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Abstract

The system of all the closure operators on a set V forms a lattice [3]. This lattice is isomorphic to the lattice of all the topped intersection structures on V. This paper discribes basic properties of these lattices, and gives a method of listing up all the members of these lattices.

1 Closure operators

A map $C : \mathcal{P}(V) \to \mathcal{P}(V)$ is called *a colosure operator on* V if it satisfies, for $A, B \subseteq V$,

(i) $A \subseteq C(A)$,

(ii) $A \subseteq B \Rightarrow C(A) \subseteq C(B)$,

(iii) $C(C(A)) = C(A)$.

We denote by $\mathbf{CO}(V)$ the set of all the closure operators.

Let C be a closure operator on V. A subset A of V is called *closed with respect to* C if it is a fixed point of C. We define

$$\mathcal{A}_C = \{A \subseteq V \mid C(A) = A\}.$$

Proposition 1 [1] Let C be a closure operator on V. Then for each $A \subseteq V$, $C(A) = \bigcap \{B \in \mathcal{A}_C \mid B \supseteq A\}$.

Consequently, for each family $\mathcal{A}$ of subsets of V, there is at most one closure operator $C_{\mathcal{A}}$ whose set of closed sets is equal to $\mathcal{A}$.

2 Topped intersection structures

A family $\mathcal{A}$ of subsets of V is called *an intersection structure on* V if $\bigcap \mathcal{S} \in \mathcal{A}$ whenever $\mathcal{S} \subseteq \mathcal{A}$, and $\mathcal{A}$ is called *topped* if $V \in \mathcal{A}$. We denote by

$\mathbf{TIStr}(V)$ the set of all the topped intersection structures on V. Obviously the minimum element of $\mathbf{TIStr}(V)$ is $\{V\}$, and the maximum element is $\mathcal{P}(V)$.

Proposition 2 [2] The map $\beta : C \mapsto \mathcal{A}_C$ is a bijection of $\mathbf{CO}(V)$ onto $\mathbf{TIStr}(V)$, with the inverse map given by $\beta^{-1}(\mathcal{A}) = C_{\mathcal{A}}$.

Lemma 3 Let $\mathcal{A}$ be a topped intersection structure on V, X a member of $\mathcal{A}$, and $\mathcal{B} = \mathcal{A} \setminus \{X\}$. Then,

$$X \neq \bigcap \{B \in \mathcal{B} \mid B \supseteq X\} \iff \mathcal{B} \in \mathbf{TIStr}(V).$$

Proof.
($\Longrightarrow$) Let $X \neq \bigcap \{B \in \mathcal{B} \mid B \supseteq X\}$, and let $\mathcal{S}$ be a subset of $\mathcal{B}$. Since $\mathcal{S} \subseteq \mathcal{B} \subseteq \mathcal{A}$, $\bigcap \mathcal{S} \in \mathcal{A}$. We prove that $\bigcap \mathcal{S} \neq X$ by contradiction. Suppose $\bigcap \mathcal{S} = X$. Because $S \supseteq X$ for each $S \in \mathcal{S}$, we have $\mathcal{S} \subseteq \{B \in \mathcal{B} \mid B \supseteq X\}$. Hence $\bigcap \mathcal{S} \supseteq \bigcap \{B \in \mathcal{B} \mid B \supseteq X\} \supseteq X$. This implies $\bigcap \{B \in \mathcal{B} \mid B \supseteq X\} = X$, a contradicton. As a result, $\bigcap \mathcal{S} \in \mathcal{A} \setminus \{X\} = \mathcal{B}$.
($\Longleftarrow$) Let $\mathcal{B} \in \mathbf{TIStr}(V)$. Since $\{B \in \mathcal{B} \mid B \supseteq X\} \subseteq \mathcal{B}$, we have $\bigcap \{B \in \mathcal{B} \mid B \supseteq X\} \in \mathcal{B}$. Because $X \notin \mathcal{B}$, it follows $X \neq \bigcap \{B \in \mathcal{B} \mid B \supseteq X\}$.

q.e.d.

Proposition 4 $\mathbf{TIStr}(V)$ is a topped intersection structure on $\mathcal{P}(V)$.

Proof. Obviously $\mathcal{P}(V) \in \mathbf{TIStr}(V)$. Take $\mathbf{Sub} \subseteq \mathbf{TIStr}(V)$. We prove that $\bigcap \mathbf{Sub} \in \mathbf{TIStr}(V)$. For each $\mathcal{A} \in \mathbf{Sub}$, $V \in \mathcal{A}$ because $\mathcal{A} \in \mathbf{TIStr}(V)$. Hence $V \in \bigcap \mathbf{Sub}$. Let $\mathcal{S} \subseteq \bigcap \mathbf{Sub}$. For each $\mathcal{A} \in \mathbf{Sub}$, $\bigcap \mathcal{S} \in \mathcal{A}$ since $\mathcal{S} \subseteq \bigcap \mathbf{Sub} \subseteq \mathcal{A}$ and $\mathcal{A} \in \mathbf{TIStr}(V)$. Hence $\bigcap \mathcal{S} \in \bigcap \mathbf{Sub}$. Therefore we obtain $\bigcap \mathbf{Sub} \in \mathbf{TIStr}(V)$.

q.e.d.

3 Lattices of closure operators

We define a binary relation $\preceq$ on $\mathbf{CO}(V)$ as the pointwise order, and define $\prec\!\!\prec$ as the coverring relation of the relation $\preceq$. i.e.,

$$\begin{aligned} C \preceq D &\iff C(A) \subseteq D(A) \text{ for all } A \subseteq V, \\ C \prec\!\!\prec D &\iff C \prec D \text{ and } C \preceq E \prec D \text{ implies } C = E. \end{aligned}$$

Obviously the relation $\preceq$ is a partial order on $\mathbf{CO}(V)$.

Proposition 5 The map $\beta : C \mapsto \mathcal{A}_C$ is an order-isomorphism from $(\mathbf{CO}(V), \preceq)$ to $(\mathbf{TIStr}(V), \supseteq)$.

Proof. Let $C \preceq D$. For $A \subseteq V$, if $D(A) = A$, then $A \subseteq C(A) \subseteq D(A) = A$, i.e., $C(A) = A$. Hence $\mathcal{A}_C \supseteq \mathcal{A}_D$. Conversely, let $\mathcal{A}_C \supseteq \mathcal{A}_D$. For each $A \subseteq V$, we have $\bigcap \{B \in \mathcal{A}_C \mid B \supseteq A\} \subseteq \bigcap \{B \in \mathcal{A}_D \mid B \supseteq A\}$, i.e., $C(A) = D(A)$. Hence $C \preceq D$.

q.e.d.

Lemma 6 $C, D \in \mathbf{CO}(V)$ and $C \prec\!\!\prec D$ implies $|\mathcal{A}_C \setminus \mathcal{A}_D| = 1$.

Proof. Let X be a maximal element of $\mathcal{A}_C \setminus \mathcal{A}_D$, and let $\mathcal{B} = \mathcal{A}_C \setminus \{X\}$. From $\mathcal{B} \subseteq \mathcal{A}_D$, it follows $\bigcap \{B \in \mathcal{B} \mid B \supseteq X\} \supseteq \bigcap \{B \in \mathcal{A}_D \mid B \supseteq X\} = D(X) \supseteq X$. Moreover, $X \notin \mathcal{A}_D$ implies $X \neq D(X)$. Hence $X \neq \bigcap \{B \in \mathcal{B} \mid B \supseteq X\}$. Using **Lemma 3.2**, we obtain $\mathcal{B} \in \mathbf{TIStr}(V)$. Since $C \prec C_{\mathcal{B}} \preceq D$ and $C \prec\!\!\prec D$, we have $C_{\mathcal{B}} = D$. Therefore $\mathcal{A}_D = \mathcal{B} = \mathcal{A}_C \setminus \{X\}$.

q.e.d.

Let $\mathcal{A}$ be a topped intersection structure on V. Since $\inf \mathcal{B} = \bigcap \mathcal{B}$ and $\sup \mathcal{B} = C_{\mathcal{A}}(\bigcup \mathcal{B})$ where $\mathcal{B} \subseteq \mathcal{A}$, $(\mathcal{A}, \subseteq)$ forms a complete lattice [2]. Moreover, $(\mathbf{TIStr}(V), \subseteq)$ and $(\mathbf{CO}(V), \preceq)$ are also complete lattices, because $\mathbf{TIStr}(V)$ is a topped intersection structure on $\mathcal{P}(V)$.

Theorem 7 Let $C = C_0 \prec\!\!\prec C_1 \prec\!\!\prec C_2 \prec\!\!\prec \ \dots \ \prec\!\!\prec C_n = D$ be a chain of $(\mathbf{CO}(V), \preceq)$. Then the length of the chain is $|\mathcal{A}_C \setminus \mathcal{A}_D|$.

Proof. Let $C = C_0 \prec\!\!\prec C_1 \prec\!\!\prec C_2 \prec\!\!\prec \ \dots \ \prec\!\!\prec C_n = D$. **Lemma 6** shows that $|\mathcal{A}_{C_i} \setminus \mathcal{A}_{C_{i+1}}| = 1$ for each i. Hence $|\mathcal{A}_C \setminus \mathcal{A}_D| = n$, i.e., the length of the chain is $|\mathcal{A}_C \setminus \mathcal{A}_D|$.

q.e.d.

We define a binary relation $<$ of $\mathcal{P}(V)$ as a linear extention of $\subset$, and for $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(V)$, we define $\mathcal{A} \ll \mathcal{B}$ if and only if there exists $X \in \mathcal{P}(V)$ such that

(i) $\mathcal{A} = \mathcal{B} \setminus \{X\}$,

(ii) X is a maximum element of $(\mathcal{P}(V) \setminus \mathcal{A}, <)$,

(iii) $X \neq \bigcap \{A \in \mathcal{A} \mid A \supseteq X\}$.

Proposition 8 Let V be a finite set, and $\mathcal{A}$ be a family of subsets of V. Then

$$\mathcal{A} \in \mathbf{TIStr}(V) \iff \mathcal{A} = \mathcal{P}(V) \text{ or } \mathcal{A} \ll \mathcal{B} \text{ for some } \mathcal{B} \in \mathbf{TIStr}(V).$$

Proof.

($\Longrightarrow$) Let $\mathcal{A} \in \mathbf{TIStr}(V)$, and $\mathcal{A} \neq \mathcal{P}(V)$. We prove that $\mathcal{A} \ll \mathcal{B}$ for some $\mathcal{B} \in \mathbf{TIStr}(V)$. Let X be the minimum element of $(\mathcal{P}(V) \setminus \mathcal{A}, <)$, and let $\mathcal{B} = \mathcal{A} \cup \{X\}$. Because $X \neq V$, we have $V \in \mathcal{B}$. Take $\mathcal{S} \subseteq \mathcal{B}$. If $X \notin \mathcal{S}$, then $\mathcal{S} \subseteq \mathcal{A}$ implies $\bigcap \mathcal{S} \in \mathcal{A}$, hence $\bigcap \mathcal{S} \in \mathcal{B}$. Otherwise, $\bigcap \mathcal{S} \subseteq X$ and X is the minimum element of $\mathcal{P}(V) \setminus \mathcal{A}$, hence $\bigcap \mathcal{S} \in \mathcal{A} \cup \{X\} = \mathcal{B}$. Therefore $\mathcal{B} \in \mathbf{TIStr}(V)$. Moreover, we have $X \neq \bigcap \{A \in \mathcal{A} \mid A \supseteq X\}$ using **Lemma 3**. Hence $\mathcal{A} \ll \mathcal{B}$.

($\Longleftarrow$) Let $\mathcal{A} \ll \mathcal{B}$, $\mathcal{B} \in \mathbf{TIStr}(V)$. Because $\mathcal{A} = \mathcal{B} \setminus \{X\}$ and $X \neq \bigcap \{A \in \mathcal{A} \mid A \supseteq X\}$, we have $\mathcal{A} \in \mathbf{TIStr}(V)$ using **Lemma 3**.

q.e.d.

Consequently, there exists a chain

$$\mathcal{A} = \mathcal{A}_0 \ll \mathcal{A}_1 \ll \ \dots \ \mathcal{A}_n = \mathcal{P}(V)$$

if and only if $\mathcal{A} \in \mathbf{TIStr}(V)$. Moreover, such a chain, if it exists, is unique.

Hence, the Hasse diagram of $(\mathbf{TIStr}(V), \lll)$ is a spanning tree of the diagram of $(\mathbf{TIStr}(V), \subset)$ (figure 1, 2). We can list up all the topped intersection structures (and all the closure operators) of V by traveling the tree (table 1).

$\lvert V\rvert$	$\lvert\mathrm{CO}(V)\rvert$
1	2
2	7
3	61
4	2480
5	1385552

Table 1: the number of all the closure operators on a finite set V

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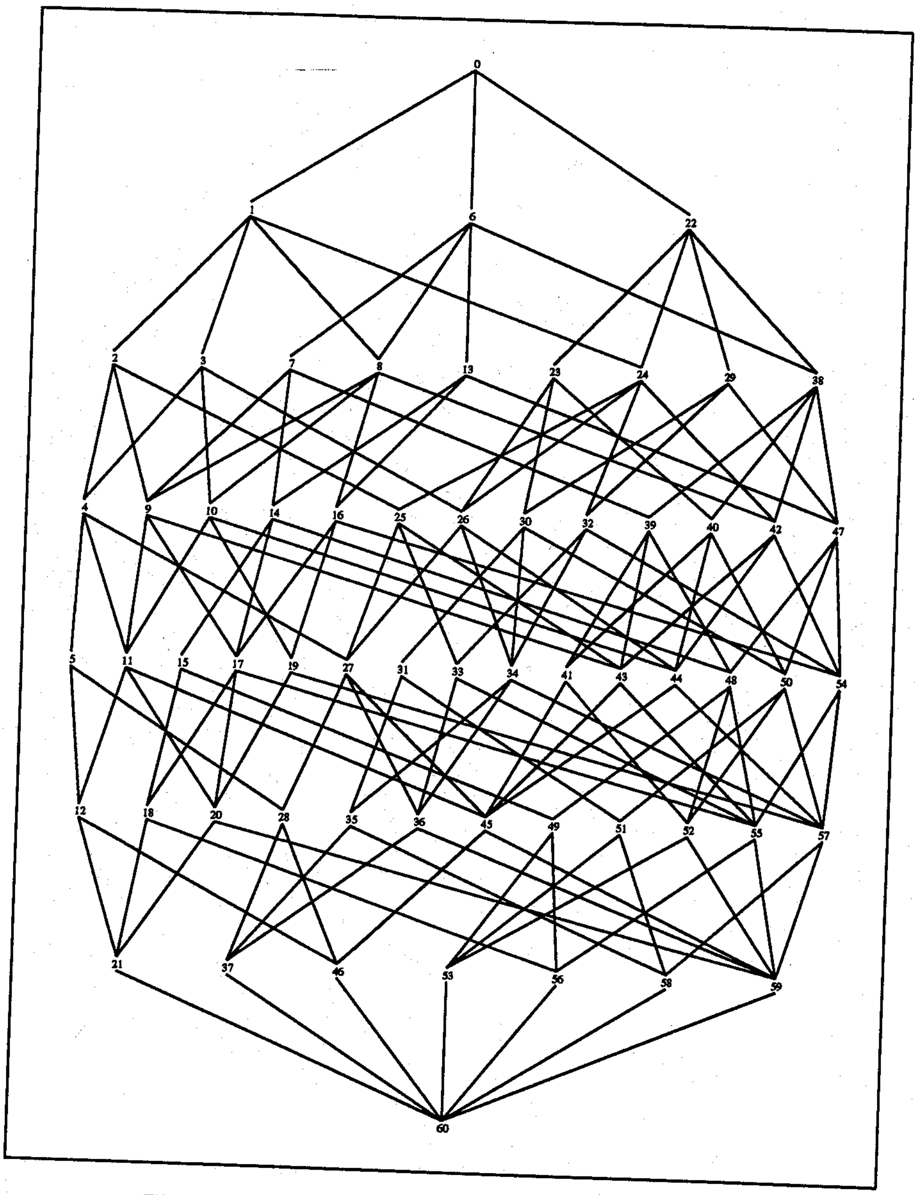

Figure 1: the diagram of $(\mathbf{TIStr}(V), \subset)$ $(|V| = 3)$

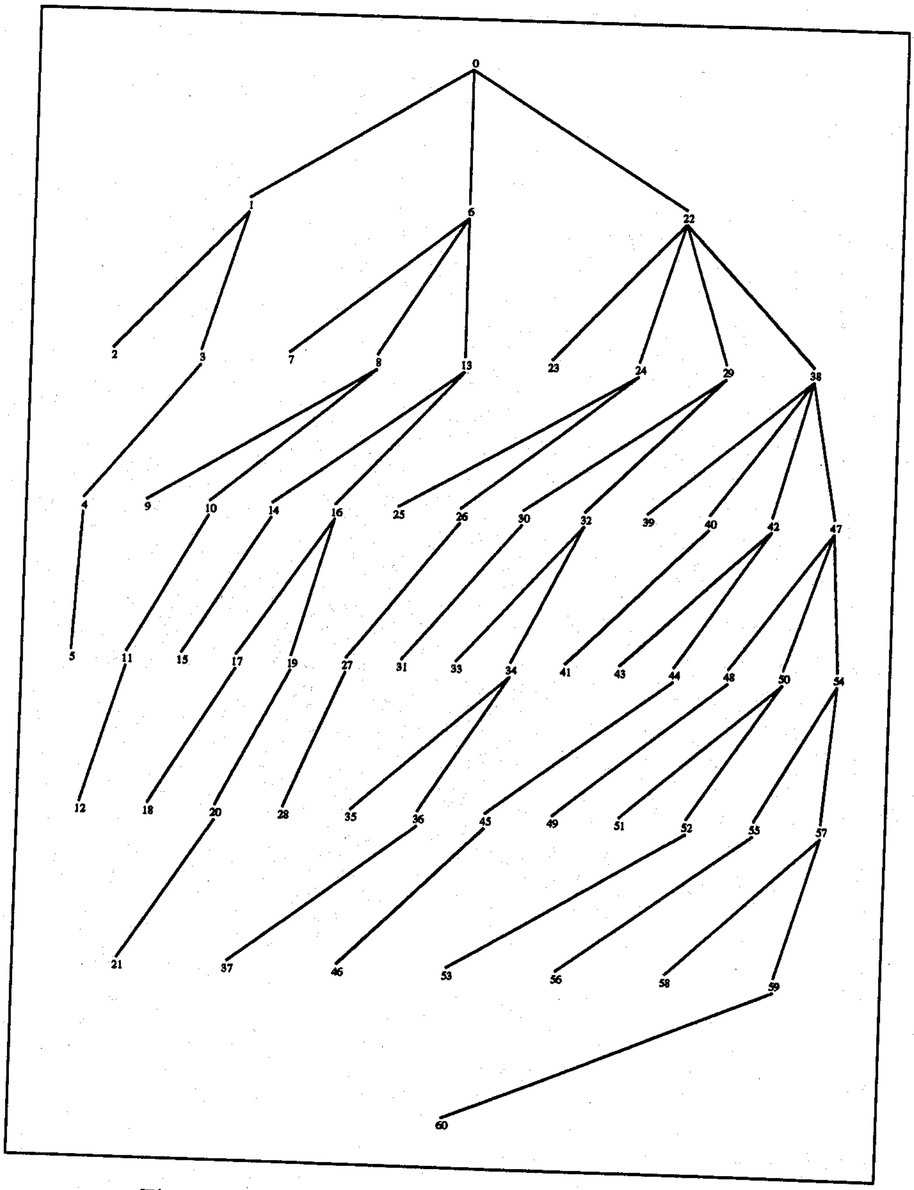

Figure 2: the diagram of $(\mathbf{TIStr}(V), \ll)$ $(|V| = 3)$