AN INFINITESIMAL APPROACH TO THE STABLE COHOMOLOGY OF THE MODULI OF RIEMANN SURFACES

Nariya Kawazumi

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AN INFINITESIMAL APPROACH TO THE STABLE COHOMOLOGY OF THE MODULI OF RIEMANN SURFACES.

NARIYA KAWAZUMI

Department of Mathematics, Faculty of Science,
Hokkaido University, Sapporo, 060 Japan

ABSTRACT. In this note we review an infinitesimal approach to the stable cohomology of the moduli of compact Riemann surfaces by means of complex analytic Gel'fand-Fuks cohomology of open Riemann surfaces. Under a hypothesis (a certain kind of the Frobenius Reciprocity Laws) we prove the (p,q)-equivariant cohomology of the dressed moduli of compact Riemann surfaces coincides with the polynomial algebra generated by the Morita-Mumford classes $e_n = \kappa_n$ $(n \geq 1)$ [Mo] [Mu] for $p \geq q$. This suggests it is reasonable to conjecture that the stable cohomology algebra of the moduli of compact Riemann surfaces would be generated by the Morita-Mumford classes e_n's. For a more detailed description the reader is referred to [Ka1,2,4,5].

Contents.

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INTRODUCTION.

The Riemann moduli space M_g of compact Riemann surfaces of genus $g \geq 2$, i.e., the complex analytic space of all isomorphism classes of (almost) complex structures on the 2-dimensional closed oriented C^∞-manifold Σ_g of genus g, is one of the most important objects not only in complex analytic geometry but in low-dimensional topology. In fact, in view of the contractibility of the Teichmüller spaces, we have a natural isomorphism of rational cohomology algebras [EE]:

$$\begin{aligned} H^*(M_g; \mathbb{Q}) &\cong H^*(B\,\mathrm{Diff}^+\,\Sigma_g; \mathbb{Q}) \\ &= \{\text{rational characteristic classes of oriented } \Sigma_g\text{-bundles}\} \end{aligned}$$

The group $\mathrm{Diff}^+\,\Sigma_g$ means the topoplogical group consisting of all orientation-preserving diffeomorphisms on the manifold Σ_g endowed with C^∞-topology.

The Harer stability theorem [H] [I1] states that the q-th rational cohomology group $H^q(M_g; \mathbb{Q})$ does not depend upon the genus g if $q < g/2$. This enables us to consider the stable cohomology group of the moduli of Riemann surfaces

$$\lim_{g\to\infty} H^*(M_g; \mathbb{Q}).$$

As was established by Miller [Mi] and Morita [Mo] independently, a polynomial algebra in countable many generators e_n's ($n \in \mathbb{N}_{\geq 1}$) is imbedded in the stable cohomology group

$$\mathbb{Q}[e_n; n \geq 1] \subset \lim_{g\to\infty} H^*(M_g; \mathbb{Q}). \tag{0.1}$$

Here $e_n = \kappa_n \in H^{2n}(M_g; \mathbb{Q})$ is the n-th Morita-Mumford characteristic class defined as follows [Mo] [Mu]. Let $C_g \to M_g$ denote the universal family of compact Riemann surfaces of genus g and let $e \in H^2(C_g)$ be the Euler class (the first Chern class) of the relative tangent bundle T_{C_g/M_g}. The n-th Morita-Mumford class e_n is defined as the fiber integral of the $(n+1)$-th power of the class e:

$$e_n := \int_{\text{fiber}} e^{n+1} \in H^{2n}(M_g; \mathbb{Q}). \tag{0.2}$$

Though it has been studied energetically by many mathematicians, the entirety of the stable cohomology algebra remains to be clarified. The purpose of this note is to review an infinitesimal approach to the stable cohomology of the moduli of compact Riemann surfaces, which gives an affirmative evidence to the conjecture that the stable cohomology algebra of the moduli of compact Riemann surfaces would be generated by the Morita-Mumford classes $e_n = \kappa_n$'s, $n \geq 1$.

Our approach is based on an observation of Kontsevich [Kon] and Beilinson-Manin-Schechtmann [BMS]. We follow the formulation given by Arbarello-DeConcini-Kac-Procesi [ADKP]. Let $\rho > 0$ be a fixed positive real number. Instead of the moduli M_g consider the moduli space $M_{g,\rho}$ of triples (C, p, z), where C is a compact Riemann surface of genus g, $p \in C$ and z is a suitable complex analytic coordinate centered at p (§1.2). The space $M_{g,\rho}$ is (homotopy equivalent to) the classifying space of the topological group $\mathrm{Diff}^+(\Sigma_{g,1})$ of all orientation-preserving diffeomorphisms of the surface $\Sigma_{g,1} := \Sigma_g -$ (interior of an imbedded disk) fixing the boundary pointwise (endowed with C^∞-topology):

$$M_{g,\rho} = B\,\mathrm{Diff}^+(\Sigma_{g,1}).$$

As a consequence of the Harer stability theorem quoted above we have

$$H^*(M_{g,\rho}; \mathbb{Q}) = H^*(M_g; \mathbb{Q}) \quad \text{for } * < g/2.$$

Beilinson-Manin-Schechtmann and Kontsevich observed that the Lie algebra $\mathfrak{d}_\rho$ of all complex analytic vector fields defined near $\{z \in \mathbb{C}; 0 < |z| \leq \rho\}$ acts on the dressed moduli $M_{g,\rho}$ infinitesimal homogeneously (§1.4). We study the stable cohomology of $M_{g,\rho}$ through this action.

Let $C^\times$ be a once punctured compact Riemann surface and $L(C^\times)$ the Lie algebra of all complex analytic vectors on $C^\times$. We compute the q-th cohomology group of $L(C^\times)$ with coefficients in the complex analytic quadratic differentials on the p-fold product space $(C^\times)^p$ for the case $p \geq q$ (Theorem 4.4.3). The cohomology group vanishes for $p > q$, and, for $p = q$, it forms a trivial constant sheaf on the dressed moduli $M_{g,\rho}$ of compact Riemann surfaces of genus g. (The stalk does not depend on the genus g.) Through the infinitesimal action of $\mathfrak{d}_\rho$ this induces a natural map of the cohomology group for $p = q$ into the (p,p)-cohomology of the moduli $M_{g,\rho}$ (§2.3). We proved the map is a stable isomorphism onto the subalgebra generated by the Morita-Mumford classes $e_n = \kappa_n$'s (Theorem 2.4). This gives an affirmative evidence for the conjecture stated above (Theorem 3.4).

In §§5 and 6 we study the (dressed) Picard variety $F_{g,\rho}$ on the universal curve over the dressed moduli $M_{g,\rho}$ following Arbarello et.al.[ADKP]. Similarly we have

$$F_{g,\rho} = B(H_1(\Sigma; \mathbb{Z}) \rtimes \mathrm{Diff}^+\,\Sigma_{g,1}).$$

Instead of $\mathfrak{d}_\rho$ the Lie algebra $\mathcal{D}_\rho$ of all complex analytic differential operators of order ≤ 1 defined near $\{z \in \mathbb{C}; 0 < |z| \leq \rho\}$ acts on the Picard variety $F_{g,\rho}$. We obtain the vanishing of the q-th cohomology of the Lie algebra of all complex analytic differential operators of order ≤ 1 on $C^\times$ with coefficients in the complex analytic alternating 1-jets of 1-forms on the p-fold product space $(C^\times)^p$ for the case $p > q$ (Theorem 5.1). Our computation for the diagonal part ($p = q$) is incomplete. In §6 we compute the stable rational cohomology of $F_{g,\rho}$ modulo that of $M_{g,\rho}$ (Theorem 6.0). Then we introduce *the generalized Morita-Mumford class* $\widetilde{m_{i,j}} \in H^{2i+2j-2}(F_{g,\rho};\mathbb{Z})$, $(i,j \geq 0,\ i+j \geq 2)$. When $j=0$, the class $\widetilde{m_{i+1,0}}$ is equal to the i-th Morita-Mumford class e_i. The class $\widetilde{m_{i,j}}$'s are contained in the image of the diagonal part $(p=q)$. It is unknown whether the diagonal part is stably isomorphic to the polynomial algebra generated by $\widetilde{m_{i,j}}$'s.

1. Homogeneous structure on the dressed moduli.

1.1. Infinitesimal action.

First of all we review the notion of an *infinitesimal action* of a Lie algebra on a manifold. Let M be a (possibly infinite dimensional) complex analytic manifold on which a Lie algebra $\mathfrak{g}$ acts complex analytically. This means a homomorphism of Lie algebras

$$\mu : \mathfrak{g} \to \mathrm{Vect}(M),$$

called the *action*, is given, where $\mathrm{Vect}(M)$ denotes the complex Lie algebra of complex analytic vector fields on M. The kernel of the composite of the evaluation map ev_x at a point $x \in M$ and the action μ

$$\mathrm{ev}_x \circ \mu : \mathfrak{g} \to \mathrm{Vect}(M) \to T_xM$$

is denoted by $\mathfrak{g}_x$ and called the *isotropy subalgebra* of $\mathfrak{g}$ at the point $x \in M$. We call the action μ *homogeneous* when the composite map $\mathrm{ev}_x \circ \mu$ is surjective at each point $x \in M$.

1.2. Dressed moduli $M_{g,\rho}$.

Now we define the dressed moduli $M_{g,\rho}$. For a non-negative real number $\rho \geq 0$ we denote

$$D_\rho := \{z \in \mathbb{C}; |z| < \rho\}, \quad \overline{D_\rho} := \{z \in \mathbb{C}; |z| \leq \rho\},$$
$$D_\rho^\times := \{z \in \mathbb{C}; 0 < |z| < \rho\} \quad \text{and} \quad \overline{D_\rho^\times} := \{z \in \mathbb{C}; 0 < |z| \leq \rho\}.$$

Fix an integer $g \geq 0$. *The dressed moduli $M_{g,\rho}$ of compact Riemann surfaces of genus g* is the space consisting of all triples (C,p,z), where C is a compact Riemann surface of genus g, p is a point of C, and z is a complex coordinate of a neighbourhood U of p satisfying the conditions

$$z(p) = 0 \quad \text{and} \quad z(U) \supset \overline{D_\rho}. \tag{1.2.1}$$

For a point $(C,p,z) \in M_{g,\rho}$, we denote

$$C^\times := C - \{p\}, \quad C_\rho := C - z^{-1}(\overline{D_\rho}) \quad \text{and}$$
$$C_{g,\rho} := \bigcup_{(C,p,z)\in M_{g,\rho}} C_\rho, \quad C_{g,\rho}^\times := \bigcup_{(C,p,z)\in M_{g,\rho}} C^\times$$
$$\overline{D_{g,\rho}^\times} := \bigcup_{(C,p,z)\in M_{g,\rho}} \overline{D_\rho^\times} = M_{g,\rho} \times \overline{D_\rho^\times} \subset C_{g,\rho}^\times.$$

We have inclusions $C_{g,\rho} \subset C_{g,\rho}^{\times}$ and $\overline{D_{g,\rho}^{\times}} \subset C_{g,\rho}^{\times}$ and a natural projection $\pi_{g,\rho}$: $C_{g,\rho}^{\times} \to M_{g,\rho}$, $(C,p,z,p_1) \mapsto (C,p,z)$. Each of these spaces has a natural fiber structure over $M_{g,\rho}$ through the projection $\pi_{g,\rho}$.

1.3. Lie algebra $\mathfrak{d}_\rho$.

Let $L(U)$ denote the topological Lie algebra consisting of all complex analytic vector fields on an open Riemann surface U with the Fréchet topology of uniform convergence on compact sets. The closed subalgebra consisting of all vector fields which have a zero at a (fixed) point $p_1 \in U$ is denoted by $L(U,p_1)$. The topological Lie algebra $\mathfrak{d}_\rho$ is defined by

$$\mathfrak{d}_\rho := \varinjlim_{\rho_1 \downarrow \rho} L(D_{\rho_1}^{\times}),$$

which is endowed with the inductive limit locally convex topology [G,TVS] [Ko] [Ko1].

1.4. $\mathfrak{d}_\rho$-action on the dressed moduli.

Now we introduce the infinitesimal action [BMS] [Kon] of the topological Lie algebra $\mathfrak{d}_\rho$ on the spaces $M_{g,\rho}$ and $C_{g,\rho}^{\times}$ following the formulation in [ADKP](3.19).

Fix $X \in \mathfrak{d}_\rho$ and $(C,p,z) \in M_{g,\rho}$ (resp. $(C,p,z,p_1) \in C_{g,\rho}^{\times}$, i.e., $(C,p,z) \in M_{g,\rho}$ and $p_1 \in C^{\times} = C - \{p\}$.) For some $\rho_1 > \rho_2 > \rho$, $z(U)$ includes $\overline{D_{\rho_1}}$, and the integral $\exp \epsilon X : D_{\rho_2}^{\times} \to D_{\rho_1}^{\times}$ of the vector field X exists for sufficiently small complex number ϵ. A complex analytic path

$$\epsilon \mapsto (C(\epsilon), p(\epsilon), z(\epsilon)) \in M_{g,\rho}, \quad (\text{resp.} \mapsto (C(\epsilon), p(\epsilon), z(\epsilon), p_1(\epsilon)) \in C_{g,\rho}^{\times})$$

is defined by

$$\begin{aligned} &C(\epsilon) := (C^{\times} \amalg D_{\rho_2})/\sim_\epsilon, \quad z \sim_\epsilon (\exp \epsilon X)(z), \quad z \in D_{\rho_2} \\ &p(\epsilon) := \text{the image of } 0 \in D_{\rho_2} \text{ on } C(\epsilon) \\ &z(\epsilon) := \text{the coordinate of } C(\epsilon) \text{ induced by the identity } D_{\rho_2} \hookrightarrow D_{\rho_2} \\ &p_1(\epsilon) := \text{the image of } p_1 \in C^{\times} \text{ on } C(\epsilon). \end{aligned}$$

Differentiating the path $(C(\epsilon), p(\epsilon), z(\epsilon))$ and $(C(\epsilon), p(\epsilon), z(\epsilon), p_1(\epsilon))$, we obtain the maps $P = P_{(C,p,z)}$ and $P = P_{(C,p,z,p_1)}$:

$$\begin{aligned} &P : \mathfrak{d}_\rho \to T_{(C,p,z)}M_{g,\rho}, \quad X \mapsto (d/d\epsilon)\big|_{\epsilon=0}(C(\epsilon), p(\epsilon), z(\epsilon)) \\ &P : \mathfrak{d}_\rho \to T_{(C,p,z,p_1)}C_{g,\rho}^{\times}, \quad X \mapsto (d/d\epsilon)\big|_{\epsilon=0}(C(\epsilon), p(\epsilon), z(\epsilon), p_1(\epsilon)), \end{aligned}$$

which define an infinitesimal action of $\mathfrak{d}_\rho$ on the dressed moduli $M_{g,\rho}$ and the universal family $C_{g,\rho}^{\times}$. These actions are compatible to the fiber structure on the space $C_{g,\rho}^{\times}$ over $M_{g,\rho}$.

1.5. Homogeneity.

Lemma 1.5.1.([ADKP] Proposition 3.19). *The sequences*

$$0 \to L(C^{\times}) \hookrightarrow \mathfrak{d}_\rho \xrightarrow{P} T_{(C,p,z)}M_{g,\rho} \to 0 \tag{1}$$

$$0 \to L(C^{\times}, p_1) \hookrightarrow \mathfrak{d}_\rho \xrightarrow{P} T_{(C,p,z,p_1)}C_{g,\rho} \to 0 \tag{2}$$

are exact. Here the algebras $L(C^{\times})$ and $L(C^{\times}, p_1)$ are regarded as subalgebras of $\mathfrak{d}_\rho$ through the coordinate z.

This gives us a presentation of the cotangent spaces of $M_{g,\rho}$ and $C_{g,\rho}$.

Corollary 1.5.2.

$$T^*_{(C,p,z)}M_{g,\rho} = (\mathfrak{d}_\rho/L(C^\times))^*$$
$$T^*_{(C,p,z,p_1)}C_{g,\rho} = (\mathfrak{d}_\rho/L(C^\times, p_1))^*$$

Here and throughout this note the asterisque $*$ means the strong dual.

2. Results.

2.1. Standard cochain complex.

We begin this section by fixing our notation on the cohomology theory for Lie algebras. Let $\mathfrak{g}$ be a complex topological Lie algebra. By a $\mathfrak{g}$-*module* we mean a complex topological vector space on which $\mathfrak{g}$ acts continuously. The standard continuous cochain complex of the topological Lie algebra $\mathfrak{g}$ with coefficients in a $\mathfrak{g}$-module N is denoted by

$$C^*(\mathfrak{g}; N) = \bigoplus_{p\geq 0} C^p(\mathfrak{g}; N),$$

where $C^p(\mathfrak{g}; N)$ is the linear space of continuous alternating multilinear mappings $c : \mathfrak{g}^{\otimes p} \to N$. The cohomology group of the complex $C^*(\mathfrak{g}; N)$ is called the (continuous) cohomology group of $\mathfrak{g}$ with coefficients in N and denoted by $H^*(\mathfrak{g}; N)$. When N is the trivial $\mathfrak{g}$-module $\mathbb{C}$, we abbreviate them to $C^*(\mathfrak{g})$ and $H^*(\mathfrak{g})$ respectively. (For details, see for example [HS].)

2.2. Main Theorem (1/2).

Theorem 2.2. *For any $\overline{x} \in M_{g,\rho}$ and $x \in C_{g,\rho}$ we have*

(2.2.1)
$$H^q((\mathfrak{d}_\rho)_{\overline{x}}; \textstyle\bigwedge^p T^*_{\overline{x}}M_{g,\rho}) = H^q((\mathfrak{d}_\rho)_x; \bigwedge^p T^*_x C_{g,\rho}) = 0 \quad (\forall p > \forall q)$$

(2.2.2)
$$\bigoplus_{p\geq 0} H^p((\mathfrak{d}_\rho)_{\overline{x}}; \textstyle\bigwedge^p T^*_{\overline{x}}M_{g,\rho}) = \mathbb{C}[\kappa_n; n \geq 1]/\mathit{relations},$$

(2.2.3)
$$\bigoplus_{p\geq 0} H^p((\mathfrak{d}_\rho)_x; \textstyle\bigwedge^p T^*_x C_{g,\rho}) = \mathbb{C}[\epsilon,\, \kappa_n; n \geq 1]/\mathit{relations}.$$

where $\kappa_n \in H^n((\mathfrak{d}_\rho)_{\overline{x}}; \bigwedge^n T^*_{\overline{x}}M_{g,\rho})$ is defined in (4.6.1) (4.4.4) and $\epsilon \in H^1((\mathfrak{d}_\rho)_x; T^*_x C_{g,\rho})$ is defined in (4.5.1) (4.4.3).

Here and throughout this paper we mean by $\bigwedge^n$ the completed n-fold alternating tensor product. Furthermore we have

Proposition 2.2.4. *The cohomology groups $H^p((\mathfrak{d}_\rho)_{\overline{x}}; \bigwedge^p T^*_{\overline{x}}M_{g,\rho})$, $\overline{x} \in M_{g,\rho}$, (resp. $H^p((\mathfrak{d}_\rho)_x; \bigwedge^p T^*_x C_{g,\rho})$, $x \in C_{g,\rho}$) form a constant sheaf on the moduli $M_{g,\rho}$ (resp. the universal curve $C_{g,\rho}$.)*

2.3. Framework for construction of cohomology classes.

Let M be a complex manifold on which a Lie algebra $\mathfrak{g}$ acts infiinitesimally as in §1.1, and $E \to M$ a complex analytic vector bundle on which the algebra $\mathfrak{g}$ acts complex analytically and compatibly with the action on M. This means $\mathfrak{g}$ acts on each $\mathcal{O}_M(E)(O)$ ($O \overset{\text{open}}{\subset} M$) such that

(1) each restriction map is $\mathfrak{g}$-equivariant, and
(2) the formula

$$X(f\sigma) = (Xf)\sigma + f(X\sigma), \quad X \in \mathfrak{g},\ f \in \mathcal{O}_M(O),\ \sigma \in \mathcal{O}_M(E)(O)$$

holds for any open subset $O \subset M$.

In the sequal we call such a vector bundle a $\mathfrak{g}$*-vector bundle over* M in short. The fiber E_x at $x \in M$ is a $\mathfrak{g}_x$-module in an obvious manner.

Let $n \in \mathbb{N}_{\geq 0}$ be a fixed non-negative integer. We assume

$$(\mathrm{A}(n)) \qquad \forall x \in M \quad \forall n' < n \quad H^{n'}(\mathfrak{g}_x; E_x) = 0.$$

Under the assumption $(\mathrm{A}(n))$ we have an exact sequence of complex analytic 'vector bundles' over M

$$0 \to E \to \coprod_{x\in M} C^1(\mathfrak{g}_x; E_x) \to \coprod_{x\in M} C^2(\mathfrak{g}_x; E_x) \to \cdots$$
$$\cdots \to \coprod_{x\in M} C^{n-1}(\mathfrak{g}_x; E_x) \to \coprod_{x\in M} Z^n(\mathfrak{g}_x; E_x) \to \coprod_{x\in M} H^n(\mathfrak{g}_x; E_x) \to 0,$$

where Z^n means the n cocycles. This exact sequence induces the n-fold composite of the connecting homomorphisms

$$D : H^0(M; \mathcal{O}_M(\coprod_{x\in M} H^n(\mathfrak{g}_x; E_x))) \to H^n(M; \mathcal{O}_M(E)).$$

The map D has a multiplicative property and a functorial property in a natural manner.

2.4. Main Theorem (2/2).

Theorem 2.4. *For $g \geq 0$ and $\rho > 0$ we have*

$$(\sqrt{-1}/2\pi)^n D(\kappa_n) = e_n \in H^{n,n}(M_{g,\rho}), \quad n \geq 1,$$

where e_n is the n-th Morita Mumford class (0.2).

As corollaries we obtain

Corollary 2.4.1. *If $\rho > 0$, the map*

$$D : \bigoplus_{n\geq 0} H^n((\mathfrak{d}_\rho)_{\overline{x}}; \bigwedge^n T^*_{\overline{x}} M_{g,\rho}) \to \bigoplus_{n\geq 0} H^{n,n}(M_{g,\rho})$$
$$(\text{resp. } D : \bigoplus_{n\geq 0} H^n((\mathfrak{d}_\rho)_x; \bigwedge^n T^*_x C_{g,\rho}) \to \bigoplus_{n\geq 0} H^{n,n}(C_{g,\rho}))$$

is a stable isomorphism onto the subalgebra generated by the Morita-Mumford classes e_n's (resp. the Euler class e and the Morita Mumford classes e_n's).

Corollary 2.4.2. *There exist no algebraic relations among the classes κ_n's, i.e., we have isomorphisms of $\mathbb{C}$-algebras*

$$\bigoplus_{n\geq 0} H^n((\mathfrak{d}_\rho)_{\overline{x}}; \bigwedge^n T^*_{\overline{x}} M_{g,\rho}) = \mathbb{C}[\kappa_n; n \geq 1]$$

$$\bigoplus_{n\geq 0} H^n((\mathfrak{d}_\rho)_x; \bigwedge^n T^*_x C_{g,\rho}) = \mathbb{C}[\epsilon, \kappa_n; n \geq 1].$$

These follow from the theorem of Miller [Mi] and Morita [Mo] quoted in (0.1).

3. Equivariant cohomology.

Now we introduce an equivariant cohomology theory for Lie algebras, which gives us a reformulation of the results stated in the preceding section. It plays an important role also in the proof of the results.

3.1. Definition of equivariant cohomology.

As in §1.1 and §2.3, let M be a (possibly infinite dimensional) complex analytic manifold on which a complex Lie algebra $\mathfrak{g}$ acts complex analytically and let E be a $\mathfrak{g}$-vector bundle over M. Here we assume that the action of $\mathfrak{g}$ on M is homogeneous, i.e., that the composite $\mathrm{ev}_x \circ \mu : \mathfrak{g} \to \mathrm{Vect}(M) \to T_x M$ is surjective for each $x \in M$.

Since $\mathcal{O}_M(E)$ is a sheaf of $\mathfrak{g}$-modules, the cochain complex of sheaves over M

$$C^*(\mathfrak{g}; \mathcal{O}_M(E)) : M \overset{\text{open}}{\supset} O \mapsto C^*(\mathfrak{g}; \mathcal{O}_M(E)(O))$$

is defined, where $C^*(\mathfrak{g}; \cdot)$ is the standard cochain complex of the Lie algebra $\mathfrak{g}$ with values in a $\mathfrak{g}$-module $\cdot$ introduced in §2.1. We denote by $H^*_{\mathfrak{g}}(M, \mathcal{O}_M(E))$ the hypercohomology group of the cochain complex of sheaves over M with respect to the functor $\Gamma(M; \cdot)$ (= the sections of $\cdot$ over M) ([G,E] ch.0, §11.4, pp.32-) and call it *the $\mathfrak{g}$-equivariant cohomology group of M with values in the $\mathfrak{g}$-vector bundle E*. Namely we define

$$H^*_{\mathfrak{g}}(M; \mathcal{O}_M(E)) := H^*(\mathrm{Total}(\Gamma(M; \mathcal{C}^{*,*})))$$

for an injective right Cartan-Eilenberg resolution $\mathcal{C}^{*,*} = (\mathcal{C}^{i,j})_{i,j\geq 0}$ of the complex $C^*(\mathfrak{g}; \mathcal{O}_M(E))$ (cf. ibid. loc. cit.). Especially, if E is the n-cotangent bundle $\bigwedge^n T^*M$, we denote

$$H^{n,*}_{\mathfrak{g}}(M) := H^*_{\mathfrak{g}}(M; \mathcal{O}_M(\bigwedge^n T^*M))$$

and call it *the $\mathfrak{g}$-equivariant $(n, *)$ cohomology of M*.

There exist two spectral sequences converging to $H^*_{\mathfrak{g}}(M; \mathcal{O}_M(E))$

$$\text{(3.1.1)} \qquad {}'E_2^{p,q} = H^p(H^q(M; C^*(\mathfrak{g}; \mathcal{O}_M(E))))$$

$$\text{(3.1.2)} \qquad {}''E_2^{p,q} = H^p(M; H^q(\mathfrak{g}; \mathcal{O}_M(E))),$$

where we denote $H^*(\mathfrak{g}; \mathcal{O}_M(E))$ is the sheaf over M defined as the cohomology of the cochain complex of sheaves $C^*(\mathfrak{g}; \mathcal{O}_M(E))$.

3.2. Finite dimensional case.

The equivariant cohomology theory on finite dimensional manifolds is essential to our computation of the cohomology groups of the isotropy subalgebras $(\mathfrak{d}_\rho)_{\overline{x}}$ and $(\mathfrak{d}_\rho)_x$. In this subsection we assume the $\mathfrak{g}$-manifold M and $\mathfrak{g}$-vector bundle E are *finite dimensional*. Then the stalk of the sheaf $H^*(\mathfrak{g}; \mathcal{O}_M(E))$ at a point $x \in M$ is given by

Lemma 3.2.1. *If $\mathfrak{g}$ is a Fréchet nuclear space, the stalk at a point $x \in M$ is naturally isomorphic to the cohomology group of the stalk $\mathcal{O}_M(E)_x$:*

$$H^*(\mathfrak{g}; \mathcal{O}_M(E))_x \cong H^*(\mathfrak{g}; \mathcal{O}_M(E)_x).$$

This follows from Lemma 4.3 in [Ka] and the exactness of the injective limit functor.

Next we look at the natural map

$$\varphi^{p,q} : {}'E_2^{p,q} = H^p(H^q(M; C^*(\mathfrak{g}; \mathcal{O}_M(E)))) \to H^p(\mathfrak{g}; H^q(M; \mathcal{O}_M(E))).$$

Especially we have the natural map

$$H^n_{\mathfrak{g}}(M; \mathcal{O}_M(E)) \to H^0(\mathfrak{g}; H^n(M; \mathcal{O}_M(E))) = H^n(M; \mathcal{O}_M(E))^{\mathfrak{g}}.$$

Proposition 3.2.2. *Let M and E be as above. We assume that the Lie algebra $\mathfrak{g}$ is topologically isomorphic to the topological vector space consisting of all complex analytic sections on a finite dimensional vector bundle over a finite dimensional paracompact complex analytic manifold, (hence $\mathfrak{g}$ is a Fréchet nuclear space), and that M, E and $H^q(M; \mathcal{O}_M(E))$ for $q \neq 0$ are all finite dimensional. Then the natural map $\varphi^{p,q}$ is an isomorphism*

$$\varphi^{p,q} : {}'E_2^{p,q} \cong H^p(\mathfrak{g}; H^q(M; \mathcal{O}_M(E))).$$

Especially if M is a finite dimensional *Stein* manifold, we have

$$H^*_{\mathfrak{g}}(M; \mathcal{O}_M(E)) = H^*(\mathfrak{g}; \mathcal{O}_M(E)(M)).$$

Hence we obtain a spectral sequence

$$\text{(3.2.3)} \qquad {}''E_2^{p,q} = H^p(M; \mathcal{H}^q) \Rightarrow H^{p+q}(\mathfrak{g}; \mathcal{O}_M(E)(M)),$$

where $\mathcal{H}^q$ is a sheaf over M whose stalk at $x \in M$ is given by

$$\mathcal{H}^q_x = H^q(\mathfrak{g}; \mathcal{O}_M(E)_x).$$

We call this sequence *the Rešetnikov spectral sequence* (see [R] and [Ka] §9).

Let E be a finite dimensional complex analytic vector bundle of over $\mathbb{C}^n$ and let $M = \mathbb{C}^n - \{0\}$. Scheja [Sc] has computed the cohomology group $H^*(\mathbb{C}^n - \{0\}; \mathcal{O}_M(E))$. So, though the cohomology group $H^{n-1}(\mathbb{C}^n - \{0\}; \mathcal{O}_M(E))$ is *not* finite dimensional, we obtain a similar result to Proposition 3.2.2 ([Ka1] Proposition 1.9):

Proposition 3.2.4. *If $\mathfrak{g}$ satisfies the condition given in (3.2.2), then the ${}'E_2$ term (3.1) converging to the equivariant cohomology $H^*_{\mathfrak{g}}(\mathbb{C}^n - \{0\}; \mathcal{O}_{\mathbb{C}^n}(E))$ is given by*

$${}'E_2^{p,q} = \begin{cases} H^p(\mathfrak{g}; \mathcal{O}_{\mathbb{C}^n}(E)(\mathbb{C}^n)), & \textit{if } q = 0, \\ H^p(\mathfrak{g}; H^{n-1}(\mathbb{C}^n - \{0\}; \mathcal{O}_{\mathbb{C}^n}(E))), & \textit{if } q = n-1, \\ 0, & \textit{otherwise.} \end{cases}$$

3.3. Frobenius Reciprocity Law.

In a *general situation* we relate the map D in §2.3 to the second spectral sequence "$E_2^{p,q}$ (3.1.2). For any $x \in M$ the evaluation map $\mathrm{ev}_x : \mathcal{O}_M(E)(O) \to E_x$, $(x \in O \subset M)$ induces a homomorphism

$$(\mathrm{ev}_x)_* : H^n(\mathfrak{g}; \mathcal{O}_M(E))_x \to H^n(\mathfrak{g}_x; E_x), \tag{3.3.1}$$

which we call *the evaluation homomorphism.*

Here we assume the action of $\mathfrak{g}$ on M is *homogeneous.* Then, taking into consideration the finite dimensional case studied by Bott [B] (although the $\mathfrak{d}_\rho$-manifolds $M_{g,\rho}$ and $C_{g,\rho}$ are infinite dimensional in our case), we may regard the $\mathfrak{g}$-module $(\mathcal{O}_M(E))_x$ as the (co-)induced module of the $\mathfrak{g}_x$-module E_x $(x \in M)$. Hence we put a general hypothesis:

Hypothesis 3.3.2.(Frobenius Reciprocity Law). *The evaluation homomorphism (3.3.1) is an isomorphism for any $x \in M$.*

It could be regarded as a certain kind of the Frobenius reciprocity laws, i.e., the Shapiro isomorphisms. Through the evaluation homomorphism (3.3.1) the vector bundle $\coprod_{x\in M} H^*(\mathfrak{g}_x; E_x)$ possesses the natural structure of a sheaf over M and we have an isomorphism

$$\text{“}E_2^{p,q} \cong H^p(M; H^q(\mathfrak{g}_x; E_x)), \tag{3.3.3}$$

where the RHS means the cohomology of M with values in the sheaf $\coprod_{x\in M} H^*(\mathfrak{g}_x; E_x)$. Thus, under the hypothesis (3.3.2), the assumption (A(n)) implies that the term '$E_2^{p,q}$ vanishes for $q < n$, so that

$$H_{\mathfrak{g}}^q(M; \mathcal{O}_M(E)) = \begin{cases} 0, & \text{if } q < n, \\ H^0(M; H^n(\mathfrak{g}_x; E_x)), & \text{if } q = n. \end{cases}$$

The map D introduced in §2.3

$$D : H^0(M; H^n(\mathfrak{g}_x; E_x)) \to H^n(M; \mathcal{O}_M(E))$$

is nothing but the composite of the above isomorphism and the natural map $H_{\mathfrak{g}}^n(M; \mathcal{O}_M(E)) \to H^n(M; \mathcal{O}_M(E))$.

3.4. Equivariant cohomology of the dressed moduli.

Consider the $\mathfrak{d}_\rho$-manifolds $M = M_{g,\rho}$, $C_{g,\rho}$ and the $\mathfrak{d}_\rho$-vector bundles $\bigwedge^n T^*M$. We assume $\rho > 0$. Then the hypothesis (3.3.2) is

$$H^*(\mathfrak{d}_\rho; \mathcal{O}_M(E)(\bigwedge^n T^*M))_x \cong H^*((\mathfrak{d}_\rho)_x; \bigwedge^n T_x^*M)$$
$$\text{for all } n \geq 0 \text{ and all } x \in M = M_{g,\rho} \text{ and } C_{g,\rho},$$

which seems to be true. But at present the author has no proof for this assertion.

Theorem 3.4. *If the Frobenius Reciprocity Law (3.3.2) holds good for the $\mathfrak{d}_\rho$-manifolds $M = M_{g,\rho}$ and $C_{g,\rho}$ and the $\mathfrak{d}_\rho$-vector bundles $\bigwedge^* T^*M$, we have*

$$\bigoplus_{p\geq q} H^{p,q}_{\mathfrak{d}_\rho}(M_{g,\rho}) = \mathbb{C}[e_n; n \geq 1] \tag{1}$$

$$\bigoplus_{p\geq q} H^{p,q}_{\mathfrak{d}_\rho}(C_{g,\rho}) = \mathbb{C}[e, e_n; n \geq 1] \tag{2}$$

for all $g \geq 0$ and $\rho > 0$, where $e = c_1(T_{C_{g,\rho}/M_{g,\rho}}) \in H^{1,1}(C_{g,\rho})$ and $e_n \in H^{n,n}(M_{g,\rho})$ is the n-th Morita Mumford class ($n \in \mathbb{N}_{\geq 1}$).

This gives an affirmative evidence for the conjecture: the stable cohomology algebra of the moduli of compact Riemann surfaces would be generated by the Morita Mumford classes e_n's.

4. Outline of the proof - Complex analytic Gel'fand-Fuks cohomology.

4.1. Köthe duality.

First of all we identify the cotangent space of the dressed moduli $M_{g,\rho}$ with the space of quadratic differential on $C_\rho = C - z^{-1}(\overline{D_\rho})$ by a generalized Köthe duality [Koe] [Ko2]. Fix a point $(C, p, z) \in M_{g,\rho}$ and $(C, p, z, p_1) \in C^\times_{g,\rho}$. We regard the disk $\overline{D_\rho}$ is regarded as a subset of C through the coordinate z. Let E be a complex analytic line bundle over C. For an open set U in C, we abbreviate $E(U) := \mathcal{O}_C(E)(U)$, which is a Fréchet space with respect to the topology of uniform convergence on compact sets. For a subset F in C, we denote by $E(F)$ the inductive limit locally convex space

$$E(F) = \varinjlim_{F\subset U} E(U),$$

where U runs over all the open neighbourhoods of F in C. The canonical bundle of C, namely, the cotangent bundle of C is denoted by K. We have $(K^{-1})(\overline{D_\rho^\times}) = \mathfrak{d}_\rho$.

Theorem 4.1.1. (generalized Köthe duality). *The map*

$$\eta : (E^{-1} \otimes K)(C_\rho) \to (E(\overline{D_\rho^\times})/E(C^\times))^*, \quad s \mapsto (f \mapsto \frac{1}{2\pi\sqrt{-1}} \oint f \cdot s)$$

is a topological isomorphism. Here $f \cdot s$ intends a 1-form defined on an annulus $\{\rho < |z| < \rho + \varepsilon\}(0 < \varepsilon \ll 1)$, and the line integral $\oint$ is carried out on the circle $\{|z| = \rho + \varepsilon/2\}$.

Denote by $Q(U)$ the Fréchet space of complex analytic quadratic differentials on a Riemann surface U:

$$Q(U) = H^0(U; \mathcal{O}_U((T^*U)^{\otimes 2})) = H^0(U; \mathcal{O}_U(K^{\otimes 2})).$$

For $\lambda \in \mathbb{Z}$ and $p_1 \in U$ we denote by $Q^\lambda(U, p_1)$ the Fréchet space of meromorphic quadratic differentials on U with a pole only at p_1 of order $\leq \lambda$.

$$Q^\lambda(U, p_1) = H^0(U; \mathcal{O}_U(K^{\otimes 2} \otimes [p_1]^{\otimes\lambda})).$$

Corollary 4.1.2. *We have topological isomorphisms*

$$T^*_{(C,p,z)}M_{g,\rho} = Q(C_\rho) \quad \text{and} \quad T^*_{(C,p,z,p_1)}C_{g,\rho} = Q^1(C_\rho, p_1).$$

These isomorphisms are respectively $(\mathfrak{d}_\rho)_{(C,p,z)} = L(C^\times)$ and $(\mathfrak{d}_\rho)_{(C,p,z,p_1)} = L(C^\times, p_1)$-equivariant by Stokes' Theorem.

4.2. Fundamental Exact Sequence.

What we have to compute are the cohomology groups of the Lie algebras $(\mathfrak{d}_\rho)_{(C,p,z)} = L(C^\times)$ and $(\mathfrak{d}_\rho)_{(C,p,z,p_1)} = L(C^\times, p_1)$ (1.5.1). These are *the complex analytic Gel'fand-Fuks cohomology groups of an open Riemann surface.* So, as is expected, it is necessary to compute the cohomology groups of the Lie algebras

$$W_1 := L(\mathbb{C}) \quad \text{and} \quad L_0 := L(\mathbb{C}, 0)$$

with coefficients in the n-fold completed tensor products $\bigwedge^n$ of

$$Q := Q(\mathbb{C}) \quad \text{and} \quad Q^1 := Q^1(\mathbb{C}, 0).$$

We should remark the cohomology group $H^*(W_1; \bigwedge^n Q)$ is embedded into the group $H^*(L_0; 1_2 \otimes \bigwedge^{n-1} Q)$:

$$H^*(W_1; \bigwedge^n Q) \subset H^*(L_0; 1_2 \otimes \bigwedge^{n-1} Q) \tag{4.2.1}$$

by the Shapiro Lemma. Here we denote

$$1_\nu := (T_0^*\mathbb{C})^{\otimes\nu}, \quad (\nu \in \mathbb{Z}),$$

on which the Lie algebra L_0 acts by the Lie derivative.

In general we denote by τ_ν^λ, $\lambda, \nu \in \mathbb{Z}$, the line bundle $(T^*\mathbb{C})^{\otimes\nu} \otimes [0]^{\otimes\lambda}$ over the complex line $\mathbb{C}$, where $[0]$ is the line bundle induced by the divisor $0 \in \mathbb{C}$. If $\lambda = 0$, we denote $\tau_\nu := \tau_\nu^0$. We have $T_\nu^\lambda = H^0(\mathbb{C}; \mathcal{O}_\mathbb{C}(\tau_\nu^\lambda))$, $T_\nu = H^0(\mathbb{C}; \mathcal{O}_\mathbb{C}(\tau_\nu))$ and $T_\nu^\times = H^0(\mathbb{C} - \{0\}; \mathcal{O}_\mathbb{C}(\tau_\nu))$. The line bundle over $\mathbb{C}^n$ given by

$$(\mathrm{pr}_1^*\tau_{\nu_1}^{\lambda_1}) \otimes \cdots \otimes (\mathrm{pr}_n^*\tau_{\nu_n}^{\lambda_n}) \quad (\lambda_1, \ldots, \lambda_n, \nu_1, \ldots, \nu_n \in \mathbb{Z})$$

is denoted by $\tau_{\nu_1,\ldots,\nu_n}^{\lambda_1,\ldots,\lambda_n}$, where $\mathrm{pr}_i : \mathbb{C}^n \to \mathbb{C}$ is the i- th projection. If $\lambda_1 = \cdots = \lambda_n = \lambda$ and $\nu_1 = \cdots = \nu_n = \nu$, we abbreviate $(\tau_\nu^\lambda)^n := \tau_{\nu_1,\ldots,\nu_n}^{\lambda_1,\ldots,\lambda_n}$. By the diagonal Lie derivative action, $1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}(\tau_{\nu_1,\ldots,\nu_n}^{\lambda_1,\ldots,\lambda_n})$ $(\nu_0 \in \mathbb{Z})$ is a sheaf of L_0-modules. From the nuclear theorem we have L_0-isomorphisms

$$1_{\nu_0} \otimes \bigotimes_{i=1}^n T_{\nu_i}^{\lambda_i} = H^0(\mathbb{C}^n; 1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}(\tau_{\nu_1,\ldots,\nu_n}^{\lambda_1,\ldots,\lambda_n})),$$

and so on. We denote

$$T_\nu^\times := H^0(\mathbb{C} - \{0\}; \mathcal{O}_\mathbb{C}((T^*\mathbb{C})^{\otimes\nu})).$$

Substituting $\mathfrak{g} = L_0$ and $E = \tau_{\nu_1,\ldots,\nu_n}^{\lambda_1,\ldots,\lambda_n}$ into Proposition 3.2.4, we have

$$\begin{aligned} \cdots \to H^{q-n}(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n (T_{\nu_i}^\times / T_{\nu_i}^{\lambda_i})) &\xrightarrow{d_n} H^q(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n T_{\nu_i}^{\lambda_i}) \\ &\to H^q_{L_0}(\mathbb{C}^n - \{0\}; 1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}(\tau_{\nu_1,\ldots,\nu_n}^{\lambda_1,\ldots,\lambda_n})) \to \cdots, \end{aligned} \tag{4.2.2}$$

which we call *the fundamental exact sequence for the L_0 module $1_{\nu_0} \otimes \bigotimes_{i=1}^n T_{\nu_i}^{\lambda_i}$.*

As an application we can compute the cohomology algebra of W_1 with coefficients in the Fréchet space $F(\mathbb{C}^n)$ consisting of all complex analytic functions on $\mathbb{C}^n$:

Theorem 4.2.3.

$$H^*(W_1; F(\mathbb{C}^n)) = H^*(P_n) \otimes \mathbb{C}[v_2, \ldots, v_n] \otimes \bigwedge^*(\nabla_0^1, \ldots, \nabla_0^n)$$

Here the (holomorphic) de Rham cohomology algebra of P_n, $H^*(P_n)$, is mapped into the algebra $H^*(L(\mathbb{C}); F(\mathbb{C}^n))$ in an obvious way ([Ka] Corollary 9.9). The 2 cocycle v_i is defined by

$$v_i(\xi_1(z)\frac{d}{dz}, \xi_2(z)\frac{d}{dz}) := \int_{z_{i-1}}^{z_i} \det \begin{pmatrix} \xi_1'(z) & \xi_2'(z) \\ \xi_1''(z) & \xi_2''(z) \end{pmatrix} dz$$

and the 1 *cocycle ∇_0^j by $\nabla_0^j(\xi(z)\frac{d}{dz}) := \xi'(z_j)$ for $\xi(z)\frac{d}{dz}, \xi_1(z)\frac{d}{dz}$ and $\xi_2(z)\frac{d}{dz} \in W_1$.*

Taking the $\mathfrak{S}_n$-invariant part of the exact sequence (4.2.2), we have

$$(4.2.4) \quad \cdots \to H^{q-n}(L_0; 1_{\nu_0} \otimes S^n(T_\nu^\times/T_\nu^\lambda)) \xrightarrow{d_n} H^q(L_0; 1_{\nu_0} \otimes \bigwedge^n T_\nu^\lambda)$$
$$\to H^q_{L_0}(\mathbb{C}^n - \{0\}; 1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}((\tau_\nu^\lambda)^n))^{\mathfrak{S}_n,\mathrm{alt}} \to \cdots,$$

which we call *the fundamental exact sequence for the L_0-module* $1_{\nu_0} \otimes \bigwedge^n T_\nu^\lambda$. Here S^n means the completed n fold symmetric tensor product of a topological vector space.

4.3. Vanishing Theorem.

Now we consider the cohomology groups $H^*(W_1; \bigwedge^n Q)$ and $H^*(L_0; \bigwedge^n Q^1)$. By a straightforward computation we have

Lemma 4.3.1.

$$H^0(L_1; S^*(T_2^\times/Q)) = \mathbb{C}[q_1] \quad (\textit{the polynomial algebra}),$$

where $q_1 := z^{-1}dz^2 \in T_2^\times$ and L_1 is the subalgebra of L_0 generated by $z^{k+1}\frac{d}{dz}$ $(k \geq 1)$.

By an inductive argument on n with (4.2.1), the fundamental sequence (4.2.4) and Lemma 4.3.1 we have

Theorem 4.3.2.(Vanishing Theorem). *If $q < n$,*

$$H^q(W_1; \bigwedge^n Q) = 0, \quad \textit{and} \quad H^q(L_0; \bigwedge^n Q^1) = 0.$$

4.4. Rešetnikov spectral sequence.

We define a sheaf $\mathcal{Q}_n$ over $C_\rho{}^n$ by

$$(4.4.1) \qquad \mathcal{Q}_n := \mathcal{O}_{C_\rho{}^n}(\bigotimes_{i=1}^n \mathrm{pr}_i{}^*(T^*C_\rho)^{\otimes 2}),$$

where $\mathrm{pr}_i : C_\rho{}^n \to C_\rho$ is the i-th projection $(1 \leq i \leq n)$. The symmetric group $\mathfrak{S}_n$ acts on the sheaf $\mathcal{Q}_n$ by the permutation of the i-th components $(1 \leq i \leq n)$ twisted by the sign. In view of the nuclear theorem, we have

$$\mathcal{Q}_n(C_\rho{}^n) = Q(C_\rho)^{\otimes n} \quad \text{and} \quad \mathcal{Q}_n(C_\rho{}^n)^{\mathfrak{S}_n} = \bigwedge^n Q(C_\rho) = \bigwedge^n T^*_{(C,p,z)} M_{g,\rho}.$$

Taking the $\mathfrak{S}_n$-invariant part of the Rešetnikov spectral sequence (3.2.3) converging to $H^*(L(C^\times); Q(C_\rho)^{\otimes n})$, we obtain a spectral sequence

(4.4.2) $$E_2^{p,q} = H^p({C_\rho}^n; H^q(L(C^\times); \mathcal{Q}_n))^{\mathfrak{S}_n} \Rightarrow H^{p+q}(L(C^\times); \bigwedge^n Q(C_\rho)),$$

Here $H^q(L(C^\times); \mathcal{Q}_n)$ is a sheaf over ${C_\rho}^n$ whose stalk at $z \in {C_\rho}^n$ is the q-th cohomology of $L(C^\times)$ with values in the stalk $(\mathcal{Q}_n)_z$ of the sheaf $\mathcal{Q}_n$. From Decomposition Theorem ([Ka] Theorem 5.3) and Vanishing Theorem (4.3.2) we can compute each stalk of the $\mathfrak{S}_n$-invariant part of the sheaf $H^q(L(C^\times); \mathcal{Q}_n)$ for $q \leq n$. Consequently we have

Theorem 4.4.3. *For $x = (C, p, z) \in M_{g,\rho}$, we have*

$$H^q(\mathfrak{d}_{\rho_x}; \bigwedge^n T_x^* M_{g,\rho}) = H^q(L(C^\times); \bigwedge^n Q(C_\rho)) = \begin{cases} 0, & \text{if } q < n, \\ H^n(W_1; \bigwedge^n Q), & \text{if } q = n. \end{cases}$$

Decomposition Theorem quated above is the reason why the n-th cohomopogy group does *not* depend on the genus g. It is derived from the Bott-Segal Addition Theorem [BS] [Ka].

One deduces the following in a similar way to $M_{g,\rho}$.

Theorem 4.4.4. *For $x = (C, p, z, p_1) \in C_{g,\rho}$, we have*

$$\begin{aligned} &H^q(\mathfrak{d}_{\rho_x}; \bigwedge^n T_x^* C_{g,\rho}) \\ &\quad = H^q(L(C^\times, p_1); \bigwedge^n Q^1(C_\rho, p_1)) = \begin{cases} 0, & \text{if } q < n, \\ H^n(L_0; \bigwedge^n Q^1), & \text{if } q = n. \end{cases} \end{aligned}$$

Thus our study is reduced to that of the cohomology groups $H^n(W_1; \bigwedge^n Q)$ and $H^n(L_0; \bigwedge^n Q^1)$.

4.5. Euler class.

Here we reconstruct the (relative) Euler class of the universal curve $C_{g,\rho} \to M_{g,\rho}$, i.e., the first Chern class of the relative tangent bundle with respect to the canonical trivialization induced by the coordinate z on $\overline{D_{g,\rho}^\times}$

$$e := c_1(T_{C_{g,\rho}^\times / M_{g,\rho}}, \frac{d}{dz}) \in H^{1,1}(C_{g,\rho}^\times, \overline{D_{g,\rho}^\times})$$

in our framework.

We define $\epsilon \in H^1(L_0; Q^1)$ by

(4.5.1)
$$\epsilon := d(z^{-2}dz^2) \quad \text{i.e.,} \quad \epsilon(f(z)\frac{d}{dz}) := 2(z^{-2}f'(z) - z^{-3}f(z))dz^2, \quad f(z)\frac{d}{dz} \in L_0.$$

This extends to a nowhere-vanishing section of the sheaf $\coprod_{x \in C_{g,\rho}} H^1((\mathfrak{d}_\rho)_x; T_x^* C_{g,\rho})$, which we denote also by ϵ. The cohomology class $D\epsilon \in H^{1,1}(C_{g,\rho})$ is induced by an extension of vector bundles

$$0 \to Q^1(C_\rho, p_1) \to Q^2(C_\rho, p_1) \overset{\mathrm{Res}}{\to} \mathbb{C} \to 0, \quad (x \in C_{g,\rho})$$

over $C_{g,\rho}$. Here Res means the residue of a meromorphic quadratic differential at the point p_1. So, in view of a theory of complex analytic connections on complex analytic fiber bundles [At], we have

Theorem 4.5.2.

$$(\sqrt{-1}/2\pi)D\epsilon = e \in H^{1,1}(C_{g,\rho}).$$

If $x = (C, p, z, p_1) \in \overline{D^{\times}_{g,\rho}} \subset C^{\times}_{g,\rho}$, the coordinate z induces a canonical decomposition of $(\mathfrak{d}_{\rho})_x (= L(C^{\times}, p_1))$-modules

$$T^*_x C^{\times}_{g,\rho} = T^*_{\pi_{g,\rho}(x)} M_{g,\rho} \oplus T^*_{p_1} C^{\times}, \tag{4.5.3}$$

which implies $\epsilon^{n+1} = 0 \in H^{n+1}((\mathfrak{d}_{\rho})_x; \bigwedge^{n+1} T^*_x C^{\times}_{g,\rho})$ for $\forall x \in \overline{D^{\times}_{g,\rho}}$ and $\forall n \geq 1$. Here it should be remarked Köthe duality (4.1.2) does not hold for the case $x \in \overline{D^{\times}_{g,\rho}}$.

Thus we have

Proposition 4.5.6. *If $n \geq 1$,*

$$(\sqrt{-1}/2\pi)^{n+1} D(\epsilon^{n+1}) = e^{n+1} \in H^{n+1,n+1}(C^{\times}_{g,\rho}, \overline{D^{\times}_{g,\rho}}).$$

4.6. Algebraic fiber integrals.

To construct an algebraic analogue of the Morita-Mumford classes e_n i.e., the fiber integral of the $(n+1)$-th power e^{n+1} in our cohomology group $H^n(W_1; \bigwedge^n Q)$ we introduce a map called the algebraic fiber integral.

We introduce a L_0-module $1_1 := T^*_0\mathbb{C}$ and W_1-modules

$$K := T_1 = \mathcal{O}_{\mathbb{C}}(T^*\mathbb{C})(\mathbb{C}), \quad Q^{\times} := T_2^{\times} = \mathcal{O}_{\mathbb{C}}((T^*\mathbb{C})^{\otimes 2})(\mathbb{C} - \{0\}) \quad \text{and}$$

$$(K \otimes \bigwedge^n Q)^{\times} := \mathcal{O}_{\mathbb{C}^{n+1}}(\mathrm{pr}_0{}^* T^*\mathbb{C} \otimes \bigotimes_{i=1}^{n} \mathrm{pr}_i{}^*(T^*\mathbb{C})^{\otimes 2})(O_n),$$

where $\mathrm{pr}_i : \mathbb{C}^{n+1} \to \mathbb{C}$ is the i-th projection $(0 \leq i \leq n)$ and

$$O_n := \{(t, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}; t \neq z_i \quad (1 \leq \forall i \leq n)\}.$$

Then the Shapiro Lemma gives a natural isomorphism

$$\psi_n : H^*(W_1; (K \otimes \bigwedge^n Q)^{\times}/(K \otimes \bigwedge^n Q)) \xrightarrow{\cong} H^*(L_0; 1_1 \otimes (\bigwedge^n Q^{\times}/\bigwedge^n Q)).$$

On the other hand the map $\oint_t$ defined by

$$\oint_t : (K \otimes \bigwedge^n Q)^{\times} \to \bigwedge^n Q$$

$$\omega = f(t, z_1, \ldots, z_n) dt dz_1{}^2 \cdots dz_n{}^2 \mapsto \oint_t \omega$$

$$\oint_t \omega := \lim_{R \to +\infty} \left(\frac{1}{2\pi\sqrt{-1}} \int_{|t|=R} f(t, z_1, \ldots, z_n) dt\right) dz_1{}^2 \cdots dz_n{}^2$$

induces a W_1-homomorphism $\oint_t : (K \otimes \bigwedge^n Q)^{\times}/(K \otimes \bigwedge^n Q) \to \bigwedge^n Q$. We call the composite map

$$\int_{\text{fiber}} := \oint_t \circ (\psi_n)^{-1} : H^*(L_0; 1_1 \otimes (\bigwedge^n Q^{\times}/\bigwedge^n Q)) \to H^*(W_1; \bigwedge^n Q)$$

the algebraic fiber integral.

Let $n \geq 1$. From (4.3.2) we have $H^n(L_0; 1_1 \otimes \bigwedge^n Q^\times) = 0$ and so

$$0 \to H^n(L_0; 1_1 \otimes (\bigwedge^n Q/\bigwedge^n Q^\times)) \xrightarrow{\delta^*} H^{n+1}(L_0; 1_1 \otimes \bigwedge^n Q) \xrightarrow{j^*} H^{n+1}(L_0; 1_1 \otimes \bigwedge^n Q^\times) \quad \text{(exact)}.$$

The L_0-homorphism $\bigwedge^{n+1} Q^1 \to \bigwedge^{n+1} Q^1/\bigwedge^{n+1} Q = 1_1 \otimes \bigwedge^n Q$ maps the cohomology class ϵ^{n+1} into $\operatorname{Ker} j^* = \operatorname{Im} \delta^*$. Hence we can define

$$\kappa_n := \int_{\text{fiber}} (\delta^*)^{-1} \epsilon^{n+1} \in H^n(W_1; \bigwedge^n Q). \tag{4.6.1}$$

This is an algebraic analogue of the n-th Morita-Mumford class $e_n \in H^{2n}(M_g)$.

From Vanishing Theorem 4.3.2 we obtain an algebraic analogue of the Harer stability theorem:

$$\begin{aligned} &\bigoplus_{n\geq 0} H^n(L_0; \bigwedge^n Q^1) = (\bigoplus_{n\geq 0} H^n(W_1; \bigwedge^n Q))[\epsilon], \quad \text{and} \\ &H^n(L_0; 1_1 \otimes (\bigwedge^n Q^\times/\bigwedge^n Q)) = \bigoplus_{i=1}^n ((\delta^*)^{-1}\epsilon^{i+1}) H^{n-i}(W_1; \bigwedge^{n-i} Q). \end{aligned} \tag{4.6.2}$$

On the other hand, the fiber integral $\int_{\text{fiber}}$ has a natural right-inverse. Therefore we obtain

$$\bigoplus_{n\geq 0} H^n(W_1; \bigwedge^n Q) = \mathbb{C}[\kappa_n; n \geq 1]/\text{relations}. \tag{4.6.3}$$

Theorem 2.2 follows from (4.4.3), (4.4.4), (4.6.2) and (4.6.3).

Comparing our algebraic integrals with the original ones, we obtain Theorem 2.4:

$$(\sqrt{-1}/2\pi)^n D(\kappa_n) = e_n \in H^{n,n}(M_{g,\rho}), \quad n \geq 1.$$

5. Relative Picard variety on the universal family.

Arbarello et.al.[ADKP] has studied also the (dressed) relative Picard variety $F_{g,\rho}$ on the universal curve over the dressed moduli $M_{g,\rho}$. For an integer $g \geq 0$ and a real number $\rho \geq 0$ we define the (dressed) relative Picard variety $F_{g,\rho}$ by the moduli of quintuples $(C, p, z, L, [\phi])$, where $(C, p, z) \in M_{g,\rho}$, L is a holomorphic line bundle on C of degree $g-1$, and ϕ is a complex analytic local trivialization of L near $z^{-1}(\overline{D_\rho})$. Furthermore $[\phi]$ is the equivalent class of ϕ under the equivalent relation

$$[\phi] = [\phi'] \iff \exists \lambda \in \mathbb{C}^\times \quad \phi' = \lambda\phi \quad \text{near } z^{-1}(\overline{D_\rho}).$$

(The space $F_{g,0}$ is equal to the space $\widehat{\mathcal{F}_{g-1}}$ in [ADKP].)

We introduce a topological Lie algebra

$$\mathcal{D}_\rho := \varinjlim_{\rho_1 \downarrow \rho} \Delta(D^\times_{\rho_1}).$$

Here we denote by $\Delta(U)$ the topological Lie algebra consisting of all complex analytic differential operators of order ≤ 1 on an open Riemann surface U. When we denote by $F(U)$ the $L(U)$-module consisting of all compex analytic functions on U, we have a decomposition

$$\Delta(U) = F(U) \rtimes L(U).$$

So we denote an element of $\Delta(U)$ (and $\mathcal{D}_\rho$) by $f + X \in \Delta(U)$, $f \in F(U)$, $X \in L(U)$ in the sequel.

The Lie algebra $\mathcal{D}_\rho$ acts on the manifold $F_{g,\rho}$ infinitesimally. In fact, for any $f+X \in \mathcal{D}_\rho$, $(C, p, z, L, [\phi]) \in F_{g,\rho}$ and sufficient small complex numbers ϵ, a complex analytic path $(C(\epsilon), p(\epsilon), z(\epsilon), L(\epsilon), [\phi(\epsilon)]) \in F_{g,\rho}$ is defined by $(C(\epsilon), p(\epsilon), z(\epsilon))$ as in §1.4 and

$$\begin{aligned}
&L(\epsilon) := (L|_{C^\times} \amalg D \times \mathbb{C})/\sim_\epsilon \\
&\qquad (z, u) \sim_\epsilon ((\exp \epsilon X)(z), u(\exp \epsilon f(z))\phi(\exp \epsilon X)) \quad (z, u) \in D \times \mathbb{C} \\
&\phi(\epsilon) := \text{the local trivialization induced by the identity } D \times \mathbb{C} \hookrightarrow D \times \mathbb{C}.
\end{aligned}$$

Differentiating the path $(C(\epsilon), p(\epsilon), z(\epsilon), L(\epsilon), [\phi(\epsilon)])$ we obtain an infinitesimal action P of $\mathcal{D}_\rho$ on the manifold $F_{g,\rho}$. As was proved in [ADKP] Proposition 3.19, we have an exact sequence

$$0 \to \Delta(C^\times) \hookrightarrow \mathcal{D}_\rho \xrightarrow{P} T_{(C,p,z,L,[\phi])}F_{g,\rho} \to 0.$$

Especially the action P is homogeneous.

Our main result in this section is

Theorem 5.1. *For all $x \in F_{g,\rho}$ we have*

$$H^q((\mathcal{D}_\rho)_x; \textstyle\bigwedge^p T^*_x F_{g,\rho}) = 0 \quad \textit{if } q < p,$$

and the diagonal part $H^p((\mathcal{D}_\rho)_x; \bigwedge^p T^*_x F_{g,\rho})$ does not depend on the genus $g \geq 0$.

As a corollary

Corollary 5.2. *If the Frobenius Reciprocity Law (3.3.2) holds for the $\mathcal{D}_\rho$-manifold $M = F_{g,\rho}$ and the $\mathcal{D}_\rho$-vector bundles $\bigwedge^* T^*M$, we have*

$$H^{p,q}_{\mathcal{D}_\rho}(F_{g,\rho}) = 0, \quad \text{if } q < p,$$

for all $g \geq 0$ and $\rho > 0$.

At present our computation for the diagonal part is incomplete. In the next section we discuss the cohomology algebra $H^*(F_{g,\rho}; \mathbb{Q})$.

Fix a point $x = (C, p, z, L, [\phi]) \in F_{g,\rho}$. The Köthe duality (4.1.1) gives a $(\mathcal{D}_\rho)_x$-isomorphism

$$T^*_x F_{g,\rho} = (\mathcal{D}_\rho/\Delta(C^\times))^* = JQ(C_\rho),$$

where we denote by $JQ(U)$ the space consisting of all complex analytic 1-jets of 1-forms on an open Riemann surface U:

$$JQ(U) := \mathcal{O}_U(J^1T^*U)(U) = K(U) \ltimes Q(U).$$

Thus we have

$$H^*((\mathcal{D}_\rho)_x; \bigwedge\nolimits^n T^*_x F_{g,\rho}) = H^*(\Delta(C^\times); \bigwedge\nolimits^n JQ(C_\rho)).$$

Similar results to those for $L(U)$ (Rešetnikov spectral sequence, Decomposition Theorem, Vanishing Theorem and so on) hold also for the Lie algebra $\Delta(U)$. So we obtain Theorem 5.1.

6. Stable cohomology of the extended mapping class groups.

Let $g \geq 2$, $r, s \geq 0$ be integers. Let $\Sigma^s_{g,r}$ denote a 2-dimensional oriented C^∞ manifold (i.e., oriented surface) of genus g with r boundary components and (ordered) s punctures. The group of path-components $\pi_0(\mathrm{Diff}^+(\Sigma^s_{g,r}))$ is denoted by $\Gamma^s_{g,r}$ (or $\mathcal{M}^s_{g,r}$) and called the mapping class group of genus g with r boundary components and (ordered) s punctures. Here $\mathrm{Diff}^+(\Sigma^s_{g,r})$ denotes the topological group (endowed with C^∞ topology) consisting of all orientation preserving diffeomorphisms of $\Sigma^s_{g,r}$ which fix all the boundary points and the punctures pointwise. When $s = 0$, we drop the indices: $\Sigma_{g,r} = \Sigma^0_{g,r}$, $\Gamma_{g,r} = \Gamma^0_{g,r}$ and similarly $\Sigma_g = \Sigma^0_{g,0}$, $\Gamma_g = \Gamma^0_{g,0}$. By the contractibility of the Teichmüller space we have $B\,\mathrm{Diff}^+(\Sigma^s_{g,r}) = B\Gamma^s_{g,r}$ [EE]. We denote

$$H := H_1(\Sigma_{g,1}; \mathbb{Z}) = H_1(\Sigma_g) = H^1(\Sigma_g) = H^1(\Sigma_{g,1}),$$

on which the mapping class group Γ_g acts in an obvious way. By the *extended mapping class group* we mean the semi-direct product

$$\widetilde{\Gamma^s_{g,r}} := H \rtimes \Gamma^s_{g,r}.$$

The relative Picard variety $F_{g,\rho}$ is (homotopy equivalent to) the classifying space of $\widetilde{\Gamma_{g,1}}$. So we have $H^*(F_{g,\rho}; \mathbb{Z}) = H^*(\widetilde{\Gamma^s_{g,r}}; \mathbb{Z})$. Our main result in this section is

Theorem 6.0. *For $* < g/2$ we have*

$$H^*(F_{g,\rho};\mathbb{Q}) = H^*(\widetilde{\Gamma_{g,1}};\mathbb{Q}) = H^*(\Gamma_{g,1};\mathbb{Q}) \otimes \mathbb{Q}[\widetilde{m_{i,j}}],$$

where $\widetilde{m_{i,j}} \in H^{2i+2j-2}(\widetilde{\Gamma_{g,1}};\mathbb{Z})$ is the generalized Morita-Mumford class and the integers i and j run over the domain $\{(i,j) \in \mathbb{Z} \times \mathbb{Z};\quad i \geq 0,\, j \geq 1 \,\text{and}\, i+j \geq 2\}$.

6.1. Cohomology of pairs of groups.

Let G be a group, K a subgroup of G, and M a G-module. We denote by $H^*(G,K;M)$ the cohomology group of the kernel of the restriction map $C^*(G;M) \to C^*(K;M)$ and call it *the (relative) cohomology group of the pair of groups* (G,K) *with values in the G-module M.* Here we denote by $C^*(G;M)$ the normalized cochain complex of a group G with values in a G-module M.

Let $N \triangleleft G$ be a normal subgroup satisfying the condition

$$KN = G. \tag{6.1.1}$$

Then we have the following Lyndon-Hochshild-Serre (LHS) spectral sequence [HS1]:

$$E_2^{p,q} = H^p(G/N; H^q(N, N\cap K; M)) \Rightarrow H^*(G,K;M), \tag{6.1.2}$$

which is multiplicative in a usual manner.

6.2. Generalized Morita-Mumford classes.

First we remark the surface $\Sigma^1_{g,1}$ is obtained by glueing the surfaces $\Sigma_{g,1}$ and $\Sigma^1_{0,2}$ along the boundaries. So the diffeomorphism of $\Sigma_{g,1}$ is naturally extended to that of $\Sigma^1_{g,1}$. The infinite cyclic group $\mathbb{Z}$ acts on the surface $\Sigma^1_{0,2}$ by rotating the puncture and fixing the boundaries pointwise. Similarly this action is extended to that on $\Gamma^1_{g,1}$ in a natural way. Thus we obtain a natural homomorphism $\Gamma_{g,1} \times \mathbb{Z} \to \Gamma^1_{g,1}$, which is injective ([Is] §5). In the sequel we regard the group $\Gamma_{g,1} \times \mathbb{Z}$ as a subgroup of $\Gamma^1_{g,1}$ through the injection. Especially we may consider the cohomology group $H^*(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; M)$ for an arbitrary $\Gamma^1_{g,1}$-module M.

The LHS spectral sequence (6.1.2) of the forgetful extension

$$1 \to \pi_1(\Sigma_{g,1}) \to \Gamma^1_{g,1} \xrightarrow{\pi} \Gamma_{g,1} \to 1 \tag{6.2.1}$$

induces a Gysin exact sequence

$$\begin{aligned} \cdots \to H^{q-1}(\Gamma_{g,1};M) &\to H^{q+1}(\Gamma_{g,1}; H\otimes M) \\ &\to H^{q+2}(\Gamma^1_{g,1}, \Gamma_{g,1}\times\mathbb{Z}; M) \xrightarrow{\pi_!} H^q(\Gamma_{g,1};M) \to \cdots \end{aligned} \tag{6.2.2}$$

for any $\Gamma_{g,1}$-module M. The map $\pi_!$ is called *the Gysin map* or *the fiber integral.* Similarly we can define *the fiber integral* $\widetilde{\pi}_! : H^p(\widetilde{\Gamma^1_{g,1}}, \widetilde{\Gamma_{g,1}} \times \mathbb{Z}) \to H^{p-2}(\widetilde{\Gamma_{g,1}})$.

Choose a simple curve l inside the subsurface $\Sigma^1_{0,2}$ connecting the puncture to a point on the boundary. A 1-cocycle $\omega_l \in Z^1(\Gamma^1_{g,1}, \Gamma_{g,1}\times\mathbb{Z}; H)$ and a 2-cocycle $\widetilde{\omega}_l \in Z^2(\widetilde{\Gamma^1_{g,1}}, \widetilde{\Gamma_{g,1}} \times \mathbb{Z}; \mathbb{Z})$ are given by

$$\begin{aligned} &\omega_l(\gamma) = \gamma l - l \in H, \quad \gamma \in \Gamma^1_{g,1}, \\ &\widetilde{\omega}_l(u_1\gamma_1, u_2\gamma_2) := \gamma_1(\gamma_2 l - l)\cdot u_1, \quad u_1,u_2 \in H,\ \gamma_1,\gamma_2 \in \Gamma^1_{g,1}, \end{aligned} \tag{6.2.3}$$

where $\gamma l - l$ is regarded as a closed curve on $\Sigma_{g,1}$. Let $e \in H^2(\Gamma^1_g; \mathbb{Z})$ be the Euler class of the central extension $0 \to \mathbb{Z} \to \Gamma_{g,1} \to \Gamma^1_g \to 1$.

For integers $i, j \geq 0$ satsifying $i + j \geq 2$ we define

$$m_{i,j} := \pi_!(e^i \omega_l{}^j) \in H^{2i+j-2}(\Gamma_{g,1}; \bigwedge^j H), \tag{6.2.4}$$
$$\widetilde{m_{i,j}} := \tilde{\pi}_!(e^i \tilde{\omega_l}{}^j) \in H^{2i+2j-2}(\widetilde{\Gamma_{g,1}}).$$

Clearly $m_{i+1,0}$ and $\widetilde{m_{i+1,0}}$ are equal to (the image of) the i-th Morita-Mumford (tautological) class $e_i(= \kappa_i) \in H^{2i}(\Gamma_g; \mathbb{Z})$ [Mo][Mu]. So we call these classes $m_{i,j}$'s and $\widetilde{m_{i,j}}$'s *the generalized Morita-Mumford classes.*

6.3. Stable cohomology of the extended mapping class groups.

Independently from our construction Looijenga [L] proved the rational stable cohomology group of the mapping class group Γ_g (with no boundary and no punctures) with coefficients in any irreducible representation of the complex symplectic group is a free module over the rational stable cohomology algebra of the mapping class group, and described its free basis. As a consequence he determined the rational stable cohomology of $\widetilde{\Gamma_g}$ modulo that of Γ_g. His computation is involved with geometric consideration on the moduli orbifold of complex algebraic curves including a theorem on Hodge theory [D]. Here it is remarkable that his results are based only on the Harer stability theorem [H] [I] with trivial coefficients.

Following Looijenga [L] we have proved the following results from the Harer stability theorem with trivial coefficients, but geometric considerations including Hodge theory do not fit our situation where the surface is not closed. We use the Lyndon-Hochschild-Serre spectral sequence for a pair of groups introduced in §6.1 instead.

Theorem 6.3.1. *If $s \geq 0$, $r \geq 1$ and $n \geq 0$, we have*

$$H^*(\Gamma^s_{g,r}; H^1(\Sigma^s_{g,r}; \mathbb{Z})^{\otimes n}) = H^*(\Gamma_{g,1}; H^{\otimes n}) \otimes_{H^*(\Gamma_{g,1};\mathbb{Z})} H^*(\Gamma^s_{g,r}; \mathbb{Z})$$

for $* \leq g/2 - n$. Furthermore the cohomology group $H^*(\Gamma_{g,1}; H^{\otimes n})$ is stably a free module over the algebra $H^*(\Gamma_{g,1}; \mathbb{Z})$ and combinations of the (modified) generalized Morita-Mumford classes give its free basis.

Taking the alternating part of the second half of this theorem, we obtain

Theorem 6.3.2.

$$H^*(\Gamma_{g,1}; \bigwedge^* H^1(\Sigma_{g,1}; \mathbb{Q})) = H^*(\Gamma_{g,1}; \mathbb{Q}) \otimes \mathbb{Q}[m_{i,j}]$$

if the total degree is smaller than $g/2$. Here the integers i and j run over the domain given in Theorem 6.0.

As a corollary we obtain Theorem 6.0.

Finally we mention how these classes $\widetilde{m_{i,j}}$'s are related to the equivariant cohomology of $F_{g,\rho}$. The class $\tilde{\omega}_l$ corresponds to the Abelian differential of the third kind on a compact Riemann surface. Especially it has a position in the complex analytic Gel'fand-Fuks cohomology. This implies the classes $\widetilde{m_{i,j}}$'s are contained in the image of the equivariant (p,p)-cohomology. The author does not know whether the equivariant (p,p)-cohomology is stably isomorphic to the polynomial algebra generated by $\widetilde{m_{i,j}}$'s.

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