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WHICH CONTAINS
PSEUDOCIRCLES**

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A TIME-LIKE SURFACE IN MINKOWSKI 3-SPACE WHICH CONTAINS PSEUDOCIRCLES

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ABSTRACT. Simple characterizations of a pseudosphere or a plane in Minkowski 3-space by the existence of pseudocircles are given.

1. INTRODUCTION

One of the simple characterizations of sphere in Euclidian 3-space $\mathbb{E}^3$ is as follows:

(*) A circle in $\mathbb{E}^3$ of arbitrary given radius can be pressed entirely on an arbitrary position of a surface.

The condition (*) is quite natural because an observer is an inhabitant of an ambient space, however, (*) requires a very large quantity of information because of its condition "an arbitrary position". In [2], Ogiue and Takagi give a much practical condition for a compact surface to be a sphere.

Theorem ([2], Ogiue-Takagi). *Let S be a surface in Euclidian 3-space $\mathbb{E}^3$. Suppose that, through each point $p \in S$, there exist two circles of $\mathbb{E}^3$ such that*

- (1) *they are contained in S in a neighbourhood of p ,*
- (2) *they are tangent to each other at p .*

Then S is locally a plane or a sphere.

In this paper we consider characterizations for a pseudosphere $S_1^2(r, a)$ in Minkowski 3-space $\mathbb{M}^3$ (for definition, see Section 2) analogous to the Ogiue-Takagi's result. The normal vector field on a pseudosphere in $\mathbb{M}^3$ is space-like, so that we stick to a surface S in $\mathbb{M}^3$ such that the normal vector field on S is space-like. Such a surface is called a *time-like surface*. There is another important class of surfaces in Minkowski 3-space $\mathbb{M}^3$ which is called *space-like surfaces*. In this case the induced metric on the surface is positively definite, so that we can give characterizations of a hyperbolic surface $H_1^2(r, a)$ by exactly the same arguments as those in [2]. Thus we do not consider this case in this paper.

The notion of pseudocircles in Minkowski 3-space is given analogous to that of circles in Euclidian 3-space. Let $\gamma(s)$ be a time-like or a space-like curve (cf., Section 2). We can define the principal normal $N(s)$, the binormal $B(s)$, the curvature $k(s)$ and the torsion $\tau(s)$, so that the Frenet-Serret type formula for such a curve holds (cf., Section 2). We say that $\gamma(s)$ is a *pseudocircle* if the torsion is equal to zero and the curvature is positive constant along the curve.

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Firstly, we have a simple characterization for pseudosphere in M^3 which is peculiar to pseudosphere in Minkowski 3-space.

Theorem A. *Let S be a time-like surface in Minkowski 3-space M^3 . Suppose that, through each point $p \in S$, there exist two pseudocircles γ_1, γ_2 of M^3 such that*

- (1) γ_1, γ_2 are contained in S in a neighbourhood of p ,
- (2) γ_1, γ_2 are tangent to each other at p ,
- (3) $\delta(\gamma_1(s))\delta(\gamma_2(s)) = -1$, where $\delta(\gamma_1(s)) = \langle N(s), N(s) \rangle$.

Then S is locally a pseudosphere.

If we remove the condition (3) in the assumption in Theorem A, we have the following theorem analogous to the Ogiue-Takagi's result.

Theorem B. *Let S be a time-like surface in Minkowski 3-space M^3 . Suppose that, through each point $p \in S$, there exist two pseudocircles γ_1, γ_2 of M^3 such that*

- (1) γ_1, γ_2 are contained in S in a neighbourhood of p ,
- (2) γ_1, γ_2 are tangent to each other at p .

Then S is locally a time-like-plane or a pseudosphere.

We remark that an intersection of a pseudosphere with an arbitray plane is a pseudocircle or a pair of light-like lines. We have given characterizations of a pseudosphere by the existence of light-like lines in [1]. The method of the proofs of the theorems we used here is analogous to that of Theorem 1 in [2].

All surfaces and maps considered here are of class C^∞ unless stated otherwise.

2. BASIC NOTIONS

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3) | x_1, x_2, x_3 \in \mathbb{R}\}$ be the usual oriented 3-dimensional vector space and differential manifold, which is oriented by $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ and given the euclidian differentiable structure. *Minkowski 3-space* is defined by $M^3 = \{\mathbb{R}^3, I_{(2,1)}\}$, where $I_{(2,1)} = dx_1^2 + dx_2^2 - dx_3^2$. Thus the metric tensor is given by $\langle X, Y \rangle = x_1y_1 + x_2y_2 - x_3y_3$, where $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$. A vector X in M^3 is called *light-like* if $\langle X, X \rangle = 0$, *space-like* if $\langle X, X \rangle > 0$ and *time-like* if $\langle X, X \rangle < 0$. A curve γ is called *light-like* if its tangent vector field $\dot{\gamma}$ is always light-like. We also say that a curve γ is *non-light-like* if its tangent vector field $\dot{\gamma}$ is always space-like or time-like. *The pseudosphere* is defined to be

$$S_1^2(r, a) = \{(x_1, x_2, x_3) | (x_1 - a_1)^2 + (x_2 - a_2)^2 - (x_3 - a_3)^2 = r^2\},$$

where $a = (a_1, a_2, a_3)$ is the center and $r > 0$ is the radial of $S_1^2(r, a)$.

The Levi-Civita connection of M^3 is denoted by $\tilde{\nabla}$. Let S be a surface in M^3 . We say that S is *time-like* if the normal vector field to S is space-like. On the time-like surface S we have the Levi-Civita connection corresponding to the induced Lorenzian metric on S . We denote it ∇ . *The normal connection* on S is denoted by $\nabla^\perp$. Let σ be a second fundamental form of S in M^3 which is given by

$$\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y,$$

where $\mathbf{X}$ and $\mathbf{Y}$ are vector field tangent to S . Since S is a time-like surface, there exists (at least locally) a unit normal vector field ξ on S . We have the following formula:

$$\langle \sigma(\mathbf{X}, \mathbf{Y}), \xi \rangle = \langle -\tilde{\nabla}_{\mathbf{X}}\xi, \mathbf{Y} \rangle,$$

so that $-\tilde{\nabla}_{\mathbf{X}}\xi$ is the shape operator in this case. We denote that $A\mathbf{X} = -\tilde{\nabla}_{\mathbf{X}}\xi$. It is well-known that the shape operator is self-adjoint with respect to $\langle, \rangle$ (i.e., $\langle A\mathbf{X}, \mathbf{Y} \rangle = \langle \mathbf{X}, A\mathbf{Y} \rangle$). The covariant derivative $\nabla'_{\mathbf{X}}\sigma$ of σ is defined by

$$\nabla'_{\mathbf{X}}\sigma(\mathbf{Y}, \mathbf{Z}) = \nabla_{\mathbf{X}}\sigma(\mathbf{Y}, \mathbf{Z}) - \sigma(\nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}) - \sigma(\mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z}).$$

We say that a point $p \in S$ is *umbilic* if there is a normal vector $\mathbf{Z} \in T_p S^\perp$ such that $\sigma(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y} \rangle \mathbf{Z}$ for all $\mathbf{X}, \mathbf{Y} \in T_p S$. A time-like surface S is *totally umbilic* provided every point of S is umbilic. It is well-known that S is totally umbilic if and only if its shape operator is scalar (cf., §4, Lemma 21 in [3]). The following result gives a characterization of a pseudo sphere or a time-like plane in $\mathbb{M}^3$ (cf., §4, Propositions 13, 36 in [3]).

Proposition 2.1. *If S is totally umbilic, then S is an open set in a pseudosphere or a time-like plane.*

In order to define the notion of pseudocircles, we introduce the notion of *Lorenzian exterior products*. For any vectors $\mathbf{X} = (x_1, x_2, x_3), \mathbf{Y} = (y_1, y_2, y_3) \in \mathbb{M}^3$, the *Lorenzian exterior product* (with respect to $\langle, \rangle$) is defined to be

$$\mathbf{X} \wedge \mathbf{Y} = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

Let γ be a non-light-like curve. We may assume that γ is parameterized by arclength s . Thus we have $\langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle = \varepsilon(\gamma(s)) = \pm 1$. We call $\varepsilon(\gamma(s))$ the *causal character* of γ . We can also define the notion of the *curvature* $k(s)$ of the curve γ by $k(s) = \sqrt{|\langle \ddot{\gamma}(s), \ddot{\gamma}(s) \rangle|}$. We define the *principal normal vector* $\mathbf{N}(s)$ by $\ddot{\gamma}(s) = k(s)\mathbf{N}(s)$ except the point where $k(s) = 0$. We call $\delta(\gamma(s)) = \langle \mathbf{N}(s), \mathbf{N}(s) \rangle$ the *second causal character* of $\gamma(s)$. We also define the *binormal vector* $\mathbf{B}(s)$ by $\mathbf{B}(s) = \mathbf{T}(s) \wedge \mathbf{N}(s)$, where $\mathbf{T}(s) = \dot{\gamma}(s)$. We can show that $\dot{\mathbf{B}}(s) = \tau(s)\mathbf{N}(s)$ for some function $\tau(s)$. We call $\tau(s)$ the *torsion* of γ at s . By the exactly same arguments as in the case for a curve in Euclidian 3-space, we have the following Frenet-Serret formula (cf., [4]):

$$(2.1) \quad \begin{cases} \tilde{\nabla}_{\mathbf{T}(s)}\mathbf{T}(s) = k(s)\mathbf{N}(s) \\ \tilde{\nabla}_{\mathbf{T}(s)}\mathbf{N}(s) = -\varepsilon(\gamma(s))\delta(\gamma(s))k(s)\mathbf{T}(s) + \varepsilon(\gamma(s))\tau(s)\mathbf{B}(s) \\ \tilde{\nabla}_{\mathbf{T}(s)}\mathbf{B}(s) = \tau(s)\mathbf{N}(s). \end{cases}$$

We remark that if the curvature k never vanish then the second causal character $\delta(s)$ is constant along the curve γ . When the γ is a plane curve, then we have $\tau(s) \equiv 0$.

We now define the notion of pseudocircle analogous to the notion of circles in Euclidian space $\mathbb{E}^3$. Let γ be a non-light-like curve in $\mathbb{M}^3$. We say that γ is a *pseudocircle* if the torsion τ is always vanish and the curvature k is positive constant along the curve. By Frenet-Serret

formula, the curve γ is a pseudocircle if and only if there exists a positive real number k such that

$$(2.2) \quad \begin{cases} \tilde{\nabla}_{\mathbf{T}(s)} \mathbf{T}(s) = k \mathbf{N}(s) \\ \tilde{\nabla}_{\mathbf{T}(s)} \mathbf{N}(s) = -\varepsilon(\gamma(s)) \delta(\gamma(s)) k \mathbf{T}(s). \end{cases}$$

We call γ a *space-like pseudocircle* if $\varepsilon(\gamma) = 1$ and a *time-like pseudocircle* if $\varepsilon(\gamma) = -1$. We also call γ a *positive pseudocircle* if $\delta(\gamma) = 1$ and γ a *negative pseudocircle* if $\delta(\gamma) = -1$. On a time-like plane, the causal characters of the tangent vector and the normal vector are different, so that we only have three cases those are the positive space-like pseudocircles, the positive time-like pseudocircles or the negative space-like pseudocircles. We have the following characterization of a pseudocircle in $\mathbb{M}^3$.

Lemma 2.2. *Let $\gamma = \gamma(s)$ be a non-light-like curve in $\mathbb{M}^3$ parameterized by arclength s . Then $\gamma = \gamma(s)$ is a pseudocircle of curvature k if and only if*

$$(2.3) \quad \tilde{\nabla}_{\dot{\gamma}} \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} + \varepsilon(\gamma) \delta(\gamma) k^2 \dot{\gamma} = 0$$

Proof. Suppose that γ is a pseudocircle of curvature k . It follows from (2.2) that

$$\tilde{\nabla}_{\dot{\gamma}} \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} = \tilde{\nabla}_{\dot{\gamma}} (k \mathbf{N}) = k \tilde{\nabla}_{\dot{\gamma}} \mathbf{N} = -\varepsilon(\gamma) \delta(\gamma) k^2 \dot{\gamma}.$$

If γ satisfies the condition (2.3), by the Frenet-Serret formula (2.1), we have

$$\begin{aligned} \tilde{\nabla}_{\dot{\gamma}(s)} \tilde{\nabla}_{\dot{\gamma}(s)} \dot{\gamma}(s) &= \tilde{\nabla}_{\dot{\gamma}(s)} (k(s) \mathbf{N}(s)) \\ &= (\tilde{\nabla}_{\dot{\gamma}(s)} k(s)) \mathbf{N}(s) + k(s) \tilde{\nabla}_{\dot{\gamma}(s)} \mathbf{N}(s) \\ &= (\tilde{\nabla}_{\dot{\gamma}(s)} k(s)) \mathbf{N}(s) - \varepsilon(\gamma) \delta(\gamma(s)) k(s)^2 \dot{\gamma}(s) + \varepsilon(\gamma) k(s) \tau(s) \mathbf{B}(s). \end{aligned}$$

It follows from (2.3) that

$$\tilde{\nabla}_{\dot{\gamma}(s)} \nabla_{\dot{\gamma}(s)} \dot{\gamma}(s) = -\varepsilon(\gamma) \delta(\gamma) k^2 \dot{\gamma}(s).$$

Thus we have

$$\begin{aligned} \tilde{\nabla}_{\dot{\gamma}(s)} k(s) &= 0, \\ -\varepsilon(\gamma) \delta(\gamma(s)) k(s)^2 &= -\varepsilon(\gamma) \delta(\gamma(s)) k^2, \\ \varepsilon(\gamma) k(s) \tau(s) &= 0, \end{aligned}$$

so that, $k(s) = k$ and $\tau(s) \equiv 0$.

We have the following result which is useful for the proof of theorems.

Lemma 2.3. Let $\gamma = \gamma(s)$ be a non-light-like curve in $\mathbb{M}^3$ parameterized by arclength s . The $\gamma = \gamma(s)$ is a pseudocircle of curvature k if and only if it satisfies

$$(2.4) \quad \begin{cases} \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} \dot{\gamma} + \varepsilon(\gamma) \delta(\gamma) k^2 \dot{\gamma} + \tilde{\nabla}_{\dot{\gamma}} \sigma(\dot{\gamma}, \dot{\gamma}) - \nabla_{\dot{\gamma}}^{\perp} \sigma(\dot{\gamma}, \dot{\gamma}) = 0 \\ \nabla'_{\dot{\gamma}} \sigma(\dot{\gamma}, \dot{\gamma}) + 3\sigma(\dot{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma}) = 0. \end{cases}$$

Proof. For any γ , we have

$$\begin{aligned} & \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} \dot{\gamma} + \varepsilon(\gamma) \delta(\gamma) k^2 \dot{\gamma} + \tilde{\nabla}_{\dot{\gamma}} \sigma(\dot{\gamma}, \dot{\gamma}) - \nabla_{\dot{\gamma}}^{\perp} \sigma(\dot{\gamma}, \dot{\gamma}) \\ &= \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} \dot{\gamma} + \varepsilon(\gamma) \delta(\gamma) k^2 \dot{\gamma} + \tilde{\nabla}_{\dot{\gamma}} (\tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} - \nabla_{\dot{\gamma}} \dot{\gamma}) - \nabla_{\dot{\gamma}}^{\perp} (\tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} - \nabla_{\dot{\gamma}} \dot{\gamma}) \\ &= \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} \dot{\gamma} + \varepsilon(\gamma) \delta(\gamma) k^2 \dot{\gamma} + \tilde{\nabla}_{\dot{\gamma}} \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} - \nabla_{\dot{\gamma}} \dot{\gamma} - \nabla_{\dot{\gamma}}^{\perp} \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} \\ &= \tilde{\nabla}_{\dot{\gamma}} \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} + \varepsilon(\gamma) \delta(\gamma) k^2 \dot{\gamma} - \nabla_{\dot{\gamma}}^{\perp} \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma}. \end{aligned}$$

If γ is a pseudocircle, it follows from (2.3) that $\nabla_{\dot{\gamma}}^{\perp} \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} = 0$, so that the first equation of (2.4) holds.

We also have

$$\begin{aligned} & \nabla'_{\dot{\gamma}} \sigma(\dot{\gamma}, \dot{\gamma}) + 3\sigma(\dot{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma}) \\ &= \nabla_{\dot{\gamma}}^{\perp} (\sigma(\dot{\gamma}, \dot{\gamma})) - 2\sigma(\nabla_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma}) + 3\sigma(\dot{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma}) \\ &= \nabla_{\dot{\gamma}}^{\perp} (\sigma(\dot{\gamma}, \dot{\gamma})) + \sigma(\dot{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma}) \\ &= \nabla_{\dot{\gamma}}^{\perp} (\tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} - \nabla_{\dot{\gamma}} \dot{\gamma}) + \tilde{\nabla}_{\dot{\gamma}} \nabla_{\dot{\gamma}} \dot{\gamma} - \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} \dot{\gamma} \\ &= \nabla_{\dot{\gamma}}^{\perp} (\tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} - \nabla_{\dot{\gamma}} \dot{\gamma}) + \nabla_{\dot{\gamma}}^{\perp} \nabla_{\dot{\gamma}} \dot{\gamma} \\ &= \nabla_{\dot{\gamma}}^{\perp} \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma}. \end{aligned}$$

It follows from (2.3) that $\nabla_{\dot{\gamma}}^{\perp} \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} = 0$.

Suppose that the equalities (2.4) hold for some positive number k , then we have

$$\begin{aligned} & \tilde{\nabla}_{\dot{\gamma}} \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} + \varepsilon(\gamma) \delta(\gamma) k^2 \dot{\gamma} \\ &= \tilde{\nabla}_{\dot{\gamma}} (\nabla_{\dot{\gamma}} \dot{\gamma} + \sigma(\dot{\gamma}, \dot{\gamma})) + \varepsilon(\gamma) \delta(\gamma) k^2 \dot{\gamma} \\ &= \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} \dot{\gamma} + \sigma(\dot{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma}) + \tilde{\nabla}_{\dot{\gamma}} \sigma(\dot{\gamma}, \dot{\gamma}) - \nabla_{\dot{\gamma}}^{\perp} \sigma(\dot{\gamma}, \dot{\gamma}) + \nabla_{\dot{\gamma}}^{\perp} \sigma(\dot{\gamma}, \dot{\gamma}) + \varepsilon(\gamma) \delta(\gamma) k^2 \dot{\gamma} \\ &= \sigma(\dot{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma}) + \nabla_{\dot{\gamma}}^{\perp} \sigma(\dot{\gamma}, \dot{\gamma}) \\ &= \sigma(\dot{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma}) + \nabla'_{\dot{\gamma}} \sigma(\dot{\gamma}, \dot{\gamma}) + 2\sigma(\nabla_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma}) \\ &= \nabla'_{\dot{\gamma}} \sigma(\dot{\gamma}, \dot{\gamma}) + 3\sigma(\dot{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma}) = 0 \end{aligned}$$

It follows from Lemma 2.2 that γ is a pseudocircle of curvature k .

We can show that the intersection of a pseudosphere $S_1^2(r, a)$ with a non-degenerate plane is a pseudocircle or a pair of light-like lines.

3. PROOF OF RESULTS

In this section we give proofs of the theorems. The method of the proofs almost follows along the line of the proof of Theorem 1 in [2], however, there are some differences from the original proof caused by the index of the metric tensor. There are common parts of the proofs for Theorems A and B, so we firstly start to prove Theorem A.

Proof of Theorem A. Let k_i be the curvature of γ_i ($i = 1, 2$). Let X_p be the common unit tangent vector of γ_1 and γ_2 at p . Then $X : p \mapsto X_p$ defines a vector field on S , which may not be continuous. Let Y be a unit vector field on S which satisfies $\langle X, Y \rangle = 0$ and $\langle X \wedge Y, \xi \rangle > 0$, where ξ is the unit normal vector field on S . Let X_1, X_2 be smooth unit vector fields on a neighbourhood of $\gamma_1 \cup \gamma_2$ such that X_i is equal to the unit tangent vector of γ_i on γ_i and $X_i(p) = X_p$ ($i = 1, 2$). It follows that $\langle \nabla_{X_i} X_i, X_i \rangle = 0$. Thus there exist real numbers c_i ($i = 1, 2$) such that $\nabla_{X_i} X_i(p) = c_i Y(p)$. Suppose that $c_1 = c_2$. Since $\tilde{\nabla}_{X_i} X_i = \nabla_{X_i} X_i + \sigma(X_i, X_i)$ ($i = 1, 2$), and $X_i(p) = X_p$, we have $\tilde{\nabla}_{X_1} X_1(p) = \tilde{\nabla}_{X_2} X_2(p)$. The last equation means that $k_1 = k(\gamma_1(p)) = k(\gamma_2(p)) = k_2$, so that $\gamma_1 = \gamma_2$ on a neighbourhood of p . This contradicts to the assumption (i.e., $c_1 \neq c_2$).

It follows from Lemma 2.3 that

$$(3.1) \quad \begin{cases} \nabla'_X \sigma(X, X) + 3\sigma(X, c_1 Y) = 0 \\ \nabla'_X \sigma(X, X) + 3\sigma(X, c_2 Y) = 0 \end{cases}$$

at each point. Therefore we have $\sigma(X, Y) = 0$. It follows that $\langle AX, Y \rangle = \langle \sigma(X, Y), \xi \rangle = 0$. Since $\dim S = 2$, we see that AX is parallel to X . Thus X is (and hence Y is also) a principal vector at each point. Let λ and μ be principal curvature and η_1 and η_2 the corresponding principal vectors so that $A\eta_1 = \lambda\eta_1$ and $A\eta_2 = \mu\eta_2$. Put $S_0 = \{p \in S | \lambda(p) \neq \mu(p)\}$. If $S_0 = \emptyset$, then S is totally umbilic. Therefore we suppose that $S_0 \neq \emptyset$. Then η_1 and η_2 are smooth vector fields in some neighbourhood of each point of S_0 . Since $\langle \eta_1, \eta_2 \rangle = 0$, we may assume that η_1 is space-like and η_2 is time-like. Since $\langle \eta_1, \eta_2 \rangle = 1$, we have

$$(3.2) \quad \langle \nabla_{\eta_i} \eta_1, \eta_1 \rangle = 0 \quad (i = 1, 2),$$

so that we may put

$$(3.3) \quad \nabla_{\eta_1} \eta_1 = \alpha \eta_2 \text{ and } \nabla_{\eta_2} \eta_1 = \beta \eta_2.$$

It follows from $\langle \eta_1, \eta_2 \rangle = 0$ that

$$(3.4) \quad \langle \nabla_{\eta_i} \eta_1, \eta_2 \rangle + \langle \eta_1, \nabla_{\eta_i} \eta_2 \rangle = 0 \quad (i = 1, 2).$$

We substitute (3.30 to (4.3), then we have $\langle \eta_1, \nabla_{\eta_1} \eta_2 \rangle = \alpha$ and $\langle \eta_1, \nabla_{\eta_2} \eta_2 \rangle = \beta$. Since η_1 is space-like, we have

$$(3.5) \quad \nabla_{\eta_1} \eta_2 = \alpha \eta_1 \text{ and } \nabla_{\eta_2} \eta_2 = \beta \eta_1.$$

Put $S_{0,i} = \{p \in S | X(p) = \eta_i(p)\}$ ($i = 1, 2$). Then $S_0 = S_{0,1} \cup S_{0,2}$, and it is easily seen that $S_0 \subset \overline{S_{0,1}} \cup \text{Int} S_{0,2}$ and $S_0 \subset \overline{S_{0,2}} \cup \text{Int} S_{0,1}$ and hence that $S_{0,1}$ or $S_{0,2}$ has interior points, or $S_{0,1}$ or $S_{0,2}$ is dense in S_0 . We now distinguish the following two cases.

Case 1) $S_{0,1}$ has interior points or $S_{0,1}$ is dense in S_0 .

Case 2) $S_{0,2}$ has interior points or $S_{0,2}$ is dense in S_0 .

The situation is different between two cases. In the case 1) both of two pseudo circles through p are space-like. On the other hand, both of the pseudocircles are positive time-like in the case 2), so we only consider the case 1).

In this case $X(p)$ is space-like. Since $\sigma(X, Y) = 0$, we obtain

$$\begin{aligned}\nabla'_{\eta_1} \sigma(\eta_1, \eta_1) &= \nabla_{\eta_1}^\perp \sigma(\eta_1, \eta_1) - 2\sigma(\eta_1, \nabla_{\eta_1} \eta_1) \\ &= \nabla_{\eta_1}^\perp (\lambda \xi) - 2\alpha \sigma(\eta_1, \eta_2) \\ &= \nabla_{\eta_1}^\perp (\lambda \xi) \\ &= (\nabla_{\eta_1}^\perp \lambda) \xi + \lambda \nabla_{\eta_1}^\perp \xi \\ &= (\nabla_{\eta_1}^\perp \lambda) \xi.\end{aligned}$$

It follows from (3.1) and $\sigma(X, Y) = 0$ that

$$(3.6) \quad \nabla'_X \sigma(X, X) = 0.$$

Therefore we have $\nabla_X \lambda = 0$ on $S_{0,1}$. If p is an interior point of $S_{0,1}$, then

$$(3.7) \quad \nabla_{\eta_1} \lambda = 0$$

holds in some neighbourhood of p . If $S_{0,1}$ is dense in S_0 , then, by continuity, (3.7) holds on S_0 .

We choose an orthonormal frame field e_1, e_2 in a sufficiently small tubular neighbourhood of γ_1 in such a way that $e_1 = \dot{\gamma}_1$ along γ_1 , and put $\nabla_{e_1} e_1 = a e_2$ and $\nabla_{e_2} e_1 = b e_2$ so that $\nabla_{e_1} e_2 = a e_1$ and $\nabla_{e_2} e_2 = b e_1$. Let (h_{ij}) be the matrix of the shape operator with respect to e_1 and e_2 (i.e., $-\nabla_{e_i} \xi = h_{i1} e_1 + h_{i2} e_2$ ($i = 1, 2$)).

On the other hand, we have

$$\tilde{\nabla}_{e_1} \sigma(e_1, e_1) - \nabla_{e_1}^\perp \sigma(e_1, e_1) = -(h_{11}^2 e_1 + h_{11} h_{12} e_2)$$

and

$$\nabla_{e_1} \nabla_{e_1} e_1 = a^2 e_1 + (\nabla_{e_1} a) e_2.$$

By Lemma 2.3, we have

$$\{a^2 e_1 + (\nabla_{e_1} a) e_2\} + \delta(\gamma_1) k^2 e_1 - \{h_{11}^2 e_1 + h_{11} h_{12} e_2\} = 0,$$

so that, along γ_1 ,

$$(3.8) \quad \delta(\gamma_1) k^2 = -a^2 + h_{11}^2$$

$$(3.9) \quad \nabla_{e_1} a = h_{11} h_{12}.$$

It also follows from the second equation of Lemma 2.3 that

$$\nabla_{e_1}^\perp \sigma(e_1, e_1) + \sigma(e_1, \nabla_{e_1} e_1) = 0.$$

Since

$$\nabla_{e_1}^\perp \sigma(e_1, e_1) = \nabla_{e_1}^\perp \{ \langle -\nabla_{e_1} \xi, e_1 \rangle \xi \} = (\nabla_{e_1}^\perp h_{11}) \xi$$

and

$$\sigma(e_1, \nabla_{e_1} e_1) = \sigma(e_1, ae_2) = -ah_{12}\xi,$$

we have

$$(3.10) \quad \nabla_{e_1} h_{11} - ah_{12} = 0.$$

We remark that ξ_1, e_1 are space-like and ξ_2, e_2 are time-like. We define θ such that

$$(3.11) \quad \begin{cases} e_1 = \eta_1 \cosh \theta + \eta_2 \sinh \theta \\ e_2 = \eta_1 \sinh \theta + \eta_2 \cosh \theta. \end{cases}$$

So we have

$$\begin{cases} Ae_1 = \lambda \eta_1 \cosh \theta + \mu \eta_2 \sinh \theta \\ Ae_2 = \lambda \eta_1 \sinh \theta + \mu \eta_2 \cosh \theta, \end{cases}$$

we get

$$(3.12) \quad \begin{cases} h_{11} = \lambda \cosh^2 \theta - \mu \sinh^2 \theta \\ h_{12} = -(\lambda - \mu) \cosh \theta \sinh \theta \\ h_{22} = -\lambda \sinh^2 \theta + \mu \cosh^2 \theta. \end{cases}$$

By differentiating (3.11) with respect to e_1 , we have

$$\nabla_{e_1} e_1 = \eta_1 \{ \alpha \cosh \theta + \beta \sinh \theta + \nabla_{e_1} \theta \} \sinh \theta + \eta_2 \{ \alpha \cosh \theta + \beta \sinh \theta + \nabla_{e_1} \theta \} \cosh \theta.$$

Since $\nabla_{e_1} e_1 = ae_2 = a\eta_1 \sinh \theta + a\eta_2 \cosh \theta$, we obtain

$$(3.13) \quad a = \alpha \cosh \theta + \beta \sinh \theta + \nabla_{e_1} \theta.$$

We also have $(\nabla_{\eta_1} A)(\eta_2) = \nabla_{\eta_1} (A\eta_2) - A\nabla_{\eta_1} \eta_2 = (\nabla_{\eta_1} \mu)\eta_2 + \alpha(\mu - \lambda)\eta_1$ and $(\nabla_{\eta_2} A)(\eta_1) = (\nabla_{\eta_2} \lambda)\eta_1 + \beta(\lambda - \mu)\eta_2$. By the equation of Codazzi $(\nabla_{\eta_1} A)(\eta_2) - (\nabla_{\eta_2} A)(\eta_1) = 0$, we have

$$(3.14) \quad \begin{cases} \alpha(\lambda - \mu) = \nabla_{\eta_2} \lambda \\ \beta(\lambda - \mu) = \nabla_{\eta_1} \mu. \end{cases}$$

Therefore by (3.7), (3.10), (3.12), (3.13) and (3.14) we obtain

$$(3.15) \quad \{ 3(\lambda - \mu) \cosh \theta \cdot \nabla_{e_1} \theta - \sinh^2 \theta \cdot \nabla_{\eta_2} \mu \} \sinh \theta = 0.$$

We remark that $\theta = 0$ at p . Then the point p has one of the following properties:

(A) There exists no sequence $\{p_n \in \gamma_1 | \theta(p_n) \neq 0\}$ with $\lim p_n = p$.

(B) There exists a sequence $\{p_n \in \gamma_1 | \theta(p_n) \neq 0\}$ with $\lim p_n = p$.

If p is a point which has the property (A), then it is clear that the integral curve of η_1 through p coincides with γ_1 on the connected component of $\{q \in \gamma_1 | \theta(q) = 0\}$ containing p .

If p is a point with the property (B), it follows from (3.15) that

$$(3.16) \quad 3(\lambda - \mu) \cosh \theta \cdot \nabla_{e_1} \theta - \sinh^2 \theta \cdot \nabla_{\eta_2} \mu = 0.$$

Since $\lambda \neq \mu$, taking to the limit of (3.16), we obtain

$$(3.17) \quad \nabla_{e_1} \theta = 0.$$

Applying ∇_{e_1} to (3.15) and substituting the relation (3.16) into the equation on the set $\{p_n \in \gamma_1 | \theta(p_n) \neq 0\}$, obtain

$$\begin{aligned} & \frac{\sinh \theta \cdot \nabla_{\eta_2} \mu}{3(\lambda - \mu)} \{3(\lambda - \mu) \cosh \theta \cdot \nabla_{e_1} \theta - \sinh^2 \theta \cdot \nabla_{\eta_2} \mu\} \\ & + \{3(\nabla_{e_1} \lambda - \nabla_{e_1} \mu) \cosh \theta \cdot \nabla_{e_1} \theta + 3(\lambda - \mu) \sinh \theta \cdot (\nabla_{e_1} \theta)^2 \\ & + 3(\lambda - \mu) \cosh \theta \cdot \nabla_{e_1} \nabla_{e_1} \theta - 2 \sinh \theta \cosh \theta \cdot \nabla_{e_1} \nabla_{\eta_2} \mu - \sinh^2 \theta \cdot \nabla_{e_1} \nabla_{\eta_2} \mu\} = 0. \end{aligned}$$

Taking the limit of the above equation we have

$$(3.18) \quad \nabla_{e_1} \nabla_{e_1} \theta = 0,$$

since $\sinh \theta = 0$ and $\cosh \theta = 1$. It is clear that the (3.17) and (3.18) hold even if p has the property (A).

It follows from (3.13) and (3.17) that $a(p) = \alpha(p)$. This, together with (3.8) and (3.12), yields

$$(3.19) \quad \delta(\gamma_1)k^2 = -(\alpha(p))^2 + (\lambda(p))^2,$$

which means that the curvature of γ_1 is $\sqrt{\delta\{-(\alpha(p))^2 + (\lambda(p))^2\}}$.

For γ_2 , by exactly the same arguments as those of γ_1 , we obtain

$$(3.20) \quad \delta(\gamma_2)k_2^2 = -(\alpha(p))^2 + (\lambda(p))^2,$$

where k_2 is the curvature of γ_2 . Until here we have not used the condition that $\delta(\gamma_1)\delta(\gamma_2) = -1$. It follows from (3.19), (3.18) and the assumption $\delta(\gamma_1)\delta(\gamma_2) = -1$ that we have

$$(-(\alpha(p))^2 + (\lambda(p))^2)^2 < 0$$

This is a contradiction, so that $S_0 = \emptyset$ and S is totally umbilic. Since the time-like plane does not satisfy the condition that $\delta(\gamma_1)\delta(\gamma_2) = -1$, S is an open set of a pseudosphere.

Proof of Theorem B. Since we already proved Theorem A, we may consider the case that $\delta(\gamma_1)\delta(\gamma_2) = 1$. We use the same notations as those of the proof of Theorem A. So our purpose is to show that $S_0 = \emptyset$. Therefore we suppose that $S_0 \neq \emptyset$, so that we also distinguish the following two cases:

Case 1) $S_{0,1}$ has interior points or $S_{0,1}$ is dense in S_0 .

Case 2) $S_{0,2}$ has interior points or $S_{0,2}$ is dense in S_0 .

Both of the pseudocircles through p are space-like in the case 1) and are positive time-like in the case 2). For a positive time-like curve, we have $\varepsilon\delta = -1$, so that the equations (2.3) and (2.4) have the same form as equations for a negative space-like curve. It follows that the almost all arguments for positive time-like curves are same as those for negative space-like curve. Hence, we only consider the case 1).

We may use the equations (3.1)-(3.20) for the proof of Theorem B. Since $\delta(\gamma_1)\delta(\gamma_2) = 1$, γ_1 and γ_2 are space-like pseudocircles with the common same curvature $\sqrt{-(\alpha(p))^2 + (\lambda(p))^2}$.

Applying ∇_{e_1} to (3.13), we obtain $\nabla_{\eta_1} a = \nabla_{\eta_1} \alpha$ at p because of (3.17) and (3.18).

On the other hand, from (3.9) we get $\nabla_{\eta_1} a = \nabla_{e_1} a = h_{11}h_{12} = 0$ at p . Therefore we have $\nabla_{\eta_1} \alpha = 0$ at p . Since p is arbitrary, we get $\nabla_{\eta_1} \alpha = 0$ on $S_{0,1}$. If p is an interior point of $S_{0,1}$, then

$$(3.21) \quad \nabla_{\eta_1} \alpha = 0$$

holds in some neighbourhood of p . If $S_{0,1}$ is dense in S_0 , then, by continuity, (3.21) holds on S_0 . It follows from (3.21) that

$$\nabla_{\eta_1} \nabla_{\eta_1} \eta_1 = \nabla_{\eta_1} (\alpha \eta_2) = \alpha^2 \eta_1.$$

Moreover, since η_1 is a principal vector, we get

$$\begin{aligned} & \langle \tilde{\nabla}_{\eta_1} \sigma(\eta_1, \eta_1) - \nabla_{\eta_1}^\perp \sigma(\eta_1, \eta_1), V \rangle = \langle \tilde{\nabla}_{\eta_1} \sigma(\eta_1, \eta_1), V \rangle - \langle \nabla_{\eta_1}^\perp \sigma(\eta_1, \eta_1), V \rangle \\ & = \langle \tilde{\nabla}_{\eta_1} \sigma(\eta_1, \eta_1), V \rangle = \langle \tilde{\nabla}_{\eta_1} \xi, V \rangle = \langle (\nabla_{\eta_1} \lambda) \xi + \lambda \tilde{\nabla}_{e_1 \eta_1} \xi, V \rangle = \lambda \langle \tilde{\nabla}_{\eta_1} \xi, V \rangle \\ & = \lambda \langle -A \eta_1, V \rangle = -\lambda^2 \langle \eta_1, V \rangle \end{aligned}$$

for any tangent vector field V on S . Therefore we obtain

$$(3.22) \quad \nabla_{\eta_1} \nabla_{\eta_1} \eta_1 + (-\alpha^2 + \lambda^2) \eta_1 + \tilde{\nabla}_{\eta_1} \sigma(\eta_1, \eta_1) - \nabla_{\eta_1}^\perp \sigma(\eta_1, \eta_1) = 0.$$

Furthermore, since $\nabla_{\eta_1} \eta_1 = \alpha \eta_2$, it follows from $\sigma(X, Y) = 0$ and (3.6) that $(\nabla'_{\eta_1} \sigma)(\eta_1, \eta_1) + 3\sigma(\eta_1, \nabla_{\eta_1} \eta_1) = 0$ on $S_{0,1}$. If p is an interior point of $S_{0,1}$,

$$(3.23) \quad (\nabla'_{\eta_1} \sigma)(\eta_1, \eta_1) + 3\sigma(\eta_1, \nabla_{\eta_1} \eta_1) = 0$$

holds in some neighbourhood of p . If $S_{0,1}$ is dense in S_0 , then, by continuity, (3.23) holds on S_0 . By (3.22), (3.23) and Lemma 2.3, we see that the integral curve of η_1 through p is a space-like pseudocircle of curvature $\sqrt{\delta(\eta_1)\{-(\alpha(p))^2 + (\lambda(p))^2\}}$, where $\delta(\eta_1) = \langle \eta_1, \eta_1 \rangle$. By (3.7) and (3.21), the above curvature is constant along the curve. It is clear that $\delta(\eta_1) = \delta(\gamma_1)$.

Since we can apply the same arguments to γ_2 , letting θ_i ($i = 1, 2$) be the function defined the same as the function of (3.11), we consider the following cases:

(A)_i: There exists no sequence $\{p_n \in \gamma_i | \theta_i(p_n) \neq 0\}$ with $p = \lim p_n$.

(B)_i: There exists a sequence $\{p_n \in \gamma_i | \theta_i(p_n) \neq 0\}$ with $p = \lim p_n$.

It is clear that (A)₁ and (A)₂ does not occur. If (B)₁ and (A)₂, then γ_1 and γ_2 have the same curvature and the integral curve of η_1 thorough p coincide with γ_2 . Then we have

$$\tilde{\nabla}_{\eta_1} \eta_1 = \nabla_{\eta_1} \eta_1 + \sigma(\eta_1, \eta_1) = \alpha \eta_2 + \lambda \xi.$$

$$\tilde{\nabla}_{e_1} e_1 = \nabla_{e_1} e_1 + \sigma(e_1, e_1) = a e_2 + h_{11} \xi.$$

Since $d = a$, $\lambda = h_{11}$ and $\eta_2 = e_2$ at p , we obtain $\tilde{\nabla}_{\eta_1} \eta_1 = \tilde{\nabla}_{e_1} e_1$. This equation means that γ_1 and γ_2 coincide in a neighbourhood of p . This contradicts the assumption. By the same arguments as the above case, the case (A)₁ and (B)₂ contradicts the assumption. If (B)₁ and (B)₂, then γ_1 , γ_2 and the integral curve of η_1 through p have the same curvature and hence, we can show that normal vectors for these vectors are the same. It follows that γ_1 and γ_2 coincide in a neighbourhood of p . This contradicts the assumption, so that $S_0 = \emptyset$. This completes the proof.

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