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Title	Residue formulas for singular foliations defined by meromorphic functions on surfaces
Author(s)	Honda, T.; Suwa, T.
Citation	Hokkaido University Preprint Series in Mathematics, 306, 1-19
Issue Date	1995-09-01
DOI	https://doi.org/10.14943/83453
Doc URL	https://hdl.handle.net/2115/69057
Type	departmental bulletin paper
File Information	re306.pdf



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Series #306. September 1995

HOKKAIDO UNIVERSITY
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Residue formulas for singular foliations defined by meromorphic functions on surfaces

Tomoaki Honda and Tatsuo Suwa

Abstract

For singular foliations defined by meromorphic functions on complex surfaces, we compute the Baum-Bott residues and describe the residue theorem. Applying these to the case of foliations arising from polynomials in two variables, we obtain various formulas including a generalization of a formula of Lê and a “Milnor number formula” for a holomorphic map with non-reduced fibers.

Let X be a complex manifold of dimension n . A dimension one (singular) foliation $\mathcal{E}$ on X is defined locally by a holomorphic vector field and its singular set $S(\mathcal{E})$ is defined by patching the zero sets of the vector fields defining $\mathcal{E}$ together. For a compact connected component Z of $S(\mathcal{E})$ (of codimension in X greater than one) and a symmetric and homogeneous polynomial ψ of degree n , there is the “Baum-Bott residue” $\text{Res}_\psi(\mathcal{E}, Z)$, which is a complex number determined by the behavior of the vector field near Z . If X is compact, the sum of these residues over the components of $S(\mathcal{E})$ is equal to the characteristic class $\psi(TX - E)$ of the virtual bundle $TX - E$, where TX denotes the holomorphic tangent bundle of X and E the line bundle associated with $\mathcal{E}$ ([BB1] and [BB2]).

When X is a complex surface ($n = 2$), the foliation $\mathcal{E}$ is also defined locally by a holomorphic 1-form. Thus if we have a meromorphic function φ on X , considering its differential $d\varphi$, we have a naturally defined foliation (see section 2 below) whose leaves are the level sets of φ and whose singular points include the critical points and the indeterminacy points of φ . In this article, we compute the above residues and examine the residue formula for

foliations defined this way. In section 1, we recall basic facts about the Baum-Bott residue and write down the global invariant in the residue formula in terms of the line bundle F (the “annihilator” of E). Since the dimension of X is two, we have essentially two kinds of residues, one for $\psi = \sigma_1^2$ and the other for $\psi = \sigma_2$, where σ_i denotes the elementary symmetric function of degree i . The residue for σ_2 at a singular point gives the index of the vector field defining $\mathcal{E}$ near the point and if $\mathcal{E}$ is defined by a global vector field, the residue formula reduces to the Poincaré-Hopf theorem. We describe, in section 2, how a foliation is defined from a given meromorphic function φ on X . If the critical points of φ (away from its pole divisor) are isolated, the line bundle F turns out to be the dual of the bundle determined by the pole divisor D of $d\varphi$ (Lemma 2.1). It is not difficult to find the residues at a singular point away from the pole divisor of φ . It is also possible to compute the residue for σ_1^2 at a singular point on the pole divisor explicitly. The residue formula in this case is simply a formula to express the self-intersection number of D as a sum of local contributions at the singular points (on D) of the foliation. We may compute the residue for σ_2 at a point on the pole divisor in some cases which will be useful in the following section. In section 3, we apply these results to foliations on the projective plane $\mathbf{P}^2$ or on its modifications arising from a polynomial f in two variables. Let φ_0 denote the rational function on $\mathbf{P}^2$ obtained by extending f . First, considering the foliation on $\mathbf{P}^2$ defined by φ_0 , we obtain a generalization of a formula of D.T. Lê (Theorem 3.2). Then, if we remove the indeterminacy of φ_0 by a sequence of blowing-ups of $\mathbf{P}^2$, φ_0 is modified to a meromorphic function φ which gives a fibration of the blown-up surface X over $\mathbf{P}^1$. Considering the foliation on X defined by φ , we obtain a formula (Theorem 3.8), which may be interpreted as a “Milnor number formula” for a map with multiple fibers.

We would like to thank Lê Dũng Tráng for suggesting the problem and for helpful conversations.

1 Baum-Bott residues of singular foliations on surfaces

Let X be a complex analytic manifold of dimension two (a complex surface). A dimension one complex analytic singular foliation $\mathcal{E}$ on X is deter-

mined by a system $\{(U_\alpha, v_\alpha)\}$, where $\{U_\alpha\}$ is an open covering of X and, for each α , v_α is a holomorphic vector field on U_α , such that $v_\beta = e_{\alpha\beta}v_\alpha$ on $U_\alpha \cap U_\beta$ for some non-vanishing holomorphic function $e_{\alpha\beta}$ on $U_\alpha \cap U_\beta$. We denote the set of zeros of v_α on U_α by $S(v_\alpha)$; $S(v_\alpha) = \{p \in U_\alpha \mid v_\alpha(p) = 0\}$, and call it the singular set of v_α . Since $S(v_\alpha)$ and $S(v_\beta)$ coincide in $U_\alpha \cap U_\beta$, the union $\cup_\alpha S(v_\alpha)$ is an analytic set in X , which we call the singular set of the foliation $\mathcal{E}$ and denote by $S(\mathcal{E})$. We say that $\mathcal{E}$ is reduced if $S(\mathcal{E})$ consists of isolated points. Since the system $\{e_{\alpha\beta}\}$ satisfies the cocycle condition, it determines a line bundle, which we denote by E .

Singular foliations can also be defined in terms of holomorphic 1-forms. Thus a codimension one complex analytic singular foliation $\mathcal{F}$ on X is determined by a system $\{(U_\alpha, \omega_\alpha)\}$, where ω_α is a holomorphic 1-form on U_α , such that $\omega_\beta = f_{\alpha\beta}\omega_\alpha$ on $U_\alpha \cap U_\beta$ for some non-vanishing holomorphic function $f_{\alpha\beta}$ on $U_\alpha \cap U_\beta$. As in the case of vector fields, we can define the singular set $S(\mathcal{F})$ by patching the singular (zero) sets $S(\omega_\alpha)$ of ω_α together and we may talk about the reducibility of $\mathcal{F}$. We denote by F the line bundle determined by the cocycle $\{f_{\alpha\beta}\}$.

The two definitions above are equivalent, as long as we consider reduced foliations, in the sense that there is a natural one-to-one correspondence between the reduced dimension one foliations and the reduced codimension one foliations ([Sw]). In fact, the correspondence is given by taking the annihilator of each other, namely, if $\mathcal{E} = \{(U_\alpha, v_\alpha)\}$ is a (reduced) dimension one foliation, it corresponds to the (reduced) codimension one foliation $\mathcal{F} = \{(U_\alpha, \omega_\alpha)\}$ with $\langle v_\alpha, \omega_\alpha \rangle = 0$ on U_α and vice versa. Note that in the above correspondence, we have $S(\mathcal{E}) = S(\mathcal{F})$ and the integral curves of the vector field v_α are the solutions of the differential equation $\omega_\alpha = 0$. In what follows we consider only reduced foliations.

Let $\mathcal{E}$ be a dimension one singular foliation on X . For each point p in $S(\mathcal{E})$ and a homogeneous and symmetric polynomial ψ of degree two, we have the Baum-Bott residue $\text{Res}_\psi(\mathcal{E}, p)$ of $\mathcal{E}$ at p for ψ , which is a complex number given as follows ([BB1], [BB2]).

Suppose that U is a coordinate neighborhood and that p is the origin of a coordinate system (x, y) on U and is an isolated zero of the vector field v defining $\mathcal{E}$ on U . We write

$$v = a(x, y) \frac{\partial}{\partial x} + b(x, y) \frac{\partial}{\partial y}$$

with a and b holomorphic functions on U , and let A be the Jacobian matrix $\frac{\partial(a,b)}{\partial(x,y)}$. For elementary symmetric polynomials σ_i , $i = 1, 2$, in two variables, we set

$$\sigma_1(A) = \text{tr} A \quad \text{and} \quad \sigma_2(A) = \det A.$$

If ψ is a homogeneous and symmetric polynomial of degree two, it is written as $\psi = \tilde{\psi}(\sigma_1, \sigma_2)$, for some polynomial $\tilde{\psi}$. We set $\psi(A) = \tilde{\psi}(\sigma_1(A), \sigma_2(A))$. Then the Baum-Bott residue $\text{Res}_\psi(\mathcal{E}, p)$ is given by the Grothendieck residue symbol

$$\text{Res}_\psi(\mathcal{E}, p) = \text{Res}_p \left[\frac{\psi(A) dx \wedge dy}{a, b} \right],$$

which is represented by the integral

$$\left(\frac{1}{2\pi\sqrt{-1}} \right)^2 \int_\Gamma \frac{\psi(A) dx \wedge dy}{ab}.$$

In the above, Γ is the 2-cycle in U defined by

$$\Gamma = \{ (x, y) \mid |a(x, y)| = |b(x, y)| = \varepsilon \},$$

for a sufficiently small positive number ε , and is oriented so that the form $d(\arg a) \wedge d(\arg b)$ is positive.

In particular, if $\psi = \sigma_2$, since $\sigma_2(A) dx \wedge dy = da \wedge db$, the residue $\text{Res}_{\sigma_2}(\mathcal{E}, p)$ is the intersection number $(a, b)_p$ at p of the divisors defined by a and b ([GH] Ch.5), which is equal to the index of the vector field v at p .

Now we denote by TX the holomorphic tangent bundle of X . A section of the bundle E is represented by a collection $\{s_\alpha\}$ of functions satisfying $s_\alpha = e_{\alpha\beta} s_\beta$. Thus the collection $\{s_\alpha v_\alpha\}$ defines a section of TX . Hence we have a bundle map $E \rightarrow TX$, which is injective exactly on $X - S(\mathcal{E})$.

We have the following residue theorem.

Theorem 1.1 (Baum-Bott) *If the complex manifold X is compact, then*

$$\sum_{p \in S(\mathcal{E})} \text{Res}_\psi(\mathcal{E}, p) = \psi(TX - E) \cap [X],$$

where, denoting by $c_1 = c_1(TX - E)$ and $c_2 = c_2(TX - E)$ the first and second Chern classes of the virtual bundle $TX - E$, we set $\psi(TX - E) = \tilde{\psi}(c_1, c_2)$.

Recall that, if we denote by $c(X)$ and $c(E)$ the total Chern classes of TX and E , the total Chern class of $TX - E$ is given by $c(TX - E) = c(X)/c(E)$. Hence we have

$$\begin{aligned} c_1(TX - E) &= c_1(X) - c_1(E), \\ c_2(TX - E) &= c_2(X) - c_1(X)c_1(E) + c_1^2(E). \end{aligned}$$

If we denote by $\mathcal{F}$ the codimension one foliation corresponding to $\mathcal{E}$ and by F the associated line bundle, we have the following lemma.

Lemma 1.2 *We have $F = E \otimes K$, where K denotes the canonical bundle of X .*

Proof. We assume that each U_α is a coordinate neighborhood and let (x_α, y_α) be a coordinate system on U_α . If we write $v_\alpha = a_\alpha \frac{\partial}{\partial x_\alpha} + b_\alpha \frac{\partial}{\partial y_\alpha}$, then $\mathcal{F}$ is defined by $\omega_\alpha = b_\alpha dx_\alpha - a_\alpha dy_\alpha$ on U_α . Let the systems $\{e_{\alpha\beta}\}$ and $\{f_{\alpha\beta}\}$ be defined by $v_\beta = e_{\alpha\beta}v_\alpha$ and $\omega_\beta = f_{\alpha\beta}\omega_\alpha$, as before, so that they define the bundles E and F , respectively. From $v_\beta = e_{\alpha\beta}v_\alpha$, we have

$$a_\beta = e_{\alpha\beta} \left(a_\alpha \frac{\partial x_\beta}{\partial x_\alpha} + b_\alpha \frac{\partial x_\beta}{\partial y_\alpha} \right), \quad b_\beta = e_{\alpha\beta} \left(a_\alpha \frac{\partial y_\beta}{\partial x_\alpha} + b_\alpha \frac{\partial y_\beta}{\partial y_\alpha} \right).$$

Substituting these in $\omega_\beta = b_\beta dx_\beta - a_\beta dy_\beta$, we get

$$\omega_\beta = e_{\alpha\beta} \det \left(\frac{\partial(x_\beta, y_\beta)}{\partial(x_\alpha, y_\alpha)} \right) \omega_\alpha,$$

which proves the lemma. ■

For line bundles L_1 and L_2 on a compact complex manifold X , we denote $(c_1(L_1) \smile c_1(L_2)) \smile [X]$ by $L_1 \cdot L_2$. Then, noting that $c_1(X) = -c_1(K)$, we have, from Theorem 1.1 and Lemma 1.2, the following formulas.

Proposition 1.3 *If X is compact,*

$$\begin{aligned} \sum_{p \in S(\mathcal{E})} \text{Res}_{\sigma_1^2}(\mathcal{E}, p) &= F^2, \\ \sum_{p \in S(\mathcal{E})} \text{Res}_{\sigma_2}(\mathcal{E}, p) &= \chi(X) + F^2 - K \cdot F, \end{aligned}$$

where $\chi(X)$ denotes the Euler number of X .

2 Singular foliations defined by meromorphic functions

Let φ be a meromorphic function on a complex surface X . We take a coordinate covering $\{U_\alpha\}$ of X so that, on each U_α , the differential $d\varphi$ of φ is written as

$$d\varphi = \varphi_\alpha \omega_\alpha,$$

where φ_α is a meromorphic function and ω_α is a holomorphic 1-form with isolated zeros on U_α . Then the system $\{(U_\alpha, \omega_\alpha)\}$ defines a (reduced) codimension one singular foliation $\mathcal{F}$. The associated line bundle F is defined by the cocycle $\{f_{\alpha\beta}\}$, $f_{\alpha\beta} = \varphi_\alpha / \varphi_\beta$. The leaves of this foliation are the level sets of φ .

We denote by $D^{(0)}$ and $D^{(\infty)}$, respectively, the zero and pole divisors of φ ; $(\varphi) = D^{(0)} - D^{(\infty)}$. Let $D^{(0)} = \sum_{j=1}^s n_j D_j^{(0)}$ and $D^{(\infty)} = \sum_{i=1}^r m_i D_i^{(\infty)}$ be the irreducible decompositions with n_i and m_j positive integers. For a divisor D on X , we denote by $|D|$ the support of D and by $[D]$ the line bundle determined by D .

Lemma 2.1 *If the critical points of φ in $X - |D^{(\infty)}|$ are all isolated, then we have*

$$F = [-\sum_{i=1}^r (m_i + 1) D_i^{(\infty)}].$$

Proof. First note that the assumption implies that $D^{(0)}$ is reduced ($n_j = 1$ for all j). At each point p in X , we express the germ φ as $\varphi = f/g$ with f and g relatively prime holomorphic function germs defining $D^{(0)}$ and $D^{(\infty)}$, respectively, at p . Let $g = g_1^{m_1} \cdots g_r^{m_r}$ be a decomposition so that $D_i^{(\infty)}$ is defined by g_i . Note that f or g_i may be a unit or may be reducible but they are reduced (i.e., not divisible by the square of a non-unit). We compute

$$d\varphi = \frac{1}{g_1^{m_1+1} \cdots g_r^{m_r+1}} \omega,$$

where ω is the holomorphic 1-form germ given by

$$\omega = g_1 \cdots g_r df - f \left(\sum_{i=1}^r m_i g_1 \cdots \hat{g}_i \cdots g_r dg_i \right).$$

We claim that the zero of ω is (at most) isolated and thus $\mathcal{F}$ is defined by ω near the point p . In fact, if $\omega = h\omega'$ for some non-unit h and a holomorphic 1-form germ ω' , then h must be divisible by a factor g' of some g_i which is a non-unit. This implies that $dg_i = g'\theta$ for some holomorphic 1-form germ θ , which is a contradiction, since g_i is reduced. ■

Remark 2.2 *Under the assumption of Lemma 2.1, the singular points of $\mathcal{F}$ in $X - |D^{(\infty)}|$ are the critical points of φ and the singular points of $\mathcal{F}$ in $|D^{(\infty)}|$ include the intersection points of $D^{(0)}$ and $D_i^{(\infty)}$ (indeterminacy points), the intersection points of $D_i^{(\infty)}$ and $D_j^{(\infty)}$, $i \neq j$, and the singular points of $D_i^{(\infty)}$.*

Hereafter throughout this section, we assume that the critical points of φ on $X - |D^{(\infty)}|$ are all isolated. We denote by $\mathcal{E}$ the dimension one foliation corresponding to $\mathcal{F}$.

Lemma 2.3 *For a singular point p of $\mathcal{E}$ in $X - |D^{(\infty)}|$, we have*

$$\text{Res}_{\sigma_1^2}(\mathcal{E}, p) = 0 \quad \text{and} \quad \text{Res}_{\sigma_2}(\mathcal{E}, p) = \mu_p(\varphi),$$

where $\mu_p(\varphi)$ denotes the Milnor number of φ at p .

Proof. By the assumption, near a point p in $X - |D^{(\infty)}|$, $\mathcal{F}$ is defined by $d\varphi$. Hence, if we denote by (x, y) a coordinate system near p , $\mathcal{E}$ is defined by the holomorphic vector field $v = \frac{\partial \varphi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \varphi}{\partial x} \frac{\partial}{\partial y}$. The Jacobian matrix A of v is

$$A = \begin{pmatrix} \frac{\partial^2 \varphi}{\partial x \partial y} & \frac{\partial^2 \varphi}{\partial y^2} \\ -\frac{\partial^2 \varphi}{\partial x^2} & -\frac{\partial^2 \varphi}{\partial y \partial x} \end{pmatrix}.$$

Thus, since $\sigma_1(A) = \text{tr}(A) = 0$, the residue $\text{Res}_{\sigma_1^2}(\mathcal{E}, p)$ is equal to 0. To compute $\text{Res}_{\sigma_2}(\mathcal{E}, p)$, put $a = \frac{\partial \varphi}{\partial y}$ and $b = -\frac{\partial \varphi}{\partial x}$. Then, since $\sigma_2(A) = \det(A)$,

$$\text{Res}_{\sigma_2}(\mathcal{E}, p) = \left(\frac{1}{2\pi\sqrt{-1}} \right)^2 \int_{\Gamma} \frac{da \wedge db}{ab} = (a, b)_p = \mu_p(\varphi). \quad \blacksquare$$

In what follows, we denote by $(D_1 \cdot D_2)_p$ the intersection number of divisors D_1 and D_2 at a point p and by $D_1 \cdot D_2$ the (total) intersection number, which is equal to $[D_1] \cdot [D_2]$.

Lemma 2.4 For a singlar point p of $\mathcal{E}$ in $|D^{(\infty)}|$, we have

$$\text{Res}_{\sigma_1^2}(\mathcal{E}, p) = \sum_{i=1}^r \frac{(m_i + 1)^2}{m_i} (D^{(0)} \cdot D_i^{(\infty)})_p - \sum_{1 \leq i < j \leq r} \frac{(m_i - m_j)^2}{m_i m_j} (D_i^{(\infty)} \cdot D_j^{(\infty)})_p.$$

Thus if p is not an intersection point of $D^{(0)}$ and $D_i^{(\infty)}$ or of $D_i^{(\infty)}$ and $D_j^{(\infty)}$ with $m_i \neq m_j$, we have $\text{Res}_{\sigma_1^2}(\mathcal{E}, p) = 0$.

Proof If we denote by (x, y) a coordinate system near p , $\mathcal{E}$ is defined by the vector field $a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}$ with

$$\begin{aligned} a &= g_1 \cdots g_r \frac{\partial f}{\partial y} - f \left(\sum_{i=1}^r m_i g_1 \cdots \hat{g}_i \cdots g_r \frac{\partial g_i}{\partial y} \right), \\ b &= -g_1 \cdots g_r \frac{\partial f}{\partial x} + f \left(\sum_{i=1}^r m_i g_1 \cdots \hat{g}_i \cdots g_r \frac{\partial g_i}{\partial x} \right) \end{aligned}$$

(see the proof of Lemma 2.1). For the matrix $A = \frac{\partial(a, b)}{\partial(x, y)}$, we compute

$$\text{tr} A = - \sum_{i=1}^r (m_i + 1) G_i \det \frac{\partial(f, g_i)}{\partial(x, y)} + \sum_{1 \leq i < j \leq r} (m_i - m_j) f G_{ij} \det \frac{\partial(g_i, g_j)}{\partial(x, y)},$$

where $G_i = g_1 \cdots \hat{g}_i \cdots g_r$ and $G_{ij} = g_1 \cdots \hat{g}_i \cdots \hat{g}_j \cdots g_r$. We have

$$(2.5) \quad \text{Res}_{\sigma_1^2}(\mathcal{E}, p) = \text{Res}_p \left[\begin{array}{c} (\text{tr} A)^2 dx \wedge dy \\ a, b \end{array} \right] = \text{Res}_p \left[\begin{array}{c} \text{tr} A \cdot \tau \\ a, b \end{array} \right],$$

where

$$\tau = - \sum_{i=1}^r (m_i + 1) \tau_i + \sum_{1 \leq i < j \leq r} (m_i - m_j) \tau_{ij}$$

with $\tau_i = G_i df \wedge dg_i$ and $\tau_{ij} = f G_{ij} dg_i \wedge dg_j$. Now, for a fixed i , we may write

$$\begin{pmatrix} a \\ b \end{pmatrix} = C \begin{pmatrix} f \\ g_i \end{pmatrix}, \quad C = \begin{pmatrix} -\sum_{k=1}^r m_k G_k \frac{\partial g_k}{\partial y}, & G_i \frac{\partial f}{\partial y} \\ \sum_{k=1}^r m_k G_k \frac{\partial g_k}{\partial x}, & -G_i \frac{\partial f}{\partial x} \end{pmatrix}.$$

Also, for $j \neq i$, we may write

$$\begin{pmatrix} a \\ b \end{pmatrix} = C_j \begin{pmatrix} f \\ g_i \end{pmatrix}, \quad C_j = \begin{pmatrix} -\sum_{k \neq j} m_k G_k \frac{\partial g_k}{\partial y}, & G_i \frac{\partial f}{\partial y} - f m_j G_{ij} \frac{\partial g_j}{\partial y} \\ \sum_{k \neq j} m_k G_k \frac{\partial g_k}{\partial x}, & -G_i \frac{\partial f}{\partial x} + f m_j G_{ij} \frac{\partial g_j}{\partial x} \end{pmatrix}.$$

Then we compute

$$\text{tr} A \cdot G_i = - \left(\frac{m_i + 1}{m_i} + \sum_{j \neq i} \frac{m_i - m_j}{m_i m_j} \right) \det C + \sum_{j \neq i} \frac{m_i - m_j}{m_i m_j} \det C_j.$$

Thus, using one of the properties of the Grothendieck residue symbol, we have

$$(2.6) \quad \text{Res}_p \left[\begin{array}{c} \text{tr} A \cdot \tau_i \\ a, b \end{array} \right] = - \frac{m_i + 1}{m_i} \text{Res}_p \left[\begin{array}{c} df \wedge dg_i \\ f, g_i \end{array} \right].$$

Next, for i and j with $i < j$, we may write

$$\begin{pmatrix} a \\ b \end{pmatrix} = C' \begin{pmatrix} g_i \\ g_j \end{pmatrix}, \quad C' = \begin{pmatrix} G_i \frac{\partial f}{\partial y} - f \sum_{k \neq i} m_k G_{ik} \frac{\partial g_k}{\partial y}, & -f m_i G_{ij} \frac{\partial g_i}{\partial y} \\ -G_i \frac{\partial f}{\partial x} + f \sum_{k \neq i} m_k G_{ik} \frac{\partial g_k}{\partial x}, & f m_i G_{ij} \frac{\partial g_i}{\partial x} \end{pmatrix}$$

and

$$\begin{pmatrix} a \\ b \end{pmatrix} = C'' \begin{pmatrix} g_i \\ g_j \end{pmatrix}, \quad C'' = \begin{pmatrix} -f m_j G_{ij} \frac{\partial g_j}{\partial y}, & G_j \frac{\partial f}{\partial y} - f \sum_{k \neq j} m_k G_{jk} \frac{\partial g_k}{\partial y} \\ f m_j G_{ij} \frac{\partial g_j}{\partial x}, & -G_j \frac{\partial f}{\partial x} + f \sum_{k \neq j} m_k G_{jk} \frac{\partial g_k}{\partial x} \end{pmatrix}.$$

Also, for $\ell \neq i, j$, we may write

$$\begin{pmatrix} a \\ b \end{pmatrix} = C'_\ell \begin{pmatrix} g_i \\ g_j \end{pmatrix}$$

with

$$C'_\ell = \begin{pmatrix} -f (m_j G_{ij} \frac{\partial g_j}{\partial y} + m_\ell G_{i\ell} \frac{\partial g_\ell}{\partial y}), & G_j \frac{\partial f}{\partial y} - f \sum_{k \neq j, \ell} m_k G_{jk} \frac{\partial g_k}{\partial y} \\ f (m_j G_{ij} \frac{\partial g_j}{\partial x} + m_\ell G_{i\ell} \frac{\partial g_\ell}{\partial x}), & -G_j \frac{\partial f}{\partial x} + f \sum_{k \neq j, \ell} m_k G_{jk} \frac{\partial g_k}{\partial x} \end{pmatrix}.$$

Then we compute

$$\begin{aligned} \text{tr} A \cdot f G_{ij} &= \frac{m_i + 1}{m_i} \det C' + \left(\sum_{\ell \neq i, j} \frac{m_\ell + 1}{m_\ell} - \frac{m_j + 1}{m_j} \right) \det C'' \\ &\quad - \sum_{\ell \neq i, j} \frac{m_\ell + 1}{m_\ell} \det C'_\ell. \end{aligned}$$

Thus we have

$$(2.7) \quad \text{Res}_p \left[\begin{array}{c} \text{tr} A \cdot \tau_{ij} \\ a, b \end{array} \right] = - \frac{m_i - m_j}{m_i m_j} \text{Res}_p \left[\begin{array}{c} dg_i \wedge dg_j \\ g_i, g_j \end{array} \right].$$

From (2.5), (2.6) and (2.7), we have the lemma. ■

In what follows, we set $D = \sum_{i=1}^r (m_i + 1) D_i^{(\infty)}$, which may be called the pole divisor of $d\varphi$. If X is compact, by Lemmas 2.1, 2.3 and 2.4, the first formula in Proposition 1.3 becomes

$$(2.8) \quad D^2 = \sum_p \left(\sum_{i=1}^r \frac{(m_i + 1)^2}{m_i} I_i(p) - \sum_{1 \leq i < j \leq r} \frac{(m_i - m_j)^2}{m_i m_j} I_{ij}(p) \right),$$

where $I_i(p) = (D^{(0)} \cdot D_i^{(\infty)})_p$ and $I_{ij}(p) = (D_i^{(\infty)} \cdot D_j^{(\infty)})_p$, and the sum for p is taken over the intersection points of $D^{(0)}$ and $D_i^{(\infty)}$ and of $D_i^{(\infty)}$ and $D_j^{(\infty)}$. Note that (2.8) also follows from the fact that $D^{(0)} - D^{(\infty)}$ is linearly equivalent to 0. Also, from the second formula in Proposition 1.3, we have the following formula.

Proposition 2.9 *Let φ be a meromorphic function on a compact complex surface X . If the critical points of φ (away from the pole divisor) are all isolated, we have*

$$\sum_{p \in S(\mathcal{E}) \cap (X - |D|)} \mu_p(\varphi) + \sum_{p \in S(\mathcal{E}) \cap |D|} \text{Res}_{\sigma_2}(\mathcal{E}, p) = \chi(X) + D^2 + K \cdot D.$$

Remark 2.10 *Following [K], we call the quantity $\frac{1}{2}(D^2 + K \cdot D) + 1$ the virtual genus of a divisor D . Then we may define the “virtual Euler number” $\chi'(D)$ of D by*

$$\chi'(D) = -(D^2 + K \cdot D).$$

With this, the right hand side of the formula in Proposition 2.9 is written as $\chi(X) - \chi'(D)$.

Now we compute $\text{Res}_{\sigma_2}(\mathcal{E}, p)$ for a singular point p of $\mathcal{E}$ in $|D| = |D^{(\infty)}|$ in some special cases.

(I) Let p be an intersection point of $D^{(0)}$ and $D_i^{(\infty)}$ and assume that $D_i^{(\infty)}$ is non-singular at p with no other components of $D^{(\infty)}$ passing through p . We may take a coordinate system (x, y) near p so that $g_i(x, y) = x$. We may write $\varphi = \frac{f}{x^{m_i}}$ with f defining $D^{(0)}$ near p and see that the holomorphic 1-form

$$\omega = \left(x \frac{\partial f}{\partial x} - m_i f \right) dx + x \frac{\partial f}{\partial y} dy$$

defines the foliation near p (see the proof of Lemma 2.1). We have

$$\text{Res}_{\sigma_2}(\mathcal{E}, p) = \left(x \frac{\partial f}{\partial y} m_i f - x \frac{\partial f}{\partial x} \right)_p = I_1 + I_2,$$

where

$$I_1 = \left(x, m_i f - x \frac{\partial f}{\partial x} \right)_p \quad \text{and} \quad I_2 = \left(\frac{\partial f}{\partial y}, m_i f - x \frac{\partial f}{\partial x} \right)_p.$$

First we have

$$(2.11) \quad I_1 = (x, f)_p = (D^{(0)} \cdot D_i^{(\infty)})_p.$$

In order to calculate I_2 , let h be a (local) irreducible component of $\frac{\partial f}{\partial y}$ at p and $\pi(t) = (x(t), y(t))$ a uniformization of the curve $h = 0$. Then, since $\frac{\partial f}{\partial y}(\pi(t)) = 0$, we have

$$(2.12) \quad \frac{df}{dt}(\pi(t)) = \frac{\partial f}{\partial x}(\pi(t)) \frac{dx}{dt}.$$

Lemma 2.13 *The germs f and $\frac{\partial f}{\partial y}$ are relatively prime at p .*

Proof. Since f is reduced and is regular in y , we have the lemma by the Weierstrass preparation theorem. ■

Lemma 2.14 *The germs $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and the germs x and $\frac{\partial f}{\partial y}$ are relatively prime at p .*

Proof. By Lemma 2.13, $f(\pi(t)) \not\equiv 0$. Hence $\frac{\partial f}{\partial x}(\pi(t)) \not\equiv 0$ and $\frac{dx}{dt} \not\equiv 0$ by (2.12). ■

Now if we write

$$f(\pi(t)) = \sum_{n \geq q} a_n t^n, \quad \frac{\partial f}{\partial x}(\pi(t)) = \sum_{n \geq r} b_n t^n \quad \text{and} \quad x(t) = \sum_{n \geq s} c_n t^n,$$

with $a_q \neq 0$, $b_r \neq 0$ and $c_s \neq 0$, from (2.12), we get

$$q = r + s \quad \text{and} \quad n a_n = \sum_{k=s}^{n-r} k b_{n-k} c_k, \quad n \geq q.$$

Thus we may write

$$\left(m_i f - x \frac{\partial f}{\partial x}\right)(\pi(t)) = \sum_{n \geq q} (da_n - \sum_{k=s}^{n-r} b_{n-k} c_k) t^n.$$

If we denote the order of this power series by $q + \delta$ with δ a non-negative integer, we have

$$\left(h, m_i f - x \frac{\partial f}{\partial x}\right)_p = q + \delta.$$

Now let $\frac{\partial f}{\partial y} = h_1^{n_1} \dots h_\ell^{n_\ell}$ be the irreducible decomposition and apply the previous argument for each h_k , $k = 1, \dots, \ell$. Then writing q and δ for h_k by q_k and δ_k and recalling that $q_k = (h_k, x \frac{\partial f}{\partial x})_p$, we get

$$\begin{aligned} I_2 &= \left(\frac{\partial f}{\partial y}, x \frac{\partial f}{\partial x}\right)_p + \delta_p = \left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right)_p + \left(\frac{\partial f}{\partial y}, x\right)_p + \delta_p \\ &= \mu_p(f) + (D^{(0)} \cdot D_i^{(\infty)})_p - 1 + \delta_p, \end{aligned}$$

where $\delta_p = \sum_{k=1}^{\ell} n_k \delta_k$. Combined with (2.11), we get the following proposition.

Proposition 2.15 *Let φ be a meromorphic function on a complex surface X whose critical points in $X - |D^{(\infty)}|$ are all isolated. For an intersection point p of $D^{(0)}$ and $D_i^{(\infty)}$ such that $D_i^{(\infty)}$ is non-singular at p with no other components of $D^{(\infty)}$ passing through p , we have*

$$\text{Res}_{\sigma_2}(\mathcal{E}, p) = \mu_p(f) + \delta_p + 2(D^{(0)} \cdot D_i^{(\infty)})_p - 1,$$

where f is an defining equation of $D^{(0)}$ near p .

Note that, in general, we have $\delta_p = 0$.

(II) Let p be an intersection point of $D_i^{(\infty)}$ and $D_j^{(\infty)}$ and assume that $D_i^{(\infty)}$ and $D_j^{(\infty)}$ intersect transversally at p with $D^{(0)}$ or any other component of $D^{(\infty)}$ not passing through p . We may take a coordinate system (x, y) near p so that $g_i(x, y) = x$ and $g_j(x, y) = y$. We may write $\varphi = \frac{1}{xy}$ and see that the holomorphic 1-form

$$\omega = ydx + xdy$$

defines the foliation near p . Then we have

$$(2.16) \quad \text{Res}_{\sigma_2}(\mathcal{E}, p) = 1.$$

3 Foliations arising from polynomials

In this section we apply the formulas in the previous section to the case of foliations on the two dimensional projective space $\mathbf{P}^2$ or on its modifications which are defined by compactifying polynomials in two variables.

Let $f(x, y)$ be a polynomial of degree d with complex coefficients. We regard $f(x, y)$ as a function on $\mathbf{C}^2$ and extend it to a meromorphic (rational) function φ_0 on $\mathbf{P}^2$. If we denote by $(\zeta_0 : \zeta_1 : \zeta_2)$ homogeneous coordinates on $\mathbf{P}^2$, the rational function φ_0 is given by

$$\varphi_0(\zeta_0 : \zeta_1 : \zeta_2) = f(\zeta_1/\zeta_0, \zeta_2/\zeta_0) = \frac{\tilde{f}(\zeta_0, \zeta_1, \zeta_2)}{\zeta_0^d},$$

where, denoting by f_k the homogeneous piece of f of degree k ,

$$\tilde{f}(\zeta_0, \zeta_1, \zeta_2) = \zeta_0^d f_0 + \zeta_0^{d-1} f_1(\zeta_1, \zeta_2) + \cdots + \zeta_0 f_{d-1}(\zeta_1, \zeta_2) + f_d(\zeta_1, \zeta_2).$$

We assume that the critical points of f are all isolated. Thus the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are relatively prime and the polynomial f is reduced.

Let $\mathcal{F}$ denote the singular foliation on $\mathbf{P}^2$ determined by φ_0 . If we denote by L_∞ the "infinite line" $\zeta_0 = 0$, then the pole divisor of φ_0 is dL_∞ . Thus the line bundle F associated with $\mathcal{F}$ is given by $F = [-(d+1)L_\infty]$ (Lemma 2.1).

Let U_i denote the coordinate neighborhood $\{\zeta_i \neq 0\}$ in $\mathbf{P}^2$, for $i = 0, 1, 2$. On the "finite part" U_0 , $\mathcal{F}$ is defined by df , since, by assumption, the critical points of f are all isolated. We have $S(\mathcal{F}) \cap U_0 = C(f)$, the set of critical points of f in $U_0 = \mathbf{C}^2$. Now we find 1-forms defining $\mathcal{F}$ on the infinite parts of $\mathbf{P}^2$. We work on the coordinate neighborhood U_2 , however, it is similar on U_1 . In fact, changing the coordinate system on $\mathbf{C}^2$, if necessary, we may assume that $f_d(x, y)$ is not divisible by y . Then the singular points of $\mathcal{F}$ on L_∞ are all in U_2 . We take $(u, v) = \left(\frac{\zeta_0}{\zeta_2}, \frac{\zeta_1}{\zeta_2}\right)$ as a coordinate system on U_2 . Then the function φ_0 is written as, on U_2 ,

$$\varphi_0 = \frac{\hat{f}(u, v)}{u^d},$$

where $\hat{f}(u, v) = \tilde{f}(u, v, 1)$. Hence, on U_2 , $\mathcal{F}$ is defined by

$$\omega = \left(u \frac{\partial \hat{f}}{\partial u} - d \cdot \hat{f}\right) du + u \frac{\partial \hat{f}}{\partial v} dv$$

(see the proof of Lemma 2.1). Note that the points in $S(\omega) \cap L_\infty$ are given by

$$u = 0 \quad \text{and} \quad f_d(v, 1) = 0.$$

Thus, if $f_d(x, y) = \prod_{i=1}^k (b_i x - a_i y)^{d_i}$, $\sum_{i=1}^k d_i = d$, is the factorization of f_d , there are k singular points $p_i = (0 : a_i : b_i)$, $i = 1, \dots, k$, of $\mathcal{F}$ on L_∞ . We call d_i the multiplicity of f at a point at infinity p_i and denote it by $m_{p_i}(f)$. It is equal to the intersection number of the divisor of $\hat{f}$ and L_∞ at p_i .

Let $\mathcal{E}$ be the dimension one foliation corresponding to $\mathcal{F}$. On the finite part U_0 of $\mathbf{P}^2$, the vector field $v_0 = \frac{\partial f}{\partial y} \frac{\partial}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial}{\partial y}$ defines $\mathcal{E}$ and on the infinite part U_2 , the vector field $v_2 = u \frac{\partial \hat{f}}{\partial v} \frac{\partial}{\partial u} - (u \frac{\partial \hat{f}}{\partial u} - d \cdot \hat{f}) \frac{\partial}{\partial v}$ defines $\mathcal{E}$.

For a singular point p in $L_\infty \cap U_2$, we have, from Proposition 2.15,

$$(3.1) \quad \text{Res}_{\sigma_2}(\mathcal{E}, p) = \mu_p(\hat{f}) + \delta_p + 2m_p(f) - 1,$$

where δ_p is a non-negative integer defined as in Proposition 2.15 with f replaced by $\hat{f}$. The number δ_p is referred to as the “value of a jump in Milnor number at infinity” by D.T. Lê. Recall that, in general, we have $\delta_p = 0$.

Since $\chi(\mathbf{P}^2) = 3$, $D = (d+1)L_\infty$, $K = -3L_\infty$, $L_\infty^2 = 1$ and $\sum m_p(f) = d$, from Proposition 2.9 and (3.1), we have the following formula.

Theorem 3.2 *For a polynomial f of degree d , we have*

$$\sum_{p \in C(f)} \mu_p(f) + \sum_{i=1}^k (\mu_{p_i}(\hat{f}) + \delta_{p_i} - 1) = d^2 - 3d + 1,$$

where, letting $f_d(x, y) = \prod_{i=1}^k (b_i x - a_i y)^{d_i}$ be the factorization of the highest degree homogeneous piece f_d of f , p_i denotes the point on L_∞ given by $p_i = (0 : a_i : b_i)$.

This formula is also obtained by D.T. Lê in the case f has no critical points (unpublished).

For the other residue $\text{Res}_{\sigma_1^2}(\mathcal{E}, p)$ at p in $S(\mathcal{E}) \cap L_\infty$, we have, by Lemma 2.4,

$$\text{Res}_{\sigma_1^2}(\mathcal{E}, p) = \frac{(d+1)^2 m_p(f)}{d}$$

and (2.8) becomes

$$(3.3) \quad (d+1)^2 = \sum_{p \in S(\mathcal{E}) \cap L_\infty} \frac{(d+1)^2 m_p(f)}{d}.$$

This is a tautology, since $\sum m_p(f) = d$. However, it is interpreted as a formula to allocate the self-intersection number of the pole divisor $(d+1)L_\infty$ of $d\varphi$ to the singular points of the foliation $\mathcal{E}$.

Example 3.4 For the polynomial $f(x, y) = x - x^2y$, the singular points of $\mathcal{E}$ are $p_1 = (0 : 1 : 0)$ and $p_2 = (0 : 0 : 1)$. We have the following table:

	$m(f)$	$\mu(\hat{f})$	δ	Res_{σ_2}	$\text{Res}_{\sigma_1^2}$
p_1	1	0	0	1	$\frac{16}{3}$
p_2	2	3	0	6	$\frac{32}{3}$

Example 3.5 For $f(x, y) = y^n - x^m$ ($2 \leq n \leq m$), the singular points of $\mathcal{E}$ are $p_1 = (1 : 0 : 0)$ and $p_2 = (0 : 0 : 1)$.

	$m(f)$	$\mu(\hat{f})$	δ	Res_{σ_2}	$\text{Res}_{\sigma_1^2}$
p_1				$(n-1)(m-1)$	0
p_2	m	$m^2 + mn - 2m + n + 1$	0	$m^2 - mn + n$	$(m+1)^2$

Next we consider the compactification $\pi : X \rightarrow \mathbf{P}^2$ of f as constructed by D.T. Lê and C. Weber in [LW]. Following [LW], the set $A(f)$ of atypical values of f is expressed as $A(f) = D(f) \cup I(f)$, where $D(f)$ is the set of critical values of f and $I(f)$ is determined by the behavior of f at infinity. Then the compactification $\pi : X \rightarrow \mathbf{P}^2$ is obtained from $\mathbf{P}^2$ by a finite sequence of blowing-ups of “points at infinity” and has the following properties ([LW]):

- (1) X is a compact complex surface and π is a proper holomorphic map inducing a biholomorphic map of $X - \pi^{-1}(L_\infty)$ onto $\mathbf{P}^2 - L_\infty = \mathbf{C}^2$.
- (2) $\pi^{-1}(L_\infty)$ is a union of projective lines with normal crossings.
- (3) The meromorphic function $\varphi = \varphi_0 \circ \pi$ does not have indeterminacy points, where $\varphi_0 = \frac{f}{\zeta_0^d}$. Thus we may think of $\varphi : X \rightarrow \mathbf{P}^1$ as a holomorphic map.
- (4) For $\lambda \in \mathbf{C} - I(f)$, π gives an imbedded resolution of the singularities of the curve $C_\lambda : \hat{f} - \lambda \zeta_0^d = 0$ on L_∞ .

Moreover, if we denote by $\mathcal{A}$ and $\mathcal{A}_\infty$, respectively, the intersection graphs of the divisor $\pi^{-1}(L_\infty)$ and the pole divisor of φ ,

- (5) $\mathcal{A}$ is a connected tree and $\mathcal{A}_\infty$ is a connected sub-tree of $\mathcal{A}$.
- (6) Each connected component of $\mathcal{A} - \mathcal{A}_\infty$ is a bamboo which contains a unique dicritical component (a component of $\pi^{-1}(L_\infty)$ on which φ is not constant).

Let $\mathcal{E}$ be the foliation on X determined by φ and let $D^{(\infty)} = \sum_{i=1}^r m_i D_i^{(\infty)}$ be the pole divisor of φ . For simplicity, we assume that the critical points of φ (away from $|D^{(\infty)}|$, even on $|\pi^{-1}(L_\infty)| - |D^{(\infty)}|$) are all isolated. Thus for any finite value λ (even for an atypical value of f), the divisor $\varphi = \lambda$ is reduced. Note that this assumption is satisfied if each component of $\mathcal{A} - \mathcal{A}_\infty$ contains only one vertex (the dicritical component). Then there are two types of singularities of $\mathcal{E}$:

- (a) critical points of φ on $X - |D^{(\infty)}|$,
- (b) intersection points in $D^{(\infty)}$.

If p is a singular point of type (a), the residues are given by Lemma 2.3. Note that if p is in $X - \pi^{-1}(L_\infty) \simeq \mathbf{P}^2 - L_\infty = \mathbf{C}^2$, we have $\mu_p(\varphi) = \mu_{\pi(p)}(f)$.

Let p be a singular point of type (b). If p is the intersection point of $D_i^{(\infty)}$ and $D_j^{(\infty)}$, we have, by Lemma 2.4 and (2.16),

$$(3.6) \quad \text{Res}_{\sigma_1}(\mathcal{E}, p) = -\frac{(m_i - m_j)^2}{m_i m_j}, \quad \text{Res}_{\sigma_2}(\mathcal{E}, p) = 1.$$

If we again set $D = \sum_{i=1}^r (m_i + 1) D_i^{(\infty)}$ (the pole divisor of $d\varphi$), the residue formula (2.8) becomes

$$(3.7) \quad D^2 = - \sum_{1 \leq i < j \leq r} \frac{(m_i - m_j)^2}{m_i m_j} \delta_{ij},$$

where $\delta_{ij} = 1$, if $D_i^{(\infty)}$ meets $D_j^{(\infty)}$, and $\delta_{ij} = 0$ otherwise.

If we recall that the critical values of φ are atypical values of f ([LW]), we see that the sum of the residues for σ_2 over the singular points p of type (a) may be expressed as $\sum_{\lambda \in A(f)} \mu(X_\lambda)$, where X_λ denotes the (reduced) curve $\varphi = \lambda$ and $\mu(X_\lambda)$ its total Milnor number. Denoting by ℓ the number of intersection points in $D^{(\infty)}$, which is the number of 1-simplices in $\mathcal{A}_\infty$, we have, from (3.6) and Proposition 2.9, the following formula.

Theorem 3.8 *In the above situation, we have*

$$\sum_{\lambda \in A(f)} \mu(X_\lambda) + \ell = \chi(X) - \chi'(D).$$

In the above, the sum $\sum_{\lambda \in A(f)} \mu(X_\lambda)$ can also be written as

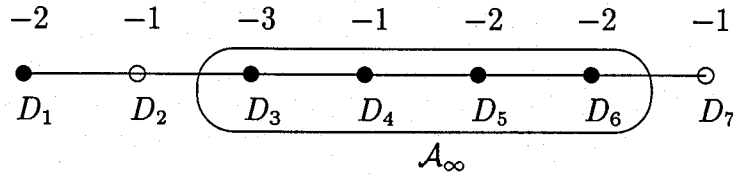
$$\sum_{p \in C(f)} \mu_p(f) + \sum_{p \in C(\varphi) \cap |\pi^{-1}(L_\infty)|} \mu_p(\varphi),$$

where $C(\varphi)$ denotes the critical set of φ restricted to $X - |D^{(\infty)}|$. Note that the value of φ at a point p in the second sum is in $I(f)$. If we denote by n the number of blowing-ups to obtain X (=number of vertices in $\mathcal{A}$) $- 1$, we have $\chi(X) = n + 3$. Recall also that $\chi'(D) = -(D^2 + K \cdot D)$ (Remark 2.10). We may represent K by a divisor with support in $\pi^{-1}(L_\infty)$.

Remark 3.9 *The formula in Theorem 3.8 may be thought of as a “Milnor number formula” in the presence of multiple (non-reduced) fibers. In fact, if $D^{(\infty)}$ is reduced, the formula coincides with the Milnor number formula ([F] Example 14.1.5) for the map $\varphi : X \rightarrow \mathbf{P}^1$, since, noting that $D = 2D^{(\infty)}$ in this case and recalling $(D^{(\infty)})^2 = 0$, we have $\chi'(D) = 2\chi'(D^{(\infty)}) = \chi(\mathbf{P}^1)\chi(X_t)$, where X_t is the (non-singular) curve $\varphi = t$ for $t \in \mathbf{C} - A(f)$.*

The following example of compactification is due to D.T. Lê. The residue formulas in this case are examined in [Sg].

Example 3.10 *Let $f(x, y) = x - x^2y$. The polynomial f has no critical points and the rational function $\varphi_0(\zeta_0 : \zeta_1 : \zeta_2) = \zeta_1(\zeta_1\zeta_2 - \zeta_0^2)/\zeta_0^3$ has indeterminacy points at $(0 : 0 : 1)$ and $(0 : 1 : 0)$. We see that $A(f) = I(f) = \{0\}$. The intersection graph $\mathcal{A}$ of $\pi^{-1}(L_\infty)$ is as follows:*



The integers in the first row denote the self-intersection numbers. D_2 and D_7 are the dicritical components, D_4 is the proper transform of L_∞ and the value of φ on D_1 is 0, which is atypical. We have $D^{(\infty)} = D_3 + 3D_4 + 2D_5 + D_6$ and

$K = -D_1 - 2D_2 - 2D_3 - 3D_4 - 2D_5 - D_6$. Hence $D = 2D_3 + 4D_4 + 3D_5 + 2D_6$. Thus we have $D^2 = -2$ and $K \cdot D = -2$. Therefore, $\chi(X) - \chi'(D) = 3 + 6 - 2 - 2 = 5$. On the other hand, the foliation defined by φ has 5 singular points p_i , $i = 1, \dots, 5$, where p_1 and p_2 are the critical points of φ and are on D_1 and p_i is the intersection point of D_i and D_{i+1} for $i = 3, 4, 5$. We compute the residues and obtain the following table:

	p_1	p_2	p_3	p_4	p_5
$\text{Res}_{\sigma_1^2}$	0	0	$-\frac{4}{3}$	$-\frac{1}{6}$	$-\frac{1}{2}$
Res_{σ_2}	1	1	1	1	1

Thus we see that (3.7) and the formula in Theorem 3.8 are satisfied.

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