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Title	Ginzburg-Landau equation with magnetic effect non-simply-connected domains
Author(s)	Jimbo, S.; Zhai, J.
Citation	Hokkaido University Preprint Series in Mathematics, 303, 1-21
Issue Date	1995-07-01
DOI	https://doi.org/10.14943/83450
Doc URL	https://hdl.handle.net/2115/69054
Type	departmental bulletin paper
File Information	re303.pdf



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WITH MAGNETIC EFFECT:
NON-SIMPLY-CONNECTED
DOMAINS**

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Series #303. July 1995

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GINZBURG-LANDAU EQUATION WITH MAGNETIC EFFECT: NON-SIMPLY-CONNECTED DOMAINS

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ABSTRACT. This paper is a continuation of the paper [15] and [16]. Ginzburg-Landau equation with the magnetic effect is studied in the case of a superconductor which is not simply-connected. There are multiple stable steady state solutions, which correspond to many kinds of permanent current of electrons.

§1. Introduction and preliminaries

Ginzburg-Landau equation is a model of the low-temperature superconductivity. The total energy for a state (Φ, A) for a superconductor $\Omega \subset \mathbb{R}^3$ where the external (applied) magnetic field is absent, is given by the following functional:

$$(1.1) \quad \mathcal{H}_\lambda(\Phi, A) = \int_{\Omega} \left(\frac{1}{2} |(\nabla - iA)\Phi|^2 + \frac{\lambda}{4} (1 - |\Phi|^2)^2 \right) dx + \int_{\mathbb{R}^3} \frac{1}{2} |\operatorname{rot} A|^2 dx.$$

Φ ($\mathbb{C}$ -valued) is the wave function of the electrons in the material, A is the vector potential of the magnetic field and $\lambda > 0$ is the GL parameter. We note that the first and second terms correspond to the energy of the electrons in the material and that of the magnetic field which occurs in the whole space $\mathbb{R}^3$, respectively. A superconducting steady state (Φ, A) is formulated as a local minimizer of $\mathcal{H}_\lambda$ and hence it is a solution to the following variational equation (GL equation):

$$(1.2) \quad \begin{cases} (\nabla - iA)^2 \Phi + \lambda(1 - |\Phi|^2)\Phi = 0 & \text{in } \Omega, \\ \frac{\partial \Phi}{\partial \nu} - i\langle A \cdot \nu \rangle \Phi = 0 & \text{on } \partial\Omega, \\ \operatorname{rot} \operatorname{rot} A + (i(\bar{\Phi} \nabla \Phi - \Phi \nabla \bar{\Phi})/2 + |\Phi|^2 A) \Lambda_\Omega = 0 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product of vectors in $\mathbb{R}^3$ and Λ_Ω is the characteristic function of Ω , i.e. $\Lambda_\Omega(x) = 1$ in Ω and $\Lambda_\Omega(x) = 0$ in $\mathbb{R}^3 \setminus \Omega$. We are interested in the existence of non-trivial solutions of (1.2) and their stability (local minimizer). In [15], it was proved that there are many kinds of stable steady state solutions for large $\lambda > 0$ in the case of the ring-shaped (rotational) domain $\Omega \subset \mathbb{R}^3$. The purpose of this paper is to extend this result for more general domain Ω (cf. Fig.1 for an example). Precisely speaking we will prove the existence of many kinds of stable solutions for a bounded domain $\Omega \subset \mathbb{R}^3$ provided that Ω is not simply-connected. In [16] the GL equation without magnetic effects was studied. It is the case of the functional

(1.3) and its variational equation (1.4) which are obtained by putting $A \equiv 0$ in (1.1) and (1.2), respectively.

$$(1.3) \quad \mathcal{H}_\lambda^0(\Phi) = \int_{\Omega} \left(\frac{1}{2} |\nabla \Phi|^2 + \frac{\lambda}{4} (1 - |\Phi|^2)^2 \right) dx,$$

$$(1.4) \quad \Delta \Phi + \lambda(1 - |\Phi|^2)\Phi = 0 \quad \text{in } \Omega, \quad \partial \Phi / \partial \nu = 0 \quad \text{on } \partial \Omega \quad (\mathbb{C} - \text{valued}).$$

Denote the set of the continuous maps from $\overline{\Omega}$ into $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ by $\mathcal{M}$, i.e. $\mathcal{M} = C^0(\overline{\Omega}; S^1)$. In view of the functional (1.4), the absolute value of a local minimizer Φ_λ may approach 1 if λ is very large. It suggests that Φ_λ approaches a certain map in $\mathcal{M}$. Actually it was proved in [16] that for any given homotopy class in $\mathcal{M}$, there exists a stable solution Φ_λ of (1.4) for large $\lambda > 0$ which uniformly approaches the harmonic map ($\in \mathcal{M}$) belonging to that homotopy class as $\lambda \rightarrow \infty$. In certain physical situations, (1.4) is regarded as an approximate model equation of (1.2) and so it is a natural problem to compare the solutions of (1.2) with those of (1.4). Let Φ be fixed and consider $\mathcal{H}_\lambda(\Phi, A)$ in the variable A . We see that this functional is convex in the variable A and so it admits only a global minimizer (essentially unique). This implies that when we seek for a local minimizer, Φ is more important variable than A because A is uniquely determined by Φ and in other words, $\mathcal{H}_\lambda(\Phi, A)$ is controlled by the variable Φ . This suggests that the situation of local minimizers for $\mathcal{H}_\lambda(\Phi, A)$ may be similar to that of $\mathcal{H}_\lambda^0(\Phi)$ for large $\lambda > 0$ and we can expect that a similar method as that in [16] is applicable to (1.1)-(1.2). We will consider a solution (Φ, A) of (1.2) such that Φ topologically behaves like a solution of (1.4). For the construction of solutions, we first deal with the limit problem $\lambda = \infty$ of (1.1) and (1.2) and next consider $0 \ll \lambda < \infty$ as a perturbation. We consider the stability problem by the limiting behaviors of the certain eigenvalues. Anyway we apply the method developed in [15] and [16].

We remark that a similar work on the existence of local minimizers of (1.1) with their topological characterization for non-simply connected superconductors is done independently by Rubinstein and Sternberg [22] recently. They are dealing with only the case that Ω has the same topological type as a solid torus. The stability inequality such as (2.3) in our main theorem is not given there.

There have been many important works on the solutions Ginzburg-Landau equations with or without magnetic effects in different situations. See [3], [4], [5], [6], [9], [10], [11], [13], [17], [21], [23], [24] and the references therein.

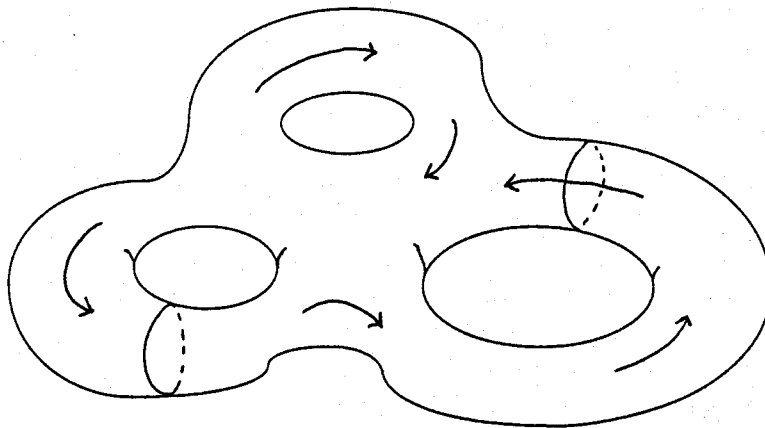


Figure 1 : $\Omega \subset \mathbb{R}^3$ (Doughnut with 3 holes)

We formulate the problem more precisely. We consider the functional (1.1) for (Φ, A) satisfying

$$(1.5) \quad A \in L^6(\mathbb{R}^3; \mathbb{R}^3), \quad \nabla A \in L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3}), \quad \Phi \in H^1(\Omega; \mathbb{C}).$$

The conditions concerning A follow from the following consideration. From (1.1) $\text{rot} A$ should belong to L^2 . Taking account of the gauge transformation, which will be mentioned later, we can assume without loss of generality that $\text{div} A = 0$ and it leads to $\nabla A \in L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$. The last line of (1.2) can be regarded as a part of the stationary Maxwell equation,

$$(1.6) \quad \begin{cases} \text{rot rot } A = J \\ J = - (i(\bar{\Phi} \nabla \Phi - \Phi \nabla \bar{\Phi})/2 + |\Phi|^2 A) \wedge \Omega, \end{cases}$$

where J naturally corresponds to the electric current. If J is a given vector valued function satisfying $\text{div } J = 0$ in $\mathbb{R}^3$, the function defined by

$$(1.7) \quad A(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{J(y)}{|x-y|} dy$$

satisfies $\text{rot } A = J$. In our situation if there exists a C^1 solution (Φ, A) to (1.2), J has a compact support and satisfies $\text{div } J = 0$ in $\mathbb{R}^3$ (in the distribution sense) and A is expressed as in (1.7) and it implies $A(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Therefore using the Sobolev's inequality (cf. Prop.6 in §3) with $\nabla A \in L^2$, we get $A \in L^6(\mathbb{R}^3; \mathbb{R}^3)$.

The formulation for the stability of the solution is completely similar to that in [15]. Note that the functional (1.1) as well as the equation (1.2) is invariant under the following (gauge) transformation:

$$(1.8) \quad \begin{cases} (\Phi, A) \mapsto (\Phi', A') \\ \Phi' = e^{i\rho} \Phi, \quad A' = A + \nabla \rho \quad (\rho: \mathbb{R} - \text{valued function in } \mathbb{R}^3). \end{cases}$$

Taking various ρ , we get a continuum of solutions from one solution (Φ, A) . We note that all (Φ', A') 's in this continuum corresponds to one physical state. In view of this, to see the stability of a solution (Φ, A) , we have to study the variation of $\mathcal{H}_\lambda$ in the transversal direction to the tangent space of the continuum at (Φ, A) which is described as

$$T(\Phi, A) = \{(i\xi\Phi, \nabla\xi) \mid \xi: \mathbb{R} - \text{valued function on } \mathbb{R}^3\}.$$

We take a space $N(\Phi, A)$ which is transversal to $T(\Phi, A)$ and satisfies $T(\Phi, A) \cap N(\Phi, A) = \{(0, 0)\}$ and consider whether $\mathcal{L}_\lambda(\Phi, A, \cdot, \cdot)$ is positive definite in $N(\Phi, A)$. We prepare the formulation for the stability of the solution. Let $\Phi \in C^1(\bar{\Omega}; \mathbb{C})$, $A \in C^1(\mathbb{R}^3; \mathbb{R}^3)$ and $\nabla A \in L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$. We note that the solution (Φ, A) which we will construct in §4 satisfies these conditions. For convenience, we sometimes deal with Φ -component in terms of real functions by taking its real and imaginary parts. We put $\Phi = u + vi$ and $\Psi = \phi + \psi i$. Hereafter we also denote $\mathcal{H}_\lambda(\Phi, A)$, $\mathcal{L}_\lambda(\Phi, A, \Psi, B)$, $T(\Phi, A)$ and $N(\Phi, A)$ by $\mathcal{H}_\lambda(u, v, A)$, $\mathcal{L}_\lambda(u, v, A, \phi, \psi, B)$, $T(u, v, A)$ and $N(u, v, A)$, respectively. Put

$$Z = \{B \in L^6(\mathbb{R}^3; \mathbb{R}^3) \mid \nabla B \in L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})\}.$$

The tangent space $T(\Phi, A)$ is defined as follows,

$$(1.9) \quad T(\Phi, A) = T(u, v, A) = \{(-v\xi, u\xi, \nabla\xi) \mid \xi \in L^2_{loc}(\mathbb{R}^3), \nabla\xi \in Z\}.$$

To define a subspace $N(\Phi, A) = N(u, v, A)$ which is transversal to $T(\Phi, A)$, we use the Helmholtz decomposition (cf. [19]). It is known that $L^6(\Omega; \mathbb{R}^3)$ and $L^6(\mathbb{R}^3; \mathbb{R}^3)$ have the decompositions:

$$L^6(\Omega; \mathbb{R}^3) = X_1 \oplus X_2, \quad L^6(\mathbb{R}^3; \mathbb{R}^3) = Y_1 \oplus Y_2,$$

where

$$\begin{aligned} X_1 &= \{\nabla\xi \mid \xi \in L^6(\Omega), \nabla\xi \in L^6(\Omega; \mathbb{R}^3)\}, \\ X_2 &= \{B \in L^6(\Omega; \mathbb{R}^3) \mid \operatorname{div} B = 0 \text{ in } \Omega, \langle B \cdot \nu \rangle = 0 \text{ on } \partial\Omega\}, \\ Y_1 &= \{\nabla\xi \mid \xi \in L^6_{loc}(\mathbb{R}^3), \nabla\xi \in L^6(\mathbb{R}^3; \mathbb{R}^3)\}, \\ Y_2 &= \{B \in L^6(\mathbb{R}^3; \mathbb{R}^3) \mid \operatorname{div} B = 0 \text{ in } \mathbb{R}^3\}. \end{aligned}$$

Let P and $\tilde{P}$ be the projectors of $L^6(\Omega; \mathbb{R}^3)$ and $L^6(\mathbb{R}^3; \mathbb{R}^3)$ onto X_2 and Y_2 , respectively. Let us define

$$\begin{aligned} \overline{N}(u, v, A) &= \left\{ (\phi, \psi, B) \in H^1(\Omega)^2 \times Z \mid \int_{\Omega} (v\phi - u\psi) dx = 0, B|_{\Omega} \in X_2 \right\}, \\ N(u, v, A) &= \left\{ (\phi, \psi, B) \in H^1(\Omega)^2 \times Z \mid \int_{\Omega} (v\phi - u\psi) dx = 0, B \in Y_2 \right\}. \end{aligned}$$

For these spaces, we have the properties in the following propositions. Their proofs can be carried out quite similarly as in §2 in [15] and so we omit them.

Proposition 1.

$$(1.10) \quad H^1(\Omega) \times H^1(\Omega) \times Z = T(u, v, A) + \overline{N}(u, v, A).$$

Proposition 2.

$$(1.11) \quad H^1(\Omega) \times H^1(\Omega) \times Z = T(u, v, A) \oplus N(u, v, A).$$

Formula of the second variation of $\mathcal{H}_{\lambda}$

$$\begin{aligned} \mathcal{L}_{\lambda}(u, v, A, \phi, \psi, B) &= \frac{d^2}{d\epsilon^2} \mathcal{H}_{\lambda}(u + \epsilon\phi, v + \epsilon\psi, A + \epsilon B)|_{\epsilon=0} = \\ &= \int_{\Omega} \{ |\nabla\phi + \psi A|^2 + |\nabla\psi - \phi A|^2 - \lambda(1 - u^2 - v^2)(\phi^2 + \psi^2) + 2\lambda(u\phi + v\psi)^2 \} dx \\ &\quad + \int_{\mathbb{R}^3} |\operatorname{rot} B|^2 dx + \int_{\Omega} (u^2 + v^2) B^2 dx \\ &\quad + 4 \int_{\Omega} \langle A \cdot B \rangle (u\phi + v\psi) dx - 2 \int_{\Omega} \{ \phi \langle \nabla v \cdot B \rangle - \psi \langle \nabla u \cdot B \rangle + u \langle \nabla \psi \cdot B \rangle - v \langle \nabla \phi \cdot B \rangle \} dx \end{aligned}$$

This is calculated directly from (1.1).

Proposition 3. Let $(\Phi, A) = (u, v, A) \in H^1(\Omega)^2 \times Z$ be a C^1 -solution of (1.2). Then

$$(1.12) \quad \mathcal{L}_{\lambda}(u, v, A, \phi, \psi, B) = \mathcal{L}_{\lambda}(u, v, A, \phi', \psi', B')$$

provided that $(\phi, \psi, B), (\phi', \psi', B') \in H^1(\Omega)^2 \times Z$ and $(\phi - \phi', \psi - \psi', B - B') \in T(u, v, A)$.

§2. Main results

In this section we present the main results. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^3 boundary. We impose the following topological condition on the domain Ω .

(A) There exists a continuous map of $\overline{\Omega}$ into $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ which is not homotopic to a constant valued map.

Under this assumption, there are at least infinitely many homotopy classes in the continuous mappings of $\overline{\Omega}$ into S^1 . Because if θ_0 is a map satisfying (A), then all the maps $\theta_0(x)^2, \theta_0(x)^3, \theta_0(x)^4, \dots$ belong to distinct homotopy classes of $\mathcal{M}$. Here S^1 is regarded as a group. We also remark that in our situation that Ω is a domain in $\mathbb{R}^3$, it is known that the above condition (A) is equivalent to that Ω is not simply-connected (see Appendix in [16]).

We construct non-trivial solutions to (1.2) for this Ω and prove their stability.

We seek for a solution (Φ, A) in the form $\Phi(x) = W(x)e^{i\theta(x)}$ where $W = W(x) > 0$ and θ is a continuous map of Ω into S^1 with an arbitrarily prescribed homotopy type.

The following theorem is the main result of this paper.

Theorem 4. Assume (A). For any $\theta_0 \in \mathcal{M} = C^0(\overline{\Omega}; S^1)$, there exists a $\lambda_0 > 0$ such that (1.2) has a solution $(\Phi_\lambda, A_\lambda)$ for any $\lambda \geq \lambda_0$ with the properties that

$$(2.1) \quad \Phi_\lambda(x) = W_\lambda(x)e^{i\theta_\lambda(x)}$$

$$(2.2) \quad \lim_{\lambda \rightarrow \infty} \sup_{x \in \Omega} |W_\lambda(x) - 1| = 0$$

and the map $\theta_\lambda(x) : \overline{\Omega} \rightarrow S^1 = \mathbb{R}/2\pi\mathbb{Z}$ is homotopic to θ_0 . Moreover it is stable in the sense that there exists a constant $c > 0$ such that

$$(2.3) \quad \mathcal{L}_\lambda(\Phi_\lambda, A_\lambda, \Psi, B) \geq c \left(\|\Psi\|_{H^1(\Omega; \mathbb{C})}^2 + \|B\|_{L^2(\Omega; \mathbb{R}^3)}^2 + \|\nabla B\|_{L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})}^2 \right)$$

for $(\Psi, B) \in N(\Phi_\lambda, A_\lambda)$ and $\lambda \geq \lambda_0$.

We will prove this theorem in the following sections. In §3 we deal with the equation for the limit case $\lambda = \infty$ and prove the existence solutions by the variational method. In §4 we deal with (1.2) for large $\lambda > 0$ and construct solutions as a "perturbation" of the limit case $\lambda = \infty$. In §5 we prove the stability by the spectral analysis on the linearized problem.

§3. Existence of solutions for $\lambda = \infty$

We consider the limit problem of (1.1), (i.e. $\lambda = \infty$). By putting $\Phi(x) = w(x)e^{i\theta(x)}$ where w is a positive function and θ is a $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ -valued function, the functional (1.1) is rewritten as

$$(3.1) \quad \mathcal{H}_\lambda(\Phi, A) = \int_\Omega \frac{1}{2} |w(\nabla\theta - A)|^2 dx + \int_{\mathbb{R}^3} \frac{1}{2} |\text{rot} A|^2 dx \\ + \int_\Omega \left(\frac{1}{2} |\nabla w|^2 + \frac{\lambda}{4} (1 - w^2)^2 \right) dx.$$

In this paper we are seeking for a local minimizer $(\Phi, A) = (we^{i\theta}, A)$ of (1.1) with $\Phi \neq 0$ and so if λ is very large, w might approach 1 because of the large $\lambda > 0$. The order of the convergece will be $w - 1 = O(1/\lambda)$ (this turns to be true in §4). This consideration suggests us the following functional as the limit case $\lambda = \infty$ of (1.1).

$$(3.2) \quad \mathcal{H}_\infty(\theta, A) = \frac{1}{2} \int_\Omega |\nabla\theta - A|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |\text{rot} A|^2 dx,$$

where θ is a S^1 -valued function in Ω and A is a $\mathbb{R}^3$ -valued function in θ . Remark that $\nabla\theta$ can be naturally be as a $\mathbb{R}^3$ -valued function. This choice of the functional will turn out to be nice afterwards. The Euler-Lagrange equation of (3.2) is

$$(3.3) \quad \begin{cases} \text{div}(\nabla\theta - A) = 0 & \text{in } \Omega, \quad \theta \sim \theta_0 \text{ (homotopy equivalent)} \\ \langle \nabla\theta - A, \nu \rangle = 0 & \text{on } \partial\Omega, \\ \text{rot rot } A + (A - \nabla\theta)\Lambda_\Omega = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

We will construct a solution (θ, A) such that θ is homotopic to θ_0 of any given homotopy type in $\mathcal{M} = C^0(\overline{\Omega}, S^1)$. For a technical reason, we use $\mathbb{R}$ -valued functions in place of θ . We can assume without loss of generality that θ_0 is C^3 because we can mollify θ_0 without changing the homotopy class. Let $\widehat{\Omega}$ be the universal covering space of Ω which is endowed with the canonical metric, i.e. the covering map $\iota_1 : \widehat{\Omega} \rightarrow \Omega$ is locally isometric. On the other hand $\mathbb{R}$ is the universal covering of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Let $\iota_2 : \mathbb{R} \rightarrow S^1$ be the covering map. Fix a point $p \in \Omega$ and let $q = \theta_0(p) \in S^1$. Take $\widehat{p} \in \widehat{\Omega}$ and $\widehat{q} \in \mathbb{R}$ such that $\iota_1(\widehat{p}) = p$, $\iota_2(\widehat{q}) = q$. For any $\theta \in C^0(\Omega, S^1)$ such that $\theta(p) = q$, there exists a unique continuous map:

$$\widehat{\theta} : \widehat{\Omega} \rightarrow \mathbb{R} = \widehat{S^1},$$

such that

$$\theta(\iota_1(z)) = \iota_2(\widehat{\theta}(z)) \quad (\forall z \in \overline{\Omega}), \quad \widehat{\theta}(\widehat{p}) = \widehat{q}.$$

$$\begin{array}{ccc} \widehat{\Omega} & \xrightarrow{\widehat{\theta}} & \mathbb{R} = \widehat{S^1} \\ \iota_1 \downarrow & & \downarrow \iota_2 \\ \Omega & \xrightarrow{\theta} & S^1 \end{array}$$

Proposition 5 ([16]). For any $\theta \in C^0(\Omega, S^1)$ such that $\theta \sim \theta_0$ and $\theta(p) = q$, the function $\eta(z) = \widehat{\theta}(z) - \widehat{\theta}_0(z)$ in $\widehat{\Omega}$ can be identified with a $\mathbb{R}$ -valued function in Ω . That is, there exists a unique $\mathbb{R}$ -valued function η' in Ω such that $\eta(z) = \eta'(\iota_1(z))$. On the other hand, for any $\mathbb{R}$ -valued continuous function η' on Ω , there exists a unique $\theta \in C^0(\Omega, S^1)$ such that $\theta(p) = q$ and $\eta'(\iota_1(z)) = \widehat{\theta}(z) - \widehat{\theta}_0(z)$ in $\widehat{\Omega}$.

By the identification in Proposition 5, we denote η' also by η . Translating the unknown function θ into η , we can rewrite the functional (3.2) and the equation (3.3),

$$(3.4) \quad \mathcal{H}'_\infty(\eta, A) = \frac{1}{2} \int_\Omega |\nabla \eta + X_0 - A|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |\operatorname{rot} A|^2 dx,$$

$$(3.5) \quad \begin{cases} \operatorname{div}(\nabla \eta + X_0 - A) = 0 & \text{in } \Omega, \quad \eta(p) = 0, \\ \langle \nabla \eta + X_0 - A, \nu \rangle = 0 & \text{on } \partial\Omega, \\ \operatorname{rot} \operatorname{rot} A + (A - \nabla \eta - X_0) \Lambda_\Omega = 0 & \text{in } \mathbb{R}^3, \end{cases}$$

where $X_0 = \nabla \theta_0$ is a C^2 class $\mathbb{R}^3$ -valued function. The important point is that η is a $\mathbb{R}$ -valued function in Ω . Clearly (3.5) is the Euler-Lagrange equation of (3.4).

We consider the minimizing problem for (3.4) in the space:

$$D = \{(\eta, A) \in H^1(\Omega) \times L^6(\mathbb{R}^3; \mathbb{R}^3) \mid \operatorname{rot} A \in L^2(\mathbb{R}^3; \mathbb{R}^3)\}.$$

Proposition 6. There exists a minimizer $(\eta_\infty, A_\infty) \in D$ of (3.4) such that

$$(3.6) \quad \operatorname{div} A = 0 \quad \text{in } \mathbb{R}^3, \quad \int_\Omega \eta dx = 0.$$

Moreover (η_∞, A_∞) belongs to $C^{2+\gamma}(\overline{\Omega}) \times C^{1+\gamma}(\mathbb{R}^3; \mathbb{R}^3)$ for any $\gamma \in [0, 1)$ and it is a solution of (3.5).

Hereafter in this section we will prove this proposition. First we recall the Sobolev imbedding theorem which we will use later.

Lemma 7 ([12]). There exists a constant $C > 0$ such that

$$\|\varphi\|_{L^6(\mathbb{R}^3)} \leq C \|\nabla \varphi\|_{L^2(\mathbb{R}^3)} \quad (\forall \varphi \in L^6(\mathbb{R}^3)).$$

(Proof of Proposition 6) Let $\{(\eta_n, A_n)\}_{n=1}^\infty$ be a minimizing sequence in D . Using the Helmholtz decomposition in L^6 (cf. [19]), each A_n is decomposed as follows:

$$(3.7) \quad \begin{cases} A_n = B_n + \nabla \xi_n & \text{in } L^6(\mathbb{R}^3; \mathbb{R}^3), \\ \operatorname{div} B_n = 0 & \text{in } \mathbb{R}^3, \\ \int_\Omega (\eta_n - \xi_n) dx = 0. \end{cases}$$

From the boundedness of

$$\mathcal{H}'_\infty(\eta_n, A_n) = \frac{1}{2} \int_\Omega |\nabla(\eta_n - \xi_n) + X_0 - B_n|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |\operatorname{rot} B_n|^2 dx,$$

$$(3.8) \quad \int_{\mathbb{R}^3} |\nabla B_n|^2 dx = \int_{\mathbb{R}^3} |\operatorname{rot} B_n|^2 dx + \int_{\mathbb{R}^3} |\operatorname{div} B_n|^2 dx$$

is bounded and hence $\{B_n\}$ is bounded in $L^6(\mathbb{R}^3; \mathbb{R}^3)$ from Proposition 7. At the same time, we have the boundedness of $\int_\Omega |\nabla(\eta_n - \xi_n)|^2 dx$ ($n \geq 1$). Using Poincaré inequality and the last line of (3.7), we obtain the boundedness of $\{\eta_n - \xi_n\}$ in $H^1(\Omega)$. From this consideration we obtain $(\eta_\infty, A_\infty) \in H^1(\Omega) \times L^6(\mathbb{R}^3; \mathbb{R}^3)$ and weakly convergent subsequence (which we use the same notation), such that

$$(3.9) \quad \begin{cases} \eta_n - \xi_n \rightarrow \eta_\infty & \text{weakly in } H^1(\Omega), \\ B_n \rightarrow A_\infty & \text{weakly in } L^6(\mathbb{R}^3; \mathbb{R}^3), \\ \nabla B_n \rightarrow \nabla A_\infty & \text{weakly in } L^2(\mathbb{R}^2; \mathbb{R}^{3 \times 3}), \end{cases}$$

as $n \rightarrow \infty$. From the lower semi-continuity of the norm under the weak convergence, we get

$$\liminf_{n \rightarrow \infty} \mathcal{H}'_\infty(\eta_n, A_n) \geq \mathcal{H}'_\infty(\eta_\infty, A_\infty).$$

$(\eta_\infty, A_\infty) \in H^1(\Omega) \times (H^1_{loc}(\mathbb{R}^3; \mathbb{R}^3) \cap L^6(\mathbb{R}^3; \mathbb{R}^3))$ is the minimizer of (3.3) and satisfies $\operatorname{div} A_\infty = 0$ in $\mathbb{R}^3$. Moreover from the regularity argument of weak solutions in the theory of the elliptic boundary value problem, $\eta_\infty \in H^2(\Omega)$ (cf. chap. 3 in [20]). Applying the Schauder estimate we obtain the desired regularity.

$$(\eta_\infty, A_\infty) \in C^{2+\gamma}(\overline{\Omega}) \times C^{1+\gamma}(\mathbb{R}^3; \mathbb{R}^3) \quad \text{for any } \gamma \in [0, 1).$$

Adding an adequate constant so that $\eta_\infty(p) = 0$, we obtain a solution of (3.5). Using again Proposition 5, we get $\theta_\infty \in C^{2+\gamma}(\overline{\Omega}; S^1)$ such that $\theta_\infty(p) = p$ and $\widehat{\theta}_\infty = \widehat{\theta}_0 + \eta_\infty$. $\square$

We can prove that the solution to (3.5) is essentially unique. Precisely we have the following result.

Proposition 8. *The solution (η_∞, A_∞) of (3.5) is unique under the condition (3.6).*

(Proof of Proposition 8) Suppose that $(\eta, A), (\eta', A')$ are two solutions of (3.5) which satisfy (3.6) and let $\tilde{\eta} = \eta - \eta', \tilde{A} = A - A'$. Clearly, $(\tilde{\eta}, \tilde{A})$ satisfies

$$\begin{cases} \operatorname{rot} \operatorname{rot} \tilde{A} + (\tilde{A} - \nabla \tilde{\eta}) \Lambda_\Omega = 0 & \text{in } \mathbb{R}^3, \\ \operatorname{div}(\tilde{A} - \nabla \tilde{\eta}) = 0 & \text{in } \Omega, \\ \langle \tilde{A} - \nabla \tilde{\eta}, \nu \rangle = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\tilde{\eta} \in C^2(\overline{\Omega})$ and $\partial\Omega$ is C^3 we can extend $\tilde{\eta}$ to $\tilde{\eta}_* \in C^2(\mathbb{R}^3)$ with a compact support. Thus we get

$$\operatorname{rot} \operatorname{rot}(A - \nabla \tilde{\eta}_*) + (A - \nabla \tilde{\eta}_*) \Lambda_\Omega = 0.$$

Multiplying the equation by $A - \nabla \tilde{\eta}_*$ and integrating it over $\mathbb{R}^3$ and using partial integration, we get

$$\int_{\mathbb{R}^3} |\operatorname{rot}(A - \nabla \tilde{\eta}_*)|^2 dx + \int_{\Omega} |A - \nabla \tilde{\eta}_*|^2 dx = 0.$$

Then, we have

$$(3.10) \quad \operatorname{rot}(A - \nabla \tilde{\eta}_*) \equiv 0 \quad \text{in } \mathbb{R}^3, \quad A - \nabla \tilde{\eta}_* \equiv 0 \quad \text{in } \Omega.$$

On the other hand $A - \nabla \tilde{\eta}_* \in L^6(\mathbb{R}^3; \mathbb{R}^3)$, there exists ξ such that $\nabla \tilde{\eta} \in L^6(\mathbb{R}^3; \mathbb{R}^3)$ and

$$(3.11) \quad \operatorname{div}(A - \nabla \tilde{\eta}_* - \nabla \xi) = 0 \quad \text{in } \mathbb{R}^3.$$

$\operatorname{rot}(A - \nabla \tilde{\eta}_* - \nabla \xi) = 0$ follows from (3.10) and it yields with (3.11) that

$$A - \nabla(\tilde{\eta}_* + \xi) = 0.$$

From $\operatorname{div} A = 0$, we have $A \equiv 0$ in $\mathbb{R}^3$ and $\tilde{\eta}_* \equiv 0$ in Ω .

This completes the proof of Proposition 8. $\square$

Remark. If we replace the condition $\int_{\Omega} \eta dx = 0$ by $\eta(p) = 0$ in (3.6), the conclusion in Proposition 8 is still true.

§4. Existence of solutions for $0 \ll \lambda < \infty$

In this section we construct a solution $(\Phi_\lambda, A_\lambda)$ to (1.2) for large $\lambda > 0$. We seek for a solution such that Φ -component has the form: $\Phi(x) = w(x)e^{i\theta(x)}$ where

$$w : \Omega \rightarrow (0, \infty), \quad \theta : \Omega \rightarrow S^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

The functional (1.1) is rewritten in terms of (w, θ, A) (cf. (3.1))

$$(4.1) \quad \begin{aligned} \mathcal{H}_\lambda(\Phi, A) = & \int_{\Omega} \frac{1}{2} |w(\nabla\theta - A)|^2 dx + \int_{\mathbb{R}^3} \frac{1}{2} |\text{rot} A|^2 dx \\ & + \int_{\Omega} \left(\frac{1}{2} |\nabla w|^2 + \frac{\lambda}{4} (1 - w^2)^2 \right) dx. \end{aligned}$$

We consider a local minimizer (w, θ, A) such that θ homotopy equivalent to θ_0 and $\theta(p) = q$ where $\theta_0 \in \mathcal{M}$ is a given map. As in §3 we use the change of the unknown variable θ to η by Proposition 5 (Recall $\eta(\iota_1(z)) = \hat{\theta}(z) - \hat{\theta}_0(z)$ in $\hat{\Omega}$). The functional $\mathcal{H}_\lambda$ is transformed into the following one:

$$(4.2) \quad \begin{aligned} \mathcal{H}'_\lambda(w, \eta, A) = & \int_{\Omega} \frac{1}{2} |w(\nabla\eta + X_0 - A)|^2 dx + \int_{\mathbb{R}^3} \frac{1}{2} |\text{rot} A|^2 dx \\ & + \int_{\Omega} \left(\frac{1}{2} |\nabla w|^2 + \frac{\lambda}{4} (1 - w^2)^2 \right) dx, \end{aligned}$$

where $X_0 = \nabla\theta_0$. The Euler-Lagrange equation of (4.2) is the following system of equations (4.3)-(4.4)-(4.5):

$$(4.3) \quad \begin{cases} \Delta w + (\lambda(1 - w^2) - |\nabla\eta + X_0 - A|^2)w = 0 & \text{in } \Omega, \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

$$(4.4) \quad \begin{cases} \text{div}(w^2(\nabla\eta + X_0 - A)) = 0 & \text{in } \Omega, \\ \langle \nabla\eta + X_0 - A, \nu \rangle = 0 & \text{on } \partial\Omega, \end{cases}$$

$$(4.5) \quad \text{rot rot} A + (A - \nabla\eta - X_0)w^2\Lambda_\Omega = 0 \quad \text{in } \mathbb{R}^3.$$

We recall the solution (η_∞, A_∞) of the equation:

$$(4.6) \quad \begin{cases} \text{div}(\nabla\eta_\infty + X_0 - A_\infty) = 0 & \text{in } \Omega, \\ \langle \nabla\eta_\infty + X_0 - A_\infty, \nu \rangle = 0 & \text{on } \partial\Omega, \\ \eta_\infty(p) = 0. \end{cases}$$

$$(4.7) \quad \text{rot rot} A_\infty + (A_\infty - \nabla\eta_\infty - X_0)\Lambda_\Omega = 0, \quad \text{div} A_\infty = 0 \quad \text{in } \mathbb{R}^3.$$

We construct a solution (w, η, A) to the above system (4.3)-(4.5) by the aid of the Schauder fixed point theorem. More precisely speaking, for a certain given (η, A) we find a unique solution $w = w(\eta, A) > 0$ of (4.3). This map will be proved to be continuous and compact. Next for this $w(\eta, A)$ we consider (4.4)-(4.5) and get a unique solution $(\tilde{\eta}(\eta, A), \tilde{A}(\eta, A))$, we denote this map T_λ and this map will be proved to be continuous. T_λ is the composition of these two maps. We can apply the Schauder fixed point theorem. Let us define precisely the set E and F for the definition of the map T_λ .

$$F(\delta) = \{w \in C^{1+\alpha}(\bar{\Omega}) \mid \|w - 1\|_{C^{1+\alpha}(\bar{\Omega})} \leq \delta\}$$

$$E = \{(\eta, A) \in C^{2+\alpha}(\bar{\Omega}) \times (C^{1+\alpha}(\bar{\Omega}; \mathbb{R}^3) \cap L^6(\mathbb{R}^3; \mathbb{R}^3)) \mid (\eta, A) \text{ satisfies (4.8)}\}$$

$$(4.8) \quad \begin{cases} \eta(p) = 0, \|\eta - \eta_\infty\|_{C^{1+\alpha}(\bar{\Omega})} \leq 1, \quad \operatorname{div} A = 0 \quad \text{in } \mathbb{R}^3, \\ \|A - A_\infty\|_{C^{1+\alpha}(\bar{\Omega}; \mathbb{R}^3)} \leq 1, \quad \|A - A_\infty\|_{L^6(\mathbb{R}^3; \mathbb{R}^3)} \leq 1. \end{cases}$$

Given (η, A) , we can get a unique positive solution to (4.3) with some detailed asymptotic properties for large $\lambda > 0$. Actually we have the following result. An almost same argument is seen in [16]. So we omit the proof.

Proposition 9. *Let $\delta > 0$. There exist $\lambda_0 = \lambda_0(\delta) > 0$ and $c_0 > 0$ such that (4.3) has a unique positive solution $w(\lambda, \eta, A) \in F(\delta)$ for any $(\eta, A) \in E$, which satisfies*

$$(4.9) \quad \begin{cases} 1 - \frac{c_0}{\lambda} < w(\lambda, \eta, A; x) \leq 1 \quad \text{for } x \in \Omega \quad (\lambda \geq \lambda_0) \\ \|w(\lambda, \eta, A) - 1\|_{C^\alpha(\bar{\Omega})} \leq \frac{c_0}{\lambda} (\|\nabla \eta\|_{C^\alpha(\bar{\Omega})}^2 + \|X_0\|_{C^\alpha(\bar{\Omega})}^2 + \|A\|_{C^\alpha(\bar{\Omega})}^2) \\ \lim_{\lambda \rightarrow \infty} \sup_{(\eta, A) \in E} \|w(\lambda, \eta, A) - 1\|_{C^{2+\alpha}(\bar{\Omega})} = 0. \end{cases}$$

From (4.9), there exists $\lambda_1 = \lambda_1(\delta) > 0$ such that $w(\lambda, \eta, A) \in F$ for any $\lambda \geq \lambda_1$ and $(\eta, A) \in E$ and moreover the image of this map $E \rightarrow F(\delta)$ is compact. By a standard argument we can show that the map is also continuous.

Proposition 10. *For any $w \in F$, there exists a unique solution $(\eta(w), A(w)) \in H^1(\Omega) \times (H_{loc}^1 \cap L^6(\mathbb{R}^3; \mathbb{R}^3))$ to (4.4)-(4.5) with $\operatorname{div} A(w) = 0$ in $\mathbb{R}^3$ and $\eta(w; p) = 0$. Moreover, $(\eta(w), A(w)) \in C^{2+\alpha}(\bar{\Omega}) \times C^{1+\alpha}(\bar{\Omega}; \mathbb{R}^3)$.*

We prove that the image of the map defined in $F(\delta)$ is included in E . Let (η, A) denote the unique solution of (4.4)-(4.5) with $\eta(p) = 0$ and $\operatorname{div} A = 0$ in $\mathbb{R}^3$ for $w \in F(\delta)$. Using (4.4)-(4.6) and (4.5)-(4.7), we consider the equations of $\eta - \eta_\infty$ and $A - A_\infty$ respectively.

$$(4.10) \quad \begin{cases} \operatorname{div} (w^2 (\nabla (\eta - \eta_\infty) - (A - A_\infty))) \\ + \operatorname{div} ((w^2 - 1) (\nabla \eta_\infty + X_0 - A_\infty)) = 0 \quad \text{in } \Omega, \\ \langle \nabla (\eta - \eta_\infty) - (A - A_\infty), \nu \rangle = 0 \quad \text{on } \partial\Omega, \end{cases}$$

$$(4.11) \quad \begin{cases} \operatorname{rot} \operatorname{rot} (A - A_\infty) + (A - A_\infty - \nabla(\eta - \eta_\infty))w^2 \Lambda_\Omega \\ + (w^2 - 1)(A_\infty - \nabla\eta_\infty - X_0)\Lambda_\Omega = 0 \quad \text{in } \mathbb{R}^3. \end{cases}$$

From these equations we get the following integral equalities.

$$(4.12) \quad \begin{cases} \int_{\mathbb{R}^3} |\operatorname{rot} (A - A_\infty)|^2 dx + \int_\Omega (A - A_\infty - \nabla(\eta - \eta_\infty))(A - A_\infty)w^2 dx \\ = \int_\Omega (1 - w^2)(A_\infty - \nabla\eta_\infty - X_0)(A - A_\infty)dx, \end{cases}$$

$$(4.13) \quad \begin{cases} \int_\Omega (|\nabla(\eta - \eta_\infty)|^2 - (A - A_\infty)\nabla(\eta - \eta_\infty))w^2 dx \\ = - \int_\Omega (1 - w^2)(A_\infty - \nabla\eta_\infty - X_0)\nabla(\eta - \eta_\infty)dx. \end{cases}$$

Adding (4.11) to (4.12), we get

$$(4.14) \quad \begin{cases} \int_{\mathbb{R}^3} |\operatorname{rot} (A - A_\infty)|^2 dx + \int_\Omega |A - A_\infty - \nabla(\eta - \eta_\infty)|^2 w^2 dx \\ = \int_\Omega (1 - w^2)(A_\infty - \nabla\eta_\infty - X_0)(A - A_\infty - \nabla(\eta - \eta_\infty))dx \\ \leq \left(\int_\Omega |A - A_\infty - \nabla(\eta - \eta_\infty)|^2 dx + \int_\Omega |A_\infty - \nabla\eta_\infty - X_0|^2 dx \right) \\ \times \frac{1}{2} \|1 - w^2\|_{L^\infty(\Omega)}^2. \end{cases}$$

Using Lemma 7 and Proposition 16 (§5) we see that there exists $\delta_0 > 0$ such that

$$(4.15) \quad \|A - A_\infty\|_{L^6(\mathbb{R}^3; \mathbb{R}^3)}^2 + \|\nabla(A - A_\infty)\|_{L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})}^2 + \|\nabla(\eta - \eta_\infty)\|_{L^2(\Omega; \mathbb{R}^3)}^2 \leq \gamma_1(\delta)$$

for and $w \in F(\delta)$ and $\delta \in (0, \delta_0)$. Here $\gamma_1(\delta) > 0$ is a function with $\lim_{\delta \rightarrow 0} \gamma_1(\delta) = 0$.

From (4.11) and $A, A_\infty \in L^6(\mathbb{R}^3; \mathbb{R}^3)$ and $\operatorname{div} A \equiv 0$, $\operatorname{div} A_\infty \equiv 0$, $A - A_\infty$ has the following expression by the aid of the fundamental solution of the Laplacian

$$(4.16) \quad \begin{cases} A(x) - A_\infty(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{J(y)}{|x - y|} dy \\ J(x) = -(A - A_\infty - \nabla(\eta - \eta_\infty))w^2 \Lambda_\Omega \\ + (1 - w^2)(A_\infty - \nabla\eta_\infty - X_0)\Lambda_\Omega. \end{cases}$$

Applying the estimates of the singular integral operator (cf. [2], [7]), we have the following estimate,

$$(4.17) \quad \|A - A_\infty\|_{H^2(|x| \leq M; \mathbb{R}^3)} \leq c_M \|J\|_{L^2(\Omega; \mathbb{R}^3)},$$

where $M > 0$ is a fixed constant so that $\overline{\Omega} \subset \{|x| < M\}$. On the other hand it is easy to see

$$(4.18) \quad \begin{aligned} \|J\|_{L^2(\Omega; \mathbb{R}^3)} &\leq \|A - A_\infty\|_{L^2(\Omega; \mathbb{R}^3)} + \|\nabla(\eta - \eta_\infty)\|_{L^2(\Omega; \mathbb{R}^3)} \\ &\quad + \|w^2 - 1\|_{L^\infty(\Omega)} \|A_\infty - \nabla\eta_\infty - X_0\|_{L^2(\Omega; \mathbb{R}^3)} \end{aligned}$$

Thus we have

$$\|A - A_\infty\|_{H^2(|x| \leq M; \mathbb{R}^3)} \leq \gamma_2(\delta)$$

where $\gamma_2(\delta) > 0$ is a function with $\lim_{\delta \rightarrow 0} \gamma_2(\delta) = 0$. Using this estimate, we apply the elliptic estimates in (4.10) with the condition $\eta(p) = 0$, we get

$$\|\eta - \eta_\infty\|_{H^2(\Omega)} \leq c\|A - A_\infty\|_{H^2(|x| \leq M; \mathbb{R}^3)}.$$

Applying repeatedly such a regularity argument to (4.10) and (4.11) we finally get the following estimates:

$$\|A - A_\infty\|_{C^{1+\alpha}(\overline{\Omega}; \mathbb{R}^3)} \leq \gamma_3(\delta), \quad \|\eta - \eta_\infty\|_{C^{2+\alpha}(\overline{\Omega})} \leq \gamma_4(\delta).$$

Thus if $\delta > 0$ is taken small and fixed and next $\lambda > 0$ is taken larger than a, then T_λ is well-defined and $T_\lambda(E) \subset E$. The continuity of the map T_λ can be proved in a similar way. Let (η_j, A_j) be the solution of (4.3)-(4.4) for $w_j \in E$ ($j = 1, 2$). By a similar calculation in (4.11), (4.12), (4.13), we obtain

$$(4.19) \quad \begin{cases} \int_{\mathbb{R}^3} |\text{rot}(A_1 - A_2)|^2 dx + \int_{\Omega} (A_1 - A_2 - \nabla(\eta_1 - \eta_2))(A_1 - A_2) w_1^2 dx \\ + \int_{\Omega} (A_2 - \nabla\eta_2 - X_0)(A_1 - A_2)(w_1^2 - w_2^2) dx = 0, \end{cases}$$

$$(4.20) \quad \begin{cases} \int_{\Omega} (|\nabla(\eta_1 - \eta_2)|^2 - (A_1 - A_2)\nabla(\eta_1 - \eta_2)) w_1^2 dx \\ + \int_{\Omega} (\nabla\eta_2 + X_0 - A_2)\nabla(\eta_1 - \eta_2)(w_1^2 - w_2^2) dx = 0, \end{cases}$$

and we get

$$(4.21) \quad \begin{cases} \int_{\mathbb{R}^3} |\text{rot}(A_1 - A_2)|^2 dx + \int_{\Omega} |A_1 - A_2 - \nabla(\eta_1 - \eta_2)|^2 w_1^2 dx \\ + \int_{\Omega} (A_2 - \nabla\eta_2 - X_0)(A_1 - A_2 - \nabla(\eta_1 - \eta_2))(w_1^2 - w_2^2) dx = 0. \end{cases}$$

Consequently we have proved that the map $T_\lambda : E \rightarrow E$ is well-defined for large $\lambda > 0$ and it is compact and continuous. Therefore from the Schauder fixed point theorem, we have a fixed point in E and we get a solution $(w_\lambda, \eta_\lambda, A_\lambda)$ to (4.3) - (4.4) - (4.5) for large $\lambda > 0$.

§5. Stability of $(\Phi_\lambda, A_\lambda)$

In this section we prove the stability of $(\Phi_\lambda, A_\lambda)$, which we constructed in §3 and §4. The procedure below is almost similar as was done in [15]. We estimate the second variation of the functional $\mathcal{H}_\lambda$ at $(\Phi_\lambda, A_\lambda) = (u_\lambda, v_\lambda, A_\lambda)$. We express Φ_λ in terms of real valued functions, i.e., we put $u_\lambda(x) = W_\lambda(x) \cos \theta_\lambda$, $v_\lambda(x) = W_\lambda(x) \sin \theta_\lambda$. Let us consider $\mathcal{L}_\lambda(u_\lambda, v_\lambda, A_\lambda, \phi, \psi, B)$ on $\overline{N}(\Phi_\lambda, A_\lambda) = \overline{N}(u_\lambda, v_\lambda, A_\lambda)$. Hereafter we denote $\mathcal{L}_\lambda(u_\lambda, v_\lambda, A_\lambda, \phi, \psi, B)$ by $\mathcal{L}_\lambda(\phi, \psi, B)$ for simplicity. We have

$$(5.1) \quad \mathcal{L}_\lambda(\phi, \psi, B) = I_1(\phi, \psi) + I_2(B) + I_3(\phi, \psi, B)$$

where

$$I_1(\phi, \psi) = \int_{\Omega} (|\nabla \phi + \psi A_\lambda|^2 + |\nabla \psi - \phi A_\lambda|^2 - \lambda(1 - u_\lambda^2 - v_\lambda^2)(\phi^2 + \psi^2) + 2\lambda(u_\lambda \phi + v_\lambda \psi)^2) dx$$

$$I_2(B) = \int_{\mathbb{R}^3} |\text{rot} B|^2 dx + \int_{\Omega} (u_\lambda^2 + v_\lambda^2) B^2 dx,$$

$$I_3(\phi, \psi, B) = 4 \int_{\Omega} \langle A_\lambda \cdot B \rangle (u_\lambda \phi + v_\lambda \psi) dx \\ - 2 \int_{\Omega} \{ \phi \langle \nabla v_\lambda \cdot B \rangle - \psi \langle \nabla u_\lambda \cdot B \rangle + u_\lambda \langle \nabla \psi \cdot B \rangle - v_\lambda \langle \nabla \phi \cdot B \rangle \} dx.$$

We change the variables ϕ, ψ into $\widehat{\phi}, \widehat{\psi}$ by

$$(5.2) \quad \begin{pmatrix} \widehat{\phi}(x) \\ \widehat{\psi}(x) \end{pmatrix} = R(-\theta_\lambda(x)) \begin{pmatrix} \phi(x) \\ \psi(x) \end{pmatrix} \quad \text{where} \quad R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

and we express I_1 and I_3 in terms of $\widehat{\phi}, \widehat{\psi}, B$. I_1 and I_3 are rewritten as follows,

$$(5.3) \quad I_1(\phi, \psi) = \widehat{I}_1(\widehat{\phi}, \widehat{\psi}) = \int_{\Omega} (|\nabla \widehat{\phi} + (A_\lambda - \nabla \theta_\lambda) \widehat{\psi}|^2 + |\nabla \widehat{\psi} - (A_\lambda - \nabla \theta_\lambda) \widehat{\phi}|^2 \\ - \lambda(1 - W_\lambda^2)(\widehat{\phi}^2 + \widehat{\psi}^2) + 2\lambda W_\lambda^2 \widehat{\phi}^2) dx$$

$$(5.4) \quad I_3(\phi, \psi, B) = \widehat{I}_3(\widehat{\phi}, \widehat{\psi}, B) \\ = 4 \int_{\Omega} \langle A_\lambda \cdot B \rangle W_\lambda \widehat{\phi} dx - 4 \int_{\Omega} (W_\lambda \langle \nabla \theta_\lambda \cdot B \rangle \widehat{\phi} - \langle \nabla W_\lambda \cdot B \rangle \widehat{\psi}) dx$$

We used that $\text{div} B = 0$ in Ω and $\langle B \cdot \nu \rangle = 0$ on $\partial\Omega$ since $(\phi, \psi, B) \in \overline{N}(u_\lambda, v_\lambda, A_\lambda)$.

To investigate the coerciveness of I_1 , we consider the eigenvalue problem,

$$(5.5) \quad \begin{cases} \Delta \begin{pmatrix} \phi \\ \psi \end{pmatrix} + \begin{pmatrix} \text{div}((A_\lambda - \nabla \theta_\lambda) \psi) + \langle (A_\lambda - \nabla \theta_\lambda) \cdot \nabla \psi \rangle \\ -\text{div}((A_\lambda - \nabla \theta_\lambda) \phi) - \langle (A_\lambda - \nabla \theta_\lambda) \cdot \nabla \phi \rangle \end{pmatrix} \\ + (\lambda(1 - W_\lambda^2) - |A_\lambda - \nabla \theta_\lambda|^2) \begin{pmatrix} \phi \\ \psi \end{pmatrix} - 2\lambda W_\lambda^2 \begin{pmatrix} \phi \\ 0 \end{pmatrix} + \mu \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \nu} = \frac{\partial \psi}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let

$$\{\mu_\ell(\lambda)\}_{\ell=1}^\infty \quad \text{and} \quad \left\{ \begin{pmatrix} \phi_{\ell,\lambda} \\ \psi_{\ell,\lambda} \end{pmatrix} \right\}_{\ell=1}^\infty \subset L^2(\Omega) \times L^2(\Omega)$$

be the eigenvalues arranged in increasing order (with counting multiplicity) and a complete system of the corresponding orthonormal eigenfunctions of (5.5). We can apply quite a similar argument as in [15] or [16] for such type of the eigenvalue problem as (5.5) and we get their asymptotic behaviors of the eigenvalues and eigenfunctions. So we give the results without proofs.

Lemma 11.

$$(5.6) \quad \lim_{\lambda \rightarrow \infty} \mu_\ell(\lambda) = \mu_\ell \quad (\ell \geq 1),$$

$$(5.7) \quad \lim_{\lambda \rightarrow \infty} \left(\|\nabla \phi_{\ell,\lambda}\|_{L^2(\Omega)}^2 + \lambda \|\phi_{\ell,\lambda}\|_{L^2(\Omega)}^2 \right) = 0 \quad (\ell \geq 1),$$

where $\{\mu_\ell\}_{\ell=1}^\infty$ is the set of the eigenvalues arranged in increasing order (with counting multiplicity) of the following eigenvalue problem:

$$(5.8) \quad \begin{cases} \Delta \psi + \mu \psi = 0 & \text{in } \Omega, \\ \frac{\partial \psi}{\partial \nu} = 0 & \text{on } \partial \Omega. \end{cases}$$

We prepare an auxiliary result concerning a complete orthonormal basis of the product of two Hilbert spaces, which we use in the proof of Lemma 14.

Lemma 12 ([15]). Let H_1 and H_2 be two real Hilbert spaces with inner products $(\cdot, \cdot)_{H_1}$ and $(\cdot, \cdot)_{H_2}$, respectively and let H be the product Hilbert space $H_1 \times H_2$ with the inner product:

$$\left(\begin{pmatrix} \phi \\ \psi \end{pmatrix}, \begin{pmatrix} \phi' \\ \psi' \end{pmatrix} \right)_H \equiv (\phi, \phi')_{H_1} + (\psi, \psi')_{H_2} \quad \text{for} \quad \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \begin{pmatrix} \phi' \\ \psi' \end{pmatrix} \in H.$$

If there exists an orthonormal basis $\left\{ \begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix} \right\}_{n=1}^\infty \subset H$, then

$$(5.9) \quad \begin{cases} \sum_{n=1}^\infty (\phi, \phi_n)_{H_1} (\psi, \psi_n)_{H_2} = 0 & \text{for any} \quad \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in H, \\ \|\phi\|_{H_1}^2 = \sum_{n=1}^\infty (\phi, \phi_n)_{H_1}^2, \quad \|\psi\|_{H_2}^2 = \sum_{n=1}^\infty (\psi, \psi_n)_{H_2}^2. \end{cases}$$

The following lemma directly follows from the above lemmas.

Lemma 13. For any $(\phi, \psi) \in L^2(\Omega) \times L^2(\Omega)$, the following equalities hold.

$$\sum_{\ell=1}^{\infty} (\phi \cdot \phi_{\ell, \lambda})_{L^2(\Omega)} (\psi \cdot \psi_{\ell, \lambda})_{L^2(\Omega)} = 0,$$

$$\|\phi\|_{L^2(\Omega)}^2 = \sum_{\ell=1}^{\infty} (\phi \cdot \phi_{\ell, \lambda})_{L^2(\Omega)}^2, \quad \|\psi\|_{L^2(\Omega)}^2 = \sum_{\ell=1}^{\infty} (\psi \cdot \psi_{\ell, \lambda})_{L^2(\Omega)}^2.$$

$\widehat{I}_1(\widehat{\phi}, \widehat{\psi})$ is expressed in terms the Fourier coefficients of $\widehat{\phi}, \widehat{\psi}$:

$$(5.10) \quad \widehat{I}_1(\widehat{\phi}, \widehat{\psi}) = \sum_{\ell=1}^{\infty} \mu_{\ell}(\lambda) g_{\ell}^2$$

where

$$(5.11) \quad g_{\ell} = (\phi_{\ell, \lambda} \cdot \widehat{\phi})_{L^2(\Omega)} + (\psi_{\ell, \lambda} \cdot \widehat{\psi})_{L^2(\Omega)}.$$

We remark that $\phi_{1, \lambda}(x) = 0$, $\psi_{1, \lambda}(x) = e_{\lambda} W_{\lambda}(x)$, $\mu_1(\lambda) = \mu_1 = 0$, $\mu_2 > 0$, $e_{\lambda} \neq 0$ is a certain real number, which satisfies $\lim_{\lambda \rightarrow \infty} e_{\lambda}^2 = 1/|\Omega|$.

We have the following coercive inequality.

Lemma 14. For any $c > 0$ and $\eta > 0$, there exists a constant $\lambda_1 = \lambda_1(c, \eta) > 0$ and $c' = c'(c, \eta) > 0$ such that

$$(5.12) \quad \widehat{I}_1(\widehat{\phi}, \widehat{\psi}) \geq c \|\widehat{\phi}\|_{L^2(\Omega)}^2 + (\mu_2(\lambda) - \eta) \|\widehat{\psi}\|_{L^2(\Omega)}^2 - c' (\widehat{\psi} \cdot W_{\lambda})_{L^2(\Omega)}^2,$$

for any $\widehat{\phi}, \widehat{\psi} \in H^1(\Omega)$ and $\lambda \geq \lambda_1$. Equivalently,

$$(5.13) \quad I_1(\phi, \psi) \geq c \|\phi \cos \theta_{\lambda} + \psi \sin \theta_{\lambda}\|_{L^2(\Omega)}^2 + (\mu_2(\lambda) - \eta) \|\phi \sin \theta_{\lambda} - \psi \cos \theta_{\lambda}\|_{L^2(\Omega)}^2$$

$$- c' \left(\int_{\Omega} (\phi v_{\lambda} - \psi u_{\lambda}) dx \right)^2,$$

for any $\phi, \psi \in H^1(\Omega)$ and $\lambda \geq \lambda_1$.

(Proof of Lemma 14) In view of the properties of the eigenvalues of (5.8) and Lemma 11, for given $c > 0$, we can take a natural number N so that $\mu_{\ell}(\lambda) \geq c + 1$ for $\ell > N$ and for any large $\lambda > 0$. Thus we have,

$$\widehat{I}_1(\widehat{\phi}, \widehat{\psi}) \geq \sum_{\ell=1}^N \mu_{\ell}(\lambda) g_{\ell}^2 + (c + 1) \sum_{\ell=N+1}^{\infty} g_{\ell}^2.$$

Substituting (5.11), we have

$$\widehat{I}_1(\widehat{\phi}, \widehat{\psi}) \geq \sum_{\ell=1}^N \mu_{\ell}(\lambda) (\widehat{\phi} \cdot \phi_{\ell, \lambda})_{L^2(\Omega)}^2 + (c + 1) \sum_{\ell=N+1}^{\infty} (\widehat{\phi} \cdot \phi_{\ell, \lambda})_{L^2(\Omega)}^2$$

$$\begin{aligned}
& + 2 \sum_{\ell=1}^N \mu_{\ell}(\lambda) (\widehat{\phi} \cdot \phi_{\ell, \lambda})_{L^2(\Omega)} (\widehat{\psi} \cdot \psi_{\ell, \lambda})_{L^2(\Omega)} + 2(c+1) \sum_{\ell=N+1}^{\infty} (\widehat{\phi} \cdot \phi_{\ell, \lambda})_{L^2(\Omega)} (\widehat{\psi} \cdot \psi_{\ell, \lambda})_{L^2(\Omega)} \\
& + \sum_{\ell=1}^N \mu_{\ell}(\lambda) (\widehat{\psi} \cdot \psi_{\ell, \lambda})_{L^2(\Omega)}^2 + (c+1) \sum_{\ell=N+1}^{\infty} (\widehat{\psi} \cdot \psi_{\ell, \lambda})_{L^2(\Omega)}^2.
\end{aligned}$$

From Lemma 13,

$$\begin{aligned}
(5.14) \quad I_1(\widehat{\phi}, \widehat{\psi}) & \geq (c+1) \|\widehat{\phi}\|_{L^2(\Omega)}^2 + \sum_{\ell=1}^N (\mu_{\ell}(\lambda) - c - 1) (\widehat{\phi} \cdot \phi_{\ell, \lambda})_{L^2(\Omega)}^2 + \\
& \sum_{\ell=1}^N 2(\mu_{\ell}(\lambda) - c - 1) (\widehat{\phi} \cdot \phi_{\ell, \lambda})_{L^2(\Omega)} (\widehat{\psi} \cdot \psi_{\ell, \lambda})_{L^2(\Omega)} + \mu_2(\lambda) \left(\|\widehat{\psi}\|_{L^2(\Omega)}^2 - (\widehat{\psi} \cdot \psi_{1, \lambda})_{L^2(\Omega)}^2 \right).
\end{aligned}$$

We used $\mu_1(\lambda) = 0$, $\phi_{1, \lambda} = 0$, $\psi_{1, \lambda} = e_{\lambda} W_{\lambda}$. Using Lemma 11-(5.7) in the right hand side of the above inequality, we see that the terms including $\phi_{\ell, \lambda}$ can be absorbed in the ones including $\|\widehat{\phi}\|_{L^2(\Omega)}^2$ and $\|\widehat{\psi}\|_{L^2(\Omega)}^2$ for large $\lambda > 0$. We have,

$$\widehat{I}_1(\widehat{\phi}, \widehat{\psi}) \geq c \|\widehat{\phi}\|_{L^2(\Omega)}^2 + (\mu_2(\lambda) - \eta) \|\widehat{\psi}\|_{L^2(\Omega)}^2 - c' (\widehat{\psi} \cdot W_{\lambda})_{L^2(\Omega)}^2$$

for large $\lambda > 0$. We obtain (5.12). (5.13) follows immediately. $\square$

Lemma 15. For any $c > 0$ and $\eta > 0$, there exist $\delta > 0$, $\lambda_1 > 0$ and $c' > 0$ such that

$$\begin{aligned}
(5.15) \quad \widehat{I}_1(\widehat{\phi}, \widehat{\psi}) & \geq \delta (\|\nabla \widehat{\phi}\|_{L^2(\Omega)}^2 + \|\nabla \widehat{\psi}\|_{L^2(\Omega)}^2) + c \|\widehat{\phi}\|_{L^2(\Omega)}^2 \\
& + (\mu_2(\lambda) - \eta) \|\widehat{\psi}\|_{L^2(\Omega)}^2 - c' (W_{\lambda} \cdot \widehat{\psi})_{L^2(\Omega)}^2.
\end{aligned}$$

for any $\widehat{\phi}, \widehat{\psi} \in H^1(\Omega)$ and $\lambda > \lambda_1$,

$$\begin{aligned}
(5.16) \quad I_1(\phi, \psi) & \geq \delta (\|\nabla \phi\|_{L^2(\Omega)}^2 + \|\nabla \psi\|_{L^2(\Omega)}^2) + c \|\phi \cos \theta_{\lambda} + \psi \sin \theta_{\lambda}\|_{L^2(\Omega)}^2 \\
& + (\mu_2(\lambda) - \eta) \|\phi \sin \theta_{\lambda} - \psi \cos \theta_{\lambda}\|_{L^2(\Omega)}^2 - c' \left(\int_{\Omega} (-\phi v_{\lambda} + \psi u_{\lambda}) dx \right)^2
\end{aligned}$$

for any $\phi, \psi \in H^1(\Omega)$ and $\lambda > \lambda_1$.

(Proof of Lemma 15) First recall that there exists a $M > 0$ independent of $\lambda > 0$, such that $|\nabla \theta_{\lambda}(x)| \leq M$ in Ω . From (5.3), for any $c > 0$ there exist constants $\lambda_1 > 0$ and $c'' > 0$ such that

$$(5.17) \quad \widehat{I}_1(\widehat{\phi}, \widehat{\psi}) \geq \frac{1}{2} (\|\nabla \widehat{\phi}\|_{L^2(\Omega)}^2 + \|\nabla \widehat{\psi}\|_{L^2(\Omega)}^2) + c \|\widehat{\phi}\|_{L^2(\Omega)}^2 - c'' \|\widehat{\psi}\|_{L^2(\Omega)}^2 \quad \text{for } \widehat{\phi}, \widehat{\psi} \in H^1(\Omega),$$

for $\lambda > \lambda_1$. Using

$$\begin{aligned}
|\nabla \widehat{\phi}|^2 + |\nabla \widehat{\psi}|^2 & = |\nabla \phi|^2 + |\nabla \psi|^2 + 2(-\psi \nabla \theta_{\lambda} \nabla \phi + \phi \nabla \theta_{\lambda} \nabla \psi) + |\nabla \theta_{\lambda}|^2 (\phi^2 + \psi^2) \\
& \geq \frac{1}{2} (|\nabla \phi|^2 + |\nabla \psi|^2) - |\nabla \theta_{\lambda}|^2 (\phi^2 + \psi^2) = \frac{1}{2} (|\nabla \phi|^2 + |\nabla \psi|^2) - |\nabla \theta_{\lambda}|^2 (\widehat{\phi}^2 + \widehat{\psi}^2).
\end{aligned}$$

in (5.17), we have

$$(5.18) \quad I_1(\phi, \psi) \geq \frac{1}{4}(\|\nabla \phi\|_{L^2(\Omega)}^2 + \|\nabla \psi\|_{L^2(\Omega)}^2) + (c - M^2/2)\|\widehat{\phi}\|_{L^2(\Omega)}^2 - (c'' + M^2/2)\|\widehat{\psi}\|_{L^2(\Omega)}^2$$

for $\widehat{\phi}, \widehat{\psi} \in H^1(\Omega)$. Using (5.12) and (5.17), we obtain,

$$\begin{aligned} \widehat{I}_1(\widehat{\phi}, \widehat{\psi}) &= (1 - \epsilon)\widehat{I}_1(\widehat{\phi}, \widehat{\psi}) + \epsilon\widehat{I}_1(\widehat{\phi}, \widehat{\psi}) \\ &\geq \frac{\epsilon}{2}(\|\nabla \widehat{\phi}\|_{L^2(\Omega)}^2 + \|\nabla \widehat{\psi}\|_{L^2(\Omega)}^2) + c\|\widehat{\phi}\|_{L^2(\Omega)}^2 \\ &\quad + ((1 - \epsilon)(\mu_2(\lambda) - \eta) - \epsilon c'')\|\widehat{\psi}\|_{L^2(\Omega)}^2 - (1 - \epsilon)c'(\widehat{\psi} \cdot W_\lambda)_{L^2(\Omega)}^2. \end{aligned}$$

where $\epsilon \in (0, 1)$. By taking $\epsilon > 0$ small, we obtain (5.15).

Similarly, with the aid of (5.14) and (5.18), we obtain,

$$\begin{aligned} I_1(\phi, \psi) &= (1 - \epsilon)I_1(\phi, \psi) + \epsilon I_1(\phi, \psi) \\ &\geq \frac{\epsilon}{4}(\|\nabla \widehat{\phi}\|_{L^2(\Omega)}^2 + \|\nabla \widehat{\psi}\|_{L^2(\Omega)}^2) + (c - \epsilon M^2/2)\|\phi \cos \theta_\lambda + \psi \sin \theta_\lambda\|_{L^2(\Omega)}^2 \\ &\quad + ((1 - \epsilon)(\mu_2(\lambda) - \eta) - \epsilon(c'' + M^2/2))\|\widehat{\psi}\|_{L^2(\Omega)}^2 - (1 - \epsilon)c' \left(\int_{\Omega} (-\phi v_\lambda + \psi u_\lambda) dx \right)^2. \end{aligned}$$

We complete the proof of (5.16) by taking $\epsilon > 0$ small. $\square$

Proposition 16. For $B \in L^2_{loc}(\mathbb{R}^3; \mathbb{R}^3)$ such that $\nabla B \in L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$,

$$\|\nabla B\|_{L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})}^2 = \|\operatorname{div} B\|_{L^2(\mathbb{R}^3)}^2 + \|\operatorname{rot} B\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)}^2$$

(Proof of Proposition 16) This equality is proved by the Fourier Transform. $\square$

Now we estimate $\mathcal{L}_\lambda(\phi, \psi, B)$ from below.

Proof of Theorem 4

First we consider the case $(\phi, \psi, B) \in \overline{N}(u_\lambda, v_\lambda, A_\lambda)$.

$$\mathcal{L}_\lambda(\phi, \psi, B) = \widehat{I}_1(\widehat{\phi}, \widehat{\psi}) + I_2(B) + \widehat{I}_3(\widehat{\phi}, \widehat{\psi}, B)$$

We prove that $|\widehat{I}_3(\widehat{\phi}, \widehat{\psi}, B)|$ is dominated by I_1 and I_2 .

$$\begin{aligned} |\widehat{I}_3(\widehat{\phi}, \widehat{\psi}, B)| &\leq 4 \int_{\Omega} (|\langle A_\lambda - \nabla \theta_\lambda \cdot B \rangle W_\lambda \widehat{\phi}| dx + 4 \int_{\Omega} |\langle \nabla W_\lambda \cdot B \rangle \widehat{\psi}| dx \\ &\leq 2 \sup_{x \in \Omega} |W_\lambda(A_\lambda - \nabla \theta_\lambda)| \int_{\Omega} (\epsilon B^2 + \widehat{\phi}^2/4\epsilon) dx + 2 \sup_{x \in \Omega} |\nabla W_\lambda| \int_{\Omega} (B^2 + \widehat{\psi}^2) dx \end{aligned}$$

There exist $M > 0$ and $\lambda_2 > 0$ such that

$$\sup_{x \in \Omega} |W_\lambda(A_\lambda - \nabla \theta_\lambda)| \leq M \quad \text{for } \lambda \geq \lambda_2.$$

Next put $\epsilon = 1/(8M)$ and $c = M/(2\epsilon) + 1$ in Lemma 15, we can take a large positive number $\lambda_3 > 0$

$$(5.19) \quad \begin{aligned} \mathcal{L}_\lambda(\phi, \psi, B) &\geq \delta \left(\|\widehat{\phi}\|_{L^2(\Omega)}^2 + \|\widehat{\psi}\|_{L^2(\Omega)}^2 + \|B\|_{L^2(\Omega; \mathbb{R}^3)}^2 + \|\operatorname{rot} B\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)}^2 \right) \\ &= \delta \left(\|\phi\|_{L^2(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2 + \|B\|_{L^2(\Omega; \mathbb{R}^3)}^2 + \|\operatorname{rot} B\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)}^2 \right) \quad \text{for } \lambda \geq \lambda_3, \end{aligned}$$

for $\delta = \min(\mu_2/2, 1) > 0$. We used $(\widehat{\psi} \cdot W_\lambda)_{L^2(\Omega)} = 0$ for $(\phi, \psi, B) \in \overline{N}(u_\lambda, v_\lambda, A_\lambda)$.

We will get a similar inequality on $N(u_\lambda, v_\lambda, A_\lambda)$. Take any $(\phi, \psi, B) \in N(u_\lambda, v_\lambda, A_\lambda)$ and we have the decomposition

$$(\phi, \psi, B) = (-v_\lambda \xi, u_\lambda \xi, \nabla \xi) + (\overline{\phi}, \overline{\psi}, \overline{B}) \in T(u, v, A) + \overline{N}(u, v, A) \quad (\text{cf. (1.10)}).$$

Note that

$$\mathcal{L}_\lambda(\phi, \psi, B) = \mathcal{L}_\lambda(\overline{\phi}, \overline{\psi}, \overline{B}). \quad \text{cf. Proposition 3}$$

We will prove that there exists a $c > 0$ which is independent of (ϕ, ψ, B) and large $\lambda > 0$ such that

$$(5.20) \quad \begin{aligned} &\|\overline{\phi}\|_{H^1(\Omega)}^2 + \|\overline{\psi}\|_{H^1(\Omega)}^2 + \|\overline{B}\|_{L^2(\Omega; \mathbb{R}^3)}^2 + \|\operatorname{rot} \overline{B}\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)}^2 \\ &\geq c \left(\|\phi\|_{H^1(\Omega)}^2 + \|\psi\|_{H^1(\Omega)}^2 + \|B\|_{L^2(\Omega; \mathbb{R}^3)}^2 + \|\operatorname{rot} B\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)}^2 \right). \end{aligned}$$

If this is not true, there exists a sequence $(\phi_n, \psi_n, B_n) \in T(u_\lambda, v_\lambda, A_\lambda)$ and ξ_n ($n \geq 1$) such that

$$(5.21) \quad (\phi_n, \psi_n, B_n) = (-v_\lambda \xi_n, u_\lambda \xi_n, \nabla \xi_n) + (\overline{\phi}_n, \overline{\psi}_n, \overline{B}_n) \in T(u_\lambda, v_\lambda, A_\lambda) + \overline{N}(u_\lambda, v_\lambda, A_\lambda).$$

with

$$(5.22) \quad \|\overline{\phi}_n\|_{H^1(\Omega)}^2 + \|\overline{\psi}_n\|_{H^1(\Omega)}^2 + \|\overline{B}_n\|_{L^2(\Omega; \mathbb{R}^3)}^2 + \|\operatorname{rot} \overline{B}_n\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)}^2 \rightarrow 0 \quad (n \rightarrow \infty),$$

$$(5.23) \quad \|\phi_n\|_{H^1(\Omega)}^2 + \|\psi_n\|_{H^1(\Omega)}^2 + \|B_n\|_{L^2(\Omega; \mathbb{R}^3)}^2 + \|\operatorname{rot} B_n\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)}^2 = 1 \quad (n \geq 1).$$

(5.21) yields

$$(5.24) \quad \int_{\Omega} (u_\lambda^2 + v_\lambda^2) \xi_n(x) dx = 0, \quad \Delta \xi_n = 0 \text{ in } \Omega, \quad \frac{\partial \xi_n}{\partial \nu} = \langle B \cdot \nu \rangle \text{ on } \partial \Omega,$$

$$(5.25) \quad \operatorname{rot} B_n = \operatorname{rot} \overline{B}_n \quad \text{in } \mathbb{R}^3.$$

There exist $\phi, \psi \in H^1(\Omega)$ such that

$$\phi_n \rightarrow \phi \quad \psi_n \rightarrow \psi \quad \text{weakly in } H^1(\Omega) \quad \text{as } n \rightarrow \infty.$$

From (5.22) and (5.25) with Proposition 16, $\|\nabla B_n\|_{L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} \rightarrow 0$ as $n \rightarrow \infty$. Using the Sobolev imbedding theorem, we have $\lim_{n \rightarrow \infty} \|B_n\|_{L^6(\mathbb{R}^3; \mathbb{R}^3)} = 0$. Using $B_n = \nabla \xi_n + \overline{B}_n$ and $\langle \overline{B}_n \cdot \nu \rangle = 0$ on $\partial \Omega$, we have

$$\int_{\Omega} B_n^2 dx = \int_{\Omega} |\nabla \xi_n|^2 dx + \int_{\Omega} \overline{B}_n^2 dx.$$

This inequality and (5.24) yields $\|\xi_n\|_{H^1(\Omega)} \rightarrow 0$ for $n \rightarrow \infty$. By considering $H^1(\Omega)$ convergence of $(\phi_n, \psi_n) = (-v_\lambda \xi_n, u_\lambda \xi_n) + (\overline{\phi}_n, \overline{\psi}_n)$ for $n \rightarrow \infty$ and we obtain $\phi_n, \psi_n \rightarrow 0$ in $H^1(\Omega)$, we see that (5.23) is impossible. This completes the proof of the existence of a certain $c > 0$ in (5.20). We complete the proof of Theorem 4. $\square$

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