in D(T^k)$. Hence, for all $\psi \in D(e^T)$, we have $ge^T\psi = \sum_{k=0}^N (-1)^k (T^*)^k g\psi/k!$. Hence $ge^T\psi \in D((e^T)^*)$ and

$$(e^T)^* ge^T\psi = \sum_{j=0}^N \sum_{k=0}^N \frac{(-1)^k (T^*)^{j+k}}{j!k!} g\psi = g\psi.$$

Thus the desired result follows. ∎

A special case is of some importance in applications. Let us consider the case

$$\mathcal{H}_0 = \cdots = \mathcal{H}_{d-1} = \mathcal{K}$$

and S be a densely defined linear operator on $\mathcal{K}$. For a $d \times d$ matrix $f = (f_{\mu\nu})_{\mu,\nu=0,1,\cdots,d-1}$, we define a $d \times d$ matrix

$$\tilde{f} = (f^{\mu}{}_{\nu})_{\mu,\nu=0,\cdots,d-1} \tag{4.3}$$

by

$$f^{\mu}{}_{\nu} = g^{\mu\lambda} f_{\lambda\nu}. \tag{4.4}$$

Let

$$S_f = \tilde{f} S, \tag{4.5}$$

a linear operator on $\bigoplus_{\mu=0}^{d-1} \mathcal{K}$.

PROPOSITION 4.4. *Let S be a symmetric operator on $\mathcal{K}$ and $f \in M_d^{\mathrm{as}}(\mathrm{R})$. Then the following (i) and (ii) hold:*

(i) *If S is bounded, then e^{S_f} is a bounded operator-valued Lorentz transformation on $\bigoplus_{\mu=0}^{d-1} \mathcal{K}$.*

(ii) *Suppose that $\tilde{f}^{N+1} = 0$ for some $N \in \mathrm{N}$. Then $S_f^{N+1} = 0$ and e^{S_f} is an operator-valued Lorentz transformation on $\bigoplus_{\mu=0}^{d-1} \mathcal{K}$ with components*

$$\left(e^{S_f}\right)^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \sum_{k=1}^{N} \frac{S^k}{k!} f^{\mu}{}_{\mu_1} f^{\mu_1}{}_{\mu_2} \cdots f^{\mu_{k-1}}{}_{\nu},\ \mu,\nu = 0,\cdots,d-1, \tag{4.6}$$

where $\delta^{\mu}{}_{\nu}$ is the Kronecker delta.

PROOF: It is easy to see that $(\tilde{f})^* g = -g\tilde{f}$. Using this property, we can show that $S_f^* g \supset -gS_f$. Hence Propositions 4.2 and 4.3 give part (i) and (ii), respcetively. ∎

V. Operational calculus on the self-adjoint operators ax, bp and $\overline{L}_f$

Let $(a,b) \in \mathrm{M}_0$. Let $f \in \mathcal{F}_a \cap \mathcal{F}_b$ and $u \in \mathrm{B}_{\mathrm{real}}(\mathbf{R}^2)$. Then, by Proposition 3.2(ii), $u(ax,bp)$ and $\overline{L}_f$ strongly commute. Hence, putting

$$D^\infty_{f,u} := \bigcap_{j,k\in \mathrm{N}_0} D(u(ax,bp)^j \overline{L}_f^k), \tag{5.1}$$

the operator

$$M(u, L_f) := \overline{[u(ax,bp)\overline{L}_f] \restriction D^\infty_{f,u}} \tag{5.2}$$

is self-adjoint.

For $f \in M_d^{\mathrm{as}}(\mathbf{R})$ and $u \in L^\infty_{\mathrm{real}}(\mathbf{R}^2)$, we define a bounded linear operator $\Lambda(f,u)$ on the Hilbert space

$$[L^2(\mathbf{R}^d)] := \bigoplus_{\mu=0}^{d-1} L^2(\mathbf{R}^d) \tag{5.3}$$

by

$$\Lambda(f,u) := e^{-\widetilde{f}u(ax,bp)} = \sum_{k=0}^\infty \frac{(-1)^k \widetilde{f}^k u(ax,bp)^k}{k!}, \tag{5.4}$$

where $\widetilde{f}$ is defined by (4.3).

Lemma 5.1. *The operator $\Lambda(f,u)$ is a bounded Lorentz transformation on $[L^2(\mathbf{R}^d)]$. Moreover, the following (i) and (ii) hold:*

(i) *If $f \in \mathcal{F}_a$, then*

$$\Lambda(f,u)^\mu{}_\nu a^\nu = a^\mu, \quad \mu = 0,1,\cdots,d-1,, \tag{5.5}$$

on $L^2(\mathbf{R}^d)$.

(ii) *If $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$, then each component $\Lambda(f,u)^\mu{}_\nu$ leaves $D(p_\lambda)$ invariant ($\lambda = 0,\cdots,d-1$) and, for all $\psi \in D(p_\lambda)$,*

$$p_\lambda \Lambda(f,u)^\mu{}_\nu \psi = -ia_\lambda \partial_1 u(ax,bp)\Lambda(f,u)^\mu{}_\rho f^\rho{}_\nu \psi + \Lambda(f,u)^\mu{}_\nu p_\lambda \psi. \tag{5.6}$$

Proof: The first assertion follows from Proposition 4.4(i). By the property $f^\mu{}_\nu a^\nu = 0$ and (5.4), we obtain (5.5).

To prove part (ii), we note that, for all $\phi, \psi \in D(p_\lambda)$,

$$(p_\lambda \phi, \Lambda(f,u)^\mu{}_\nu \psi) = \sum_{k=0}^\infty \frac{(\widetilde{f}^k)^\mu{}_\nu}{k!}(p_\lambda\phi, u(ax,bp)^k\psi).$$

By Lemma 2.4, $u(ax,bp)^k\psi \in D(p_\lambda)$ and

$$p_\lambda u(ax,bp)^k\psi = ika_\lambda \partial_1 u(ax,bp)\cdot u(ax,bp)^{k-1}\psi + u(ax,bp)^k p_\lambda \psi. \tag{5.7}$$

Hence

$$(p_\lambda\phi, \Lambda(f,u)^\mu{}_\nu\psi) = (\phi, ia_\lambda\partial_1 u(ax,bp)\Lambda(f,u)^\mu{}_\rho f^\rho{}_\nu\psi + \Lambda(f,u)^\mu{}_\nu p_\lambda\psi).$$

Since this equation holds for all $\phi \in D(p_\lambda)$, the desired result follows. ∎

The first of the main results of this section is the following.

THEOREM 5.2. *Let $f \in \mathcal{F}_a \cap \mathcal{F}_b$ and $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. Then, for all $\psi \in D(N)$, $e^{-iM(u,L_f)}\psi$ is in $D(p_\mu), \mu = 0,\cdots,d-1$, and*

$$e^{iM(u,L_f)}p^\mu e^{-iM(u,L_f)}\psi = \Lambda(f,u)^\mu{}_\nu\left(p^\nu + a^\nu\partial_1 u(ax,bp)\overline{L}_f\right)\psi \tag{5.8}$$

REMARK: By (5.5), the RHS of (5.8) can be written

$$\Lambda(f,u)^\mu{}_\nu\left(p^\nu + a^\nu\partial_1 u(ax,bp)\overline{L}_f\right)\psi = \Lambda(f,u)^\mu{}_\nu p^\nu\psi + a^\mu\partial_1 u(ax,bp)\overline{L}_f\psi. \tag{5.9}$$

To prove this theorem, we need some preliminaries. For two linear operators A and B acting in a Hilbert space, we define $(\mathrm{ad}\,A)^k B, k \in \mathbf{N}_0$, by

$$\begin{aligned}&(\mathrm{ad}\,A)^0 B = B,\\ &(\mathrm{ad}\,A)^k B = [A, (\mathrm{ad}\,A)^{k-1}B], \quad k \geq 1.\end{aligned} \tag{5.10}$$

It is well known (or easy to see) that

$$(\mathrm{ad}\,A)^k B\psi = \sum_{j=0}^{k}\frac{(-1)^{k-j}k!}{j!(k-j)!}A^j B A^{k-j}\psi \tag{5.11}$$

for all $\psi \in \bigcap_{j=0}^k D(A^j B A^{k-j})$.

LEMMA 5.3. *Let $f \in \mathcal{F}_a \cap \mathcal{F}_b$ and $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. Let $\psi \in \mathcal{S}(\mathbf{R}^d)$. Then, for all $j,k \in \mathbf{N}_0$ and $\mu = 0,1,\cdots,d-1$, we have $M(u,L_f)^k\psi \in D(p_\mu)$ and $p_\mu M(u,L_f)^k\psi \in D(M(u,L_f)^j)$. Moreover,*

$$(\mathrm{ad}\ M(u,L_f))^k p^\mu\psi = \begin{cases} -ia^\mu\partial_1 u(ax,bp)L_f\psi + iu(ax,bp)f^\mu{}_\nu p^\nu\psi, & k=1,\\ i^k(\widetilde{f^k})^\mu{}_\nu u(ax,bp)^k p^\nu\psi, & k\geq 2.\end{cases} \tag{5.12}$$

PROOF: Since $L_f^k\psi$ is in $\mathcal{S}(\mathbf{R}^d) \subset D(p_\mu)$, Lemma 2.4 and (5.7) imply that $u(ax,bp)^k L_f^k\psi \in D(p_\mu)$ and

$$p_\mu u(ax,bp)^k L_f^k\psi = ika_\mu \partial_1 u(ax,bp)\cdot u(ax,bp)^{k-1}L_f^k\psi + u(ax,bp)^k p_\mu L_f^k\psi.$$

It follows from this equation and the strong commutativity of $u(ax,bp), \partial_1 u(ax,bp)$ with $\overline{L}_f$ that $p_\mu u(ax,bp)L_f^k\psi = p_\mu M(u,L_f)^k\psi$ is in $D(M(u,L_f)^j)$ and

$$\begin{aligned}&M(u,L_f)^j p_\mu M(u,L_f)^k\psi\\ &= ika_\mu\partial_1 u(ax,bp)M(u,L_f)^{j+k-1}L_f\psi + M(u,L_f)^j u(ax,bp)^k p_\mu L_f^k\psi.\end{aligned} \tag{5.13}$$

Thus the first half of the lemma follows.

By (5.13) and the commutation relations

$$[p^\mu, L_f]\phi = -if^\mu{}_\nu p^\nu\phi, \quad \phi\in\mathcal{S}(\mathbf{R}^d), \mu = 0,\cdots,d-1, \tag{5.14}$$

we obtain (5.12) with $k=1$. Hence we have

$$\begin{aligned}(\mathrm{ad}\, M(u,L_f))^2 p^\mu\psi &= iu(ax,bp)f^\mu{}_\nu[M(u,L_f),p^\nu]\psi\\ &= iu(ax,bp)f^\mu{}_\nu(-ia^\nu\partial_1 u(ax,bp)L_f + iu(ax,bp)f^\nu{}_\lambda p^\lambda)\psi\\ &= i^2(\widetilde{f}^2)^\mu{}_\nu u(ax,bp)^2p^\nu\psi.\end{aligned}$$

This implies that

$$(\mathrm{ad}\, M(u,L_f))^3 p^\mu\psi = i^3(\widetilde{f}^3)^\mu{}_\nu u(ax,bp)^3 p^\nu\psi.$$

Repeating this process, we obtain (5.12). ∎

We next estimate $\|L_f^k\psi\|$ ($f\in M_d^{\mathrm{as}}(\mathbf{R}), k\in\mathbf{N}$) for suitable vectors ψ. For this purpose, it is convenient to rewrite L_f in terms of the *annihilation* and *creation operators* deifned by

$$c_\mu = \frac{1}{\sqrt{2}}(x^\mu - ip_\mu), \quad c_\mu^\dagger = \frac{1}{\sqrt{2}}(x^\mu + ip_\mu), \tag{5.15}$$

respectively. Obviously c_μ and $c_\mu^\dagger$ leave $\mathcal{S}(\mathbf{R}^d)$ invariant and satisfy the commutation relations:

$$[c_\mu,c_\nu]\psi = 0, \quad [c_\mu,c_\nu]\psi = 0, \quad [c_\mu,c_\nu^\dagger]\psi = \delta_{\mu\nu}\psi, \quad \psi\in\mathcal{S}(\mathbf{R}^d). \tag{5.16}$$

Let

$$\Phi_0(x) = \pi^{-d/4}e^{-|x|^2/2}, \quad x\in\mathbf{R}^d, \tag{5.17}$$

and

$$\Phi_\alpha = \frac{1}{\sqrt{\alpha_0!\cdots\alpha_{d-1}!}}(c_{\mu_0}^\dagger)^{\alpha_0}\cdots(c_{\mu_{d-1}}^\dagger)^{\alpha_{d-1}}\Phi_0, \tag{5.18}$$

where $\alpha = (\alpha_0, \cdots, \alpha_{d-1}) \in \mathrm{N}_0^d$ is a multi-index ($\alpha_\mu \in \mathrm{N}_0, \mu = 0, \cdots, d-1$). It is well known that $\{\Phi_\alpha\}_\alpha$ is a complete orthonormal system of $L^2(\mathbf{R}^d)$ (note that Φ_α is a tensor product of Hermite functions). It is easy to see that

$$c_\mu^\dagger \Phi_\alpha = \sqrt{\alpha_\mu + 1}\Phi_{(\alpha_0, \cdots, \alpha_\mu+1, \cdots, \alpha_{d-1})}$$

and

$$\begin{aligned} c_\mu \Phi_\alpha &= \sqrt{\alpha_\mu}\Phi_{(\alpha_0, \cdots, \alpha_\mu-1, \cdots, \alpha_{d-1})}, \quad \alpha_\mu \geq 1, \\ c_\mu \Phi_{(\alpha_0, \cdots, \alpha_{\mu-1}, 0, \alpha_{\mu+1} \cdots, \alpha_{d-1})} &= 0. \end{aligned}$$

We denote by $c_\mu^\#$ either c_μ or $c_\mu^\dagger$. It follows from the above relations that

$$\|c_{\mu_1}^\# \cdots c_{\mu_k}^\# \Phi_\alpha\| \leq \sqrt{(|\alpha|+1)\cdots(|\alpha|+k)}, \tag{5.19}$$

where $|\alpha| = \sum_{\mu=0}^{d-1} \alpha_\mu$.

We have

$$\begin{aligned} \|L_f^k \Phi_\alpha\| &\leq |f_{\mu_1\nu_1}| \cdots |f_{\mu_k\nu_k}| \|p^{\mu_1}(x^{\nu_1} - q^{\nu_1}) \cdots p^{\mu_k}(x^{\nu_k} - q^{\nu_k})\Phi_\alpha\| \\ &\leq \frac{1}{2^k}|f_{\mu_1\nu_1}| \cdots |f_{\mu_k\nu_k}| \sum_{6^k \text{ terms}} \|c_{\mu_1}^\# b_{\nu_1}^\# \cdots c_{\mu_k}^\# b_{\nu_k}^\# \Phi_\alpha\|, \end{aligned}$$

where $b_\nu^\#$ denotes $c_\nu^\#$ either $\sqrt{2}q^\nu$. Hence

$$\|L_f^k \Phi_\alpha\| \leq C(f,q)^k \sqrt{(|\alpha|+1)\cdots(|\alpha|+2k)}, \tag{5.20}$$

where

$$C(f,q) = 3\left(\sum_{\mu,\nu=0}^{d-1} |f_{\mu\nu}|\right)(\sqrt{2}|q|_\infty + 1) \tag{5.21}$$

with $|q|_\infty := \max_{\nu=0,\cdots,d-1} |q^\nu|$. Similarly we have for all $r \in \mathrm{N}$ and $\mu_j = 0, \cdots, d-1, j = 1, \cdots, r$

$$\|p_{\mu_1} \cdots p_{\mu_r} L_f^k \Phi_\alpha\| \leq \frac{1}{2^{r/2}} C(f,q)^k \sqrt{(|\alpha|+1)\cdots(|\alpha|+2k+r)}. \tag{5.22}$$

Let

$$\mathcal{S}_H(\mathbf{R}^d) = \mathcal{L}\{\Phi_\alpha | \ \alpha \in \mathrm{N}_0^d\}, \tag{5.23}$$

where $\mathcal{L}\{\cdots\}$ denotes the subspace algebraically spanned by the vectors in the set $\{\cdots\}$. For $f \in \mathcal{F}_a \cap \mathcal{F}_b$ and $u \in L^\infty_{\text{real}}(\mathbf{R}^2)$, we define

$$D_f(u) = \mathcal{L}\{u(ax, bp)^j L_f^k \psi_\ell | \psi_\ell \in \mathcal{S}_H(\mathbf{R}^d), j, k, \ell \in \mathrm{N}_0\}. \tag{5.24}$$

Obviously we have $\mathcal{S}_H(\mathbf{R}^d) \subset D_f(u)$.

LEMMA 5.4. *Let $f \in \mathcal{F}_a \cap \mathcal{F}_b, u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$ and set*

$$r_0 = \frac{1}{2\|u\|_\infty C(f,q)}, \tag{5.25}$$

where we set $r_0 := \infty$ if $u = 0$ or $f = 0$. Let $|t| < r_0$ ($t \in \mathbf{R}$) and $\psi \in D_f(u)$. Then, for all $m \in \mathbf{N}$, multi-indices α, and $\mu = 0, \cdots, d-1$, the vector ψ is in $D(\overline{L}_f^m)$ and $u(ax,bp)^m \overline{L}_f^m \psi$ is in $D(p_\mu)$. Moreover, $e^{-itM(u,L_f)}\psi$ is in $D(p_\mu)$ and

$$e^{-itM(u,L_f)}\psi = \sum_{m=0}^{\infty} \frac{(-it)^m u(ax,bp)^m \overline{L}_f^m \psi}{m!}, \tag{5.26}$$

$$p_\mu e^{-itM(u,L_f)}\psi = \sum_{m=0}^{\infty} \frac{(-it)^m p_\mu u(ax,bp)^m \overline{L}_f^m \psi}{m!}. \tag{5.27}$$

PROOF: It is sufficient to prove the assertion of the present lemma for $\psi = u(ax,bp)^j L_f^k \Phi_\alpha$ ($j,k \in \mathbf{N}_0$). The fact that $\mathcal{S}(\mathbf{R}^d) \subset \cap_{j=0}^{\infty} D(L_f^j)$ and the strong commutativty of $\overline{L}_f$ and $u(ax,bp)$ imply that $\psi \in D(\overline{L}_f^m)$. We have $u(ax,bp)^m \overline{L}_f^m \psi = u(ax,bp)^{m+j} L_f^{m+k} \Phi_\alpha$. Hence, by (5.20), we have

$$\|(-it)^m u(ax,bp)^m \overline{L}_f^m \psi\| \le |t|^m \|u\|_\infty^{m+j} C(f,q)^{m+k} \sqrt{(|\alpha|+1)\cdots(|\alpha|+2m+2k)},$$

which implies that, for $|t| < r_0$, the infinite series $\sum_{m=0}^{\infty} \|(-it)^m u(ax,bp)^m \overline{L}_f^m \psi/m!\|$ converges and so does $\Psi := \sum_{m=0}^{\infty} (-it)^m u(ax,bp)^m \overline{L}_f^m \psi/m!$. By the strong commutativity of $u(ax,bp)$ and $\overline{L}_f$, we can write $u(ax,bp)^m \overline{L}_f^m \psi = M(u,L_f)^m \psi$. Hence $\Psi = e^{-itM(u,L_f)}\psi$, which gives (5.26).

Let

$$\Psi_N = \sum_{m=0}^{N} \frac{(-it)^m}{m!} u(ax,bp)^m \overline{L}_f^m \psi = \sum_{m=0}^{N} \frac{(-it)^m}{m!} u(ax,bp)^{m+k} L_f^{m+k} \Phi_\alpha.$$

Then, by Lemma 5.3, $\Psi_N \in D(p_\mu)$ and

$$p_\mu \Psi_N = \sum_{m=0}^{N} \frac{(-it)^m p_\mu u(ax,bp)^{m+j} L_f^{m+k} \Phi_\alpha}{m!}.$$

By (5.7) and (5.22), we see that $p_\mu \Psi_N$ converges as $N \to \infty$. Since $\Psi_N \to e^{-itM(u,L_f)}\psi$ as $N \to \infty$, it follows from the closedness of p_μ that $e^{-itM(u,L_f)}\psi \in D(p_\mu)$ and (5.27) holds. ∎

Proof of Theorem 5.2

Throughout the proof, we set $M = M(u, L_f)$. We first consider the case $|t| < r_0$. Let $\psi, \phi \in D_f(u)$. Then, by Lemma 5.4, (5.11) and Lemma 5.3, we have

$$\begin{aligned}(\phi, e^{itM} p^\mu e^{-itM}\psi) &= (e^{-itM}\phi, p^\mu e^{-itM}\psi) = \sum_{j=0}^\infty \sum_{k=0}^\infty \frac{(it)^j(-it)^k}{j!k!}(M^j\phi, p^\mu M^k\psi) \\ &= \sum_{m=0}^\infty \frac{(it)^m}{m!}(\phi, (\mathrm{ad}\, M)^m p^\mu \psi) \\ &= (\phi, ta^\mu \partial_1 u(ax, bp)\overline{L}_f\psi + \Lambda(tf, u)^\mu{}_\nu p^\nu \psi).\end{aligned}$$

Since $D_f(u)$ is dense in $L^2(\mathbf{R}^d)$, we obtain

$$\begin{aligned}e^{itM} p^\mu e^{-itM}\psi &= ta^\mu \partial_1 u(ax, bp)\overline{L}_f\psi + \Lambda(tf, u)^\mu{}_\nu p^\nu \psi \\ &= \Lambda(tf, u)^\mu{}_\nu \left(p^\nu + ta^\nu \partial_1 u(ax, bp)\overline{L}_f\right)\psi. \qquad (5.28)\end{aligned}$$

Let $|s| < r_0$ and

$$\psi_N = \sum_{k=0}^N \frac{(-is)^k M^k \psi}{k!}.$$

Then $\psi_N \in D_f(u)$ and $\psi_N \to e^{-isM}\psi$ as $N \to \infty$. By (5.28), we have

$$e^{itM} p^\mu e^{-itM}\psi_N = \Lambda(tf, u)^\mu{}_\nu \left(p^\nu + ta^\nu \partial_1 u(ax, bp)\overline{L}_f\right)\psi_N.$$

By Lemma 5.4, we have

$$p^\nu \psi_N \to p^\nu e^{-isM}\psi \ (N \to \infty).$$

Note that

$$\overline{L}_f \psi_N = \sum_{k=0}^N \frac{(-is)^k M^k \overline{L}_f \psi}{k!}$$

and $\overline{L}_f\psi \in D_f(u)$. Hence $L_f\psi_N \to e^{-isM}\overline{L}_f\psi$ $(N \to \infty)$. Since $\partial_1 u(ax, bp)$ is bounded, it follows that

$$\begin{aligned}ta^\nu \partial_1 u(ax, bp)\overline{L}_f\psi_N &\to ta^\nu \partial_1 u(ax, bp) e^{-isM}\overline{L}_f\psi \\ &= e^{-isM} ta^\nu \partial_1 u(ax, bp)\overline{L}_f\psi \quad (N \to \infty).\end{aligned}$$

Hence

$$e^{itM} p^\mu e^{-itM}\psi_N \to \Lambda(tf, u)^\mu{}_\nu \left(p^\nu e^{-isM} + e^{-isM} ta^\nu \partial_1 u(ax, bp)\overline{L}_f\right)\psi.$$

It is clear that $e^{-itM}\psi_N \to e^{-itM}e^{-isM}\psi = e^{-i(t+s)M}\psi$ $(N \to \infty)$. Hence, $e^{-i(t+s)M}\psi \in D(p^\mu)$ and

$$e^{itM} p^\mu e^{-i(t+s)M}\psi = \Lambda(tf, u)^\mu{}_\nu \left(p^\nu e^{-isM} + e^{-isM} ta^\nu \partial_1 u(ax, bp)\overline{L}_f\right)\psi.$$

Hence we obtain

$$e^{i(t+s)M}p^\mu e^{-i(t+s)M}\psi = \Lambda(tf,u)^\mu{}_\nu \left(e^{isM}p^\nu e^{-isM} + ta^\nu\partial_1 u(ax,bp)\overline{L}_f\right)\psi,$$

where we have used the commutativity of e^{isM} and $\Lambda(tf,u)^\mu{}_\nu$. By using (5.28) with $t=s$, we obtain (5.28) with t replaced by $t+s$. Repeating this process, we can show that (5.8) holds for all $\psi \in D_f(u)$.

By (3.9) and (3.10), the RHS of (5.8) with $\psi \in D_f(u)$ is dominated by $C\|(N+1)\psi\|$ ($C>0$ is a constant). As is well known, each Φ_α is an eigenfunction of N with eigenvalue $2|\alpha|+d$. Hence $\mathcal{S}_H(\mathbf{R}^d)$ ($\subset D_f(u)$) is a core of N. By a simple limiting argument, one can easily show that (5.8) established for $\psi \in D_f(u)$ extends to all $\psi \in D(N)$, at the same time, proving that $e^{-iM}\psi \in D(p^\mu)$ for all $\psi \in D(N)$. This completes the proof of Theorem 5.2. ∎

Let $u,v \in \mathbf{B}_{\text{real}}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$. Then, by Proposition 3.2, $u(ax,bp)$ and $M(v,L_f)$ strongly commute. Hence $u(ax,bp)+M(v,L_f)$ is essentially self-adjoint. We denote its closure by $M(u;v,L_f)$:

$$M(u;v,L_f) = \overline{u(ax,bp)+M(v,L_f)}. \tag{5.29}$$

We have

$$e^{itM(u;v,L_f)} = e^{itu(ax,bp)}e^{itM(v,L_f)} = e^{itM(v,L_f)}e^{itu(ax,bp)}, \quad t \in \mathbf{R}. \tag{5.30}$$

THEOREM 5.5. *Let $u,v \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$. Then, for all $\psi \in D(N)$ and $\mu = 0,\cdots,d-1$, $e^{-iM(u;v,L_f)}\psi \in D(p^\mu)$ and*

$$\begin{aligned} &e^{iM(u;v,L_f)}p^\mu e^{-iM(u;v,L_f)}\psi \\ &= \Lambda(f,v)^\mu{}_\nu \left\{p^\nu + a^\nu(\partial_1 v(ax,bp)\overline{L}_f + \partial_1 u(ax,bp))\right\}\psi. \end{aligned} \tag{5.31}$$

PROOF: Under the present assumption, $\partial_1 u(ax,bp)$ is a bounded self-adjoint operator. Hence, $p^\mu + a^\mu\partial_1 u(ax,bp)$ is self-adjoint. Then, Theorem 2.5 implies the operator eqaulity

$$e^{iu(ax,bp)}p^\mu e^{-iu(ax,bp)} = p^\mu + a_\mu\partial_1 u(ax,bp).$$

Hence

$$e^{iM(v,L_f)}e^{iu(ax,bp)}p^\mu e^{-iu(ax,bp)}e^{-iM(v,L_f)} = e^{iM(v,L_f)}p^\mu e^{-iM(v,L_f)} + a^\mu\partial_1 u(ax,bp),$$

where we have used the strong commutativity of $\partial_1 u(ax,bp)$ and $e^{\pm iM(v,L_f)}$. By (5.30) and Theorem 5.2, we obtain (5.31). ∎

We next consider the transformation of the free d'Alembertian by the unitary operator $e^{iM(u;v,L_f)}$. For this purpose, we prepare some lemmas.

LEMMA 5.6. *Let $u \in \mathfrak{B}^{2,\infty}(\mathbf{R}^2)$. Then, for each $\mu,\nu = 0,1,\cdots,d-1$, $u(ax,bp)$ leaves $D(p_\nu p_\mu)\cap D(p_\mu p_\nu)$ invariant and, for all $\psi \in D(p_\nu p_\mu)\cap D(p_\mu p_\nu)$,*

$$\begin{aligned}&p_\nu p_\mu u(ax,bp)\psi\\&= u(ax,bp)p_\nu p_\mu\psi + i\partial_1 u(ax,bp)(a_\nu p_\mu + a_\mu p_\nu)\psi - a_\nu a_\mu \partial_1^2 u(ax,bp)\psi\end{aligned}$$

PROOF: Application of Lemma 2.4(i). ∎

LEMMA 5.7. *Let $u \in \mathfrak{B}^{\infty,2}(\mathbf{R}^2)$. Then, for each $\mu,\nu = 0,1,\cdots,d-1$, $u(ax,bp)$ leaves $D(x_\nu x_\mu)\cap D(x_\mu x_\nu)$ invariant and, for all $\psi \in D(x_\nu x_\mu)\cap D(x_\mu x_\nu)$,*

$$\begin{aligned}&x_\nu x_\mu u(ax,bp)\psi\\&= u(ax,bp)x_\nu x_\mu\psi - i\partial_2 u(ax,bp)(b_\nu x_\mu + x_\nu b_\mu)\psi - b_\nu b_\mu \partial_2^2 u(ax,bp)\psi.\end{aligned}$$

PROOF: Application of Lemma 2.4(ii). ∎

Let $r,s \in \mathbf{N}_0$. We say that a function u on $\mathbf{R}^2$ is in the set $\mathfrak{B}^{r,s}(\mathbf{R}^2)$ if, for all $x_2 \in \mathbf{R}$, $u(\cdot,x_2) \in C^r_{\text{real}}(\mathbf{R})$ and $\partial_1^j u \in \mathfrak{B}^{\infty,s}(\mathbf{R}^2)\cap C(\mathbf{R}^2)$ for $j = 0,\cdots,r$ with $\partial_2^k\partial_1^j u = \partial_1^j\partial_2^k u$, $j = 0,\cdots,r$, $k = 0,\cdots,s$, where $C(\mathbf{R}^2)$ denotes the space of continuous functions on $\mathbf{R}^2$.

LEMMA 5.8. *Let $u \in \mathfrak{B}^{2,2}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$. Then $u(ax,bp)\overline{L}_f$ maps $D(N^2)$ into $D(N)$ and, for all $\psi \in D(N^2)$,*

$$||Nu(ax,bp)\overline{L}_f\psi|| \le C||(N+1)^2\psi||, \tag{5.32}$$

where $C > 0$ is a constant.

PROOF: By (3.10) and the fact that $\mathcal{S}_H(\mathbf{R}^d)$ is a core for $(N+1)^2$, it is sufficient to prove (5.32) for $\psi \in \mathcal{S}_H(\mathbf{R}^d)$. Let $\psi \in \mathcal{S}_H(\mathbf{R}^d)$. Then $\overline{L}_f\psi = L_f\psi \in \mathcal{S}_H(\mathbf{R}^d) \subset D(N) = D(\Delta)\bigcap D(|x|^2)$. Hence, by Lemmas 5.6 and 5.7, $u(ax,bp)\overline{L}_f\psi \in D(N)$ and

$$||Nu(ax,bp)L_f\psi|| \le C_0(||L_f\psi|| + \sum_{\mu=0}^{d-1}(||a_\mu p_\mu L_f\psi|| + ||b_\mu x_\mu L_f\psi||) + ||NL_f\psi||,$$

where C_0 is a constant. By using (5.19), we can show that

$$\begin{aligned}||a_\mu p_\mu L_f\psi|| &\le C_1||(N+1)^{3/2}\psi||,\\ ||b_\mu x_\mu L_f\psi|| &\le C_1||(N+1)^{3/2}\psi||,\\ ||NL_f\psi|| &\le C_1||(N+1)^2\psi||,\end{aligned}$$

where C_1 is a constant. By these estimates and (3.10), we obtain (5.32) with $\psi \in \mathcal{S}_H(\mathbf{R})$. ∎

The following lemma is well known (or easy to prove) (use (5.19) and a limiting argument).

LEMMA 5.9. *For each $\mu = 0, \cdots, d-1$, p_μ maps $D(N^{3/2})$ into $D(N)$ and*

$$||Np_\mu\psi|| \le C||(N+1)^{3/2}\psi||, \quad \psi \in D(N^{3/2}),$$

where C is a constant.

For $u, v \in \mathfrak{C}^1(\mathbf{R}^2)$, we define

$$\begin{aligned}\Box_{f,u,v} &= (p + a(\partial_1 u(ax, bp) + \partial_1 v(ax, bp)\overline{L}_f))^2 \\ &= (p_0 + a_0(\partial_1 u(ax, bp) + \partial_1 v(ax, bp)\overline{L}_f))^2 \\ &\quad - \sum_{j=1}^{d-1}(p_j + a_j(\partial_1 u(ax, bp) + \partial_1 v(ax, bp)\overline{L}_f))^2 \end{aligned} \tag{5.33}$$

with $D(\Box_{f,u,v}) = \cap_{\mu=0}^{d-1} D\left((p_\mu + a_\mu(\partial_1 u(ax, bp) + \partial_1 v(ax, bp)\overline{L}_f))^2\right)$.

THEOREM 5.10. *Let $u, v \in \mathfrak{B}^{3,2}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$. Then, for each $\mu = 0, \cdots, d-1$, $e^{-iM(u;v,L_f)}$ maps $D(N^2)$ into $D((p^\mu)^2)$ and, for all $\psi \in D(N^2)$,*

$$\begin{aligned}&e^{iM(u;v,L_f)}(p^\mu)^2 e^{-iM(u;v,L_f)}\psi \\ &= \left(\Lambda(f,v)^\mu{}_\nu[p^\nu + a^\nu(\partial_1 u(ax, bp) + \partial_1 v(ax, bp)\overline{L}_f)]\right)^2 \psi.\end{aligned} \tag{5.34}$$

In particular, $D(N^2) \subset D(\Box_{f,u,v})$ and

$$e^{iM(u;v,L_f)}\Box e^{-iM(u;v,L_f)}\psi = \Box_{f,u,v}\psi, \quad \psi \in D(N^2). \tag{5.35}$$

PROOF: Let $\phi \in D(p_\mu^2)$ and $\psi \in D(N^2)$. Set $M = M(u; v, L_f)$. Then, by Theorem 5.5, $e^{-iM}\psi \in D(p^\mu)$ and we have

$$\begin{aligned}((p^\mu)^2\phi, e^{-iM}\psi) &= (p^\mu\phi, p^\mu e^{-iM}\psi) \\ &= (p^\mu\phi, \eta),\end{aligned}$$

where

$$\eta = e^{-iM}\Lambda(f,v)^\mu{}_\nu\{p^\nu + a^\nu(\partial_1 u(ax, bp) + \partial_1 v(ax, bp)\overline{L}_f)\}\psi.$$

By Lemmas 5.6–5.9, the vector $\{p^\nu + a^\nu(\partial_1 u(ax, bp) + \partial_1 v(ax, bp)\overline{L}_f)\}\psi$ is in $D(N)$. Note that e^{-iM} commutes with $\Lambda(f,v)^\mu{}_\nu$. Hence, by Theorem 5.5, $\eta \in D(p^\mu)$. Hence we obtain

$$((p^\mu)^2\phi, e^{-iM}\psi) = (\phi, p^\mu\eta).$$

This implies that $e^{-iM}\psi \in D((p^\mu)^2)$ and $(p^\mu)^2 e^{-iM}\psi = p^\mu\eta$. Thus (5.34) follows. Equation (5.35) follows from (5.34) and the fact that $\Lambda(f,v)$ is a bounded Lorentz transformation on $[L^2(\mathbf{R}^d)]$ (Lemma 5.1). ∎

For $u, v \in \mathbf{B}_{\text{real}}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$, we define a self-adjoint operator $H_f(u,v)$ by

$$H_f(u,v) := e^{iM(u;v,L_f)}H_0 e^{-iM(u;v,L_f)}, \tag{5.36}$$

where H_0 is given by (2.14). By Theorem 5.10, we obtain the following result.

COROLLARY 5.11. *Let $u,v \in \mathfrak{B}^{3,2}(\mathbf{R})$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$. Then $H_f(u,v)$ is a self-adjoint extension of $\Box_{f,u,v} \restriction D(N^2)$.*

REMARK: It is an open problem to clarify whether the symmetric operator $\Box_{f,u,v} \restriction D(N^2)$ in Corollary 5.11 is essentially self-adjoint or not.

VI. INTEGRAL KERNELS OF THE UNITARY GROUPS GENERATED BY PERTURBED D'ALEMBERTIANS

Let H_0 be given by (2.9). It is well known (or easy to see) that e^{isH_0} ($s \in \mathbf{R} \setminus \{0\}$) is an integral operator in the sense that

$$(e^{isH_0}\psi)(x) = \int_{\mathbf{R}^d} \Delta_s(x,y)\psi(y)dy, \quad \psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d), \tag{6.1}$$

with

$$\Delta_s(x,y) = \frac{e^{i\epsilon(s)\pi(d-2)/4}}{2^d\pi^{d/2}|s|^{d/2}} e^{i(x-y)^2/4s}, \tag{6.2}$$

where $\epsilon(s)$ is the sign function: $\epsilon(s) = 1$ if $s > 0$ and $\epsilon(s) = -1$ if $s < 0$.

Throughout this section, we assume that $(a,b) \in \mathbf{M}_0$. Let $H_f(u,v)$ be as in (5.36) with $u,v \in \mathbf{B}_{\rm real}(\mathbf{R}^2)$. In this section, we show that $e^{isH_f(u,v)}$ is an integral operator and compute the integral kernel of it.

We first consider a simple case. Let $u \in \mathbf{B}_{\rm real}(\mathbf{R}^2)$ and

$$H(u) = e^{iu(ax,bp)}H_0e^{-iu(ax,bp)}. \tag{6.3}$$

Note that, if $u \in \mathfrak{B}^{2,\infty}(\mathbf{R}^2)$, then $H(u) = \overline{\Box}_u$ (see Theorem 2.7.)

A vector $x \in \mathbf{M}^d$ satisfying $x^2 = 0$ is called a null vector. We denote by $\mathcal{N}_d$ the set of null vectors in $\mathbf{M}^d$.

THEOREM 6.1. *Let $u \in \mathbf{B}_{\rm real}(\mathbf{R}^2)$ and $a,b \in \mathcal{N}_d$. Then, for all $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ and $s \in \mathbf{R} \setminus \{0\}$,*

$$(e^{isH(u)}\psi)(x) = \int_{\mathbf{R}^d} e^{i[u(ax,(by-bx)/2s)-u(ay,(by-bx)/2s)]}\Delta_s(x,y)\psi(y)dy. \tag{6.4}$$

To prove this theorem, we need a lemma.

LEMMA 6.2. *Let K be a bounded integral operator on $L^2(\mathbf{R}^d)$ with kernel $k(x,y)$ such that, for all $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$,*

$$(K\psi)(x) = \int_{\mathbf{R}^d} k(x,y)\psi(y)dy. \tag{6.5}$$

Suppose that $|k(x,y)|$ is bounded on $\mathbf{R}^d \times \mathbf{R}^d$ and, for all $\xi_j \in \mathbf{R}$,

$$k(x - \sum_{j=1}^{r} \xi_j b_j, y) = e^{i\sum_{j=1}^{r} \xi_j \theta_j(x,y)} k(x,y), \quad \text{a.e.}(x,y) \in \mathbf{R}^d \times \mathbf{R}^d, \tag{6.6}$$

where $r \in \mathbf{N}, b_j \in \mathbf{M}^d$ $(j = 1, \cdots, r)$ is a constant vector with property $(a, b_j) \in \mathbf{M}_0$ and $\theta_j(x,y)$ is a real-valued Borel measurable function on $\mathbf{R}^d \times \mathbf{R}^d$. Further, assume that, for all null sets B in $\mathbf{R}$ w.r.t. the one-dimensional Lebesgue measure, $\{(x,y) \in \mathbf{R}^d \times \mathbf{R}^d | \theta_j(x,y) \in B\}$ is a null set w.r.t. the 2d-dimensional Lebesgue measure for all $j = 1, \cdots, r$. Let $F \in L^\infty(\mathbf{R}^{r+1})$. Then, for all $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$,

$$(F(ax, b_1 p, \cdots, b_r p) K\psi)(x) = \int_{\mathbf{R}^d} F(ax, \theta_1(x,y), \cdots, \theta_r(x,y)) k(x,y) \psi(y) dy. \tag{6.7}$$

PROOF: For $F \in \mathcal{S}(\mathbf{R}^{r+1})$, we have

$$F(ax, b_1 p, \cdots, b_r p) = \frac{1}{(2\pi)^{r/2}} \int_{\mathbf{R}} \tilde{F}(ax, \xi_1, \cdots, \xi_r) e^{i\sum_{j=1}^{r} \xi_j b_j p} d\xi_1 \cdots d\xi_r,$$

where

$$\tilde{F}(ax, \xi_1, \cdots, \xi_r) = \frac{1}{(2\pi)^{r/2}} \int_{\mathbf{R}} e^{-i\sum_{j=1}^{r} t_j \xi_j} F(ax, t_1, \cdots, t_r) dt_1 \cdots dt_r$$

in the operator norm toplogy. Let $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$. Then we have

$$(e^{i\sum_{j=1}^{r} \xi_j b_j p} K\psi)(x) = (K\psi)(x - \sum_{j=1}^{r} \xi_j b_j) = \int_{\mathbf{R}^d} e^{i\sum_{j=1}^{r} \xi_j \theta_j(x,y)} k(x,y) \psi(y) dy. \tag{6.8}$$

Using these representations, we see that (6.7) holds.

For any $F \in L^\infty(\mathbf{R}^{r+1})$, there exists a sequence $\{F_m\}_{m=1}^{\infty} \subset \mathcal{S}(\mathbf{R}^{r+1})$ such that $\sup_{m \geq 1} \|F_m\|_\infty < \infty$ and, for a.e. $(x_1, x_2, \cdots, x_{r+1}) \in \mathbf{R}^{r+1}$, $F_m(x_1, \cdots, x_{r+1}) \to F(x_1, \cdots, x_{r+1})$ as $m \to \infty$. Then, by the functional calculus, $F_m(ax, b_1 p, \cdots, b_r p) \to F(ax, b_1 p, \cdots, b_r p)$ strongly as $m \to \infty$. By the assumption on $\theta_j(x,y)$,

$$F_m(ax, \theta_1(x,y), \cdots, \theta_r(x,y)) \to F(ax, \theta_1(x,y), \cdots, \theta_r(x,y))$$

for a.e.(x,y) as $m \to \infty$. Hence, by the dominated convergence theorem, we have for all $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$

$$\begin{aligned} &\lim_{m \to \infty} \int_{\mathbf{R}^d} F_m(ax, \theta_1(x,y), \cdots, \theta_r(x,y)) k(x,y) \psi(y) dy \\ &= \int_{\mathbf{R}^d} F(ax, \theta_1(x,y), \cdots, \theta_r(x,y)) k(x,y) \psi(y) dy, \quad \text{a.e. } x. \end{aligned}$$

Thus we obtain (6.7) with $F \in L^\infty(\mathbf{R}^{r+1})$. ∎

LEMMA 6.3. *Let $F \in L^\infty(\mathbf{R}^{r+1}), a \in \mathrm{M}^d$, and $(a, b_j), (b_j, b_k) \in \mathrm{M}_0, j, k = 1, \cdots, r$. Then, for all $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ and $s \in \mathbf{R} \setminus \{0\}$,*

$$
\begin{aligned}
&(F(ax, b_1 p, \cdots, b_r p) e^{isH_0}\psi)(x) \\
&= \int_{\mathbf{R}^d} F\left(ax, \frac{b_1 y - b_1 x}{2s}, \frac{b_2 y - b_2 x}{2s}, \cdots, \frac{b_r y - b_r x}{2s}\right) \Delta_s(x, y)\psi(y)dy.
\end{aligned}
$$

PROOF: By (6.2) and the condition $b_j b_k = 0$, we have

$$
\Delta_s(x - \sum_{j=1}^r \xi_j b_j, y) = e^{i\sum_{j=1}^r \xi_j (b_j y - b_j x)/2s} \Delta_s(x, y).
$$

It is easy to see that $\theta_j(x, y) = (b_j y - b_j x)/2s$ satisfies the assumption of Lemma 6.2. Hence we can apply Lemma 6.2 to obtain the desired result. ∎

Proof of Theorem 6.1

We can write

$$
e^{isH(u)} = e^{iu(ax, bp)} e^{-iu_s} e^{isH_0},
$$

where $u_s = e^{isH_0} u(ax, bp) e^{-isH_0}$. Let

$$
X(s) = e^{isH_0} ax e^{-isH_0}.
$$

and

$$
x^\mu(s) = e^{isH_0} x^\mu e^{-isH_0}.
$$

Then it is easy to see that $x^\mu(s) = x^\mu + 2sp^\mu$ on $\mathcal{S}(\mathbf{R}^d)$. Hence it follows that $X(s) = ax + 2sap$ on $\mathcal{S}(\mathbf{R}^d)$. By applying Nelson's commutator theorem as in the proof of Proposition 3.1, we can show that $ax + 2sap$ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$. Hence $X(s) = \overline{ax + 2sap}$. By the condition $a^2 = 0$, $X(s)$ strongly commutes with ax, ap and bp. Also note that p^μ strongly commutes with H_0. Hence, by the functional calculus, we have $u_s = u(\overline{ax + 2sap}, bp)$. This implies also that $u(ax, bp)$ and u_s strongly commute. Hence

$$
e^{isH(u)} = e^{i[u(ax, bp) - u(\overline{ax + 2sap}, bp)]} e^{isH_0}.
$$

We can apply Lemma 6.3 with $F(x_1, x_2, x_3) = e^{i[u(x_1, x_3) - u(x_1 + 2sx_2, x_3)]}$ to obtain (6.4). ∎

We next consider the integral-kernel representation of the unitary group generated by the self-adjoint operator

$$
H_f(u) := e^{iM(u, L_f)} H_0 e^{-iM(u, L_f)}, \tag{6.9}
$$

where $u \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$. It follows from Corollary 5.11 that, if $u \in \mathfrak{B}^{3,2}(\mathbf{R}^2)$, then $H_f(u)$ is a self-adjoint extension of the symmetric operator $(p + a\partial_1 u(ax, bp)\overline{L}_f)^2 \restriction D(N^2)$.

For a matrix $T = (T_{\mu\nu})_{\mu,\nu=0,\cdots,d-1}$ and $x, y \in \mathbf{M}^d$, we set

$$xTy = T_{\mu\nu}x^\mu y^\nu.$$

For $u \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$, we define a function $\Phi_{u,f}$ on $\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R} \setminus \{0\}$ by

$$\begin{aligned}&\Phi_{u,f}(x, y; s)\\ &= \frac{1}{2s}(y - q)\left(1 - e^{-[u(ax,(by-bx)/2s)-u(ay,(by-bx)/2s)]f}\right)(x - q).\end{aligned} \tag{6.10}$$

THEOREM 6.4. *Let $u, v \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}), f \in \mathcal{F}_a \cap \mathcal{F}_b$ and $a, b \in \mathcal{N}_d$. Then, for all $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ and $s \in \mathbf{R} \setminus \{0\}$,*

$$(e^{isH_f(u)}\psi)(x) = \int_{\mathbf{R}^d} e^{i\Phi_{u,f}(x,y;s)}\Delta_s(x, y)\psi(y)dy. \tag{6.11}$$

To prove this theorem, we prepare two lemmas. For a function h on $\mathbf{R} \times \mathbf{R}^d$ and $\alpha \in \mathbf{C}$, we define

$$\Psi_h^s(x, y; \alpha) = \frac{1}{2s}(y - q)\left(1 - e^{-\alpha h(ax,y)f}\right)(x - q). \tag{6.12}$$

LEMMA 6.5. *Let $f \in \mathcal{F}_a$. Suppose that , for each $y \in \mathbf{R}^d$, $h(t, y)$ is continuously differentiable in $t \in \mathbf{R}$. Then, for all $x, y \in \mathbf{R}^d$ and $\alpha, \beta \in \mathbf{C}$,*

$$\sum_{k=0}^{\infty} \frac{(i\alpha)^k}{k!} h(ax, y)^k (L_f^x)^k e^{i\Psi_h^s(x,y;\beta)}\Delta_s(x, y) = e^{i\Psi_h^s(x,y;\alpha+\beta)}\Delta_s(x, y), \tag{6.13}$$

where L_f^x denotes the operator L_f with respect to x variable.

PROOF: We first prove (6.13) with $\beta = 0$:

$$\sum_{k=0}^{\infty} \frac{(i\alpha)^k}{k!} h(ax, y)^k (L_f^x)^k \Delta_s(x, y) = e^{i\Psi_h^s(x,y;\alpha)}\Delta_s(x, y). \tag{6.14}$$

It is obvious that the function

$$E(\alpha) := e^{i\Psi_h^s(x,y;\alpha)}\Delta_s(x, y)$$

of α is entire. Hence, to prove (6.14), it is sufficient to show that the following equality holds:

$$i^k h(ax, y)^k (L_f^x)^k \Delta_s(x, y) = \left.\frac{\partial^k}{\partial\alpha^k} E(\alpha)\right|_{\alpha=0}. \tag{6.15}$$

We prove this by induction on k. The case $k=0$ is trivial. Suppose that (6.15) holds for k. Since $h(ax,y)$ and L_f^x commute, we have

$$\begin{aligned}
& i^{k+1}h(ax,y)^{k+1}(L_f^x)^{k+1}\Delta_s(x,y) \\
&= ih(ax,y)L_f^x \frac{\partial^k}{\partial\alpha^k}E(\alpha)\bigg|_{\alpha=0} \\
&= -h(ax,y)f_{\mu\nu}(x^\nu - q^\nu)\left[\frac{\partial^k}{\partial\alpha^k}\frac{\partial}{\partial x_\mu}E(\alpha)\right]_{\alpha=0} \\
=& -\frac{i}{2s}\sum_{r=0}^{k}(-1)^{r+1}{}_kC_r(y-q)(h(ax,y)f)^{r+1}(x-q)\frac{\partial^{k-r}E(\alpha)}{\partial\alpha^{k-r}}\bigg|_{\alpha=0},
\end{aligned} \tag{6.16}$$

where we have used the property that $f_{\mu\nu}a^\nu = 0, \mu = 0, \cdots, d-1$. It is easy to see that the last quantity in (6.16) is equal to

$$\frac{\partial^{k+1}}{\partial\alpha^{k+1}}E(\alpha)\bigg|_{\alpha=0} = \left[\frac{\partial^k}{\partial\alpha^k}\left(\frac{\partial}{\partial\alpha}E(\alpha)\right)\right]_{\alpha=0}.$$

Thus (6.15) holds with k replaced by $k+1$. Hence we obtain (6.14).

As for (6.13), by the same reason as above, it is sufficient to show that the following equality holds:

$$i^k h(ax,y)^k (L_f^x)^k e^{i\Psi_h^s(x,y;\beta)}\Delta_s(x,y) = \frac{\partial^k}{\partial\alpha^k}E(\alpha+\beta)\bigg|_{\alpha=0}.$$

This can be done in quite the same way as in the preceding case. ∎

LEMMA 6.6. *For all $f \in M_d^{\mathrm{as}}(\mathbf{R})$, $\overline{L}_f$ strongly commutes with H_0.*

PROOF: Let $\psi \in \mathcal{S}(\mathbf{R}^d)$. Then $e^{isH_0}\psi \in \mathcal{S}(\mathbf{R}^d)$ and, by (6.1), we have

$$(\overline{L}_f e^{isH_0}\psi)(x) = -\frac{1}{2s}\int_{\mathbf{R}^d} f_{\mu\nu}(x^\nu - q^\nu)(x^\mu - y^\mu)\Delta_s(x,y)\psi(y)dy.$$

By the antisymmetry of $f_{\mu\nu}$, we can write

$$\begin{aligned}
f_{\mu\nu}(x^\nu - q^\nu)(x^\mu - y^\mu)\Delta_s(x,y) &= f_{\mu\nu}(x^\nu - q^\nu)(q^\mu - y^\mu)\Delta_s(x,y) \\
&= f_{\mu\nu}(x^\nu - y^\nu)(q^\mu - y^\mu)\Delta_s(x,y) \\
&= 2sif_{\mu\nu}(q^\mu - y^\mu)\frac{\partial\Delta_s(x,y)}{\partial y_\nu}.
\end{aligned}$$

Then, by integration by parts, we obtain

$$\overline{L}_f e^{isH_0}\psi = e^{isH_0}\overline{L}_f\psi.$$

By Proposition 3.1, this equality implies that $e^{isH_0}\overline{L}_f \subset \overline{L}_f e^{isH_0}$. Thus, by Proposition 2.1, $\overline{L}_f$ and H_0 strongly commute. ∎

Proof of Theorem 6.4

We first consider the case where $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. We can write

$$e^{isH_f(u)} = e^{iM(u,L_f)}e^{-iM_s(u,L_f)}e^{isH_0},$$

where

$$M_s(u, L_f) = e^{isH_0}M(u, L_f)e^{-isH_0}.$$

By Lemma 6.6 and the proof of Theorem 6.1, we have

$$M_s(u, L_f) = u(\overline{ax + 2sap}, bp)\overline{L}_f$$

on $D(\overline{L}_f)$. Hence, putting

$$V(x_1, x_2, x_3) = u(x_1, x_3) - u(x_1 + 2sx_2, x_3), \quad x_1, x_2 \in \mathbf{R},$$

and denoting by M the closure of $V(ax, ap, bp)\overline{L}_f$, we obtain

$$e^{isH_f(u)} = e^{iM}e^{isH_0}.$$

Let $\phi \in \mathcal{S}_H(\mathbf{R}^d)$ (see (5.23)) and $\psi \in C_0^\infty(\mathbf{R}^d)$. Let $\delta = (2\|V\|_\infty C(f, q))^{-1}$ and $|t| < \delta$. Then, as in the proof of Lemma 5.4, we have

$$e^{-itM}\phi = \sum_{k=0}^{\infty} \frac{(-it)^k}{k!} V(ax, ap, bp)^k L_f^k \phi.$$

Hence

$$(\phi, e^{itM}e^{isH_0}\psi) = \sum_{k=0}^{\infty} \frac{t^k}{k!}(\phi, i^k V(ax, ap, bp)^k L_f^k e^{isH_0}\psi).$$

Using the strong commutativity and Lemma 6.2, we can write

$$(\phi, i^k V(ax, ap, bp)^k L_f^k e^{isH_0}\psi) = \int_{\mathbf{R}^d} \phi(x)^* \left(\int_{\mathbf{R}^d} i^k h(ax, y)^k (L_f^x)^k \Delta_s(x, y)\psi(y)dy\right) dx,$$

where we set

$$h(ax, y) = V(ax, (ay - ax)/2s, (by - bx)/2s).$$

By (6.15) and Cauchy's estimate, we have for all $r > 0$

$$\begin{aligned} |i^k h(ax, y)^k (L_f^x)^k \Delta_s(x, y)| &\le k! \frac{\sup_{|\alpha|=r} |E(\alpha)|}{r^k} \\ &\le \frac{k!}{r^k} H_r(x, y), \end{aligned}$$

where

$$H_r(x,y) = \frac{1}{2^d \pi^{d/2}|s|^{d/2}} \exp\left(\frac{1}{2|s|}(1+e^{r\|h\|_\infty C(f)})|x-q||y-q|\right),$$

Hence, for all $r > |t|$ and $N \in \mathbf{N}$, we obtain

$$\left|\sum_{k=0}^{N} \frac{t^k}{k!} i^k h(ax,y)^k (L_f^x)^k \Delta_s(x,y)\right| \leq \left(1-\frac{|t|}{r}\right)^{-1} H_r(x,y).$$

Since $\phi(x)$ is of the form $P(x)e^{-|x|^2/2}$ with $P(x)$ a polynomial (see (5.17)) and ψ has compact support, it follows that $|\phi(x)^* H_r(x,y)\psi(y)|$ is in $L^1(\mathbf{R}^d \times \mathbf{R}^d)$. Hence, by Fubini's theorem and the Lebesgue dominated convergence theorem, we obtain

$$(\phi, e^{itM}e^{isH_0}\psi) = \int_{\mathbf{R}^d\times\mathbf{R}^d} \phi(x)^* e^{i\Psi_h^s(x,y;t)} \Delta_s(x,y)\psi(y)dx\,dy.$$

Thus

$$(e^{itM}e^{isH_0}\psi)(x) = \int_{\mathbf{R}^d} e^{i\Psi_h^s(x,y;t)}\Delta_s(x,y)\psi(y)dy. \tag{6.17}$$

Let $|\tau| < \delta$. Then we have

$$(\phi, e^{i(\tau+t)M}e^{isH_0}\psi) = \sum_{k=0}^{\infty} \frac{\tau^k}{k!}(\phi, i^k V(ax,ap,bp)^k L_f^k e^{itM}e^{isH_0}\psi). \tag{6.18}$$

Using (6.17) and Lemma 6.5, we can show in the same way as above that the RHS of (6.18) is equal to

$$\int_{\mathbf{R}^d\times\mathbf{R}^d} \phi(x)^* e^{i\Psi_h^s(x,y;t+\tau)}\Delta_s(x,y)\psi(y)dx\,dy.$$

Hence (6.17) holds with t replaced by $t+\tau$. Repeating this procedure, we see that (6.17) holds for all $t \in \mathbf{R}$. In the present case, we have $\Psi_h^s(x,y;1) = \Phi_{u,f}(x,y;s)$. Thus (6.11) with $\psi \in C_0^\infty(\mathbf{R}^d)$ follows. To extend this result to all $\psi \in L^1(\mathbf{R}^d)\bigcap L^2(\mathbf{R}^d)$, we need only to make a simple limiting argument, noting that there exists a sequence $\{\psi_k\}_{k=1}^\infty \subset C_0^\infty(\mathbf{R}^d)$ such that, for $p = 1,2$, $\|\psi_k - \psi\|_{L^p(\mathbf{R}^d)} \to 0$ as $k \to \infty$.

Finally we consider the case where $u \in L^\infty_{\mathrm{real}}(\mathbf{R}^2)$. Then there exist sequences $\{u_k\}_k$ in $\mathfrak{B}^{1,\infty}(\mathbf{R}^2)$ such that $\sup_{k\geq 1}\|u_k\|_\infty < \infty$ and, for a.e. $(x_1,x_2) \in \mathbf{R}^2$, $u_k(x_1,x_2) \to u(x_1,x_2)$ as $k \to \infty$. By the preceding result, we have

$$(e^{isH_f(u_k)}\psi)(x) = \int_{\mathbf{R}^d} e^{i\Phi^s_{u_k,f}(x,y)}\psi(y)dy.$$

By the Lebesgue dominated convergence theorem, we easily see that

$$\lim_{k\to\infty}\int_{\mathbf{R}^d} e^{i\Phi^s_{u_k,f}(x,y)}\psi(y)dy = \int_{\mathbf{R}^d} e^{i\Phi_{u,f}(x,y;s)}\psi(y)dy.$$

On the other hand, for all $\psi \in D^{\infty}_{f,u}$ (see (5.1)), $M(u_k, L_f)\psi \to M(u, L_f)\psi$ as $k \to \infty$. Since $D^{\infty}_{f,u}$ is a common core for $M(u_k, L_f)$ and $M(u, L_f)$, it follows by standard convergence theorems ([R-S1, §VIII.7]) that, for all $t \in \mathbf{R}$,

$$\mathrm{s} - \lim_{k\to\infty} e^{itM(u_k,L_f)} = e^{itM(u,L_f)}.$$

Hence

$$\mathrm{s} - \lim_{k\to\infty} e^{isH_f(u_k)} = e^{isH_f(u)}.$$

Thus (6.11) with $u \in L^{\infty}_{\mathrm{real}}(\mathbf{R}^2)$ holds. By a limiting argument similar to the one just given, we can extend (6.11) with $u \in L^{\infty}_{\mathrm{real}}(\mathbf{R}^2)$ to the case where $u \in \mathrm{B}_{\mathrm{real}}(\mathbf{R}^2)$. ∎

We are now ready to derive the integral-kernel representation of the unitary group $e^{isH_f(u,v)}$ (see (5.36)).

THEOREM 6.7. *Let $u, v \in \mathrm{B}_{\mathrm{real}}(\mathbf{R}^2)$, $f \in \mathcal{F}_a \cap \mathcal{F}_b$ and $a, b \in \mathcal{N}_d$. Then, for all $\psi \in L^1(\mathbf{R}^d) \bigcap L^2(\mathbf{R}^d)$ and $s \in \mathbf{R} \setminus \{0\}$,*

$$(e^{isH_f(u,v)}\psi)(x) = \int_{\mathbf{R}^d} e^{i[u(ax,(by-bx)/2s)-u(ay,(bx-by)/2s)]+i\Phi^s_{v,f}(x,y)}\Delta_s(x,y)\psi(y)dy. \quad (6.19)$$

PROOF: As in the case of proof of Theorems 6.1 or Theorem 6.4, we need only to prove (6.19) for the case where $u \in L^{\infty}_{\mathrm{real}}(\mathbf{R}^2)$ and $v \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. In the same way as in the case of $H(u)$ or $H_f(v)$, we can show that

$$e^{isH_f(u,v)} = e^{i[u(ax,bp)-u(\overline{ax+2sap},bp)]}e^{isH_f(v)}.$$

The kernel

$$k(x,y) = e^{i\Phi^s_{v,f}(x,y)}\Delta_s(x,y)$$

of $e^{isH_f(v)}$ satisfies the assumption of Lemma 6.2 with

$$k(x - \xi_1 a - \xi_2 b, y) = e^{i\xi_1(ay-ax)/2s+i\xi_2(by-bx)/2s}k(x,y).$$

Thus, by Theorem 6.4 and Lemma 6.2, we obtain (6.19). ∎

VII. Application to the external field problem

In this section we apply the operator theory developed in the preceding sections to the external field problem mentioned in the Intorduction.

We consider a quantum system of a charged particle moving in the Minkowski space M^d under the influence of an electromagnetic field $F = (F_{\mu\nu})_{\mu,\nu=0,\cdots,d-1}$, a tensor field on

$\mathbf{M}^d$. A vector potential $A = (A_0, \cdots, A_{d-1})$ of the electromagnetic field is a vector field on $\mathbf{M}^d$ such that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{7.1}$$

The gauge covariant momentum operator $(\pi_0, \cdots, \pi_{d-1})$ is defined by

$$\pi_\mu = p_\mu - A_\mu \tag{7.2}$$

with $D(\pi_\mu) = D(p_\mu) \cap D(A_\mu)$, where we take the physical unit system such that $\hbar$ (the Planck constant divided by 2π) $= c$ (the light speed) $= 1$ and the electric charge of the particle is equal to one.

To apply our theory, we introduce a class of vector potentials: we assume that the vector potential A is of the form

$$A_\mu(x) = Q_\mu(x)W'(ax), \quad , \mu = 0, \cdots, d-1, \tag{7.3}$$

where $a \in \mathbf{M}^d$ is a constant vector, $W \in C^1_{\mathrm{real}}(\mathbf{R})$, and Q_μ is given by (3.2) with $f \in M_d^{\mathrm{as}}(\mathbf{R})$. If $W \in C^2_{\mathrm{real}}(\mathbf{R})$, then

$$F_{\mu\nu}(x) = -2f_{\mu\nu}W'(ax) + (a_\mu f_{\nu\lambda} - a_\nu f_{\mu\lambda})(x^\lambda - q^\lambda)W''(ax). \tag{7.4}$$

PROPOSITION 7.1. *Let $W \in \mathfrak{B}^2(\mathbf{R})$. Then each π_μ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$.*

PROOF: It is obvious that π_μ is symmetric and $C_0^\infty(\mathbf{R}^d) \subset D(\pi_\mu)$. Let N be the self-adjoint operator given by (3.6) and $\psi \in C_0^\infty(\mathbf{R}^d)$. Then, by (3.7) and (3.9), we have

$$||\pi_\mu \psi|| \le C||(N+1)^{1/2}\psi|| \le C||(N+1)\psi||,$$

where C is a constant. Moreover, we have

$$\begin{aligned} &[\pi_\mu, N]\psi \\ &= 2ix^\mu\psi - i\sum_{\nu=0}^{d-1}\left\{[f_{\mu\nu}W'(ax) + Q_\mu a_\nu W''(ax)]p_\nu\psi + p_\nu[f_{\mu\nu}W'(ax) + Q_\mu a_\nu W''(ax)]\psi\right\}. \end{aligned}$$

Hence it follows that

$$|(\pi_\mu\psi, N\psi) - (N\psi, \pi_\mu\psi)| \le C'||(N+1)^{1/2}\psi||^2,$$

where C' is a constant. Thus we can apply Nelson's commutator theorem ([R-S2, Theorem X.37]) to obtain the desired result. ∎

For technical reasons as well as for some interest, we first consider a deformation of A.

7.1 A class of deformed vector potentials with a parameter $\varepsilon > 0$.

Let $\varepsilon > 0$ be a parameter and $u_\varepsilon = u_\varepsilon(t)$ be a function in $C^1_{\mathrm{real}}(\mathbf{R})$, depending on ε with the following properties:

(i) $tu_\varepsilon \in \mathfrak{B}^1(\mathbf{R})$.

(ii)

$$\sup_{t\in\mathbf{R}} |tu_\varepsilon(t)| \le C \tag{7.5}$$

with C a constant independent of ε.

(iii)

$$\lim_{\varepsilon\to 0} u_\varepsilon(t) = \frac{1}{t}, \quad t \in \mathbf{R}\setminus\{0\}. \tag{7.6}$$

REMARK: A simple example of u_ε is the following one:

$$u_\varepsilon(t) = \frac{t}{t^2+\varepsilon^2}.$$

In what follows, we assume that $a \in \mathcal{N}_d$, i.e., $a^2 = 0$. For $W \in C^1_{\mathrm{real}}(\mathbf{R})$, we define

$$A^\varepsilon_\mu = apu_\varepsilon(ap)A_\mu + \frac{1}{2}a_\mu u_\varepsilon(ap)[1 - apu_\varepsilon(ap)](ax-aq)^2 W'(ax)^2. \tag{7.7}$$

The following lemma shows that the operator A^ε_μ is a deformation of the multiplication operator A_μ.

LEMMA 7.2. *For all* $\psi \in \left[\cap_{\mu=0}^{d-1} D(A_\mu)\right] \cap D((ap)^{-1}(ax-aq)^2 W'(ax)^2)$,

$$\lim_{\varepsilon\to 0} A^\varepsilon_\mu \psi = A_\mu\psi, \quad \mu = 0, \cdots, d-1.$$

PROOF: By the functional calculus, (7.5) and (7.6), we have

$$\lim_{\varepsilon\to 0} apu_\varepsilon(ap)\phi = \phi, \quad \phi \in L^2(\mathbf{R}^d). \tag{7.8}$$

Let ψ be as above. Then, by (7.8), we have

$$\lim_{\varepsilon\to 0} apu_\varepsilon(ap)A_\mu\psi = A_\mu\psi$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0} u_\varepsilon(ap)\left(apu_\varepsilon(ap) - 1\right)(ax-aq)^2 W'(ax)^2 \psi \\
&= \lim_{\varepsilon\to 0} apu_\varepsilon(ap)\left(apu_\varepsilon(ap) - 1\right)(ap)^{-1}(ax-aq)^2 W'(ax)^2 \psi \\
&= 0.
\end{aligned}$$

Thus, by (7.7), the desired result follows. ∎

For $f \in \mathcal{F}_a$ and $u, v \in L^\infty_{\mathrm{real}}(\mathbf{R}^2)$, we define

$$P_\mu(u, v; f) = p_\mu + f_{\mu\nu} u(ax, ap) p^\nu + v(ax, ap) Q_\mu. \tag{7.9}$$

LEMMA 7.3. *Suppose that $u, v \in \mathfrak{B}^{1,1}(\mathbf{R}^2)$. Then, each $P_\mu(u, v; f)$ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$.*

PROOF: For all $\psi, \phi \in C_0^\infty(\mathbf{R}^d)$, we have $(\psi, f_{\mu\nu} u(ax, ap) p^\nu \phi) = (f_{\mu\nu} u(ax, ap)\psi, p^\nu \phi)$. By Lemma 2.4, $u(ax, ap)\psi \in D(p_\nu)$ and $p^\nu u(ax, ap)\psi = i a^\nu \partial_1 u(ax, ap)\psi + u(ax, ap) p^\nu \psi$. Using $f_{\mu\nu} a^\nu = 0$, we see that $f_{\mu\nu} u(ax, ap) p^\nu$ is symmetric on $C_0^\infty(\mathbf{R}^d)$. Similarly, we can show that $v(ax, ap) Q_\mu$ is symmetric on $C_0^\infty(\mathbf{R}^d)$. Hence $P_\mu(u, v; f)$ is symmetric on $C_0^\infty(\mathbf{R}^d)$. We have

$$||P_\mu(u, v; f)\psi|| \le C||(N+1)^{1/2}\psi|| \le C||(N+1)\psi||, \quad \psi \in C_0^\infty(\mathbf{R}^d),$$

where C is a constant. Similarly we can show that

$$|(P_\mu(u, v; f)\psi, N\psi) - (N\psi, P_\mu(u, v; f)\psi)| \le D||(N+1)^{1/2}\psi||^2,$$

where D is a constant. Thus, by Nelson's commutator theorem, we obtain the desired result. ∎

Corresponding to the deformation of A given by (7.7), we have a deformation of π_μ:

$$\pi_\mu(\varepsilon) = p_\mu - A_\mu^\varepsilon \tag{7.10}$$

with $D(\pi_\mu(\varepsilon)) = D(p_\mu) \cap D(A_\mu^\varepsilon) = D(p_\mu) \cap D(A_\mu) \cap D((ax - aq)^2 W'(ax)^2)$.

LEMMA 7.4. *Suppose that $W \in \mathfrak{B}^2(\mathbf{R})$ and $(t - aq)W' \in L^\infty(\mathbf{R})$. Then $\pi_\mu(\varepsilon)$ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$.*

PROOF: Under the present assumption for W, the second term on the RHS of (7.7) is bounded. Hence it is sufficient to show that

$$\widetilde{\pi}_\mu(\varepsilon) := p_\mu - apu_\varepsilon(ap) A_\mu$$

is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$. Note that $\widetilde{\pi}_\mu = P_\mu(0, -W' \otimes tu_\varepsilon; f)$, where, for two functions u, v on $\mathbf{R}$, $u \otimes v$ is a function on $\mathbf{R}^2$ defined by

$$(u \otimes v)(s, t) = u(s)v(t), \quad s, t \in \mathbf{R}. \tag{7.11}$$

Hence Lemma 7.3 gives the desired result. ∎

For $W \in C^1_{\text{real}}(\mathbf{R})$ such that

$$Y(t) := W(t) + (t - aq)W'(t) \tag{7.12}$$

is bounded, we can define a bounded operator-valued Lorentz transformation on $[L^2(\mathbf{R}^d)]$ by

$$\Lambda(\varepsilon) = e^{\widetilde{f}Y(ax)u_\varepsilon(ap)}, \tag{7.13}$$

where $f \in M_d^{\text{as}}(\mathbf{R})$ (see Proposition 4.4).

Let

$$\hat{\pi}^\mu(\varepsilon) = \Lambda(\varepsilon)^\mu{}_\nu \pi^\nu(\varepsilon). \tag{7.14}$$

We say that W is in the set $\mathfrak{W}_r$ ($r \in \mathbf{N}$) if $W \in \mathfrak{B}^r(\mathbf{R})$ and $(t - aq)(d^kW/dt^k) \in L^\infty(\mathbf{R}), k = 1, \cdots, r$.

We define a subset of $\mathcal{F}_a$ by

$$\mathcal{G}_a = \{f \in \mathcal{F}_a | f^\mu{}_\lambda f^\lambda{}_\nu = a^\mu a_\nu, \mu, \nu = 0, \cdots, d-1\}. \tag{7.15}$$

Note that every $f \in \mathcal{G}_a$ satisfies

$$\widetilde{f}^k = 0, \quad k \geq 3. \tag{7.16}$$

LEMMA 7.5. *Let $W \in \mathfrak{W}_2$, $a \in \mathcal{N}_d$, and $f \in \mathcal{G}_a$. Then $\hat{\pi}^\mu(\varepsilon)$ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$.*

PROOF: By direct computations, we have for all $\psi \in C_0^\infty(\mathbf{R}^d)$

$$\hat{\pi}_\mu(\varepsilon)\psi = [S_\mu + V_\mu(\varepsilon)]\psi,$$

where

$$\begin{aligned} S_\mu &= P_\mu(Y \otimes u_\varepsilon, -W' \otimes tu_\varepsilon; f) - a_\mu apu_\varepsilon(ap)^2 Y(ax)(ax - aq)W'(ax), \\ V_\mu(\varepsilon) &= \frac{1}{2} a_\mu apu_\varepsilon(ap)^2 Y(ax)^2 \\ &\quad + \frac{1}{2} a_\mu u_\varepsilon(ap)(apu_\varepsilon(ap) - 1)(ax - aq)^2 W'(ax)^2. \end{aligned}$$

It is easy to see that u_ε and Y are in $\mathfrak{B}^1(\mathbf{R})$. Hence $V_\mu(\varepsilon)$ is bounded. In the same way as in the proof of Lemma 7.3, we can show that S_μ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$. Thus the desired result follows. ∎

For $W \in C^1_{\text{real}}(\mathbf{R})$, we define

$$\Omega(t) = \int_{aq}^t (s - aq)^2 W'(s)^2 ds, \quad t \in \mathbf{R}, \tag{7.17}$$

and put

$$U_\varepsilon = \exp\left\{-iM\left(\frac{\Omega\otimes u_\varepsilon}{2}; W\otimes u_\varepsilon, L_f\right)\right\}, \tag{7.18}$$

where $M(\,\cdot\,;\,\cdot\,, L_f)$ is defined by (5.29).

We say that $f \in \mathcal{G}_a$ is in the set $\mathcal{G}_a^0$ if

$$f_{\mu\nu}a_\lambda + f_{\nu\lambda}a_\mu + f_{\lambda\mu}a_\nu = 0,\ \mu,\nu,\lambda = 0,1,\cdots,d-1, \tag{7.19}$$

THEOREM 7.6. *Let $a \in \mathcal{N}_d$, $f \in \mathcal{G}_a^0$ and $W \in \mathfrak{W}_2$. Assume that $(t-aq)W' \in L^2(\mathbf{R})$. Then the following operator equality holds:*

$$U_\varepsilon p^\mu U_\varepsilon^{-1} = \overline{\hat{\pi}^\mu(\varepsilon)}.$$

PROOF: Let $\psi \in \mathcal{S}(\mathbf{R}^d)$. By the assumption on W, Ω is in $\mathfrak{B}^1(\mathbf{R})$. Hence we can apply Theorem 5.5 and Lemma 5.1 to obtain

$$\begin{aligned}&U_\varepsilon p^\mu U_\varepsilon^{-1}\psi\\ &= \Lambda(\varepsilon)^\mu{}_\nu\left\{\Lambda(f,(Y-W)\otimes u_\varepsilon)^\nu{}_\lambda p^\lambda\psi - a^\nu u_\varepsilon(ap)W'(ax)L_f\psi - \frac{a^\nu u_\varepsilon(ap)\Omega'(ax)}{2}\psi\right\}.\end{aligned} \tag{7.20}$$

By Proposition 4.4(ii), we have

$$\begin{aligned}&\Lambda(f,(Y-W)\otimes u_\varepsilon)^\nu{}_\lambda p^\lambda\psi\\ &= p^\nu\psi - f^\nu{}_\lambda u_\varepsilon(ap)(Y(ax)-W(ax))p^\lambda\psi + \frac{1}{2}a^\nu apu_\varepsilon(ap)^2(Y(ax)-W(ax))^2\psi.\end{aligned} \tag{7.21}$$

By (7.19), we can write

$$a^\nu L_f\psi = apQ^\nu\psi - f^\nu{}_\lambda p^\lambda(ax-aq)\psi.$$

Putting this relation and (7.21) into (7.20), we obtain

$$U_\varepsilon p^\mu U_\varepsilon^{-1}\psi = \hat{\pi}^\mu(\varepsilon)\psi.$$

This equation and Lemma 7.5 imply the desired result. ∎

REMARK: Theorem 7.6 shows that the closure of the Lorentz-transformation of $\pi(\varepsilon) = (\pi^\mu(\varepsilon))$ by $\Lambda(\varepsilon)$ is unitarily equivalent to the canonical momentum operator $p = (p^\mu)$. This may be an interesting fact from a view-point of representation of "partially broken canonical commutation relations" (see Section VIII).

Let

$$H_\varepsilon = U_\varepsilon H_0 U_\varepsilon^{-1}. \tag{7.22}$$

As a corollary of Theorem 7.6, we have the following:

COROLLARY 7.7. *Let $a \in \mathcal{N}_d, f \in \mathcal{G}_a^0, W \in \mathfrak{W}_3$ and assume that $u_\varepsilon \in \mathfrak{B}^2(\mathbf{R})$ in addtion to the given properties of u_ε. Then H_ε is a self-adjoint extension of $\pi(\varepsilon)^2 \restriction D(N^2)$.*

PROOF: One easily sees that $W \in \mathfrak{W}_3$ implies $\Omega \in \mathfrak{B}^3(\mathbf{R})$. Hence, by Theorem 7.6 and the mapping property of $\exp(-iM(\,\cdot\,;\,\cdot\,,L_f))$ on the domain $D(N^2)$ given in Theorem 5.10, we obtain the desired result. ∎

We have the following integral-kernel representation of H_ε.

THEOREM 7.8. *Let $W \in C^1_{\mathrm{real}}(\mathbf{R})$ and $a \in \mathcal{N}_d, f \in \mathcal{F}_a$. Then, for all $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ and $s \in \mathbf{R} \setminus \{0\}$,*

$$(e^{isH_\varepsilon}\psi)(x) = \int_{\mathbf{R}^d} e^{-i\Theta_\varepsilon(x,y;s)}\Delta_s(x,y)\psi(y)dy, \tag{7.23}$$

where

$$\begin{aligned}\Theta_\varepsilon(x,y;s) =& \frac{1}{2}u_\varepsilon\left(\frac{ay-ax}{2s}\right)[\Omega(ax)-\Omega(ay)]\\ &-\frac{1}{2s}(y-q)\left[1-\exp\left(u_\varepsilon\left(\frac{ay-ax}{2s}\right)(W(ax)-W(ay))f\right)\right](x-q).\end{aligned} \tag{7.24}$$

PROOF: We need only to apply Theorem 6.7 with $u = -\Omega \otimes u_\varepsilon/2, v = -W \otimes u_\varepsilon$. ∎

7.2 The limit $\varepsilon \to 0$.

We next consider the limit $\varepsilon \to 0$. Let

$$u_{-1}(t) = \frac{1}{t}, \quad t \in \mathbf{R} \setminus \{0\}. \tag{7.25}$$

For $W \in C^1_{\mathrm{real}}(\mathbf{R})$, we define

$$M = M\left(\frac{\Omega \otimes u_{-1}}{2}; W \otimes u_{-1}, L_f\right) \tag{7.26}$$

and set

$$U = e^{-iM}. \tag{7.27}$$

LEMMA 7.9.

$$\mathrm{s}-\lim_{\varepsilon \to 0} U_\varepsilon = U, \quad \mathrm{s}-\lim_{\varepsilon \to 0} U_\varepsilon^{-1} = U^{-1}. \tag{7.28}$$

PROOF: We set

$$M_\varepsilon = M\left(\frac{\Omega \otimes u_\varepsilon}{2}; W \otimes u_\varepsilon, L_f\right).$$

Let $D = \cap_{j,k=0}^{\infty} D((ap)^{-j}\overline{L}_f^k)$. Then, for all $\psi \in D$, we have

$$M_\varepsilon\psi = \frac{1}{2}u_\varepsilon(ap)\Omega(ax)\psi + u_\varepsilon(ap)W(ax)\overline{L}_f\psi,$$
$$M\psi = \frac{1}{2}\Omega(ax)(ap)^{-1}\psi + W(ax)(ap)^{-1}\overline{L}_f\psi.$$

By the functional calculus, one can easily show that $\lim_{\varepsilon\to 0} M_\varepsilon\psi = M\psi$. Note that D is a common core for M_ε and M. Hence, we can apply general convergence theorems ([R-S1, Theorem VIII.25, Theorem VIII.21]) to obtain (7.28). ∎

For $f \in \mathcal{G}_a$ and $W \in \mathfrak{W}_1$, we define an operator Λ acting in $[L^2(\mathbf{R}^d)]$ by

$$\Lambda = e^{\widetilde{f}(ap)^{-1}Y(ax)}. \tag{7.29}$$

Then, by by Proposition 4.4, Λ is an operator-valued Lorentz transformation on $[L^2(\mathbf{R}^d)]$ and

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + f^\mu{}_\nu(ap)^{-1}Y(ax) + \frac{1}{2}f^\mu{}_\lambda f^\lambda{}_\nu((ap)^{-1}Y(ax))^2 \tag{7.30}$$

on $D(\Lambda^\mu{}_\nu) := D(((ap)^{-1}Y(ax))^2)$.

We prepare a general lemma.

LEMMA 7.10. *Let S_k ($k \in \mathrm{N}$) and S be self-adjoint operators on a Hilbert space $\mathcal{H}$ such that S_k converges to S in the strong resolvent sense as $k \to \infty$. Suppose that there exist a dense domain D of $\mathcal{H}$ and a symmetric operator S_0 on $\mathcal{H}$ such that $D \subset D(S_k) \cap D(S_0)$ for all $k \in \mathrm{N}$ and, for all $\psi \in D$, $\lim_{k\to\infty} S_k\psi = S_0\psi$. Then, $S_0 \restriction D \subset S$, i.e., S is a self-adjoint extension of $S_0 \restriction D$.*

PROOF: Using the equality $(S_k - z)(S_k - z)^{-1} = I$, $z \in \mathbf{C} \setminus \mathbf{R}$ (I:identity), we have

$$((S_k - z^*)\psi, (S_k - z)^{-1}\phi) = (\psi, \phi), \quad \psi \in D, \phi \in \mathcal{H}.$$

Taking the limit $k \to \infty$ gives

$$((S_0 - z^*)\psi, (S - z)^{-1}\phi) = (\psi, \phi),$$

which implies that, for all $\eta \in D(S)$, $(S_0\psi, \eta) = (\psi, S\eta)$. Hence, $\psi \in D(S^*) = D(S)$ and $S\psi = S_0\psi$. Thus the desired result follows. ∎

Let

$$\mathcal{D}_a = \cap_{\mu=0}^{d-1} D((ap)^{-1}x_\mu) \cap D(x_\mu) \cap D(p_\mu) \cap D((ap)^{-1}p_\mu) \cap D(p_\mu(ap)^{-1}) \cap D((ap)^{-2}p_\mu) \tag{7.31}$$

Then $\mathcal{D}_a \subset D(\Lambda^\mu{}_\nu \pi^\nu)$ and

$$\begin{aligned}&\Lambda^\mu{}_\nu \pi^\nu \\ &= \pi^\mu + Y(ax)(ap)^{-1} f^\mu{}_\nu p^\nu - Y(ax)(ap)^{-1} f^\mu{}_\nu A^\nu + \frac{1}{2} Y(ax)^2 f^\mu{}_\lambda f^\lambda{}_\nu (ap)^{-2} p^\nu.\end{aligned} \tag{7.32}$$

on $\mathcal{D}_a$.

We set

$$\hat{p}_\mu := U p^\mu U^{-1}, \quad \mu = 0, \cdots, d-1. \tag{7.33}$$

THEOREM 7.11. *Let W, a and f be as in Theorem 7.6. Then*

$$\hat{p}^\mu \supset [\Lambda^\mu{}_\nu \pi^\nu] \restriction \mathcal{D}_a, \tag{7.34}$$

i.e, $\hat{p}^\mu$ is a self-adjoint extension of $[\Lambda^\mu{}_\nu \pi^\nu] \restriction \mathcal{D}_a$.

PROOF: By Theorem 7.6, we have for all $z \in \mathbf{C} \setminus \mathbf{R}$

$$U_\varepsilon (p^\mu - z)^{-1} U_\varepsilon^{-1} = \left(\overline{\hat{\pi}^\mu(\varepsilon)} - z\right)^{-1}.$$

Hence, by Lemma 7.9,

$$\begin{aligned}\mathrm{s} - \lim_{\varepsilon \to 0} \left(\overline{\hat{\pi}^\mu(\varepsilon)} - z\right)^{-1} &= U(p^\mu - z)^{-1} U^{-1} \\ &= (\hat{p}^\mu - z)^{-1}.\end{aligned}$$

Hence $\overline{\hat{\pi}^\mu(\varepsilon)}$ converges to $\hat{p}^\mu$ in the strong resolvent sense as $\varepsilon \to 0$. On the other hand, it is straight forward to see that, for all $\psi \in \mathcal{D}_a$,

$$\lim_{\varepsilon \to 0} \hat{\pi}^\mu(\varepsilon)\psi = \Lambda^\mu{}_\nu \pi^\nu \psi.$$

Thus, applying Lemma 7.10, we obtain (7.34). ∎

Let H_0 be given by (2.13) and

$$H = U H_0 U^{-1} \tag{7.35}$$

LEMMA 7.12. *Let $W \in C^1_{\mathrm{real}}(\mathbf{R})$ and $f \in \mathcal{F}_a$. Then H_ε converges to H in the strong resolvent sense as $\varepsilon \to 0$.*

PROOF: This can be proven by using Lemma 7.9. ∎

THEOREM 7.13. *Let $W \in \mathfrak{W}_3$, $a \in \mathcal{N}_d$, and $f \in \mathcal{G}_a^0$. Then H is a self-adjoint extension of $\pi^2 \restriction D(N^2)$.*

PROOF: By Corollary 7.7 and Lemma 7.4, $D(N^2) \subset D(H_\varepsilon)$ and, for all $\psi \in D(N^2)$,

$$\lim_{\varepsilon \to 0} H_\varepsilon \psi = \pi^2 \psi,$$

where we take u_ε such that $u_\varepsilon \in \mathfrak{B}^2(\mathbf{R})$. Hence, by Lemma 7.12 and Lemma 7.10, we obtain the desired result. ∎

THEOREM 7.14. *Let W, a and f be as in Theorem 7.8. Then, for all $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ and $s \in \mathbf{R} \setminus \{0\}$,*

$$(e^{isH}\psi)(x) = \int_{\mathbf{R}^d} e^{-i\Theta(x,y;s)} \Delta_s(x,y)\psi(y)dy, \tag{7.36}$$

where

$$\begin{aligned}\Theta(x,y;s) = &- \frac{s[\Omega(ax) - \Omega(ay)]}{ax - ay} \\ &- \frac{1}{2s}(y-q)\left[1 - \exp\left(-2s\frac{W(ax) - W(ay)}{ax - ay} f\right)\right](x-q).\end{aligned} \tag{7.37}$$

PROOF: This follows from an application of Theorem 6.7. ∎

REMARK: (i) As easily seen,

$$\lim_{\varepsilon \to 0} \Theta_\varepsilon(x,y;s) = \Theta(x,y;s), \quad x \neq y.$$

(ii) If $f \in \mathcal{G}_a$, then we have

$$\begin{aligned}\Theta(x,y;s) = &- \frac{s[\Omega(ax) - \Omega(ay)]}{ax - ay} \\ &+ \frac{W(ax) - W(ay)}{ax - ay}(y-q)f(x-q) + s\left(\frac{W(ax) - W(ay)}{ax - ay}\right)^2 (y-q)f^2(x-q).\end{aligned}$$

Note that we can be write

$$\frac{W(ax) - W(ay)}{ax - ay}(y-q)f(x-q) = \int_x^y A_\mu(x')dx'^\mu,$$

where the integral on the RHS is taken along the straight line from x to y.

(iii) Formula (7.36) is more general than the corresponding formula in [V-S-H] (Equation (70) there). In [V-S-H] only a special value of the kernel $e^{-i\Theta(x,y;s)}\Delta_s(x,y)$, i.e., $e^{-i\Theta(x,q;s)}\Delta_s(x,q)$ (the case $y=q$ and $f \in \mathcal{G}_a^0$) is computed.

Finally we consider the implications of the preceding results for the Green functions (propagators) of d'Alembertians or Kelin-Gordon operators with perturbations. By Theorem 7.13, the self-adjoint operator $H+m^2$ with $m \geq 0$ (a constant) may be regarded as a generalization of the perturbed Klein-Gordon operator π^2+m^2. The propagator of $H+m^2$ may be defined as the limit $\varepsilon \to 0$ of

$$G_{\pm,\varepsilon} := (H+m^2 \pm i\varepsilon)^{-1} \tag{7.38}$$

in a suitable sense, where $\varepsilon > 0$ is a constant parameter.

Let $\sigma > 0$ be a constant and

$$G^\sigma_{\pm,\varepsilon}(x,y) = \mp i \int_\sigma^\infty e^{-s\varepsilon \pm ism^2 - i\Theta(x,y;\pm s)}\Delta_{\pm s}(x,y)ds. \tag{7.39}$$

These integrals are absolutely convergent.

REMARK: The function $|\Delta_s(x,y)|$ as a function of s has singularity of order $d/2$ at $s=0$ (see (6.2)). This is the reason why the cutoff parameter σ is introduced in the integral in (7.39).

LEMMA 7.15. *Let W, a and f be as in Theorem 7.8. Then, for all $\phi, \psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$,*

$$(\phi, G_{\pm,\varepsilon}\psi) = \lim_{\sigma\to 0} \int_{\mathbf{R}^d \times \mathbf{R}^d} G^\sigma_{\pm,\varepsilon}(x,y)\phi(x)^*\psi(y)dxdy, \tag{7.40}$$

PROOF: By the functional calculus, we have

$$G_{\pm,\varepsilon} = \lim_{\sigma\to 0} \mp i \int_\sigma^\infty e^{\pm is(m^2\pm i\varepsilon)} e^{\pm isH} ds, \tag{7.41}$$

where the integral and the limit $\sigma \to 0$ are taken in the operator norm topology. Hence

$$(\phi, G_{\pm,\varepsilon}\psi) = \lim_{\sigma\to 0} \mp i \int_\sigma^\infty e^{\pm is(m^2\pm i\varepsilon)}(\phi, e^{\pm isH}\psi)ds.$$

Using Theorem 7.14 and applying Fubini's theorem to interchange the ds-integral and the space integral, we obtain the desired result. ∎

REMARK: An expression similar to (7.40) can derived for H_δ ($\delta > 0$) too (see (7.22)).

Let $d \geq 3$. Then, for all $\sigma > 0$, the integrals

$$G^\sigma_\pm(x,y) := \int_\sigma^\infty e^{\pm ism^2 - i\Theta(x,y;\pm s)}\Delta_{\pm s}(x,y)ds \tag{7.42}$$

are absolutely convergent (see (6.2)).

THEOREM 7.16. *Let $d \geq 3$. Let W, a and f be as in Theorem 7.8. Then there exist unique $G_\pm \in \mathcal{S}'(\mathbf{R}^d \times \mathbf{R}^d)$ (the space of tempered distributions on $\mathbf{R}^d \times \mathbf{R}^d$), respectively, such that, for all $\phi, \psi \in \mathcal{S}(\mathbf{R}^d)$,*

$$G_\pm(\phi \otimes \psi) = \lim_{\varepsilon \to 0}(\phi^*, G_{\pm,\varepsilon}\psi). \tag{7.43}$$

Moreover,

$$G_\pm(\phi \otimes \psi) = \lim_{\sigma \to 0} \int_{\mathbf{R}^d \times \mathbf{R}^d} G_\pm^\sigma(x, y)\phi(x)\psi(y)dxdy. \tag{7.44}$$

PROOF: Let $\phi, \psi \in \mathcal{S}(\mathbf{R}^d)$. By Lemma 7.15 and its proof, we can write

$$(\phi^*, G_{\pm,\varepsilon}\psi) = T_{\pm,\varepsilon}^\delta(\phi, \psi) + S_{\pm,\varepsilon}^\delta(\phi, \psi)$$

with

$$T_{\pm,\varepsilon}^\delta(\phi, \psi) = \int_{\mathbf{R}^d \times \mathbf{R}^d} G_{\pm,\varepsilon}^\delta(x, y)\phi(x)\psi(y)dxdy,$$
$$S_{\pm,\varepsilon}^\delta(\phi, \psi) = \mp i \int_0^\delta e^{-s\varepsilon \pm ism^2}(\phi^*, e^{\pm isH}\psi)ds,$$

where $\delta > 0$ is arbitrary. We have

$$|G_{\pm,\varepsilon}^\delta(x, y)\phi(x)\psi(y)| \leq C_\delta|\phi(x)||\psi(y)|, \tag{7.45}$$

where

$$C_\delta = \frac{1}{2^d\pi^{d/2}} \int_\delta^\infty \frac{1}{s^{d/2}} ds < \infty,$$

and

$$\lim_{\varepsilon \to 0} G_{\pm,\varepsilon}^\delta(x, y)\phi(x)\psi(y) = G_\pm^\delta(x, y)\phi(x)\psi(y).$$

Hence, by the dominated convergence theorem,

$$T_\pm^\delta(\phi, \psi) := \lim_{\varepsilon \to 0} T_{\pm,\varepsilon}^\delta(\phi, \psi)$$

exist and

$$T_\pm^\delta(\phi, \psi) = \int_{\mathbf{R}^d \times \mathbf{R}^d} G_\pm^\delta(x, y)\phi(x)\psi(y)dxdy. \tag{7.46}$$

Estimate (7.45) implies that

$$|T_\pm^\delta(\phi, \psi)| \leq C_\delta||\phi||_{L^1(\mathbf{R}^d)}||\psi||_{L^1(\mathbf{R}^d)}. \tag{7.47}$$

As for $S^{\delta}_{\pm,\varepsilon}(\phi,\psi)$, we have

$$|e^{-s\varepsilon\pm ism^2}(\phi^*, e^{\pm isH}\psi)| \le ||\phi||_{L^2(\mathbf{R}^d)}||\psi||_{L^2(\mathbf{R}^d)}$$

and

$$\lim_{\varepsilon\to 0} e^{-s\varepsilon\pm ism^2}(\phi^*, e^{\pm isH}\psi) = e^{\pm ism^2}(\phi^*, e^{\pm isH}\psi).$$

Hence, by the dominated convergence theorem,

$$S^{\delta}_{\pm}(\phi,\psi) := \lim_{\varepsilon\to 0} S^{\delta}_{\pm,\varepsilon}(\phi,\psi)$$

exist and

$$S^{\delta}_{\pm}(\phi,\psi) = \mp i\int_0^{\delta} e^{\pm ism^2}(\phi^*, e^{\pm isH}\psi)ds \tag{7.48}$$

with

$$|S^{\delta}_{\pm}(\phi,\psi)| \le \delta||\phi||_{L^2(\mathbf{R}^d)}||\psi||_{L^2(\mathbf{R}^d)}. \tag{7.49}$$

Thus

$$\widetilde{G}_{\pm}(\phi,\psi) := \lim_{\varepsilon\to 0}(\phi^*, G_{\pm,\varepsilon}\psi)$$

exist and

$$\widetilde{G}_{\pm}(\phi,\psi) = T^{\delta}_{\pm}(\phi,\psi) + S^{\delta}_{\pm}(\phi,\psi). \tag{7.50}$$

By (7.47) and (7.49), the bilinear functionals $\widetilde{G}_{\pm}(\,\cdot\,,\,\cdot\,)$ on $\mathcal{S}(\mathbf{R}^d)\times\mathcal{S}(\mathbf{R}^d)$ are jointly continuous. Hence, by the nuclear theorem, there exist unique $G_{\pm}\in\mathcal{S}'(\mathbf{R}^d\times\mathbf{R}^d)$, respectively, such that, for all $\phi,\psi\in\mathcal{S}(\mathbf{R}^d)$, (7.43) holds. Taking the limit $\delta\to 0$ in (7.50) and using (7.46) and (7.49), we obtain (7.44). ∎

VIII. Concluding Remarks – A View-Point in Connection with Representation of the Canonical Commutation Relations

In this paper we have developed an operator theory concerning a family of strongly commuting self-adjoint operators in $L^2(\mathbf{R}^d)$ which are associated with some objects in the Minkowski space $\mathbf{M}^d$. We have constructed a class of self-adjoint operators in $L^2(\mathbf{R}^d)$ (see (5.36)) which may be regarded as perturbed d'Alembertians in the sense of unitary groups and whose unitary groups have integral-kernel representations in explicit forms (Theorem 6.7). We have shown that the operator theory given here can be applied to the external field problem of a charged particle for a class of vector potentials and clarifies an algebraic-analytic structure behind it. We expect that the framework of the present work can be extended to a more general one that enables us to treat the external field problem for a Dirac particle, where Dirac operators on $\mathbf{M}^d$ are the main objects to be analyzed. This aspect will be considered in a separate paper.

In concluding this paper, we want to give a brief remark on the mathematical meaning of what is done in Section VII. The position operator $x = (x^\mu)$ and the gauge covariant momentum operator π satisfy the following commutation relations on a suitable domain:

$$[x^\mu, x^\nu] = 0, \tag{8.1}$$

$$[x^\mu, \pi_\nu] = -i\delta^\mu_\nu, \tag{8.2}$$

$$[\pi_\mu, \pi_\nu] = -iF_{\mu\nu}, \quad \mu, \nu = 0, \cdots, d-1. \tag{8.3}$$

If $F_{\mu\nu} \equiv 0$, then these commutation relations are just the canonical commutation relations (CCR) with d degrees of freedom. Hence (8.1)–(8.3) may be regarded as a deformation of the CCR in the sense that π_μ's are not necessarily commutative ; or we may say that (8.1)–(8.3) are a representation of "partially broken CCR"(PB-CCR). Theorem 7.11 shows that, for a class of vector potentials, there exists an operator-valued Lorentz transformation Λ on $L^2(\mathbf{R}^d)$ such that $\Lambda\pi$ is unitarily equivalent on a dense domain to the canonical momentum operator p. From a representation-theoretical point of view of CCR, this can be rephrased as follows: Let U be defined by (7.27). Then the set $\{Ux^\mu U^{-1}, \Lambda_{\mu\nu}\pi^\nu\}_{\mu=0}^{d-1}$ of symmetric operators satisfy the CCR with d degrees of freedom. If $\Lambda_{\mu\nu}\pi^\nu$ is essentially self-adjoint on $\mathcal{D}_a$ (see (7.31))(unfortunately, we have not been able to prove this), then the representation $\{Ux^\mu U^{-1}, \overline{\Lambda_{\mu\nu}\pi^\nu \restriction \mathcal{D}_a}\}_{\mu=0}^{d-1}$ of CCR is equivalent to the Schrödinger representation $\{x^\mu, p_\mu\}_{\mu=0}^{d-1}$. Thus, for a class of vector potentials, the PB-CCR (8.1)–(8.3) are related, via an operator-valued Lorentz transformation, to the CCR with d degrees of freedom. This may be a remarkable mechanism to be further investigated. It is an interesting problem to find wider classes of vector potentials for which such a mechanism exists.

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