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SINGULARITIES FOR PROJECTIONS OF CONTOUR LINES OF SURFACES ONTO PLANES

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ABSTRACT. We study the visions for contour lines of surfaces when one looks at it from a distant view in some direction. The study of such a landscape (i.e. so-called "topography") is reduced to the study of a certain divergent diagram of the smooth mappings $\mathbb{R} \leftarrow M \to \mathbb{R}^2$, where M is a smooth surface. We give a generic semi-local classification of such divergent diagrams.

1. FORMULATIONS AND RESULTS

In this paper we give a generic semi-local classification for the singularities of orthogonal projections of contour lines of surfaces onto planes.

Let M be a surface in $\mathbb{R}^3 = \{(x,y,z)\}$ and let E_d be a hyperplane in $\mathbb{R}^3$ with the normal direction d such that $E_d \cap M = \phi$. We denote by $Emb(M, \mathbb{R}^3)$ the space of all embeddings $M \to \mathbb{R}^3$ endowed with the Whitney C^∞-topology. Let $\pi_d : \mathbb{R}^3 \to E_d$ be a orthogonal projection along the direction d. Now consider a level set of the height function $z : i(M) \to \mathbb{R}$, that is $i(M) \cap \{z = \text{constant}\}$ for $i \in Emb(M, \mathbb{R}^3)$. We call the set *a contour line* on $i(M)$. If one looks at a contour line on $i(M)$ from a distant view in some direction d, then one will get $\pi_d(i(M) \cap \{z = c\})$ as the viewing image. We study such a landscape as one of the problems in the vision theory. That is, our subject is a semi-local classification of singularities for one parameter families $\{\pi_d(i(M) \cap \{z = c\})\}_{c \in \mathbb{R}}$ which is called *a topography* of $i(M)$ with respect to a direction d.

Let us formulate our theorems. Throughout this paper we shall suppose that all mappings, map germs and manifolds are of class C^∞ unless otherwise stated. Now, without loss of generality we can suppose that $d = (0, \cos\xi, \sin\xi) \in S^2 \cap \{x = 0\}$

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where $0 < \xi \leqq \frac{\pi}{2}$. For a direction d, by the transformation of $\frac{\pi}{2} - \xi$ rotation around x-axis in $\mathbb{R}^3$, we choose a new coordinate (x', y', z'). Then the direction d becomes $(0, 0, 1)$ and the height function z is expressed by $-y' \sin(\frac{\pi}{2} - \xi) + z' \cos(\frac{\pi}{2} - \xi)$ in the new coordinate. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be a projection defined by $\pi(u, v, w) = (u, v)$. We call the following divergent diagram of mappings *a topographic diagram* of $i(M)$, which is denoted by (μ_i, g) :

$$\mathbb{R} \xleftarrow{\mu_i} M \xrightarrow{g} \mathbb{R}^2,$$

where $\mu_i = -i_2 \sin\theta + i_3 \cos\theta \, (0 \leqq \theta < \frac{\pi}{2}), g = \pi \circ i$.

Since our concern is to describe the discriminant set of $\pi \circ i$ (the outline of $i(M)$) and the bifurcation of $\pi \circ i(\mu_i^{-1}(c))$ along the parameter $c \in \mathbb{R}$ in semi-local situation, we introduce the following definitions. Let $i \in Emb(M, \mathbb{R}^3)$ and let $\{p_1, \dots, p_r\}$ be a subset of M whose elements are all distinct points in M such that $\pi \circ i(p_1) = \cdots = \pi \circ i(p_r)$, where r is a positive integer. Then the multigerm of a topographic diagram at $\{p_1, \dots, p_r\}$ which is denoted by ${}_rT_i$

$$\begin{array}{ccccc} (\mathbb{R}, 0) & \xleftarrow{\mu_1} & (\mathbb{R}^2, 0) & \xrightarrow{g_1} & (\mathbb{R}^2, 0) \\ (\mathbb{R}, 0) & \xleftarrow{\mu_2} & (\mathbb{R}^2, 0) & \nearrow_{g_2} & \\ & \vdots & & \vdots\, g_r & \\ (\mathbb{R}, 0) & \xleftarrow{\mu_r} & (\mathbb{R}^2, 0) & \nearrow & \end{array},$$

where μ_k, g_k are germs of $\mu_i, \pi \circ i$ at p_k respectively $(k = 1, \dots, r)$, is called *a topographic multigerm of* i. Let ${}_rT_i$ and ${}_rT_{i'}$ be topographic multigerms. Then ${}_rT_i$ and ${}_rT_{i'}$ are said to be *equivalent* if there exist diffeomorphism germs $\lambda_k : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, $\psi_k : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, where $k = 1, \dots, r$, and $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $\lambda_k \circ \mu_k = \mu'_k \circ \psi_k, \phi \circ g_k = g'_k \circ \psi_k$.

We shall state a genericity theorem for topographic multigerms.

Theorem A. *There exists a residual subset (hence dense)* ${}_r\mathcal{O}$ *in* $Emb(M, \mathbb{R}^3)$ *such that for any* $i \in {}_r\mathcal{O}$ *the topographic multigerms* ${}_rT_i \, (1 \leq r \leq 3)$ *is one of the following types:*

In the case $r = 1$.

(I) μ_1 *is a submersion and* g_1 *is regular.*

(II) μ_1 *is a Morse type and* g_1 *is regular.*

(III) μ_1 *is a submersion,* g_1 *is a fold,* μ_1 *restricted to the singular set of* g_1 *is regular and* $(\mu_1, g_1) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *is regular.*

(IV) μ_1 *is a submersion,* g_1 *is a fold,* μ_1 *restricted to the singular set of* g_1 *is a Morse type and* $(\mu_1, g_1) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *is regular.*

(V) μ_1 *is a submersion,* g_1 *is a fold,* $(\mu_1, g_1) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *is a Whitney's umbrella such that the line of double points of which is transversal at* 0 *to the direction* $\{0\} \times \mathbb{R}^2$ *in* $\mathbb{R}^3$.

(VI) μ_1 *is a submersion,* g_1 *is a cusp and* $(\mu_1, g_1) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *is regular.*

In the case $r = 2$.

$(I, I)_0$:

$$(\mu_1, g_1), (\mu_2, g_2) \text{ are both of type (I)}$$
$$\text{and } (\mu_1 \circ {g_1}^{-1}, \mu_2 \circ {g_2}^{-1}) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \text{ is regular.}$$

$(I, I)_1$:

$$(\mu_1, g_1), (\mu_2, g_2) \text{ are both of type (I)}$$
$$\text{and } (\mu_1 \circ {g_1}^{-1}, \mu_2 \circ {g_2}^{-1}) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \text{ is a fold.}$$

$(I, I)_2$:

$$(\mu_1, g_1), (\mu_2, g_2) \text{ are both of type (I)}$$
$$\text{and } (\mu_1 \circ {g_1}^{-1}, \mu_2 \circ {g_2}^{-1}) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \text{ is a cusp.}$$

(II, I) :

$$(\mu_1, g_1) \text{ is of type (II), } (\mu_2, g_2) \text{ is of type (I)}$$
$$\text{and } (\mu_1 \circ {g_1}^{-1}, \mu_2 \circ {g_2}^{-1}) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \text{ is a fold.}$$

$(III, I)^0$:

$$(\mu_1, g_1) \text{ is of type (III), } (\mu_2, g_2) \text{ is of type (I)}$$
$$\text{and the discriminant set of } g_1 \text{ and } g_2(\mu_2^{-1}(0)) \text{ are transversal.}$$

$(III, I)^1$:

$$(\mu_1, g_1) \text{ is of type (III), } (\mu_2, g_2) \text{ is of type (I)}$$
$$\text{and the discriminant set of } g_1 \text{ and } g_2(\mu_2^{-1}(0)) \text{ have two point contact.}$$

(IV, I) :

$$(\mu_1, g_1) \textit{ is of type (IV), } (\mu_2, g_2) \textit{ is of type (I)}$$
$$\textit{and the discriminant set of } g_1 \textit{ and } g_2(\mu_2^{-1}(0)) \textit{ are transversal.}$$

(V, I) :

$$(\mu_1, g_1) \textit{ is of type (V), } (\mu_2, g_2) \textit{ is of type (I),}$$
$$\textit{the discriminant set of } g_1 \textit{ and } g_2(\mu_2^{-1}(0)) \textit{ are transversal.}$$

(VI, I) :

$$(\mu_1, g_1) \textit{ is of type (VI), } (\mu_2, g_2) \textit{ is of type (I)}$$
$$\textit{and the tangent cone of the discriminant set of } g_1 \textit{ and } g_2(\mu_2^{-1}(0)) \textit{ are transversal.}$$

(III, III) :

$$(\mu_1, g_1), (\mu_2, g_2) \textit{ are both of type (III)}$$
$$\textit{and the discriminant sets of } g_1, g_2 \textit{ are transversal.}$$

In the case $r = 3$.

(I, I, I)$_{1,1}$: $(\mu_j, g_j; \mu_k, g_k)$ is of type (I, I)$_1$ for $1 \leqq j < k \leqq 3$.

(III, I, I)$_1^{0,0}$: $(\mu_1, g_1; \mu_2, g_2)$ is of type (III, I)^0 and $(\mu_2, g_2; \mu_3, g_3)$ is of type (I, I)$_1$.

(III, III, I)^{0,0} : $(\mu_1, g_1; \mu_2, g_2)$ is of type (III, III) and $(\mu_j, g_j; \mu_3, g_3)$ is of type (III, I)^0 for $j = 1, 2$.

Besides the following nine types:

(I, I, I)$_{0,0}$, (II, I, I)$_0^{0,0}$, (III, I, I)$_0^{0,0}$, (IV, I, I)$_0^{0,0}$, (V, I, I)$_0^{0,0}$, (VI, I, I)$_0^{0,0}$: $(\mu_1, g_1; \mu_k, g_k)$ *is of type* (I, I)$_0$, (ν, I), ν=II,...,VI *for* $k = 2, 3$ *and* $(\mu_2, g_2; \mu_3, g_3)$ *is of type* (I, I)$_0$.

(I, I, I)$_{1,0}$, (I, I, I)$_{2,0}$, (III, I, I)$_0^{1,0}$: $(\mu_1, g_1; \mu_2, g_2)$ *is of type* (I, I)$_1$, (I, I)$_2$, (III, I)^1 *and* $(\mu_2, g_2; \mu_3, g_3)$ *is of type* (I, I)$_0$.

Remark 1.1. In the case $r = 2$, the generic condition of (I, I)$_1$ (resp. (I, I)$_2$) means that $g_1({\mu_1}^{-1}(0))$ and $g_2({\mu_2}^{-1}(0))$ have second (resp. third) order contact.

Remark 1.2. In the case $r \geqq 4$ all of generic types are essentially same as the case $r = 3$. That is, we can add only type (I) to the each type in the case $r = 3$ such that $(\mu_j, g_j; \mu_r, g_r)$ for $j = 1, \ldots, r-1$ is not of type (*, I)$_1$, (*, I)$_2$ in the list of the case $r=2$. In the same sense for the case $r=3$, the generic types except three types $(\mathrm{I}, \mathrm{I}, \mathrm{I})_{1,1}$, $(\mathrm{III}, \mathrm{I}, \mathrm{I})_1^{0,0}$, $(\mathrm{III}, \mathrm{III}, \mathrm{I})^{0,0}$ are essentially same as in the case $r=2$.

Next we shall give a normal form for each type stated as above. Denote by $\mathcal{E}_{x_1,\dots,x_n}$ the ring of all smooth function germs on $\mathbb{R}^n$ at 0 with a coordinate $(x_1,\cdots,x_n)$ and denote by $\mathcal{M}_{x_1,\dots,x_n}$ the unique maximal ideal of $\mathcal{E}_{x_1,\dots,x_n}$.

Theorem B. *The topographic multigerms of each type are equivalent to one of the following multigerms* ${}_rT_i = (\mu_1, g_1; \cdots; \mu_r, g_r)$*:*

In the case $r=1$.

(I)

$$\mu_1 = y_1, \qquad g_1 = (x_1, y_1).$$

(II)

$$\mu_1 = x_1^2 \pm y_1^2, \qquad g_1 = (x_1, y_1).$$

(III)

$$\mu_1 = x_1 + y_1, \qquad g_1 = (x_1, y_1^2).$$

(IV)

$$\mu_1 = x_1^2 + y_1, \qquad g_1 = (x_1, y_1^2).$$

(V)

$$\mu_1 = x_1 + x_1y_1 + y_1^3, \qquad g_1 = (x_1, y_1^2).$$

(VI)

$$\mu_1 = y_1 + \alpha \circ g_1, \qquad g_1 = (x_1, y_1^3 + x_1y_1),$$

where $\alpha \in \mathcal{M}_{u,v}$.

In the case $r=2$.

$(I,I)_0$

$$\mu_1 = y_1, \qquad g_1 = (x_1, y_1);$$

$$\mu_2 = x_2, \qquad g_2 = (x_2, y_2).$$

$(I,I)_1$

$$\mu_1 = y_1, \qquad g_1 = (x_1, y_1);$$

$$\mu_2 = x_2^2 + y_2, \qquad g_2 = (x_2, y_2).$$

$(I, I)_2$

$$\mu_1 = y_1, \quad g_1 = (x_1, y_1);$$

$$\mu_2 = x_2^3 + x_2 y_2 + y_2, \quad g_2 = (x_2, y_2).$$

(II, I)

$$\mu_1 = x_1^2 \pm y_1^2, \quad g_1 = (x_1, y_1);$$

$$\mu_2 = x_2, \quad g_2 = (x_2, y_2).$$

$(III, I)^0$

$$\mu_1 = x_1 + y_1, \quad g_1 = (x_1, y_1^2);$$

$$\mu_2 = x_2 + \theta(x_2, y_2), \quad g_2 = (x_2, y_2),$$

$$\textit{where } \theta \in \mathcal{M}_{x_2,y_2} \textit{ with } \frac{\partial \theta}{\partial x_2}(0) = 0.$$

$(III, I)^1$

$$\mu_1 = x_1 + y_1, \quad g_1 = (x_1, y_1^2);$$

$$\mu_2 = y_2 + \theta(x_2, y_2), \quad g_2 = (x_2, y_2),$$

$$\textit{where } \theta \in \mathcal{M}^2_{x_2,y_2} \textit{ with } \frac{\partial^2 \theta}{\partial x_2^2}(0) \neq 0.$$

(IV, I)

$$\mu_1 = x_1^2 + y_1, \quad g_1 = (x_1, y_1^2);$$

$$\mu_2 = x_2 + \theta(x_2, y_2), \quad g_2 = (x_2, y_2),$$

$$\textit{where } \theta \in \mathcal{M}_{x_2,y_2} \textit{ with } \frac{\partial \theta}{\partial x_2}(0) = 0.$$

(V, I)

$$\mu_1 = x_1 + x_1 y_1 + y_1^3, \quad g_1 = (x_1, y_1^2);$$

$$\mu_2 = x_2 + \theta(x_2, y_2), \quad g_2 = (x_2, y_2),$$

$$\textit{where } \theta \in \mathcal{M}_{x_2,y_2} \textit{ with } \frac{\partial \theta}{\partial x_2}(0) = 0.$$

(VI, I)

$$\mu_1 = y_1 + \alpha \circ g_1, \quad g_1 = (x_1, y_1^3 + x_1 y_1);$$

$$\mu_2 = x_2 + \theta(x_2, y_2), \quad g_2 = (x_2, y_2),$$

$$\textit{where } \alpha \in \mathcal{M}_{u,v}, \theta \in \mathcal{M}_{x_2,y_2} \textit{ with } \frac{\partial \theta}{\partial x_2}(0) = 0.$$

(III, III)

$$\mu_1 = y_1 + \alpha_1 \circ g_1, \quad g_1 = (x_1, y_1^2);$$

$$\mu_2 = x_2 + \alpha_2 \circ g_2, \quad g_2 = (x_2^2, y_2),$$

$$\textit{where } \alpha_1, \alpha_2 \in \mathcal{M}_{u,v} \textit{ with } \tfrac{\partial \alpha_1}{\partial u}(0) \neq 0, \tfrac{\partial \alpha_2}{\partial v}(0) \neq 0.$$

Remark 1.3. The normal forms of type $(\mathrm{I, I, I})_{1,1}$, $(\mathrm{III, I, I})_1^{0,0}$, $(\mathrm{III, III, I})^{0,0}$ are the following:

$(\mathrm{I, I, I})_{1,1}$: $\mu_1, g_1; \mu_2, g_2$ have the same form as the normal form of type $(\mathrm{I, I})_1$,

$$\mu_3 = y_3 + a{x_3}^2 + \theta(x_3, y_3), \quad g_3 = (x_3, y_3)$$

$$\text{where } a \in \mathbb{R} - \{0, 1\},\ \theta \in \mathcal{M}^2_{x_3, y_3} \text{ with } \tfrac{\partial^2 \theta}{\partial {x_3}^2}(0) = 0.$$

$(\mathrm{III, I, I})_1^{0,0}$:

$$\mu_1 = x_1 + y_1, \quad g_1 = (x_1, {y_1}^2);$$

$$\mu_2 = x_2 + \alpha(x_2, y_2), \quad g_2 = (x_2, y_2);$$

$$\mu_3 = x_3 + \beta(x_3, y_3), \quad g_3 = (x_3, y_3),$$

where $\alpha \in \mathcal{M}_{x_2, y_2}, \beta \in \mathcal{M}_{x_3, y_3}$ with $\frac{\partial \alpha}{\partial x_2}(0) = \frac{\partial \beta}{\partial x_3}(0) = 0$, $\frac{\partial \alpha}{\partial y_2}(0) = \frac{\partial \beta}{\partial y_3}(0)$ and $\frac{\partial^2 \alpha}{\partial {y_2}^2}(0) - \frac{\partial \alpha}{\partial y_2}(0)\frac{\partial^2 \alpha}{\partial x_2 \partial y_2}(0) \neq \frac{\partial^2 \beta}{\partial {y_3}^2}(0) - \frac{\partial \beta}{\partial y_3}(0)\frac{\partial^2 \beta}{\partial x_3 \partial y_3}(0)$.

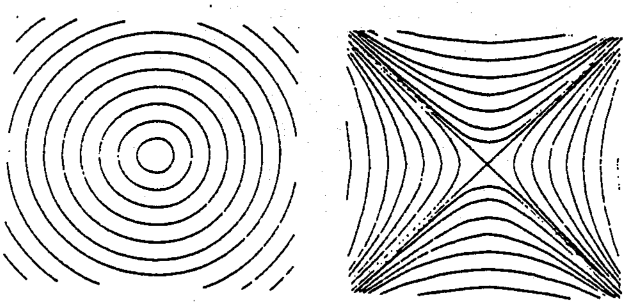
$(\mathrm{III, III, I})^{0,0}$: $\mu_1, g_1; \mu_2, g_2$ have the same form as the normal form of type (III, III),

$$\mu_3 = x_3 + y_3 + \theta(x_3, y_3), \quad g_3 = (x_3, y_3) \text{ where } \theta \in \mathcal{M}^2_{x_3, y_3}.$$

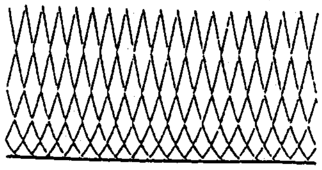
Remark 1.4. In order to understand our classification of topographies geometrically, let us describe the level curves $\{g_k({\mu_k}^{-1}(c))\}$ and the discriminant set of $g_k, 1 \leqq k \leqq r\ (r = 1, 2)$ for each type.

(I)

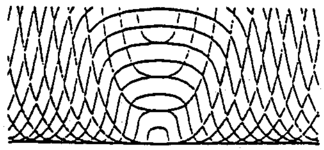
(II)



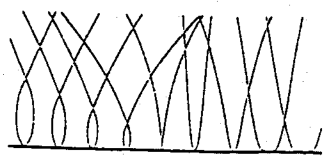
(III)



(IV)



(V)



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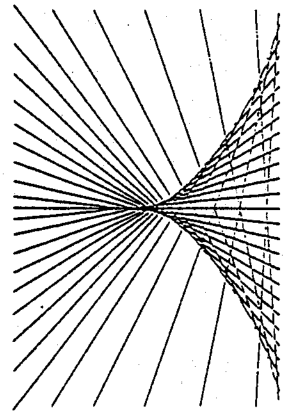
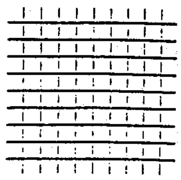
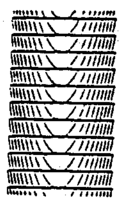


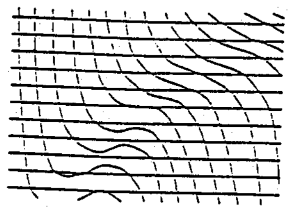
$(\mathrm{I}, \mathrm{I})_0$



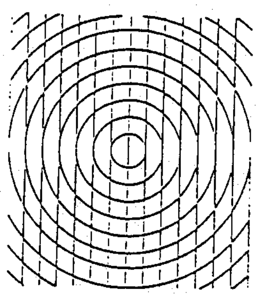
$(\mathrm{I}, \mathrm{I})_1$



$(\mathrm{I}, \mathrm{I})_2$



(II, I)



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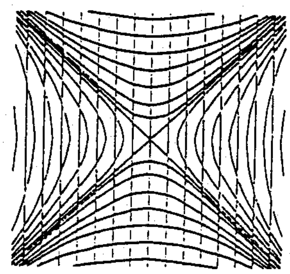
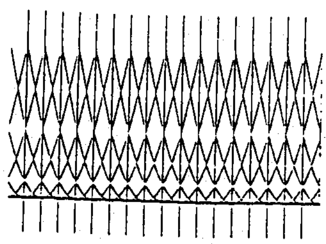
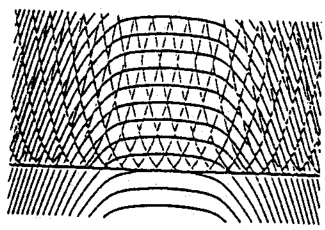


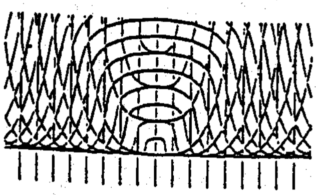
$(\mathrm{III}, \mathrm{I})^0$



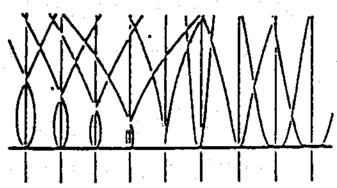
$(\mathrm{III}, \mathrm{I})^1$



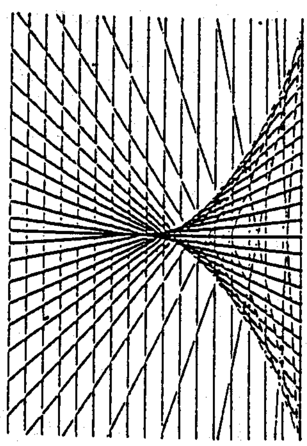
(IV, I)



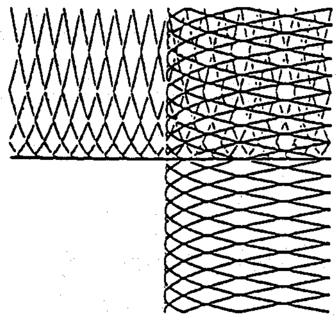
(V, I)



(VI, I)



(III, III)



Remark 1.5. The normal forms in Theorem B depend on arbitrary functions with some conditions, that is so-called "functional moduli" appear in the normal forms. For the type (VI) the functional moduli have been characterized and the complete invariant has been detected ([11], [15]). For other types which appear functional moduli, however we can not obtain a characterization of the functional moduli in this paper. We remark that the topographies have "d-web"(a configuration of d foliations) structure. It is known that the only one functional moduli appear in the local normal form of 3-webs which consist of 3 curvilinear foliations in $\mathbb{R}^2$ ([12]). So we observe that it is natural the only one functional moduli appear in our normal forms of the types which have 3-web structures. Also we remark that two functional moduli which appear in our normal form of type (III, III) deeply connect with the 4-web structure which the topographies of type (III, III) have. We can not obtain the result that whether the two functional moduli are reduced to only one or not in this paper.

Remark 1.6. The divergent diagram $(\mathbb{R}, 0) \leftarrow (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ have been studied by Arnol'd [1], Carneiro [5], Dufour [10] from the viewpoint of envelope, stability theory. Also, the normal forms of each type stated in Theorem A for $r = 1$ has been obtained by Arnol'd [1], Dufour [10].(The classification of Arnol'd is not C^∞ case but formal case.) For $r = 2$, essentially types $(I, I)_k$ $(k = 0, 1, 2)$, (II, I) have been studied by Dufour [7, 8, 9] from the viewpoint of "bi-stability" and its normal forms have been obtained. On the other hand, singularities for certain visual images have been studied by several authors [4, 6, 13, 16, 17]. In particular, Dufour and Tueno have investigated in [13] the pattern of illuminance due to a point source of light which coincide with our Theorem A in the case $r = 1$.

2. Genericity

2.1. Preliminary of proof of Theorem A.

In order to describe the genericity we shall prepare a transversality lemma. Let $M^{(r)} = \{(p_1, \ldots, p_r) \in M^r \,|\, p_j \neq p_k \text{ for } 1 \leq j < k \leq r\}$, where M^r is the r-fold product of M and let ${}_rJ^l(M, \mathbb{R}^3)$ denote the multi jet space from M to $\mathbb{R}^3$ of multiplicity r and order l. For a map $f : M \rightarrow \mathbb{R}^3$, denote by ${}_rj^lf : M^{(r)} \rightarrow {}_rJ^l(M, \mathbb{R}^3)$ the multi l-extension of f. Denote by ${}_rJ^l_{emb}(M, \mathbb{R}^3)$ the set of all multi jets of $i \in Emb(M, \mathbb{R}^3)$. Note that ${}_rJ^l_{emb}(M, \mathbb{R}^3)$ is open in ${}_rJ^l(M, \mathbb{R}^3)$. We suppose that l is a positive integer with $l \geqq 3$.

Consider the map ${}_r\Phi : {}_rJ^l_{emb}(M, \mathbb{R}^3) \rightarrow {}_rJ^l_{emb}(M, \mathbb{R}^3)$ defined by

$$ {}_r\Phi(j^li(p_1), \ldots, j^li(p_r)) = (j^lT_i(p_1), \ldots, j^lT_i(p_r)), $$

where $T_i = (\mu_i, \pi \circ i) \in Emb(M, \mathbb{R}^3)$ defined in section 1. We can easily see that the map ${}_r\Phi$ is a diffeomorphism.

The following is our transversality lemma. Let S be a smooth submanifold of ${}_rJ^l_{emb}(M, \mathbb{R}^3)$. Then by the multi jet transversality theorem we can easily have

Lemma 2.1. *If* $codim S > 2r$

$$ \mathcal{O}_S := \{i \in Emb(M, \mathbb{R}^3) \,|\, {}_rj^lT_i(M^{(r)}) \cap S = \phi\} $$

is residual in $Emb(M, \mathbb{R}^3)$.

Remark 2.2. Any algebraic set $A \subset \mathbb{R}^N$ can be decomposed into a finite disjoint union of analytic manifolds with dimensions less than or equal to $dim A$ (see [3]). Hence our transversality lemma is valid even if $\mathcal{S}$ is an algebraic set in ${}_rJ^l(\mathbb{R}^2, \mathbb{R}^3)$.

According to Dufour[10], we now define subvariety of $J^l(2,3)$ as follows.

$\Sigma_a = \{j^3(\mu, g)(0) \,|\, g$ is a fold map at 0 and $\mu|_{S_g}$ is degenerate at $0\}$,

$\Sigma_b = \{j^3(\mu, g)(0) \,|\, g$ is a fold map at 0 and (μ, g) is Whitney's umbrella such that the line of double points of which is non $-$ transversal at 0 to the direction $\{0\} \times \mathbb{R}^2$ in $\mathbb{R}^3\}$,

$\Sigma_c = \{j^3(\mu, g)(0) \,|\, g$ is a cusp at 0 and (μ, g) is singular at $0\}$,

$\Sigma_d = \{j^3(\mu, g)(0) \,|\, \mu$ is singular and degenerate at $0\}$,

where $\mu : \mathbb{R}^2, 0 \to \mathbb{R}, 0, g : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$.

For Σ_d, by its definition, Σ_d is a subvariety of codimension 3 in $J^l(2,3)$.

Lemma 2.3(Dufour[10]). *$\Sigma_a, \Sigma_b, \Sigma_c$ are subvarieties of codimension 3 in $J^l(2,3)$.*

2.2. Proof of Theorem A in the case $r = 1$.

In the case $r = 1$, as was shown in [10], by Lemma 2.1 and Lemma 2.3 we obtain Theorem A.

2.3. Proof of Theorem A in the case $r = 2$.

Denote by ${}_r\Delta$ the diagonal set in r-fold product of $\mathbb{R}^2$. Then ${}_r\Delta$ is codimension $2r - 2$ submanifold of $\mathbb{R}^2 \times \cdots \times \mathbb{R}^2$ (r-times). Set

$$\mathcal{S}_* = \mathbb{R}^{2^{(2)}} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \times \Sigma_* \times J^l(2,3),$$

which is algebraic sets in ${}_2J^l(\mathbb{R}^2, \mathbb{R}^3)$, where $* = a, b, c, d$ and Σ_* is as above. By Lemma 2.3, $codim \mathcal{S}_* > 4$. Hence, by Lemma 2.1 a residual subset $\mathcal{O}_0 = \mathcal{O}_{\mathcal{S}_a} \cup \mathcal{O}_{\mathcal{S}_b} \cup \mathcal{O}_{\mathcal{S}_c} \cup \mathcal{O}_{\mathcal{S}_d}$ in $Emb(M, \mathbb{R}^3)$ has the following property: for any $i \in \mathcal{O}_0$, (μ_1, g_1) is one of the types $(I), \ldots, (VI)$, (μ_2, g_2) is of type (I).

Moreover we need the following to see relations between type $(I), \ldots, (VI)$ and type (I).

Proposition 2.4. *There exist residual subsets $\mathcal{O}_{(\nu, I)}$ in $Emb(M, \mathbb{R}^3)$ such that for any $i \in \mathcal{O}_{(\nu,I)}$, ${}_2T_i$ is of type (ν, I), where $\nu = I, \ldots, VI$ and of type (I, I) means* *that of type* $(I,I)_0,(I,I)_1,(I,I)_2$ *and similarly of type* (III,I) *means that of type* $(III,I)^0,(III,I)^1$.

Proof. Define $\Sigma_{fold}, \Sigma_{cusp} \subset J^l(2,2)$ by

$\Sigma_{fold} = \{j^l g(0) \mid g \text{ is a fold map at } 0\}$,

$\Sigma_{cusp} = \{j^l g(0) \mid g \text{ is a cusp map at } 0\}$.

It is a well-known fact that Σ_{fold} is a codimension 1 subvariety and Σ_{cusp} is a codimension 2 subvariety in $J^l(2,2)$.

Proof of the case $\nu = III, IV, V$. Let $g : M \to \mathbb{R}^2$ and let $p_1, p_2 \in M$ with $g(p_1) = g(p_2)$. Suppose that p_1 is a fold point of g. We take a local coordinate (x_1, y_1) of M around p_1 and (u,v) of the target space of g such that $g(x_1,y_1) = (x_1, y_1^2)$. Then, to describe the contact of the discriminant set $g_1(S_{g_1})$ and the curve $g_2(\mu_2^{-1}(0))$, we define a function germ $K : (\mathbb{R},0) \to (\mathbb{R},0)$ by

$$K(u) = \mu_2 \circ g_2^{-1}(u,0).$$

By direct calculations, we see that $\{\frac{dK}{du}(0)=0\}, \{\frac{d^2K}{du^2}(0)=0\}$ are algebraic sets in $J^l(2,3)$.

Define algebraic sets $S_{(\nu,I)}$ in ${}_2J^l(\mathbb{R}^2,\mathbb{R}^3)$ by

$$S_{(III,I)} = \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \times J^l(2,1) \times \Sigma_{fold} \times \{\frac{dK}{du}(0) = \frac{d^2K}{du^2}(0) = 0\},$$

$$S_{(IV,I)} = \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \times \{\frac{\partial \mu}{\partial x_1}(0) = 0\} \cap \{\frac{\partial^2 \mu}{\partial {x_1}^2}(0) \neq 0\} \times \Sigma_{fold} \times \{\frac{dK}{du}(0) = 0\},$$

$$\begin{aligned} S_{(V,I)} = \mathbb{R}^{2(2)} &\times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \\ &\times \{(\mu, g) \text{ is Whitney's umbrella such that} \\ &\text{the line of double points and } \{0\} \times \mathbb{R}^2 \text{ are transversal at } 0\} \\ &\times \Sigma_{fold} \times \{\frac{dK}{du}(0) = 0\}. \end{aligned}$$

Since $codim S_{(\nu,I)} = 5$, by Lemma 2.1 ${}_2T_i$ is of type (ν, I) for any element i of the residual set $\mathcal{O}_{S_{(\nu,I)}}$. □

Proof of the case $\nu = VI$. Suppose that g_1 is a cusp map. We take a coordinate (x_1, y_1) such that $g_1 = (x_1, {y_1}^3 + x_1 y_1)$. Then the function defined as above $K(u) =$

$\mu_2 \circ g_2^{-1}(u,0)$ describe the contact of the tangent cone of the discriminant set $g_1(S_{g_1})$ and the curve $g_2(\mu_2{}^{-1}(0))$. Define an algebraic set in ${}_2J^l(\mathbb{R}^2,\mathbb{R}^3)$ by

$$\mathcal{S}_{(VI,I)} = \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \times J^l(2,1) \times \Sigma_{cusp} \times \{\frac{dK}{du}(0) = 0\}.$$

Since $codim\mathcal{S}_{(VI,I)} = 5$, by Lemma 2.1 the residual set $\mathcal{O}_{\mathcal{S}_{(VI,I)}}$ has the required properties. $\square$

Proof of the case $\nu = I$. Suppose that $(\mu_1, g_1), (\mu_2, g_2)$ are both of type (I). Let (u,v) be a coordinate in the target space of g_1 and g_2. Since $\mu_1 \circ g_1{}^{-1}$ is a submersion, we may suppose that $\frac{\partial(\mu_1 \circ g_1{}^{-1})}{\partial u}(0) \neq 0$. So that consider the coordinate transformation $(u,v) \mapsto (u',v') = (\mu_1 \circ g_1{}^{-1}, v)$. Then in the new coordinate (u',v'), the map germ $(\mu_1 \circ g_1{}^{-1}, \mu_2 \circ g_2{}^{-1}) : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ becomes the following form:

$$(u', G(u',v')) \text{ where } G(u',v') = \mu_2 \circ g_2{}^{-1}(u(u',v'),v').$$

Define algebraic sets V_1, V_2 in $J^l(2,3) \times J^l(2,3)$ by

$$V_1 = \{\frac{\partial G}{\partial v'}(0) = \frac{\partial^2 G}{\partial v'^2}(0) = \frac{\partial^3 G}{\partial v'^3}(0) = 0\},$$

$$V_2 = \{\frac{\partial G}{\partial v'}(0) = \frac{\partial^2 G}{\partial v'^2}(0) = \frac{\partial^2 G}{\partial u' \partial v'}(0) = 0\}.$$

By direct caluculations, we see that the codimensions of V_1, V_2 in $J^l(2,3) \times J^l(2,3)$ are both three. As is well-known $(j^l\mu_1(0), j^l g_1(0), j^l\mu_2(0), j^l g_2(0)) \in J^l(2,3) \times J^l(2,3) - (V_1 \cup V_2)$ if and only if the map germ $(\mu_1 \circ g_1{}^{-1}, \mu_2 \circ g_2{}^{-1})$ is regular or a fold or a cusp.

We now define an algebraic set $\mathcal{S}_{(I,I)}$ by

$$\mathcal{S}_{(I,I)} = \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \times V_1 \cup V_2.$$

Since $codim\mathcal{S}_{(I,I)} = 5$, by Lemma 2.1 the residual set $\mathcal{O}_{\mathcal{S}_{(I,I)}}$ has the required properties. $\square$

Proof of the case $\nu = II$. Let (μ_1, g_1) is of type (I), and let (μ_2, g_2) is of type (II). By the same argument as in the case (I, I), we see that $(\mu_1 \circ g_1{}^{-1}, \mu_2 \circ g_2{}^{-1}) =$

$(u', G(u', v'))$ for some coordinate (u', v'). Note that by the direct caluculation we have

$$\{\frac{\partial \mu}{\partial x}(0) = \frac{\partial \mu}{\partial y}(0) = 0\} \subset \{\frac{\partial G}{\partial v'}(0) = 0\}.$$

Define an algebraic set $\mathcal{S}_{(II,I)}$ in ${}_2J^l(\mathbb{R}^2, \mathbb{R}^3)$ by

$$\begin{aligned}\mathcal{S}_{(II,I)} = \mathbb{R}^{2(2)} &\times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \\ &\times \{\frac{\partial \mu}{\partial x}(0) = \frac{\partial \mu}{\partial y}(0) = 0\} \cap \{\frac{\partial G}{\partial v'}(0) = 0\} \cap \{\frac{\partial^2 G}{\partial v'^2}(0) \neq 0\} \\ &\times \{\frac{\partial^2 \mu}{\partial x^2}(0)\frac{\partial^2 \mu}{\partial y^2}(0) - 2\frac{\partial^2 \mu}{\partial x \partial y}(0) = 0\}.\end{aligned}$$

Since $codim \mathcal{S}_{(II,I)} = 5$, by Lemma 2.1 the residual set $\mathcal{O}_{\mathcal{S}_{(II,I)}}$ has the required properties. □

Thus this completes the proof of Proposition 2.4. □

Next we shall prove the genericity of type (III, III). The following lemma is obtained by a theorem of Whitney and Lemma 2.1.

Lemma 2.5. *There exists a residual set $\mathcal{O}_1$ in $Emb(M, \mathbb{R}^3)$ such that for any $i \in \mathcal{O}_1$, the topographic multigerm ${}_2T_i$ is the following type: g_1 and g_2 are both fold map germs such that the discriminant sets of g_1, g_2 are transversal at 0 each other.*

Proposition 2.6. *There exists a residual set $\mathcal{O}_{(III,III)}$ in $Emb(M, \mathbb{R}^3)$ such that for any $i \in \mathcal{O}_{(III,III)}$, ${}_2T_i$ is of type (III, III).*

Proof. Suppose that g_1 is a fold map germ. We can take a coordinate (x, y) such that $g = (x, y^2)$. In this coordinate, $\mu|_{S_g}$ is singular at 0, which means that $\frac{\partial \mu}{\partial x}(0) = 0$ and $(\mu, g) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is singular at 0, which means that $\frac{\partial \mu}{\partial y}(0) = 0$. So that we shall define algebraic sets $\mathcal{S}_1, \mathcal{S}_2$ in ${}_2J^l(\mathbb{R}^2, \mathbb{R}^3)$ as follows.

$$\mathcal{S}_1 := \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \times \{\frac{\partial \mu}{\partial x}(0) = 0\} \times \Sigma_{fold} \times J^l(2, 1) \times \Sigma_{fold},$$

$$\mathcal{S}_2 := \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \times \{\frac{\partial \mu}{\partial y}(0) = 0\} \times \Sigma_{fold} \times J^l(2, 1) \times \Sigma_{fold}.$$

Since the codimensions of $\mathcal{S}_1, \mathcal{S}_2$ are both five, by Lemma 2.1 $\mathcal{O}_{(III,III)} := \mathcal{O}_{\mathcal{S}_1} \cap \mathcal{O}_{\mathcal{S}_2} \cap \mathcal{O}_1$ has required properties. □

Proof of Theorem A in the case $r = 2$. Set $\mathcal{O} = \cup_{\nu=I}^{VI}(\mathcal{O}_{(\nu,I)} \cap \mathcal{O}_0) \cup \mathcal{O}_{(III,III)}$. Then by Proposition 2.4 and Proposition 2.6, $\mathcal{O}$ has required properties in Theorem A. This completes the proof in the case $r = 2$. □

2.4. Proof of Theorem A in the case $r = 3$.

We shall give the proof which is likewise in the case $r = 2$ by using Lemma 2.1. So that we admit only $codim S \cap (\{0\} \times (J^l(2,3))^3) \leqq 2$ in $(J^l(2,3))^3$, where $\mathcal{S} \subset \mathbb{R}^{2(3)} \times \mathbb{R}^3 \times {}_3\Delta \times (J^l(2,3))^3$, because $codim_3\Delta = 4$ in ${}_3J^l(\mathbb{R}^2, \mathbb{R}^3)$.

Proposition 2.7. *There exist residual sets $\mathcal{O}_n$ in $Emb(M, \mathbb{R}^3)$ such that for any $i \in \mathcal{O}_n$, the topographic multigerm ${}_3T_i$ is one of the following types:*

(1) (I, I, I)$_{0,0}$, (I, I, I)$_{1,0}$, (I, I, I)$_{1,1}$, (I, I, I)$_{2,0}$ *if $n = 1$;*

(2) (III, I, I)$_0^{0,0}$, (III, I, I)$_0^{1,0}$, (III, I, I)$_1^{0,0}$ *if $n = 2$;*

(3) (III, III, I)^{0,0} *if $n = 3$.*

Proof of the case $n = 1$. Let (μ_k, g_k) be of type (I) and let $F_k = \mu_k \circ g_k{}^{-1}$. Suppose that the map germ $(F_j, F_k) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is regular or a fold or a cusp, where $1 \leqq j < k \leqq 3$. We may take a coordinate (u, v) of the source space of (F_j, F_k) such that $\frac{\partial F_j}{\partial u}(0) \neq 0$. Hence it follows that $(F_j, F_k) = (u', G_{j,k}(u', v'))$ for some coordinate (u', v'), where $G_{j,k}(u', v') = F_k(u(u', v'), v')$. Then define an algebraic set $W_{j,k}$ in $(J^l(2,3))^3$ by

$$W_{j,k} = \{\frac{\partial G_{j,k}}{\partial v'}(0) = 0\} \cap \{\frac{\partial^2 G_{j,k}}{\partial v'^2}(0) \neq 0\}.$$

By the direct caluculation, we see that

$$W_{1,2} \cap W_{1,3} \subset \{\frac{\partial G_{2,3}}{\partial v'}(0) = 0\}.$$

Hence $codim W_{1,2} \cap W_{1,3} \cap W_{2,3} = 2$. Therefore we see that combinations of types (F_j, F_k) are voided of the following two cases in generic: for any two (F_j, F_k), both types are cusp; one is a cusp and the other is a fold. This implies the assertion.

Proof of the case $n = 2$. We use the same notations in the proof of Proposition 2.3 and also in the proof of genericity of type (I, I). Suppose that g_1 is a fold map germ. Let us take coordinates $(x_1, y_1), (u, v)$ such that $g_1(x_1, y_1) = (x_1, y_1^2)$. Then define an algebraic set $\mathcal{S}_{(III,I,I)}$ in ${}_3J^l(\mathbb{R}^2, \mathbb{R}^3)$ by

$$\begin{aligned} \mathcal{S}_{(III,I,I)} &= \mathbb{R}^{2(3)} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times {}_3\Delta \\ &\quad \times J^l(2,1) \times \Sigma_{fold} \times \{\frac{dK}{du}(0) = 0\} \cap \frac{d^2K}{du^2}(0) \neq 0\} \\ &\quad \cap \{\frac{\partial G}{\partial v'}(0) = 0\} \cap \{\frac{\partial^2 G}{\partial v'^2}(0) \neq 0\}. \end{aligned}$$

Since $codimS_{(III,I,I)} = 7$, by Lemma 2.1 it follows the assertion.

Proof of the case $n = 3$. It is clear that the condition for type (III, III, I) is generic. This implies the assertion. $\square$

Clearly the other types in the case $r = 3$ are also generic. Thus these fact implies the Theorem A in the case $r = 3$.

3. Normal forms

3.1. Proof of Theorem B in the case $r = 1$.

For the case $r = 1$, the normal forms have been obtained by Dufour [10] except type (II). For type (II), by Morse's lemma we immediately obtain the normal form.

3.2. Proof of Theorem B in the case $r = 2$.

In the case $r = 2$, we shall detect normal forms as follows.

Case of type $(III, I)^0, (III, I)^1$. We can suppose that

$$\mu_1 = x_1 + y_1,\ g_1 = (x_1, {y_1}^2);\ (\mu_2, g_2) \text{ is of type (I)}.$$

By the coordinate change g_2^{-1} in the source space of g_2 and μ_2, $(\mu_1, g_1; \mu_2, g_2)$ is equivalent to $(\mu_1, g_1; \mu_2', g_2')$, where

$$\mu_2'(x_2', y_2') = \mu_2 \circ g_2^{-1}(x_2', y_2'),\ g_2'(x_2', y_2') = (x_2', y_2').$$

Then in the case $(III, I)^0$, the generic condition about transversality means that $\frac{\partial \mu_2'}{\partial x_2'}(0) \neq 0$. By the coordinate change $(\frac{\partial \mu_2'}{\partial x_2'}(0))^{-1}$ which is a scalar multiple in the target space of μ_2', $(\mu_1, g_1; \mu_2', g_2')$ is equivalent to $(\mu_1, g_1; \mu_2'', g_2')$, where

$$\mu_2'(x_2'', y_2') = x_2' + \theta(x_2', y_2'),\ \theta \in \mathcal{M}_{x_2', y_2'} \text{ with } \frac{\partial \theta}{\partial x_2'}(0) = 0.$$

On the other hand, in the case $(III, I)^1$, the non-transversality condition means that $\frac{\partial \mu_2'}{\partial x_2'}(0) = 0$. Since $\mu_2' = \mu_2 \circ g_2^{-1}$ is a submersion, we have $\frac{\partial \mu_2'}{\partial y_2'}(0) \neq 0$. Moreover, from the generic condition of the second order contact, we have $\frac{\partial^2 \mu_2'}{\partial {x_2'}^2}(0) \neq 0$. Hence, by the coordinate change $(\frac{\partial \mu_2'}{\partial y_2'}(0))^{-1}$ which is a scalar multiple in the target space of μ_2', $(\mu_1, g_1; \mu_2', g_2')$ is equivalent to $(\mu_1, g_1; \mu_2'', g_2')$, where

$$\mu_2''(x_2', y_2') = y_2' + \theta(x_2', y_2'),$$

$$\theta \in \mathcal{M}_{x_2', y_2'} \text{ with } \frac{\partial \theta}{\partial x_2'}(0) = \frac{\partial \theta}{\partial y_2'}(0) = 0 \text{ and } \frac{\partial^2 \theta}{\partial {x_2'}^2}(0) \neq 0.$$

Thus we obtained normal forms of type $(III, I)^0, (III, I)^1$.

Case of type $(IV, I), (V, I), (VI, I)$. By the same argument in the case of type $(III, I)^0$, we obtain the normal forms in Theorem B.

Case of type (III, III). Let $g = (x, y^2) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a fold map germ. At first we shall determine coordinate changes which preserve g in the source space and the target space of g.

Since g is even with respect to y-variable, we can easily have

Lemma 3.1. *Let $A, B \in \mathcal{E}_{u,v}$. If $A \circ g + yB \circ g = 0$ in $\mathcal{E}_{x,y}$, then $A \circ g = B \circ g = 0$ in $\mathcal{E}_{x,y}$. That is, $\mathcal{E}_{x,y}$ is a free $\mathcal{E}_{u,v}$-module via g^*.*

Proposition 3.2. *Let $H : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, $K : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be diffeomorphism germs. Then $g \circ H = K \circ g$ if and only if H, K have the following forms:*

$$H = (\alpha \circ g, y\beta \circ g), \; K = (\alpha, v\beta^2),$$

where $\alpha, \beta \in \mathcal{E}_{u,v}$ with $\alpha(0) = 0$, $\frac{\partial \alpha}{\partial u}(0) \neq 0$, $\beta(0) \neq 0$.

Proof. The sufficiency is clear, by direct caluculations. We shall prove the necessity. Suppose that $g \circ H = K \circ g$. Set $H = (h_1, h_2), K = (k_1, k_2)$. Then by the assumption, $h_1 = k_1 \circ g$ and ${h_2}^2 = k_2 \circ g$. On the other hand, using the Malgrange preparation theorem we have $h_2(x, y) = A \circ g(x, y) + yB \circ g(x, y)$ for some $A, B \in \mathcal{E}_{u,v}$. Thus $(A^2 + vB^2 - k_2) \circ g + y(2AB) \circ g = 0$. By Lemma 3.1, we have $AB \circ g = 0$. Note that $B(0) \neq 0$ and $\frac{\partial k_1}{\partial u}(0) \neq 0$, because $H = (k_1 \circ g, A \circ g + yB \circ g)$ is a diffeomorphism germ. Thus $B \circ g$ does not vanish on $(\mathbb{R}^2, 0)$. Hence $A \circ g = 0$, and $h_2 = yB \circ g$. Consequently a form of H is determined to

$$H = (k_1 \circ g, yB \circ g).$$

Then by the form of H and the assumption $g \circ H = K \circ g$, the form of K is

$$K(u, v) = (k_1(u, v), vB(u, v)^2).$$

These forms of H, K are the required forms. $\square$

Now we detect the normal form of g_1 and g_2 of type (III, III) and next we shall detect the normal form of type (III, III) by using coordinate changes which preserve the normal form of g_1 and g_2.

Lemma 3.3. *Let* $(\mathbb{R}^2,0) \xrightarrow{g_1} (\mathbb{R}^2,0) \xleftarrow{g_2} (\mathbb{R}^2,0)$ *be a bi-germ such that* g_1, g_2 *are both fold map germs and that the discriminant sets of* g_1, g_2 *are transversal at* 0 *each other. Then we can express the bi-germ for some coordinates as follows:*

$$g_1(x_1,y_1) = (x_1, {y_1}^2),\ g_2(x_2,y_2) = ({x_2}^2, y_2).$$

Proof. We can choose coordinates (x_1,y_1) in the source space of g_1, (u,v) in the target space of g_1 and g_2 such that $g_1(x_1,y_1) = (x_1,{y_1}^2)$. Let (x_2,y_2) be a coordinate in the source space of g_2. By direct caluculations, the transversality condition for the discriminant sets of g_1, g_2 means that $\frac{\partial(v\circ g_2)}{\partial x_2}(0) \neq 0$ or $\frac{\partial(v\circ g_2)}{\partial y_2}(0) \neq 0$. If $\frac{\partial(v\circ g_2)}{\partial x_2}(0) \neq 0$ (resp. $\frac{\partial(v\circ g_2)}{\partial y_2}(0) \neq 0$), then by using the coordinate transformation in the source space of g_2

$$(x_2,y_2) \mapsto (y_2, v\circ g_2)\ (\text{resp.}\ (x_2,y_2) \mapsto (x_2, v\circ g_2)),$$

we can suppose that

$$g_2(x_2,y_2) = (u\circ g_2, y_2).$$

Since g_2 is a fold map germ, from the Malgrange preparation theorem, $\mathcal{E}_{x_2,y_2}$ is generated by $1, x_2$ as $\mathcal{E}_{u,v}$-module via ${g_2}^*$. Thus there exist $A, B \in \mathcal{E}_{u,v}$ such that

$$(*)\qquad \begin{aligned} {x_2}^2 &= A\circ g_2 + 2(B\circ g_2)x_2 \\ &= A(u\circ g_2, y_2) + 2B(u\circ g_2, y_2)x_2. \end{aligned}$$

By simple caluculations, we see that $A(0) = B(0) = 0$, $\frac{\partial A}{\partial u}(0) \neq 0$. Now we can define coordinate transformations ϕ_1, ψ, ϕ_2 by

$$\phi_1(x_1,y_1) = (A\circ g_1(x_1,y_1) + B^2\circ g_1(x_1,y_1), y_1),$$

$$\psi(u,v) = (A(u,v) + B(u,v)^2, v),$$

$$\phi_2(x_2,y_2) = (x_2 - B(u\circ g_2(x_2,y_2), y_2), y_2).$$

By the definitions of ψ, ϕ_1, it is clear that

$$\psi \circ g_1 = g_1 \circ \phi_1.$$

Set $\phi_2(x_2, y_2) = (x_2', y_2'), \psi(u, v) = (u', v')$. Then by using (*) we see that

$$\begin{aligned}(x_2')^2 &= (x_2 - B(u \circ g_2, y_2))^2 \\ &= A(u \circ g_2, y_2) + B(u \circ g_2, y_2)^2 \\ &= u' \circ g_2.\end{aligned}$$

Therefore, in these new coordinates preserving g_1, we have

$$(u' \circ g_2)(x_2', y_2') = (x_2')^2, \ (v' \circ g_2)(x_2', y_2') = y_2'.$$

This completes the proof of Lemma 3.3 $\square$

Proposition 3.4. *Let* $(\mathbb{R}^2, 0) \xrightarrow{g_1} (\mathbb{R}^2, 0) \xleftarrow{g_2} (\mathbb{R}^2, 0)$ *be a bi-germ such that* $g_1(x_1, y_1) = (x_1, {y_1}^2)$, $g_2(x_2, y_2) = ({x_2}^2, y_2)$. *Let* $H_1, H_2, K : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *be diffeomorphism germs. Then* $g_1 \circ H_1 = K \circ g_1$ *and* $g_2 \circ H_2 = K \circ g_2$ *if and only if* H_1, H_2, K *have the following forms:*

$$H_1(x_1, y_1) = (x_1 \alpha^2 \circ g_1, y_1 \beta \circ g_1),$$

$$H_2(x_2, y_2) = (x_2 \alpha \circ g_2, y_2 \beta^2 \circ g_2),$$

$$K(u, v) = (u\alpha^2, v\beta^2),$$

where $\alpha, \beta \in \mathcal{E}_{u,v}$ *with* $\alpha(0) \neq 0, \beta(0) \neq 0$.

Proof. By the same argument as in Proposition 3.2, the following fact holds: $g_2 \circ H_2 = K \circ g_2$ if and only if

$$H_2(x_2, y_2) = (x_2 \alpha_2 \circ g_2, \beta_2 \circ g_2), \ K(u, v) = (u{\alpha_2}^2, \beta_2),$$

where $\alpha_2, \beta_2 \in \mathcal{E}_{u,v}$ with $\alpha_2(0) \neq 0, \beta_2(0) = 0, \frac{\partial \beta_2}{\partial v}(0) \neq 0$.

Combining this fact and Proposition 3.2, we can deduce Proposition 3.4. $\square$

We are now ready to detect a normal form of type (III, III).

Let $(\mu_1, g_1; \mu_2, g_2)$ be of type (III, III). By Lemma 3.3, we can suppose that

$$g_1(x_1, y_1) = (x_1, {y_1}^2),$$

$$g_2(x_2, y_2) = ({x_2}^2, y_2).$$

Applying the Malgrange preparation theorem, μ_1, μ_2 have the following forms:

$$\mu_1 = A_1 \circ g_1 + y_1 B_1 \circ g_1,$$

$$\mu_2 = A_2 \circ g_2 + x_2 B_2 \circ g_2,$$

where $A_k, B_k \in \mathcal{E}_{u,v}\,(k = 1,2)$. Note that $B_k(0) \neq 0\,(k = 1,2)$ because the map germ (μ_k, g_k) is non-singular. Thus we can define coordinate transformations H_1, H_2, K by

$$H_1(x_1, y_1) = (x_1 {B_2}^2 \circ g_1, y_1 B_1 \circ g_1),$$

$$H_2(x_2, y_2) = (x_2 B_2 \circ g_2, y_2 {B_1}^2 \circ g_2),$$

$$K(u, v) = (u{B_2}^2, v{B_1}^2).$$

In the new coordinates by H_1, H_2, K, it follows from Proposition 3.4 that $(\mu_1, g_1; \mu_2, g_2)$ is equivalent to $(\mu_1', g_1; \mu_2', g_2)$, where

(**)
$$\mu_1'(x_1', y_1') = \alpha_1 \circ g_1(x_1', y_1') + y_1',$$

$$\mu_2'(x_2', y_2') = \alpha_2 \circ g_2(x_2', y_2') + x_2',$$

where $\alpha_k = A_k \circ K^{-1} \in \mathcal{M}_{u,v}\,(k = 1,2)$.

Since $\mu_1'|_{S_{g_1}}$ is non-singular, we have $\frac{\partial \alpha_1}{\partial u}(0) \neq 0$. Similarly $\frac{\partial \alpha_2}{\partial v}(0) \neq 0$. Conversely, for any $\alpha_1, \alpha_2 \in \mathcal{M}_{u,v}$ with $\frac{\partial \alpha_1}{\partial u}(0) \neq 0$, $\frac{\partial \alpha_2}{\partial v}(0) \neq 0$, a germ whose form is given by (**) is of type (III, III). Therefore we obtain the normal form of type (III, III).

Case of (I, I)$_0$, (I, I)$_1$, (I, I)$_2$. The normal forms of this case have been detected essentially by Dufour [7, 8, 9], however, we shall give a process of detecting the normal forms.

Let $(\mu_1, g_1; \mu_2, g_2)$ be a topographic multigerm. If g_k $(k = 1, 2)$ is a diffeomorphism, then by the coordinate transformation ${g_k}^{-1}$ in the source space of μ_k

and g_k, $(\mu_1, g_1; \mu_2, g_2)$ is equivalent to $(F, g_1'; G, g_2')$, where $F = \mu_1 \circ g_1{}^{-1}$, $G = \mu_2 \circ g_2{}^{-1}$, $g_k' = id_{\mathbb{R}^2}$ $(k = 1, 2)$. From now on, we only consider the coordinate transformations preserving both $g_1' = id$ and $g_2' = id$. Hence it is sufficiant to detect normal forms of the following divergent diagrams up to coordinate transformations:

$$(\mathbb{R}, 0) \xleftarrow{F} (\mathbb{R}^2, 0) \xrightarrow{G} (\mathbb{R}, 0).$$

Let (x, y) be a coordinate of the source space of F, G. Then we detect the norml form of each type by using the deformation theory of functions ([2]) as follows.

In the case of type (I, I)$_0$: From the generic condition, $(F, G) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is a diffeomorphism germ. Hence, clearly we obtain the normal form $F = x, G = y$.

In the case of type (I, I)$_1$: From the generic condition, G is a submersion. Thus we may suppose that $G(x, y) = y$. Then the generic condition such that the map germ $(F, G) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is a fold and besides F is a submersion means that

$$\frac{\partial F}{\partial x}(0) = 0\,,\ \frac{\partial^2 F}{\partial x^2}(0) \neq 0\,;\ \frac{\partial F}{\partial y}(0) \neq 0.$$

Namely $F(x, 0)$ has A_1-singularity and F is a deformation of $F(x, 0)$. So that let $\psi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be the diffeomorphism germ such that $F(\psi(x), 0) = \pm x^2$. Then trivially the diagram (F, y) is equivalent to (F', y') , where $F'(x', y') = \pm F(\psi(x'), y')$. We can easily verify that F' is an R-infinitessimaly versal (hence R-versal) deformation of x'^2. On the other hand, $x^2 + y$ is also an R-versal deformation of x^2. Therefore by the uniqueness of the R-versal deformation, the following holds: there exist a diffeomorphism germ $\lambda \in \mathcal{M}_y$ and $h \in \mathcal{M}_{x,y}$ with $h(x, 0) = x$ such that the two diagrams $(F', y'), (x^2 + y, y)$ are equivalent transforming the coordinates in the target space of y' by λ and in the source space of both F' and y' by $(x, y) \mapsto (h(x, y), \lambda(y))$. Thus we obtain the normal form of type (I, I)$_1$.

In the case of type (I, I)$_2$. We may suppose that $G(x, y) = y$. The generic condition on this type means that

$$\frac{\partial F}{\partial x}(0) = \frac{\partial^2 F}{\partial x^2}(0) = 0\,,\ \frac{\partial^3 F}{\partial x^3}(0) \neq 0\,,\ \frac{\partial^2 F}{\partial x \partial y}(0) \neq 0\,;\ \frac{\partial F}{\partial y}(0) \neq 0.$$

Hence $F(x, 0)$ has A_2-singularity. So that let $\psi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be the diffeomorphism germ such that $F(\psi(x), 0) = \pm x^3$. Then trivially the diagram (F, y)

is equivalent to (F', y') , where $F'(x', y') = \pm F(\psi(x'), y')$. Due to the condition $\frac{\partial^2 F}{\partial x \partial y}(0) \neq 0$, we can easily verify that F' is a R^+-infinitessimaly versal (hence R^+-versal) deformation of x'^3. On the other hand, $x^3 + xy$ is also an R^+-versal deformation of x^3. Therefore by the uniqueness of versal deformation, the diagram (F', y') is equivalent to

$$(x^3 + xy + \alpha(y), y)$$

for some $\alpha \in \mathcal{M}_y$. Here we need the following lemma which was shown by Dufour in [7].

Lemma 3.5. *For any $\alpha \in \mathcal{M}_y$, the diagram $(x^3 + xy + \alpha(y), y)$ is equivalent to $(x^3 + xy, y)$.*

That is, we obtain the normal form.

In the case of type (II, I): We may suppose that $G(x, y) = y$. By the generic condition, since the map germ $(F, G) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is a fold and F is a Morse germ, we have

$$\frac{\partial F}{\partial x}(0) = 0\,, \ \frac{\partial^2 F}{\partial x^2}(0) \neq 0\,; \ \frac{\partial F}{\partial y}(0) = 0.$$

Thus F is an R^+-infinitesimally versal (hence R^+-versal) deformation of $F(x, 0)$ which has A_1-singularity. So that we see the diagram (F, y) is equivalent to (F', y'), where F' is an R^+-versal deformation of x'^2. On the other hand the trivial deformation of x^2 is also R^+-versal. Therefore the diagram (F, y) is equivalent to $(x^2 + \alpha(y), y)$, where $\alpha \in \mathcal{M}_y$. Moreover by the generic condition, $x^2 + \alpha(y)$ must be a Morse germ. That is

$$\frac{\partial \alpha}{\partial y}(0) = 0\,, \ \frac{\partial^2 \alpha}{\partial y^2}(0) \neq 0.$$

Hence $\alpha(y) = \pm \psi(y)^2$ for some coordinate transformation ψ in y-variable. Consequently by the coordinate transformation $(x, y) \mapsto (x, \psi(y))$ we obtain the normal form $(x^2 \pm y^2, y)$.

This completes the proof of Theorem B in the case $r = 2$.

3.3. Normal form in the case $r = 3$.

We shall detect the normal forms of types $(\mathrm{I}, \mathrm{I}, \mathrm{I})_{1,1}$, $(\mathrm{III}, \mathrm{I}, \mathrm{I})_1^{0,0}$, $(\mathrm{III}, \mathrm{III}, \mathrm{I})^{0,0}$ stated in Remark 1.3.

Case of type (I, I, I)$_{1,1}$. Let $(\mu_1, g_1; \mu_2, g_2; \mu_3, g_3)$ be of type $(\mathrm{I}, \mathrm{I}, \mathrm{I})_{1,1}$. Set $F_k = \mu_k \circ g_k{}^{-1}$ $(k = 1,2,3)$. From the generic condition such that the map germ (F_1, F_2) is a fold, we may suppose that $F_1 = y, F_2 = y + x^2$. Moreover since (F_1, F_3) is a fold, F_3 can be written

$$F_3 = y + ax^2 + \theta(x,y),$$

where $a \neq 0$, $\theta \in \mathcal{M}_{x,y}^2$ with $\frac{\partial^2\theta}{\partial x^2}(0) = 0$. Then to see the condition such that (F_3, F_2) is a fold, we consider the coordinate transformation $(x,y) \mapsto (x, F_2)$. In the new coordinate the map germ (F_3, F_2) has the following form: $(G(x,y), y)$, where $G(x,y) = y - x^2 + ax^2 + \theta(x,y)$. Thus the condition, (F_3, F_2) is a fold, means that $\frac{\partial G}{\partial x}(0) = 0$, $\frac{\partial^2 G}{\partial x^2}(0) \neq 0$. That is, $\frac{\partial \theta}{\partial x}(0) = 0$, $a \neq 1$ since $\frac{\partial^2\theta}{\partial x^2}(0) = 0$. Therefore by adding this condition to the above F_3, we obtain the normal form.

Case of type (III, I, I)$_1^{0,0}$. Let $(\mu_1, g_1; \mu_2, g_2; \mu_3, g_3)$ be of type $(\mathrm{III}, \mathrm{I}, \mathrm{I})_1^{0,0}$. We may suppose that (μ_1, g_1) is the normal form of type (III) and that $(\mu_2, g_2; \mu_3, g_3) = (F_2, id\ ;\ F_3, id)$, where $F_k = x_k + \theta_k(x_k, y_k)$ $(k = 1,2)$, $\theta_k \in \mathcal{M}_{x_k,y_k}$ with $\frac{\partial \theta_k}{\partial x_k}(0) = 0$. We remark that the generic condition such that (F_2, F_3) is a fold is equivalent to the condition such that $F_2{}^{-1}(0)$ and $F_3{}^{-1}(0)$ have second order contact at 0. Hence by the direct caluculation we get the condition θ_2, θ_3 so that we obtain the normal form stated in Remark 1.3.

Case of type (III, III, I)^{0,0}. Let $(\mu_1, g_1; \mu_2, g_2; \mu_3, g_3)$ be of type $(\mathrm{III}, \mathrm{III}, \mathrm{I})^{0,0}$. We may suppose that the form of $(\mu_1, g_1; \mu_2, g_2)$ is the normal form of type (III, III) and that $(\mu_3, g_3) = (F_3, id)$, where $F_3 = \mu_3 \circ g_3{}^{-1}$. Then by the generic condition, F_3 can be written $F_3 = ax_3 + by_3 + \theta(x,y)$, where $a \neq 0$, $b \neq 0$, $\theta \in \mathcal{M}_{x,y}^2$. Consider the following six coordinate transformations preserving g_1, g_2 and $g_3 = id$:

$$(x_1, y_1) \mapsto (ax_1, \sqrt{b}y_1) \text{ in the source space of } g_1,$$

$$(x_2, y_2) \mapsto (\sqrt{a}x_2, by_2) \text{ in the source space of } g_2,$$

$$(x_3, y_3) \mapsto (ax_3, by_3) \text{ in the source space of } g_3,$$

$$(u, v) \mapsto (au, bv) \text{ in the common target space of } g_1,\ g_2 \text{ and } g_3,$$

$\sqrt{b}$, $\sqrt{a}$ which are scalar multiples in the target spaces of μ_1, μ_2 respectively. Then in these new coordinates we obtain the normal form.

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