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Abstract

For an automorphism α of a UHF algebra it is shown that α has the Rohlin property if and only if α^m is uniformly outer for any $m \neq 0$. It is also shown that the automorphisms of a UHF algebra with the Rohlin property are outer conjugate with each other.

1 Introduction

Let α be an automorphism of a unital C^*-algebra A. First we recall some definitions [13, 6, 17, 15].

Definition 1.1 The automorphism α has the Rohlin property if for any $k \in \mathbf{N}$ there are positive integers $k_1, \ldots, k_m \geq k$ satisfying the following condition: For any finite subset F of A and $\epsilon > 0$ there are projections $e_{i,j}$; $i = 1, \ldots, m$, $j = 0, \ldots, k_i - 1$ in A such that

$$\sum_{i=1}^{m} \sum_{j=0}^{k_i-1} e_{i,j} = 1,$$
$$\|\alpha(e_{i,j}) - e_{i,j+1}\| < \epsilon,$$
$$\|[x, e_{i,j}]\| < \epsilon$$

for $i = 1, \ldots, m$, $j = 0, \ldots, k_i - 1$, and $x \in F$ where $e_{i,k_i} = e_{i,0}$.

Definition 1.2 The automorphism α is uniformly outer if for any $a \in A$, any projection $p \in A$, and any $\epsilon > 0$, there are a finite number of projections $p_1, \ldots, p_n$ in A such that

$$p = \sum_{i=1}^{n} p_i,$$
$$\|p_i a \alpha(p_i)\| < \epsilon, \ i = 1, \ldots, n.$$

If α has the Rohlin property then α^m is uniformly outer for any $m \neq 0$. The converse is shown for Cuntz algebras $A = O_n$ in [15]. In this note we shall show the converse for UHF algebras (and for certain unital simple AF algebras). This generalizes the results on shifts on UHF algebras M_{n^∞} [6, 15] and on quasi-free automorphisms of the CAR algebras [4]. Note that if A is UHF, the property that α^m is uniformly outer for any $m \neq 0$ is equivalent to that the crossed product $A \times_\alpha \mathbf{Z}$ has a unique tracial state [15]. To be more precise, the former condition implies, by [15], Lemma 4.3, that any tracial state of $A \times_\alpha \mathbf{Z}$ is invariant under the dual action $\hat{\alpha}$, and hence that $A \times_\alpha \mathbf{Z}$ has a unique tracial state; which implies in turn, by (the proof of) [15], Lemma 4.4, that $\overline{\alpha}^m$ is outer for any $m \neq 0$, where $\overline{\alpha}$ denotes the extension of α to an automorphism of the weak closure of A in the tracial representation; which implies, by [15], Theorem 4.5, the condition on α we started with.

If an automorphism α of a UHF algebra A has the Rohlin property, then for any $\epsilon > 0$ there is a unitary $u \in A$ such that $\|u - 1\| < \epsilon$ and $\mathrm{Ad}u \circ \alpha$ is of infinite tensor product type [18, 4]. (This follows from the proof of [4], Theorem 1.7, which asserts the same conclusion for certain UHF algebras based on a special form of Rohlin property. Adopting the present form of Rohlin property as suggested by Rørdam [17], the conclusion holds for any UHF algebras.) Furthermore $A \times_\alpha \mathbf{Z}$ is approximately divisible [3] and is isomorphic to a unique simple $\mathbf{AT}$ algebra independent of α (as far as α has the Rohlin property) [4]. ($K_*(A \times_\alpha \mathbf{Z}) \cong K_0(A)$ by the Pimsner-Voiculescu exact sequence [1]; $A \times_\alpha \mathbf{Z}$ is an inductive limit of C^*-algebras of the form $C(\mathbf{T}) \otimes M_n$, called an $\mathbf{AT}$ algebra; $A \times_\alpha \mathbf{Z}$ has real rank zero [2]: These properties imply the above assertion by Elliott's theory [12].) Note that if we just assume that any non-zero power of α is outer (instead of uniformly outer), we cannot expect in general the closeness, in norm, of α to an automorphism of infinite tensor product type (cf. [8]). Summing up such results we obtain the following result; the corresponding result for quasi-free automorphisms of the CAR algebra is given in [4], Theorem 1.1:

Theorem 1.3 *Let A be a UHF algebra, i.e., the inductive limit of an increasing sequence $\{M_{n_1} \otimes \cdots \otimes M_{n_k}\}$ of full matrix algebras with natural embeddings. If α is an automorphism of A, the following conditions are equivalent:*

1. *α has the Rohlin property.*
2. *α^m is uniformly outer for any $m \neq 0$.*
3. *$A \times_\alpha \mathbf{Z}$ has a unique tracial state.*
4. *$A \times_\alpha \mathbf{Z}$ has real rank zero.*
5. *There exists a unitary $u \in A$ such that $(A, \mathrm{Ad}\, u \circ \alpha)$ is isomorphic to*

$$(\otimes_{k=1}^{\infty} M_{m_k}, \otimes_{k=1}^{\infty} \mathrm{Ad}\, u_k),$$

where u_k is a unitary of M_{m_k} and the eigenvalues of $\otimes_{k=l}^{\infty} u_k$ are uniformly distributed for any $l \in \mathbf{N}$.

6. *$A \times_\alpha \mathbf{Z}$ is the inductive limit of an increasing sequence $\{B_k\}$ of C^*-algebras such that*

$$B_k \equiv C(\mathbf{T}) \otimes M_{n_1} \otimes \cdots \otimes M_{n_k} \cong M_{n_k}(B_{k-1})$$

and the embedding of B_k into B_{k+1} is given by

$$f \mapsto f \oplus \beta_\theta(f) \oplus \beta_{2\theta}(f) \oplus \cdots \oplus \beta_{(n_{k+1}-1)\theta}(f),$$

where θ is irrational and $\beta_\theta(f)(t) = f(t-\theta)$ for $f \in B_k \cong C(\mathbf{T}, M_{n_1} \otimes \cdots \otimes M_{n_k})$.

When (1) holds, (6) is obtained by choosing a special α by using the uniqueness of $A \times_\alpha \mathbf{Z}$ mentioned above; (6) implies (3) (by an easy computation) and (4) (by [2]). See Section 5 for details of (5); (5) implies (1) by the condition of uniform distribution (Lemma 5.2). We have noted before the theorem that (5) $\Leftarrow$ (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3). We shall show that (4) implies (3). A C^*-algebra has real rank zero if and only if the self-adjoint elements with finite spectra are dense in the set of all self-adjoint elements [7]. Hence if (4) holds, the linear span of projections is dense in $A \times_\alpha \mathbf{Z}$ and so any tracial state on $A \times_\alpha \mathbf{Z}$ is invariant under the dual action of $\hat{\mathbf{Z}} \cong \mathbf{T}$ which is connected. Since A has only one tracial state, this shows that (4) implies (3). Sections 2–4 are devoted to the proof of the remaining implication (2) $\Rightarrow$ (1).

In Section 2 we give a simple main technical lemma concerning the shift automorphism on the compact operators on $l^2(\mathbf{Z})$. This asserts that this automorphism has a kind of Rohlin property. In Section 3 using Connes' result [9] on the Rohlin property for automorphisms of the injective type II_1 factor, we show that we can *simulate* a system as in Section 2 in certain systems (A, α) if the automorphism α satisfies the property that α^m is uniformly outer for any $m \neq 0$. If the C^*-algebra A is UHF, this simulation can be done in the relative commutant of any finite-dimensional subalgebra. Combining these results we conclude, in Section 4, that α has the *approximate* Rohlin property [5]. Using the methods given in [15] we then conclude that α has the Rohlin property.

In Section 5 we shall show (cf. [14]):

Theorem 1.4 *Let α and β be automorphisms of a UHF algebra A. If α and β have the Rohlin property, then for any $\epsilon > 0$ there exists an automorphism γ of A such that $\|\alpha - \gamma \circ \beta \circ \gamma^{-1}\| < \epsilon$.*

The conclusion of this theorem is equivalent to saying that α and β are outer conjugate. Since an automorphism of a unital simple C^*-algebra is inner if it is close to the identity in norm, the outer conjugacy follows from the conclusion. If $\mathrm{Ad}\, u \circ \alpha = \gamma \circ \beta \circ \gamma^{-1}$ and $\|u - v^*\alpha(v)\| < \epsilon$ for some unitary $v \in A$, then $\|\alpha - \mathrm{Ad}\, v \circ \gamma \circ \beta \circ \gamma^{-1} \circ \mathrm{Ad}\, v^*\| < 2\epsilon$ follows; the existence of the v above follows from the Rohlin property by [13]. Hence the uniqueness of $A \times_\alpha \mathbf{Z}$ asserted by using Elliott's result [12] is also a consequence of this theorem.

In Section 6 we shall show by similar methods that an automorphism α of a simple non-commutative torus which comes from the gauge action has the Rohlin property if any non-zero power of α is outer. Since the crossed product by α depends on α in general [11], a result like the above theorem cannot hold in this case.

2 Shift on $l^2(\mathbf{Z})$

Let $\{E_{ij};\ i,j \in \mathbf{Z}\}$ be matrix units:

$$(E_{ij})^* = E_{ji},\ E_{ij}E_{kl} = \delta_{jk}E_{il}.$$

Let K be the closed linear span of these E_{ij} with unique C*-norm; K is the compact operators on $l^2(\mathbf{Z})$ by identifying E_{ii} with the one-dimensional projection onto the functions supported by $\{i\} \subset \mathbf{Z}$. Define an automorphism σ of K by $\sigma(E_{i,j}) = E_{i+1,j+1}$. For any $N \in \mathbf{N}$ let $P_N = \sum_{i=0}^{N-1} E_{ii}$.

Lemma 2.1 *Let K, σ, P_N, etc be as above. For any $\epsilon > 0$ and $n \in \mathbf{N}$ there exist $N \in \mathbf{N}$ and projections $e_0, e_1, \ldots, e_{n-1}$ in K such that*

$$\begin{aligned}
&\sum_{i=0}^{n-1} e_i \le P_N, \\
&\|\sigma(e_i) - e_{i+1}\| < \epsilon,\ i = 0, \ldots, n-1, \\
&\frac{n \dim e_0}{N} > 1 - \epsilon,
\end{aligned}$$

where $e_n = e_0$.

Proof. We shall choose $k, l \in \mathbf{N}$ such that $1 \ll k \ll l$. Define

$$\begin{aligned}
f = \ & \sum_{m=1}^{k-1} \{\frac{m}{k} E_{nm,nm} + \frac{k-m}{k} E_{n(m+k+l),n(m+k+l)} \\
& + \frac{\sqrt{m(k-m)}}{k} E_{nm,n(m+k+l)} + \frac{\sqrt{m(k-m)}}{k} E_{n(m+k+l),nm}\} \\
& + \sum_{m=k}^{k+l} E_{nm,nm}.
\end{aligned}$$

Then f is a projection such that

$$f \le \sum_{m=1}^{2k+l-1} E_{nm,nm}$$

and the dimension of f is equal to $k+l$. For $m = 0, \ldots, n-1$ let $e_m = \sigma^{m-n}(f)$. Let $N = n(2k+l-1)$. Then $\{e_m;\ m = 0, \ldots, n-1\}$ are mutually orthogonal projections of K such that $e_0 + \cdots + e_{n-1} \le P_N$. Since $\sigma(e_{n-1}) - e_0 = f - e_0$ is equal to

$$\begin{aligned}
& -\frac{1}{k}\sum_{m=0}^{k-1} E_{nm,nm} + \frac{1}{k}\sum_{m=0}^{k-1} E_{n(m+k+l),n(m+k+l)} \\
& + \sum_{m=0}^{k-1} \frac{1}{k}[\sqrt{m(k-m)} - \sqrt{(m+1)(k-m-1)}]\{E_{nm,n(m+k+l)} + E_{n(m+k+l),nm}\}
\end{aligned}$$

and for $m = 0, \ldots, k-1$

$$|\sqrt{m(k-m)} - \sqrt{(m+1)(k-m-1)}| < \sqrt{k},$$

we have that

$$\|\sigma(e_{n-1}) - e_0\| < \frac{1}{k} + \frac{1}{\sqrt{k}}.$$

Hence if k is so large that $k^{-1} + k^{-1/2} < \epsilon$, then we have that $\|\sigma(e_i) - e_{i+1}\| < \epsilon$ for $i = 0, \ldots, n-1$ with $e_n = e_0$. Since

$$\frac{n \dim e_0}{N} = \frac{k+l}{2k+l-1},$$

if l is so large that

$$\epsilon > \frac{k-1}{2k+l-1}$$

then we have that

$$\frac{n \dim e_0}{N} > 1 - \epsilon.$$

This concludes the proof.

3 Simulation

We consider a certain class of AF algebras which includes UHF algebras. Let A be a unital infinite-dimensional simple AF algebra. Note that $K_0(A)$ is of finite rank r and has no infinitesimal elements if and only if $K_0(A)$ is order isomorphic to a dense subgroup of $\mathbf{R}^d$ for some $d \leq r$, provided with the relative strict order [10]. (The strict order on $\mathbf{R}^d$ is defined as follows: $a \leq b$ if $a = b$ or $a_i < b_i$ for $i = 1, \ldots, d$.) In this case A has d extreme tracial states.

Lemma 3.1 *Let A be a unital simple AF algebra such that $K_0(A)$ is of finite rank and has no infinitesimal elements. Let α be an automorphism of A such that α^m is uniformly outer for any $m \neq 0$. For any $m \in \mathbf{N}, \epsilon > 0$, and any finite-dimensional C^*-subalgebra B of A there exists an orthogonal family $\{e_i;\ i = 0, \ldots, m\}$ of projections in $A \cap B'$ such that*

$$\|\alpha(e_i) - e_{i+1}\| < \epsilon,\ i = 0, \ldots, m-1,$$
$$[e_0] = [e_1] = \cdots = [e_m],$$
$$m[1] \leq (m+k)(m+1)[e_0]$$

where $[\cdot]$ denotes the equivalence class in $K_0(A \cap B')$ and k is the period of α_* on $K_0(A)$.

Proof. Let $\{\tau_1, \ldots, \tau_d\}$ be the set of extreme tracial states of A. For each $i = 1, \ldots, d$ let k_i be the minimal positive integer such that $\tau_i \circ \alpha^{k_i} = \tau_i$ and let k be the least common multiple of $k_1, \ldots, k_d$. Then k is the minimal positive integer such that $\tau \circ \alpha^k = \tau$ for any tracial state τ on A or equivalently $\alpha_*^k = id$ on $K_0(A)$.

Let $\rho = \oplus_{i=1}^d \pi_{\tau_i}$. Then there is a unique automorphism $\overline{\alpha}$ of $\rho(A)''$ such that $\overline{\alpha} \circ \rho = \rho \circ \alpha$. Since $\pi_{\tau_i}(A)''$ is an injective type II_1 factor and any non-zero power of $\overline{\alpha}^{k_i}$ on $\pi_{\tau_i}(A)''$ is outer [15], Theorem 4.5, we can apply [9] to $\overline{\alpha}^{k_i}|\pi_{\tau_i}(A)''$ to conclude that $\overline{\alpha}^{k_i}|\pi_{\tau_i}(A)''$ has the (von Neumann version of) Rohlin property. Thus we have, for any $l \in \mathbf{N}$, a central sequence $\{(E_0^{(j)}, \ldots, E_{k_il-1}^{(j)})\}$ of orthogonal families of projections in $\oplus_{s=0}^{k_i-1} \pi_{\tau_i \circ \alpha^s}(A)''$ such that

$$\sum_{i=0}^{k_il-1} E_i^{(j)} \to 1,$$
$$\overline{\alpha}(E_i^{(j)}) - E_{i+1}^{(j)} \to 0, \ i = 0, \ldots, k_il-1,$$

strongly as $j \to \infty$, where $E_{k_il}^{(j)} = E_0^{(j)}$. Since $\rho(A)''$ is the finite direct sum of such von Neumann algebras, we can conclude that for any $l \in \mathbf{N}$ there exists a central sequence $\{(E_0^{(j)}, \ldots, E_{kl-1}^{(j)})\}$ of orthogonal families of projections in $\rho(A)''$ such that

$$\sum_{i=0}^{kl-1} E_i^{(j)} \to 1,$$
$$\overline{\alpha}(E_i^{(j)}) - E_{i+1}^{(j)} \to 0, \ i = 0, \ldots, kl-1,$$

strongly as $j \to \infty$, where $E_{kl}^{(j)} = E_0^{(j)}$.

Then we can replace $\{E_i^{(j)}\}$ by a central sequence $\{e_i^{(j)}\}$ of orthogonal families of projections in A such that

$$\rho(\sum_{i=0}^{kl-1} e_i^{(j)}) \to 1,$$
$$\|\alpha(e_i^{(j)}) - e_{i+1}^{(j)}\| \to 0, \ i = 0, \ldots, kl-2$$

as $j \to \infty$. (See [15], Section 4 for details.) Actually what we do is as follows: First find a central sequence $\{e_j\}$ of projections of A such that $\rho(e_j) - E_0^{(j)} \to 0$ strongly. Second, noting that $\rho(e_j(\sum_{s=1}^{kl-1} \alpha^s(e_j))e_j) \to 0$, find a subprojection p_j of e_j such that $\{p_j\}$ is a central sequence and

$$\|p_j(\sum_{s=1}^{kl-1} \alpha^s(e_j))p_j\| \to 0,$$
$$\rho(e_j - p_j) \to 0.$$

The condition of centrality can be satisfied by assuming that the algebra generated by $e_j, \alpha(e_j), \ldots, \alpha^{kl-1}(e_j)$ lies in the relative commutant of a prescribed finite-dimensional

C^*-subalgebra and then by choosing p_j from this commutant. Then the projections $p_j, \alpha(p_j), \ldots, \alpha^{kl-1}(p_j)$ will be nearly orthogonal, and hence we can construct, by a standard technique, an orthogonal family $\{f_0^{(j)}, \ldots, f_{kl-1}^{(j)}\}$ of projections such that $f_s^{(j)} \approx \alpha^s(p_j)$. Furthermore we can assume that all $f_s^{(j)}$ are in $A \cap B'$.

Given $m \in \mathbf{N}$ let $m_0 \geq m$ be the smallest integer such that $m_0 \equiv 0 \bmod k$. Let $l = m_0 + 1$ and

$$e_i^{(j)} = \sum_{s=0}^{k-1} f_{i+s(m_0+1)}, \; i = 0, \ldots, m.$$

Then $\{e_i^{(j)};\ i = 0, \ldots, m\}$ is a central sequence in j and $[e_i^{(j)}]$ is independent of i in $K_0(A \cap B')$ for a sufficiently large j by Lemma 3.2 below. It also follows that

$$\begin{aligned} &e^{(j)} = \sum_{i=0}^{m} e_i^{(j)} \leq 1, \\ &\|\alpha(e_i^{(j)}) - e_{i+1}^{(j)}\| \to 0, \; i = 0, \ldots, m-1, \\ &\tau_s(e^{(j)}) \to \frac{m+1}{m_0+1}, \; s = 1, \ldots, d. \end{aligned}$$

Since τ_s is factorial, it follows that for any minimal central projection E of B,

$$\tau_s(e^{(j)}E) \to \frac{m+1}{m_0+1}\tau_s(E).$$

Since the traces $\tau_1, \ldots, \tau_d$ on $(A \cap B')E$ determine the order of the dimension group of $(A \cap B')E$, which is order isomorphic to $K_0(A)$, it follows that

$$(m_0+1)[e^{(j)}] \geq m[1]$$

in $K_0(A \cap B')$ if j is sufficiently large. Since $m_0 \leq m + k - 1$, this concludes the proof.

Lemma 3.2 *In the situation of Lemma 3.1 let $\{p_j\}$ be a central sequence of projections in A and let $\beta = \alpha^{kl}$ for some $l \in \mathbf{N}$. Then there is a central sequence $\{x_j\}$ in A such that $x_j^* x_j = p_j$ and $x_j x_j^* = \beta(p_j)$.*

Proof. Let D be a finite-dimensional subalgebra of A and let v be a unitary of A such that $\beta|D = \operatorname{Ad} v|D$. Then for a sufficiently large j, p_j almost commutes with D and $\beta(p_j)$ is close to $\operatorname{Ad} v^* \circ \beta(p_j)$. Since $\operatorname{Ad} v^* \circ \beta|D = id$, it follows that there is a unitary $u_j \in A \cap D'$ such that $\operatorname{Ad} v^* \circ \beta(p_j) \approx \operatorname{Ad} u_j(p_j)$. Thus we can find x_j with $x_j^* x_j = p_j$, $x_j x_j^* = \beta(p_j)$ in a small neighbourhood of $u_j p_j$, which implies that x_j almost commutes with D. Since D is arbitrary, we obtain the conclusion.

4 The Rohlin property

Theorem 4.1 *Let A be a unital simple AF algebra such that $K_0(A)$ is of finite rank and has no infinitesimal elements. Let α be an automorphism of A. If α satisfies the property that α^m is uniformly outer for any $m \neq 0$, then α has the Rohlin property.*

We note the following consequence obtained in [18]: In the above theorem further suppose that $\alpha_* = id$ on $K_0(A)$. Then for any $\epsilon > 0$ there are a unitary $u \in A$ and an increasing sequence $\{A_n\}$ of finite-dimensional subalgebras of A such that $\|u - 1\| < \epsilon$, $\cup A_n$ is dense in A, and $\mathrm{Ad}\, u \circ \alpha(A_n) = A_n$ for all n.

Before giving the proof we introduce the following definition [5].

Definition 4.2 An automorphism α of a unital AF algebra A has the approximate Rohlin property if for any $m, n \in \mathbf{N}$, any $\epsilon > 0$, and any finite-dimensional C^*-subalgebra B of A there is an orthogonal family $\{e_i;\ i = 0, \ldots, m-1\}$ of projections in $A \cap B'$ such that

$$\begin{aligned}
&\|\alpha(e_i) - e_{i+1}\| < \epsilon,\ i = 0, \ldots, m-1,\\
&[e_0] = [e_1] = \cdots = [e_{m-1}],\\
&[e_0] \geq n[1 - \sum_{i=0}^{m-1} e_i],
\end{aligned}$$

where $[\cdot]$ denotes the equivalence class in $K_0(A \cap B')$ and $e_m = e_0$.

Theorem 4.1 follows from the following two lemmas.

Lemma 4.3 *Under the situation of Theorem 4.1 suppose that α^m is uniformly outer for any $m \neq 0$. Then α has the approximate Rohlin property.*

Proof. This follows from 2.1 and 3.1. Let k be the period of α_* on $K_0(A)$ as in the proof of 3.1. Let $m \in \mathbf{N}$, $\epsilon > 0$, and B a finite-dimensional C^*-subalgebra of A. By 2.1 one finds $N \in \mathbf{N}$ and projections $e_0, \ldots, e_{m-1}$ in K such that

$$\begin{aligned}
&e_0 + \cdots + e_{m-1} \leq P_N,\\
&\|\sigma(e_i) - e_{i+1}\| < \epsilon,\\
&\frac{m \dim e_0}{N} > 1 - \epsilon,
\end{aligned}$$

where $e_m = e_0$. By 3.1 one finds $N+1$ projections $p_0, \ldots, p_N$ in $A \cap B'$ such that

$$\begin{aligned}
&p_0 + \cdots + p_N \leq 1,\\
&\|\alpha(p_i) - p_{i+1}\| < \epsilon/N,\ i = 0, \ldots, N-1,
\end{aligned}$$

and $[p_0] = \cdots = [p_N]$, $N[1] \leq (N+k)(N+1)[p_0]$ in $K_0(A \cap B')$. Then there is a unitary $u \in A$ such that $\|u - 1\| < 4\epsilon$ and $\mathrm{Ad}\, u \circ \alpha(p_i) = p_{i+1}$ for $i = 0, \ldots, N-1$. Let w be a partial isometry in $A \cap B'$ such that $w^*w = p_0$ and $ww^* = p_1$. Let $\alpha_1 = \mathrm{Ad}\, u \circ \alpha$ and let C_1 (resp. C) be the C^*-algebra generated by $w, \alpha_1(w), \ldots, \alpha_1^{N-2}(w)$ (resp. and $\alpha_1^{N-1}(w)$). Then $M_N \cong C_1 \subset C \cong M_{N+1}$. Define a homomorphism Φ of C into K (as in 2.1) by $\Phi(\alpha_1^i(w)) = E_{i+1,i}$. Then Φ satisfies that $\sigma \circ \Phi | C_1 = \Phi \circ \alpha_1 | C_1$ and $\Phi(C_1) = P_N K P_N$. Thus, denoting $\Phi^{-1}(e_i)$ by e_i, we obtain projections $e_0, \ldots, e_{m-1}$ in $C_1 \subset A \cap B'$ such that

$$\begin{aligned}
&e_0 + \cdots + e_{m-1} \leq 1, \\
&\|\alpha(e_i) - e_{i+1}\| < 9\epsilon, \ i = 0, \ldots, m-1, \\
&m[e_0] \geq \frac{N^2}{(N+k)(N+1)}(1-\epsilon)[1],
\end{aligned}$$

where $e_m = e_0$ and the last inequality is in $K_0(A \cap B') \otimes \mathbf{R}$. For a sufficiently large N and a sufficiently small $\epsilon > 0$ we can conclude the proof.

Lemma 4.4 *Under the situation of Theorem 4.1 suppose that α has the approximate Rohlin property. Then α has the Rohlin property.*

Proof. This follows by the arguments in the proof of Theorem 2.1 of [15]. We just outline the proof. The approximate Rohlin property is not good enough only because

$$e = \sum_{i=0}^{m} e_i$$

is not 1, where e_i's are as in Definition 4.2. But $1 - e$ is small compared with e_0 (if n is large), which is the fact we shall use below. Assuming $\epsilon > 0$ is sufficiently small, let us assume that $\alpha(e_i) = e_{i+1}$ and so $\alpha^m(e_0) = e_0$. We use the approximate Rohlin property for $(e_0 A e_0, \alpha^m)$ to obtain projections $p_1, \ldots, p_n$ in $A \cap B'$ such that

$$\begin{aligned}
&p_1 + \cdots + p_n \leq e_0, \\
&\alpha^m(p_i) \approx p_{i+1}, \ i = 1, \ldots, n-1,
\end{aligned}$$

and $[p_1] = [1 - e]$ in $K_0(A \cap B')$. Let v be a partial isometry in $A \cap B'$ such that $v^*v = 1 - e$ and $vv^* = p_1$ and let

$$w' = n^{-1/2} \sum_{i=0}^{n-1} \alpha^{mi}(v).$$

Then w' is close to a partial isometry and we will obtain a partial isometry w in a small neighbourhood of w' such that $w^*w = 1 - e$, $ww^* \leq e_0$, w is almost invariant under α^m (if n is large), and w is almost central (if B is large). Then the C^*-algebra D generated by $w, \alpha(w), \ldots, \alpha^{m-1}(w)$ is isomorphic to M_{m+1}, and $\alpha | D$ is close to $\mathrm{Ad}\, U | D$, where U is a unitary in D whose eigenvalues are

$$1, \ e^{2\pi i j/m}; j = 0, 1, \ldots, m-1.$$

Since the eigenvalues of U are almost uniformly distributed (if m is large), one can find projections $f_0, \ldots, f_{l-1}; g_0, \ldots, g_l$ in D (for a prescribed l) such that they add up to the identity of D and

$$\alpha(f_i) \approx f_{i+1};\ \alpha(g_i) \approx g_{i+1}$$

where $f_l = f_0$ and $g_{l+1} = g_0$. Hence the projections

$$f_0, \ldots, f_{l-1};\ g_0, \ldots, g_l;\ e_0 - ww^*, \ldots, e_{m-1} - \alpha^{m-1}(ww^*)$$

satisfy the required properties.

5 Outer conjugacy

Definition 5.1 Let S_k be a finite sequence $\{s_{k1}, \ldots, s_{ka_k}\}$ in $\mathbf{T}$ for each $k \in \mathbf{N}$. The sequence $\{S_k\}$ is uniformly distributed on $\mathbf{T}$ if for any continuous function f on $\mathbf{T}$

$$\lim_k a_k{}^{-1} \sum_{i=1}^{a_k} f(s_{ki}) = \int_{\mathbf{T}} f(\lambda) d\lambda$$

where $d\lambda$ is normalized Haar measure on $\mathbf{T}$.

See [4] for other equivalent conditions.

Lemma 5.2 *Let α be an automorphism of a UHF algebra A such that $\alpha = \otimes_{k=1}^{\infty} \mathrm{Ad}\, u_k$ on $A = \otimes_{k=1}^{\infty} M_{m_k}$, where u_k is a unitary in the k-th factor M_{m_k}. For $k, l \in \mathbf{N}$ with $l \leq k$ let $S_{l,k}$ be a sequence consisting of the eigenvalues of the unitary $\otimes_{i=l}^{k} u_i$ repeated as often as multiplicity indicates. Then α has the Rohlin property if and only if $\{S_{l,k}\}_{k=l}^{\infty}$ is uniformly distributed for any $l \in \mathbf{N}$.*

Proof. This is almost contained in [14, 4]. Suppose that α has the Rohlin property, and that a sufficiently large $N \in \mathbf{N}$ is given for k in Definition 1.1. Then there exist $N_1, \ldots, N_t \geq N$ such that for any $\epsilon > 0$ there are projections e_{ij}; $i = 1, \ldots, t$, $j = 0, \ldots, N_i - 1$ which are adding up to 1 and satisfy that $\|\alpha(e_{i,j}) - e_{i,j+1}\| < \epsilon$, where $e_{i,k_i} = e_{i,0}$. This shows that for a sufficiently large $k \in \mathbf{N}$, the unitary $\otimes_{i=1}^{k} u_i$ is almost unitarily equivalent to

$$\oplus_{i=1}^{t} U_i \otimes V_i$$

where U_i is a unitary of M_{N_i} with eigenvalues

$$e^{2\pi i j/N_i};\ j = 0, \ldots, N_i - 1$$

and V_i is a unitary of $M_{m_1 \cdots m_k \tau_i}$ with $\tau_i = \tau(e_{i0})$. This shows that the eigenvalues of $\otimes_{i=1}^{k} u_i$ is almost uniformly distributed whatever V_i's are, i.e., $\{S_{1,k}\}$ is uniformly distributed. This argument applies to the restriction of α to the subalgebra $\otimes_{k=l}^{\infty} M_{m_k}$ of A for any $l \in \mathbf{N}$, concluding that $\{S_{l,k}\}_{k=l}^{\infty}$ is uniformly distributed.

Suppose that $\{S_{l,k}\}_{k=l}^{\infty}$ is uniformly distributed for any $l \in \mathbf{N}$. Then it is easy to construct the required projections as in Definition 1.1 with $F = \emptyset$ in $\otimes_{i=l}^{k} M_{m_i}$ for a sufficiently large k. (This is a linear algebra problem.) Hence the Rohlin property follows (see [4, 16] for details).

When $\{S_{l,k}\}_{k=l}^{\infty}$ is uniformly distributed in Lemma 5.2, we say that the eigenvalues of $\otimes_{i=l}^{\infty} u_i$ are uniformly distributed.

Proof of Theorem 1.4. As noted in Section 1, if α and β have the Rohlin property, for any $\epsilon > 0$ there exist unitaries $u_1, u_2 \in A$ such that $\|u_1 - 1\| < \epsilon$, $\|u_2 - 1\| < \epsilon$ and $\mathrm{Ad}\, u_1 \circ \alpha$ and $\mathrm{Ad}\, u_2 \circ \beta$ are of infinite tensor product type. Hence we may assume that there is a sequence $\{m_k\}$ of positive integers such that (A, α) is isomorphic to

$$(M_{m_1} \otimes \bigotimes_{i=1}^{\infty} (M_{m_{2i}} \otimes M_{m_{2i+1}}), \bigotimes_{i=0}^{\infty} \mathrm{Ad} u_{2i})$$

where $u_{2i} \in M_{m_{2i}} \otimes M_{m_{2i+1}}$, with $M_{m_0} = \mathbf{C}$ and (A, β) is isomorphic to

$$(\bigotimes_{i=1}^{\infty} (M_{m_{2i-1}} \otimes M_{m_{2i}}), \bigotimes_{i=0}^{\infty} \mathrm{Ad}\, u_{2i-1})$$

where $u_{2i-1} \in M_{m_{2i-1}} \otimes M_{m_{2i}}$. Since the eigenvalues of $\otimes_{i=1}^{\infty} u_{2i-1}$ are uniformly distributed, it follows that for a sufficiently large k, $\otimes_{i=1}^{k} u_{2i-1}$ is almost equal to $u_0 \otimes V$ in norm (up to conjugacy) where V is a unitary in $M_{m_2 \cdots m_{2k}}$. Then replacing M_{m_2} by $M_{m_2 \cdots m_{2k}}$ we may assume that there is a unitary $w_1 \in M_{m_1} \otimes M_{m_2}$ such that $\|w_1 - 1\| < \epsilon/2$, $w_1 u_1 = u_0 \otimes v_2$ where v_2 is a unitary of M_{m_2}. Note that (A, α) is still of the same form. In the same way we may assume that there is a unitary $w_2 \in M_{m_2} \otimes M_{m_3}$ such that $\|w_2 - 1\| < \epsilon/2^2$, $w_2 u_2 = v_2 \otimes v_3$ where v_3 is a unitary of M_{m_3}. Repeating this procedure we obtain unitaries $w_i \in M_{m_i} \otimes M_{m_{i+1}}$ and $v_i \in M_{m_i}$ such that

$$\begin{aligned} \|w_{2i} - 1\| &< 2^{-i}\epsilon, \\ w_i u_i &= v_i \otimes v_{i+1}, \end{aligned}$$

for $i = 1, 2, \ldots$ where $v_1 = u_0$. Let

$$w_e = \otimes_{i=0}^{\infty} w_{2i}, \ w_o = \otimes_{i=0}^{\infty} w_{2i-1}.$$

Then w_e, w_o are unitaries of A with $\|w_e - 1\| < \epsilon$, $\|w_o - 1\| < \epsilon$ and

$$\begin{aligned} \mathrm{Ad}\, w_e \circ \alpha &= \otimes_{i=0}^{\infty} \mathrm{Ad}\, w_{2i} u_{2i} = \otimes_{i=0}^{\infty} \mathrm{Ad}\, v_i, \\ \mathrm{Ad}\, w_o \circ \beta &= \otimes_{i=0}^{\infty} \mathrm{Ad}\, w_{2i-1} u_{2i-1} = \otimes_{i=0}^{\infty} \mathrm{Ad} v_i. \end{aligned}$$

This completes the proof.

6 Non-commutative tori

Let $\Theta = (\theta_{ij}) \in M_n(\mathbf{R})$ be such that $\theta_{ij} = -\theta_{ji}$ and let A_Θ be the universal C^*-algebra generated by n unitaries $u_1, \ldots, u_n$ with relations $u_i u_j u_i^* u_j^* = e^{2\pi i\theta_{ij}}1$; A_Θ is called a non-commutative torus. We call Θ to be completely irrational if for any $x \in \mathbf{Z}^n \setminus \{0\}$ there is a $y \in \mathbf{Z}^n$ with $(\Theta x, y) \notin \mathbf{Z}$, i.e., $u^x = u_1^{x_1} \cdots u_n^{x_n}$ is not in the center of A_Θ. It follows that A_Θ is simple if and only if Θ is completely irrational. In this case A_Θ has a unique tracial state. Define an action α of $\mathbf{T}^n$ on A_Θ by $\alpha_\lambda(u_i) = \lambda_i u_i$ with $\lambda = (\lambda_1, \ldots, \lambda_n)$. The action α is ergodic and the above tracial state is defined as the average over this action. For a $\lambda \in \mathbf{T}^n$ α_λ is inner if and only if $\lambda_j = e^{2\pi i(\Theta x)_j}$ for some $x \in \mathbf{Z}^n$, i.e., α_λ is implemented by u^x for some $x \in \mathbf{Z}^n$.

Theorem 6.1 *Let A_Θ be as above and suppose that A_Θ is a simple C^*-algebra. Let $\lambda \in \mathbf{T}^n$ be such that α_{λ^m} is outer for any $m \neq 0$. Then α_λ has the Rohlin property.*

Proof. We may assume that θ_{12} is irrational and $\lambda_1^m \notin \{e^{2\pi i(\Theta x)_1};\ x \in \mathbf{Z}^n\}$ for all $m \neq 0$. Let $\rho = e^{2\pi i\theta_{12}}$. Then, by using the proof of Lemma 4.6 of [3], for any $\epsilon > 0$ there are $p, q \in \mathbf{N}$ such that

$$\begin{aligned}
&|\rho^p - 1|,\ |\lambda_1^p - \lambda_1|,\ |e^{2\pi i p\theta_{k1}} - 1|\ (k = 3, \ldots, n),\\
&|\rho^q - 1|,\ |\lambda_2^q - 1|,\ |e^{2\pi i q\theta_{k1}} - 1|\ (k = 3, \ldots, n),\\
&|\rho^{pq} - \rho|
\end{aligned}$$

are all smaller than ϵ. (First choose a sufficiently large p such that the values in the first line above are all smaller than ϵ, and then choose q such that $|\rho^q - \rho^{1/p}|$, $|\lambda_2^q - 1|$, $|e^{2\pi i q\theta_{k1}} - 1|$ are sufficiently small, where $\rho^{1/p} = e^{2\pi i\theta_{12}/p}$. Then the estimate on $|\rho^q - \rho^{1/p}|$ will yield both $|\rho^q - 1| < \epsilon$ and $|\rho^{pq} - \rho| < \epsilon$.) For $\epsilon = 1/m$ let $v_m = u_1^p$ and $w_m = u_2^q$. Then $\{v_m, w_m\}$ forms a central sequence of pairs of unitaries in A_Θ such that

$$\begin{aligned}
&v_m w_m v_m^* w_m^* = \rho_m 1,\ \rho_m \to \rho,\\
&\alpha_\lambda(v_m) = \mu_m v_m,\ \mu_m \to \lambda_1,\\
&\alpha_\lambda(w_m) = \nu_m w_m,\ \nu_m \to 1.
\end{aligned}$$

By applying Lemma 6.3 below we can conclude the proof.

Lemma 6.2 *Let θ be an irrational number and let A_θ be the non-commutative torus corresponding to $\Theta \in M_2(\mathbf{R})$ with $\theta_{12} = \theta$. Let $\lambda \in \mathbf{T}^2$ be such that any non-zero power of α_λ is outer. Then α_λ has the approximate Rohlin property in the sense that for any $m \in \mathbf{N}$ there is a central sequence of orthogonal families $\{e_i^{(j)};\ i = 0, \ldots, m-1\}$ of projections in A_θ such that*

$$\begin{aligned}
&\|\alpha_\lambda(e_i^{(j)}) - e_{i+1}^{(j)}\| \to 0,\\
&\tau(e_0^{(j)}) \to 1/m,
\end{aligned}$$

where $e_m^{(j)} = e_0^{(j)}$ and τ is the unique tracial state of A_θ.

Proof. Note that A_θ is a nuclear C^*-algebra of real rank zero [3] and that $K_0(A_\theta) \cong \mathbf{Z}+\theta\mathbf{Z}$ and A_θ has cancellation [1]. We can use the proof of Lemma 4.3 to obtain such a sequence if we drop the requirement of centralness. We needed the AF property there to make the sequence central; othewise the above mentioned properties would suffice. To obtain a central sequence of such families, we use the argument in the proof of Theorem 6.1; assuming that $(2\pi)^{-1}\log\lambda_1$ is rationally independent of θ, we obtain a central sequence $\{v_k, w_k\}$ of pairs of unitaries in A_θ such that

$$\begin{aligned} &v_k w_k v_k^* w_k^* = \rho_k 1, \ \rho_k \to e^{2\pi i\theta}, \\ &\alpha_\lambda(v_k) = \mu_k v_k, \ \mu_k \to \lambda_1, \\ &\alpha_\lambda(w_k) = \nu_k w_k, \ \nu_k \to 1. \end{aligned}$$

On the other hand, for any $\epsilon > 0$ and $m \in \mathbf{N}$ we have an orthogonal family $\{e_i;\ i = 0,\ldots,m-1\}$ of projections in A_θ such that

$$\begin{aligned} &\|\alpha_{(\lambda_1,1)}(e_i) - e_{i+1}\| < \epsilon, \\ &m\tau(e_0) > 1-\epsilon, \end{aligned}$$

where $e_m = e_0$. Then we obtain a continuous field of such families $\{e_i(t);\ i = 0,\ldots,m-1\}$ in A_t with t in a small neighbourhood of θ [11]. Since the action α of $\mathbf{T}^2$ acts continuously on this field, we have that for t in a small neighbourhood of θ and for μ in a small neighbourhood of $(\lambda_1, 1)$,

$$\begin{aligned} &\|\alpha_\mu(e_i(t)) - e_{i+1}(t)\| < \epsilon, \\ &m\tau(e_i(t)) > 1-\epsilon, \end{aligned}$$

where $e_m(t) = e_0(t)$. By identifying the C^*-algebra B_k generated by v_k, w_k with A_{θ_k} with $\rho_k = e^{2\pi i\theta_k}$, we obtain the family $\{e_i(\theta_k)\}$ of projections in B_k. Since they form a central sequence, we can now conclude the proof.

Lemma 6.3 *Let A_θ, α_λ be as in Lemma 6.2. Then for any $k \in \mathbf{N}$ and any $\epsilon > 0$ there are $k_1,\ldots,k_m \geq k$ and projections $e_{i,j}$; $i = 1,\ldots,m$, $j = 0,\ldots,k_i - 1$ in A_θ such that*

$$\begin{aligned} &\sum_{i=1}^{m}\sum_{j=0}^{k_i-1} e_{i,j} = 1, \\ &\|\alpha_\lambda(e_{i,j}) - e_{i,j+1}\| < \epsilon, \end{aligned}$$

where $e_{i,k_i} = e_{i,0}$.

Proof. This can be proved as Lemma 4.4, by using the fact that if two projections in A_θ have the same trace value then they are equivalent. Furthermore, in Definition 1.1 of the Rohlin property, we can choose $m = 2$, $k_1 = k$, $k_2 = k+1$ as in the case of UHF algebras (cf. [17]). This follows by noting that we can choose, in Lemma 6.2, matrix units $\{e_{i,k}^{(j)}\}$ such that $\|\alpha_\lambda(e_{i,k}^{(j)}) - e_{i+1,k+1}^{(j)}\| < \epsilon$, $\tau(e_{0,0}) \to 1/m$ instead of just families of projections (see the proof of Theorem 2.1 in [15] for details).

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