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ABSTRACT

We show that the first exit times of small random perturbations of dynamical systems from a bounded domain $D(\subset \mathbf{R}^d)$ weakly converge to the explosion time of an explosive diffusion process and that the mean first exit times converge to the mean explosion time, as random perturbations disappear, when they are appropriately scaled. We consider the case when D contains only one equilibrium point o of dynamical systems and when o is polynomially unstabe and is repulsive.

1. INTRODUCTION.

Let us consider the following stochastic differential equation: for $t > 0$, $x \in \mathbf{R}^d$ and $\varepsilon > 0$,

$$\begin{aligned} dX^\varepsilon(t,x) &= b(X^\varepsilon(t,x))dt + \varepsilon^{1/2}\sigma(X^\varepsilon(t,x))dW(t), \\ X^\varepsilon(0,x) &= x, \end{aligned} \qquad (1.1).$$

where $b(\cdot) = (b^i(\cdot))_{i=1}^d : \mathbf{R}^d \mapsto \mathbf{R}^d$ is bounded and globally Lipschitz continuous, where $\sigma(\cdot) = (\sigma^{ij}(\cdot))_{i,j=1}^d : \mathbf{R}^d \mapsto M_d(\mathbf{R})$ is bounded, globally Lipschitz continuous, and uniformly nondegenerate, and where $W(\cdot)$ is a d-dimensional Wiener process[9]. $X^\varepsilon(t,x)$ can be considered as the small random perturbations of $X^0(t,x)$ for small ε[4,5,15].

Let $D(\subset \mathbf{R}^d)$ be a bounded domain which contains the origin o and suppose that $b(x) = o$ iff $x = o$. Then the asymptotic behavior, as $\varepsilon \to 0$, of the first exit time $\tau_D^\varepsilon(x)$ of $X^\varepsilon(t,x)$ from D defined by

$$\tau_D^\varepsilon(x) \equiv \inf\{t > 0; X^\varepsilon(t,x) \notin D\} \qquad (1.2).$$

has been studied by many authors.

When $X^0(t,x) \to o$ as $t \to \infty$ for all $x \in \overline{D}$, it is studied by Freidlin and Wentzell[15], Day[1], and Fleming and James[3].

When o is a hyperbolic equilibrium point[7] of $X^0(t,x)$ and when it is unstable, it is considered by Kifer[10], Mikami[13], and Day[2].

Mikami[14] considered the following case.

(A.0). $D(\subset \mathbf{R}^d)$ is a bounded domain which contains o. For any $x \in \overline{D}\backslash\{o\}$ there exists $s = s(x) \geq 0$ such that $X^0(t,x) \notin \overline{D}$ for $t > s$ and such that $X^0(t,x) \in D$ for $t < s$, and $X^0(t,x) \to o$ as $t \to -\infty$.

(A.1). There exist positive constants ℓ, C_0 and δ_0 such that $U_{\delta_0}(o) \equiv \{y \in \mathbf{R}^d; |y| < \delta_0\} \subset D$ and that

$$\begin{aligned} |b(x)| \leq C_0 |x|^{\ell+1} \quad & \text{for all } x \in D, \\ < x, b(x) > \geq |x|^{\ell+2}/C_0 \quad & \text{for all } x \text{ for which } |x| < \delta_0. \end{aligned} \tag{1.3}$$

Remark 1. Kifer[10], Mikami[13], and Day[2] considered the case when $|b(x)| \sim |x|$ as $x \to o$ and when $b(x)$ is differentiable at o. If $d = 1$, $D = (\alpha, \beta)$ $(\alpha < 0 < \beta)$ and

$$b(x) = \begin{cases} x(2 + \sin(1/x)) & ; \text{if } x \neq o, \\ o & ; \text{if } x = o, \end{cases}$$

then (A.0)-(A.1) hold with $\ell = 0$, $C_0 = 3$, and $|b(x)| \sim |x|$ as $x \to o$, but $b(x)$ is not differentiable at o. The case when $b(x)$ is not differentiable at o has not been studied yet. Mikami[14] and this paper consider the case $|b(x)| \sim |x|^{\ell+1}$ as $x \to o$ for $\ell > 0$ (see Example 1 in section 2). In this case $Db(o)$ is a zero matrix, and henceforth o is not a hyperbolic equilibrium point[7] of $X^0(t,x)$.

The following theorem is known.

Theorem 1. (I). Suppose that (A.0)-(A.1) hold. Then for any $\delta > 0$,

$$\lim_{\varepsilon \to 0} P(\varepsilon^{-(1-\delta)\ell/(\ell+2)} \leq \tau_D^\varepsilon(o) \leq \varepsilon^{-(1+\delta)\ell/(\ell+2)}) = 1 \tag{1.4}$$

(see Corollary 1.2[14]).

(II).[5] Suppose that (A.0) holds. Then for any $x \in D\backslash\{o\}$ and $\delta > 0$,

$$\lim_{\varepsilon \to 0} P(|\tau_D^\varepsilon(x) - \tau_D^0(x)| < \delta) = 1. \tag{1.5}$$

Theorem 2 (Theorem 1.3[14] and Corollary 1.4[14]). Suppose that (A.0)-(A.1) hold. Then

$$\lim_{\varepsilon \to 0} \{\log E[\tau_D^\varepsilon(o)]\} / \log(\varepsilon^{-\ell/(\ell+2)}) = 1, \tag{1.6}$$

and for any $x \in D\backslash\{o\}$,

$$\lim_{\varepsilon \to 0} E[\tau_D^\varepsilon(x)] = \tau_D^0(x). \tag{1.7.}$$

In this paper we show that $\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)$ weakly converge, as $\varepsilon \to 0$, to the explosion time $\tau(o)$ of an explosive diffusion process and that $E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)]$ converge, as $\varepsilon \to 0$, to $E[\tau(o)]$, under the stronger assumption than (A.0)-(A.1).

In section 2 we state our result and prove it in section 4, using lemmas which are stated and proved in section 3.

2. MAIN RESULT.

In this section we state our result. Let us first introduce the assumption.
(H.0)=(A.0).
(H.1). There exists $\ell > 0$ such that $b(x)$ can be written as follows; $b(x) = B(x)+R(x)$ for $x \in D$. Here $B(x)$ and $R(x)$ are locally Lipschitz continuous functions which satisfy the following;

$$B(tx) = t^{\ell+1}B(x) \quad \text{for all } t > 0 \text{ and } x \in \mathbf{R}^d, \tag{2.1.}$$

and there exist C_1 and $\gamma_1 \in (0, 1]$ such that

$$R(x) \le C_1|x|^{\ell+1+\gamma_1} \quad \text{for all } x \in \mathbf{R}^d. \tag{2.2.}$$

(H.2).

$$\inf_{|x|=1} < B(x), x >> 0. \tag{2.3.}$$

Remark 2. (H.1) holds with $\gamma_1 = 1$ when $\ell \in \mathbf{N}$, $b \in C_b^{\ell+2}(\mathbf{R}^d; \mathbf{R}^d)$, and

$$\partial^k b^i(o)/\partial x_1^{j_1} \cdots \partial x_d^{j_d} = 0$$

for all $i = 1, \cdots, d$, $k = 0, \cdots, \ell$ and $(j_1, \cdots, j_d)$ for which $j_1 + \cdots + j_d = k$ and $j_m \ge 0$ $(m = 1, \cdots, d)$, by Taylor's expansion. It also holds if $b(x) = |x|^\ell x (\notin C_b^{\ell+1}(\mathbf{R}^d; \mathbf{R}^d))$ for $\ell \in \mathbf{N}$.

Let $Y^\varepsilon(t, x)$ $(t \ge 0, x \in \mathbf{R}^d, \varepsilon > 0)$ be the solution of the following stochastic differential equation, up to the explosion time[9]:

$$\begin{aligned}&dY^\varepsilon(t,x)=B(Y^\varepsilon(t,x))dt+\varepsilon^{1/2}\sigma(o)dW(t),\\&Y^\varepsilon(0,x)=x,\end{aligned} \tag{2.4}$$

and put

$$\tau(x)\equiv\inf\{t>0;\ \sup_{0\le s\le t}|Y^1(s,x)|=\infty\} \tag{2.5}$$

Then we can prove the following.

Theorem 3. Suppose that (H.0)-(H.2) hold. Then $\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)$ weakly converges to $\tau(o)$ as $\varepsilon\to 0$, and

$$\lim_{\varepsilon\to 0}E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)]=E[\tau(o)]<\infty. \tag{2.6}$$

As a corollary to Theorem 3, we get the following by which we answer Professor A. A. Novikov's question.

Corollary 1. Suppose that (H.0)-(H.2) hold. Then $\tau_D^\varepsilon(o)/E[\tau_D^\varepsilon(o)]$ weakly converges to $\tau(o)/E[\tau(o)]$ as $\varepsilon\to 0$.

Remark 3. (H.2) implies that $P(\tau(x)<\infty)=1$ for all $x\in\mathbf{R}^d$ by Has'minskii's test(8,12). It is easy to see that (H.1)-(H.2) imply (A.1) in section 0, and hence that (1.4)-(1.7) holds under (H.0)-(H.2).

Let us give an example which shows that (H.1)-(H.2) is stronger than (A.1).

Example 1. Let $d=1$ and $D=(\alpha,\beta)$ $(\alpha<0<\beta)$ and put for $\ell>0$

$$b(x)=\begin{cases}|x|^\ell x(2+\sin(1/x)) & ;\text{ if } x\neq o \text{ and } \alpha<x<\beta,\\ o & ;\text{ if } x=o.\end{cases} \tag{2.7}$$

Then this $b(x)$ satisfies (A.1) with $C_0=3$, but does not satisfy (H.1). In fact, $b(x)/x^{\ell+1}$ does not converge as $x\downarrow 0$, though

$$\lim_{x\downarrow 0}b(x)/x^{\ell+1}=B(1) \tag{2.8}$$

if (H.1) holds.

3. LEMMAS.

In this section we state and prove lemmas.

Let us give some notation.

Put, for $t \geq 0$, $x \in \mathbf{R}^d$ and $\varepsilon > 0$,

$$\begin{aligned} Z_1^\varepsilon(t,x) &\equiv \varepsilon^{-1/(\ell+2)} X^\varepsilon(\varepsilon^{-\ell/(\ell+2)}t, \varepsilon^{1/(\ell+2)}x), \\ Z_2^\varepsilon(t,x) &\equiv \varepsilon^{-1/(\ell+2)} Y^\varepsilon(\varepsilon^{-\ell/(\ell+2)}t, \varepsilon^{1/(\ell+2)}x) \end{aligned} \tag{3.1}$$

(see Eqs (1.1) and (2.4)).

The following lemma plays a crucial role in the proof of Theorem 3.

Lemma 1. Suppose that (H.0)-(H.1) hold and that there exist $r, t > 0$, $x \in \mathbf{R}^d$ and $\varepsilon(< 1)$ for which $U_{r\varepsilon^{1/(\ell+2)}}(o) \subset D$ such that $\max(|Z_1^\varepsilon(s,x)|, |Z_2^\varepsilon(s,x)|) \leq r$ for all $s \in [0,t]$. Then there exist positive constants $C(r)$ and $C(\sigma)$, and a one-dimensional Wiener process $\tilde{W}(\cdot)$ such that for all $s \in [0,t]$,

$$\begin{aligned} &|Z_1^\varepsilon(s,x) - Z_2^\varepsilon(s,x)| \\ &\leq \exp(C(r)t)\varepsilon^{\gamma_1/(\ell+2)}(1 + tC_1 r^{\ell+1+\gamma_1} + C(\sigma)^2 r^2 t/2 \\ &\quad + C(\sigma) r \sup_{0 \leq u \leq t} |\tilde{W}(u)|). \end{aligned} \tag{3.2}$$

Proof. There exists a one-dimensional Wiener process $\tilde{W}(\cdot)$ such that for $s \in [0,t]$,

$$\begin{aligned} &|Z_1^\varepsilon(s,x) - Z_2^\varepsilon(s,x)| \\ &\leq \varepsilon^{1/(\ell+2)} + tC_1 r^{\ell+1+\gamma_1}\varepsilon^{\gamma_1/(\ell+2)} + C(\sigma)^2 r^2 t \varepsilon^{1/(\ell+2)}/2 \\ &\quad + C(\sigma) r \varepsilon^{1/(\ell+2)} \sup_{0 \leq u \leq s} |\tilde{W}(u)| + \int_0^s C(r)|Z^\varepsilon(u,x) - Z(u,x)|du \end{aligned} \tag{3.3}$$

from (H.1), which will be proved later. Here we put

$$C(r) \equiv \sup\{|B(x') - B(x)|/|x' - x|; x' \neq x, |x'|, |x| \leq r\},$$

$$C(\sigma) \equiv \sup\{[\sum_{i,j=1}^d |\sigma^{ij}(x) - \sigma^{ij}(o)|^2]^{1/2}/|x|; x \neq o, x \in \mathbf{R}^d\}.$$

By Gronwall's inequality[7], the proof is over from (3.3) since $\varepsilon < 1$ and $\gamma_1 \in (0,1]$ from (H.1).

Let us prove (3.3). For any $s \in [0, \varepsilon^{-\ell/(\ell+2)}t]$, putting $y = \varepsilon^{1/(\ell+2)}x$

$$
\begin{aligned}
&|X^\varepsilon(s,y) - Y^\varepsilon(s,y)| \qquad\qquad (3.4). \\
&\le \varepsilon^{2/(\ell+2)} + \int_0^s |B(X^\varepsilon(u,y)) - B(Y^\varepsilon(u,y))|du + \int_0^s |R(X^\varepsilon(u,y))|du \\
&\quad + \varepsilon^{1/2}| \int_0^s < (X^\varepsilon(u,y) - Y^\varepsilon(u,y))(|X^\varepsilon(u,y) - Y^\varepsilon(u,y)|^2 \\
&\qquad + \varepsilon^{4/(\ell+2)})^{-1/2}, [\sigma(X^\varepsilon(u,y)) - \sigma(o)]dW(u) > | \\
&\quad + (\varepsilon^{\ell/(\ell+2)}/2) \int_0^s \sum_{i,j=1}^d |\sigma^{ij}(X^\varepsilon(u,y)) - \sigma^{ij}(o)|^2 du,
\end{aligned}
$$

since by the Ito formula[9],

$$
\begin{aligned}
&(|X^\varepsilon(s,y) - Y^\varepsilon(s,y)|^2 + \varepsilon^{4/(\ell+2)})^{1/2} \\
&= \varepsilon^{2/(\ell+2)} \\
&\quad + \int_0^s < (X^\varepsilon(u,y) - Y^\varepsilon(u,y))(|X^\varepsilon(u,y) - Y^\varepsilon(u,y)|^2 + \varepsilon^{4/(\ell+2)})^{-1/2} \\
&\qquad , B(X^\varepsilon(u,y)) - B(Y^\varepsilon(u,y)) + R(X^\varepsilon(u,y)) > du \\
&\quad + \varepsilon^{1/2} \int_0^s < (X^\varepsilon(u,y) - Y^\varepsilon(u,y))(|X^\varepsilon(u,y) - Y^\varepsilon(u,y)|^2 \\
&\qquad + \varepsilon^{4/(\ell+2)})^{-1/2}, [\sigma(X^\varepsilon(u,y)) - \sigma(o)]dW(u) > \\
&\quad + (\varepsilon/2) \int_0^s \sum_{i,j=1}^d |\sigma^{ij}(X^\varepsilon(u,y)) - \sigma^{ij}(o)|^2 \\
&\qquad \times (|X^\varepsilon(u,y) - Y^\varepsilon(u,y)|^2 + \varepsilon^{4/(\ell+2)})^{-1/2} du \\
&\quad - (\varepsilon/2) \int_0^s |(\sigma(X^\varepsilon(u,y)) - \sigma(o))^*(X^\varepsilon(u,y) - Y^\varepsilon(u,y))|^2 \\
&\qquad \times (|X^\varepsilon(u,y) - Y^\varepsilon(u,y)|^2 + \varepsilon^{4/(\ell+2)})^{-3/2} du.
\end{aligned}
$$

Here we used that $X^\varepsilon(u,y) \subset D$ $(0 \le u \le s)$ from the assumption since $U_{r\varepsilon^{1/(\ell+2)}}(o) \subset D$, and denote by $\sigma(x)^*$ the transposed matrix of $\sigma(x)$.

From (3.4) and (3.5)-(3.7) below, we get (3.3).

For $s \in [0,t]$ and $y = \varepsilon^{1/(\ell+2)}x$

$$
\varepsilon^{-1/(\ell+2)} \int_0^{\varepsilon^{-\ell/(\ell+2)}s} |B(X^\varepsilon(u,y)) - B(Y^\varepsilon(u,y))|du \qquad (3.5).
$$

$$= \int_0^s |B(Z_1^\varepsilon(u,x)) - B(Z_2^\varepsilon(u,x))|du \quad \text{(from (2.1))}$$
$$\leq C(r) \int_0^s |Z_1^\varepsilon(u,x) - Z_2^\varepsilon(u,x)|du$$

from the assumption; and

$$\varepsilon^{-1/(\ell+2)} \int_0^{\varepsilon^{-\ell/(\ell+2)}s} |R(X^\varepsilon(u,y))|du \tag{3.6.}$$
$$\leq \varepsilon^{-(1+\ell)/(\ell+2)} s C_1 (r\varepsilon^{1/(\ell+2)})^{\ell+1+\gamma_1} = C_1 s r^{\ell+1+\gamma_1} \varepsilon^{\gamma_1/(\ell+2)}$$

from (2.2) and the assumption; and

$$\varepsilon \int_0^{\varepsilon^{-\ell/(\ell+2)}s} \sum_{i,j=1}^d |\sigma^{ij}(X^\varepsilon(u,y)) - \sigma^{ij}(o)|^2 du \tag{3.7.}$$
$$\leq \varepsilon\varepsilon^{-\ell/(\ell+2)} s C(\sigma)^2 (r\varepsilon^{1/(\ell+2)})^2 = \varepsilon^{4/(\ell+2)} s C(\sigma)^2 r^2$$

from the assumption.

Q.E.D.

Put for $x \in D$, and $r > 0$ and $\varepsilon > 0$ for which $r\varepsilon^{1/(\ell+2)} < \delta_0$ (see (A.1)),

$$T_{r,\varepsilon}(x) \equiv \{t > 0; |X^\varepsilon(t,x)| \geq r\varepsilon^{1/(\ell+2)}\}, \tag{3.8.}$$
$$T_\varepsilon(x) \equiv \{t > 0; |X^\varepsilon(t,x)| \geq \delta_0\}. \tag{3.9.}$$

Then the following lemma can be proved by the strong Markov property[9] of $X^\varepsilon(t,x)$.

Lemma 2. Suppose that (H.0) holds. Then for $x \in D$, $T > 0$, $\delta \in (0,T)$, $r > 0$ and $\varepsilon > 0$ for which $r\varepsilon^{1/(\ell+2)} < \delta_0$,

$$P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(x) \leq T) \tag{3.10.}$$
$$\geq \inf_{|y|\leq r} P(\sup_{0\leq s\leq T-\delta} |Z_1^\varepsilon(s,y)| \geq r)$$
$$\times \inf_{|y|\geq r\varepsilon^{1/(\ell+2)}} P(\sup_{0\leq s\leq \varepsilon^{-\ell/(\ell+2)}\delta/2} |X^\varepsilon(s,y)| \geq \delta_0)$$
$$\times \inf_{|y|\geq\delta_0} P(\tau_D^\varepsilon(y) \leq \varepsilon^{-\ell/(\ell+2)}\delta/2).$$

In particular,

$$
\begin{aligned}
&P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) \le T) && (3.11).\\
&\ge P(\sup_{0\le s\le T-\delta} |Z_1^\varepsilon(s,o)| \ge r)\\
&\quad \times \inf_{|y|\ge r\varepsilon^{1/(\ell+2)}} P(\sup_{0\le s\le \varepsilon^{-\ell/(\ell+2)}\delta/2} |X^\varepsilon(s,y)| \ge \delta_0)\\
&\quad \times \inf_{|y|\ge\delta_0} P(\tau_D^\varepsilon(y) \le \varepsilon^{-\ell/(\ell+2)}\delta/2).
\end{aligned}
$$

Proof. For $x \in D$, $T > 0$, $\delta \in (0,T)$, $r > 0$ and $\varepsilon > 0$ for which $r\varepsilon^{1/(\ell+2)} < \delta_0$,

$$
\begin{aligned}
&P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(x) \le T) && (3.12).\\
&\ge P(\sup_{0\le s\le\varepsilon^{-\ell/(\ell+2)}(T-\delta)} |X^\varepsilon(s,x)| \ge r\varepsilon^{1/(\ell+2)},\\
&\qquad \sup_{T_{r,\varepsilon}(x)\le s\le T_{r,\varepsilon}(x)+\varepsilon^{-\ell/(\ell+2)}\delta/2} |X^\varepsilon(s,x)| \ge \delta_0,\\
&\qquad \tau_D^\varepsilon(X^\varepsilon(T_\varepsilon(X^\varepsilon(T_{r,\varepsilon}(x),x)),X^\varepsilon(T_{r,\varepsilon}(x),x))) \le \varepsilon^{-\ell/(\ell+2)}\delta/2).
\end{aligned}
$$

By the strong Markov property of $X^\varepsilon(t,x)$, the proof is over.

Q.E.D.

For the sake of completeness, we give the proof of the two lemmas which will be used in Lemma 5.

Lemma 3. Let $f(t)$, $u(t)$ $(t \ge 0)$ be positive continuous functions such that there exist positive constants C and T for which the following holds; for all $t \in [0,T]$

$$f(t) \ge C + \int_0^t u(s)f(s)ds. \qquad (3.13).$$

Then for all $t \in [0,T]$,

$$f(t) \ge C\exp(\int_0^t u(s)ds). \qquad (3.14).$$

Proof. From (3.13), for all $t \in [0,T]$

$$d[\int_0^t u(s)f(s)ds\exp(-\int_0^t u(s)ds)]/dt \tag{3.15}$$
$$= u(t)\exp(-\int_0^t u(s)ds)(f(t)-\int_0^t u(s)f(s)ds)$$
$$\geq Cd[-\exp(-\int_0^t u(s)ds)]/dt.$$

Integrating both sides of (3.15) in t, we get (3.14) from (3.13).

Q. E. D.

Lemma 4. Let $f(t)$ $(t \geq 0)$ be a positive continuous function such that there exist positive constants C', $C"$ and T for which the following holds; for all $t \in [0, T]$

$$f(t) \geq C'\exp(C"\int_0^t f(s)ds). \tag{3.16}$$

Then

$$C'C"T < 1. \tag{3.17}$$

Proof. From (3.16),

$$-d[\exp(-C"\int_0^t f(s)ds)]/dt \geq C'C". \tag{3.18}$$

Integrating both sides of (3.18) in t, we get

$$1 > 1 - \exp(-C"\int_0^T f(s)ds) \geq C'C"T. \tag{3.19}$$

Q.E.D.

From Lemmas 3 and 4, we get the following which will be used in section 4.

Lemma 5. Suppose that (A.0)-(A.1) hold. Then for any $\delta > 0$, $r \geq (2^{\ell+1}C_0/[\ell\delta])^{1/\ell}$ and y for which $|y| \geq r\varepsilon^{1/(\ell+2)}$,

$$P(\sup_{0\leq s\leq \varepsilon^{-\ell/(\ell+2)}\delta/2} |X^\varepsilon(s,y)| < \delta_0) \tag{3.20}$$
$$\leq P(\sup\{|\varepsilon^{1/2}\int_0^t (|X^\varepsilon(s,y)|^2+1/n)^{-1/2} < X^\varepsilon(s,y)$$
$$,\sigma(X^\varepsilon(s,y))dW(s) > |; 0 \leq t \leq \varepsilon^{-\ell/(\ell+2)}\delta/2, n \geq 1\} \geq |y|/2).$$

Proof.

$$P(\sup_{0\le s\le \varepsilon^{-\ell/(\ell+2)}\delta/2}|X^\varepsilon(s,y)|<\delta_0) \qquad (3.21).$$

$$\le P(\sup\{|\varepsilon^{1/2}\int_0^t (|X^\varepsilon(s,y)|^2+1/n)^{-1/2} < X^\varepsilon(s,y)$$

$$,\sigma(X^\varepsilon(s,y))dW(s)>|;0\le t\le \varepsilon^{-\ell/(\ell+2)}\delta/2, n\ge 1\}\ge |y|/2)$$

$$+P(\sup_{0\le s\le \varepsilon^{-\ell/(\ell+2)}\delta/2}|X^\varepsilon(s,y)|<\delta_0, (|X^\varepsilon(t,y)|^2+1/n)^{1/2}$$

$$\ge |y|/2+\int_0^t < X^\varepsilon(s,y), b(X^\varepsilon(s,y)) > (|X^\varepsilon(s,y)|^2+1/n)^{-1/2}ds$$

$$\text{for all } n\ge 1, t\in[0,\varepsilon^{-\ell/(\ell+2)}\delta/2]).$$

In fact, if $\sup_{0\le s\le \varepsilon^{-\ell/(\ell+2)}\delta/2}|X^\varepsilon(s,y)|<\delta_0$, then for $n\ge 1$ and $t\in[0,\varepsilon^{-\ell/(\ell+2)}\delta/2]$, by the Ito formula[9],

$$\begin{aligned}
&(|X^\varepsilon(t,y)|^2+1/n)^{1/2}\\
&=(|y|^2+1/n)^{1/2}\\
&\quad+\int_0^t < X^\varepsilon(s,y), b(X^\varepsilon(s,y)) > (|X^\varepsilon(s,y)|^2+1/n)^{-1/2}ds\\
&\quad+\varepsilon^{1/2}\int_0^t (|X^\varepsilon(s,y)|^2+1/n)^{-1/2} < X^\varepsilon(s,y), \sigma(X^\varepsilon(s,y))dW(s) >\\
&\quad+(\varepsilon/2)\int_0^t (|X^\varepsilon(s,y)|^2+1/n)^{-1/2}[Trace[a(X^\varepsilon(s,y))]\\
&\qquad - < a(X^\varepsilon(s,y))X^\varepsilon(s,y), X^\varepsilon(s,y) > /(|X^\varepsilon(s,y)|^2+1/n)]ds\\
&\ge |y| - |\varepsilon^{1/2}\int_0^t (|X^\varepsilon(s,y)|^2+1/n)^{-1/2} < X^\varepsilon(s,y), \sigma(X^\varepsilon(s,y))dW(s) > |\\
&\quad+\int_0^t < X^\varepsilon(s,y), b(X^\varepsilon(s,y)) > (|X^\varepsilon(s,y)|^2+1/n)^{-1/2}ds.
\end{aligned}$$

Here we put $a(x)=\sigma(x)\sigma(x)^*$.

Let us show that the second probability on the right hand side of (3.21) is zero for $r\ge (2^{\ell+1}C_0/[\ell\delta])^{1/\ell}$ and y for which $|y|\ge r\varepsilon^{1/(\ell+2)}$.

Suppose that $\sup_{0\le s\le \varepsilon^{-\ell/(\ell+2)}\delta/2}|X^\varepsilon(s,y)|<\delta_0$ and that the following holds for all $n\ge 1$ and $t\in[0,\varepsilon^{-\ell/(\ell+2)}\delta/2]$;

$$\begin{aligned}
&(|X^\varepsilon(t,y)|^2+1/n)^{1/2}\\
&\ge |y|/2+\int_0^t < X^\varepsilon(s,y), b(X^\varepsilon(s,y)) > (|X^\varepsilon(s,y)|^2+1/n)^{-1/2}ds.
\end{aligned}$$

Then we get for $t \in [0, \varepsilon^{-\ell/(\ell+2)}\delta/2]$, by Fatou's lemma,

$$|X^\varepsilon(t,y)| \geq |y|/2 + \int_0^t C_0^{-1}|X^\varepsilon(s,y)|^{\ell+1}ds$$

from (A.1), and henceforth

$$|X^\varepsilon(t,y)| \geq (|y|/2)\exp(\int_0^t C_0^{-1}|X^\varepsilon(s,y)|^\ell ds) \qquad (3.22).$$

from Lemma 3.

From (3.22), we get for $t \in [0, \varepsilon^{-\ell/(\ell+2)}\delta/2]$,

$$|X^\varepsilon(t,y)|^\ell \geq (|y|/2)^\ell \exp(\ell C_0^{-1} \int_0^t |X^\varepsilon(s,y)|^\ell ds),$$

and from Lemma 4, for y for which $|y| \geq r\varepsilon^{1/(\ell+2)}$,

$$1 > (|y|/2)^\ell \ell C_0^{-1}\varepsilon^{-\ell/(\ell+2)}\delta/2 \geq [\ell\delta/(2^{\ell+1}C_0)]r^\ell$$

which is a contradiction since $r \geq (2^{\ell+1}C_0/[\ell\delta])^{1/\ell}$.

Q.E.D.

4. PROOF OF MAIN RESULT.

In this section, we prove Theorem 3 in section 2. We devide the proof into the four steps; for any $T > 0$,

$$\limsup_{\varepsilon\to 0} P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) \le T) \le P(\tau(o) \le T), \qquad (4.1).$$

$$\liminf_{\varepsilon\to 0} P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) \le T) \ge P(\tau(o) \le T), \qquad (4.2).$$

$$\liminf_{\varepsilon\to 0} E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)] \ge E[\tau(o)], \qquad (4.3).$$

$$\limsup_{\varepsilon\to 0} E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)] \le E[\tau(o)] < \infty. \qquad (4.4).$$

Let us first prove (4.1).

Proof of (4.1). For $r > 0$ and $\varepsilon(< 1)$ for which $U_{r\varepsilon^{1/(\ell+2)}}(o) \subset D$,

$$\begin{aligned} &P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) \le T) \qquad (4.5).\\ &\le P(\sup_{0\le s\le \varepsilon^{-\ell/(\ell+2)}T} |X^\varepsilon(s,o)| \ge r\varepsilon^{1/(\ell+2)})\\ &\le P(\sup_{0\le s\le T} |Z_2^\varepsilon(s,o)| \ge r-1)\\ &\quad + P(\sup_{0\le s\le T} |Z_1^\varepsilon(s,o)| \ge r,\ \sup_{0\le s\le T} |Z_2^\varepsilon(s,o)| \le r-1) \end{aligned}$$

(see Eq (3.1)).

The first probability in the last part of (4.5) is shown to converge to $P(\tau \le T)$ as $r \to \infty$;

$$P(\sup_{0\le s\le T} |Z_2^\varepsilon(s,o)| \ge r-1) = P(\sup_{0\le s\le T} |Y^1(s,o)| \ge r-1) \to P(\tau(o) \le T), \qquad (4.6).$$

as $r \to \infty$ since the probability law of $Z_2^\varepsilon(s,o)$ $(0 \le s)$ is the same as that of $Y^1(s,o)$ $(0 \le s)$ from (H.1) (see Eqs (2.4) and (3.1)).

The second probability in the last part of (4.5) converges to 0 as $\varepsilon \to 0$ for any r and $T > 0$. Let us prove this. Put for r, $\varepsilon > 0$, $y \in \mathbf{R}^d$ and $i = 1, 2$

$$\tau_r^{i,\varepsilon}(y) \equiv \{t > 0; |Z_i^\varepsilon(t,y)| \ge r\} \qquad (4.7).$$

(see Eq (3.1)). Suppose that $U_{r\varepsilon^{1/(\ell+2)}}(o) \subset D$. Then from Lemma 1, there exists a one-dimensional Wiener process $\tilde{W}(\cdot)$ such that

$$
\begin{aligned}
&P(\sup_{0\le s\le T}|Z_1^\varepsilon(s,o)|\ge r,\ \sup_{0\le s\le T}|Z_2^\varepsilon(s,o)|\le r-1) \qquad (4.8).\\
&\le P(\tau_r^{1,\varepsilon}(o)\le T, 1\le |Z_1^\varepsilon(\tau_r^{1,\varepsilon}(o),o)-Z_2^\varepsilon(\tau_r^{1,\varepsilon}(o),o)|)\\
&\le P(1\le \exp(C(r)T)\varepsilon^{\gamma_1/(\ell+2)}(1+TC_1r^{\ell+1+\gamma_1}+C(\sigma)^2r^2T/2\\
&\quad + C(\sigma)r\sup_{0\le u\le T}|\tilde{W}(u)|))\\
&\to 0 \quad \text{as } \varepsilon\to 0.
\end{aligned}
$$

Q.E.D.

Next we prove (4.2).

Proof of (4.2). From Lemma 2, for $\delta\in(0,T)$, $r>0$ and $\varepsilon>0$ for which $r\varepsilon^{1/(\ell+2)}<\delta_0$,

$$
\begin{aligned}
&P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)\le T) \qquad (4.9).\\
&\ge P(\sup_{0\le s\le T-\delta}|Z_1^\varepsilon(s,o)|\ge r)\\
&\quad\times \inf_{|y|\ge r\varepsilon^{1/(\ell+2)}} P(\sup_{0\le s\le \varepsilon^{-\ell/(\ell+2)}\delta/2}|X^\varepsilon(s,y)|\ge\delta_0)\\
&\quad\times \inf_{|y|\ge\delta_0} P(\tau_D^\varepsilon(y)\le\varepsilon^{-\ell/(\ell+2)}\delta/2).
\end{aligned}
$$

The following (4.10)-(4.12) which are proved later complete the proof.

$$
\liminf_{\delta\to0}\liminf_{r\to\infty}\liminf_{\varepsilon\to0}P(\sup_{0\le s\le T-\delta}|Z_1^\varepsilon(s,o)|\ge r)\ge P(\tau(o)\le T) \qquad (4.10).
$$

; and for any $\delta\in(0,T)$,

$$
\lim_{r\to\infty}\liminf_{\varepsilon\to0}\inf_{|y|\ge r\varepsilon^{1/(\ell+2)}}P(\sup_{0\le s\le\varepsilon^{-\ell/(\ell+2)}\delta/2}|X^\varepsilon(s,y)|\ge\delta_0)=1 \qquad (4.11).
$$

; and for any $\delta\in(0,T)$,

$$
\lim_{\varepsilon\to0}\inf_{|y|\ge\delta_0}P(\tau_D^\varepsilon(y)\le\varepsilon^{-\ell/(\ell+2)}\delta/2)=1. \qquad (4.12).
$$

Let us first prove Eq (4.12). From (H.0), there exist $\delta_1>0$ and $T_1>0$ such that

$$\inf_{|x|\geq\delta_0} \sup\{dist(X^0(t,x),D); 0\leq t\leq T_1\} \geq 2\delta_1. \qquad (4.13).$$

Therefore

$$\begin{aligned} &P(\tau_D^\varepsilon(y) \leq \varepsilon^{-\ell/(\ell+2)}\delta/2) \qquad (4.14).\\ &\geq P(\sup_{0\leq t\leq T_1} |X^\varepsilon(s,y) - X^0(s,y)| < \delta_1) \to 1 \quad \text{as } \varepsilon\to 0,\end{aligned}$$

uniformly in y for which $|y| \geq \delta_0^{(5)}$.

Next we prove (4.10).

For any r and $\delta \in (0,T)$

$$\begin{aligned} &P(\sup_{0\leq s\leq T-\delta} |Z_1^\varepsilon(s,o)| \geq r) \qquad (4.15).\\ &\geq P(\sup_{0\leq s\leq T-\delta} |Z_2^\varepsilon(s,o)| \geq r+1)\\ &\quad - P(\sup_{0\leq s\leq T-\delta} |Z_2^\varepsilon(s,o)| \geq r+1, \sup_{0\leq s\leq T-\delta} |Z_1^\varepsilon(s,o)| < r).\end{aligned}$$

The first probability on the right hand side of (4.15) converges to $P(\tau(o) \leq T)$ as $r\to\infty$, and then $\delta \to 0$, in the same way as in (4.6).

Let us prove that the second probability on the right hand side of (4.15) converges to 0 as $\varepsilon \to 0$ for any $r > 0$ and $\delta \in (0,T)$, which can be done in the same way as in (4.8). Suppose that $U_{r\varepsilon^{1/(\ell+2)}}(o) \subset D$. Then from Lemma 1, there exists a one-dimensional Wiener process $\tilde{W}(\cdot)$ such that

$$\begin{aligned} &P(\sup_{0\leq s\leq T-\delta} |Z_2^\varepsilon(s,o)| \geq r+1, \sup_{0\leq s\leq T-\delta} |Z_1^\varepsilon(s,o)| < r) \qquad (4.16).\\ &\leq P(\tau_{r+1}^{2,\varepsilon}(o) \leq T, 1 \leq |Z_2^\varepsilon(\tau_{r+1}^{2,\varepsilon}(o),o) - Z_1^\varepsilon(\tau_{r+1}^{2,\varepsilon}(o),o)|)\\ &\leq P(1 \leq \exp(C(r+1)T)\varepsilon^{\gamma_1/(\ell+2)}(1+TC_1[r+1]^{\ell+1+\gamma_1}\\ &\qquad + C(\sigma)^2(r+1)^2T/2 + C(\sigma)(r+1)\sup_{0\leq u\leq T}|\tilde{W}(u)|))\\ &\to 0 \quad \text{as } \varepsilon\to 0.\end{aligned}$$

Finally we prove Eq (4.11) from Lemma 5. Since (H.1)-(H.2) is stronger than (A.1) (see Remark 3 in section 2), we can suppose that (A.1) holds.

For $C_0 > 0$ in (A.1), $r \geq (2^{\ell+1}C_0/[\ell\delta])^{1/\ell}$ and y for which $|y| \geq r\varepsilon^{1/(\ell+2)}$,

$$
\begin{aligned}
&P(\sup_{0\le s\le \varepsilon^{-\ell/(\ell+2)}\delta/2} |X^\varepsilon(s,y)| < \delta_0) \qquad (4.17).\\
&\le P(\sup\{|\varepsilon^{1/2}\int_0^t (|X^\varepsilon(s,y)|^2+1/n)^{-1/2} < X^\varepsilon(s,y), \sigma(X^\varepsilon(s,y))dW(s) > |\\
&\quad ; 0\le t\le \varepsilon^{-\ell/(\ell+2)}\delta/2, n\ge 1\} \ge |y|/2)
\end{aligned}
$$

from Lemma 5; and by the time change[9], there exists a one dimensional Wiener process $\tilde{W}$ such that

$$
\begin{aligned}
&P(\sup\{|\varepsilon^{1/2}\int_0^t (|X^\varepsilon(s,y)|^2+1/n)^{-1/2} < X^\varepsilon(s,y) \qquad (4.18).\\
&\qquad , \sigma(X^\varepsilon(s,y))dW(s) > |; 0\le t\le \varepsilon^{-\ell/(\ell+2)}\delta/2, n\ge 1\} \ge |y|/2)\\
&\le P(C_1(\sigma)(2\delta)^{1/2}/r \sup_{0\le s\le 1} |\tilde{W}(s)| \ge 1)\\
&\to 0 \quad \text{as } r\to\infty.
\end{aligned}
$$

Here we put $C_1(\sigma) = \sup_{x\in\mathbf{R}^d}(\sum_{i,j=1}^d \sigma^{ij}(x)^2)^{1/2}$.

Q.E.D.

Let us prove (4.3).

Proof of (4.3). For any $T > 0$,

$$
\liminf_{\varepsilon\to 0} E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)] \ge \int_0^T P(\tau(o) > t)dt \qquad (4.19).
$$

from (4.1) by Fatou's lemma, since

$$
\begin{aligned}
E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)] &= \int_0^\infty P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > t)dt \qquad (4.20).\\
&\ge \int_0^T P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > t)dt.
\end{aligned}
$$

Let $T\to\infty$ in (4.19). Then we get (4.3).

Q.E.D.

Finally we prove (4.4).

Proof of (4.4). We only have to show the following to complete the proof;

$$\lim_{n\to\infty} \limsup_{\varepsilon\to 0} E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) - n; \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > n] = 0 \qquad (4.21).$$

since for any $n \in \mathbf{N}$, in the same way as in (4.20),

$$\begin{aligned}
&E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)] &(4.22).\\
&= E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o); \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > n]\\
&\qquad - nP(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > n) + \int_0^n P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > t)dt\\
&= E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) - n; \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > n] + \int_0^n P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > t)dt
\end{aligned}$$

; and

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_0^n P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > t)dt &(4.23).\\
&\le \int_0^n P(\tau(o) > t)dt \quad \text{(from (4.2) by Fatou's lemma)}\\
&\to E[\tau(o)] \quad \text{as } n \to \infty
\end{aligned}$$

; and for $n \in \mathbf{N}$, from (4.3) and (4.22)-(4.23),

$$\begin{aligned}
E[\tau(o)] &\le \liminf_{\varepsilon\to 0} E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)] &(4.24).\\
&\le \limsup_{\varepsilon\to 0} E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) - n; \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > n]\\
&\qquad + \int_0^n P(\tau(o) > t)dt.
\end{aligned}$$

To prove (4.21), we only have to prove

$$\limsup_{\varepsilon\to 0} [\sup_{x\in D} P(1 < \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(x))] < 1 \qquad (4.25).$$

since

$$\begin{aligned}
&E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) - n; \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > n]\\
&= \sum_{k=n}^{\infty} E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) - n; k < \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) \le k+1]
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=n}^{\infty}(k+1-n)P(k<\varepsilon^{\ell/(\ell+2)}\tau_D^{\varepsilon}(o)\leq k+1)\\
&\leq \sum_{k=n}^{\infty}P(k<\varepsilon^{\ell/(\ell+2)}\tau_D^{\varepsilon}(o))\\
&\leq \sum_{k=n}^{\infty}\{\sup_{x\in D}P(1<\varepsilon^{\ell/(\ell+2)}\tau_D^{\varepsilon}(x))\}^k\\
&=\{\sup_{x\in D}P(1<\varepsilon^{\ell/(\ell+2)}\tau_D^{\varepsilon}(x))\}^n(1-\sup_{x\in D}P(1<\varepsilon^{\ell/(\ell+2)}\tau_D^{\varepsilon}(x)))^{-1}.
\end{aligned}
$$

Though (4.25) can be shown to be true in the same way as in the proof of (4.2), we prove it for the sake of completeness.

From Lemma 2, for $r>0$ and $\varepsilon>0$ for which $r\varepsilon^{1/(\ell+2)}<\delta_0$, and $x\in D$,

$$
\begin{aligned}
&P(\varepsilon^{\ell/(\ell+2)}\tau_D^{\varepsilon}(x)\leq 1) \qquad (4.26).\\
&\geq \inf_{|y|\leq r}P(\sup_{0\leq s\leq 1/2}|Z_1^{\varepsilon}(s,y)|\geq r)\\
&\quad\times \inf_{|y|\geq r\varepsilon^{1/(\ell+2)}}P(\sup_{0\leq s\leq \varepsilon^{-\ell/(\ell+2)}/4}|X^{\varepsilon}(s,y)|\geq\delta_0)\\
&\quad\times \inf_{|y|\geq\delta_0}P(\tau_D^{\varepsilon}(y)\leq\varepsilon^{-\ell/(\ell+2)}/4).
\end{aligned}
$$

Take sufficiently large $r_0>0$ so that

$$
\liminf_{\varepsilon\to 0}\inf_{|y|\geq r_0\varepsilon^{1/(\ell+2)}}P(\sup_{0\leq s\leq\varepsilon^{-\ell/(\ell+2)}/4}|X^{\varepsilon}(s,y)|\geq\delta_0)\geq 1/2, \qquad (4.27).
$$

which is possible from (4.11).

From (4.27) and (4.12), we only have to show the following;

$$
\liminf_{\varepsilon\to 0}\inf_{|y|\leq r_0}P(\sup_{0\leq s\leq 1/2}|Z_1^{\varepsilon}(s,y)|\geq r_0)>0, \qquad (4.28).
$$

which can be proved in the same way as in (4.15)-(4.16).

In fact, for any y for which $|y|\leq r_0$

$$
\begin{aligned}
&P(\sup_{0\leq s\leq 1/2}|Z_1^{\varepsilon}(s,y)|\geq r_0) \qquad (4.29).\\
&\geq P(\sup_{0\leq s\leq 1/2}|Z_2^{\varepsilon}(s,y)|\geq r_0+1)\\
&\quad -P(\sup_{0\leq s\leq 1/2}|Z_2^{\varepsilon}(s,y)|\geq r_0+1,\ \sup_{0\leq s\leq 1/2}|Z_1^{\varepsilon}(s,y)|<r_0).
\end{aligned}
$$

The first probability on the right hand side of (4.29) can be shown to be bounded from below by a positive constant, uniformly in $\varepsilon > 0$ and y for which $|y| \leq r_0$;

$$\begin{aligned}
&P(\sup_{0\leq s\leq 1/2} |Z_2^\varepsilon(s,y)| \geq r_0 + 1) \\
&= P(\sup_{0\leq s\leq 1/2} |Y^1(s,y)| \geq r_0 + 1) \\
&\geq \inf_{|z|\leq r_0} P(\sup_{0\leq s\leq 1/2} |Y^1(s,z)| \geq r_0 + 1) > 0.
\end{aligned}$$

This is true since $u(t,x) \equiv P(\sup_{0\leq s\leq t} |Y^1(s,x)| \geq r_0 + 1)$ is a smooth solution of the following P.D.E. from (H.1) (see p. 383[6], problems 1 and 2);

$$\begin{aligned}
\partial u(t,x)/\partial t &= [\sum_{i,j=1}^{d} a^{ij}(o)\partial^2 u(t,x)/\partial x_i \partial x_j]/2 \\
&\quad + \sum_{i=1}^{d} B^i(x)\partial u(t,x)/\partial x_i \quad \text{for } t > 0, |x| < r_0 + 1, \\
u(0,x) &= 0 \quad \text{for } |x| < r_0 + 1, \\
u(t,x) &= 1 \quad \text{for } t \geq 0, |x| = r_0 + 1.
\end{aligned}$$

Let us prove that the second probability on the right hand side of (4.29) converges to 0 as $\varepsilon \to 0$, uniformly in y for which $|y| \leq r_0$. From Lemma 1, there exists a one-dimensional Wiener process $\tilde{W}(\cdot)$ such that for y for which $|y| \leq r_0$

$$\begin{aligned}
&P(\sup_{0\leq s\leq 1/2} |Z_2^\varepsilon(s,y)| \geq r_0 + 1, \sup_{0\leq s\leq 1/2} |Z_1^\varepsilon(s,y)| < r_0) \\
&\leq P(\tau_{r_0+1}^{2,\varepsilon}(y) \leq 1/2, 1 \leq |Z_2^\varepsilon(\tau_{r_0+1}^{2,\varepsilon}(y),y) - Z_1^\varepsilon(\tau_{r_0+1}^{2,\varepsilon}(y),y)|) \\
&\leq P(1 \leq \exp(C(r_0+1)/2)\varepsilon^{\gamma_1/(\ell+2)}(1 + C_1[r_0+1]^{\ell+1+\gamma_1}/2 \\
&\qquad + C(\sigma)^2(r_0+1)^2/4 + C(\sigma)(r_0+1)\sup_{0\leq u\leq 1/2} |\tilde{W}(u)|) \\
&\to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Q.E.D.

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