ON THE NONLINEAR SCHRÖDINGER EQUATIONS OF DERIVATIVE TYPE

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ON THE NONLINEAR SCHRÖDINGER EQUATIONS OF DERIVATIVE TYPE

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Dedicated to Professor Kôji Kubota on his sixtieth birthday

Abstract. This paper studies the Cauchy problem both at finite and infinite times for a class of nonlinear Schrödinger equations with coupling of derivative type. The proof uses gauge transformations which reduce the original equations to systems of equations without coupling of derivative type. Concerning the Cauchy problem at finite times, we give sufficient conditions for the global well-posedness in the energy space. Concerning the Cauchy problem at infinity, we construct modified wave operators on small and sufficiently regular asymptotic states.

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1. Introduction

We consider the nonlinear Schrödinger equations of the form

$$i\partial_t u + \partial^2 u = i\lambda|u|^2\partial u + i\mu u^2\partial\bar{u} + f(u) \tag{1.1}$$

where u is a complex valued function of $(t,x) \in \mathbb{R}\times\mathbb{R}$, $\partial_t = \partial/\partial t$, $\partial = \partial/\partial x$, λ, $\mu \in \mathbb{R}$, and f is a nonlinear interaction, a typical form of which is the so-called double power interaction

$$f(u) = \nu_1|u|^{p_1-1}u + \nu_2|u|^{p_2-1}u \tag{1.2}$$

with $1 \le p_1 \le p_2 < \infty$ and ν_1, $\nu_2 \in \mathbb{R}$. The following examples are special cases of (1.1).

Example 1. When $\lambda = 2\mu$ and $f \equiv 0$, (1.1) becomes

$$i\partial_t u + \partial^2 u = i\mu\partial(|u|^2 u) \tag{1.3}$$

which is known as the derivative nonlinear Schrödinger equation and is studied in [17, 19, 21, 22, 31, 32, 38, 39, 40, 41, 49, 50, 52].

Example 2. When $\mu = 0$ and $f \equiv 0$, (1.1) becomes

$$i\partial_t u + \partial^2 u = i\lambda|u|^2\partial u \tag{1.4}$$

which is studied in [2, 36].

Example 3. When $\lambda = 0$ and $f(u) = \nu_1|u|^2u + \nu_2|u|^4u$, (1.1) becomes

$$i\partial_t u + \partial^2 u = i\mu u^2\partial\bar{u} + \nu_1|u|^2u + \nu_2|u|^4u \tag{1.5}$$

which is studied in [8, 36].

Example 4. When $\lambda = \mu$ and $f \equiv 0$, (1.1) becomes

$$i\partial_t u + \partial^2 u = i\lambda(\partial|u|^2)u = 2i\lambda(\mathrm{Re}\, u\partial\bar{u})u \tag{1.6}$$

which is studied in [48, 51].

Example 5. When $\lambda = -\mu$ and $f \equiv 0$, (1.1) becomes

$$i\partial_t u + \partial^2 u = i\lambda(\bar{u}\partial u - u\partial\bar{u})u = 2\lambda(\operatorname{Im} u\partial\bar{u})u \tag{1.7}$$

which is studied in [2].

Example 6. When $f(u) = \nu_1|u|^2 u + \nu_2|u|^4 u$, (1.1) is written as

$$i\partial_t u + \partial^2 u = i(\lambda - \mu)|u|^2\partial u + i\mu(\partial|u|^2)u + \nu_1|u|^2 u + \nu_2|u|^4 u \tag{1.8}$$

which is studied in [15, 25, 26, 37, 47]. This model generalizes (1.3)-(1.7).

The first purpose in this paper is to study the Cauchy problem for (1.1) with data given at finite times. A standard way to construct local solutions for (1.1) with data u_0 at time $t = t_0$ consists in solving the integral equation

$$u(t) = U_0(t - t_0)u_0 - i\int_{t_0}^{t} U_0(t - \tau)(i\lambda|u|^2\partial u + i\mu u^2\partial\bar{u} + f(u))(\tau)d\tau \tag{1.9}$$

where $U_0(t) = \exp(it\partial^2)$ is the free propagator, namely the unitary group which solves the free Schrödinger equation. A major difficulty is caused by the loss of derivatives in the usual iteration scheme as long as the method depends exclusively on the unitarity of the free propagator and on the smoothing estimates of Strichartz type applied directly on (1.9). Recent developments on the nonlinear Schrödinger equations with coupling of derivative type tell that the difficulty may be avoided in the Sobolev spaces H^m with $m \geq 3$ without imposing specific structure of nonlinearity [3, 4, 20, 33]. When restricted to (1.1), however, it is shown in [47] that the equation (1.9) has a unique global solution in H^1, the Sobolev space directly related to the energy of the equation (1.1). The method in [47] depends on the gauge transformation technique developed in [17, 19, 21, 22, 31, 36-38, 52]. Here our main attention is on the generalization of results in [17, 19, 22, 47] as regards the nonlinearity f and the size of the Cauchy problem of the derivative nonlinear Schrödinger equation.

The interaction f will satisfy assumptions to be taken from the following list.

(H1) f is decomposed as $f = f_1 + f_2$ with $f_j \in C^1(\mathbb{C};\mathbb{C})$, $f_j(0) = 0$, $f_j'(0) = 0$, and for some p_j with $1 \leq p_1 \leq p_2 < \infty$

$$|f_j'(z_1) - f_j'(z_2)| \leq \left\{ \begin{array}{ll} C(|z_1|^{p_j - 2} + |z_2|^{p_j - 2})\,|z_1 - z_2| & \text{if } p_j \geq 2, \\ C|z_1 - z_2|^{p_j - 1} & \text{if } p_j < 2, \end{array} \right\} \tag{1.10}$$

for all z_1, $z_2 \in \mathbb{C}$, where f_j' denotes any of $\partial f_j/\partial z$ and $\partial f_j/\partial \bar{z}$ and $j = 1, 2$.

(H2)(Gauge Invariance) For all $z \in \mathbb{C}$ and $\theta \in \mathbb{R}$, $f(e^{i\theta}z) = e^{i\theta}f(z)$ and $f(\mathbb{R}) \subset \mathbb{R}$. Equivalently, if $f \in C(\mathbb{C};\mathbb{C})$, there exists $F \in C^1(\mathbb{C};\mathbb{R})$ such that $F(0) = 0$, $F(e^{i\theta}z) = F(z)$ for all $z \in \mathbb{C}$ and $\theta \in \mathbb{R}$, and $f = \partial F/\partial \bar{z}$.

(H3) F satisfies the estimate

$$\int F(\varphi)dx \geq -N(||\varphi||_2) - \nu_0||\varphi||_6^6$$

for all $\varphi \in H^1$, where $N \in C(\mathbb{R}_+;\mathbb{R}_+)$, $\nu_0 \geq 0$, and $||\cdot||_p$ denotes the norm in the standard Lebesgue space $L^p = L^p(\mathbb{R})$.

In order to state the basic existence and uniqueness result for the Cauchy problem in the energy space for (1.9) we introduce additional notations. For a nonnegative integer m and $1 \leq p \leq \infty$, let $W^{m,p}$ be the usual Sobolev space defined by

$$W^{m,p} = \{\varphi \in L^p; ||\varphi||_{W^{m,p}} \equiv \sum_{j=0}^{m} ||\partial^j \varphi||_p < \infty\}$$

and let $H^m = W^{m,2}$, as usual. The scalar product in L^2 is denoted by $< \cdot, \cdot >$ depending linearly and antilinearly in the first and second component, respectively.

Theorem 1. *Let f satisfy (H1), (H2) and (H3). Let*

$$\kappa = (\lambda + \mu)(5\mu - 3\lambda)/48.$$

(1) If $0 \leq \nu_0 \leq \kappa$, then for any $u_0 \in H^1$ and $t_0 \in \mathbb{R}$ the equation (1.9) has a unique solution u such that

$$u \in C(\mathbb{R}; H^1) \cap L^4_{\mathrm{loc}}(\mathbb{R}; W^{1,\infty})$$

with $u(t_0) = u_0$. That solution satisfies

$$||u(t)||_2 = ||u_0||_2, \tag{1.11}$$

$$E(u(t)) = E(u_0), \tag{1.12}$$

for all $t \in \mathbb{R}$, where

$$E(\varphi) = ||\partial\varphi||_2^2 + \frac{\lambda+\mu}{2}\mathrm{Im}\langle|\varphi|^2\varphi, \partial\varphi\rangle + \frac{(\lambda+\mu)\mu}{6}||\varphi||_6^6 + \int F(\varphi)dx. \tag{1.13}$$

Moreover, the map $u_0 \mapsto u$ *is continuous from* H^1 *to* $L^\infty(-T,T;H^1) \cap L^4(-T,T;W^{1,\infty})$ *for any* $T > 0$.

(2) If $\kappa < \nu_0$, *then for any* $u_0 \in B \equiv \{\varphi \in H^1; ||\varphi||_2^2 < \pi/(2(\nu_0 - \kappa)^{1/2})\}$ *and* $t_0 \in \mathbb{R}$ *the equation (1.9) has a unique solution* u *such that*

$$u \in C(\mathbb{R}; H^1) \cap L^4_{\mathrm{loc}}(\mathbb{R}; W^{1,\infty})$$

with $u(0) = u_0$. *That solution satisfies (1.11) and (1.12) for all* $t \in \mathbb{R}$. *Moreover, the map* $u_0 \mapsto u$ *is continuous from* B *with topology induced from* H^1 *to* $L^\infty(-T,T;H^1) \cap L^4(-T,T;W^{1,\infty})$ *for any* $T > 0$.

Remark 1. Double power interaction (1.2) satisfies (H1), (H2) and (H3) in the following cases: (i) $1 \le p_1 < p_2 < 5$ with $\nu_0 > 0$ arbitrarily small. (ii) $1 \le p_1 < p_2 = 5$ with $\nu_2 + 6\nu_0 > 0$. (iii) $1 \le p_1 < p_2 = 5$ with $\nu_2 + 6\nu_0 = 0$, $\nu_1 \ge 0$. (iv) $1 \le p_1 < 5 < p_2$ with $\nu_0 > 0$ arbitrarily small and $\nu_2 \ge 0$. (v) $5 = p_1 < p_2$ with $\nu_1 + 6\nu_0 \ge 0$, $\nu_2 \ge 0$. (vi) $5 < p_1 \le p_2$ with $\nu_1, \nu_2 \ge 0 = \nu_0$.

Remark 2. Let $(\lambda+\mu)(5\mu - 3\lambda) < 0$. Then under the double interaction (1.2) the bound of the size of the Cauchy data u_0 may be taken as $||u_0||_2^2 < \pi/(2|\kappa|^{1/2})$ in Part (2) of Theorem 1 in the following cases: (a) $1 \le p_1 < p_2 < 5$. (b) $1 \le p_1 < 5 \le p_2$ with $\nu_2 \ge 0$. (c) $5 \le p_1 \le p_2$ with $\nu_1, \nu_2 \ge 0$.

Remark 3. The condition $(\lambda + \mu)(5\mu - 3\lambda) < 0$ holds for instance in the cases: $\lambda = 2\mu$; $\mu = 0$. See Examples 1 and 2. In particular, for (1.3) and (1.4) the global well-posedness in H^1 holds provided that the L^2 norm of the Cauchy data u_0 is less than $(2\pi/|\mu|)^{1/2}$ and $(2\pi/|\lambda|)^{1/2}$, respectively, the former of which has been known for (1.3) [19, 22].

Remark 4. The condition $(\lambda + \mu)(5\mu - 3\lambda) \ge 0$ holds for instance in the cases: $\lambda = 0$; $\lambda = \mu$; $\lambda = -\mu$. See Examples 3,4, and 5. In particular, as for the global well-posedness in H^1 for (1.6) we need not impose the smallness assumption on the data (compare with [51]).

Remark 5. When $\lambda = -\mu$, the energy (1.13) is exactly the same as that of the usual nonlinear Schrödinger equation, namely (1.1) with $\lambda = \mu = 0$. In particular, the equation

(1.7) has the energy identical to that of the free Schrödinger equation, though (1.7) is known as a non-integrable equation [2].

The method of proof of Theorem 1 is an extension of that used in [22]. We consider the system of nonlinear Schrödinger equations

$$i\partial_t\varphi + \partial^2\varphi = -i(\lambda-\mu)\varphi^2\bar{\psi} - ((\lambda-2\mu)\mu/4)|\varphi|^4\varphi + f(\varphi), \tag{1.14}$$

$$\begin{aligned} &i\partial_t\psi + \partial^2\psi \\ &= i(\lambda-\mu)\psi^2\bar{\varphi} - ((\lambda-2\mu)\mu/4)(3|\varphi|^4\psi + 2\varphi^3\bar{\varphi}\bar{\psi}) + \partial_\varphi f(\varphi)\psi + \partial_{\bar{\varphi}} f(\varphi)\bar{\psi}, \end{aligned} \tag{1.15}$$

with Cauchy data $(\varphi(t_0), \psi(t_0)) = (\varphi_0, \psi_0)$. The system of equations (1.14) and (1.15) is derived formally from (1.1) through the transformation

$$\varphi(t,x) = \exp(-i\frac{\lambda}{2}\int_{-\infty}^{x}|u(t,y)|^2dy)u(t,x), \tag{1.16}$$

$$\psi(t,x) = \exp(-i\frac{\lambda}{2}\int_{-\infty}^{x}|u(t,y)|^2dy)(\partial u(t,x) - i\frac{\mu}{2}|u|^2u(t,x)). \tag{1.17}$$

By definition new unknown functions φ and ψ are subject to the constraint

$$\psi = \partial\varphi - i((\lambda-\mu)/2)|\varphi|^2\varphi. \tag{1.18}$$

Conversely, for solutions (φ, ψ) satisfying (1.14), (1.15), and (1.18), a solution of (1.1) is given by

$$u(t,x) = \exp(i\frac{\lambda}{2}\int_{-\infty}^{x}|\varphi(t,y)|^2dy)\varphi(t,x). \tag{1.19}$$

In this sense the original equation (1.1) and the system of equations (1.14) and (1.15) with constraint (1.18) are gauge equivalent. From the point of view of solving the Cauchy problem, the system (1.14), (1.15) has the advantage of fitting into the contraction argument based on the Strichartz estimates without apparent loss of derivatives in the iteration scheme. We solve the Cauchy problem for (1.1) with data u_0 at initial time t_0 in the following way: First we define (φ_0, ψ_0) in terms of u_0 through the relations (1.16) and (1.17) at $t = t_0$. This provides the constraint (1.18) at $t = t_0$. With data (φ_0, ψ_0) at $t = t_0$ we solve (1.14) and (1.15) in the integral form as in (1.9) locally in time on the basis of the

Strichartz estimates. Then we prove that the constraint (1.18) is preserved under the time evolution generated by the system (1.14), (1.15). The proof uses the differential inequality on $||\psi - \partial\varphi + i((\lambda-\mu)/2)|\varphi|^2\varphi||_2^2$ derived from (1.14) and (1.15) with integration by parts. The constraint (1.18) enables to make the problem back to (1.1) to provide a local solution u by the relation (1.19).

In order to implement the scheme above under the minimal regularity assumption $u_0 \in H^1$, equivalently $(\varphi_0, \psi_0) \in H^1 \times L^2$, we need an approximation procedure with continuous dependence of solutions of (1.14), (1.15) on the data.

Concerning the globalization of local solutions to (1.1), we follow the standard method depending on a priori estimates derived from the conservation laws of the L^2 norm (1.11) and of the energy (1.12). A difficulty in obtaining H^1 control from the energy (1.13) arises from the second term of the energy since it has an indefinite sign. Here we absorb the second term by writing

$$E(u) = ||\partial G u||_2^2 + ((\lambda+\mu)(5\mu - 3\lambda)/48)||u||_6^6 + \int F(u)dx$$

with

$$G(t,x) = \exp(-i((\lambda+\mu)/4)\int_{-\infty}^{x} |u(t,y)|^2 dy).$$

This makes it natural to distinguish the cases according to the sign of $(\lambda+\mu)(5\mu-3\lambda)/48 - \nu_0$.

Related results on the Cauchy problem for the nonlinear Schrödinger equations with coupling of derivative type have been obtained in [3, 4, 5, 6, 16, 18, 28, 33, 43, 46, 51].

The second purpose in this paper is to study the long time behavior of solutions of the equation (1.1) from the point of view of construction of wave operators. The problem of the existence of wave operators for the nonlinear Schrödinger equations is formulated as follows. Let $U(t)$ be the free propagator. Let X be a Hilbert space continuously embedded in L^2 and invariant under the actions of two propagators $U_0(t)$ and $U(t)$. We consider the following problem: Given $u_+ \in X$, find $u_0 \in X$ such that $U_0(-t)U(t)u_0 \to u_+$ as $t \to +\infty$. If that is the case the map $\Omega_+ : u_+ \mapsto u_0$ is well defined in X and is called the wave operator for positive times. We call u_+ and u_0 asymptotic and interacting states, respectively. The wave operator Ω_+ corresponds the prescribed free evolution, which is described as a straight flow in X, with the total evolution asymptotic to it for large times. A similar problem can be considered for negative times, though we restrict our attention mainly to positive times for definiteness. As in the linear scattering theory, the existence of Ω_+ requires the integrability in time of a suitable norm of the interaction term. Since we suppose the solution behaving like free solutions, the required integrability is reasonably measured by the behavior of the nonlinear interaction caused by the free solutions. This implies that the problem of existence of wave operators turns difficult as the degree of nonlinearity decreases. In one space dimension cubic nonlinearity emerges as the borderline case, where there are no nontrivial solutions behaving like free solutions. The same situation happens in the Coulomb scattering where the corresponding Cook integral just fails to converge. In [10, 11, 21, 23, 42] we have developed a nonlinear analogue to the theory of long range potential scattering. In order to take the long range effect of the cubic nonlinearity into account, we make a phase modification in the free evolution to describe the right comparison dynamics. The modified wave operators are introduced as maps from asymptotic states to interacting states on the basis of the modified free evolution instead of $U_0(t)$, accordingly, a major step consists in finding proper phase functions. The theory covers the cases: (1) $\lambda = \mu = 0$, $\nu_1 = 3$, $\nu_2 = 5$ [42]. (2) $\lambda = 2\mu$, $f \equiv 0$ [21]. Every cubic nonlinearity, however, need not require a special treatment within the framework of long range scattering. Indeed, Y. Tsutsumi [51] has recently proved the existence of small amplitude solutions of (1.6) behaving asymptotically free without modification on phase. In [51] Y. Tsutsumi proposed the null gauge condition, which ensures small amplitude solutions with asymptotically free behavior. The null gauge condition for the nonlinear Schrödinger equations is reminiscent of the null condition for the wave equations of massless and massive case [7, 14, 24, 27, 34, 35] (see also [44, 45]). It should be emphasized that the null gauge condition with some consideration related to energy severely restricts the structure of nonlinearity, and in fact, the equation (1.6) is the only one that falls within the scope. Therefore, the results in [51] give a reason to believe that (1.6) is exceptional in a general setting for small amplitude solutions. In this paper we describe the long time behavior of solutions of (1.1) in the framework of the long range scattering and clarify the reason why the equation (1.6) is exceptional regarding the construction of modified wave operators.

We now introduce modified free evolutions for (1.1). For simplicity we assume $f(u) = \nu_1|u|^2u + \nu_2|u|^4u$. For any asymptotic state u_+ we define the phase function

$$S^{\pm}(t,\xi) = \pm(((\lambda-\mu)\xi - \nu_1)/2)|\hat{u}_+(\xi)|^2 \log|t| \tag{1.20}$$

where $\hat{u}_+$ is the Fourier transform of u_+ according to the normalization

$$\hat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) = (2\pi)^{-1/2}\int \exp(-ix\xi)\varphi(x)dx.$$

We introduce the modified free evolutions

$$v_1^{\pm}(t) = U_0(t)\exp(iS^{\pm}(t,-i\partial))u_+, \tag{1.21}$$

$$v_2^{\pm}(t) = U_0(t)M(-t)\exp(iS^{\pm}(t,-i\partial))u_+, \tag{1.22}$$

$$v_3^{\pm}(t) = \exp(iS^{\pm}(t,x/2t))U_0(t)u_+, \tag{1.23}$$

where $M(t) = \exp(ix^2/4t)\cdot$ and the phase factors in (1.21) and (1.22) are defined as Fourier multipliers. Let $H^{m,s}$ be the weighted Sobolev space defined for $m, s \in \mathbb{R}$ by

$$H^{m,s} = \{\varphi \in \mathcal{S}'; ||\varphi||_{m,s} \equiv ||(1+x^2)^{s/2}(1-\partial^2)^{m/2}\varphi||_2 < \infty\}$$

where $\mathcal{S}'$ is the space of temperate distributions. Let $U(t)$ be the nonlinear propagator for (1.1) given by Theorem 1, namely the map $U(t) : u(0) \mapsto u(t)$. The following theorem summarizes the basic existence and uniqueness result for (1.1) with asymptotic states given at infinity.

Theorem 2. *Let $f(u) = \nu_1|u|^2u + \nu_2|u|^4u$. Let $u_+ \in H^{4,0} \cap H^{0,4}$ with $||(1+\xi^2)\hat{u}_+||_\infty$ sufficiently small. Then there exists a unique $u_0 \in H^2$ such that for any θ with $1/4 < \theta < 1$ and any $j = 1,2,3$*

$$\sup_{t\geq 1} t^\theta ||U(t)u_0 - v_j^+(t)||_{H^{2,0}} < \infty. \tag{1.24}$$

Moreover, the solution $u \equiv U(\cdot)u_0$ of (1.1) satisfies $u \in L^q_{\mathrm{loc}}(\mathbb{R}; L^r)$ and

$$\sup_{t\geq 1} t^\theta ||u - v_j^+||_{L^q(t,\infty;W^{2,r})} < \infty \tag{1.25}$$

for any q, r, θ with $0 \leq 2/q = 1/2 - 1/r \leq 1/2$, $1/4 < \theta < 1$, and $j = 1,2,3$. A similar result holds for negative times.

Remark 6. By Theorem 2, the modified wave operators $\Omega_\pm : u_\pm \mapsto u_0$ is well-defined on the set

$$X_\varepsilon = \{\varphi \in H^{4,0} \cap H^{0,4}; \|(1+\xi^2)\hat{\varphi}_+\|_\infty < \varepsilon\}$$

with $\varepsilon > 0$ sufficiently small. The free propagator $U_0(t)$ leaves X_ε invariant and therefore Theorem 2 applies to any state on the flow $\{U_0(t)\varphi\}_{t=-\infty}^{\infty}$ for $\varphi \in X_\varepsilon$ with $\varepsilon > 0$ sufficiently small.

We state the intertwining property of the modified wave operators $\Omega_\pm$.

Theorem 3. *Let $\varepsilon > 0$ satisfy the smallness condition of Theorem 2. Then for any $\varphi \in X_\varepsilon$ and $t \in \mathbb{R}$*

$$U(t)\Omega_\pm\varphi = \Omega_\pm U_0(t)\varphi. \tag{1.26}$$

The results in Theorems 2 and 3 extend those of [21, 42]. As for the equation (1.6) the corresponding phase functions $S^\pm$ vanish and the modified wave operators $\Omega_\pm$ reduce to the usual wave operators without modification on the phase factor. Conversely, if $S^\pm \equiv 0$, then $\lambda = \mu$ and $\nu_1 = 0$, which is precisely the case of (1.6). This proves that the equation (1.6) is exceptional in the long time behavior of solutions in the framework of the long range scattering for (1.1).

This paper is organized as follows. Section 2 is devoted to the local Cauchy problem for (1.1). We first give a formal calculation of the gauge equivalence between the equation (1.1) and the system of equations (1.14) and (1.15) in order to justify the reduction of the problem to the system. We solve the system (1.14), (1.15) in the spaces $H^1 \times H^1$ and $H^2 \times H^2$ (Proposition 2.2) and prove the invariance of the constraint (1.18) under the time evolution (Proposition 2.3). This provides the H^1 control of φ in terms of the L^2 norm of ψ, by which we prove that the system (1.14), (1.15) makes sense in the space $H^1 \times L^2$ under the constraint (1.18) (Proposition 2.4). Combining these results we prove the local well-posedness of (1.1) in H^1 (Proposition 2.5). Section 3 is devoted to the global Cauchy problem for (1.1). We derive the conservation laws of the L^2 norm and of the energy (Proposition 3.1) and an a priori estimate of the H^1 norm of the local solution (Proposition 3.1). Combining these results we prove the global well-posedness of (1.1) in

H^1 (Proof of Theorem 1). Section 4 is devoted to the Cauchy problem at infinity for (1.1). We solve (1.1) around the modified free evolution v_2^+ defined by (1.22) and prove the existence of modified wave operators (Proof of Theorem 2) and the corresponding intertwining property (Proof of Theorem 3).

We conclude this section by giving some of the notation used in this paper. For any r with $1 \le r \le \infty$ we denote by r' the conjugate exponent defined by $1/r + 1/r' = 1$. For definiteness we restrict our attention to positive times and the time variable t is taken to be positive and might be omitted when no confusion could arise. For any interval I and any Banach space X we denote by $C(I;X)$ the space of strongly continuous functions from I to X and by $L^q(I;X)$ [resp. $L^q_{\rm loc}(I;X)$] the space of measurable functions u from I to X such that $\|u(\cdot)\|_X \in L^q(I;X)$ [resp. $\|u(\cdot)\|_X \in L^q_{\rm loc}(I;X)$]. The abbreviation such as $L^q_t L^r_x = L^q_t(I;L^r_x) = L^q(I;L^r(\mathbb{R}))$ will often be made when this cause no confusion. Different positive constants in the estimates below might be denoted by the same letter C and if necessary, by $C(*,\cdots,*)$ in order to indicate the dependence on the quantities appearing in parentheses.

2. Local Cauchy Problem

We start on formal calculations for the reduction of the equation (1.1) to the system of equations (1.14) and (1.15). For simplicity we set

$$L = i\partial_t + \partial^2.$$

Let u be a solution to (1.1) with sufficient smoothness in space-time and integrability in space. It suffices to assume that $u \in C^1(\mathbb{R};L^2)\cap C(\mathbb{R};H^2)$ to justify the calculations below in the space L^2. We introduce the function θ by

$$\theta(t,x) = \int_{-\infty}^{x} |u(t,y)|^2 dy. \tag{2.1}$$

For $\alpha \in \mathbb{R}$ we define

$$v_\alpha = e^{i\alpha\theta}u. \tag{2.2}$$

By definition, $|v_\alpha| = |u|$ and $e^{i\alpha\theta}\partial u = \partial v_\alpha - i\alpha|v_\alpha|^2 v_\alpha$. By (1.1), we have $\partial_t|u|^2 = \partial(2\,\mathrm{Im}\,u\partial\bar{u} + ((\lambda+\mu)/2)|u|^4)$ and therefore

$$L\theta = 2u\partial\bar{u} + i((\lambda+\mu)/2)|u|^4 = 2v_\alpha\partial\bar{v}_\alpha + i(2\alpha + (\lambda+\mu)/2)|v_\alpha|^4. \tag{2.3}$$

By (1.1), (2.2), (2.3), and (H2), we obtain

$$
\begin{aligned}
Lv_\alpha &= e^{i\alpha\theta}(Lu + i\alpha u L\theta + 2i\alpha|u|^2\partial u - \alpha^2|u|^4u) \\
&= i(\lambda + 2\alpha)|v_\alpha|^2\partial v_\alpha + i(\mu + 2\alpha)v_\alpha^2\partial\bar{v}_\alpha \\
&\quad + ((\lambda - 3\mu - 2\alpha)\alpha/2)|v_\alpha|^4 v_\alpha + f(v_\alpha).
\end{aligned} \tag{2.4}
$$

In particular, $v \equiv v_{-\mu/2}$ and $\varphi \equiv v_{-\lambda/2}$ solve

$$
Lv = i(\lambda - \mu)|v|^2\partial v - ((\lambda - 2\mu)\mu/4)|v|^4 v + f(v), \tag{2.5}
$$

$$
L\varphi = -i(\lambda - \mu)\varphi^2\partial\bar{\varphi} - ((2\lambda - 3\mu)\lambda/4)|\varphi|^4\varphi + f(\varphi), \tag{2.6}
$$

respectively. Moreover, $\psi \equiv e^{-i((\lambda-\mu)/2)\theta}\partial v = e^{-i((\lambda-\mu)/2)\theta}\partial(e^{-i(\mu/2)\theta}u)$ satisfies

$$
\begin{aligned}
L\varphi &= e^{i((\lambda-\mu)/2)\theta}(L\partial v + i((\lambda - \mu)/2)\partial v \cdot L\theta + i(\lambda - \mu)|v|^2\partial^2 v - ((\lambda - \mu)/2)^2|v|^4\partial v) \\
&= e^{-i((\lambda-\mu)/2)\theta}(i(\lambda - \mu)\bar{v}(\partial v)^2 - ((\lambda - 2\mu)\mu/4)\partial(|v|^4 v) + \partial(f(v))),
\end{aligned} \tag{2.7}
$$

where we have used (2.5) and (2.3) with $\alpha = -\mu/2$. By definition,

$$
\partial\varphi = \psi - i((\lambda - \mu)/2)|\varphi|^2\varphi. \tag{2.8}
$$

By (2.8) and the definitions of φ and ψ, (2.6) and (2.7) lead to

$$
L\varphi = -i(\lambda - \mu)\varphi^2\bar{\psi} - ((\lambda - 2\mu)\mu/4)|\varphi|^4\varphi + f(\varphi) \equiv F_1(\varphi, \psi), \tag{2.9}
$$

$$
\begin{aligned}
L\psi &= i(\lambda - \mu)\psi^2\bar{\varphi} - ((\lambda - 2\mu)\mu/4)(3|\varphi|^4\psi + 2\varphi^3\bar{\varphi}\bar{\psi}) \\
&\quad + \partial_\varphi f(\varphi)\psi + \partial_{\bar{\varphi}} f(\varphi)\bar{\psi} \equiv F_2(\varphi, \psi),
\end{aligned} \tag{2.10}
$$

as was to be shown.

Conversely, let (φ, ψ) satisfy the system of equations (2.9) and (2.10) with constraint (2.8). Then we define

$$
u = e^{i(\lambda/2)\Theta}\varphi
$$

where

$$
\Theta(t, x) = \int_{-\infty}^{x} |\varphi(t, y)|^2 dy.
$$

By (2.8) and (2.9), we have (2.6), so that $\partial_t|\varphi|^2 = \partial(2\,\mathrm{Im}\,\varphi\partial\bar{\varphi} - ((\lambda-\mu)/2)|\varphi|^4)$. This yields

$$\begin{aligned} Lu &= e^{i(\lambda/2)\Theta}(L\varphi + i(\lambda/2)\varphi L\Theta + i\lambda|\varphi|^2\partial\varphi - (\lambda^2/4)|\varphi|^4\varphi) \\ &= e^{i(\lambda/2)\Theta}(i\lambda|\varphi|^2\partial\varphi + i\mu\varphi^2\partial\bar{\varphi} - ((\lambda-\mu)\lambda/2)|\varphi|^4\varphi + f(\varphi)) \\ &= i\lambda|u|^2\partial u + i\mu u^2\partial\bar{u} + f(u), \end{aligned}$$

where we have used the relation $\partial\varphi = e^{-i(\lambda/2)\Theta}\partial u - i(\lambda/2)|u|^2u$ and (H2).

On the basis of the formal calculations above, we solve the local Cauchy problem for (1.1) by making a reduction to the system of equations (2.9) and (2.10). We make an essential use of the following well-known estimates on the free propagator $U_0(t)$ [1, 9, 13, 29, 30, 53].

Lemma 2.1. *(1) Let q and r satisfy $0 \le 2/q = 1/2 - 1/r \le 1/2$ and let $\varphi \in L^2$. Then $U_0(\cdot)\varphi$ satisfies the estimate*

$$||U_0(\cdot)\varphi||_{L^q(\mathbb{R};L^r)} \le C||\varphi||_2.$$

(2) Let (q_1, r_1) and (q_2, r_2) satisfy $0 \le 2/q_j = 1/2 - 1/r_j \le 1/2$, $j = 1, 2$. Then for any interval $I \subset \mathbb{R}$, possibly unbounded, and for any $s \in \bar{I}$, the closure of I in $\mathbb{R} \cup \{\pm\infty\}$ with the natural topology, the operator F_s defined by

$$(F_s v)(t) = \int_s^t U_0(t-\tau)v(\tau)d\tau$$

satisfies the estimate

$$||F_s v||_{L^{q_1}(I;L^{r_1})} \le C||v||_{L^{q_2'}(I;L^{r_2'})}$$

with constant C independent of I and s.

By the standard method based on Lemma 2.1, we have the following proposition (see, for instance, [1, 9, 12, 29, 30]).

Proposition 2.2. *Let f satisfy (H1). Then for any $\rho > 0$ there exist two positive constants $T_j(\rho)$, $j = 1, 2$, depending only on ρ, λ, μ, f with the following properties. For any $t_0 \in \mathbb{R}$ and for any $(\varphi_0, \psi_0) \in H^j \times H^j$ with $||\varphi_0||_{H^j} + ||\psi_0||_{H^j} \le \rho$ the system of equations (2.9) and (2.10) has a unique pair of solutions (φ, ψ) on the time interval $I_j \equiv [t_0 - T_j(\rho), t_0 + T_j(\rho)]$ such that $(\varphi(t_0), \psi(t_0)) = (\varphi_0, \psi_0)$ and φ, $\psi \in C(I_j; H^j) \cap L^4(I_j; W^{j,\infty})$.*

Remark. In general, $0 < T_2(\rho) \leq T_1(\rho)$. If we impose the smoothness condition that $f \in C^2(\mathbb{C};\mathbb{C})$, then by the regularity result of Kato [29, 30] we obtain $T_1(\rho) = T_2(\rho)$. In this paper, however, we never use the C^2 assumption on the nonlinearity.

Proposition 2.3. *Le f satisfy (H1) and (H2). Let $\varphi_0 \in H^3$ and let*

$$\psi_0 = \partial\varphi_0 + i((\lambda-\mu)/2)|\varphi_0|^2\varphi_0.$$

Let (φ,ψ) be a pair of H^2 solutions of (2.9) and (2.10) given by Proposition 2.2. Then

$$\psi = \partial\varphi + i((\lambda-\mu)/2)|\varphi|^2\varphi$$

on the time interval I_2 for local existence of H^2 solution gien by Proposition 2.2.

Proof. The proof is very similar to that of [22, Proposition 2.4]. Let $\tilde{\psi} = \partial\varphi + i((\lambda - \mu)/2)|\varphi|^2\varphi$. By (2.9) and (2.10) we compute $L(\tilde{\psi}-\psi)$, multiply the resulting equation by $\bar{\tilde{\psi}}-\bar{\psi}$, take the imaginary part, and integrate over the space with integration by parts. This gives a differential inequality for $||\tilde{\psi}-\psi||_2^2$, which is integrated by Gronwall's inequality to give $||\tilde{\psi}-\psi||_2^2 \equiv 0$ since it vanishes at $t = t_0$. Details are omitted. QED

Proposition 2.4. *Let f satisfy (H1) and (H2). Then for any $\rho > 0$ there exists a positive constant $T_0(\rho)$ depending only on ρ, λ, μ, f with the following properties. For any $t_0 \in \mathbb{R}$ and any $(\varphi_0,\psi_0) \in H^3 \times H^2$ with $\psi_0 = \partial\varphi_0 + i((\lambda-\mu)/2)|\varphi_0|^2\varphi_0$ and $||\varphi_0||_2 + ||\psi_0||_2 \leq \rho$, the H^2 solutions (φ,ψ) of (2.9) and (2.10) given by Proposition 2.2 satisfy the estimate*

$$|||\varphi||| + |||\partial\varphi||| + |||\psi||| \leq C(\rho,T)$$

for any T with $0 < T \leq T_0(\rho)$, where

$$|||v||| = ||v||_{L^\infty(t_0-T,t_0+T;L^2)} + ||v||_{L^4(t_0-T,t_0+T;L^\infty)}.$$

Moreover, let $(\tilde{\varphi}_0,\tilde{\psi}_0) \in H^3 \times H^2$ with $\tilde{\psi}_0 = \partial\tilde{\varphi}_0 + i((\lambda-\mu)/2)|\tilde{\varphi}_0|^2\tilde{\varphi}_0$ and $||\tilde{\varphi}_0||_2 + ||\tilde{\psi}_0||_2 \leq \rho$ and let $(\tilde{\varphi},\tilde{\psi})$ be the corresponding solutions of (2.9) and (2.10) given by Proposition 2.2. Then

$$|||\varphi-\tilde{\varphi}||| + |||\partial\varphi - \partial\tilde{\varphi}||| + |||\psi-\tilde{\psi}||| \leq C(\rho,T)(||\varphi_0-\tilde{\varphi}_0||_2 + ||\psi_0 - \tilde{\psi}_0||_2)$$

for any T with $0 < T \le T_0(\rho)$.

Proof. By Proposition 2.3, we have

$$\psi = G^{-1}\partial G\varphi$$

where

$$G(t,x) = \exp\left(i\frac{\lambda-\mu}{2}\int_{-\infty}^{x} |\varphi(t,y)|^2 dy\right).$$

By the Gagliardo-Niremberg inequality

$$||\varphi||_6^3 = ||G\varphi||_6^3 \le C||G\varphi||_2^2||\partial G\varphi||_2 = C||\varphi||_2^2||\psi||_2$$

so that

$$||\partial\varphi||_2 = ||\psi - i((\lambda-\mu)/2)|\varphi|^2\varphi||_2 \le ||\psi||_2 + C||\varphi||_2^2||\psi||_2. \tag{2.11}$$

Similarly, by introducing $\tilde{G}$ by

$$\tilde{G}(t,x) = \exp\left(i\frac{\lambda-\mu}{2}\int_{-\infty}^{x} |\tilde{\varphi}(t,y)|^2 dy\right),$$

we have

$$\begin{aligned}|||\varphi|^2\varphi - |\tilde{\varphi}|^2\tilde{\varphi}||_2 &\le C(||G\varphi||_\infty^2 + ||\tilde{G}\tilde{\varphi}||_\infty^2)||\varphi - \tilde{\varphi}||_2 \\ &\le C(||\varphi||_2||\psi||_2 + ||\tilde{\varphi}||_2||\tilde{\psi}||_2)||\varphi - \tilde{\varphi}||_2.\end{aligned}$$

This gives

$$||\partial\varphi - \partial\tilde{\varphi}||_2 \le ||\psi - \tilde{\psi}||_2 + C(||\varphi||_2||\psi||_2 + ||\tilde{\varphi}||_2||\tilde{\psi}||_2)||\varphi - \tilde{\varphi}||_2. \tag{2.12}$$

By (2.3) and (2.10),

$$\begin{aligned}\varphi(t) &= U_0(t-t_0)\varphi_0 - i\int_{t_0}^{t} U_0(t-\tau)F_1(\varphi,\psi)(\tau)d\tau, \\ \psi(t) &= U_0(t-t_0)\psi_0 - i\int_{t_0}^{t} U_0(t-\tau)F_2(\varphi,\psi)(\tau)d\tau.\end{aligned}$$

By Lemma 2.1, the Hölder and Gagliardo-Nirenberg inequalities, and (2.11), we obtain with $p_0 \equiv 5$

$$\begin{aligned}
|||\varphi||| &\le C||\varphi_0||_2 + C||\varphi^2\psi||_{L_t^1L_x^2} + C||\varphi^5||_{L_t^1L_x^2} + C||f(\varphi)||_{L_t^1L_x^2} \\
&\le C||\varphi_0||_2 + CT||\varphi||^2_{L_t^\infty L_x^\infty}||\psi||_{L_t^\infty L_x^2} + CT\sum_{j=0}^{2}||\varphi||^{p_j-1}_{L_t^\infty L_x^\infty}||\varphi||_{L_t^\infty L_x^2} \\
&\le C||\varphi_0||_2 + CT(1+||\varphi||^2_{L_t^\infty L_x^2})||\varphi||_{L_t^\infty L_x^2}||\psi||^2_{L_t^\infty L_x^2} \\
&\qquad + CT\sum_{j=0}^{2}(1+||\varphi||^{p_j-1}_{L_t^\infty L_x^2})||\varphi||^{(p_j+1)/2}_{L_t^\infty L_x^2}||\psi||^{(p_j-1)/2}_{L_t^\infty L_x^2}, \\
|||\psi||| &\le C||\psi_0||_2 + C||\psi^2\varphi||_{L_t^{4/3}L_x^1} + C||\varphi^4\psi||_{L_t^1L_x^2} + C||f'(\varphi)\psi||_{L_t^1L_x^2} \\
&\le C||\psi_0||_2 + CT^{1/2}||\psi||^2_{L_t^\infty L_x^2}||\varphi||_{L_t^4L_x^\infty} + CT\sum_{j=0}^{2}||\varphi||^{p_j-1}_{L_t^\infty L_x^\infty}||\psi||_{L_t^\infty L_x^2} \\
&\le C||\psi_0||_2 + CT^{1/2}||\psi||^2_{L_t^\infty L_x^2}||\varphi||_{L_t^4L_x^\infty} \\
&\qquad + CT\sum_{j=0}^{2}(1+||\varphi||^{p_j-1}_{L_t^\infty L_x^2})||\varphi||^{(p_j-1)/2}_{L_t^\infty L_x^2}||\psi||^{(p_j+1)/2}_{L_t^\infty L_x}.
\end{aligned}$$

Therefore, $m(T) \equiv |||\varphi||| + |||\psi|||$ satisfies the inequality

$$m(T) \le C(||\varphi_0||_2 + ||\psi_0||_2) + C(T^{1/2}+T)m(T)^3 + CT\sum_{j=0}^{2}(m(T)^{p_j} + m(T)^{2p_j-1}).$$

Taking $T > 0$ sufficiently small, we have

$$m(T) \le C(\rho, T). \tag{2.13}$$

A similar estimate and (2.13) imply, for $T > 0$ sufficiently small,

$$|||\varphi - \tilde{\varphi}||| + |||\psi - \tilde{\psi}||| \ge C(\rho,T)(||\varphi_0 - \tilde{\varphi}_0||_2 + ||\psi_0 - \tilde{\psi}_0||_2). \tag{2.14}$$

By (2.11), (2.12), (2.13), (2.14), we have

$$\begin{aligned}
&||\partial\varphi||_{L_t^\infty L_x^2} \le m(T) + m(T)^3 \le C(\rho,T), \\
&||\partial\varphi - \partial\tilde{\varphi}||_{L_t^\infty L_x^2} \le |||\psi - \tilde{\psi}||| + C(\rho,T)|||\varphi - \tilde{\varphi}||| \le C(\rho,T)(||\varphi_0 - \tilde{\varphi}_0||_2 + ||\psi_0 - \tilde{\psi}_0||_2).
\end{aligned}$$

By Proposition 2.3 and the Galiardo-Nirenberg inequality,

$$\begin{aligned}||\partial\varphi||_{L_t^4 L_x^\infty} &\le ||\psi||_{L_t^4 L_x^\infty} + C||\varphi^3||_{L_t^4 L_x^\infty}\\ &\le |||\psi||| + C||\, ||G\varphi||_2^{3/2}||\partial G\varphi||_2^{3/2}||_{L_t^4}\\ &\le |||\psi||| + CT^{1/4}||\varphi||^{3/2}_{L_t^\infty L_x^2}||\psi||^{3/2}_{L_t^\infty L_x^2} \le C(\rho, T).\end{aligned}$$

Similarly,

$$\begin{aligned}||\partial\varphi - \partial\tilde{\varphi}||_{L_t^4 L_x^\infty} &\le ||\psi - \tilde{\psi}||_{L_t^4 L_x^\infty} + C(||\varphi||^2_{L_t^\infty L_x^\infty} + ||\tilde{\varphi}||^2_{L_t^\infty L_x^\infty})||\varphi - \tilde{\varphi}||_{L_t^4 L_x^\infty}\\ &\le |||\psi - \tilde{\psi}||| + C(||\varphi||_{L_t^\infty L_x^2}||\psi||_{L_t^\infty L_x^2} + ||\tilde{\varphi}||_{L_t^\infty L_x^2}||\tilde{\psi}||_{L_t^\infty L_x^2})|||\varphi - \tilde{\varphi}|||\\ &\le C(\rho, T)(||\psi_0 - \tilde{\psi}_0||_2 + ||\varphi_0 - \tilde{\varphi}_0||_2).\end{aligned}$$

Collecting these estimates proves the proposition. QED

Proposition 2.5. *Let f satisfy (H1) and (H2). For any $\rho > 0$ there exists a positive constant $T(\rho)$ depending only on ρ, λ, μ, f with the following properties. For any $t_0 \in \mathbb{R}$ and any $u_0 \in H^1$ with $||u_0||_{H^1} \le \rho$ the equation (1.9) has a unique solution u on the interval $I = [t_0 - T(\rho), t_0 + T(\rho)]$ such that $u \in C(I; H^1) \cap L^4(I; W^{1,\infty})$. Moreover, for any T with $0 < T < T(\rho)$ the map $u_0 \mapsto u$ is locally Lipschitz from H^1 to $L^\infty(I'; H^1) \cap L^4(I'; W^{1,\infty})$ where $I' = [t_0 - T, t_0 + T]$.*

Proof. For $u_0 \in H^1$, we define $(\varphi_0, \psi_0) \in H^1 \times L^2$ by

$$\begin{aligned}\varphi_0(x) &= \exp(-i\frac{\lambda}{2}\int_{-\infty}^{x} |u_0(y)|^2 dy) u_0(x),\\ \psi_0 &= \partial\varphi_0 + i((\lambda - \mu)/2)|\varphi_0|^2\varphi_0.\end{aligned}$$

Let $\{\varphi_0^{(n)}\}$ be a sequence in H^3 such that $\varphi_0^{(n)} \to \varphi_0$ in H^1 as $n \to \infty$ and let $\psi_0^{(n)} = \partial\varphi_0^{(n)} + i((\lambda - \mu)/2)|\varphi_0^{(n)}|^2\varphi_0^{(n)}$. Let $(\varphi^{(n)}, \psi^{(n)})$ be the solutions to (2.9) and (2.10) with $(\varphi^{(n)}(t_0), \psi^{(n)}(t_0)) = (\varphi_0^{(n)}, \psi_0^{(n)})$, given by Proposition 2.2. By Proposition 2.4, there exist positive constants T and C independent of m and n such that

$$|||\varphi^{(n)}||| + |||\partial\varphi^{(n)}||| + |||\psi^{(n)}||| \le C, \tag{2.15}$$

$$|||\varphi^{(m)} - \varphi^{(n)}||| + |||\partial\varphi^{(m)} - \partial\varphi^{(n)}||| + |||\psi^{(m)} - \psi^{(n)}||| \le C||\varphi_0^{(m)} - \varphi_0^{(n)}||_{H^1}. \tag{2.16}$$

We define $u^{(n)}$ and $u_0^{(n)}$ by

$$u^{(n)}(t,x) = \exp(i\frac{\lambda}{2}\int_{-\infty}^{x} |\varphi^{(n)}(t,y)|^2 dy)\varphi^{(n)}(t,x), \tag{2.17}$$

$$u_0^{(n)}(x) = \exp(i\frac{\lambda}{2}\int_{-\infty}^{x} |\varphi_0^{(n)}(y)|^2 dy)\varphi_0^{(n)}(x). \tag{2.18}$$

Since $\psi^{(n)} = \partial\varphi^{(n)} + i((\lambda-\mu)/2)|\varphi^{(n)}|^2\varphi^{(n)}$ by Proposition 2.3, we obtain

$$\begin{aligned} u^{(n)}(t) &= U_0(t-t_0)u_0^{(n)} \\ &\quad - i\int_{t_0}^{t} U_0(t-t_0)(i\lambda|u^{(n)}|^2\partial u^{(n)} + i\mu(u^{(n)})^2\partial\bar{u}^{(n)} + f(u^{(n)}))(\tau)d\tau. \end{aligned} \tag{2.19}$$

By (2.15), (2.16), (2.17), (2.18), we have

$$|||u^{(n)}||| + |||\partial u^{(n)}||| \le C, \tag{2.20}$$

$$|||u^{(m)} - u^{(n)}||| + |||\partial u^{(m)} - \partial u^{(n)}||| \le C||\varphi_0^{(m)} - \varphi_0^{(n)}||_{H^1}. \tag{2.21}$$

By (2.20) and (2.21) there exists $u \in C([t_0-T, t_0+T]; H^1) \cap L^4(t_0-T, t_0+T; W^{1,\infty})$ with $T > 0$ independent of n such that

$$|||u^{(n)} - u||| + |||\partial u^{(n)} - \partial u||| \to 0$$

as $n \to \infty$. Letting $n \to \infty$ in (2.19), we have

$$u(t) = U_0(t-t_0)u_0 - i\int_{t_0}^{t} U_0(t-\tau)(i\lambda|u|^2\partial u + i\mu u^2\partial\bar{u} + f(u))(\tau)d\tau. \tag{2.22}$$

Uniqueness of solutions of (1.9) follows from that of the system of equations (2.9) and (2.10). Continuous dependence of u on u_0 follows in the same way as above. QED

3. Global Cauchy Problem

In this section we prove the existence of global solutions of the equation (1.1) by making an essential use of the conservation laws of the L^2 norm and of the energy. We first derive those conservation laws.

Proposition 3.1. *Let f satisfy (H1) and (H2). Let $u_0 \in H^1$, let $t_0 \in \mathbb{R}$, and let u be the solution of (1.1) with $u(t_0) = u_0$, given by Proposition 2.5. Then*

$$||u(t)||_2 = ||u_0||_2, \tag{3.1}$$

$$E(u(t)) = E(u_0) \tag{3.2}$$

on the time interval of local existence, where

$$E(u) = ||\partial u||_2^2 + ((\lambda+\mu)/2)\,\mathrm{Im}\langle |u|^2 u, \partial u\rangle + ((\lambda+\mu)\mu/6)||u||_6^6 + \int F(u)dx.$$

Proof. In view of the continuous dependence in the argument in the proof of Proposition 2.5, it suffices to prove (3.1) and (3.2) by a formal calculation. To simplify the exposition we first prove (3.1) and (3.2) for solutions of (1.1) with $\mu = 0$ and then we recover the general case by a gauge transformation. To be more specific, we prove the identities

$$\frac{d}{dt}||v||_2^2 = 0, \tag{3.3}$$

$$\frac{d}{dt}(||\partial v||_2^2 + (\lambda/2)\,\mathrm{Im}\langle |v|^2 v, \partial v\rangle + \int H(v)dx) = 0 \tag{3.4}$$

for v satisfying the equation

$$i\partial_t v + \partial^2 v = i\lambda |v|^2 \partial v + h(v) \tag{3.5}$$

where $h(v) = \partial H/\partial \bar{v}$ satisfying (H1) and (H2), and then we prove the proposition by replacing λ, h, H by $\lambda - \mu$, $((\lambda - 2\mu)\mu/4)|v|^4 v + f(v)$, $-((\lambda-\mu)\mu/12)|v|^6 + H(v)$ and by using the gauge transformation

$$v(t,x) = \exp(-i\frac{\mu}{2}\int_{-\infty}^{x} |u(t,y)|^2 dy) u(t,x).$$

By (3.5) and gauge invariance of $h(v)$, we have

$$\partial_t |v|^2 = \partial(2\,\mathrm{Im}\, v\partial\bar{v} + (\lambda/2)|v|^4), \tag{3.6}$$

which implies (3.1). For (3.4) we compute

$$\begin{aligned}
&\partial_t(|\partial v|^2 + H(v)) \\
&\quad = 2\,\mathrm{Re}\,\partial_t\partial\bar{v}\partial v + (\partial H/\partial v)\partial_t v + (\partial H/\partial\bar{v})\partial_t\bar{v} \\
&\quad = \partial(2\,\mathrm{Re}\,\partial_t\bar{v}\partial v) - 2\,\mathrm{Re}\,\partial_t\bar{v}\partial^2 v + \overline{h(v)}\partial_t v + h(v)\partial_t\bar{v} \\
&\quad = \partial(2\,\mathrm{Re}\,\partial_t\bar{v}\partial v) + 2\,\mathrm{Re}\,\partial_t\bar{v}(i\partial_t v - i\lambda|v|^2\partial v - h(v)) - 2\,\mathrm{Re}\, h(v)\partial_t\bar{v} \\
&\quad = \partial(2\,\mathrm{Re}\,\partial_t\bar{v}\partial v) + 2\lambda|v|^2\,\mathrm{Im}\,\partial_t\bar{v}\partial v,
\end{aligned} \tag{3.7}$$

where we have used (3.5) and gauge invariance of $h(v)$ again. For the second term on the RHS of the last equality of (3.7), we compute

$$\begin{aligned}
&|v|^2 \operatorname{Im} \partial_t \bar{v} \partial v \\
&\quad = \partial_t(|v|^2 \operatorname{Im} \bar{v}\partial v) - (\partial_t |v|^2) \operatorname{Im} \bar{v}\partial v - |v|^2 \operatorname{Im} \bar{v}\partial_t \partial v \\
&\quad = \partial_t(|v|^2 \operatorname{Im} \bar{v}\partial v) - (\partial(2 \operatorname{Im} v\partial\bar{v} + (\lambda/2)|v|^4)) \operatorname{Im} \bar{v}\partial v \\
&\qquad - \partial(|v|^2 \operatorname{Im} \bar{v}\partial_t v) + (\partial |v|^2) \operatorname{Im} \bar{v}\partial_t v + |v|^2 \operatorname{Im} \partial\bar{v}\partial_t v \\
&\quad = \partial_t(|v|^2 \operatorname{Im} \bar{v}\partial v) + \partial((\operatorname{Im} \bar{v}\partial v)^2 - |v|^2 \operatorname{Im} \bar{v}\partial_t v) - (\lambda/2)(\partial |v|^4) \operatorname{Im} \bar{v}\partial v \\
&\qquad + (\partial |v|^2) \operatorname{Im} \bar{v}(i\partial^2 v + \lambda |v|^2 \partial v - ih(v)) - |v|^2 \operatorname{Im} \partial_t \bar{v}\partial v \\
&\quad = \partial_t(|v|^2 \operatorname{Im} \bar{v}\partial v) + \partial((\operatorname{Im} \bar{v}\partial v)^2 - |v|^2 \operatorname{Im} \bar{v}\partial_t v) \\
&\qquad + (\partial |v|^2) \operatorname{Re} \bar{v}\partial^2 v - (\partial |v|^2) \operatorname{Re} \bar{v} h(v) - |v|^2 \operatorname{Im} \partial_t \bar{v}\partial v,
\end{aligned} \tag{3.8}$$

where we have used (3.5), (3.6), and gauge invariance of $h(v)$. For the RHS of the last equality of (3.8), we compute

$$\begin{aligned}
&(\partial |v|^2) \operatorname{Re} \bar{v}\partial^2 v \\
&\quad = \partial((1/4)(\partial |v|^2)^2 - |v|^2 |\partial v|^2 + 2|v|^2 \operatorname{Re} \partial\bar{v}\partial^2 v \\
&\quad = -\partial((\operatorname{Im} \bar{v}\partial v)^2) + 2|v|^2 \operatorname{Re} \partial\bar{v}(-i\partial_t v + i\lambda |v|^2 \partial v + h(v)) \\
&\quad = -\partial((\operatorname{Im} \bar{v}\partial v)^2) + 2|v|^2 \operatorname{Im} \partial\bar{v}\partial_t v + 2|v|^2 \operatorname{Re} \partial\bar{v} h(v) \\
&\quad = -\partial((\operatorname{Im} \bar{v}\partial v)^2) - 2|v|^2 \operatorname{Im} \partial_t \bar{v}\partial v + (\partial |v|^2) \bar{v} h(v),
\end{aligned} \tag{3.9}$$

where we have used the identities $\operatorname{Re} \bar{v}\partial^2 v = (1/2)\partial^2 |v|^2 - |\partial v|^2$, $(\operatorname{Im} \bar{v}\partial v)^2 + (1/4)(\partial |v|^2)^2 = |v|^2 |\partial v|^2$, $(\partial |v|^2)\bar{v}h = 2|v|^2 \operatorname{Re} \partial\bar{v}h$, and (3.5). Combining (3.8) and (3.9), we have

$$|v|^2 \operatorname{Im} \partial_t \bar{v}\partial v = (1/4)\partial_t(|v|^2 \operatorname{Im} \bar{v}\partial v) - (1/4)\partial(|v|^2 \operatorname{Im} \bar{v}\partial_t v). \tag{3.10}$$

By (3.7) and (3.10),

$$\partial_t(|\partial v|^2 + (\lambda/2)|v|^2 \operatorname{Im} v\partial\bar{v} + H(v)) = \partial(2 \operatorname{Re} \partial_t \bar{v}\partial v - (\lambda/2)|v|^2 \operatorname{Im} \bar{v}\partial_t v),$$

which implies (3.4), as required. QED

We now prove under (H2) that the conservation laws of the L^2 norm and of the energy provide an a priori estimate of the H^1 norm of the local solution constructed in the preceding section.

Proposition 3.2. *Let f satisfy (H1), (H2), and (H3). Let $u_0 \in H^1$, let $t_0 \in \mathbb{R}$, and let u be the solution of (1.1) with $u(t_0) = u_0$, given by Proposition 2.5. Let $\kappa = (\lambda+\mu)(5\mu-3\lambda)/48$.*
(1) If $0 \le \nu_0 \le \kappa$, then on the time interval of local existence u satisfies the estimate

$$||\partial u||_2^2 \le C(1 + ||u_0||_2^4)(E(u_0) + N(||u_0||_2)), \tag{3.11}$$

where C is independent of t, t_0, u.
(2) If $\kappa < \nu_0$ and $||u_0||_2^2 < \pi/(2(\nu_0 - \kappa)^{1/2})$, then on the time interval of local existence u satisfies the estimate

$$||\partial u||_2^2 \le C(1 - (4(\nu_0 - \kappa)/\pi^2)||u_0||_2^4)^{-1}(1 + ||u_0||_2^4)(E(u_0) + N(||u_0||_2)). \tag{3.12}$$

where C is independent of t, t_0, u.

Proof. We define G by

$$G(t,x) = \exp(-i\frac{\lambda+\mu}{4}\int_{-\infty}^{x} |u(t,y)|^2 dy).$$

Then the energy $E(u)$ is written as

$$E(u) = ||\partial G u||_2^2 + \kappa||u||_6^6 + \int F(u)dx.$$

By Proposition 3.1 and (H3),

$$E(u_0) + N(||u_0||_2) \ge ||\partial G u||_2^2 + (\kappa - \nu_0)||u||_6^6. \tag{3.13}$$

By the Gagliardo-Nirenberg inequality with sharp constant,

$$||u||_6^6 = ||Gu||_6^6 \le (4/\pi^2)||Gu||_2^4||\partial G u||_2^2 = (4/\pi^2)||u_0||_2^4||\partial G u||_2^2. \tag{3.14}$$

Moreover,

$$\begin{aligned} ||\partial u||_2 = ||\partial G^{-1} G u||_2 \le ||(\partial G^{-1})Gu||_2 + ||\partial G u||_2 &\le C||Gu||_6^3 + ||\partial G u||_2 \\ &\le C(1 + ||u_0||_2^2)||\partial G u||_2^2. \end{aligned} \tag{3.15}$$

Collecting (3.13), (3.14), (3.15) yields the proposition. QED

Proof of Theorem 1. Since $E(u_0)$ is estimated in terms of the H^1 norm of u_0, the theorem follows from Proposition 2.5, 3.1, 3.2 and the standard continuation argument. QED

4. Modified Wave Operators

In this section we prove the existence of modified wave operators for the equation (1.1) with $f(u) = \nu_1|u|^2u+\nu_2|u|^4u$. We first look for the phase function which provides sufficient approximation for the long range effect of derivative type nonlinearities. We set $v = v_2$ as in (1.22), namely

$$v(t) = U_0(t)M(-t)\exp(iS(t,-i\partial))u_+ \tag{4.1}$$

where $M(t) = \exp(ix^2/4t)\cdot$ and subscripts have been omitted for simplicity except u_+. The modified free evolution v is written as

$$\begin{aligned} v(t) &= \exp(iS(t,x/2t))M(t)D(t)\hat{u}_+ \\ &= \exp(iS(t,x/2t))U_0(t)M(-t)u_+ \end{aligned} \tag{4.2}$$

where $D(t)$ is a modified dilation operator defined by

$$(D(t)\varphi)(x) = (2it)^{-1/2}\varphi(x/2t),$$

since the free propagator $U_0(t)$ has the factorization

$$U_0(t) = M(t)D(t)\mathcal{F}M(t). \tag{4.3}$$

By a straightforward calculation,

$$\begin{aligned} (i\partial_t + \partial^2)U_0M^{-1}\mathcal{F}^{-1} = U_0 i\partial_t M^{-1}\mathcal{F} &= U_0M^{-1}(i\partial_t - x^2/(4t^2))\mathcal{F}^{-1} \\ &= U_0M^{-1}\mathcal{F}^{-1}(i\partial_t + (1/4t^2)\partial^2), \end{aligned}$$

so that

$$\begin{aligned} (i\partial_t + \partial^2)v &= U_0M^{-1}\mathcal{F}^{-1}(i\partial_t + (1/4t^2)\partial^2)e^{iS}\hat{u}_+ \\ &= U_0M^{-1}\mathcal{F}^{-1}(-(\partial_t S) + (1/4t^2)\partial^2)e^{iS}\hat{u}_+ \\ &= -MD(\partial_t S)e^{iS}\hat{u}_+ + MD(1/4t^2)\partial^2(e^{iS}\hat{u}_+). \end{aligned} \tag{4.4}$$

Another simple calculation gives

$$\begin{aligned} \partial v &= U_0M^{-1}\mathcal{F}^{-1}(i\xi + (1/2t)\partial)e^{iS}\hat{u}_+ \\ &= MDi\xi e^{iS}\hat{u}_+ + MD(1/2t)e^{iS}(\partial\hat{u}_+ + i(\partial S)\hat{u}_+) \end{aligned}$$

and

$$
\begin{aligned}
& i\lambda|v|^2\partial v + i\mu v^2\partial\bar{v} + f(v) \\
& \quad = MD(-(\lambda/2t)\xi|\hat{u}_+|^2 e^{iS}\hat{u}_+) + MD(i\lambda/4t^2)|\hat{u}_+|^2 e^{iS}(\partial\hat{u}_+ + i(\partial S)\hat{u}_+) \\
& \quad + MD((\mu/2t)\xi|\hat{u}_+|^2 e^{iS}\hat{u}_+) + MD(i\mu/4t^2)(\hat{u}_+)^2 e^{-iS}(\overline{\partial\hat{u}_+} - i(\partial S)\overline{\hat{u}_+}) \\
& \quad + MD(\nu_1/2t)|\hat{u}_+|^2 e^{iS}\hat{u}_+ + MD(\nu_2/4t^2)|\hat{u}_+|^4 e^{iS}\hat{u}_+.
\end{aligned} \tag{4.5}
$$

By (4.4) and (4.5),

$$
\begin{aligned}
& i\partial_t v + \partial^2 v - i\lambda|v|^2\partial v - i\mu v^2\partial\bar{v} - f(v) \\
& \quad = -MDe^{iS}[(\partial_t S) - (((\lambda-\mu)\xi - \nu_1)/2t)|\hat{u}_+|^2]\hat{u}_+ \\
& \quad + (1/4t^2)MD[\partial^2(e^{iS}\hat{u}_+) - i\lambda e^{iS}|\hat{u}_+|^2(\partial\hat{u}_+ + i(\partial S)\hat{u}_+) \\
& \quad - i\mu e^{-iS}(\hat{u}_+)^2(\overline{\partial\hat{u}_+} - i(\partial S)\overline{\hat{u}_+}) - \nu_2 e^{iS}|\hat{u}_+|^4\hat{u}_+].
\end{aligned} \tag{4.6}
$$

This indicates the choice

$$
S(t,\xi) = (((\lambda-\mu)\xi - \nu_1)/2)|\hat{u}_+(\xi)|^2 \log t \tag{4.7}
$$

and the choice in (4.6) yields the estimate

$$
||i\partial_t v + \partial^2 v - i\lambda|v|^2\partial v - i\mu v^2\partial\bar{v} - f(v)||_{H^k} \le C(k, ||u_+||_{H^{k+2}}, ||\hat{u}_+||_{H^{k+2}})t^{-2}(\log t)^2 \tag{4.8}
$$

for any $t \ge 2$ and any nonnegative integer k, where C is independent of t.

Proof of Theorem 2. As in the preceding sections, we make a reduction to the system. For a given asymptotic state u_+ we define the modified free evolution v by (4.2) and (4.7) and we solve (1.1) around v. According to the reduction to the system, we transform v and u to $(\varphi^{(0)}, \psi^{(0)})$ and (φ, ψ), respectively, as follows. We define

$$
\varphi^{(0)} = G_\lambda^{(0)} v, \ \psi^{(0)} = G_{\lambda-\mu}^{(0)}\partial(G_\mu^{(0)} v), \tag{4.9}
$$

$$
\varphi = G_\lambda u, \ \psi = G_{\lambda-\mu}\partial(G_\mu u), \tag{4.10}
$$

where

$$
G_\nu^{(0)}(t,x) = \exp(-i\frac{\nu}{2}\int_{-\infty}^{x} |v(t,y)|^2 dy),
$$

$$
G_\nu(t,x) = \exp(-i\frac{\nu}{2}\int_{-\infty}^{x} |u(t,y)|^2 dy).
$$

Now we consider (see (2.9) and (2.10))

$$L\varphi = F_1(\varphi, \psi), \tag{4.11}$$

$$L\psi = F_2(\varphi, \psi), \tag{4.12}$$

with

$$(\varphi - \varphi^{(0)}, \psi - \psi^{(0)}) \in X_\theta(T) \tag{4.13}$$

for any θ with $1/4 < \theta < 1$ and $T > 0$ sufficiently large, where

$$\begin{aligned} X_\theta(T) = \{(w^{(1)}, w^{(2)}); w^{(j)} \in C([T,\infty); H^1) \cap L^4(T,\infty; W^{1,\infty}), \\ \|w^{(j)}\|_X \equiv \sup_{t\geq T} t^\theta (\|w^{(j)}(t)\|_{H^1} + \|w^{(j)}\|_{L^4(t,\infty;W^{1,\infty})}) < \infty, j = 1,2\}. \end{aligned}$$

We solve the problem in the integral form

$$\varphi(t) = \varphi^{(0)}(t) + i\int_t^\infty U_0(t-\tau)(F_1(\varphi,\psi) - F_1(\varphi^{(0)}, \psi^{(0)}) - R_1)(\tau)d\tau, \tag{4.14}$$

$$\psi(t) = \psi^{(0)}(t) + i\int_t^\infty U_0(t-\tau)(F_2(\varphi,\psi) - F_2(\varphi^{(0)}, \psi^{(0)}) - R_2)(\tau)d\tau, \tag{4.15}$$

with (4.13), where

$$\begin{aligned} R_1 &= G_\lambda^{(0)}(Lv - i\lambda|v|^2\partial v - i\mu v^2 \partial\bar{v} - f(v)), \\ R_2 &= G_{\lambda-\mu}^{(0)}\partial(G_\mu^{(0)}(Lv - i\lambda|v|^2\partial v - i\mu v^2\partial\bar{v} - f(v))). \end{aligned}$$

In the same way as in [10, 21], the problem (4.13), (4.14), (4.15) has a unique pair of solutions (φ, ψ) provided that $\|(1+\xi^2)\hat{u}_+\|_\infty$ is sufficiently small and T is sufficiently large since the contraction factor is proportional to $\|(1+\xi^2)\hat{u}_+\|_\infty + T^{1/2-\theta}$ in the contraction argument on (4.14) and (4.15) in the space $X_\theta(T)$. Moreover in the same way as in [21], we obtain the constraint

$$\psi = \partial\varphi - i((\lambda-\mu)/2)|\varphi|^2\varphi$$

on the time interval $[T, \infty)$ with T sufficiently large. Then we set

$$u(t,x) = \exp(i\frac{\lambda}{2}\int_{-\infty}^x |\varphi(t,y)|^2 dy)\varphi(t,x)$$

and we see that u solves (1.1) in a neighborhood of infinity in time and that $u-v \in X_\theta(T)$. This implies that $||u(T)||_2$ is sufficiently small, since

$$\begin{aligned}||u(T)||_2 &\le ||u(T)-v(T)||_2 + ||v(T)||_2 \\ &\le ||u-v||_X T^{-\theta} + ||\hat{u}_+||_2 \le ||u-v||_X T^{-\theta} + C||(1+\xi^2)\hat{u}_+||_\infty.\end{aligned}$$

By Theorem 1, we have the global solution of (1.1) with Cauchy data $u(T)$ given at $t_0 = T$. This proves an essential part of Theorem 2. The rest of the argument is the same as in [10, 11, 21, 23, 42]. QED

Proof of Theorem 3. It suffices to prove that for any $s \in \mathbb{R}$ and any $\varphi \in X_\epsilon$

$$v(s+\cdot\,;\varphi) - v(\cdot\,;U_0(s)\varphi) \in X_\theta(T) \tag{4.16}$$

provided that T is sufficiently large, where

$$v(t;\varphi) = U_0(t)\exp(iS(t,-i\partial))\varphi.$$

The proof of (4.16) is quite similar to that of [10; Proposition 3.2]. QED

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