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MARKOV PROPERTY AND COKERNELS OF LOCAL OPERATORS

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MARKOV PROPERTY AND COKERNELS OF LOCAL OPERATORS

KOICHIRO IWATA AND JÖRG SCHÄFER

ABSTRACT. We discuss Gaussian generalized random fields indexed by smooth sections of vector bundles with respect to Markov properties. We propose a new set-up which is suitable for the present question and within which new phenomena are detected naturally. In particular, we give a counterexample to the belief that locality in the RKHS implies germ Markov property. We also prove the close connection between Markov property and cokernels of local operators. Furthermore we prove the Markov property for a very degenerate Gaussian random field.

1. INTRODUCTION

In a paper by Preiss and Kotecky [8] delicate points concerning Markov properties of generalized random fields are discussed. They clearly pointed out the origin of confusions so far and they selected a presumably most suitable definition, which we shall also follow.

Let $\{X(\varphi)\,;\,\varphi \in C_0^\infty(\mathbb{R}^d)\}$ be a generalized random field defined on a complete probability space $(\Omega, \mathcal{F}, P)$. If D is an open subset of $\mathbb{R}^d$, then we define

$$\mathcal{F}_D := \sigma\{X(\varphi)\,;\,\operatorname{supp}\varphi \subset D\} \vee \mathcal{N},$$

where $\mathcal{N}$ is the trivial sub σ-field of $\mathcal{F}$. On the other hand we set

$$\mathcal{F}_C := \bigcap \{\mathcal{F}_D\,;\, C \subset D, D \text{ open}\}$$

for closed subsets C. The so defined family of sub σ-fields of $\mathcal{F}$ will be called the canonical filtration subordinate to $\{X(\varphi)\}$. We shall use $\overline{A}$ to denote the closure of a set A, sometimes we also use $\{A\}^-$.

(1.1) Definition. We say that $\{X(\varphi)\}$ is *germ Markov*, if $\mathcal{F}_{\overline{D}} \perp \mathcal{F}_{\mathbb{R}^d\setminus D} \,|\, \mathcal{F}_{\partial D}$ for any open subset D. (Two sub σ-fields $\mathcal{F}_{\overline{D}}$ and $\mathcal{F}_{\mathbb{R}^d\setminus D}$ are conditionally independent given $\mathcal{F}_{\partial D}$.)

A superstition on Gaussian generalized random fields used to be that the germ Markov property is equivalent to the locality of the reproducing kernel Hilbert space (RKHS). Roughly speaking, locality means that the inner product $(\cdot,\cdot)$ in

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the RKHS satisfies $(f,g) = 0$ whenever the support of f does not intersect the support of g. Here, f and g denote elements of the RKHS, usually realized as some function space or space of measures. Thus, speaking of supports of f and g respectively makes sense. A counterexample, given in [8], to the above assertion is a mean zero Gaussian system with

$$E[X(\varphi)^2] = \int_{\mathbb{R}} \left\{ \varphi(x)^2 + (\varphi'(x))^2 \right\} dx. \tag{1.2}$$

We can check that $\mathcal{F}_{\overline{D}} \perp \mathcal{F}_{\mathbb{R}\setminus D}$ and $\mathcal{F}_{\partial D} = \mathcal{N}$ for any open subset by examining the corresponding condition in the Sobolev space carrying the inner product (1.2). Hence $\{X(\varphi)\}$ is germ Markov. But the bilinear form for the reproducing kernel Hilbert space has an integral kernel $\frac{1}{2}e^{-|x-y|}$ and therefore it is not local.

The superstition in former days was caused by the misunderstanding that the following would hold whenever open subsets D_+, D_- cover $\mathbb{R}^d$.

$$\mathcal{F}_{D_+} = \mathcal{F}_{\mathbb{R}^d\setminus D_-} \vee \mathcal{F}_{D_+\cap D_-}. \tag{1.3}$$

However, as for the Gaussian system characterized by (1.2) we have $\mathcal{F}_{\mathbb{R}} \neq \mathcal{F}_{[0,\infty)} \vee \mathcal{F}_{(-\infty,0)}$, because the latter is independent of $\sigma\{X(f)\}$, where $f(x) = e^{-|x|}$. Thus this example also serves as a counterexample to (1.3).

Based on the discussion so far, Preiss and Kotecky [8] proposed for the so-called "MI-Markov property" which is equivalent to the following definition.

(1.4) Definition. We say that $\{X(\varphi)\}$ is *MI-Markov*, if it is germ Markov and satisfies

$$\mathcal{F}_{\mathbb{R}^d} = \mathcal{F}_{\mathbb{R}^d\setminus D} \vee \mathcal{F}_D \text{ for all open subsets } D \text{ of } \mathbb{R}^d. \tag{1.5}$$

(1.6) Remark. It is known that if the "covering" condition

$$\mathcal{F}_{D_1} \perp \mathcal{F}_{D_2} \,|\, \mathcal{F}_{D_1\cap D_2} \tag{1.7}$$

holds for any open *covering* $\{D_1, D_2\}$ of $\mathbb{R}^d$, then the MI-Markov property follows, see [2]. We shall prove (see Theorem 2a.7) that, actually, the MI-Markov property implies the former property. Referring to (1.7) we suggest to use the alternative name "c"-Markov where "c" stands for "covering" instead of MI. Postponing the proof to Appendix 2.a, however, we shall at first distinguish both properties in this note.

In order to compensate for the incompleteness of this note concerning historical developments we refer to [3], [6], [11] and references therein. The present topic was also treated in the framework of Dirichlet spaces, see [5], [9] and [10]. However, within this framework one can only handle scalar random fields, i.e. fields indexed by scalar valued *functions*.

Therefore, in Section 2 we introduce a new set-up which fits Gaussian generalized random fields indexed by sections of vector bundles and we explain the relation between a new Markov property to be defined below and the locality. Our new formulation can be proven to be equivalent to (1.4) as far as Gaussian systems are

considered, see Corollary 2.21. In addition we shall mention the connection to the results mentioned above. Within our set-up degenerate fields, i.e. with degenerate covariance are easy to handle. Actually in the context of multicomponent fields very large degeneracy naturally happens, and in contrast to the non-degenerate scalar situation genuinely new phenomena arise. This is shown in Section 3, where we report on a section-indexed Gaussian generalized random field whose reproducing kernel Hilbert space has locality but which is *not* germ Markov. This is one of the two main theorems of this note. Moreover, for a class of Gaussian generalized random fields we give a necessary and sufficient condition to be Markov (Theorem 3.16), our second main theorem. The criterion we give is clearly different from those appearing in the context of Dirichlet spaces, see [10]. In particular, we see that locality of the RKHS is only part of the necessary condition. Together with the example of Preiss and Kotecky this shows the subtleties in the relation between germ Markov property and locality. As a by-product of our presentation we clarify the relation between the germ-Markov property and the c-Markov property, see Remark 2a.8. We show that in the context of *generalized* random fields it is not the germ, but rather the c-Markov property which is natural and intuitive. Furthermore, in Chapter 4 we show that our approach is well-suited to prove the Markov property even in very degenerate situations, which by standard methods used in the past were difficult to handle. The example we give, Theorem 4.13, has its origins in constructive quantum field theory, in particular in quantum electrodynamics.

2. The Markov property and the locality

We formulate the question in terms of Hilbert space category, since we shall discuss Gaussian systems. Let $\mathbf{E} \to M$ be a real vector bundle over a paracompact C^∞-manifold. We use the notation $\mathcal{D} = \mathcal{D}(\mathbf{E})$ for the totality of C^∞-sections of $\mathbf{E}$ with compact supports. $\mathcal{D}$ is equipped with the Schwartz topology. Our objects of study are continuous linear maps

$$\gamma\colon \mathcal{D} \longrightarrow \mathcal{H}$$

with real Hilbert spaces $\mathcal{H}$ as targets, in which the images are dense. Given $\gamma\colon \mathcal{D} \longrightarrow \mathcal{H}$, we introduce a family of closed subspaces $\{\mathcal{H}(D)\}$ (resp. $\{\mathcal{H}(C)\}$) of $\mathcal{H}$ indexed by open (resp. closed) subsets of M as follows:

$$\begin{aligned} \mathcal{H}(D) &:= \overline{\{\gamma(\varphi)\,;\ \operatorname{supp}\varphi \subset D\}}, \\ \mathcal{H}(C) &:= \bigcap \{\mathcal{H}(D)\,;\ C \subset D, D \text{ open}\}. \end{aligned} \tag{2.1}$$

In the study of a Gaussian generalized random field $\{X(\varphi)\,;\ \varphi \in \mathcal{D}\}$, the map $\gamma\colon \mathcal{D} \longrightarrow \mathcal{H}$ is induced by the correspondence $\varphi \longmapsto X(\varphi)$ and therefore $\mathcal{H}$ is realized as a closed subspace of the $\mathcal{L}^2$-space on $(\Omega, \mathcal{F}, P)$. If this is the case we say that $\gamma\colon \mathcal{D} \longrightarrow \mathcal{H}$ is canonically constructed from $\{X(\varphi)\}$. We here adopt the conclusion of Lemma 3 from [3] as a definition of Markov property. In our context it reads as follows:

(2.2) Definition. The map $\gamma\colon \mathcal{D} \longrightarrow \mathcal{H}$ is called *Markov* if

$$\mathcal{H}(\overline{D}) = \mathcal{H}(\partial D) + \mathcal{H}(M \setminus D)^{\perp} \text{ for all open subsets } D,$$

where $\perp$ means the orthogonal complement.

Before relating our definition to Markov properties of the underlying random fields, see Corollary 2.21 and Theorem 2a.7, we first draw some conclusions. We recall the following fact, which is a standard tool to push the argument in the present subject and whose proof is elementary and thus omitted.

(2.3) Lemma. *Let $\mathcal{K}_0$, $\mathcal{K}_1$ and $\mathcal{K}_2$ be closed subspaces of $\mathcal{H}$.*
(i) *$\mathcal{K}_1 = \mathcal{K}_0 + \mathcal{K}_2^{\perp}$ and $\mathcal{K}_0 \subset \mathcal{K}_2$ is equivalent to that $\mathcal{K}_1 \cap \mathcal{K}_2 = \mathcal{K}_0$ and $\mathcal{K}_2^{\perp} \subset \mathcal{K}_1$.*
(ii) *$\mathcal{K}_1 = \mathcal{K}_0 + (\mathcal{K}_1 \cap \mathcal{K}_2^{\perp})$ and $\mathcal{K}_0 \subset \mathcal{K}_2$ is equivalent to that $\mathcal{K}_1 \cap \mathcal{K}_2 = \mathcal{K}_0$ and $\mathcal{K}_2^{\perp} = (\mathcal{K}_1 \cap \mathcal{K}_2^{\perp}) + (\mathcal{K}_1 + \mathcal{K}_2)^{\perp}$.*

With its help we get the following.

(2.4) Proposition. *The Markov property of the map $\gamma\colon \mathcal{D} \longrightarrow \mathcal{H}$ is equivalent to each of the following properties:*
(i) *$\mathcal{H}(M \setminus D)^{\perp} \subset \mathcal{H}(\overline{D})$ and $\mathcal{H}(\partial D) = \mathcal{H}(\overline{D}) \cap \mathcal{H}(M \setminus D)$ for all open subsets D.*
(ii) *$\mathcal{H} = \mathcal{H}(\overline{D}) + \mathcal{H}(M \setminus D)$ and $\mathcal{H}(\overline{D}) = \mathcal{H}(\partial D) + (\mathcal{H}(\overline{D}) \cap \mathcal{H}(M \setminus D)^{\perp})$ for all open subsets D.*

Proof. Suppose γ is Markov. Then we get $\mathcal{H}(\overline{D}) \cap \mathcal{H}(M \setminus D)^{\perp} = \mathcal{H}(M \setminus D)^{\perp}$ and $\mathcal{H}(\overline{D})^{\perp} \subset \mathcal{H}(M \setminus D)$. Clearly the former and (2.2) implies the second condition in (ii). On the other hand $\mathcal{H} = \mathcal{H}(\overline{D}) + \mathcal{H}(\overline{D})^{\perp} \subset \mathcal{H}(\overline{D}) + \mathcal{H}(M \setminus D)$ follows from the latter. Thus we get (ii). We shall prove the implication (ii) $\Rightarrow$ (i). Choosing $\mathcal{K}_0 = \mathcal{H}(\partial D)$, $\mathcal{K}_1 = \mathcal{H}(\overline{D})$, and $\mathcal{K}_2 = \mathcal{H}(M \setminus D)$ we apply (ii) of (2.3). Then it follows that $\mathcal{H}(\overline{D}) \cap \mathcal{H}(M \setminus D) = \mathcal{H}(\partial D)$ and $\mathcal{H}(M \setminus D)^{\perp} = \mathcal{H}(\overline{D}) \cap \mathcal{H}(M \setminus D)^{\perp} \subset \mathcal{H}(\overline{D})$. Finally we obtain the equivalence (i) $\Leftrightarrow$ (2.2) by applying (i) of (2.3). $\square$

Thus we see that the Markov property defined in (2.2) consists of two conditions: The first one reads as

$$\mathcal{H}(M \setminus D)^{\perp} \subset \mathcal{H}(\overline{D}), \tag{2.5}$$

for all open subsets D. This condition has been formulated as locality of the dual (so called reproducing kernel) Hilbert space. In our set-up locality is rather called *semi-locality*, see also (2.14), because as we shall see, it is only half of the relevant condition. Namely, the second important condition (which by reasons to be explained below we like to call *split*) is given by

$$\mathcal{H}(\partial D) = \mathcal{H}(\overline{D}) \cap \mathcal{H}(M \setminus D) \tag{2.6}$$

for all open subsets D, which so far received less attention than (2.5). The reason is probably because within the framework of Dirichlet spaces (and scalar random fields) (2.6) is trivially fulfilled, compare [10]. However, within the framework of generalized random fields indexed by sections of bundles it appears very naturally as a necessary non-trivial condition.

Now we use dualization to obtain an equivalent characterization of the Markov property which is well-suited to discuss locality. We denote the topological dual space of $\mathcal{D}$ by $\mathcal{D}'$. Therefore $\mathcal{D}'$ consists of distributional sections. Unless the elements of $\mathcal{H}$ are canonically interpreted as distributions in such a way that the composition $\mathcal{D} \xrightarrow{\gamma} \mathcal{H} \hookrightarrow \mathcal{D}'$ yields a natural inclusion $\mathcal{D} \hookrightarrow \mathcal{D}'$, there is no hope to see how the shape of the index sets is reflected in the filtration $\{\mathcal{H}(D)\}$ (resp. $\{\mathcal{H}(C)\}$). But, since the map γ has dense image in its target, the transposed map γ^* realizes the topological dual $\mathcal{H}'$ of $\mathcal{H}$ as a subspace of $\mathcal{D}'$. We shall detect the shape of the index sets in the following way. Let $\kappa\colon \mathcal{H} \longrightarrow \mathcal{H}'$ be the Riesz isomorphism given by ${}_{\mathcal{H}'}\langle \kappa(u), v\rangle_{\mathcal{H}} = (u, v)_{\mathcal{H}}$, for $u, v \in \mathcal{H}$. Therefore we end up with the commutative diagram

$$
\begin{array}{ccccc}
\mathcal{H} \times \mathcal{D} & \xrightarrow{\kappa \times \gamma} & \mathcal{H}' \times \mathcal{H} & \xrightarrow{\text{canonical pairing}} & \mathbb{R} \\
\| & & & & \| \\
\mathcal{H} \times \mathcal{D} & \xrightarrow{(\gamma^* \circ \kappa) \times \mathrm{id}} & \mathcal{D}' \times \mathcal{D} & \xrightarrow{\text{canonical pairing}} & \mathbb{R}
\end{array}
\tag{2.7}
$$

and obtain the following characterization of the relevant subspaces in terms of conditions on injections of the dual space of $\mathcal{H}$ into distributions.

(2.8) Lemma. *Let D be a open subset and C be a closed subset. Then*

$$\mathcal{H}(D)^{\perp} = \{u \in \mathcal{H};\ \mathrm{supp}\ \gamma^* \circ \kappa(u) \subset M \setminus D\},$$
$$\mathcal{H}(C)^{\perp} = \overline{\{u \in \mathcal{H};\ \mathrm{supp}\ \gamma^* \circ \kappa(u) \subset M \setminus C\}}.$$

Proof. Let F be a closed subset of M and $T \in \mathcal{D}'$. Then, by definition, $\mathrm{supp}\ T \subset F$ if and only if $\langle T, \varphi\rangle = 0$ for all $\varphi \in \mathcal{D}$ with $\mathrm{supp}\ \varphi \subset M \setminus F$. The first statement immediately follows from (2.1), (2.7), and this characterization. Considering the orthogonal complement of the defining relation (2.1) of $\mathcal{H}(C)$, we see by (2.7) that

$$\begin{aligned}
\mathcal{H}(C)^{\perp} &= \Big\{\bigcup \{\mathcal{H}(D)^{\perp};\ D \text{ open}, D \supset C\}\Big\}^{-}, \\
&= \Big\{\{u \in \mathcal{H};\ \mathrm{supp}\ \gamma^* \circ \kappa(u) \subset M \setminus D \text{ for some } D \text{ open}, D \supset C\}\Big\}^{-},
\end{aligned}$$

and hence the second statement follows. □

The power of the dualization can be seen in the proof of the following.

(2.9) Lemma. *If (2.5) holds for all open subsets D, then $\mathcal{H}(M \setminus D)^{\perp} \subset \mathcal{H}(D)$ for all open subsets D.*

Proof. By virtue of Lemma 2.8, all we have to show is that if $u \in \mathcal{H}$ satisfies $\mathrm{supp}\ \gamma^* \circ \kappa(u) \subset D$, then $u \in \mathcal{H}(D)$. To this end, we choose an open set G so that $\mathrm{supp}\ \gamma^* \circ \kappa(u) \subset G \subset \overline{G} \subset D$. We see by Lemma 2.8 that u is actually contained in $\mathcal{H}(M \setminus G)^{\perp}$. Then (2.5), with D replaced by G, tells us that $u \in \mathcal{H}(\overline{G})$. Since $\overline{G} \subset D$, we conclude that $u \in \mathcal{H}(D)$. □

(2.10) Remark. Since $\mathcal{H} = \mathcal{H}(M \setminus D)^{\perp} + \mathcal{H}(M \setminus D)$ is always valid, we see by virtue of (2.9) that if (2.5) holds for *all* open subsets D, then $\mathcal{H} = \mathcal{H}(D) + \mathcal{H}(M \setminus D)$. This condition is clearly stronger than the first condition in (ii) of (2.4).

The dual characterization of the Markov property can now easily be given.

(2.11) Lemma. (2.5) *is equivalent to*

$$\{v \in \mathcal{H}' ; \operatorname{supp} \gamma^*(v) \subset M \setminus \overline{D}\} \perp \{v \in \mathcal{H}' ; \operatorname{supp} \gamma^*(v) \subset D\}, \tag{2.12}$$

and (2.6) *is equivalent to*

$$\begin{aligned} &\{v \in \mathcal{H}' ; \operatorname{supp} \gamma^*(v) \subset M \setminus \partial D\} \subset \\ &\Big\{\{v \in \mathcal{H}' ; \operatorname{supp} \gamma^*(v) \subset M \setminus \overline{D}\} + \{v \in \mathcal{H}' ; \operatorname{supp} \gamma^*(v) \subset D\}\Big\}^- . \end{aligned} \tag{2.13}$$

Proof. Apply Lemma 2.8 and keep in mind that $\mathcal{H}(\partial D) \subset \mathcal{H}(\overline{D}) \cap \mathcal{H}(M \setminus D)$. □

Thus we see that the Markov property is equivalent to

$$\begin{aligned} &\overline{\{v \in \mathcal{H}' ; \operatorname{supp} \gamma^*(v) \subset M \setminus \partial D\}} = \\ &\overline{\{v \in \mathcal{H}' ; \operatorname{supp} \gamma^*(v) \subset M \setminus \overline{D}\}} \oplus \overline{\{v \in \mathcal{H}' ; \operatorname{supp} \gamma^*(v) \subset D\}}. \end{aligned}$$

We formulate the condition (2.12) as semi-locality. Our objects are continuous linear injections

$$\gamma^* : \mathcal{H}' \longrightarrow \mathcal{D}'$$

with real Hilbert spaces $\mathcal{H}'$ as sources. (Here we have used the notations γ^* and $\mathcal{H}'$, in order to keep the continuity of the arguments from the preceding paragraphs.)

(2.14) Definition. The map $\gamma^* : \mathcal{H}' \longrightarrow \mathcal{D}'$ is called *semi-local*, if two elements in $\mathcal{H}'$ are mutually orthogonal whenever they have disjoint supports.

Obviously the semi-locality of $\gamma^* : \mathcal{H}' \longrightarrow \mathcal{D}'$ is the same as that (2.12) holds for all open subsets. In [3] the semi-locality was named the condition (A_1), which is equivalent to the condition (A_2) in [3], provided that the dual generalized random field exists.

On the other hand one reads (2.13) as that if the support of v is contained in a union of two disjoint open sets then v can be split into two parts whose supports are contained in only one component, which might explain our choice of the terminology "split". We note that the property "split" does not directly refer to the inner product but rather to the underlying vector topology.

To relate the previous definition to the Markov property of the underlying random field we have to investigate the σ-fields carefully. Now the map $\gamma : \mathcal{D} \longrightarrow \mathcal{H}$ is canonically constructed from a Gaussian generalized random field $\{X(\varphi)\}$ indexed by $\mathcal{D}$. For a subspace $\mathcal{K}$ of $\mathcal{H}$ we use the short-hand notation $\sigma\{\mathcal{K}\}$ to denote $\sigma\{\mathcal{K}\} \vee \mathcal{N}$, where $\mathcal{K}$ is considered as a set of random variables. Recall that $\{\mathcal{F}_D\}$ is the canonical filtration subordinate to $\{X(\varphi)\}$. Then $\mathcal{F}_D = \sigma\{\mathcal{H}(D)\}$ for an open set D follows immediately, while the corresponding relation for a closed set C needs to be proved. One way to accomplish this is to use the well-known Wiener-Itô chaos decomposition of $\mathcal{L}^2(\Omega, \mathcal{F}, P)$. However we will provide a quick alternative proof. The following version of the martingale convergence theorem will be used.

(2.15) Lemma. *Let H be a Hilbert space and $\mathcal{P}$ be a set of orthogonal projectors in H. Suppose $\mathcal{P}$ is directed in the following sense: for any pair $\Pi_1, \Pi_2 \in \mathcal{P}$ there exists $\Pi_3 \in \mathcal{P}$ with $\Pi_3 < \Pi_1$ and $\Pi_3 < \Pi_2$. Then given $x \in H$ there is a sequence $\{\Pi_n\} \subset \mathcal{P}$ such that $\Pi_n x \to \inf \mathcal{P} x$, where $\inf \mathcal{P}$ is the orthogonal projector onto $\bigcap_{\Pi \in \mathcal{P}} \text{Image } \Pi$.*

Proof. We choose a sequence $\{\Pi_n\} \subset \mathcal{P}$ so that $\|\Pi_n x\|^2 \leq \inf_{\Pi \in \mathcal{P}} \|\Pi x\|^2 + n^{-1}$. Since $\mathcal{P}$ is directed, we may assume that $\Pi_n > \Pi_{n+1}$. Then it follows that $\Pi_n x \to y$ for some $y \in H$. We show that $y = \inf \mathcal{P} x$. For each $\Pi \in \mathcal{P}$, using that $\mathcal{P}$ is directed, we can choose a sequence $\{\Pi'_n\} \subset \mathcal{P}$ such that $\Pi'_n < \Pi_n$ and $\Pi'_n < \Pi$ for $n = 1, 2, \ldots$. Then $\Pi'_n x \to y$, because

$$\|\Pi_n x - \Pi'_n x\|^2 = \|\Pi_n x\|^2 - \|\Pi'_n x\|^2 \leq \inf_{\Pi \in \mathcal{P}} \|\Pi x\|^2 + \frac{1}{n} - \|\Pi'_n x\|^2 \leq \frac{1}{n}.$$

Therefore, since $\Pi > \Pi'_n$, we get $\Pi y = \lim_{n \to \infty} \Pi \Pi'_n x = \lim_{n \to \infty} \Pi'_n x = y$, which implies that $\inf \mathcal{P} y = y$. Thus we see that $y = \inf \mathcal{P} y = \lim_{n \to \infty} \inf \mathcal{P} \Pi_n x = \inf \mathcal{P} x$. □

Let $\Pi_{\mathcal{K}}$ be the orthogonal projector from $\mathcal{H}$ onto a closed subspace $\mathcal{K}$. We shall take advantage of one of the most fundamental formula for Gaussian systems, namely

$$E\left[e^{\sqrt{-1}X} \,\middle|\, \sigma\{\mathcal{K}\}\right] = e^{\sqrt{-1}\Pi_{\mathcal{K}} X - \frac{1}{2}E[(X - \Pi_{\mathcal{K}} X)^2]}, \quad \text{for } X \in \mathcal{H}. \tag{2.16}$$

In particular we immediately see that $E\left[X \,\middle|\, \sigma\{\mathcal{K}\}\right] = \Pi_{\mathcal{K}} X$.

(2.17) Lemma. (i) *Let C be a closed subset of M. Then $\mathcal{F}_C = \sigma\{\mathcal{H}(C)\}$.*
(ii) *Let $\mathcal{K}$, $\mathcal{K}_1$ and $\mathcal{K}_2$ be closed subspaces in $\mathcal{H}$. Then $\sigma\{\mathcal{K}\} = \sigma\{\mathcal{K}_1\} \vee \sigma\{\mathcal{K}_2\}$ if and only if $\mathcal{K} = \overline{\mathcal{K}_1 + \mathcal{K}_2}$,*

Proof. Let $X \in \mathcal{H}$. Thanks to the martingale convergence theorem (2.15), there exist two decreasing sequences $\{U_n^1\}$ and $\{U_n^2\}$ of open sets containing C such that $\Pi_{\mathcal{H}(U_n^1)} X \to \Pi_{\mathcal{H}(C)} X$ and $E\left[e^{\sqrt{-1}X} \,\middle|\, \mathcal{F}_{U_n^2}\right] \to E\left[e^{\sqrt{-1}X} \,\middle|\, \mathcal{F}_C\right]$ in $\mathcal{L}^2$. One easily shows that the above convergence also occurs if $\{U_n^1\}$ respectively $\{U_n^2\}$ is replaced by $\{U_n^1 \cap U_n^2\}$. Therefore invoking (2.16) with keeping in mind that $\mathcal{F}_D = \sigma\{\mathcal{H}(D)\}$ for open sets we obtain that

$$E\left[e^{\sqrt{-1}X} \,\middle|\, \mathcal{F}_C\right] = e^{\sqrt{-1}\Pi_{\mathcal{H}(C)} X - \frac{1}{2}E[(X - \Pi_{\mathcal{H}(C)} X)^2]},$$

where the right hand side is $\sigma\{\mathcal{H}(C)\}$-measurable. This shows that $\mathcal{F}_C = \sigma\{\mathcal{H}(C)\}$. On the other hand, as a consequence of the formula $E\left[X \,\middle|\, \sigma\{\mathcal{K}\}\right] = \Pi_{\mathcal{K}} X$, we see that $\sigma\{\mathcal{K}\} = \sigma\{\mathcal{K}_1 + \mathcal{K}_2\}$ if and only if $\mathcal{K} = \overline{\mathcal{K}_1 + \mathcal{K}_2}$. Thus, by checking that $\sigma\{\mathcal{K}_1 + \mathcal{K}_2\} = \sigma\{\mathcal{K}_1\} \vee \sigma\{\mathcal{K}_2\}$ in general, we reach the statement (ii). □

We shall prove now that Definition 2.2 is actually another formulation of the MI-Markov property for Gaussian generalized random fields.

(2.18) Theorem. *Let $\{X(\varphi);\ \varphi \in \mathcal{D}\}$ be a Gaussian generalized random field and the map $\gamma\colon \mathcal{D} \longrightarrow \mathcal{H}$ be canonically constructed from $\{X(\varphi)\}$. The germ Markov property is equivalent to the second condition in* (ii) *of* (2.4). *Moreover if $\{X(\varphi)\}$ is germ Markov then* (2.6) *holds.*

Proof. Assume that $\mathcal{H}(\overline{D}) = \mathcal{H}(\partial D) + (\mathcal{H}(\overline{D}) \cap \mathcal{H}(M \setminus D)^{\perp})$ holds. Let $X \in \mathcal{H}(\overline{D})$. Then $\Pi_{\mathcal{H}(M\setminus D)}X = \Pi_{\mathcal{H}(\partial D)}X$ and, due to (2.16), it follows that

$$E[e^{\sqrt{-1}X} \,|\sigma\{\mathcal{H}(M \setminus D)\}] = E[e^{\sqrt{-1}X} \,|\sigma\{\mathcal{H}(\partial D)\}]. \tag{2.19}$$

This implies the germ Markov property, since $\mathcal{F}_C = \sigma\{\mathcal{H}(C)\}$ for any closed subset. The converse implication follows from the formula $E[X \,|\sigma\{\mathcal{K}\}] = \Pi_{\mathcal{K}}X$ for $X \in \mathcal{H}$. Finally (ii) of Lemma 2.3 provides the second assertion. □

The MI-Markov property is by definition the combination of the germ Markov property and the validity of

$$\mathcal{F}_M = \mathcal{F}_D \vee \mathcal{F}_{M\setminus D} \text{ for all open subsets } D \text{ of } M. \tag{2.20}$$

(2.21) Corollary. *The MI-Markov property is equivalent to the Markov property of γ. The latter is equivalent to that both* (2.12) *and* (2.13) *hold for all open subsets.*

Proof. We see by (ii) of (2.17) that (2.20) implies $\mathcal{H} = \overline{\mathcal{H}(D) + \mathcal{H}(M \setminus D)}$ and hence the first condition in (ii) of (2.4) follows. Therefore, combining with (2.18), we can deduce the Markov property of γ from the MI-Markov property. Conversely we assume the Markov property of γ. Then the germ Markov property follows by (2.18) and (2.20) follows by (2.4), (2.10) and (ii) of (2.17). Finally we immediately get the second statement by Proposition 2.4 and Lemma 2.11. □

2.A Appendix: The Equivalence of the MI- and c-Markov Property and Relation to the Germ Markov property

In this section we prove the equivalence of the c-Markov property and the MI-Markov property. The condition $\mathcal{F}_{D_+} \perp \mathcal{F}_{D_-} \,|\, \mathcal{F}_{D_+\cap D_-}$ for any open covering $\{D_+, D_-\}$ of M defines the c-Markov property. We do emphasize that all arguments and results of this appendix are valid for non-Gaussian fields too. We first recall the following facts, the proof of which are trivial and thus omitted.

(2a.1) Lemma. *Suppose $\{\mathcal{F}_D\}$ is the canonical filtration subordinate to a generalized random field. If $\{U_\alpha\}$ is an open covering of D, then $\mathcal{F}_D = \vee\mathcal{F}_{U_\alpha}$.*

(2a.2) Lemma. *If the sub σ-fields $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ of $\mathcal{F}$ satisfy $\mathcal{F}_3 \vee \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_3 \vee \mathcal{F}_2$ and $\mathcal{F}_1 \perp \mathcal{F}_2 \,|\, \mathcal{F}_0$, then $\mathcal{F}_1 = \mathcal{F}_3 \vee \mathcal{F}_0$.*

We quote a proposition from [2], Proposition 1.14, with a slight modification.

(2a.3) Lemma. *Let $\{\mathcal{F}_D\}$ be a c-Markov filtration. Then*

$$\mathcal{F}_{C_+} \perp \mathcal{F}_{C_-} \,|\, \mathcal{F}_{C_+\cap C_-} \tag{2a.4}$$

for any closed covering $\{C_+, C_-\}$ of M. Conversely, as for the canonical filtration subordinate to a generalized random field, the c-Markov property is implied by

$$\mathcal{F}_{\overline{D_+}} \perp \mathcal{F}_{\overline{D_-}} \,|\, \mathcal{F}_{\overline{D_+}\cap\overline{D_-}} \tag{2a.5}$$

for any open covering $\{D_+, D_-\}$ of M.

(2a.6) Lemma. *Let $\{D_+, D_-\}$ be an open covering. If $D = (M\setminus\overline{D_+})\cup(D_+\cap D_-)$, then $\overline{D_-} = \overline{D}$ and $\partial D = \partial D_+ \cup \partial D_- \subset \overline{D_+}\cap\overline{D_-}$.*

The proof requires only elementary set-theory and topology. However, a formal proof is rather lengthy and thus omitted, the reader might draw a convincing picture instead or look up the proof in [12]. With the help of this lemma and Lemma 2a.3 we prove the converse of the implication: every c-Markov filtration possesses the germ Markov property and satisfies (2.20).

(2a.7) Theorem. *Every MI-Markov canonical filtration subordinate to a generalized random field has c-Markov property.*

Proof. Given an open covering $\{D_+, D_-\}$, set D as in Lemma 2a.6. We have $\mathcal{F}_{\overline{D_-}} = \mathcal{F}_{\overline{D}}$. We apply Lemma 2a.2 for $\mathcal{F}_0 = \mathcal{F}_{\partial D}$, $\mathcal{F}_1 = \mathcal{F}_{\overline{D}}$, $\mathcal{F}_2 = \mathcal{F}_{M\setminus D}$ and $\mathcal{F}_3 = \mathcal{F}_D$. (Note that $\mathcal{F}_2 \vee \mathcal{F}_3 = \mathcal{F}_M$ because of (2.20).) Then we see that $\mathcal{F}_{\overline{D}} = \mathcal{F}_D \vee \mathcal{F}_{\partial D}$. By Lemma 2a.1, the latter coincides with $\mathcal{F}_{M\setminus\overline{D_+}} \vee \mathcal{F}_{D_+\cap D_-} \vee \mathcal{F}_{\partial D}$, which is contained in $\mathcal{F}_{M\setminus D_+} \vee \mathcal{F}_{\overline{D_+}\cap\overline{D_-}}$ because of the second statement of Lemma 2a.6. Hence $\mathcal{F}_{\overline{D_-}} \subset \mathcal{F}_{M\setminus D_+} \vee \mathcal{F}_{\overline{D_+}\cap\overline{D_-}}$. Let f be $\mathcal{F}_{M\setminus D_+}$-measurable and g be $\mathcal{F}_{\overline{D_+}\cap\overline{D_-}}$-measurable. Then we see that

$$E[fg|\mathcal{F}_{\overline{D_+}}] = E[f|\mathcal{F}_{\overline{D_+}}]g = E[f|\mathcal{F}_{\partial D_+}]g.$$

The right hand side is $\mathcal{F}_{\overline{D_+}\cap\overline{D_-}}$-measurable because of the second statement of Lemma 2a.6. Thus we obtained the conditional independence (2a.5), which implies the c-Markov property by virtue of Lemma 2a.3. □

(2a.8) Remark. Crucial part of the proof is that we deduced $\mathcal{F}_{\overline{D}} = \mathcal{F}_D \vee \mathcal{F}_{\partial D}$ from the germ Markov property and (2.20). Usually one interprets a Markov property intuitively as independence of the "future" versus the "past" given the "present". This interpretation forces us to assume that the knowledge of the past and the future together with the present should equip us with complete information. In our context of generalized field this means that condition (2.20) should be satisfied. However, this together with the previous theorem shows that it is the c-Markov property which is rather natural in the context of *generalized* random fields and *not* the germ Markov property.

3. A COUNTEREXAMPLE

One of the aims of the present paper is to show by concrete examples that we really decompose Definition 2.2 into two independent parts. An example mentioned in the introduction is germ Markov and hence, according to Theorem 2.18, it satisfies (2.6), while the condition (2.5) breaks down. The other example we give in this section, see Remark 3.15 and Proposition 3.18, satisfies (2.5), while the condition (2.6) breaks down. Thus, according to Theorem 2.18 again, it is not germ Markov.

We suppose that M is compact and connected. Given a fibre metric and a Radon measure chosen from the C^∞-category, one can speak of square integrable sections of $\mathbf{E} \to M$. We will use the short-hand notation $\mathcal{L}^2(\mathbf{E})'$ to denote the collection of elements in $\mathcal{D}'(\mathbf{E})$ which are realizable as square integrable sections. Once the fiber metric and the underlying measure is specified, the relation

$$(3.1) \qquad {}_{\mathcal{D}'}\langle\beta(f),\varphi\rangle_{\mathcal{D}} = (f,\varphi)_{\mathcal{L}^2(\mathbf{E})} \text{ for all } f \in \mathcal{L}^2(\mathbf{E}), \varphi \in \mathcal{D}(\mathbf{E})$$

uniquely determines an isomorphism $\beta\colon \mathcal{L}^2(\mathbf{E}) \to \mathcal{L}^2(\mathbf{E})'$, which is a realization of the Riesz isomorphism for $\mathcal{L}^2(\mathbf{E})$. One fact we should keep in our mind is that β preserves the supports. As far as the fiber metric and the underlying measure are in the C^∞-category, whether or not a distribution is representable by a square-integrable section is independent of the individual $\mathcal{L}^2$-structure, and all $\mathcal{L}^2$-norms under discussion induce the same vector topology on $\mathcal{L}^2(\mathbf{E})'$ since M is compact. Clearly for each open subset D we have the following direct sum decomposition

$$\begin{aligned}(3.2)\qquad &\overline{\{f \in \mathcal{L}^2(\mathbf{E})';\ \mathrm{supp}\, f \subset M \setminus \partial D\}} \\ &= \overline{\{f \in \mathcal{L}^2(\mathbf{E})';\ \mathrm{supp}\, f \subset M \setminus \overline{D}\}} \oplus \overline{\{f \in \mathcal{L}^2(\mathbf{E})';\ \mathrm{supp}\, f \subset D\}}.\end{aligned}$$

If $\mathcal{L}^2(\mathbf{E})'$ in (3.2) is replaced by a subspace equipped with a stronger vector topology, the condition (2.13) might break down, as we describe now using facts of the preceding paragraph. Let W' be a 'rich' enough subspace of $\mathcal{L}^2(\mathbf{E})'$, that means that the support of $\varphi \in \mathcal{D}(\mathbf{E})$ does not meet an open set D if ${}_{\mathcal{D}'}\langle f, \varphi\rangle_{\mathcal{D}} = 0$ for all $f \in W'$ with $\mathrm{supp} f \subset D$. Typical examples of such W' are the β-images of positive order Sobolev spaces consisting of square integrable sections, where the Sobolev inner products are obtained by the usual coordinate patching. We shall construct a 'poor' but finite codimensional subspace of W', which turns out to break (2.13).

Let $\mathbf{F} \to M$ be another real vector bundle over M and $L\colon \mathcal{D}(\mathbf{F}) \to \mathcal{D}(\mathbf{E})$ be a linear differential operator with the transposed mapping $L'\colon \mathcal{D}'(\mathbf{E}) \to \mathcal{D}'(\mathbf{F})$. On the other hand, given $\mathcal{L}^2$ structures, $L^*\colon \mathcal{D}(\mathbf{E}) \to \mathcal{D}(\mathbf{F})$ denotes the formal adjoint of L with respect to the pair $\mathcal{L}^2(\mathbf{F})$ and $\mathcal{L}^2(\mathbf{E})$, that is $\beta_{\mathbf{F}}(L^*\varphi) = L'\beta_{\mathbf{E}}(\varphi)$ for $\varphi \in \mathcal{D}(\mathbf{E})$. Here $\beta_{\mathbf{F}}$ and $\beta_{\mathbf{E}}$ are the Riesz isomorphisms for $\mathcal{L}^2(\mathbf{F})$ and $\mathcal{L}^2(\mathbf{E})$ respectively described in (3.1). We assume that

(3.3) Assumption. The symbol of L defines surjections on each fibre.

Then $LL^*\colon \mathcal{D}(\mathbf{E}) \to \mathcal{D}(\mathbf{E})$ is an elliptic differential operator if and only if L satisfies (3.3). We may describe the hypoellipticity, see e.g. [1] vol.III, of L' as follows. The elements of $\ker L'$ are realizable as C^∞-sections in the sense that $\ker L' \subset \beta(\mathcal{D}(\mathbf{E}))$, where β is defined by (3.1). Therefore, if $h \in \ker L'$ and $f \in \mathcal{L}^2(\mathbf{E})'$, $(h, f)_{\mathcal{L}^2(\mathbf{E})'}$ makes sense.

We are concerned with the following subspace

$$(3.4)\qquad W'_\perp := \{f \in W';\ (h, f)_{\mathcal{L}^2(\mathbf{E})'} = 0 \text{ for all } h \in \ker L'\},$$

provided that $\ker L'$ is non-trivial. Since $(h, f)_{\mathcal{L}^2(\mathbf{E})'} = {}_{\mathcal{D}'}\langle f, \beta^{-1}(h)\rangle_{\mathcal{D}}$, it follows from the property of W' that if D is an open set

$$\begin{aligned}(3.5)\qquad &\{h \in \ker L';\ D \cap \mathrm{supp} h = \emptyset\} \\ &= \{h \in \ker L';\ (h, f)_{\mathcal{L}^2(\mathbf{E})'} = 0 \text{ for all } f \in W' \text{ with } \mathrm{supp} f \subset D\}.\end{aligned}$$

If $\ker L' \neq \{0\}$, the subspace $W'_\perp$ is 'poor' in the sense that (3.5) with W' replaced by $W'_\perp$ fails for $D = M$.

We here recall the following.

(3.6) Definition. We say L' has the unique continuation property if any $h \in \ker L'$ vanishing on a non-empty open subset vanishes throughout M.

It is known that quite a large class of elliptic differential operators have the unique continuation property. We refer to [4] for this and the relation to *strong* continuation as well as for the historical development. Suppose L' has the unique continuation property. Then the left hand side of (3.5) is trivial unless $D = \emptyset$, while the right hand side of (3.5) with W' replaced by $W'_\perp$ is $\ker L'$ always. In what follows we shall show that $W'_\perp$ is exactly what we want.

We denote the cokernel $\mathcal{D}(\mathbf{E})/\mathrm{Image}L$ by $\mathrm{coker}L$. Then $\mathrm{coker}L$ and the kernel $\ker L'$ are naturally paired by

$$_{\ker L'}\langle h, [\varphi]\rangle_{\mathrm{coker}L} = {}_{\mathcal{D}'}\langle h, \varphi\rangle_{\mathcal{D}}, \quad h \in \ker L', \varphi \in \mathcal{D}(\mathbf{E}),$$

where $\mathcal{D}(\mathbf{E}) \ni \varphi \mapsto [\varphi] \in \mathrm{coker}L$ is the canonical mapping. Since LL^* is a Fredholm operator according to the general theory of elliptic differential operators, see e.g. [1], this dualization determines the isomorphism $\ker L' \simeq \mathrm{Hom}(\mathrm{coker}L, \mathbb{R})$ of two finite dimensional vector spaces. In particular we have the following.

(3.7) Lemma. *Let $\alpha\colon V \longrightarrow \mathrm{coker}L$ be a linear mapping. α is surjective if and only if*

$$\{h \in \ker L';\ {}_{\ker L'}\langle h, \alpha(v)\rangle_{\mathrm{coker}L} = 0 \text{for all } v \in V\} = \{0\}.$$

We describe the unique continuation in terms of the following mapping Π whose definition is suggested by (3.5).

(3.8) Definition. Given an $\mathcal{L}^2$-norm on $\mathcal{D}(\mathbf{E})$, $\Pi\colon W' \to \mathrm{coker}L$ denotes the linear mapping characterized by

$$_{\ker L'}\langle h, \Pi f\rangle_{\mathrm{coker}L} = (h, f)_{\mathcal{L}^2(\mathbf{E})'}, \quad h \in \ker L', f \in W'.$$

Clearly $\ker\Pi = W'_\perp$.

By virtue of (3.5) and Lemma 3.7 we get

(3.9) Lemma. *The unique continuation property for L' with respect to a non-empty open subset D is equivalent to the surjectivity of the mapping*

$$\{f \in W';\ \mathrm{supp}\ f \subset D\} \ni f \longmapsto \Pi f \in \mathrm{coker}L.$$

(3.10) Corollary. *Let Π be as in (3.8), and let $\mathcal{H}'$ be a subspace of $\mathcal{L}^2(\mathbf{E})'$ with $\ker\Pi \subset \mathcal{H}'$. Suppose that $\mathcal{H}'$ is equipped with a vector topology such that the inclusion mapping $\mathcal{H}' \to \mathcal{L}^2(\mathbf{E})'$ is continuous and moreover L' has the unique continuation property with respect to open subsets D and $M \setminus \overline{D}$, both of which are non-empty. Then $W' \subset \mathcal{H}'$ if the following inclusion holds:*

$$\begin{aligned}(3.11)\qquad &\{v \in \mathcal{H}';\ \mathrm{supp}\ v \subset M \setminus \partial D\}^{-} \\ &\subset \overline{\{v \in \mathcal{H}';\ \mathrm{supp}\ v \subset M \setminus \overline{D}\}} + \overline{\{v \in \mathcal{H}';\ \mathrm{supp}\ v \subset D\}}.\end{aligned}$$

Proof. Given $f \in W'$, with the help of Lemma 3.9, we can choose f_1, $f_2 \in W'$ so that $\mathrm{supp} f_1 \subset M \setminus \overline{D}$ and $f_1 + f \in \ker\Pi$, respectively $\mathrm{supp} f_2 \subset D$ and $f_2 - f \in \ker\Pi$. We set $v := f_1 + f_2 \in \ker\Pi \subset \mathcal{H}'$, which has the property that $\mathrm{supp}\, v \subset M \setminus \partial D$. It then follows from (3.11) that there exist v_1, $v_2 \in \mathcal{H}'$ with $v_1 \in \overline{\{v \in \mathcal{H}';\ \mathrm{supp}\, v \subset M \setminus \overline{D}\}}$ and $v_2 \in \overline{\{v \in \mathcal{H}';\ \mathrm{supp}\, v \subset D\}}$ respectively, and $v = v_1 + v_2$. Since the closure above was considered with respect to a vector topology stronger than the $\mathcal{L}^2(\mathbf{E})'$-topology, we see that v_1 is contained in the first term of the sum of the right-hand side of (3.2) and v_2 is contained in the second one. By virtue of (3.2), we must have $v_1 = f_1$ and $v_2 = f_2$, which implies $f_1 \in \mathcal{H}'$. Thus we get $f = f_1 + f - f_1 \in \ker\Pi + \mathcal{H}' \subset \mathcal{H}'$. □

We now choose a Sobolev space of non-negative integer order as W'. Similar to the natural vector topology on $\mathcal{L}^2(\mathbf{E})'$, any of the Sobolev inner products respecting the order of W' and C^∞-structure of $\mathbf{E} \to M$ induces the same vector topology on W'.

(3.12) Lemma. *Let $\mathcal{H}'$ be a closed subspace of W'. Then* (2.13) *is equivalent to* (3.11)*, where $\gamma^*\colon \mathcal{H}' \to \mathcal{D}'(\mathbf{E})$ is the canonical mapping.*

Proof. Clearly with respect to a natural Sobolev inner product we have

$$\{v \in \mathcal{H}';\ \mathrm{supp}\, v \subset M \setminus \overline{D}\} \perp \{v \in \mathcal{H}';\ \mathrm{supp}\, v \subset D\}.$$

Thus (2.13) is equivalent to (3.11). □

We need a criterion whether a subspace $\mathcal{H}'$ is closed in W'. In the following lemma W' may be any subspace of $\mathcal{L}^2(\mathbf{E})'$.

(3.13) Lemma. *Let Π be as in* (3.8)*, and let $\mathcal{H}'$ be a subspace of W' with* $\ker\Pi \subset \mathcal{H}'$*. Suppose that W' is equipped with a vector topology such that the inclusion mapping $W' \to \mathcal{D}'(\mathbf{E})$ is continuous. Then $\mathcal{H}'$ is closed in W'.*

Proof. We set $N := \{\Pi v \in \mathrm{coker} L;\ v \in \mathcal{H}'\}$. Since $\ker\Pi \subset \mathcal{H}'$, we see that $\mathcal{H}' = \{v \in W';\ \Pi v \in N\}$. On the other hand the continuity of the inclusion mapping $W' \to \mathcal{D}'(\mathbf{E})$ implies that of Π, whose target is of finite dimension. Thus the statement follows. □

We suppose that L' has the unique continuation property. For example we consider the Laplace Beltrami operator for a compact Riemannian manifold. Let $\mathcal{H}'$ be a subspace of W' with $\ker\Pi \subset \mathcal{H}'$. Then, by Lemma 3.13, $\mathcal{H}'$ is closed in W'. Any Hilbert space structure on $\mathcal{H}'$ which forces the inclusion mapping $\mathcal{H}' \to \mathcal{D}'(\mathbf{E})$ to be continuous induces the same Fréchet topology as that induced by a natural Sobolev inner product on W', due to the closed graph theorem. Therefore all of the requirements in Corollary 3.10 and Lemma 3.12 are fulfilled. We denote the topological dual of $\mathcal{H}'$ by $\mathcal{H}$. Since the dual pair $\mathcal{D}(\mathbf{E})$, $\mathcal{D}'(\mathbf{E})$ is reflexive, a continuous linear mapping $\gamma\colon \mathcal{D}(\mathbf{E}) \longrightarrow \mathcal{H}$ is defined by

$$\langle v, \gamma(\varphi)\rangle_{\mathcal{H}} = \langle v, \varphi\rangle_{\mathcal{D}}, \quad v \in \mathcal{H}', \varphi \in \mathcal{D}(\mathbf{E}).$$

We note that the image $\gamma(\mathcal{D})$ is dense in $\mathcal{H}$ and the canonical norm is given by

$$\|\gamma(\varphi)\|_{\mathcal{H}} = \sup\{|\langle v, \varphi\rangle_{\mathcal{D}}|;\ v \in \mathcal{H}',\ \|v\|_{\mathcal{H}'} \le 1\} \quad \varphi \in \mathcal{D}(\mathbf{E}).$$

By Lemma 2.11 and Corollary 3.10 we get therefore.

(3.14) Proposition. *Suppose $\mathcal{H}' \neq W'$. Then (2.6) breaks down for any non-void and non-dense open subset D of M.*

Let us adjust the order of the Sobolev space W' to the order of the differential operator L and let us provide W' with an equivalent norm

$$\|f\|_{W'} := \|\Pi f\|_{\mathrm{coker}L} + \|L'f\|_{\mathcal{L}^2(\mathbf{F})'}.$$

Choose $\mathcal{H}'$ as $W'_\perp = \ker\Pi$ with the induced norm $\|\cdot\|_{W'}\restriction_{W'_\perp}$.

(3.15) Remark. The canonical map $W'_\perp \to \mathcal{D}'$ is semi-local (see (2.14)), because the nonlocal term $\|\Pi\cdot\|_{\mathrm{coker}L}$ is killed by the condition $\Pi v = 0$. On the other hand, $W'_\perp = W'$ if and only if $\ker L' = \{0\}$, i.e. if and only if $\mathrm{coker}L = \{0\}$. Therefore for operators L with non-trivial cokernels we get examples which satisfy (2.5) but do *not* satisfy (2.6), while as for W' (2.13) cleary holds. A concrete example is provided by Proposition 3.18 below.

In order to describe the canonical norm for $\mathcal{H}$, we introduce the "Green" operator G for L. Let $\Lambda\colon \mathcal{L}^2(\mathbf{F}) \to \mathcal{L}^2(\mathbf{F})$ denote the orthogonal projector with $\ker\Lambda = \ker L$. Then $G\colon \mathcal{D}(\mathbf{E}) \to \mathcal{D}(\mathbf{F})$ is determined by $GL\xi = \Lambda\xi$ for $\xi \in \mathcal{D}(\mathbf{F})$ and $\ker G = (\mathrm{Image}L)^\perp$, where the orthogonal complement is considered with respect to the $\mathcal{L}^2(\mathbf{E})$-inner product. We now evaluate $\|\gamma(\varphi)\|_{\mathcal{H}} = \|G\varphi\|_{\mathcal{L}^2(\mathbf{F})}$. This gives us the following theorem, cf. [11] for non-degenerate situation.

(3.16) Theorem. *A mean 0 Gaussian random field $\{X(\varphi)\}_{\varphi\in\mathcal{D}}$ with $E[X(\varphi)^2] = (G\varphi, G\varphi)_{\mathcal{L}^2(\mathbf{F})}$ is Markov if and only if* $\mathrm{coker}L = \{0\}$.

(3.17) Remark. The theorem is inspired by solving $L'X = Y$ for "Gaussian white noise" Y, see [2], [6] and [12].

To illustrate the abstract set-up developed so far, we state the following example.

(3.18) Proposition. *Let M be a compact, connected, and differentiable Riemannian manifold. Let $\Delta := dd^* + d^*d$ denote the Laplace-Beltrami operator acting on p-forms $\omega \in \Omega^p$. Denote by $g := \Delta^{-1}$ the Green operator defined through $\Delta g = g\Delta = Id - \Pi_H$, where Π_H denotes the $\mathcal{L}^2\Omega^p$ projection onto the space H of harmonic p-forms. Let $\{X(\omega)\}$ be the Gaussian generalized random field with mean 0 and covariance*

$$E[X(\omega)^2] = \int_M (\omega, g\omega)\, vol,$$

where vol denotes the Riemannian volume element and $(\cdot,\cdot)$ the canonical fibre metric induced by the Riemannian metric. Then $\{X(\omega)\}$ is c-Markov if and only if $H = \{0\}$, i.e. if there are no non-trivial harmonic p-forms. Furthermore, if $H \neq \{0\}$ then $\{X(\omega)\}$ is not even germ Markov.

Proof. In this setting the relevant objects are as follows. Let $\bigwedge^p(M)$ benote the vector bundle of alternating p-forms. We choose $\mathbf{E}$ as $\bigwedge^p(M)$ and $\mathbf{F}$ as the direct sum $\bigwedge^{p-1}(M) \oplus \bigwedge^{p+1}(M)$. Then $\mathcal{D}(\mathbf{E}) = \Omega^p$ and $\mathcal{D}(\mathbf{F}) = \Omega^{p-1} \times \Omega^{p+1}$. We introduce a differential operator $L\colon \Omega^{p-1} \times \Omega^{p+1} \longrightarrow \Omega^p$ by $(\xi,\eta) \longmapsto d\xi + d^*\eta$. One sees easily that $G\colon \Omega^p \longrightarrow \Omega^{p-1} \times \Omega^{p+1}$ defined by $\omega \longmapsto (d^*g\omega, dg\omega)$ is the Green operator for L and

$$(G\omega, G\omega)_{\mathcal{L}^2(\mathbf{F})} = (d^*g\omega, d^*g\omega)_{\mathcal{L}^2\Omega^{p-1}} + (dg\omega, dg\omega)_{\mathcal{L}^2\Omega^{p+1}} = (\omega, g\omega)_{\mathcal{L}^2\Omega^p}.$$

Since $L'\beta_{\mathbf{E}}(\omega) = \beta_{\mathbf{F}}((d^*\omega, d\omega))$ for $\omega \in \Omega^p$, L satisfies the assumption (3.3) and hence by the hypoellipticity $\ker L' = \beta_{\mathbf{E}}(H)$. Thanks to the already mentioned well-known unique continuation for the Laplace-Beltrami operator, L' also has the unique continuation property. Therefore we can apply Proposition 3.14, Remark 3.15 and Theorem 3.16. Thus the first statement follows. Finally non-existence of non-trivial harmonic p-forms is necessary for (2.6) to hold, since $\ker L' = \beta_{\mathbf{E}}(H)$ □

(3.19) Remark. In the theorems above we may replace "Markov", which actually means c-Markov, by any of the following Markovian properties: sharp Markov in the sense of Nelson, see [7], abstract Markov, see [12], or germ Markov. A proof is provided by the abstract considerations of [12].

For the future we intend to prove, that (3.16) is the "generic" case in the sense that no matter how Markov property is interpreted there exist *no* Markovian solutions of stochastic PDE's described in Remark 3.17, unless L has trivial cokernel.

4. A Markov Field with Degenerate Covariance

One might get the impression from the discussion so far that it is the degeneracy of the covariance that prevents the fields from being Markov. However, that is not always the case, things are more subtle. We shall present an example arising from quantum electrodynamics, that realizes a Markov field with very degenerate covariance. The kernel of its covariance is of infinite dimension!

For the proof we shall take advantage of the following definition and lemma.

(4.1) Definition. The map $\gamma^*: \mathcal{H}' \longrightarrow \mathcal{D}'$ is called *compatible with C^∞-cut off*, if the image is stable under the following manipulation:

Let $v \in \mathcal{H}'$. Suppose two disjoint open sets D_1, D_2 and $\chi \in C^\infty(M)$ satisfy that

$$D_1 \cup D_2 \supset \operatorname{supp} \gamma^*(v), \quad \chi = 1 \text{ on } D_1 \text{ and } \chi = 0 \text{ on } D_2. \tag{4.2}$$

Then $\chi\gamma^*(v) \in \gamma^*(\mathcal{H}')$.

We remark that the compatibility with C^∞-cut off is an aspect of the locality. Let $\mathbf{F} \to M$ be another real vector bundle over M and $L: \mathcal{D}(\mathbf{F}) \to \mathcal{D}(\mathbf{E})$ be a linear differential operator. The transposed map $L': \mathcal{D}'(\mathbf{E}) \to \mathcal{D}'(\mathbf{F})$ of L is compatible with C^∞-cut off in the sense that the commutativity $L'\chi T = \chi L'T$ holds if $T \in \mathcal{D}'(\mathbf{E})$ is in the situation of (4.1) in place of $\gamma^*(v)$. Given a continuous linear injection $\gamma^*: \mathcal{H}' \longrightarrow \mathcal{D}'$, restricting the source to the closed subspace

$$\mathcal{H}'_L := \{v \in \mathcal{H}'; L'\gamma^*(v) = 0\},$$

we get another continuous linear injection $\gamma_L^*: \mathcal{H}'_L \longrightarrow \mathcal{D}'$. Suppose that γ^* is compatible with C^∞-cut off. Then, combining with the compatibility with C^∞-cut off of L', which we have just observed, we see that γ_L^* is also compatible with C^∞-cut off.

(4.3) Lemma. *If the transposed map of $\gamma: \mathcal{D} \longrightarrow \mathcal{H}$ is semi-local and compatible with C^∞-cut off, then γ is Markov.*

Proof. We show (2.13). Suppose supp $\gamma^*(v) \subset M \setminus \partial D$. We note that $M \setminus \partial D$ consists of two disjoint open sets $M \setminus \overline{D}$ and D. Therefore we can choose two disjoint open sets D_1 and D_2 with

$$\overline{D_1} \subset M \setminus \overline{D}, \quad \overline{D_2} \subset D \text{ and } D_1 \cup D_2 \supset \text{supp } \gamma^*(v).$$

Since M is paracompact, there are C^∞-functions χ_1, χ_2 such that

$$\chi_1 = 1 \text{ on } D_1, \text{ supp } \chi_1 \subset M \setminus \overline{D} \text{ and } \chi_2 = 1 \text{ on } D_2, \text{ supp } \chi_2 \subset D.$$

Therefore, thanks to the compatibility with C^∞-cut off, there exists $v_1, v_2 \in \mathcal{H}'$ with $\chi_1\gamma^*(v) = \gamma^*(v_1)$ and $\chi_2\gamma^*(v) = \gamma^*(v_2)$ respectively. Since $\chi_1 + \chi_2 = 1$ on $D_1 \cup D_2$ and γ^* is injective, the desired decomposition $v = v_1 + v_2$ with supp $\gamma^*(v_1) \subset M \setminus \overline{D}$ and supp $\gamma^*(v_2) \subset D$ follows. □

We fix nonnegative integers d and n with $2n < d$. We shall denote the (mixed) distributional derivative of a locally integrable function f on $\mathbb{R}^d$ of order n by $\nabla^n f$. We define

$$W_n := \{f \in \mathcal{L}^{2d/(d-2n)}(\mathbb{R}^d); \nabla^n f \in \mathcal{L}^2\},$$

i.e. W_n is consisting of distributions whose derivatives of order n are in $\mathcal{L}^2(\mathbb{R}^d)$. We equip W_n with a seminorm:

$$\|f\|_{W_n} := \|\nabla^n f\|_{\mathcal{L}^2}, \text{ for } f \in W_n. \tag{4.4}$$

We quote a theorem from [1] vol.I, Theorem 4.5.9.

(4.5) Lemma. *Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and assume that $\nabla u \in \mathcal{L}^q$, where $1 < q < d$. Then there is a constant c such that $u - c \in \mathcal{L}^p$, where $\frac{1}{p} + \frac{1}{d} = \frac{1}{q}$. Moreover there is a constant $Const_{d,q}$ depending only on d and q such that*

$$\|u - c\|_{\mathcal{L}^p} \leq Const_{d,q}\|\nabla u\|_{\mathcal{L}^q}. \tag{4.6}$$

The estimate (4.6) derives from the Hardy-Littlewood-Sobolev inequality.

(4.7) Lemma. *$\|\cdot\|_{W_n}$ is actually a norm and W_n is complete with respect to this norm, and hence a Hilbert space is obtained. Moreover $\mathcal{D}(\mathbb{R}^d)$ is dense in W_n.*

Proof. Let $f \in W_n$. By iterating Lemma 4.5, we see that there exists an $\widetilde{f} \in \mathcal{L}^p(\mathbb{R}^d)$ such that $\nabla^n(f - \widetilde{f}) = 0$, and

$$\|\widetilde{f}\|_{\mathcal{L}^p} \leq \text{Const}_{d,n}\|\nabla^n f\|_{\mathcal{L}^2}.$$

Here $p = \frac{2d}{d-2n}$ and n must satisfy that $0 < 2n < d$ (transience). We shall show $h := f - \widetilde{f} = 0$. h must be a polynomial function of degree at most $n-1$, since $\nabla^n h = 0$. On the other hand, f being in $\mathcal{L}^p(\mathbb{R}^d)$, $h \in \mathcal{L}^p(\mathbb{R}^d)$. Therefore we get $h = 0$ and $f = \widetilde{f}$. Hence we get

$$\|f\|_{\mathcal{L}^p} \leq \text{Const}_{d,n}\|\nabla^n f\|_{\mathcal{L}^2}, \text{ for } f \in W_n. \tag{4.8}$$

Thus the statement follows from this estimate. Finally one obtains the last assertion by the standard smoothing and cut-off argument. □

Now let r be a positive integer with $r < d$. We use $\wedge_r\mathbb{R}^d$ to denote the r-fold exterior product of $\mathbb{R}^d$. The natural inner product of $\wedge_r\mathbb{R}^d$ shall be denoted by $(\cdot\,,\cdot)$, which is induced by the universality of the exterior algebra. We equip $W_n \otimes \wedge_r\mathbb{R}^d$ with a seminorm

$$\|(d+d^*)^n f\|_{\mathcal{L}^2}, \quad f \in W_n \otimes \wedge_r\mathbb{R}^d. \tag{4.9}$$

Since $(d+d^*)^{2n} = \Delta^n$ ($\Delta = dd^* + d^*d$), one sees with help of the Fourier transformation that $\|(d+d^*)^n f\|_{\mathcal{L}^2} = \|\nabla^n f\|_{\mathcal{L}^2}$ for $f \in \mathcal{S}(\mathbb{R}^d) \otimes \wedge_r\mathbb{R}^d$ and hence, due to Lemma 4.7, $\|(d+d^*)^n \cdot\|_{\mathcal{L}^2}$ actually introduces a Hilbert space strucure on $W_n \otimes \wedge_r\mathbb{R}^d$. We write $\mathcal{D} := C_0^\infty(\mathbb{R}^d \to \wedge_r\mathbb{R}^d)$, then a pairing $(W_n \otimes \wedge_r\mathbb{R}^d) \times \mathcal{D} \longrightarrow \mathbb{R}$ is naturally defined by $\int_{\mathbb{R}^d} (f(x), \varphi(x))\, dx$. Namely we get a continuous linear injection

$$W_n \otimes \wedge_r\mathbb{R}^d \longrightarrow \mathcal{D}',$$

which is obviously semi-local and compatible with C^∞-cut off. (The continuity follows from (4.8).) We introduce a closed subspace of $W_n \otimes \wedge_r\mathbb{R}^d$ by

$$\mathcal{H}' := \{v \in W_n \otimes \wedge_r\mathbb{R}^d ; d^*v = 0\},$$

and hence a continuous linear injection

$$\gamma^* : \mathcal{H}' \longrightarrow \mathcal{D}'.$$

We consider the transposed map $\gamma : \mathcal{D} \longrightarrow \mathcal{H}$ of γ^*, where $\mathcal{H}$ is the topological dual of $\mathcal{H}'$. We note that $\gamma(\mathcal{D})$ is dense in $\mathcal{H}$. As the dual space of $\mathcal{H}'$, $\mathcal{H}$ carries a natural Hilbert space structure given by

$$\|\gamma(\varphi)\|_{\mathcal{H}} = \sup\{|\langle \gamma^*(v), \varphi\rangle| ; v \in \mathcal{H}', \|v\|_{\mathcal{H}'} \le 1\}, \quad \varphi \in \mathcal{D}.$$

We can easily see, using the Fourier transformation, that

$$\|\gamma(\varphi)\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^d} (k \wedge \widehat{\varphi}(k), k \wedge \widehat{\varphi}(-k)) |k|^{-2(n+1)}\, dk, \quad \varphi \in \mathcal{D}. \tag{4.10}$$

Here $\widehat{\varphi}$ is the Fourier transform of φ.

Combining Lemma 4.3 and the argument before Lemma 4.3 we get:

(4.11) Theorem. *The map $\gamma : \mathcal{D} \longrightarrow \mathcal{H}$ is Markov.*

Let us return to the discussion of generalized random fields. The interior multiplication $\wedge_r\mathbb{R}^d \times \mathbb{R}^d \to \wedge_{r-1}\mathbb{R}^d$ shall be denoted by $\llcorner$. We consider a mean 0 Gaussian system $\{X(\varphi)\}$ indexed by $\mathcal{D}$ with

$$\begin{aligned} E[X(\varphi)^2] = \int_{\mathbb{R}^d} \{\, & c_1(k \wedge \widehat{\varphi}(k), k \wedge \widehat{\varphi}(-k)) \\ & + c_2(\widehat{\varphi}(k)\llcorner k, \widehat{\varphi}(-k)\llcorner k)\,\}\, |k|^{-2}\, dk, \end{aligned} \tag{4.12}$$

where c_1 and c_2 are nonnegative real numbers with $c_1 + c_2 > 0$. Combining (4.11) and known results (cf. [11]) on Gaussian generalized random fields with non-degenerate covariance, we have finally proven:

(4.13) Theorem. *$\{X(\varphi)\}$ with (4.12) is c-Markov if and only if $c_1 = c_2 > 0$ or $c_1 c_2 = 0$.*

(4.14) Remark. The motivation for the field defined as above stems from constructive quantum field theory. Within this framework $\{X(\varphi)\}$ corresponds to the so-called free quantized electromagnetic field, where c_1 and c_2 are gauge parameters.

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