BIFURCATIONS OF SHOCK WAVES FOR VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS OF ONE SPACE VARIABLE

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BIFURCATIONS OF SHOCK WAVES FOR VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS OF ONE SPACE VARIABLE

SHYUICHI IZUMIYA[1] AND GEORGIOS T. KOSSIORIS[2]

1. Introduction

In this paper we study the generation and propagation of singularities (shock waves) for viscosity solutions to the Cauchy problem of Hamilton-Jacobi equations in one space variable

$$\text{(P)} \qquad \begin{cases} \dfrac{\partial y}{\partial t} + H(\dfrac{\partial y}{\partial x}) = 0, & t > 0 \\ y(0,x) = \phi(x), & x \in \mathbb{R}, \end{cases}$$

where H and ϕ are C^∞-functions and H has isolated critical points. We follow the method introduced by M. Tsuji to study shock waves for first order equations ([24], [25], [26]) and Monge Ampère equations ([27]).

Hamilton-Jacobi equations play an important role in various fields e.g., calculus of variations (see e.g., [21]), optimal control theory (see e.g., [10]) and differential games (see e.g., [9] and references cited therein). The theory of *viscosity solutions* (see [5]) has provided the right weak setting for the study of (P). Existence and uniqueness of the solution of (P) in the viscosity sense have been established in [6]. For small time t the viscosity solution of (P) is smooth and it is determined by the classical method of characteristics until the first critical time that characteristics cross. After the characteristics cross, the continuous viscosity solution develops *shock waves* i.e., discontinuities in the gradient of the solution. The characteristic manifold given by the method of characteristics is defined beyond the first critical time and gives rise to the notion of the *geometric solution* of (P). The geometric solution coincides with the smooth viscosity solution until the first critical time and it becomes multi-valued beyond this time. M. Tsuji in ([24], [25], [26]) has studied the multi-valuedness of the geometric solution in the neighborhood of the first critical time and he constructed the weak solution by selecting a single-valued branch of the geometric solution in the case of a convex Hamiltonian for $x \in \mathbb{R}^n$, $n = 1, 2$. The shock surface corresponds to the intersection of the branches of the graph of the multi-valued geometric solution.

Following the spirit of [24] and [25], Nakane in [22] has constructed the weak semi-concave solution past the first critical time in case that H is convex, for $n \geq 1$. In the case of a general Hamiltonian, the viscosity solution of (P) in a neighborhood of the first critical time

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has been constructed in [20] (see also [19], [22]). The global in time geometric solution y of (P), for $x \in \mathbb{R}^n$, $n \geq 1$, has been recently defined in ([15], [16]) in the framework of *one-parameter Legendrian unfoldings*. The corresponding classification of the multi-valuedness of the geometric solution y has also been given in [15] (see also [17]). In [4], Bogaevskii has shown that the potential solution of the Burgers system with vanishing viscosity is given by the minimum function of a certain family of smooth functions and he has given a classification for $n = 1, 2, 3$. This solution corresponds to the viscosity solution of the Hamilton-Jacobi equation when the Hamiltonian is given by $H(p_1, \dots p_n) = \frac{1}{2}p_1^2 + \cdots + \frac{1}{2}p_n^2$. His method applies also to the case that the Hamiltonian is uniformly convex or concave. However, the situation is quite more complicated for general Hamiltonians.

In order to construct the viscosity solution away from the first critical time we follow the evolution of the intersection of the branches defining the shock. We obtain the global in time structure of the viscosity solution of (P) by solving all generic local Riemann problems at a subsequent time t. This was achieved in [19] in the case $n = 1$ under the assumption that H has only one inflection point by using techniques that are limited to this case. In the present paper we combine the classification results of [15] with techniques of viscosity solutions to construct the solutions to all possible Riemann problems that generically appear as t increases. We make no assumption on the number of inflection points of H. The case $n > 1$ will be studied in a future work.

There is an analogue of the entropy condition for conservation laws that guarantees that a piecewise smooth single-valued continuous function is the viscosity solution of (P), see Theorem 3.1. We will refer to this condition as *the viscosity criterion*. It is shown that the viscosity criterion is satisfied in a neighbourhood of the first critical time t_0 that characteristics cross (i.e., singularities of geometric solutions appear). Thus we can draw the picture of bifurcations of the graph of the geometric (i.e., multi-valued) solution and the corresponding viscosity solution around the time t_0 as follows (cf., Propositon 2.2 and Theorem 3.2):

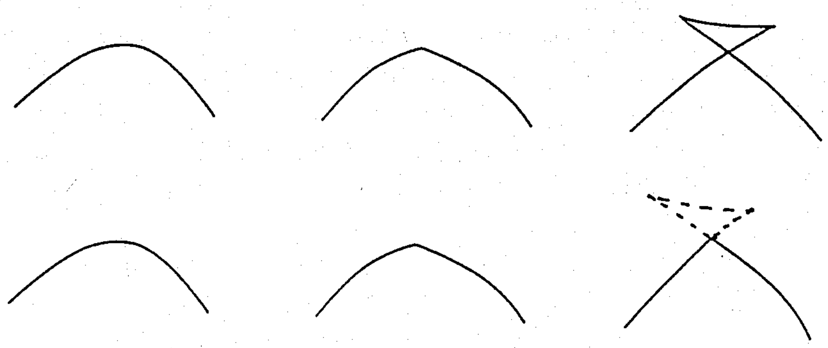

FIGURE 1

As long as the viscosity criterion is not violated, the viscosity solution is obtained by selecting the continuous branch of the geometric solution. In this case the shock (i.e., *a genuine shock*) corresponds to the intersection of the incoming characteristics. It is shown, by using the normal form of the bifurcations of the branches of the geometric solution, (cf.,Theorem 2.1 and Figure 6) that genuine shocks only interact with each other and they never bifurcate. These results are shown in the next figure where the graph of the viscosity solution is depicted by a full line.

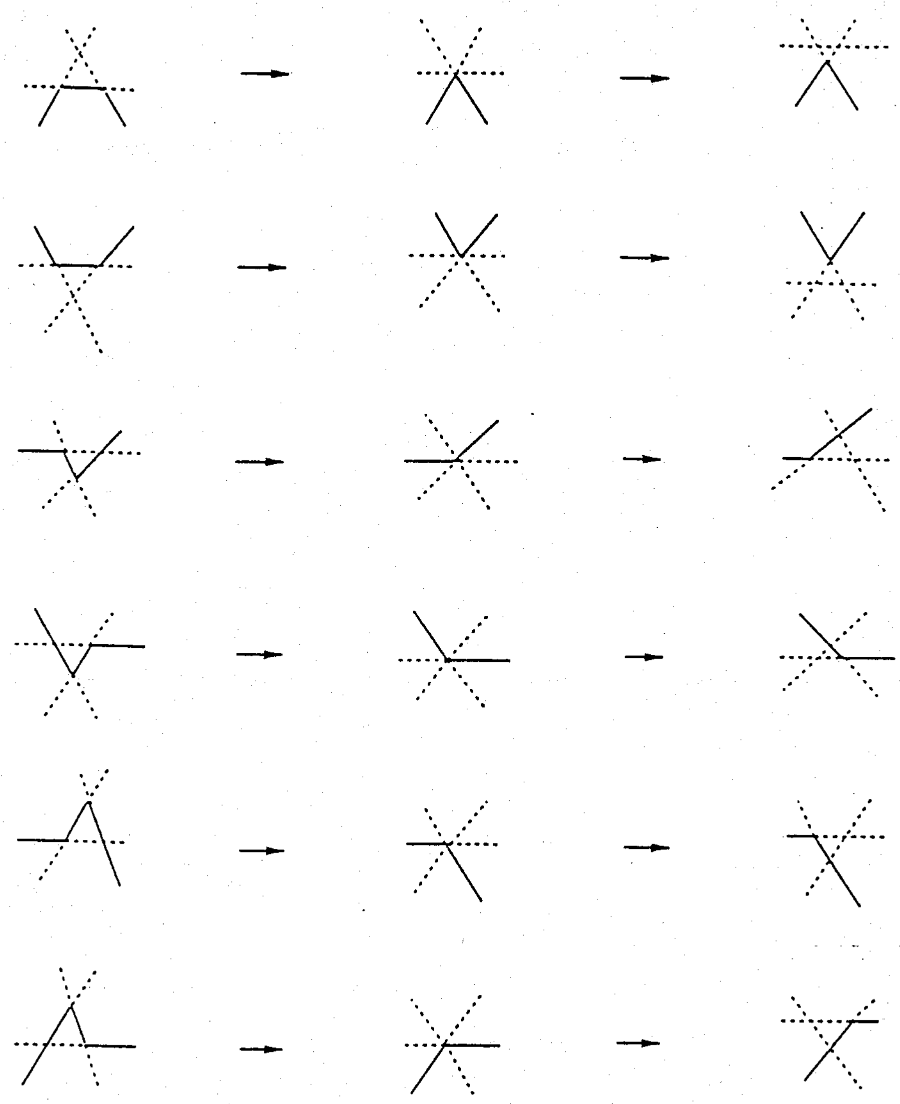

FIGURE 2

So the next step is to construct the viscosity solution around the time t_α beyond which the viscosity criterion is violated across the intersection of two branches. In [19], the second named author pursued this process under the assumption that the Hamiltonian has only one inflection point. This was the first progress for studying the case for a non-convex Hamiltonian. The assumption in [19] is too restrictive. Moreover the techniques used in [19] are limited to the case $n = 1$. In this paper we consider the problem (P) for generic Hamiltonians and generic initial fucntions. We assume that the Hamiltonian $H(p)$ satisfies the following generic conditions (cf., Section 3):

(1) $H(p)$ has no tritangent lines.

(2) Inflection points of $H(p)$ are isolated.
(3) Critical points of $H(p)$ are isolated.

Under the assumptions (1)-(3), for generic initial condition ϕ, the pictures of the bifurcations of the graph of the geometric (i.e., multi-valued) solution and the pictures of the corresponding viscosity solutions around the time t_α when the viscosity criterion is violated are depicted as follows (up to diffeomorphisms):

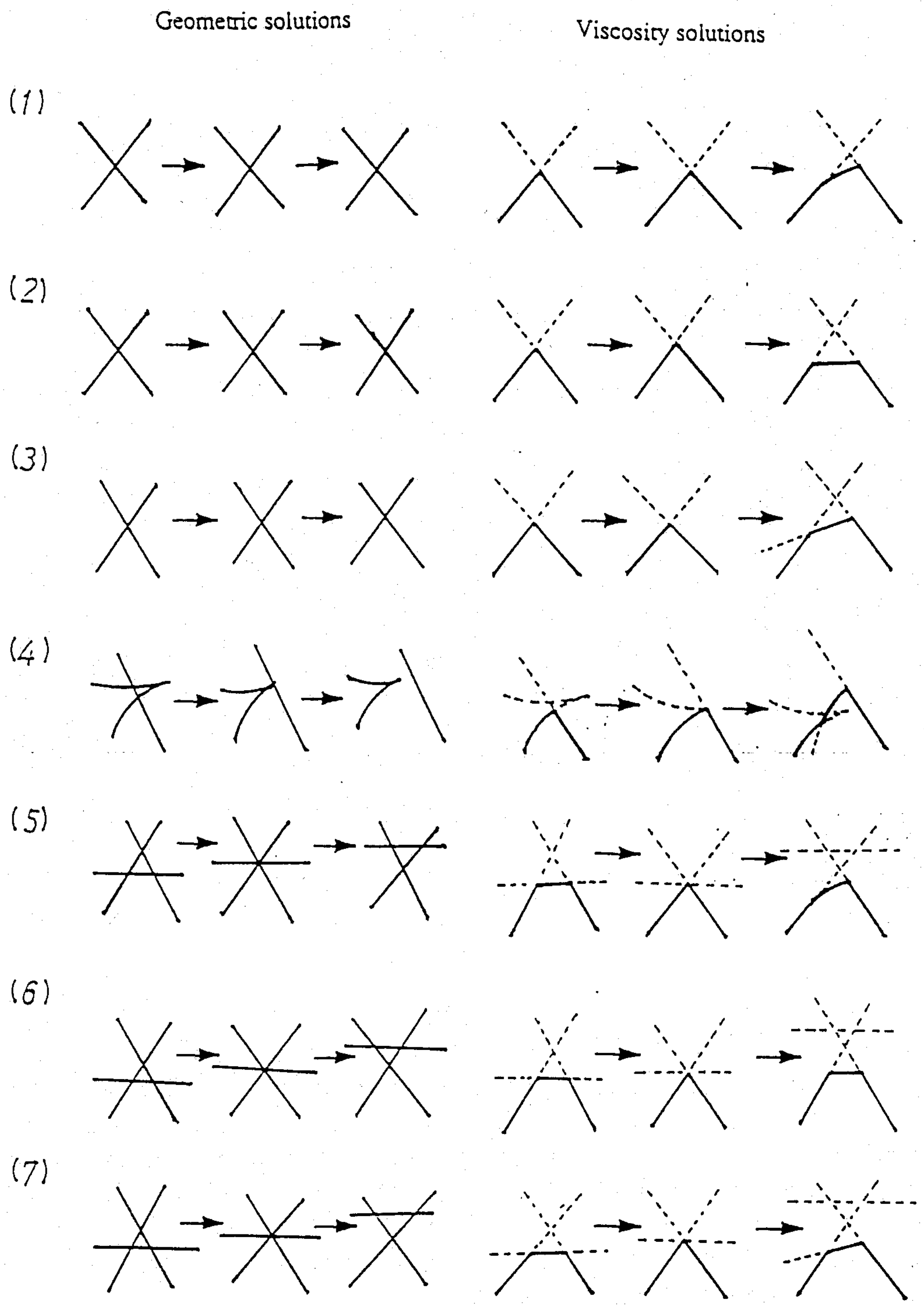

FIGURE 3

Detailed descriptions for each figure are given in Section 4.

The pictures of the generic bifurcations of the shock curves of the viscosity solutions around the time t_α on (t, x)-plane are shown in Figure 4.

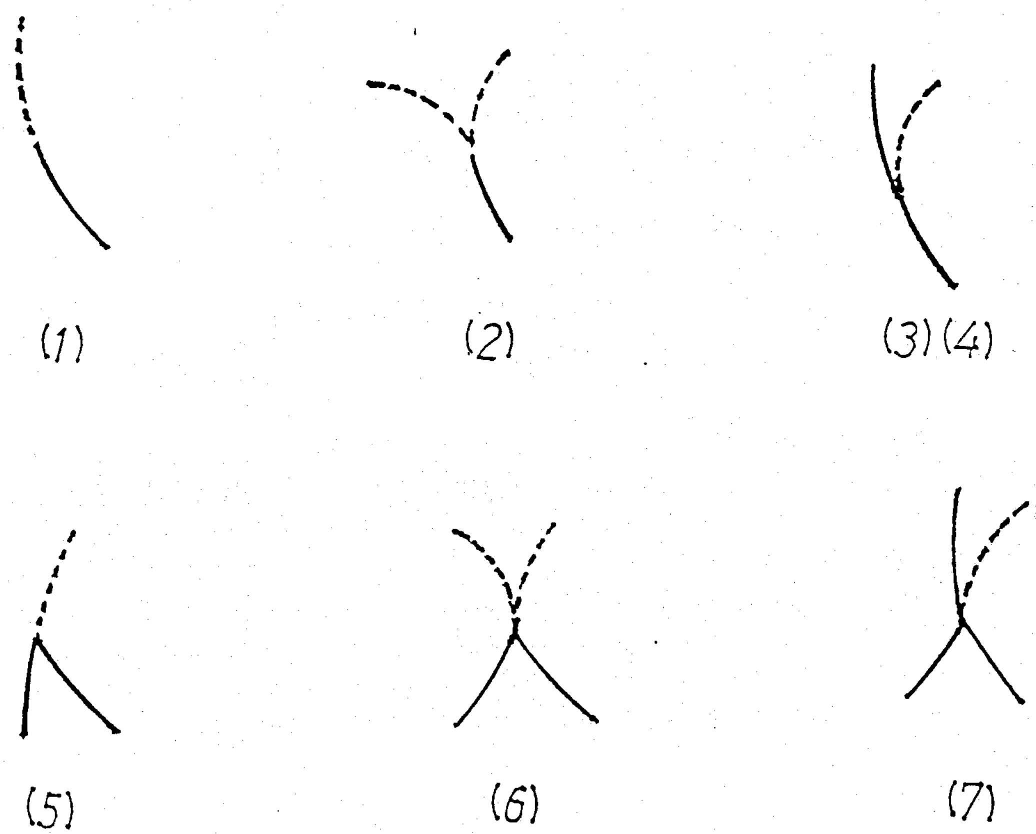

FIGURE 4

The detailed descriptions are also given in Section 4. By differentiating the viscosity solution of (P) in the region of smoothness it can be shown that the bifurcations appearing in Figure 4 are also generic in the case of the Cauchy problem for a scalar conservation law

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial H(y)}{\partial x} = 0, \quad t > 0 \\ y(0, x) = \phi(x), \quad x \in \mathbb{R}, \end{cases}$$

Such a classification complements the results of [8].

In Section 2 it is briefly introduced the framework for the geometric solutions of Hamilton-Jacobi equations (cf., Appendix). The notion of viscosity solutions and the viscosity criterion are given in Section 3. In this section it is also classified how the viscosity criterion should be violated under the assumptions (1)-(3) (cf., Theorem 3.4). The local Riemann problems for the cases of Theorem 3.4 are solved in Section 4.

All maps considered here are of class C^∞ unless otherwise stated.

2. Geometric solutions

In this section we briefly introduce the framework for the geometric solutions of Hamilton-Jacobi equations. The method of characteristics reduces to the solution of the following initial value problem

$$\begin{cases} \dfrac{dx}{dt} = H'(p),\ \dfrac{dp}{dt} = 0,\ \dfrac{dy}{dt} = -H(p) + p \cdot H'(p),\ x, p \in \mathbb{R}, \\ x(0) = u,\ p(0) = \phi'(u),\ y(0) = \phi(u),\ u \in \mathbb{R}. \end{cases}$$

The characteristics equations are explicitly solved as follows

$$\begin{cases} x(t,u) = u + tH'(\phi'(u))),\ p(u,t) = \phi'(u) \\ y(t,u) = t\{-H(\phi'(u)) + \phi'(u) \cdot H'(\phi'(u))\} + \phi(u). \end{cases}$$

We define the geometric solution to be an image of the embedding $L_\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^5$ defined by $L_\phi(t,u) = (t, x(t,u), y(t,u), -H(p(t,u)), p(t,u))$. Although the geometric solution is globally defined as a smooth manifold, there is in general a critical time t_0 beyond which the the characteristics cross in the (t,x)-plane. According to the classical theory, $W_t(L_\phi) = \{(t, x(t,u), y(t,u)) | u \in \mathbb{R}\}$ is the graph of the classical solution of (P) for $t < t_0$. After the characteristics cross W_t becomes multi-valued i.e., singularities appear. By using the above exact solutions for the characteristic equation, we can draw the picture of graphs of the geometric solution for a given initial condition by a computer.

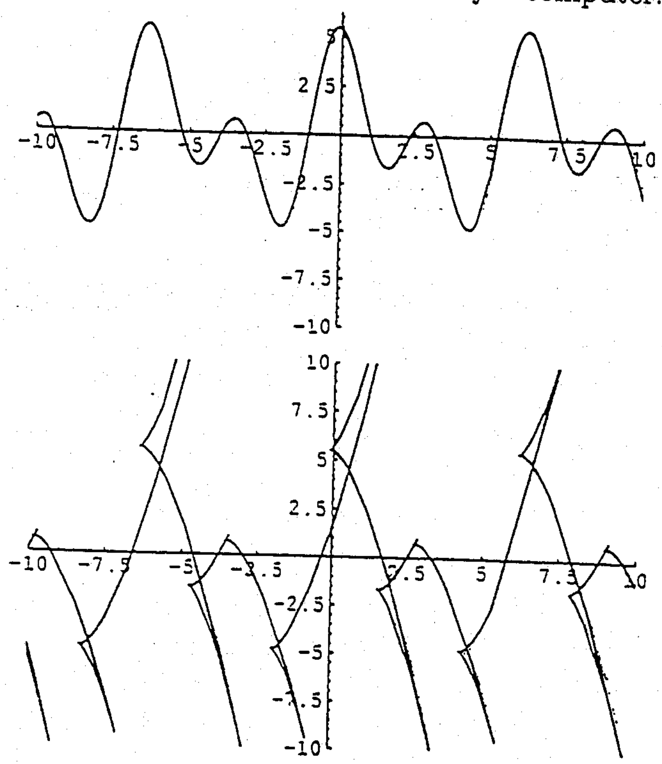

FIGURE 5

In order to study the evolution of the shock waves of the viscosity solutions of (P), we have to classify the generic types of the appearing singularities for geometric solutions i.e., how a singularity is generated, how one type can change into another and how different types of singularities interact. We study how various branches of the multi-valued graph W_t intersecting at a point bifurcate in time for an arbitrary Hamiltonian $H(p)$ in [16]. We give a brief review of the geometric framework in Appendix which has been developed in [16]. By Theorem A.1 in Appendix, we have the following theorem.

Theorem 2.1. *For generic initial data ϕ, the local pictures of the bifurcations of multi-valued graphs $W_t(L_\phi)$ of the corresponding geometric solution L_ϕ around each point $(x_0, y_0) \in \mathbb{R}\times\mathbb{R}$ is locally diffeomorphic to one of the bifurcations of images of multi-germs in the following list :*

$r = 1$;
${}^0A_1 \ : \ (t, u, 0)$;
${}^0A_2 \ : \ (t, 3u^2, 2u^3)$;
${}^1A_3 \ : \ (t, 4u^3 + 2ut, 3u^4 + u^2t)$.
$r = 2$;
${}^0({}^0A_1\,{}^0A_1) \ : \ ((t, u, -u), (t, u, u))$;
${}^1({}^0A_1\,{}^0A_1) \ : \ ((t, u, t \pm u^2), (t, u, 0))$;
${}^1A_2\,{}^0A_1 \ : \ ((t, 3u^2 - t, 2u^3), (t, u, -u))$.
$r = 3$;
${}^0A_1\,{}^0A_1\,{}^0A_1 \ : \ ((t, u, t - u), (t, u, 0), (t, u, u))$.

We can draw the pictures of the above normal forms as in Figure 6.

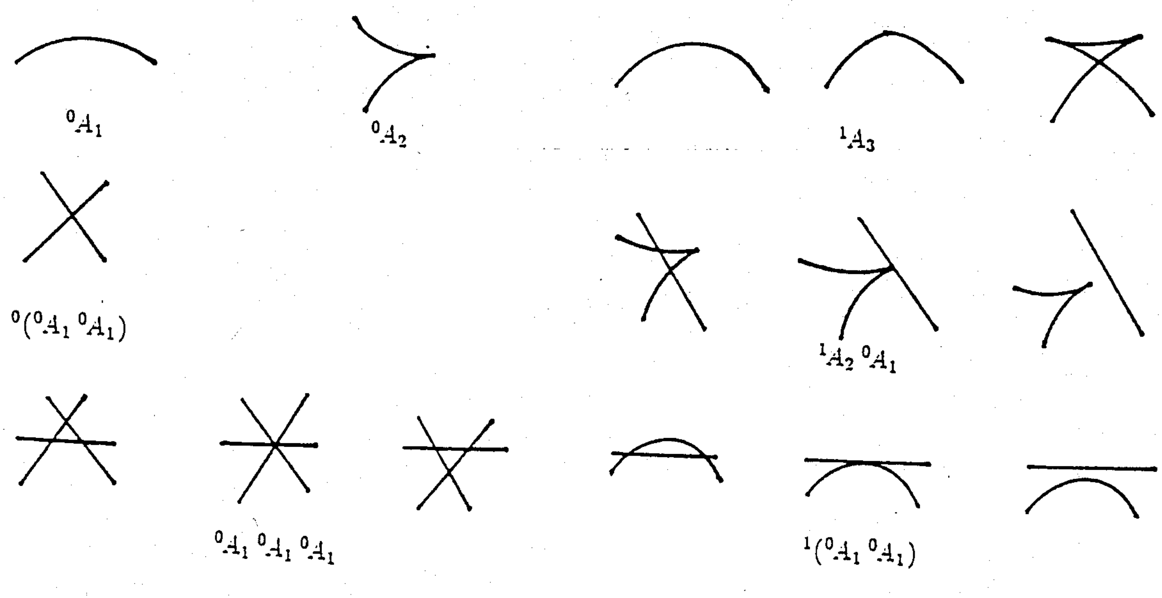

FIGURE 6

We remark that ${}^1A_3$-singularities describe how first singularities appear from a smooth solution. Proposition A.3 asserts that ${}^1A_3$-singularities appear at a non-degenerate (i.e., a convex or concave for $n = 1$) point. If we distinguish germs ${}^1A_3^+ : (t, 4u^3 + 2ut, 3u^4 + u^2t)$ and ${}^1A_3^- : (t, 4u^3 + 2ut, -(3u^4 + u^2t))$ for ${}^1A_3$-singularities, we can state a detailed assertion.

Proposition 2.2. *Let L_ϕ be the geometric solution of the Cauchy problem* (P) *for a generic initial condition ϕ. Suppose that an ${}^1A_3$-singularity appears at (t_0, u_0) as the first singularity. Then the singularity is the ${}^1A_3^+$-type (respectively, ${}^1A_3^-$-type) if and only if $H(p)$ is convex (respectively, concave) at $p_0 = p(t_0, u_0)$.*

Proof. By Theorem A.1, the corresponding normal form is given in the form ${}^1A_3^\pm$: $(t, 4u^3 + 2ut, \pm(3u^4+u^2t), \mp u^2, \pm u)$, so that we consider germs $X(t,u) = 4u^3+2ut$ and $P(t,u) = \pm u$ at the origin. For this normal form, the singularity is given at $(t,u) = (0,0)$, $\frac{\partial X}{\partial u}(0,u) = 12u^2 > 0$ except at $u = 0$ and $\frac{\partial P}{\partial u}(0,u) = \pm 1$.

On the other hand, the characteristic curve is given by $x(t,u) = u + tH'(\phi'(u))$ and $p(t,u) = \phi'(u)$ for a given initial function $\phi(u)$. Suppose that an ${}^1A_3^\pm$-singularities appears at (t_0, u_0), then the normal form of $L_\phi(t,u)$ is written in the above form by a suitable coordinate change preserving the time parameter. We also have $\frac{\partial x}{\partial u} = 1 + tH''(\phi'(u))\phi''(u)$. Since (t_0, u_0) is a singular point, we have $\frac{\partial x}{\partial u}(t_0, u_0) = 0$, so that $H''(\phi(u_0))\phi''(u_0) = -\frac{1}{t_0} < 0$.

For the initial time $t = 0$, we have $\frac{\partial x}{\partial u}(0,u) = 1 > 0$. Since t_0 is the first time at where the singularities appears, we have $\frac{\partial x}{\partial u}(t_0, u) > 0$ for near the point u_0 and except u_0. It follows from the fact $\frac{\partial p}{\partial u}(t_0, u) = \phi''(u)$ that $H''(\phi'(u))$ and $\phi''(u)$ have different signs near the point u_0.

By the observations in the first part of the proof, $H''(p)$ is convex (respectively, concave) if and only if $W_{t_0}(L_\phi)$ has an ${}^1A_3^-$-singularity (respectively, ${}^1A_3^+$-singularity). $\square$

The graph of each normal forms for the ${}^1A_3^\pm$-singularities is depicted as in Figure 7.

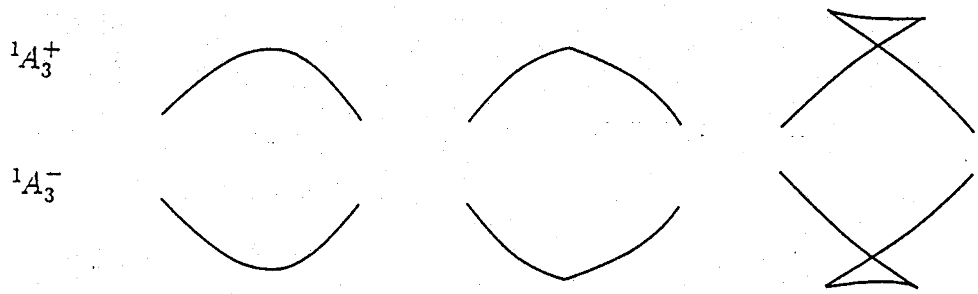

FIGURE 7

3. Viscosity solutions

The viscosity solutions for nonlinear equations of first order have been introduced by Crandall and Lions [6]. Such solutions need not be differentiable everywhere, as the only regularity required in the definition is that of continuity. The function $y_{\mathfrak{v}} \in C(\mathcal{O})$ is a viscosity solution of

$$\text{(H-J)} \qquad \frac{\partial y}{\partial t} + H(\frac{\partial y}{\partial x}) = 0$$

in the open domain $\mathcal{O} \subset \mathbb{R}^+ \times \mathbb{R}$ provided

$$\frac{\partial \psi}{\partial t}(t,x) + H(\frac{\partial \psi}{\partial x}(t,x)) \leq 0, (resp. \geq 0)$$

for any $\psi \in C^1(\mathcal{O})$ for which $y_{\mathfrak{v}} - \psi$ attains a local maximum (resp. local minimum) at the point $(t,x) \in \mathcal{O}$. The function $y_{\mathfrak{v}} \in C([0,\infty) \times \mathbb{R})$ is a viscosity solution of the Cauchy problem (P) if and only if it is a viscosity solution of (H-J) in the domain $(0,\infty) \times \mathbb{R}$ and satisfies the initial condition $\lim_{t \to 0+} y_{\mathfrak{v}}(t,x) = \phi(x)$. The above inequality will be referred as the *viscosity criterion* at the point (t,x). We next state the viscosity criterion in a form which is more useful for the construction of the solution. To this end, assume that $\mathcal{O} \subset (0,\infty) \times \mathbb{R}$ is open and that there is a smooth curve $t \to x = \chi(t),\ t \in (t_1,t_2) \subset \mathbb{R}^+$, with negative slope, which divides $\mathcal{O}$ into two open sets $\mathcal{O}^+$ and $\mathcal{O}^-$, $\mathcal{O} = \Gamma \cup \mathcal{O}^+ \cup \mathcal{O}^-, : \Gamma = \{(t,x) :\ x = \chi(t)\}$, where $\mathcal{O}^+$ lies on the right of Γ. The curve $\chi(t)$ in the neighborhood of which $y_{\mathfrak{v}}$ has the above specified properties is a shock curve.

Theorem 3.1. *Let $y_{\mathfrak{v}} \in C(\mathcal{O})$ and $y_{\mathfrak{v}} = y_{\mathfrak{v}}^+$ in $\mathcal{O}^+ \cup \Gamma, y_{\mathfrak{v}} = y_{\mathfrak{v}}^-$ in $\mathcal{O}^- \cup \Gamma$ where $y_{\mathfrak{v}}^{\pm} \in C^1(\mathcal{O}^{\pm} \cup \Gamma)$. Then $y_{\mathfrak{v}}$ is a viscosity solution of* (H-J) *in $\mathcal{O}$ if and only if the following conditions hold:*

a) $y_{\mathfrak{v}}^+$ and $y_{\mathfrak{v}}^-$ are classical solutions of (H-J) in $\mathcal{O}^+$ and $\mathcal{O}^-$ respectively,

b) If $y_{\mathfrak{v},x}^- > y_{\mathfrak{v},x}^+$ (resp. $y_{\mathfrak{v},x}^- < y_{\mathfrak{v},x}^+$) across the shock curve, then

$$H\left((1-\lambda)\frac{\partial y_{\mathfrak{v}}^+}{\partial x} + \lambda \frac{\partial y_{\mathfrak{v}}^-}{\partial x}\right) - (1-\lambda)H(\frac{\partial y_{\mathfrak{v}}^+}{\partial x}) - \lambda H(\frac{\partial y_{\mathfrak{v}}^-}{\partial x}) \leq 0 \ (resp. \ \geq 0),$$

where $\lambda \in [0,1]$. That is, the graph of H lies respectively below or above the line segment joining the points $\left(\frac{\partial y_{\mathfrak{v}}^+}{\partial x}, H(\frac{\partial y_{\mathfrak{v}}^+}{\partial x})\right)$ and $\left(\frac{\partial y_{\mathfrak{v}}^-}{\partial x}, H(\frac{\partial y_{\mathfrak{v}}^-}{\partial x})\right)$.

The proof of Theorem 3.1 is given in ([18],[19]) as a direct application of Theorem 1.3 in [7]. The condition b) will be referred in the sequel as the *viscosity criterion.* The hypersurface Γ in the neighbourhood of which $y_{\mathfrak{v}}$ has the properties specified in the above theorem is the *shock curve.* If the Hamiltonian is uniformly convex (or concave),it has been known that the unique viscosity solution of (P) is the minimum (or maximum) of the geometric solution ([2],[12]), so that the local pictures of such curves have been already known ([4]).

Throughout the remainder of the paper, we do not assume that $H(p)$ is convex or concave. In this case the situation should be quite complicated even for the one-dimensional space case.

Before the first critical time that characteristics cross in the (t, x)–plane, W_t is the graph of the viscosity solution $y_{\mathfrak{v}}$. After the characteristics cross, W_t becomes singular. Theorem 2.1 describes the generic singularities of W_t. The first singularity appears in the form of ${}^1A_3$. By Theorem 2.4, these appear at the convex or the concave points of the Hamiltonian function. Away from the singularity, the viscosity solution is given by W_t. In ([18], [19]) we have constructed the unique viscosity solution past the first critical time by selecting a single-valued branch of W_t. Assume that the singularity of type ${}^1A_3$ appears at the point (t_0, x_0, p_0). After the critical time t_0, the wave front W_t is three-valued on an interval $(x_1(t), x_2(t))$; see Figure 1b. Let y_i, $i = 1, 2, 3$ be the three branches of W_t, where y_1 is defined on a neighborhood of $x_1(t)$ and y_2 on a neighborhood of $x_2(t)$. Then y_1, y_2 intersect at one point $\chi(t) \in (x_1(t), x_2(t))$, for $t > t_0$.

We define the viscosity solution past t_0 by selecting a continuous single-valued branch of W_t as follows:

Theorem 3.2. *There exists an $\varepsilon > 0$ such that the function $y_{\mathfrak{v}}(t, x), (t, x) \in (t_0, t_0 + \varepsilon) \times (x_1(t), x_2(t))$, defined by*

$$y_{\mathfrak{v}}(t, x) = \begin{cases} y_1(t, x), \; x \leq \chi(t) \\ y_3(t, x), \; x \geq \chi(t), \end{cases} \tag{4.1}$$

is the viscosity solution of (P) in a neighborhood of x_0 past the time t_0.

In view of Theorem 2.3 the viscosity criterion (see Section 3) is satisfied across $\chi(t)$ while $y_{\mathfrak{v}}$ defined by (4.1) is a classical solution away from $\chi(t)$. Hence, by the uniqueness of the viscosity solution, (4.1) gives the viscosity solution of (P) past t_0.

By this construction, we have extended the viscosity solution beyond the first critical time t_0. According to Theorem 2.5 the shock is generated in a convex or concave domains of $H(p)$, so the viscosity criterion is automatically satisfied. The graph of the viscosity solution past the first critical time is depicted by a full line in Figure 1c, where we assume that H is convex in the neighborhood of the appearing singularity ${}^1A_3$. The shock corresponds to the intersection of the two branches and it is called a *genuine shock*. The genuine shock is defined as the intersection of two incoming characteristics (or waves) and its speed is given by the Rankine-Hugoniot condition

$$\chi'(t) = \frac{H(y^+_{\mathfrak{v},x}(t, \chi(t))) - H(y^-_{\mathfrak{v},x}(t, \chi(t)))}{y^+_{\mathfrak{v},x}(t, \chi(t)) - y^-_{\mathfrak{v},x}(t, \chi(t))},$$

where $y^{\pm}_{\mathfrak{v},x} = \frac{\partial y^{\pm}_{\mathfrak{v}}}{\partial x}$ and $\chi'(t) = \frac{d\chi}{dt}(t)$.

If the viscosity criterion is satisfied at the time $t_\alpha = t_0 + \varepsilon$, we can choose the correct branch of the graphs of the geometric solutions as viscosity solutions (see Figure 3).

We now consider the following problem.

Problem If the two branches initially defining the shock continue to cross, whether the viscosity criterion is satisfied across the intersection.

Assume that a generated shock is defined by two intersecting branches y^- and y^+. We denote by y^- (resp. y^+) the branch representing the viscosity solution for $x < \chi(t)$ (resp.

$x > \chi(t)$). If the two branches remain intersected they evolve according to ${}^0({}^0A_1\, {}^0A_1)$. We denote by $\chi(t)$ the intersection of the two branches. In the case when $H(p)$ has only one inflection point Kossioris [18] studied this problem and constructed the viscosity solutions. We consider the general situation here, however, we should avoid the pathological situations, so we consider some generic properties of Hamiltonian functions. Let $H(p)$ be a Hamiltonian function and ℓ be a line in the (p,h)-plane, where $h = H(p)$ is the graph of $H(p)$. We say that ℓ is *a tritangent line* if there exits distinct three points p_1, p_2, p_3 such that ℓ is tangent to the graph of $H(p)$ at each p_1, p_2, p_3 or there exist distinct two points of $H(p)$ at q_1, q_2 such that ℓ is tangent to the graph of $H(p)$ at each p_1, p_2 and one of them is the inflection point of $H(p)$.

As a direct application of Thom's multi-jet transversality theorem (cf., [11]), we have the following proposition.

Proposition 3.3. *There exists a residual subset $\mathcal{O} \subset C^\infty(\mathbb{R}, \mathbb{R})$ such that for any $H(p) \in \mathcal{O}$, $H(p)$ has the following properties:*
(1) *$H(p)$ has no tritangent lines.*
(2) *Inflection points of $H(p)$ are isolated.*
(3) *Critical points of $H(p)$ are isolated.*
Here,$C^\infty(\mathbb{R}, \mathbb{R})$ is the space of smooth functions equipped with Whitney C^∞-topology.

Proposition 3.3 guarantees that the following assumption is sufficiently general for smooth Hamiltonian functions.

Assumption (*). $H(p)$ satisfies the properties (1),(2) and (3) in Proposition 3.3.

Throughout the remainder of this paper, we assume the assumption (*). The following theorem describe how viscosity criterion should be violated for a generic initial function under the assumption (*).

Theorem 3.4. *There exists a residual subset $\mathcal{O} \subset C^\infty(\mathbb{R}, \mathbb{R})$ such that for any initial function $\phi \in \mathcal{O}$, if the viscosity criterion is violated at t_α, then the only following 8 cases may occur:*
(1) *The normal form is ${}^0({}^0A_1\, {}^0A_1)$ and $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at only one of the points P^+, P^- and the line is not tangent to the graph at other points between these points.*
(2) *The normal form is ${}^0({}^0A_1\, {}^0A_1)$ and $\overline{P^+P^-}$ is not tangent to the graph of $H(p)$ at each point P^+, P^- and there exists only one another point between these points at where the above line is tangent to the graph.*
(3) *The normal form is ${}^0({}^0A_1\, {}^0A_1)$ and $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at only one of the points P^+, P^- and there exists only one another point between these points at where the above line is tangent to the graph.*
(4) *The normal form is ${}^0({}^0A_1\, {}^0A_1)$ and $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at each point of P^+, P^-.*
(5) *The normal form is ${}^0({}^0A_1\, {}^0A_1)$ and $\overline{P^+P^-}$ is not tangent to the graph of $H(p)$ at each point P^+, P^- and there exists exactly two other points between these points at where the above line is tangent to the graph.*
(6) *The normal form is ${}^1A_2\, {}^0A_1$ and $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at only one of the points P^+, P^- and the line is not tangent to the graph at other points between these points.*

(7) *The normal form is* ${}^0A_1\,{}^0A_1\,{}^0A_1$ *and* $\overline{P^+P^-}$ *is tangent to the graph of* $H(p)$ *at only one of the points* P^+, P^- *and is not tangent to the graph at other points between these points.*
(8) *The normal form is* ${}^0A_1\,{}^0A_1\,{}^0A_1$ *and* $\overline{P^+P^-}$ *is not tangent to the graph of* $H(p)$ *at each point* P^+, P^- *and there exists only one another point between these points at where the above line is tangent to the graph.*

Here, $P^+ = (y_x^+(t_\alpha, \chi(t_\alpha)), H(y_x^+(t_\alpha, \chi(t_\alpha))))$, $P^- = (y_x^-(t_\alpha, \chi(t_\alpha)), H(y_x^-(t_\alpha, \chi(t_\alpha)))$ *and* $\overline{P^+P^-}$ *denotes the line through* P^+, P^- *in the* (p, h)*-plane.*

Proof. By Theorem 2.1, we may assume that the first singularities appear in the form of ${}^1A_3$. After that the singularities of the graph of the geometric solution bifurcate in the forms of ${}^0({}^0A_1\,{}^0A_1)$, ${}^1({}^0A_1\,{}^0A_1)$, ${}^1A_2\,{}^0A_1$ or ${}^0A_1\,{}^0A_1\,{}^0A_1$. Since the characteristics in $\mathbb{R}^5$ never cross, we can get rid of the case ${}^1({}^0A_1\,{}^0A_1)$.

We already mentioned that the viscosity criterion is satisfied past the first critical time t_0, so that it is satisfied until the time t_α when $\overline{P^+P^-}$ is tangent to the graph of $H(p)$. By the assumptions on the Hamiltonian $H(p)$, we may consider the case that $\overline{P^+P^-}$ is at most a double tangent line for each normal form. We now distinguish each normal form. We denote that $p^+ = y_x^+(t_\alpha, \chi(t_\alpha)) = \phi'(u_+)$ and $p^- = y_x^-(t_\alpha, \chi(t_\alpha)) = \phi'(u_-)$.

(A) ${}^0({}^0A_1\,{}^0A_1)$: In this case each branch of the graph of geometric solution is a non-singular curve. We remark that $y^\pm(t, \chi(t)) = t\{-H(\phi'(u_\pm)) + \phi'(u_\pm)H'(\phi(u_\pm))\} + \phi(u_\pm)$. Since the normal form ${}^0({}^0A_1\,{}^0A_1)$ has trivial bifurcations along the time parameter, the condition $y^+(t, \chi(t)) = y^-(t, \chi(t))$ defines a codimension 0 submanifold in the corresponding jet space, so that we may ignore this condition. Let ${}_rJ^1(\mathbb{R}, \mathbb{R})$ be a multi-1-jet space of function germ $\mathbb{R} \to \mathbb{R}$. We use the multi-jet transversality theorem for functions $\phi : \mathbb{R} \to \mathbb{R}$ Since we only use this purely geometric theorem in the proof, we do not mentioned the detail. The reader may refer to the article [11]. We now consider the following conditions which correspond to all possible cases:

(a) $\pm H'(\phi'(u_+)) = \frac{H(\phi'(u_+)) - H(\phi'(u_-))}{\phi'(u_+) - \phi'(u_-)}$, which defines a submanifold in ${}_2J^1(\mathbb{R}, \mathbb{R})$ of codimension 1. Of course, we have to consider the case that $\pm H'(\phi'(u_-)) = \frac{H(\phi'(u_+)) - H(\phi'(u_-))}{\phi'(u_+) - \phi'(u_-)}$, however this case is essentially contained in the above, so that we may ignore such non-essentially different cases in the following arguments.
(b) There exists u_0 with $u_0 \neq u_\pm$ such that $\pm H'(\phi'(u_0)) = \frac{H(\phi'(u_+)) - H(\phi'(u_-))}{\phi'(u_+) - \phi'(u_-)}$. This condition defines a submanifold in ${}_2J^1(\mathbb{R}, \mathbb{R})$ of codimension 1.
(c) There exists u_0 with $u_0 \neq u_\pm$ such that

$$\pm H'(\phi'(u_+)) = \pm H'(\phi'(u_0)) = \frac{H(\phi'(u_+)) - H(\phi'(u_-))}{\phi'(u_+) - \phi'(u_-)} = \frac{H(\phi'(u_+)) - H(\phi'(u_0))}{\phi'(u_+) - \phi'(u_0)}.$$

This condition defines a submanifold in ${}_3J^1(\mathbb{R}, \mathbb{R})$ of codimension 3.
(d) $\pm H'(\phi'(u_+)) = \pm H'(\phi'(u_-)) = \frac{H(\phi'(u_+)) - H(\phi'(u_-))}{\phi'(u_+) - \phi'(u_-)}$, which defines a submanifold in ${}_2J^1(\mathbb{R}, \mathbb{R})$ of codimension 2.

(e) There exist u_0, u_1 which are different from $u_\pm$ such that

$$\pm H'(\phi'(u_0)) = \pm H'(\phi'(u_1)) = \frac{H(\phi'(u_+)) - H(\phi'(u_-))}{\phi'(u_+) - \phi'(u_-)}$$
$$= \frac{H(\phi'(u_+)) - H(\phi'(u_0))}{\phi'(u_+) - \phi'(u_0)} = \frac{H(\phi'(u_+)) - H(\phi'(u_1))}{\phi'(u_+) - \phi'(u_1)}.$$

This condition defines a submanifold of ${}_4J^1(\mathbb{R},\mathbb{R})$ of codimension 4. Each submanifold in ${}_rJ^1(\mathbb{R},\mathbb{R})$ has at most codimension r, so that we can not avoid such conditions by the multi-jet transversality theorem.

(B) ${}^1A_2\,{}^0A_1$: In this case the normal form ${}^1A_2\,{}^0A_1$ bifurcate at the time t_α, so that we should consider the condition $y^+(t_\alpha, \chi(t_\alpha)) = y^-(t_\alpha, \chi(t_\alpha))$ for fixed t_α. It defines a submanifold in ${}_2J^1(\mathbb{R},\mathbb{R})$ of codimension 1. By the same arguments as the above, we can avoid the conditions (c), (d) and (e). So we may consider the condition (a) or (b). We now show that the condition (a) holds for the normal form ${}^1A_2\,{}^0A_1$. On the (t,x)-plane, we denote $(t,\chi(t))$ the genuine shocks for $t \le t_\alpha$. Suppose that the point u_- corresponds to the cusp point at the time t_α. Then there exists a smooth function $u(t)$ such that $\chi(t) = u(t) + tH'(\phi'(u(t)))$ for $t \le t_\alpha$ and $u(t_\alpha) = u_-$, where we choose one of the branches of the graph of the geometric solution corresponding to u_-. It follows that we have

$$\chi'(t) = u'(t)(1 + H''(\phi'(u(t))\phi''(t)) + H'(\phi'(u(t)).$$

Since the graph of the geometric solution has a singularity at t_α, we have $\frac{\partial x}{\partial u}(t_\alpha, u_-) = 1 + t_\alpha H''(\phi'(u_-))\phi''(u_-) = 0$. So we have $\chi'(t_\alpha) = \lim_{t\to t_\alpha} \chi'(t) = H'(\phi'(u_-))$.

On the other hand, by the Rankine-Hugoniot condition we have

$$\chi'(t) = \frac{H(y^+_{\mathfrak{v},x}(t,\chi(t))) - H(y^-_{\mathfrak{v},x}(t,\chi(t)))}{y^+_{\mathfrak{v},x}(t,\chi(t)) - y^-_{\mathfrak{v},x}(t,\chi(t))},$$

for $t \le t_\alpha$. Since $\lim_{t\to t_\alpha} y^\pm_{\mathfrak{v},x}(t,\chi(t))) = \phi'(u_\pm)$, we have $\chi'(t_\alpha) = \frac{H(\phi'(u_+)) - H(\phi'(u_-))}{\phi'(u_+) - \phi'(u_-)}$, so that we have $H'(\phi'(u_-)) = \frac{H(\phi'(u_+)) - H(\phi'(u_-))}{\phi'(u_+) - \phi'(u_-)}$. This condition corresponds to the case (a) and we may get rid of the case (b).

(C) ${}^0A_1\,{}^0A_1\,{}^0A_1$: In this case the normal form also bifurcate at the point t_α, so that we can get rid of the case (c), (d) and (e) by the similar reasons as those of the case (B). Since each branch of the normal form is non-singular, the remaining two cases may occur in generic. This completes the proof. □

4. Local Riemann problems

In this section we describe the evolution of the shock after the time t_α when the viscosity criterion is violated. As we already mentioned in the previous sections, the viscosity criterion is satisfied past the first critical time t_0, so that it is satisfied until the time t_α in Theorem 3.3.

After the time t_α, we solve local Riemann problem and construct the viscosity solution for the non-smooth initial data which is given in each case in Theorem 3.3.

Case (1). We assume that the graph of the viscosity solution at the time $t \le t_\alpha$ is depicted as in Figure 8a.

Without the loss of generality, we assume that $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at the point $(y_x^-(t_\alpha, \chi(t_\alpha)), H(y_x^-(t_\alpha, \chi(t_\alpha)))$. By the Rankine-Hugoniot condition, we have $\chi'(t_\alpha) = H'(\phi'(u_-(t_\alpha))$. Differentiating the both side of $\chi(t) = u_-(t) + tH'(\phi'(u_-(t)))$ with respect to t, we have

$$\chi'(t) = u_-'(t)(1 + H''(\phi'(u_-(t)))\phi''(u_-(t))) + H'(\phi'(u_-(t))).$$

It follows that we have $u_-'(t_\alpha)(1+H''(\phi'(u_-(t_\alpha)))\phi''(u_-(t_\alpha))) = 0$. Since $u_-'(t)$ is a monotone function around t_α, the above equality means that $1+H''(\phi'(u_-(t_\alpha)))\phi''(u_-(t_\alpha)) = 0$, so that $H''(\phi'(u_-(t_\alpha))) \neq 0$.

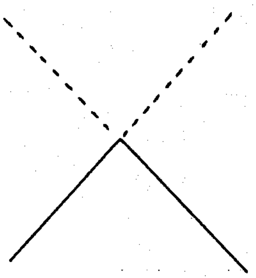

Figure 8a

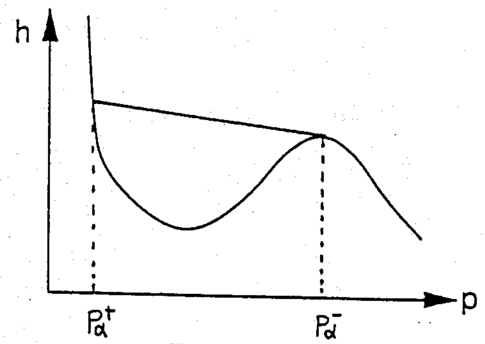

Figure 8b

Hence, we may assume that $H''((y_x^-(t_\alpha, \chi(t_\alpha))) < 0$ (see Figure 4b). As we already mentioned that the genuine shocks satisfies the Rankine-Hugoniot condition. So we should construct new characteristics which satisfies both of the Rankine-Hugoniot condition and the viscosity criterion. In this case we have

$$H'(y_x^-(t_\alpha, \chi(t_\alpha))) = \frac{H(y_x^+(t_\alpha, \chi(t_\alpha))) - H(y_x^-(t_\alpha, \chi(t_\alpha)))}{y_x^+(t_\alpha, \chi(t_\alpha)) - y_x^-(t_\alpha, \chi(t_\alpha))} = \chi'(t_\alpha).$$

We now distinguish two cases as follows:

a) If

$$H'(y_x^-(t, \chi(t))) \ge \frac{H(y_x^+(t, \chi(t))) - H(y_x^-(t, \chi(t)))}{y_x^+(t, \chi(t)) - y_x^-(t, \chi(t))}$$

for $t_\alpha \le t < t_\alpha + \varepsilon$ for sufficiently small $\varepsilon > 0$, then we can easily show that the viscosity criterion is satisfied for $t < t_\alpha + \varepsilon$. So we can choose single valued continuous branches of the geometric solution as the viscosity solution.

b) If

$$H'(y_x^-(t,\chi(t))) < \frac{H(y_x^+(t,\chi(t))) - H(y_x^-(t,\chi(t)))}{y_x^+(t,\chi(t)) - y_x^-(t,\chi(t))}$$

for $t_\alpha \le t < t_\alpha + \varepsilon$ for sufficiently small $\varepsilon > 0$, then we can easily show that the viscosity criterion is violated for $t_\alpha < t < t_\alpha + \varepsilon$, so that a new way to build the solution is required (cf., Figure 9).

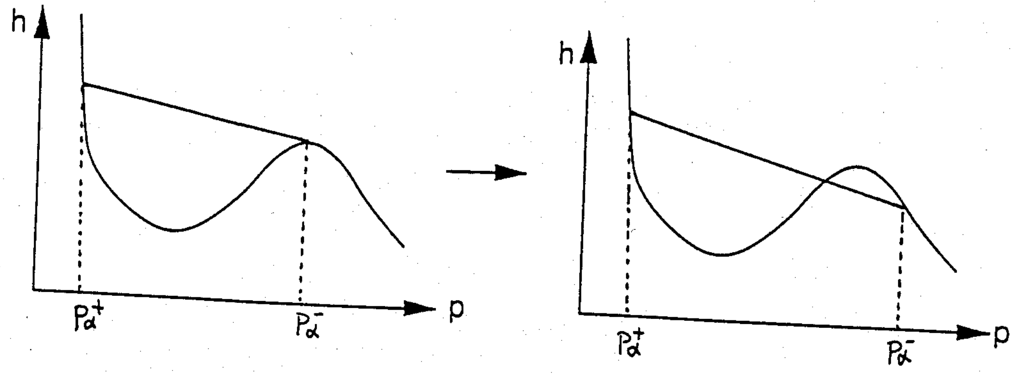

FIGURE 9

In this case we can use the techniques in [18] to construct the contact discontinuity shock curve and then obtain new characteristics. Lets consider the relation $H'(q) = \frac{H(p)-H(q)}{p-q}$ around (q_0, p_0) with $q_0 \neq p_0$, $H'(q_0) = \frac{H(p_0)-H(q_0)}{p_0-q_0}$ and $H''(q_0) \neq 0$. By the implicit function theorem, there exists a smooth function ψ around p_0 such that the above relation is equivalent to $q = \psi(p)$. We will first construct the contact discontinuity as the solution of the following initial value problem.

$$\begin{cases} \chi_c'(t) = H'(\psi(y_x^+(t,\chi_c(t)))), \\ \chi_c(t_\alpha) = \chi(t_\alpha). \end{cases}$$

The characteristic which is started at a point $(\tau, \chi_c(\tau))$ should be satisfied the following:

$$\begin{cases} x'(t) = H'(p(t)), \\ p'(t) = 0 \\ y'(t) = -H(p(t)) + p(t)H'(p(t)), \end{cases}$$

with initial condition $x(\tau) = \chi_c(\tau)$, $y(\tau) = y^+(\tau, \chi_c(\tau))$ and $p(\tau) = \psi(y_x^+(\tau, \chi_c(\tau)))$. So the solution is exactly given as follows:

$$\begin{cases} \tilde{x}(t) = \chi_c(\tau) + (t-\tau)H'(\psi(y_x^+(\tau,\chi_c(\tau)))), \\ \tilde{p}(t) = \psi(y_x^+(\tau,\chi_c(\tau))) \\ \tilde{y}(t) = y^+(\tau,\chi_c(\tau)) \\ \qquad +(t-\tau)\{-H(\psi(y_x^+(\tau,\chi_c(\tau)))) + \psi(y_x^+(\tau,\chi_c(\tau)))H'(\psi(y_x^+(\tau,\chi_c(\tau))))\}. \end{cases}$$

By definition of the contact discontinuity, we have

$$\chi_c''(t) = H''(\psi(\phi'(u_+(t))\frac{\partial \psi}{\partial p}(\phi'(u_+(t))\phi''(u_+(t))u_+'(t),$$

where $\chi_c(t) = u_+(t) + tH'(\phi(u_+(t))$. Since $\frac{\partial \psi}{\partial p} = \frac{H'(p)-H'(q)}{H''(q)(p-q)}$, we have

$$\chi_c''(t) = \frac{H'(\phi'(u_+(t)) - H'(\psi(\phi'(u_+(t))))}{\phi'(u_+(t)) - \psi(\phi'(u_+(t)))}\phi''(u_+(t))u_+'(t).$$

We also have

$$\chi'(t) = u_+'(t)\{1 + tH''(\phi'(u_+(t))\phi''(u_+(t))\} + H'(\phi'(u_+(t))).$$

It follows that

$$\chi_c''(t) = -\frac{(H'(\phi'(u_+(t))) - H'(\psi(\phi'(u_+(t)))))^2}{\phi'(u_+(t)) - \psi(\phi'(u_+(t)))}\frac{\phi''(u_+(t))}{1 + tH''(\phi'(u_+(t))\phi''(u_+(t))}.$$

Since

$$\frac{\partial x}{\partial u}(t, u_+(t)) = 1 + tH''(\phi'(u_+(t)))\phi''(u_+(t)),$$

we may assume that $1 + tH''(\phi'(u_+(t)))\phi''(u_+(t)) > 0$. So $\chi_c(t)$ is convex if and only if $\phi''(u_+(t)) > 0$. We suppose that $\phi''(u_+(t)) \leq 0$ and denote $\chi_c(t) = u_+(t) + tH'(\phi(u_+(t)) = u_-(t) + tH'(\phi(u_-(t)))$, where $u_-(t)$ (resp. $u_+(t)$) is the point corresponding to the characteristic from the right (resp. left) side of $(t, \chi_c(t))$. We distinguish two cases as follows:

b-1) If $\phi''(u_-(t)) > 0$, then ϕ' is monotone. Since $u_-'(t) < 0$, $\phi_-'(u_(t))$ moves to the left direction, so that the viscosity criterion is satisfied across χ.

b-2) If $\phi''(u_-(t)) < 0$ and the viscosity criterion is violated across χ for $t > t_\alpha$, then $1 + tH''(\phi'(u_-(t)))\phi''(u_-(t)) > 0$ near t_α. Differentiate the equality $\chi_c(t) = u_-(t) + tH'(\phi(u_-(t))$ with respect to t, then we have

$$\chi'(t) - H'(\phi'(u_-(t))) = \{1 + tH''(\phi'(u_-(t)))\phi''(u_-(t))\}u_-'(t).$$

Since

$$\chi'(t) = \frac{H(\phi'(u_+(t))) - H(\phi'(u_-(t)))}{\phi'(u_+(t)) - \phi'(u_-(t))} > H'(\phi'(u_-(t))),$$

we have $u_-'(t) > 0$, so that $u_-(t)$ is increase, which is a contradiction.

Hence, if the viscosity criterion is violated for $t > t_\alpha$, the contact discontinuity curve χ is convex and the viscosity solution can be constructed. We draw the picture which is illustrating the situations as follows :

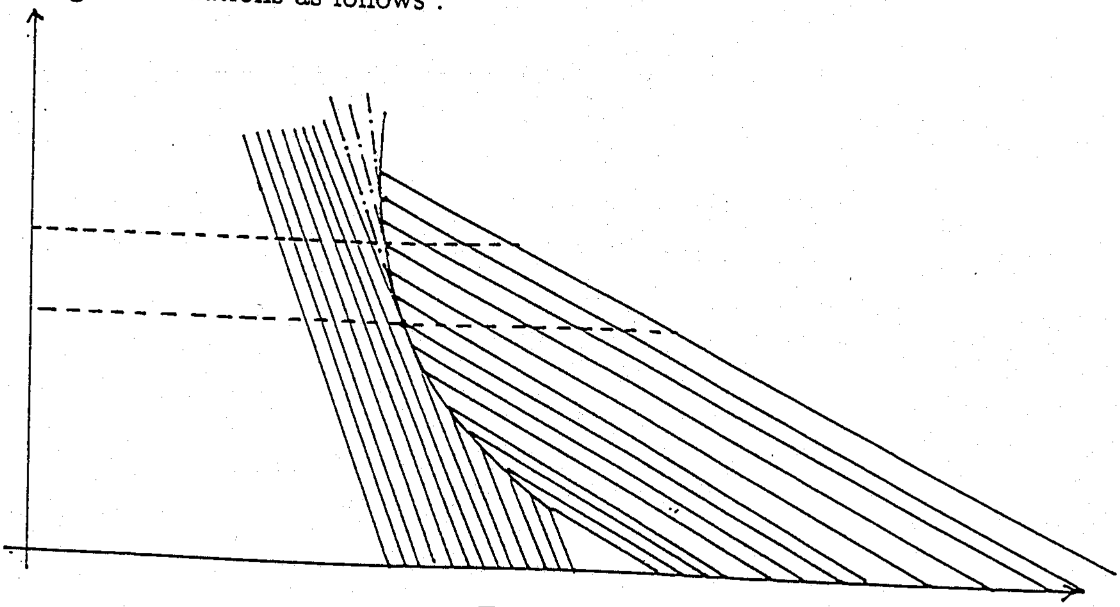

FIGURE 10

Then we can draw the picture of the graph of the viscosity solution for $t > t_\alpha$ and the shock curve around t_α.

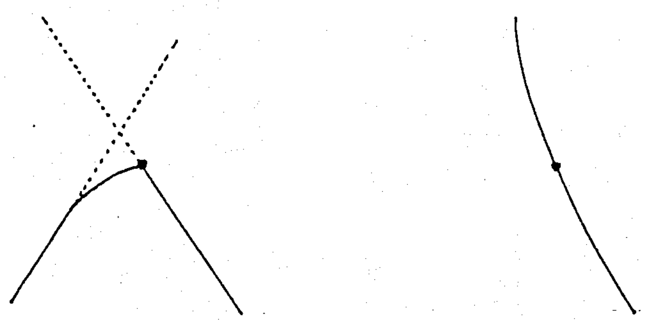

FIGURE 11

Case (2). In this case the graph of the viscosity solution at the time $t \leq t_\alpha$ is depicted as in Figure 12a.

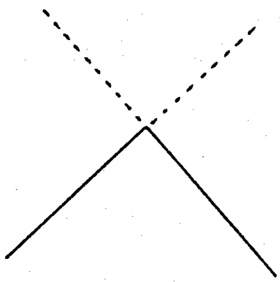

Figure 12a

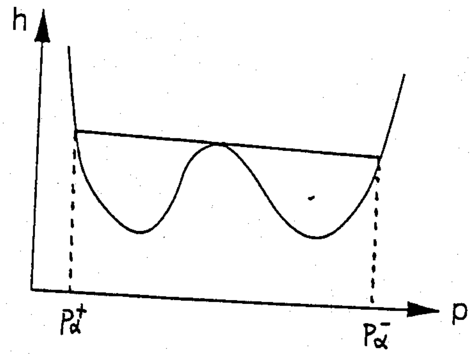

Figure 12b

We assume that $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at the point $(p_0, H(p_0))$, where $p_0 = \phi'(u_0)$ with $p^+ < p_0 < p^-$ and $u_-(t_\alpha) < u_0 < u_+(t_\alpha)$ (see, Figure 12b). The Rankine-

Hugoniot condition is give as

$$\chi'(t_\alpha) = \frac{H(\phi'(u_-(t_\alpha))) - H(\phi'(u_+(t_\alpha)))}{\phi'(u_-(t_\alpha)) - \phi'(u_+(t_\alpha))} = H'(p_0).$$

We define a function $h_{p_0}(t)$ for $t_\alpha < t < t_\alpha + \varepsilon$ by

$$h_{p_0}(t) = \frac{H(\phi'(u_-(t))) - H(\phi'(u_+(t)))}{\phi'(u_-(t)) - \phi'(u_+(t))}(p_0 - \phi'(u_+(t))) + H(\phi'(u_+(t))).$$

We now distinguish two cases.

a) If $H(p_0) \le h_{p_0}(t)$ for $t_\alpha < t < t_\alpha + \varepsilon$ for sufficiently small $\varepsilon > 0$, then the viscosity criterion is satisfied for $t < t_\alpha + \varepsilon$, (see, Figure 13a), so that we can choose single valued continuous branches of the geometric solution as the viscosity solution.

b) If If $H(p_0) > h_{p_0}(t)$ for $t_\alpha < t < t_\alpha + \varepsilon$ for sufficiently small $\varepsilon > 0$, then the viscosity criterion is violated for $t < t_\alpha + \varepsilon$, so that we should also construct the new shock and the solution (see, Figure 13b)

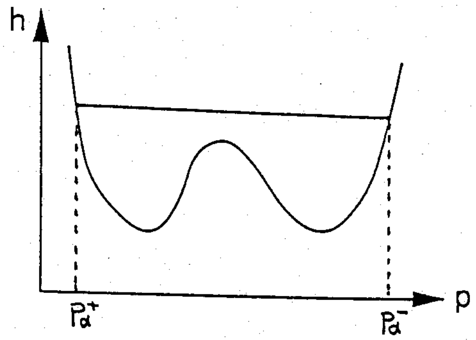

Figure 13a

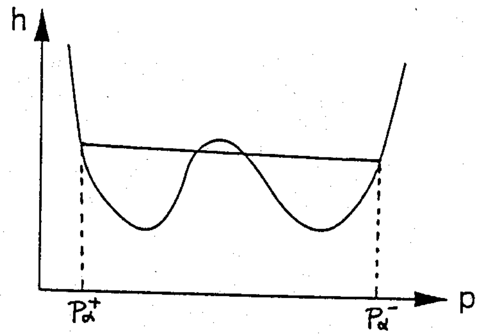

Figure 13b

In this case we also use the technique in the case (1) to construct the contact discontinuity shock curve. We use the same notations as in the case (1).

Since we have relations

$$H'(p_0) = \frac{H(p^+) - H(p_0)}{p^+ - p_0} = \frac{H(p^-) - H(p_0)}{p^- - p_0}$$

and $H''(p_0) \neq 0$, the relations $H'(q^\pm) = \frac{H(p^\pm) - H(p_0)}{p^\pm - p_0}$ are equivalent to the relation $q^\pm = \psi_\pm(p^\pm)$ for some smooth functions $\psi_\pm$ near the points $(p_0, p^\pm)$ respectively.

The contact discontinuity curves are the solutions of the following initial value problems:

$$\begin{cases} \chi'_{c\pm}(t) = H'(\psi_\pm(y_x^\pm(t, \chi_{c\pm}(t)))), \\ \chi_{c\pm}(t_\alpha) = \chi(t_\alpha). \end{cases}$$

The characteristic which is started at a point $(\tau, \chi_{c\pm}(\tau))$ is given as follows:

$$\begin{cases} \tilde{x}_\pm(t) = \chi_{c\pm}(\tau) + (t-\tau)H'(\psi_\pm(y_x^\pm(\tau, \chi_{c\pm}(\tau)))), \\ \tilde{p}_\pm(t) = \psi_\pm(y_x^\pm(\tau, \chi_{c\pm}(\tau))) \\ \tilde{y}_\pm(t) = y^\pm(\tau, \chi_{c\pm}(\tau)) \\ \qquad + (t-\tau)\{-H(\psi_\pm(y_x^\pm(\tau, \chi_{c\pm}(\tau)))) + \psi_\pm(y_x^\pm(\tau, \chi_{c\pm}(\tau)))H'(\psi_\pm(y_x^\pm(\tau, \chi_{c\pm}(\tau))))\}. \end{cases}$$

We distinguish the following three subcases.

b-1) If $\phi''(u_+(t)) > 0$ and $\phi''(u_-(t)) > 0$ for any $t_\alpha < t < \alpha + \varepsilon$, then $\chi_{c_+}(t)$ is convex and $\chi_{c_-}(t)$ is concave by the calculation in the case (1). Thus we can construct the local viscosity solution for $t_\alpha < t < \alpha + \varepsilon$. In this subcase two shocks bifurcate from the point (t_α, x_α). Both of them are the contact discontinuity shock curves $(t, \chi_{c_\pm}(t))$.

We draw the picture which is illustrating the situation as follows:

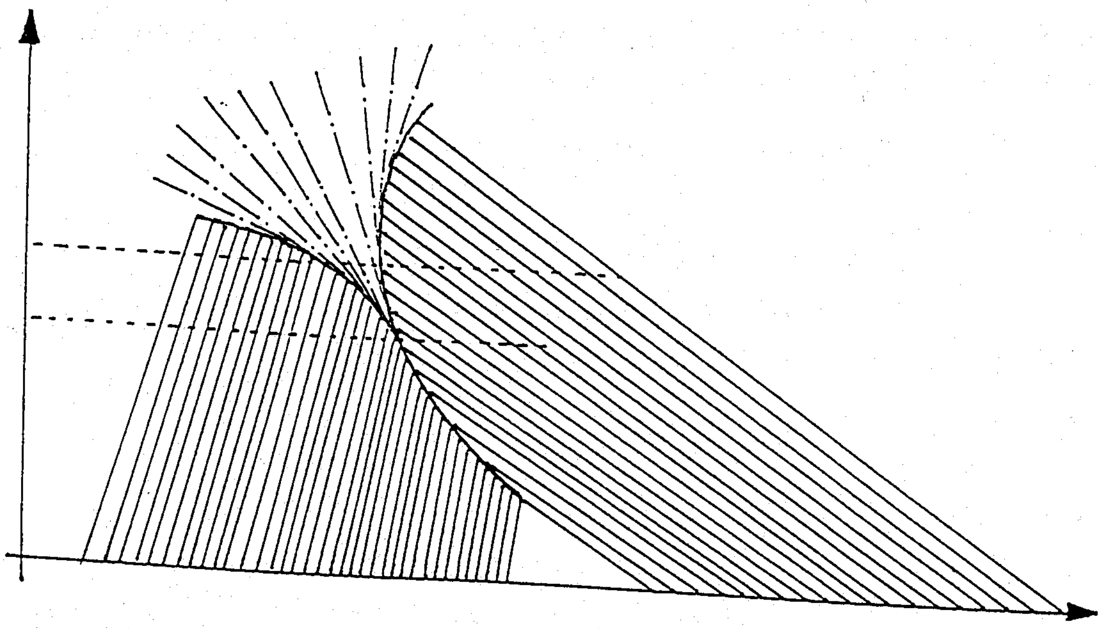

FIGURE 14

Then we can draw the picture of the graph of the viscosity solution for $t > t_\alpha$ and the shock curve around t_α.

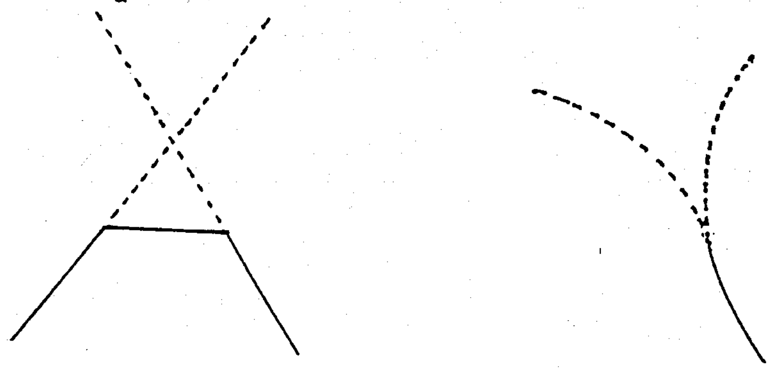

FIGURE 15

b-2) If $\phi''(u_+(t)) \leq 0$ and $\phi''(u_-(t)) \geq 0$ for any $t_\alpha < t < \alpha + \varepsilon$, then $\phi'(u_+)$ is monotone decreasing and $\phi'(u_-)$ is monotone increasing. It follows that $\phi'(u_+(t))$ is monotone decreasing and $\phi'(u_-(t))$ is monotone increasing with respect to the t-variable. This means that the viscosity criterion is satisfied for $t_\alpha < t < \alpha + \varepsilon$.

b-3) If $\phi''(u_+(t)) > 0$ and $\phi''(u_-(t)) \leq 0$ for $t_\alpha < t < \alpha + \varepsilon$, then $\chi_{c_+}(t)$ is convex and $\phi''(u_-(t))$ is monotone increasing with respect to the t-variable. In this case we adopt the contact discontinuity shock curve $(t, \chi_{c+}(t))$. Since $\phi''(u_+(t)) > 0$, $\chi_{c+}(t)$ is convex by the argument in the case (1).

We consider

$$y_-(t) = t\{-H(\phi'(u_-(t))) + \phi'(u_-(t))H'(\phi'(u_-(t)))\} + \phi(u_-(t))$$

and

$$y_\alpha(t) = t\{-H(p_0) + p_0 H'(p_0)\} + p_0,$$

where $x(t, u_0) = u_0 + tH'(p_0) = x(t, u_-(t)) = u_-(t) + tH'(\phi'(u_-(t)))$. Differentiating the last equality with respect to the t-variable, we get

$$H'(p_0) = u'_-(t) + H'(\phi'(u_-(t))) + tH''(\phi'(u_-(t))\phi'(u_-(t))u'_-(t).$$

Then

$$\begin{aligned} y'_-(t) =& - H(\phi'(u_-(t)) + \phi'(u_-(t))H'(\phi'(u_-(t))) \\ &+tH''(\phi'(u_-(t)))\phi'(u_-(t))\phi''(u_-(t))u'_-(t) + \phi'(u_-(t))u'_-(t) \\ =& - H(\phi'(u_-(t))) + \phi'(u_-(t))H'(p_0) \end{aligned}$$

and

$$y'_\alpha(t) = -H(p_0) + p_0 H'(p_0).$$

So we obtain

$$\frac{d}{dt}(y_\alpha(t) - y_-(t)) = H(\phi'(u_-(t))) - H(p_0) - H'(p_o)(\phi'(u_(t)) - p_0).$$

On the other hand, we consider a function

$$h(t) = \frac{H(\phi'(u_-(t)) - H(p_0)}{\phi'(u_-(t)) - p_0}.$$

It follows that

$$h'(t) = \frac{\phi''(u_-(t))u'_-(t)\{H'(\phi'(u_-(t))(\phi'(u_-(t)) - p_0) - (H'(\phi'(u_-(t)) - H(p_0))}{(\phi'(u_-(t)) - p_0)^2}.$$

Since $\phi''(u_-(t)) \geq 0$, $u'_-(t) < 0$ and $H'(\phi'(u_-(t)) > \frac{H(\phi'(u_-(t)) - H(p_0)}{\phi'(u_-(t)) - p_0}$, we have $h'(t) \geq 0$. By the assumption, we have $h(t_\alpha) = H'(p_0)$, so that $h(t) \geq H'(p_0)$. This inequality implies that $\frac{d}{dt}(y_\alpha(t) - y_-(t)) \geq 0$. Since $y_\alpha(t_\alpha) = y_-(t_\alpha)$, we have $y_\alpha(t) \geq y_-(t)$.

We also consider

$$y_{c\pm}(t) = t\{-H(\phi'(x_\pm(t))) + \phi'(x_\pm(t))H'(\phi'(x_\pm(t)))\} + \phi(x_\pm(t)),$$

where $\chi_{c+}(t) = x_\pm(t) + tH'(\phi'(x_\pm(t)))$, $x_-(t)$ is in the left side and $x_+(t)$ is in the right side.It is clear that $y_-(t) > y_+(t)$ for $t_\alpha < t < t_\alpha + \varepsilon$. It follows that , there exists a unique $(t, \chi_g(t))$

with $x(t, u_0) < \chi_g(t) < \chi_{c+}(t)$, such that $y_-(t, \chi_g(t)) = \tilde{y}_+(t, \chi_g(t))$. We should consider that the viscosity criterion is satisfied or not across the curve $(t, \chi_g(t))$.

Since the contact discontinuity shock is the curve define by the ordinary differential equation $\chi'_{c+}(t) = H'(\psi_+(u_+(\tau(t)))$, we already get the following formula:

$$\chi''_{c+}(t) = H''(\psi(\phi'(u_+(t))\frac{\partial\psi}{\partial p}(\phi'(u_+(t))\phi''(u_+(t))u'_+(t)$$

in the arguments for Case 1). By definition, we have

$$u_-(t) + tH'(\phi'(u_-(t))) = \chi_g((t)) = \chi_{c+}(\tau(t)) + (t - \tau(t))H'(\psi_+(\phi'(u_+(\tau(t))))).$$

Differentiating the both side of the last equality with respect t and follows from the above formula, we have

$$\begin{aligned}
\chi'_g(t) &= \chi'_{c+}(\tau(t))\tau'(t) \\
&\quad +(1 - \tau(t))H'(\psi_+(\phi'(u_+(\tau(t))) \\
&\quad +(t - \tau(t))H''(\psi_+(\phi'(u_+(\tau(t)))\frac{d\psi_+}{dp}\phi''(u_+(\tau(t))u'_+(\tau(t))\tau'(t) \\
&= H'(\psi_+(\phi'(u_+(\tau(t))))\tau'(t) + (1 - \tau'(t)))H'(\psi_+(\psi'(u_+(\tau(t)))) \\
&\quad +(t - \tau(t))\chi''_{c+}(\tau(t))\tau'(t) \\
&= H'(\psi_+(\phi'(u_+(\tau(t))) + (t - \tau(t))\chi''_{c+}(t)\tau'(t).
\end{aligned}$$

It also follows from the calculation in Case 1) and the assumption $\phi''(u_+(t)) > 0$ that $\chi''_{c+}(t) > 0$. Since $t - \tau(t) > 0$ and $\tau'(t) > 0$ for $t > t_\alpha$, we have

$$\chi'_g(t) > H'(\psi_+(\phi'(u_+(\tau(t)))$$

for $t > t_\alpha$. Since $\chi_g(t)$ is given by the intersection of characteristics, so that it satisfies the Rankine-Hugoniot condition

$$\chi'_g(t) = \frac{H(\psi_+(\phi'(u_+(\tau(t))) - H(\phi'(u_-(t)))}{\psi_+(\phi'(u_+(\tau(t))) - \phi'(u_-(t))}.$$

It follows that

$$H'(\psi_+(\phi'(u_+(\tau(t))) < \frac{H(\psi_+(\phi'(u_+(\tau(t))) - H(\phi'(u_-(t)))}{\psi_+(\phi'(u_+(\tau(t))) - \phi'(u_-(t))}.$$

Thus the viscosity criterion is satisfied across $(t, \chi_g(t))$ (see, Figure 16).

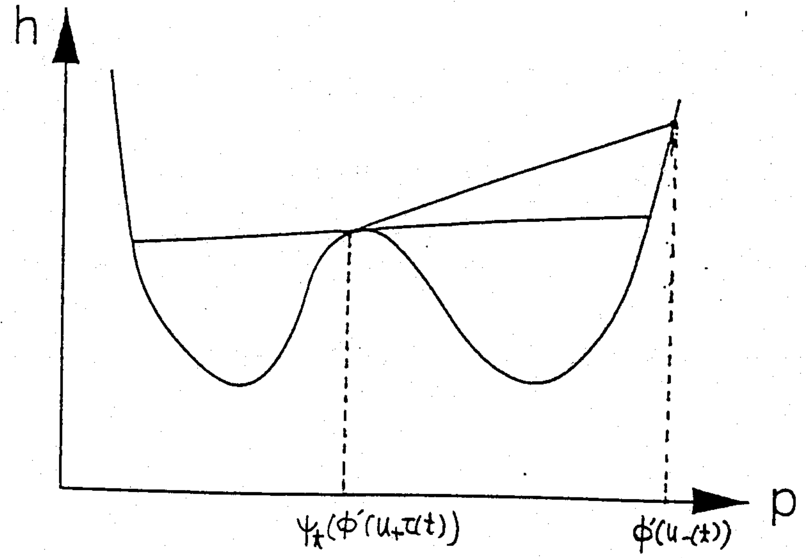

FIGURE 16

These mean that two shocks bifurcate from the point (t_α, x_α) (see, Figure 17).

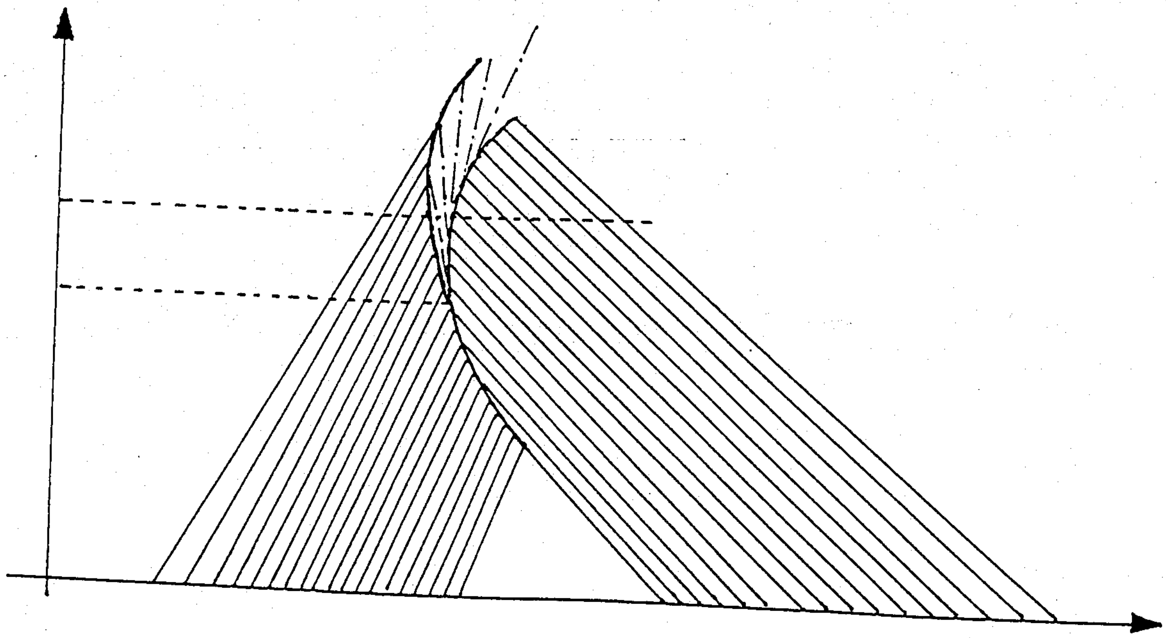

FIGURE 17

The left one is a new shock given by the intersection of the original characteristics from the left side and the new characteristics from the contact discontinuity shock curve (i.e., *the* *rarefaction waves*). We can draw the picture of the graph of the viscosity solution for $t > t_\alpha$ and the shock curve around t_α.

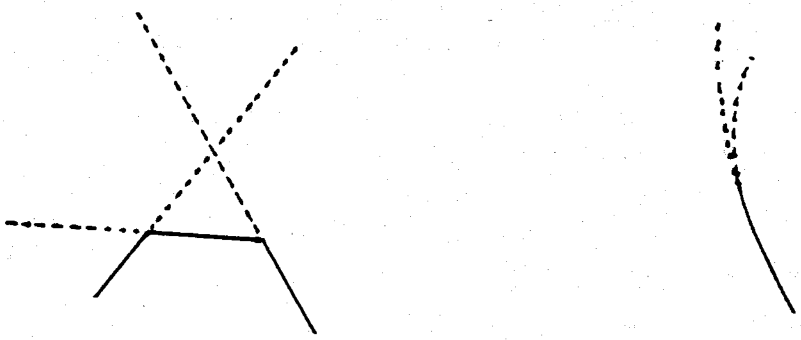

FIGURE 18

Case (3). We assume that the graph of the viscosity solution at the time $t \leq t_\alpha$ is depicted as in Figure 19a.

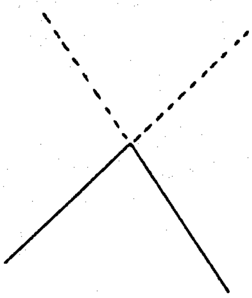

Figure 19a

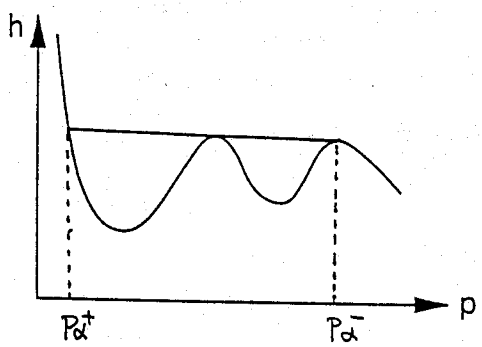

Figure 19b

By the same reasons as in Case (1), we may assume that $H''(\phi'(u_-(t_\alpha)) < 0$, so that the graph of the Hamiltonian is given as in Figure 19b.

We also distinguish two cases as follows:

a) If

$$H'(y_x^-(t,\chi(t))) \geq \frac{H(y_x^+(t,\chi(t))) - H(y_x^-(t,\chi(t)))}{y_x^+(t,\chi(t)) - y_x^-(t,\chi(t))}$$

for $t_\alpha \leq t < t_\alpha + \varepsilon$ for sufficiently small $\varepsilon > 0$, then we have exactly the same results as in Case (2).

b) If

$$H'(y_x^-(t,\chi(t))) < \frac{H(y_x^+(t,\chi(t))) - H(y_x^-(t,\chi(t)))}{y_x^+(t,\chi(t)) - y_x^-(t,\chi(t))}$$

for $t_\alpha \leq t < t_\alpha + \varepsilon$ for sufficiently small $\varepsilon > 0$, then we can easily show that the viscosity criterion is violated for $t_\alpha < t < t_\alpha + \varepsilon$. We also adopt the notion of the contact discontinuity shock like as in Case (1). In this case we have relations

$$H'(p_0) = \frac{H(p_\alpha^+) - H(p_0)}{p_\alpha^+ - p_0} = H'(p_\alpha^-) = \frac{H(p_\alpha^+) - H(p_\alpha^-)}{p_\alpha^+ - p_\alpha^-}$$

and $H''(p_0) \neq 0$, so that the relation $H'(q) = \frac{H(p^+)-H(q)}{p^+-q}$ is equivalent to the relation $q = \psi(p^+)$ for some smooth function ψ near the point (p_0, p_α^+).

Since $H'(p_0) = H'(p_\alpha^-) = \chi'(t_\alpha)$, we can define the contact discontinuity curve like as the previous cases.

We also distinguish the following two subcases like as in Case (1).

b-1) If $\phi''(u_-(t)) > 0$, then $\phi'(u_-(t))$ moves to the left direction. This case is contained in the case a), so that it has been already discussed.

b-2) If $\phi''(u_-(t) < 0$ and the viscosity criterion is violated across $\chi(t)$ for $t > t_\alpha$, then exactly the same results as in Case (1), b-1) hold.

Case (4). In thsi case we ahve the following relations:

$$H'(p_\alpha^+) = H'(p_\alpha^-) = \frac{H(p_\alpha^+) - H(p_\alpha^-)}{p_\alpha^+ - p_\alpha^-}.$$

Since $H'(\phi'(u(t))$ gives the slope of the characteristic line on the (t,x)-plane, the above realtion means that the characteristics $x(t, u_+(t_\alpha))$ and $x(t, u_-(t_\alpha))$ exactely coincide. This contradicts to the assumption that the characteristics cross transversally at the point $(t_\alpha, \chi(t_\alpha))$. Thus this case cannot occur.

Case (5). We assume that the graph of the viscosity solution at the time $t \leq t_\alpha$ is depicted as in Figure 20a.

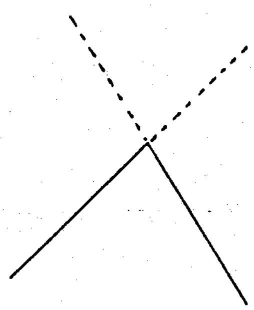

Figure 20a

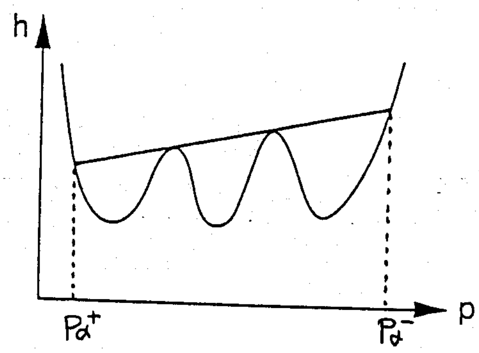

Figure 20b

We also assume that there exist p_0, p_1 such that $p_\alpha^+ < p_0 < p_1 < p_\alpha^-$, $H''(p_0) < 0$, $H''(p_1) < 0$ and

$$H'(p_0) = H'(p_1) = \frac{H(p_\alpha^+) - H(p_0)}{p_\alpha^+ - p_0} = \frac{H(p_\alpha^-) - H(p_0)}{p_\alpha^- - p_0} = \frac{H(p_\alpha^+) - H(p_\alpha^-)}{p_\alpha^+ - p_\alpha^-},$$

so that the graph of the Hamiltonian is given as in Figure 20b. We define functions $h_{p_i}(t)$ for $t_\alpha < t < t_\alpha + \varepsilon$ $(i = 0, 1)$ by

$$h_{p_i}(t) = \frac{H(\phi'(u_-(t)) - H(\phi'(u_+(t)))}{\phi'(u_-(t)) - \phi'(u_+(t))}(p_i - \phi'(u_+(t))) + H(\phi'(u_+(t))).$$

We distinguish three cases.
a) If $H(p_0) \leq h_{p_0}(t)$ and $H(p_1) \leq h_{p_1}(t)$ for $t_\alpha < t < t_\alpha + \varepsilon$ for sufficiently small $\varepsilon > 0$, then the viscosity criterion is satisfied across $(t, \chi(t))$ for $t < t_\alpha + \varepsilon$ (cf., Figure 21a).

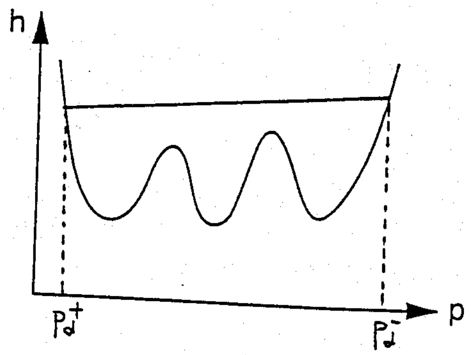

Figure 21a

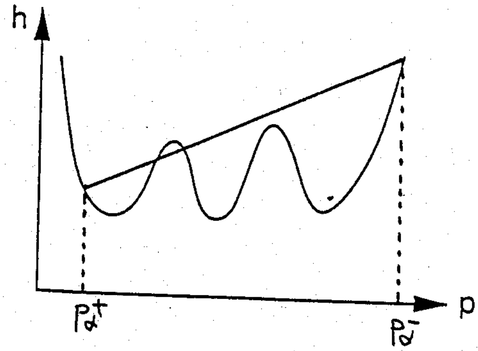

Figure 21b

b) If only one of the conditions $H(p_0) > h_{p_0}(t)$ or $H(p_1) > h_{p_1}(t)$ holds for $t_\alpha < t < t_\alpha + \varepsilon$ for sufficiently small $\varepsilon > 0$, then the viscosity criterion is violated across $(t, \chi(t)$ for $t < t_\alpha + \varepsilon$ (cf.,Figure 21b). This case is contained in the case (2).
c) If the both of the conditions $H(p_0) > h_{p_0}(t)$ and $H(p_1) > h_{p_1}(t)$ hold for $t_\alpha < t < t_\alpha + \varepsilon$ for sufficiently small $\varepsilon > 0$, then the viscosity criterion is violated across $(t, \chi(t)$ for $t < t_\alpha + \varepsilon$ (see, Figure 22a).

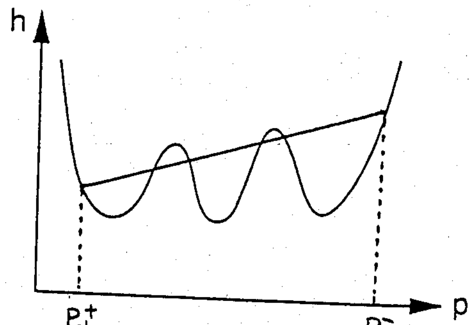

Figure 22a

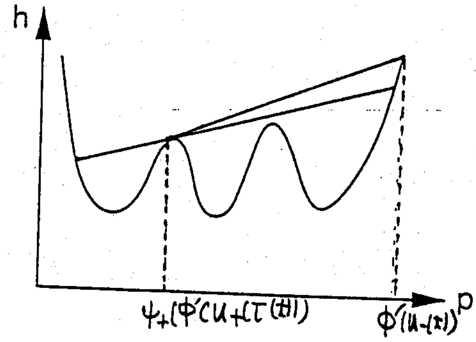

Figure 22b

In this case we also use the technique in the case (2), b). Since we have relations

$$H'(p_0) = \frac{H(p_\alpha^+) - H(p_0)}{p_\alpha^+ - p_0} = H'(p_1) = \frac{H(p_\alpha^-) - H(p_1)}{p_\alpha^- - p_1}$$

and $H''(p_i) \neq 0$, $(i = 0, 1)$, it is equivalent to the relations $q^+ = \psi_+(p^+)$ (respectively $q^- = \psi_-(p^-)$) for some smooth function $\psi_\pm$ near the point (p_0, p_α^+) (respectively, (p_1, p_α^-)). So the contact discontinuity curves are the solutions of the following initial value problems:

$$\begin{cases} \chi'_{c\pm}(t) = H'(\psi_\pm(y_x^\pm(t, \chi_{c\pm}(t)))), \\ \chi_{c\pm}(t_\alpha) = \chi(t_\alpha). \end{cases}$$

The characteristic which is started at a point $(\tau, \chi_{c\pm}(\tau))$ is given as follows:

$$\begin{cases} \tilde{x}_\pm(t) =\chi_{c\pm}(\tau) + (t-\tau)H'(\psi_\pm(y_x^\pm(\tau,\chi_{c\pm}(\tau)))), \\ \tilde{p}_\pm(t) =\psi_\pm(y_x^\pm(\tau,\chi_{c\pm}(\tau))) \\ \tilde{y}_\pm(t) =y^\pm(\tau,\chi_{c\pm}(\tau)) \\ \qquad +(t-\tau)\{-H(\psi_\pm(y_x^\pm(\tau,\chi_{c\pm}(\tau)))) + \psi_\pm(y_x^\pm(\tau,\chi_{c\pm}(\tau)))H'(\psi_\pm(y_x^\pm(\tau,\chi_{c\pm}(\tau))))\}. \end{cases}$$

We also distinguish the following three subcases.

b-1) Suppose that $\phi''(u_+(t)) > 0$ and $\phi''(u_-(t)) > 0$ for any $t_\alpha < t < \alpha+\varepsilon$, then the situation is exactly the same as that of the subcase b-1) in the case (2), so that two shocks bifurcate from the point (t_α, x_α). Both of them asre the contact discontinuity curves $(t, \chi_{c\pm}(t)$. The pictures are exactly the same as Figures 14, 15.

b-2) Suppose that $\phi''(u_+(t)) \le 0$ and $\phi''(u_-(t)) \ge 0$ for any $t_\alpha < t < \alpha+\varepsilon$. By the same reason as that of the subcase b-2) in the case (2), the viscosity criterion is satisfied across $(t, \chi(t))$ for $t_\alpha < t < t_\alpha + \varepsilon$.

b-3) Suppose that $\phi''(u_+(t)) > 0$ and $\phi''(u_-(t)) \le 0$ for $t_\alpha < t < \alpha+\varepsilon$. By the same reason as that of the subcase b-3) in the case (2), $\chi_{c\pm}(t)$ is convex and there exists a unique $(t, \chi_g(t))$ with $x(t, u_0) < \chi_g(t) < \chi_{c+}(t)$ such that $y_-(t, \chi_g(t)) = \tilde{y}_+(t, \chi_g(t))$. We should check that the viscosity criterion is satisfied across the curve $(t, \chi_g(t))$. By the same arguments as those of the subcase b-3) in the case (2), we have

$$H'(\psi_+(\phi'(u_+(\tau(t)))) < \frac{H(\psi_+(\phi'(u_+(\tau(t)))) - H(\phi'(u_-(t))))}{\psi_+(\phi'(u_+(\tau(t))) - \phi'(u_-(t))}.$$

We remark that $p_\alpha < \phi'(u_+(\tau(t))) < \phi'(u_+(t))$ and $p_- < \phi'(u_-(t))$, so that we have the following enequality

$$\psi_+(\phi'(u_+(\tau(t))) < p_0.$$

Thus we have

$$H'(p_0) < H'(\psi_+(\phi'(u_+(\tau(t))))) = \frac{H(\phi'(u_+(\tau(t)))) - H(\psi_+(\phi'(u_+(\tau(t)))))}{\phi'(u_+(\tau(t))) - \psi_+(\phi'(u_+(\tau(t))))}$$

and hence

$$\frac{H(p_1) - H(p_0)}{p_1 - p_0} = H'(p_0) < \frac{H(\psi_+(\phi'(u_+(\tau(t))))) - H(\phi'(u_-(t)))}{\psi_+(\phi'(u_+(\tau(t)))) - \phi'(u_-(t))}.$$

The last inequality means that the line between the two points $(\phi'(u_-(t)), H'(\phi'(u_-(t))))$ and $(\psi_+(\phi'(u_+(\tau(t))), H'(\psi_+(\phi'(u_+(\tau(t)))))$ never cross the graph of the Hamiltonian (see, Figure 22b). Thus the viscosity criterion is satisfied across $(t, \chi_g(t))$. So the pictures of the shock curve and the graph of the viscosity solution for $t > t_\alpha$ is the same as those of in the subcase b-3) in the case (2).

Case (6). The bifurcations of the graphs of the geometric solution at the time t_α is depicted as follows:

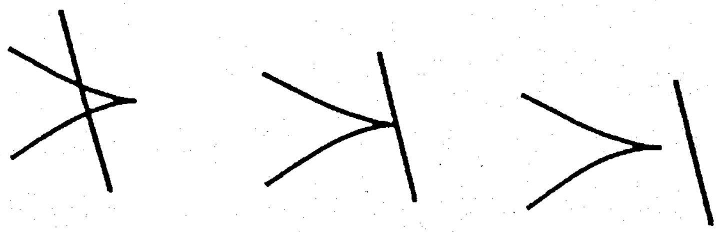

FIGURE 23

We use the same notation as the case (1). Since $u_-(t_\alpha)$ corresponds to the cusp point, we have $1 + t_\alpha H''(\phi(u_-(t_\alpha)))\phi''(u_-(t_\alpha)) = 0$. Let $(t, \sigma(t))$ be the locus of the cusps, where we denote $\sigma(t) = \sigma_-(t) + tH'(\phi'(\sigma_-(t)))$ as the family of characteristics come from the left side, so that we have $1 + tH''(\phi'(\sigma_-(t)))\phi''(\sigma_-(t)) = 0$ and $\sigma'_-(t) < 0$. It follows that $H''(\phi'(\sigma_-(t)))\phi''(\sigma_-(t)) < 0$ and $\sigma'(t) = H'(\phi'(\sigma'_-(t))$. Differentiating the equation again, we get $\sigma''(t) = H''(\phi'(\sigma_-(t))\phi''(\sigma_-(t))\sigma'_-(t) > 0$. Therefore $(t, \sigma(t))$ is strictly convex. We denote $\chi(t) = u_-(t) + tH'(\phi'(u_-(t))) = u_+(t) + tH'(\phi'(u_+(t)))$ for $t \le t_\alpha$, then we have $u_-(t_\alpha) = \sigma_-(t_\alpha)$, so that $\sigma'(t_\alpha) = H'(\phi'(\sigma_-(t_\alpha))) = H'(\phi(u_-(t_\alpha))) = \chi'(t_\alpha)$ by the proof of Theorem 3.4. We also construct the contact discontinuity $(t, \chi_c(t))$ exactly the same way as that of in the case (1). We need to examine the following two subcases.

a) Assume that $\sigma(t) \ge \chi_c(t)$ for $t \ge t_\alpha$. Since both χ_c and σ are convex near t_α, we have that $\sigma''(t_\alpha) > \chi''(t_\alpha)$.

On the other hand, we have

$$y(t, \sigma_\pm(t)) = t\{-H(\phi'(\sigma_\pm(t))) + \phi'(\sigma_\pm(t))H'(\phi'(\sigma_\pm(t)))\} + \phi(\sigma_\pm(t)),$$

$$\begin{aligned}\frac{dy}{dt}(t, \sigma_+(t)) &= -H(\phi'(\sigma_+(t))) + \phi'(\sigma_+(t))H'(\phi'(\sigma_+(t))) \\ &\quad +\phi'(\sigma_+(t))\sigma'_+(t)\{1 + tH''(\phi'(\sigma_+(t)))\phi''(\sigma_+(t))\}\end{aligned}$$

and

$$\frac{dy}{dt}(t, \sigma_-(t)) = -H(\phi'(\sigma_-(t))) + \phi'(\sigma_-(t))H'(\phi'(\sigma_-(t))).$$

Let $A(t) = y(t, \sigma_+(t)) - y(t, \sigma_-(t))$ for $t \ge t_\alpha$.

Differentiating the equality $\sigma_-(t) + tH'(\phi'(\sigma_-(t))) = \sigma_+(t) + tH'(\phi'(\sigma_+(t)))$ with respect to t, we get

$$\sigma'_+(t)\{1 + tH''(\phi'(\sigma_+(t)))\phi''(\sigma_+(t))\} = H'(\phi'(\sigma_-(t))) - H'(\phi'(\sigma_+(t))).$$

It follows that

$$A'(t) = H'(\phi(\sigma_-(t)))\{\phi'(\sigma_+(t)) - \phi'(\sigma_-(t))\} - (H(\phi'(\sigma_+(t))) - H(\phi'(\sigma_-(t)))).$$

Furthermore, we have

$$\begin{aligned}A''(t) =&(H'(\phi'(\sigma_-(t)) - H'(\phi'(\sigma_+(t)))\phi''(\sigma_+(t))\sigma'_+(t)\\ &+H''(\phi'(\sigma_-(t)))\phi''(\sigma_-(t))\{\phi'(\sigma_+(t)) - \phi'(\sigma_-(t))\}\sigma'_-(t).\end{aligned}$$

Since $\sigma'_+(t) = \frac{H'(\phi'(\sigma_-(t))-H'(\phi'(\sigma_+(t))}{1+tH''(\phi'(\sigma_+(t))\phi''(\sigma_+(t))}$ and $\sigma''(t) = H''(\phi'(\sigma_-(t)))\phi''(\sigma_-(t))\sigma'_-(t)$, we have

$$A''(t) = \frac{(H'(\phi'(\sigma_-(t)) - H'(\phi'(\sigma_+(t)))^2}{1+tH''(\phi'(\sigma_+(t))\phi''(\sigma_+(t))}\phi''(\sigma_+(t)) + \sigma''(t)\{\phi'(\sigma_+(t)) - \phi'(\sigma_-(t))\}.$$

On the other hand, as we already calculated in the case (1) that

$$\chi''_c(t) = -\frac{(H'(\phi'(u_+(t)) - H'(\psi(\phi'(u_+(t)))))^2}{\phi'(u_+(t)) - \psi(\phi'(u_+(t)))}\frac{\phi''(u_+(t))}{1+tH''(\phi'(u_+(t))\phi''(u_+(t))}.$$

At the point $t = t_\alpha$, we have $u_\pm(t_\alpha) = \sigma_\pm(t_\alpha)$ and $\psi(\phi'(\sigma_+(t_\alpha))) = \phi'(\sigma_-(t_\alpha))$, so that

$$\chi''_c(t_\alpha) = -\frac{(H'(\phi'(u_+(t_\alpha)) - H'(\psi(\phi'(u_+(t_\alpha)))))^2}{\phi'(u_+(t_\alpha)) - \psi(\phi'(u_+(t_\alpha)))}\frac{\phi''(u_+(t_\alpha))}{1+t_\alpha H''(\phi'(u_+(t_\alpha))\phi''(u_+(t_\alpha))}.$$

Thus, we have

$$A''(t_\alpha) = (\sigma''(t_\alpha) - \chi''_c(t_\alpha))(\phi'(\sigma_+(t_\alpha) - \phi'(\sigma_-(t_\alpha)).$$

Since $\sigma''(t_\alpha) > \chi''_c(t_\alpha)$ and $\phi'(\sigma_+(t_\alpha) < \phi'(\sigma_-(t_\alpha)$, we have $A''(t_\alpha) < 0$. This means that $A'(t) < 0$ near t_α, so $y(t,\sigma_+(t)) < y(t,\sigma_-(t))$.

We also consider

$$y_+(t) = t\{-H(\phi'(u_+(t))) + \phi'(u_+(t))H'(\phi'(u_+(t)))\} + \phi(u_+(t))$$

and

$$y_\alpha(t) = t\{-H(\phi'(u_\alpha)) + \phi'(u_\alpha)H'(\phi'(u_\alpha))\} + \phi(u_\alpha),$$

where $u_\alpha = \sigma_-(t_\alpha)$ and $x(t,u_\alpha) = u_\alpha + tH'(\phi'(u_\alpha)) = x(t,u_+(t)) = u_+(t) + tH'(\phi'(u_+(t)))$. Differentiating the last equality, we get

$$H'(\phi'(u_\alpha)) = u'_+(t) + H'(\phi'(u_+(t))) + tH''(\phi'(u_+(t)))\phi''(u_+(t))u'_+(t).$$

Then

$$\begin{aligned}y'_+(t) =& - H(\phi'(u_+(t)) + \phi'(u_+(t))H'(\phi'(u_+(t)))\\ &+tH''(\phi'(u_+(t)))\phi'(u_+(t))\phi''(u_+(t))u'_+(t) + \phi'(u_+(t))u'_+(t)\\ =& - H(\phi'(u_+(t))) + \phi'(u_+(t))H'(\phi'(u_\alpha)).\end{aligned}$$

So we obtain

$$\frac{d}{dt}(y_\alpha(t) - y_+(t)) = H(\phi'(u_+(t))) - H(\phi'(u_\alpha)) - H'(\phi'(u_\alpha))(\phi'(u_+(t)) - \phi'(u_\alpha)).$$

Since $\phi'(u_+(t)) < \phi'(u_\alpha)$ and $\phi'(u_\alpha)$ is in the convex region of $H(p)$, we have

$$H'(\phi'(u_\alpha)) < \frac{H(\phi'(u_+(t))) - H(\phi'(u_\alpha))}{\phi'(u_+(t)) - \phi'(u_\alpha)},$$

so that we have $\frac{d}{dt}(y_\alpha(t) - y_+(t)) < 0$. This means that $y_\alpha(t) < y_+(t)$ for $t > t_\alpha$.

These situation describes that the two branches of the multi-valued graph have intersection for $t > t - \alpha$. This contradicts to the assumption that the singularity is ${}^1A_2^0A_1$

b) Here we assume that $\sigma(t) < \chi_c(t)$ for $t > t_\alpha$. In this subcase two shocks bifurcates from the point (t_α, x_α).

See Figure 24.

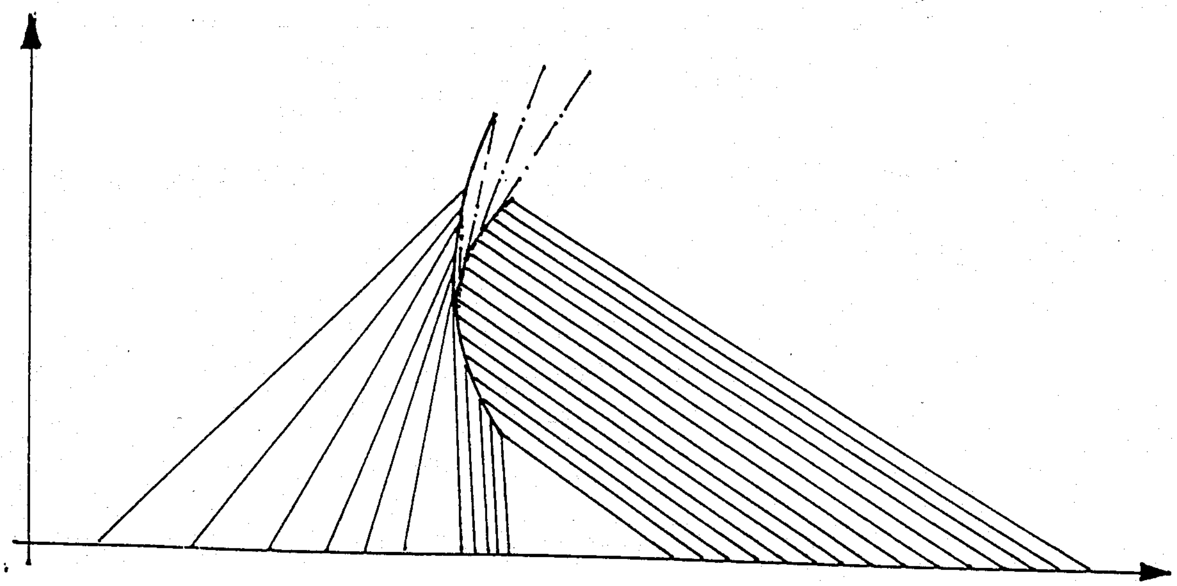

FIGURE 24

The left one is a new shock given by the intersection of the original characteristic from the left side and the new characteristic from the contact discontinuity these are also the rarefaction waves.

By definition, we have

$$\tilde{y}_\alpha(t) = (t - t_\alpha)\{-H(\psi(y_x^+(t_\alpha, \chi_c(t_\alpha)))) + \psi(y_x^+(t_\alpha, \chi_c(t_\alpha)))H''(\psi(y_x^+(t_\alpha, \chi_c(t_\alpha))))\}$$
$$+y^+(t_\alpha, \chi(t_\alpha))$$

and

$$y_-(t) = t\{-H(\phi'(u_-(t))) + \phi'(u_-(t))H'(\phi(u_-(t)))\} + \phi(u_-(t)),$$

where $u_\alpha + tH'(\phi'(u_\alpha))) = x(t, u_-(t)) = u_-(t) + tH'(\phi' u_-(t))$.

Differentiating the last equality, we get

$$u'_-(t) + H'(\phi'(u_-(t))) + tH''(\phi'(u_-(t)))\phi''(u_-(t))u'_-(t) = H'(\phi'(u_\alpha)).$$

Then

$$\begin{aligned} y'_-(t) = &- H(\phi'(u_-(t))) + \phi'(u_-(t))H'(\phi(u_(t))) \\ &+tH''(\phi'(u_-(t)))\phi'(u_(t))\phi''(u_-(t))u'_-(t) + \phi'(u_-(t))u'_-(t) \\ = &- H(\phi'(u_-(t))) + \phi'(u_-(t))H'(\phi(u_\alpha)) \end{aligned}$$

So we obtain

$$\frac{d}{dt}(\tilde{y}_\alpha(t) - y_-(t)) = H(\phi'(u_-(t))) - H(\phi'(u_\alpha)) - H'(\phi'(u_\alpha))(\phi'(u_-(t)) - \phi'(u_\alpha)).$$

Since $u_-(t) < u_\alpha$ and both of $\phi'(u_(t))$, $\phi'(u_\alpha)$ are in the convex region of $H(p)$, we have $\phi'(u_(t)) > \phi'(u_\alpha))$ and

$$H'(\phi'(u_\alpha)) > \frac{H(\phi'(u_-(t)) - H(\phi'(u_\alpha))}{\phi'(u_-(t)) - \phi'(u_\alpha)},$$

so that we obtain $\frac{d}{dt}(\tilde{y}_\alpha(t) - y_-(t)) < 0$. Since $\tilde{y}_\alpha(t_\alpha) = y_-(t_\alpha)$, the last inequality means that $\tilde{y}_\alpha(t) < y_-(t)$ for $t > t_\alpha$.

We also consider

$$\begin{aligned}\tilde{y}_\alpha(t, \sigma(t)) =&(t - \tau(t))\{-H(\psi(y_x^+(\tau(t), \chi_c(\tau(t))))\\ &+\psi(y_x^+(\tau(t), \chi_c(\tau(t))))H''(\psi(y_x^+(\tau(t), \chi_c(\tau(t)))))\}\\ &+y^+(\tau(t), \chi_c(\tau(t)))\end{aligned}$$

and

$$y_-(t, \sigma(t)) = t\{-H(\phi'(\sigma_-(t))) + \phi'(\sigma_-(t))H'(\phi(\sigma_-(t))\} + \phi(\sigma_-(t)),$$

where

$$\begin{aligned}&y^+(\tau(t), \chi_c(\tau(t))) = \tau(t)\{-H(\phi'(u_+(\tau(t)))) + \phi'(u_+(\tau(t)))H'(\phi(u_+(\tau(t)))\} + \phi(u_+(\tau(t))),\\ &y_x^+(\tau(t), \chi_c(\tau(t))) = \phi'(u_+(\tau(t)))\end{aligned}$$

and

$$\sigma(t) = \sigma_-(t)) + tH'(\phi'(\sigma_-(t)) = \chi_c(\tau(t)) + (t - \tau(t))H'(\psi(\phi'(u_+(\tau(t)))).$$

Since $\chi_c(\tau) = H'(\psi(\phi'(u_+(\tau)))$ and $1 + tH''(\phi'(\sigma_-(t)))\phi''(\sigma_-(t)) = 0$, differentiating the above equality, we get

$$\begin{aligned}H'(\phi'(\sigma_-(t))) =&H'(\psi(\phi'(u_+(\tau(t))))\\ &+(t - \tau(t))H''(\psi(\phi(u_+(\tau(t))))\frac{d\psi}{dp}\phi'(u_+(\tau(t)))u'_+(\tau(t))\tau'(t).\end{aligned}$$

Then

$$\begin{aligned}\tilde{y}'_\alpha(t, \sigma(t)) =&\tau'(t)\{-H(\phi'(u_+(\tau(t)))) + \phi'(u_+(\tau(t)))H'(\phi'(u_+(\tau(t))))\}\\ &+\tau(t)\phi'(u_+\tau(t))H''(\phi'(u_+(\tau(t))))\phi''(u_+(\tau(t)))u'_+(\tau(t))\tau'(t)\\ &+\phi'(u_+(\tau(t)))u'_+(\tau(t))\tau'(t)\\ &+(1 - \tau'(t))\{-H(\psi(\phi'(u_+(\tau(t))))) + \psi(\phi'(u_+(\tau(t)))H'(\psi(\phi'(u_+(\tau(t)))))\}\\ &+(t - \tau(t))\psi(\phi'(u_+(\tau(t))))H''(\psi(\phi'(u_+(\tau(t)))))\frac{d\psi}{dp}\phi''(u_+(\tau(t)))u'_+(\tau(t))\tau'(t)\\ =&\tau'(t)\{H(\psi(\phi'(u_+(\tau(t))))) - H(\phi'(u_+(\tau(t))))\\ &+(\phi'(u_+(\tau(t)))u_+(\tau(t)) - \psi(\phi'(u_+(\tau(t))))H'(\psi(\phi'(u_+(\tau(t)))))\}\\ &+\psi(\phi'(u_+(\tau(t))))H'(\phi'(\sigma_-(t)) - H(\psi(\phi'(u_+(\tau(t))))).\end{aligned}$$

By definition, we have

$$H'(\psi(\phi'(u_+(\tau(t))) = \frac{H(\phi'(u_+(\tau(t)))) - H(\psi(\phi'(u_+(\tau(t)))))}{\phi'(u_+(\tau(t))) - \psi(\phi'(u_+(\tau(t))))},$$

so that

$$\tilde{y}'(t,\sigma(t)) = \psi(\phi'(u_+(\tau(t))))H'(\phi'(\sigma(t))) - H(\psi(\phi'(u_+(\tau(t))))).$$

Thus we have

$$\begin{aligned}\frac{d}{dt}(\tilde{y}(t,\sigma(t)) - y_-(t,\sigma(t))) =& H'(\phi'(\sigma_-(t)) - H(\psi(\phi'(u_+(\tau(t)))) \\ & - H'(\phi'(\sigma(t)))(\phi'(\sigma_-(t)) - \psi(\phi'(u_+(\tau(t))))).\end{aligned}$$

Since both $\phi'(\sigma_-(t))$ and $\psi(\phi'(u_+(\tau(t))))$ belong to the concave region of $H(p)$, $\phi'(\sigma_-(t)) < \psi(\phi'(u_+(\tau(t))))$ for $t > t_\alpha$. Therefore

$$\frac{H'(\phi'(\sigma_-(t)) - H(\psi(\phi'(u_+(\tau(t))))}{\phi'(\sigma_-(t)) - \psi(\phi'(u_+(\tau(t))))} < H'(\phi'(\sigma(t))),$$

hence we have $\frac{d}{dt}(\tilde{y}(t,\sigma(t)) - y_-(t,\sigma(t))) > 0$. Since $\tilde{y}(t_\alpha,\sigma(t_\alpha)) = y_-(t_\alpha,\sigma(t_\alpha))$, the above inequality means that $\tilde{y}(t,\sigma(t)) > y_-(t,\sigma(t)))$ for $t > t_\alpha$. It follows that, there exists a unique $(t,\chi_r(t))$ with $x(t,u_\alpha) < \chi_r(t) < \sigma(t)$, such that $\tilde{y}(t,\chi_r(t)) = y_-(t,\chi_r(t))$.

Then we can draw the picture of the graph of the viscosity solution for $t > t_\alpha$ and the bifurcation of the shock curves around t_α. (see, Figure 25)

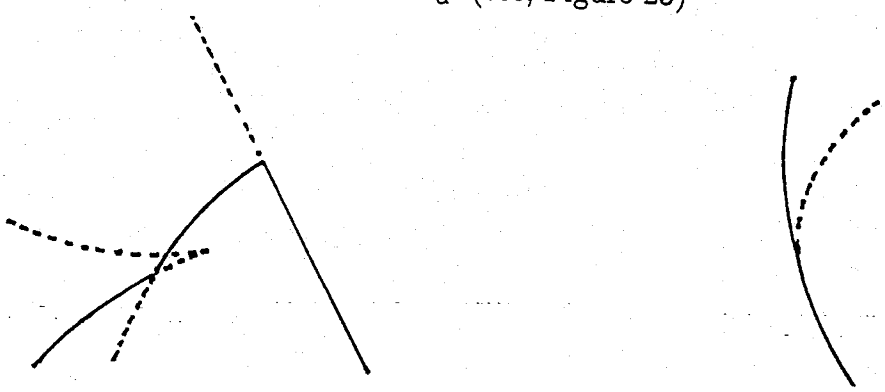

FIGURE 25

Case (7). In this case the graph of the viscosity solution at the time $t < t_\alpha$ is depicted as in Figure 26a.

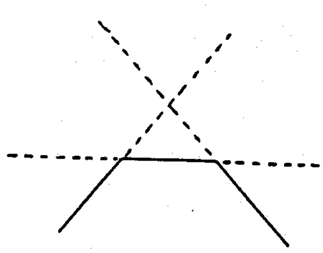

Figure 26a

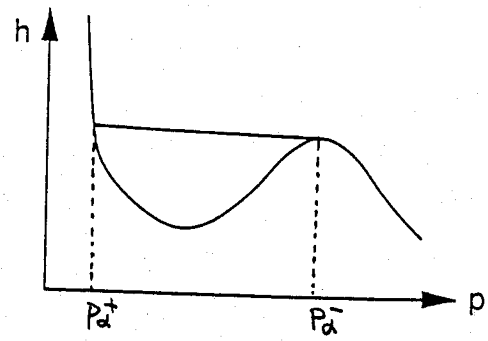

Figure 26b

At the tiem t_α, P^+P^- is tangent to the graph of $H(p)$ at hte point $(p_\alpha^-, H(p_\alpha^-))$ (see, Figure 26b). Thus we observ that two genuine shocks meet at the time t_α and after that the situation is exactly the same as that of in the case (1). So there appear two cases as follows:

a) The viscosity criterion is satisfied across the intersection curve of the characteristics, this case is already discussed in Section 1 (see, Figure 2).

b) The viscosity criterion is violated across the intersection curve of the characteristics, so that we construct the contact discontinuity shock curve for $t > t_\alpha$. We draw the picture which is illuastrating the situation as follows:

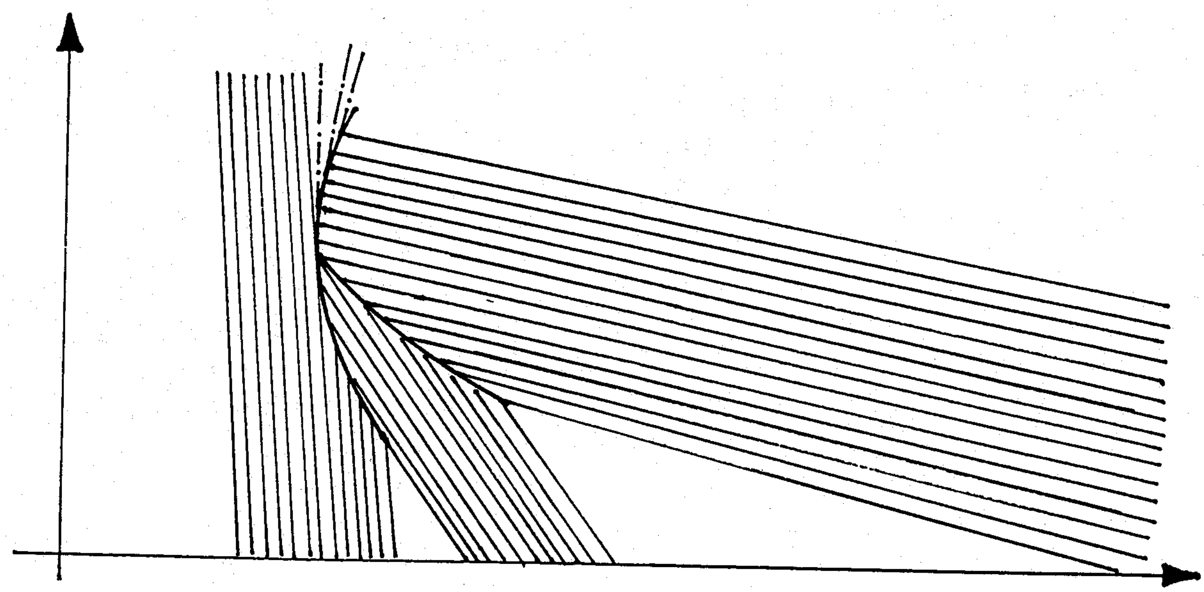

FIGURE 27

Then we can draw the pictures of the graph of the viscosity solution for $t > t_\alpha$ and the shock curve around t_α.

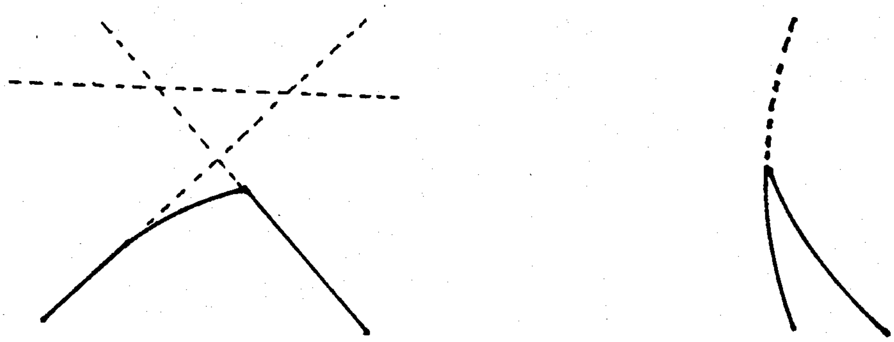

FIGURE 28

Case (8). In this case the graph of the viscosity solution at ht etime $t < t_\alpha$ is depicted as in Figure 29a.

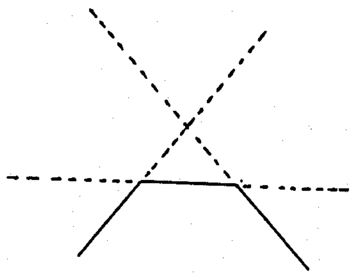

Figure 29a

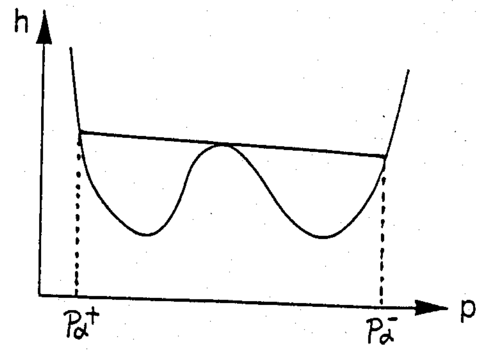

Figure 29b

By the same reasons as those of inthe case (7), the situation is exactly same the case (2) after the tim t_α (see, Figure 29b). We can also draw the picture of the situation as follows:

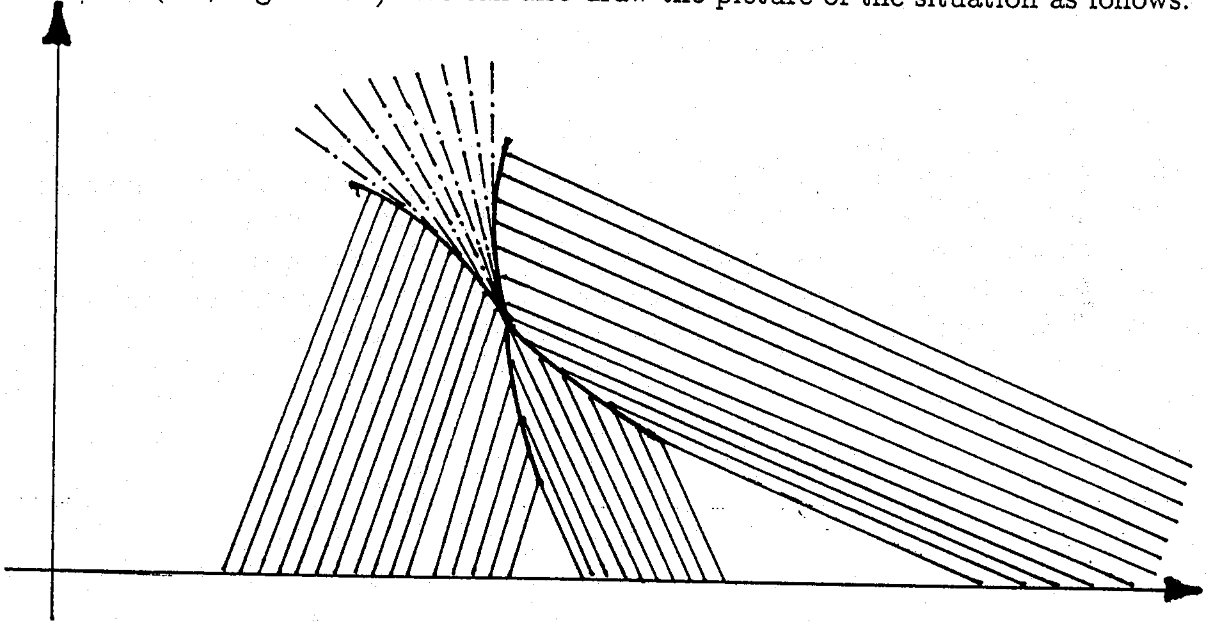

FIGURE 30

So there appear two cases except the case when the viscosity criterion is atisfied across the intersection curve of the characteristics for $t > t_\alpha$.

a) Two contact discontinuity shock curves appear after the time t_α, so that the pictures of the graph of the viscosity solution for $t > t_\alpha$ and the shock curve around t_α is depicted as follows:

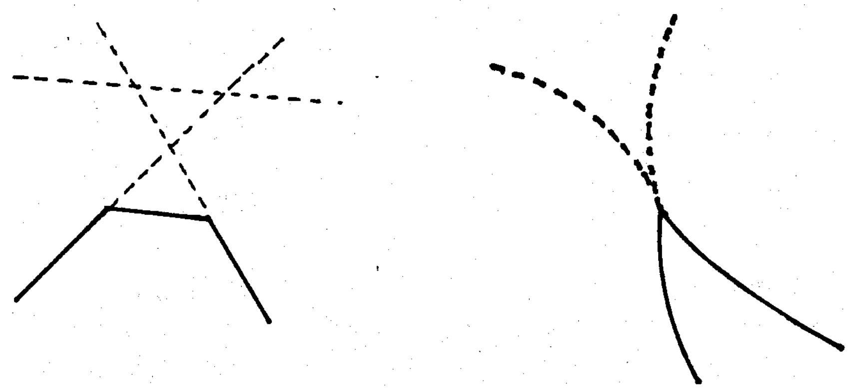

FIGURE 31

b) Two shock curves appear after the time t_α. One of them is the contact discontinuity shock curve and another one is a new shock curve given by the intersection of the original charatereistics from the initial condition and the new characteristics from the contact discontinuity shock curve (i.e., the rarefaction waves). We can also draw the pictures of the garaph of the viscosity solution for $t > t_\alpha$ and the shock curve around t_α.

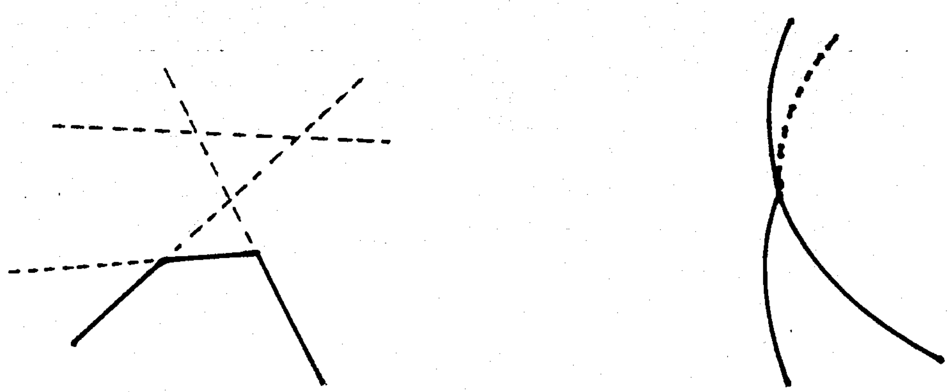

FIGURE 32

We have finished to construct the solutions for local Riemann problems corresponding to all cases in Theorem 3.4. So the bifurcations of shock curves around the time t_α when the viscosity criterion is violated can be depicted as in Figure 4 in Section 1.

Appendix. Geometric framwork

In which we give a brief review of the geometric framework for the study of singularities geometric solutions of Hamilton-Jacobi equations ([14][16][17]). We have only considered the case for one space variable in the main text, however, we describe the theory for the general case here.

Let $J^1(\mathbb{R}^n, \mathbb{R})$ be the 1-jet bundle of functions of n-variables which may be considered as $\mathbb{R}^{2n+1}$ with a natural coordinate system $(x_1, \dots, x_n, y, p_1, \dots, p_n)$, where $(x_1, \dots, x_n)$ is a coordinate system of $\mathbb{R}^n$. We also have a natural projection $\pi : J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ given by $\pi(x, y, p) = (x, y)$.

An immersion germ $i : (L_0, u_0) \to J^1(\mathbb{R}^n, \mathbb{R})$ is said to be *a Legendrian immersion germ* (i.e., Legendrian submanifold germ) if $\dim L = n$ and $i^*\theta = 0$, where $\theta = dy - \sum_{i=1}^n p_i \cdot dx_i$. The image of $\pi \circ i$ is called *the wave front set* of i and it is denoted by $W(i)$. We also consider the 1-jet bundle $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and the canonical 1-form Θ on that space. Let $(t, x_1, \dots, x_n)$ be a canonical coordinate system on $\mathbb{R} \times \mathbb{R}^n$ and $(t, x_1, \dots, x_n, y, s, p_1, \dots, p_n)$ the corresponding coordinate system on $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. Then, the canonical 1-form is given by $\Theta = dy - \sum_{i=1}^n p_i \cdot dx_i - s \cdot dt = \theta - s \cdot dt$.

We define the natural projection $\Pi : J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \to (\mathbb{R} \times \mathbb{R}^n) \times \mathbb{R}$ by $\Pi(t, x, y, s, p) = (t, x, y)$. We call the above 1-jet bundle *an unfolded 1-jet bundle.*

A Hamilton-Jacobi equation is defined to be a hypersurface

$$\text{(G-H-J)} \qquad E(H) = \left\{(t, x, y, s, p) \in J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) | s + H(t, x, p) = 0\right\}$$

in $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. *A geometric (multi-valued) solution* of $E(H)$ is a Legendrian submanifold L in $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ lying in $E(H)$. In this case the wave front set $W(i)$ is "the graph" of the geometric solution which is generally a hypersurface with singularities.

In order to study (P) we need the following framework: For any $c \in (\mathbb{R}, 0)$, we define

$$E(H)_c = \left\{(c, x, y, -H(c, x, p), p) | (x, y, p) \in J^1(\mathbb{R}^n, \mathbb{R})\right\}.$$

Then, $E(H)_c$ is a $(2n+1)$-dimensional submanifold of $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and $\Theta_c = \Theta | E(H)_c = dz - \sum_{i=1}^n p_i dx_i$ gives a contact structure on $E(H)_c$. We define a mapping $\iota_c : J^1(\mathbb{R}^n, \mathbb{R}) \to E(H)_c$ by $\iota_c(x, y, p) = (c, x, y, -H(c, x, p), p)$. The mapping i_c is a contact diffeomorphism and the following diagram is commutative:

$$\begin{array}{ccc} J^1(\mathbb{R}^n, \mathbb{R}) & \xrightarrow{\iota_c} & E(H)_c \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_c} \\ \mathbb{R}^n \times \mathbb{R} & = & \mathbb{R}^n \times \mathbb{R}. \end{array}$$

We say that *a geometric Cauchy problem (with initial condition L') associated with the time parameter(GCPT) is given for an equation $E(H)$* if there is given an n-dimensional submanifold $i : L' \subset E(H)$ with $i^*\Theta = 0$ and $i(L') \subset E(H)_c$ for some $c \in (\mathbb{R}, 0)$. Since $X_H \notin TE(H)_c$, we have $X_H \notin TL'$, where X_H is the characteristic vector field given by

$$X_H = \frac{\partial}{\partial t} + \sum_{i=1}^n \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x_i} + \left(\sum_{i=1}^n p_i \frac{\partial H}{\partial p_i} - H\right) \frac{\partial}{\partial y} - \frac{\partial H}{\partial t} \frac{\partial}{\partial s} - \sum_{i=1}^n \frac{\partial H}{\partial x_i} \frac{\partial}{\partial p_i}.$$

By using the classical characteristic method, we can show that there exists a unique geometric solutions around L'.

We remark that Cauchy problem (P) is a GCPT. The initial submanifold is given by

$$L_{\phi,0} = \left\{(0, x, \phi(x), -H(0, x, \frac{\partial \phi}{\partial x}), \frac{\partial \phi}{\partial x})|x \in \mathbb{R}^n\right\} \subset E(H)_0.$$

The problem of studying the singularities of the graph of the geometric solution is formulated as follows:

Geometric Problem. *Classify the generic bifurcations of wave fronts of*

$$\pi_t| : L \cap E(H)_t \to \mathbb{R}^n \times \mathbb{R}$$

with respect to the parameter t (i.e., the generic bifurcations of wave fronts of geometric solutions along the time parameter).

Following [16], in order to study the singularities of the geometric solution we identify geometric solutions with one-parameter Legendrian unfoldings. Let R be an $(n+1)$-dimensional smooth manifold, $\mu : (R, u_0) \to (\mathbb{R}, t_0)$ be a submersion germ and $\ell : (R, u_0) \to J^1(\mathbb{R}^n, \mathbb{R})$ be a smooth map germ. We say that the pair (μ, ℓ) is *a Legendrian family* if $\ell_t = \ell|\mu^{-1}(t)$ is a Legendrian immersion germ for any $t \in (\mathbb{R}, t_0)$. Then there exist a unique element $h \in C^\infty_{u_0}(R)$ such that $\ell^*\theta = h \cdot d\mu$, where $C^\infty_{u_0}(R)$ is the ring of smooth function germs at u_0. Define a map germ $\mathcal{L} : (R, u_0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ by

$$\mathcal{L}(u) = (\mu(u), x \circ \ell(u), y \circ \ell(u), h(u), p \circ \ell(u)).$$

We can easily show that $\mathcal{L}$ is a Legendrian immersion germ. If we fix 1-forms Θ and θ, the Legendrian immersion germ $\mathcal{L}$ is uniquely determined by the Legendrian family (μ, ℓ). We call $\mathcal{L}$ *a Legendrian unfolding associated with the Legendrian family* (μ, ℓ).

We have to study how various branches of the multi-valued graph $W_t = (\{t\} \times \mathbb{R}^n \times \mathbb{R}) \cap W(i)$ intersecting at a point bifurcate in time for an arbitrary Hamiltonian $H(t, x, p)$ in [16]. We classify the bifurcations of the branches of the graph by classifying the bifurcations of singularities of multi-Legendrian unfoldings which are expressed in terms of multi-germs.

Let $\mathcal{L}_i : (R, u_0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_i)$ $(i = 1, \ldots, r)$ be Legendrian unfoldings with $\Pi(z_i) = 0$ where $z_1, \ldots, z_r$ are distinct. We call $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ *a multi-Legendrian unfolding*. Let $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ and $(\mathcal{L}'_1, \ldots, \mathcal{L}'_r)$ be multi-Legendrian unfoldings. We say that these are $P_{(r)}$-*Legendrian equivalent* if there exist contact diffeomorphism germs

$$K_i : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_i) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z'_i) \quad (i = 1, \ldots, r)$$

of the form $K_i(t, x, y, s, p) = (\phi_1(t), \phi_2(t, x, y), \phi_3(t, x, y), \phi^i_4(t, x, y, s, p), \phi^i_5(t, x, y, s, p))$ and a diffeomorphism germ $\Psi : (R, u_0) \to (R, u'_0)$ such that $K_i \circ \mathcal{L}_i = \mathcal{L}'_i \circ \Psi$ for any $i = 1, \ldots, r$. It is clear that if two multi-Legendrian unfoldings are $P_{(r)}$-Legendrian equivalent, then there exists a diffeomorphism germ $\Phi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Phi(t, x, y) = (\phi_1(t), \phi_2(t, x, y), \phi_3(t, x, y))$ such that $\Phi(\cup^r_{i=1} W(\mathcal{L}_i)) = \cup^r_{i=1} W(\mathcal{L}_i)$. Thus the above equivalence describes how bifurcations of wavefronts (i.e. graphs of solutions) interact. We can define the notion of stability with respect to the $P_{(r)}$-Legendrian equivalence in the same way as for the ordinary Legendrian stability (see [1],[26]). Motivated by Arnol'd-Zakalyukin's theory ([1],[26]), we can construct multi-generating families of multi-Legendrian unfoldings and give a classification of $P_{(r)}$-Legendrian stable Legendrian unfoldings by using the classification of multi-families of function germs in Zakalyukin [26]. We get a list of classifications for $n = 1, 2, 3$ in [16]. However, we only present the list of classifications for $n = 1$. For the case $n = 2, 3$, see [16].

Theorem A.1 [16]. *Suppose that $n = 1$. Then a generic multi-Legendrian unfolding is $P_{(r)}$-Legendrian equivalent to one of the multi-Legendrian unfoldings in the following list :*

$r = 1$;
${}^0A_1$: $(t, u, 0, 0, 0)$;
${}^0A_2$: $(t, 3u^2, 2u^3, 0, u)$;
${}^1A_3$: $(t, 4u^3 + 2ut, 3u^4 + u^2t, -u^2, u)$.
$r = 2$;
${}^0({}^0A_1\,{}^0A_1)$: $((t, u, -u, 0, -1), (t, u, u, 0, 1))$;
${}^1({}^0A_1\,{}^0A_1)$: $((t, u, t \pm u^2, 1, \pm 2u), (t, u, 0, 0, 0))$;
${}^1A_2\,{}^0A_1$: $((t, 3u^2 - t, 2u^3, u, u), (t, u, -u, 0, -1))$.
$r = 3$;
${}^0A_1\,{}^0A_1\,{}^0A_1$: $((t, u, t - u, 1, -1), (t, u, 0, 0, 0), (t, u, u, 0, 1))$.

When we consider the geometric solution, we can get rid of the germ ${}^1({}^0A_1\,{}^0A_1)$ from the above list because the geometric solution is a one-to-one immersions into the unfolded 1-jet space. For the purpose, we need a kind of non-degeneracy condition on the Hamiltonian function. We say that a Hamiltonian function $H(t, x, p)$ is *non-degenerate at* (t_0, x_0, p_0) if it $\frac{\partial^2 H}{\partial p_i \partial p_j}(t_0, x_0, p_0) \neq 0$ for some $1 \leq i, j \leq n$. This condition is weaker than the condition that $H(t, x, p)$ is convex (or concave) with respect to $(p_1, \ldots, p_n)$-variables at (t_0, x_0, p_0) for $n \geq 2$. The following theorem is a realization theorem for generic singularities for a given Hamilton-Jacobi equation.

Theorem A.2 ([16],[17]). *Let $H(t, x, p)$ be a non-degenerate Hamiltonian function germ at (t_0, x_0, p_0) and $\mathcal{L} : (R, u_0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), (t_0, x_0, y_0, s_0, p_0))$ be a $P_{(1)}$-Legendrian stable Legendrian unfolding associated with (μ, ℓ). Then there exists a Legendrian unfolding $\mathcal{L}'$ which is a geometric solution of the Hamilton-Jacobi equation $s + H(t, x, p) = 0$ such that $\mathcal{L}$ and $\mathcal{L}'$ are $P_{(1)}$-Legendrian equivalent.*

We remark that ${}^1A_3$ singularity (even for general n) describes how the singularity appears from a smooth solution. These are $P_{(1)}$-Legendrian stable Legendrian unfoldings, so that these can be realized as geometric solutions at the non-degenerated point for a given Hamilton-Jacobi equation. We can asserts the detailed statement for the case that the Hamiltonian function depends only on $(p_1, \ldots, p_n)$-variables. In this case the Cauchy problem is given by

$$
(\mathrm{P}') \qquad \begin{cases} \dfrac{\partial y}{\partial t} + H(\dfrac{\partial y}{\partial x_1}, \cdots, \dfrac{\partial y}{\partial x_n}) = 0 \\ y(0, x_1, \cdots, x_n) = \phi(x_1, \cdots, x_n), \end{cases}
$$

where H and ϕ are C^∞-functiions. Then we have the following proposition.

Proposition A.3. *Let $s + H(p) = 0$ be a Hamilton-Jacobi equation. If a singularity of geometric solution for the Cauchy problem (P′) appears at a point (t_0, x_0, p_0), then H is non-degenrated at (t_0, x_0, p_0).*

Proof. In this case the characteristic equation is given by

$$(\mathrm{C}') \qquad \begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial H}{\partial p_i}(p) \ (i = 1, \ldots, n), \\ \dfrac{dp_i}{dt} = 0 \ (i = 1, \ldots, n), \\ \dfrac{dy}{dt} = -H(p) + \displaystyle\sum_{i=1}^{n} p_i \cdot \dfrac{\partial H}{\partial p_i}(p), \ x, p \in \mathbb{R}^n, \\ x(0) = u, \ p(0) = \dfrac{\partial \phi}{\partial u}(u), \ y(0) = \phi(u), \ u \in \mathbb{R}^n. \end{cases}$$

We can explicity solve the charcteristic equation as follows:

$$(\mathrm{S}') \qquad \begin{cases} x_i(t, u) = u_i + t\dfrac{\partial H}{\partial p_i}(\dfrac{\partial \phi}{\partial u}(u)) \ (i = 1, \ldots, n), \\ p_i(t, u) = \dfrac{\partial \phi}{\partial u_i}(u) \ (i = 1, \ldots, n), \\ y(t, u) = t\{-H(\dfrac{\partial \phi}{\partial u}(u)) + \displaystyle\sum_{i=1}^{n} \dfrac{\partial \phi}{\partial u_i}(u) \cdot \dfrac{\partial H}{\partial p_i}(\dfrac{\partial \phi}{\partial u}(u))\} + \phi(u). \end{cases}$$

The Jacobian matrix of the mapping $x : \mathbb{R}^n \to \mathbb{R}^n$ with respect to u is

$$\begin{pmatrix} 1 + tc_{11} & tc_{12} & \ldots & tc_{1n} \\ tc_{21} & 1 + tc_{22} & \ldots & tc_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ tc_{n1} & tc_{n2} & \ldots & 1 + tc_{nn} \end{pmatrix},$$

where $c_{ij} = \sum_{k=1}^{n} \frac{\partial^2 H}{\partial p_i \partial p_k}(\phi(u)) \frac{\partial^2 \phi}{\partial u_k \partial u_j}(u)$. By the definiton of singularity of geometric solutions, the point $(t_0, x_0, p_0) = (t_0, x(t_0, u_0), p(t_0, u_0))$ is a singularitiy of the geometric solution if and only if the rank of the Jacobian matrix of $x : \mathbb{R}^n \to \mathbb{R}^n$ is less than n. However, if the Hamiltonian function $H(p)$ is degenerated at the point (t_0, x_0, p_0), then the Jacobian matrix is equal to the identity matrix, so that the geometric solution is non-singular at the point. □

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