GEOMETRIC EVOLUTION BY NONSMOOTH INTERFACIAL ENERGY

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GEOMETRIC EVOLUTION BY NONSMOOTH INTERFACIAL ENERGY

MI-HO GIGA AND YOSHIKAZU GIGA

Abstract. This paper reviews the recent progress of the analytic theory for surface-energy-driven motion of curves when the interface energy is not smooth. The theory of viscosity solutions is extended for the nonsmooth energy including crystalline energy.

1. Introduction. In crystal growth problems the effect of anisotropy of material plays an important role even if the speed of phase boundaries depends only on its local geometry. Let Γ_t denote the phase boundary depending on time t. We consider a typical evolution equation for Γ_t including anisotropic effects, when Γ_t is assumed to be a curve in the plane and the speed of Γ_t depends only on its local geometry. Let $\mathbf{n}$ be the unit normal vector field of Γ_t determinig the orientation of Γ_t. Let V denote the normal velocity in the direction of the form

$$V = -\frac{1}{\beta(\mathbf{n})}\left(\sum_{i=1}^{2}\frac{\partial}{\partial x_i}(\partial_i\gamma(\mathbf{n})) + C(t)\right) \quad \text{on} \quad \Gamma_t. \tag{1}$$

Here $\gamma : \mathbf{R}^2 \to \mathbf{R}$ is of the form

$$\gamma(q) = |q|\gamma_0(q/|q|)$$

and γ_0, β are real-valued functions defined on the unit circle S^1. In (1), $\partial_i\gamma$ denotes the partial derivative $\partial\gamma/\partial q_i$ as a function on $\mathbf{R}^2$. We shall always assume that $\beta > 0$ and $\gamma_0 > 0$ (or $\gamma_0 \equiv 0$). The quantity γ_0 is called the *interface energy density*, while β is called

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the *kinetic coefficient.* The function $C(t)$ is given and physically it describes the bulk free energy of crystal relative to that of the ancillary phase. Typically, this is the difference of temperature between two phases on Γ_t or the difference of pressure. The equation (1) is derived mathematically from balances of forces and the second law of the thermodynamics by Angenent and Gurtin [AG1]. If γ_0 is a positive constant, the interface energy density γ_0 is called *isotropic.* If both γ_0 and β are isotropic, the equation (1) with $C = 0$ becomes the curve shortening equation. The presence of γ_0 and β gives anisotropic effect.

In this paper we report the recent progress in the analytic theory for the initial value problem for (1) when γ is convex (so that (1) is at least degenerate parabolic) but not regular. There are nice review articles by Taylor, Cahn and Handwerker [TCH] and by Gurtin [G] where the background of the problems is discussed. This paper does not intend to exhaust all references related to this topic. It rather cites articles directly related.

To clarify the situation of progress we list typical problems.

1.1. Initial value problem. We seek a solution of the initial value problem for (1) for a given initial curve. There are several ways to formulate the problems of the unique existence of solutions. One typical way is to construct a local or global solution in a nice class of curves reflecting some regularity when an initial data is nice. In many cases the existence of local solutions is expected. However, the solution may cease to belong to the nice class in a finite time. It is not only difficult but also impossible to obtain a global-in-time solution in this formulation in general.

Another typical way is to construct a global solution in a very big class by extending even the meaning of curves for a very general initial data. In this formulation global existence of solution is expected since we weaken the meaning of solutions. However, there are two drawbacks. It may often lose the uniqueness of solutions since the class of solutions is too large. The meaning of solution may not be clear as a geometric object. In particular one should always prove that the solution obtained here should agree with a nice local solution until the time the latter exist. The last property is called the *consistency* of two kinds of solutions. To overview the progress for these problems we consider two typical types of initial curves.

(i) Closed curve. We assume that the initial curve Γ_0 is a closed *embedded* curve in the plane. In the evolution the solution curve Γ_t may lose smoothness. This is easy to imagine if we consider an extreme case $\gamma_0 \equiv 0, \beta \equiv 1, C \equiv -1$ so that (1) becomes $V = 1$, and consider nonconvex smooth initial data.

It is often convenient to rewrite (1) using the argument θ of $\mathbf{n}$ i.e. $\mathbf{n} = (\cos\theta, \sin\theta)$. In §1.1, we always assume γ is at least C^2 to carry out calculation. In this new variable the curvature k (in the direction of $\mathbf{n}$) equals

$$k = -\operatorname{div}\mathbf{n} = \theta_x \sin\theta - \theta_y \cos\theta, \tag{2}$$

where we rewrite x_1 by x and x_2 by y and $\theta_x = \partial\theta/\partial x, \theta_y = \partial\theta/\partial y$. If we introduce

$$f(\theta) = \gamma_0(\cos\theta, \sin\theta),$$

then

$$\frac{\partial}{\partial x}(\partial_1\gamma(\mathbf{n})) + \frac{\partial}{\partial y}(\partial_2\gamma(\mathbf{n})) = -(f(\theta) + f''(\theta))k, \tag{3}$$

where f'' denotes the second derivative of f. Although the proof is elementary we shall give it for convenience. By the homogeneity of γ we see

$$\begin{aligned} \gamma &= \gamma_1 \cos\theta + \gamma_2 \sin\theta \\ 0 &= \gamma_{11} \cos\theta + \gamma_{12} \sin\theta \\ 0 &= \gamma_{21} \cos\theta + \gamma_{22} \sin\theta. \end{aligned} \tag{4}$$

where $\gamma_i = \partial_i \gamma, \gamma_{ij} = \partial_{ij}\gamma$ evaluated at $(\cos\theta, \sin\theta)$ for $1 \leqslant i, j \leqslant 2$. The left hand side I of (3) now becomes

$$\begin{aligned} I &= \gamma_{11}\theta_x(-\sin\theta) + \gamma_{12}\theta_x \cos\theta + \gamma_{21}\theta_y(-\sin\theta) + \gamma_{22}\theta_y \cos\theta \\ &= -\gamma_{11}\theta_x(\sin\theta + \cos^2\theta/\sin\theta) + \gamma_{11}\theta_y(\cos\theta\sin\theta/\sin\theta + \cos^2\theta\cos\theta/\sin^2\theta) \\ &= -\gamma_{11}k/\sin^2\theta. \end{aligned}$$

If we calculate derivatives of f, we get

$$\begin{aligned} f'(\theta) &= -\gamma_1 \sin\theta + \gamma_2 \cos\theta \\ f''(\theta) &= \gamma_{11}\sin^2\theta - 2\gamma_{12}\sin\theta\cos\theta + \gamma_{22}\cos^2\theta - (\gamma_1\cos\theta + \gamma_2\sin\theta). \end{aligned}$$

Using (4) we observe that

$$\begin{aligned} f(\theta) + f''(\theta) &= \gamma_{11}\sin^2\theta - 2\gamma_{12}\sin\theta\cos\theta + \gamma_{22}\cos^2\theta \\ &= (\sin^2\theta + 2\sin\theta\cos\theta\cos\theta/\sin\theta + \cos^2\theta\cos^2\theta/\sin^2\theta)\gamma_{11} \\ &= \gamma_{11}/\sin^2\theta \qquad \text{(cf. [FG])}. \end{aligned}$$

This now yields

$$I = -(f(\theta) + f''(\theta))k$$

which is the same as (3). Applying the identity (3) to (1) yields

$$V = \frac{1}{B(\theta)}((f(\theta) + f''(\theta))k - C(t)) \tag{5}$$

with $B(\theta) = \beta(\cos\theta, \sin\theta)$. This form of the equation (1) is used in [AG1] and [G] while the original form (1) is proposed by [CH]. It is often convenient to consider the evolution of curvature k under the Gauss parametrization of the curve Γ_t. Let θ be the argument of the normal $\mathbf{n}$ at $x \in \Gamma_t$. If the curvature of Γ_t is positive everywhere, the mapping $\theta = \theta(x)$ gives a diffeomorphism from Γ_t to the one dimensional torus $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. In the new independent variable θ, the curvature $k = k(t, \theta)$ always solves the identity

$$k_t = k^2(V_{\theta\theta} + V), \tag{6}$$

where independent variables of V are also t and θ, and $k_t = \partial k/\partial t, V_{\theta\theta} = (V_\theta)_\theta, V_\theta = \partial V/\partial\theta$. This follows from a direct calculation, see e.g. [G]. Combining (5) with (6) we obtain a PDE for k

$$k_t = k^2((\alpha(\theta)k - C(t)/B(\theta))_{\theta\theta} + \alpha(\theta)k - C(t)/B(\theta)), \quad \alpha(\theta) = \frac{1}{B(\theta)}(f(\theta) + f''(\theta)). \tag{7}$$

If (5) is the curve shortening equation $V = k$, then (7) is simply of the form

$$k_t = k^2(k_{\theta\theta} + k). \tag{8}$$

The equation (7) is equivalent to (5) if Γ_t stays as a curve of positive curvature. Note that we need to impose the closedness condition

$$\int_0^{2\pi} \frac{e^{i\theta}}{k} d\theta = 0 \tag{9}$$

so that the curve is closed ; however this condition is preserved in the evolution (7) if k satisfies (9) at $t = 0$. PDEs (7) and (8) are convenient to study evolution of curves Γ_t with positive curvature. This formulation also applies to immersed curves by replacing 2π by $2\pi m$ in the definition of $\mathbf{T}$ and the condition (9) with positive integer m.

(ii) Graph. We consider the evolution equation (1) when Γ_0 is a graph of a function of one variable. We expect the solution curve Γ_t is also represented as a graph of function $y = u(t, x), x \in \mathbf{R}^2$ at least locally in time. The equation (1) is now a PDE for u (cf. [FG].) Indeed, if $\mathbf{n}$ is taken upward, then

$$\mathbf{n} = \frac{1}{(1 + u_x^2)^{1/2}}(-u_x, 1), \quad V = \frac{u_t}{(1 + u_x^2)^{1/2}}.$$

Since $\partial_1\gamma$ is positively homogeneous of degree zero,

$$\partial_i\gamma(\mathbf{n}) = \partial_i\gamma(-u_x, 1).$$

This quantity is independent of $x_2 = y$ and depends only on $x_1 = x$, (1) now becomes

$$u_t = a(u_x)\big[\big(W'(u_x)\big)_x - C(t)\big] \tag{10}$$

with

$$\begin{cases} a(p) = (1 + p^2)^{1/2} M(p), \\ \dfrac{1}{M(p)} = \beta\left(\dfrac{-p}{(1 + p^2)^{1/2}}, \dfrac{1}{(1 + p^2)^{1/2}}\right), \\ W(p) = \gamma(-p, 1). \end{cases} \tag{11}$$

1.2. Class of interfacial energy. The set

$$\mathrm{Frank}(\gamma_0) = \{(q_1, q_2) \in \mathbf{R}^2, \gamma(q) = 1\}$$

is called the *Frank diagram* of γ_0. The shape of this curve is important to classify the types of the equation (1). Note that this set clearly agrees with the polar diagram of $1/\gamma_0$ i.e.,

$$\mathrm{Frank}(\gamma_0) = \{(q_1, q_2) \in \mathbf{R}^2; \quad |q| = 1/\gamma_0(q/|q|)\}.$$

There are three typical shapes [G]:
(i) the Frank diagram is smooth and has positive curvature;
(ii) it is smooth but nonconvex;
(iii) it is nonsmooth and nonconvex.
In the first case the equation (1) is strictly parabolic so that W is smooth and $W'' > 0$ in (10). The initial value problem for (1) is well-posed both for arbitrary smooth closed curves (e.g. [An1]) and graphs [LSU] at least locally in time. However, in the remaining cases the problem is not well posed because the equation (1) is backward parabolic in some directions of Γ_t. It is often proposed that the evolution of Γ_t is governed by (1) with γ replaced by its *convexification* γ^c. Of course the Frank diagram of γ_0^c (the restriction of γ^c on S) is the convexification of Frank(γ_0). In the case (ii) Frank(γ_0^c) may not be C^2 even if Frank(γ_0) is smooth. Generically, Frank(γ_0^c) is $C^{1,1}$ in the sense that the curvature is bounded but may have jumps. Also the curvature may be zero so that Frank(γ_0^c) contains a straight segment. In the case (iii) Frank(γ_0^c) is no longer $C^{1,1}$. It even loses C^1 regularity although it is Lipschitz by convexity. In another words Frank(γ_0^c) has corners in the sense that the direction of normals jumps there. If Frank(γ_0^c) is a polygon, the interfacial energy γ_0^c is called *crystalline* and is a typical example of (iii). In [G] the original energy γ_0 for (iii) is called the energy with sharp spots while γ_0 for (ii) is called the unstable smooth energy; γ_0 for (i) is called the stable smooth energy. In what follows we always consider the convexification of γ_0 instead of γ_0 itself. Here and hereafter we shall drop the superscript c of γ and γ_0. Based on classification (i), (ii), (iii) we consider three classes of γ_0 depending on the regularity of the Frank diagram.
(a) Frank(γ_0) is smooth and has positive curvature.We call γ_0 a *strictly convex smooth energy.*
(b) Frank(γ_0) is at most $C^{1,1}$ and C^2 except finitely many points. The curvature of Frank(γ_0) is bounded but may be zero somewhere. We call γ_0 a *singular energy without corners.*
(c) Frank(γ_0) is C^2 except finitely many points but is not C^1. The curvature of Frank(γ_0) is bounded but may be zero somewhere. We call γ_0 an *energy with corners.*

In all cases the convexity of Frank(γ_0) is assumed because it originally comes from convexification. The situation (a), (b), (c) corresponds to (i), (ii), (iii) respectively but it only handles generic or typical pictures. For example, the number of singularities of Frank(γ_0) may be infinite if γ_0 is the convexification of the energy in (ii) and (iii).

It is convenient to recall the Wulff shape W_{γ_0} of γ_0 to understand the type of γ_0 (see §2.1 for definition). Formally, the weighted curvature

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_1}(\partial_i \gamma)(\mathbf{n})$$

equals one on ∂W_{γ_0} so it is considered as a substitute of the unit circle when γ_0 is anisotropic. If γ_0 is strictly convex and smooth, so is W_{γ_0}. If Frank(γ_0) has a flat part, W_{γ_0} has a corner. If Frank(γ_0) has a corner, W_{γ_0} has a flat part called a facet. Frank(γ_0) and W_{γ_0} are dual objects in the convex geometry. In fact the indicator function of W_{γ_0} is the convex conjugate of γ and the support function of W_{γ_0} is γ_0.

We shall consider the initial value problem for (1) or (10) with γ_0 in (a), (b), (c) both for closed curves and graphs.

2. Strictly convex smooth energy. As mentioned before the equation (1) is strictly parabolic so the unique existence of a local-in-time smooth solution for a given data is known for both closed curves and graphs provided that β is smooth. Since the problem is parabolic there is a smoothing effect. For (immersed) closed curves and graph of a periodic function, C^1 regularity of initial data yields a local smooth solution [An1]. The proof depends on a clever parametrization of the curves. Note that the local existence of a smooth solution is proved also by a so-called level set method [GG2] but the proof requires that initial curve is enough regular and embedded.

2.1. Generalized evolutions of closed embedded curves. Unless C is identically equal to zero, the local smooth solution Γ_t may develop singularities in a finite time. It is by now standard to track the evolution after onset of singularities by the level set method [CGG1], [ES]. We consider the *level set equation* of (1) for $\psi = \psi(x_1, x_2, t)$:

$$\psi_t + |\nabla\psi|\frac{1}{\beta(\mathbf{m})}\left(\sum_{i=1}^{2}\frac{\partial}{\partial x_i}(\partial_i\gamma)(\mathbf{m}) + C(t)\right) = 0, \quad (x_1, x_2, t) \in \mathbf{R}^2 \times (0,\infty), \tag{12}$$
$$\mathbf{m} = -\nabla\psi/|\nabla\psi|.$$

The solution ψ of this equation has the property that *each* level set of ψ moves by (1) at least formally. Such an equation is uniquely determined by (1). For the general derivation of the level set equation see [GG1].

Unfortunately, the parabolicity of (12) is degenerate in the normal direction of each level set of ψ so existence of global-in-time smooth solution for (12) is not expected. We need the notion of weak solutions called *viscosity solutions* [CIL], which are not C^1 but enjoy a comparison principle.

Fundamental Theorem. ([CGG1]) *Assume that γ is C^2 (except at the origin) and convex and that $\beta > 0$ and C are continuous. Let D_0 be an bounded open set in $\mathbf{R}^2$ with boundary Γ_0. Then there are unique open set D and closed set Γ in $\mathbf{R}^2 \times [0,\infty)$ such that*

(i) $$D = \{(x,t);\ \psi(x,t) > 0\}, \quad \Gamma = \{(x,g);\ \psi(x,t) = 0\}$$

(ii) $$D_0 = \{x;\ \psi(x,0) > 0\} \quad \Gamma_0 = \{x;\ \psi(x,0) = 0\}$$

for some $\psi \in C(\mathbf{R}^2 \times (0,\infty))$ solving (12) to equal a negative constant outside some compact set in $\mathbf{R}^2 \times [0,T]$ for each $T > 0$. The set D (resp. Γ) is called a generalized inner (resp. interface) evolution with initial data D_0(resp.Γ_0) for (1).

Similar result is proved independently by [ES] for the mean curvature flow equation. In the Fundamental Theorem we do not assume strictly convexity of γ_0. If γ_0 is a strictly convex smooth energy, a generalized interface evolution agrees with a classical one for (1) for smooth initial data provided that the latter exists [GG2]. Thus generalized notion of solution is consistent with usual one.

We consider D rather than Γ to fix the orientation of Γ. The set D is uniquely determined by D_0. Unfortunately, Γ is not necessarily the boundary of D. It may fatten so that Γ has an interior point even if Γ_0 is a smooth curve. An example is provided by [BP] even for $V = k + 1$, where γ_0 and β are isotropic.

If C is a negative constant and D_0 is sufficiently large, the generalized inner evolution grows to $\mathbf{R}^2$ as $t \to \infty$ and the interface energy plays a little role for rough shape of generalized inner evolution. The large time asymptotics is the Wulff shape of $1/\beta$:

$$W_{1/\beta} = \{p = (p_1, p_2);\ p \cdot q \leqq 1/\beta(q) \quad \text{for all } q \text{ s.t. } |q| = 1\}$$

as proved in [S]. If C is nonnegative, both generalized interface and inner evolution extinct in a finite time [CGG1]. The key tool of both results is the containment principle, a typical form of which reads: if D_0 is contained in D_0', then the generalized inner evolution D with initial data D_0 is contained in D', the generalized inner evolution with initial data D_0'. Other property, for example, a bound of interface energy

$$\int_{\Gamma_t} \gamma_0(\mathbf{n}) ds \quad \text{(ds: length element)}$$

is obtained in [CGG2]. For $C = 0$, the extinction time is estimated from below by [Y] by extending the idea of [GY]. All results so far discussed in Section 2.1 extends to higher dimensions with no essential alternations as discussed in original articles. Moreover, the Fundamental Theorem applies to (1) even if C depends on x provided that C is Lipschitz in x uniformly in every time interval $[0, T]$ (cf. [GGI]). Note that the extinction time is computable for smooth curves but for higher dimension only estimates are available.

2.2. The case. $C = 0$. We assume that Γ_0 is a smooth embedded curve. For an isotropic case i.e. the curve shortening equation $V = k$ it is shown that
(a) the solution Γ_t stays smooth, embedded and becomes convex in a finite time [Gr],
(b) the convex curve stays convex and shrinks to a point [GH],
(c) the way of shrinking is asymptotically self-similar [GH].
The general tendency for anisotropic case is the same especially when the evolution does not depend on the orientation of the curve, i.e. $\alpha(\theta)$ in (7) is π-periodic and (5) is $V = \alpha(\theta)k$. The part (b), (c) are extended by [G2], [GL] while the part (a) is extended by H. Taniyama based on a geometric idea of [An2]. Then existence of self-similar shrinking solution is known [S] when $f(\theta)B(\theta)$ is constant, where f and B is as in (5). It is extended for general equation $V = \alpha(\theta)k$ by [GL] without assuming π-periodicity of α. Their method depends on a careful analysis of a rescaled version of the curvature evolution equation (7). This is useful to study (c). However, the existence of self-similar solution is equivalent to the existence of a positive 2π-periodic solution h of ODE

$$h'' + h = \alpha/h.$$

A direct proof is provided by [DGM]. Its uniqueness is still open unless α is π-periodic [G2], [DG]. Note that for isotropic case the only (embedded) self-similar shrinking solution is a circle [G1]; a simple direct proof is found in [DGM]. For anisotropic case the shape of self-similar solution is the Wulff shape of h (by regarding a function on S^1).

2.3. Evolution of graphs. Since the equation (10) is strictly parabolic i.e. $aW'' > 0$ everywhere (but not uniformly), we need a bound for u_x so that (10) is uniformly parabolic

[LSU]. Such *a priori* bound is provided for the Cauchy problem for a globally Lipschitz initial data by a maximum principle so the global existence of unique smooth solution is known for smooth initial data [LSU]. In other words a graph stays a graph for the evolution equation (1). For the Dirichlet problem the graph may not stay as a graph unless C equals identically zero. This is easy to observe if we note the solution curve evolves by (1).

3. Singular energy without corners. In this case our Fundamental Theorem in § 2.1 does not apply directly because γ is singular on finitely many half lines from the origin although γ is $C^{1,1}$ outside the origin. Fortunately, thanks to [OS] and [GSS] the Fundamental Theorem extends to this case provided that β is continuous. The proof in [OS] seems to be simpler than [GSS], where evolution of unbounded set is also discussed. An extension of higher dimension is found in [I]. The evolution of graph is studied in [Gi]. In [GSS] a large time behavior of solution discussed in §2.1 is extended for such γ.

We now consider a typical γ. Assume that $f(\theta) = \gamma(\cos\theta, \sin\theta)$ is a $C^{1,1}$ function which is not C^2 at $\theta = \theta_1$ and $\theta_2 > \theta_1$ and that $f'' + f = 0$ for $\theta_1 < \theta < \theta_2$. We also assume $C \equiv 0$. Consider initial curve is piecewise smooth such that the argument of normals lies outside (θ_1, θ_2) for smooth part and that its limit to the corner equals either θ_1 or θ_2. It is shown [AG2] that there is a local solution having the same property and depending on time smoothly in a reasonable sense. In [GSS] the consistency of generalized solution with their "strong" solution is discussed. If initial curve has a nonparabolic region where the argument of the normal lies in (θ_1, θ_2), then such region should decrease and disappear in a finite time; see [Gi] where curve is assumed to be a graph. Moreover, C^2 regularity of graph loses instantaneously in the evolution [Gi]. The property of evolution is also studied in [GSS], [Gi] for nonzero C.

For behavior of solutions less is known compared with smooth solutions. For example results corresponding §2.2 are not known for this type of energy. However, when $C < 0$, it is not difficult to show that the large time behavior is the same as for smooth energy [S].

4. Energy with corners. In this case the evolution equation (1) (or (5), (10)) is no longer a usual PDE. For example, if γ_0 is crystalline, W in (11) is piecewise linear that the diffusion coefficient in (10) is $a(u_x)$ times a delta type function W'' of u_x. Evolution equation corresponding (1) is proposed independently by [T] for crystalline energy and by [AG1] for more general energy.

4.1. Crystalline energy. Let γ_0 be a crystalline energy so that the Frank diagram of γ_0 is a convex polygon. By definition there is a unique finite set N on the unit circle such that the set of all verteces of Frank(γ_0) is of the form

$$\{(\gamma_0(\mathbf{m}))^{-1}\mathbf{m};\ \mathbf{m} \in N\}.$$

What is a reasonable evolution equation for such an energy? In both works [T] and [AG1] they formulate evolution equation for a restricted class of curves — admissible piecewise linear curves. Let us write down their evolution equations for closed curves. For this purpose we recall the notion of *admissible evolving crystals*. Let Γ_t be a polygon depending on time $t \in [0, T)$. Each line segment is called *facet* and its outward unit normal is called the *orientation* of the facet. We moreover assume that each orientation belongs to N and that

the orientations of adjacent facets are adjacent in N, i.e. there is no element of N between orientations of adjacent facets. If each facet of Γ_t depends on t smoothly and if there is no annihilation (and creation) of facets in Γ_t for $0 \leqslant t < T$, we say Γ_t is an *admissible evolving crystal*; see [GGu] for more detail. There are at least two reasons we restrict the class of Γ_t. First, the diffusion is so large if the orientation is contained in N that only portion having such orientation survives eventually. Heuristically, main feature of evolution is described by such portions so it is reasonable to assume that orientations are contained in a finite set N. A typical example is a polygon. Second, if orientations of adjacent facets are not adjacent in N, there is a direction $\mathbf{m}$ in N between these orientations. Because of high diffusion in the direction $\mathbf{m}$, it is reasonable to imagine that new facet of orientation of $\mathbf{m}$ is created between original two adjacent facets instantaneously in time. This is inconsistent with the assumption that Γ_t depends smoothly in time.

We are now in position to write down the evolution equation corresponding (1) for admissible evolving crystals. Let V_i be the normal velocity of the facet $F_i(t)$ of Γ_t and let $\mathbf{n}_i$ be the orientation of F_i, where i is an index of each facet. Let L_i be the length of F_i and let χ_i be the *transition number* of F_i defined as follows

$$\text{(13)} \qquad \begin{cases} \chi_i = -1 & \text{if the facets adjacent to } F_i \text{ belong to a half space } \quad K(\mathbf{n}_i), \\ \chi_i = 1 & \text{if the facets adjacent to } F_i \text{ belong to a half space } \quad K(-\mathbf{n}_i), \\ \chi_i = 0 & \text{otherwise,} \end{cases}$$

with $K(\mathbf{n}_i) = \{x \in \mathbf{R}^2;\ (x-a)\cdot \mathbf{n}_i \leqslant 0\}, \quad a \in F_i$. The proposed equation corresponding to (1) is

$$\text{(14)} \qquad V_i = \frac{1}{\beta(\mathbf{n}_i)}\left(\frac{\chi_i}{L_i}\Delta_i - C(t)\right) \quad \text{on} \quad F_i$$

with $\Delta_i = f'(\theta_i + 0) - f'(\theta_i - 0)$ where θ_i is the argument of $\mathbf{n}_i$. This yields an ODE system for $L_i(t)$ and $V_i(t)$ as derived in [AG1] and it is well-posed locally in time.

In [T] the equation (14) is derived by variational principle while in [AG1] it is derived from the balance of force and the second law of thermodynamics. However, both works postulate that a facet stays as a facet at least for a short time.

To convince readers we derive (14) from (10) for evolution of graph on one facet at least formally. Since γ_0 is crystalline, we may assume that W is piecewise linear by (11). By adding a linear function to u we may assume that $W'(p)$ has a jump at $p = 0$. For simplicity, we assume that $\beta \equiv 1$ and $C \equiv 0$ so that (10) becomes

$$\text{(15)} \qquad u_t = \sqrt{1+u_x^2}(W'(u_x))_x.$$

Suppose that u is smooth and that $u(t,\cdot)$ is constant on some closed interval $I(t)$. Suppose furthermore that the x-derivative of u has a definite sign in the left (and right) of $I(t)$ and that it is small so that $u_x(t,x)$ lies outside the jump of W'. Integrating (15) in a neighborhood $(a_1(t)-\delta, a_2(t)+\delta), \delta > 0$ of the interval $I(t) = (a_1(t), a_2(t))$ to get

$$\begin{aligned} \int_{a_1-\delta}^{a_2+\delta} u_t dx &= 1 \cdot \int_{a_1-\delta}^{a_2+\delta} (W'(u_x))_x dx \\ &= W'(u_x(a_2+\delta)) - W'(u_x(a_1-\delta)). \end{aligned}$$

We have postulated that the speed $u_t(t,x)$ is independent of x on $I(t)$. Since $V = u_t$, sending δ to zero yields

$$V = \frac{\chi\Delta}{L} \quad \text{with} \quad \Delta = W'(+0) - W'(-0),\ L = a_2 - a_1,$$

where χ is the transition number similarly defined in (13); in this case if u is convex (resp. concave) when $\chi = 1$ (resp. -1) and otherwise $\chi = 0$. By a computation in 1.1.(i) we easily observe that

$$f'(\pi/2 \pm 0) = -\gamma_1(\mp 0, 1) = W'(\pm 0)$$

so that Δ is consistent with our previous notation Δ_i. Geometrically, Δ_i is the length of the segment in Wulff shape of γ_0 with normal $\mathbf{n}_i = (\cos\theta_i, \sin\theta_i)$ [T], [G]. We thus derived (14) on the facet over $I(t)$.

4.2. Comparison principle. *Assume that Γ_t and Γ'_t are two admissible evolving crystal solving* (14) *with constant C for $(0, t_0)$. If initially Γ'_0 encloses Γ_0, then Γ'_t encloses Γ_t for $0 < t < t_0$.*

This is found in [GGu]. The proof is elementary but one should be careful to exhaust all cases of possible touching of Γ_t and Γ'_t. The proof applies to (14) for general continuous C. To track whole evolution it is natural to allow disappearance of some facets during evolution [GirK]. As noted in [GGu] 4.2 applies to such an evolution.

Let us give a few words for evolution of a graph. The notion of admissible evolving crystal easily extends to graph evolution by replacing a polygon by a graph of a piecewise linear function. Local well-posedness and comparison principle are also valid at least for motion of periodic-in-space functions whose graphs are moved by (14) with no essential alternations.

4.3. Justifications. There are a couple of attempts that the equation (14) is consistent with smooth case. In fact, when $C \equiv 0$ and Γ_t is a graph of periodic functions, the meaning of (1) or (10) is given even for general energy with corners by adapting the theory of nonlinear semigroups [FG]. In that paper a unique global weak solution is constructed. Moreover, it was shown that approximate solutions with smoothed energy converge to the weak solution. The velocity at the place where Γ_t is a facet solves (14) as shown in [FG]. Note that the theory applies to general evolution other than admissible evolving crystal and that there are no assumptions that the velocity is constant on each facet. The consistency of velocity follows *a posterior* by calculating the velocity of solution.

In [GirK] a smooth energy is approximated by crystalline energies. Then authors of [Girk] considered admissible evolving crystal for crystalline energies approximating the original energy and derived various convergence rates of approximate evolutions converging to an evolution for the original energy. They assumed that $C \equiv 0$ and that Γ_t is a graph of a function satisfying some boundary conditions. See [Gir] for closed convex curves.

Unfortunately, both works need the divergence structure of the equation so the assumption $C \equiv 0$ is essential in their methods.

In [AT] a semi-discretized implicit scheme (called flat flow) is proposed to construct solutions of (14) for admissible evolving crystals. The time is discretized and at each time

step the value of solution is given by solving a variational problem. Their scheme does not require to assume that a facet stays facet. Moreover, their way of time discretization is consistent with that for the case of smooth energy [ATW], [FK]. However, it is not known whether their approximate solution converges to the solution of (14).

Despite these works so far there was no theory for studying nonadmissible general evolution for (1) or (10) for general energy with corners when $C \not\equiv 0$.

Recently, we are successful to formulate solutions of (10) for general energy with corners by allowing nonadmissible evolution when $C \not\equiv 0$. We strengthen the theory of so-called viscosity solutions [CIL] to construct solutions satisfying comparison principle. We shall briefly discuss our results in the next section.

4.5. General energy with corners and $C \not\equiv 0$. If γ_0 is an energy with corners in the sense of §1.2 (c), W in (10) is a convex function which is C^2 except finitely many points

$$p_1 < p_2 < \cdots < p_m$$

where $\Delta_i = W'(p_i + 0) - W'(p_i - 0) > 0, i = 1, \cdots m$; W'' is bounded on $\mathbf{R} \setminus P$, where the set of all p_i denotes P. Assume that $\beta > 0$ and C in (1) are continuous so that $a > 0$ and C in (10) are continuous.

Definition (P-faceted). Let Ω be an open interval. A function ϕ in $C(\Omega)$ is called *P-faceted* at x_0 in Ω if ϕ fulfills the following conditions.

There are a closed interval I containing x_0 and p in P such that ϕ agrees with an affine function

$$\ell_p(x) = p(x - x_0) + \phi(x_0)$$

in I and $\phi(x) \neq \ell_p(x)$ for all $x \in J \setminus I$ with some neighborhood J of I. The interval I is called a *faceted region* of ϕ containing x_0. The value p is called the slope of the facet.

Definition (Weighted curvature with driving term). Let C be a constant and x_0 be a point in Ω. For $\phi \in C(\Omega)$ we set the value

$$\Lambda_W(\phi, x_0, ; C) = W''(\phi'(x_0))\phi''(x_0) + C$$

if ϕ is second differentiable at x_0 and $\phi'(x_0) \notin P$ and

$$\Lambda_W(\phi, x_0; C) = \frac{\chi}{L}\Delta_i + C$$

if ϕ in P-faceted at x_0 in Ω with slope p_i. Here $L = L(\phi, x_0)$ is the length of the faceted region I containing x_0 and $\chi = \chi(\phi, x_0)$ is the transition number defined by

$$\begin{aligned} &\chi = +1 \quad \text{if} \quad \phi \geqslant \ell_{p_i} \quad \text{in} \quad J, \\ &\chi = -1 \quad \text{if} \quad \phi \leqslant \ell_{p_i} \quad \text{in} \quad J, \\ &\chi = 0 \qquad \text{otherwise} \end{aligned}$$

for some neighborhood of the facet region I.

Remark 1. The value $\chi\Delta_i/L$ is often called the weighted curvature [T]. We rather define the sum of weighted mean curvature and C because Λ_W is more intrinsic quantity especially when C depends on x. As explain in §4.6 one cannot single out the part of weighted curvature for such C so that the comparison principle holds for (10) formally written as

$$u_t = a(u_x)\Lambda_W(u, x; -C).$$

Definition. A function $\phi \in C^2(\Omega)$ belongs to a class $C_P^2(\Omega)$ if ϕ is P-faceted at x_0 in Ω whenever $\phi'(x_0)$ belongs to P. For $\phi \in C_P^2, \Lambda_W(\phi, x, C)$ is defined for all $x \in \Omega$ so we often write it by $\Lambda_W(\phi; C)(x)$.

Definition (Space of admissible functions). For $Q = (0,T) \times \Omega$ let $A_P(Q)$ be the set of functions on Q of the form

$$\phi(x) + g(t), \quad \phi \in C_P^2(\Omega), \quad g \in C^1(0,T).$$

An element of $A_P(Q)$ is called an admissible function.

We are now in position to define our generalized solution in the viscosity sense.

Definition (Generalized solution). A real valued function u on Q is a (*viscosity*) *subsolution* of

$$u_t - a(u_x)\Lambda_W(u; -C) = 0 \tag{16}$$

if the upper semicontinuous envelope $u^* < \infty$ in $\overline{Q}$ and

$$\psi_t(t_0, x_0) - a(\psi_x(t_0, x_0))\Lambda_W(\psi(t_0, \cdot); -C(t_0))(x_0) \leqslant 0 \tag{17}$$

whenever $(\psi, (t_0, x_0)) \in A_P(Q) \times Q$ fulfills

$$\max_Q(u^* - \psi) = (u^* - \psi)(t_0, x_0).$$

A (*viscosity*) *supersolution* is defined by replacing $u^*(< \infty)$ by the lower semicontinuous envelope $u_*(> -\infty)$, max by min and the inequality in (17) by the opposite one. If u is sub- and supersolution, u is called a *viscosity solution* or a *generalized solution.* It turns out that this definition agrees with usual one [CIL] when W is C^2. Hereafter we avoid to use the word viscosity because this is nothing to do with viscosity in the sense of physics corresponding to (16). A key result is

Fundamental Comparison Theorem. Let u and v be a sub- and supersolution of (16), respectively, where Ω is a bounded open interval. Assume that $C \in C(0,T)$ and $a \in C(\mathbf{R})$ with $a \geqslant 0$. If $u^* \leqslant v_*$ on the parabolic boundary of Q, then $u^* \leqslant v_*$ on Q.

Remark 2. The idea of the proof reflect the work of [Go] and [CGG1], where singularity of the equation is handled by a careful argument by contradiction. We give a proof elsewhere because it is too long. Note that if u and v are spatially periodic with the same period independent of t, then $u^* \leqslant v_*$ in Q follows from $u^* \leqslant v_*$ time $t = 0$. This is not included

in the above theorem but the proof works for this problem, too, since $u(t,\cdot), v(t,\cdot)$ are regarded as functions on a torus which has no boundary.

Existence Theorem. *Suppose that $u_0 \in C(\mathbf{R})$ is periodic with period ω. Then there is a unique global generalized solution $u \in C([0,\infty) \times \mathbf{R})$ of (16) with $u(0,x) = u_0(x)$ and $u(t, x+\omega) = u(t,x)$.*

The proof is based on the Perron method with construction of sub- and supersolution with initial data u_0. We give it elsewhere. As noted in the case of smooth energy, the Dirichlet problem is not expected to be solved globally if we keep the value at the boundary.

Remark 3. Both comparison and existence theorems extend to more general equation

$$u_t + F(u_x, \Lambda_W(u; C)) = 0,$$

where F is degenerate elliptic, i.e., $F(q,X) \leqslant F(q,Y)$ for $X \geqslant Y, q \in \mathbf{R}$ and continuous on $\mathbf{R} \times \mathbf{R}$.

We do not mention any stability results here but it is likely. In particular we expect that if W is approximated by the smooth one, then the solution of the approximate equation converges to a solution of (16). We shall discuss this problem in one of our forthcoming paper.

4.6. Suggestions for generalizations. If C depends on a space variable, the assumption that a facet stays as a facet for a short time is not natural. If we would assume and consider our admissible evolving crystal as in § 4.1, then the equation corresponding to (14) would be

$$V_i = \frac{1}{\beta(n_i)}\left(\frac{\chi_i}{L_i}\Delta_i - \frac{1}{L_i}\int_{F_i} C(t)d\ell\right) \quad \text{on} \quad F_i \tag{18}$$

where $d\ell$ is the line element; for derivation, see the argument at the end of §4.1. Unfortunately, it is easy to construct two solutions of (18) violating the comparison principle. This explains the reason why one should not take the value of Λ_W as

$$\Lambda_W(\phi, x; C) = \frac{\chi\Delta}{L} + \frac{1}{L}\int_I C dx \tag{19}$$

if C depends on x (cf. Remark 1 in §4.5.) The question is what is the correct choice of Λ_W.

There are two answers and we hope both agree. The first one is implicit in [FG] where velocity is characterized by the canonical restriction of subdifferential of energy. It is natural to define $\Lambda_W(\phi; C)$ by a canonical restriction (or a minimal section [Ba]) of the subdifferential $-\partial E$ of

$$E(\phi) = \int (W(\phi_x(x)) - C(x)\phi(x))dx,$$

where integration is over a neighborhood of a facet region. Of course one should clarify a Hilbert space to consider subdifferentials and canonical restrictions. This idea is promising

and actually it applies to get a comparison principle for admissible surfaces evolution by crystalline energy [GGuM], where C is a constant. The second one is due to Roosen [R]. He proposed to consider a flat flow as in [AT] mentioned in §4.3 for C depending on x and calculate velocity, which is not (19) at least. It is an interesting question to generalize the theory in §4.5 for C depending on x.

Another interesting question is to extend the comparison principle in §4.5 for level set equations so that the level set method applies to energies with corners. The space dimension is at least two even if we consider evolution of curves. For a higher dimensional problem like surface evolution by energy with corners little is known. In [GGuM] a comparison principle is established for a little bit restrictive class of admissible crystal evolutions moved by crystalline energy.

For behavior of solution little is known. When $C \equiv 0$, an evolving crystal by crystalline energy shrinks, however it may not shrink a point as remarked in [AG1] so results corresponding to §2.2 is not expected for this energy. For $C < 0$ large time behavior of admissible evolving crystal (by crystalline energy) is studied in [GGu], where results parallel those for smooth energy at least qualitatively.

Recently, the uniqueness of self-similar solutions for motion by mean curvature is studied by [St].

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