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A Scaling limit of a Hamiltonian of many nonrelativistic particles interacting with a quantized radiation field

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Abstract

This paper presents a scaling limit of Hamiltonians which describe interactions of N-nonrelativistic charged particles in a scalar potential and a quantized radiation field in the Coulomb gauge with the dipole approximation. The scaling limit defines effective potentials. In one-nonrelativistic particle case, the effective potentials have been known to be Gaussian transformations of the scalar potential [J.Math.Phys.34(1993)4478-4518]. However it is shown that the effective potentials in the case of N-nonrelativistic particles are not necessary to be Gaussian transformations of the scalar potential.

1 INTRODUCTION

The main problem in this paper is to consider a scaling limit of a model in quantum electrodynamics which describes an interaction of many nonrelativistic charged particles and a quantized radiation field in the Coulomb gauge with the dipole approximation. For our discussion we may limit ourselves to the case of a fixed number N of the particles, since N dose not change in time. The model we consider is called "the Pauli-Fierz model", which has been a subject of great interests and by which real physical phenomena of charged particles and a quantized radiation field such as "Lamb shift" can be interpreted. There has been a considerable amount of literature on the Pauli-Fierz model with one-nonrelativistic charged particle, e.g.,[1,2] from points of view of physics and [3,4,5,6,7,8] mathematical points of view. In particular, the authors of [5,6] have studied a scaling limit of the Pauli-Fierz model with one-nonrelativistic charged particle. We may well extend the scaling limit of one-particle system to N-particle system.

The authors of [5,6] defined Hamiltonians of the Pauli-Fierz model as self-adjoint operators H_ρ with an ultraviolet cut-off function ρ acting in the tensor product of the Hilbert space $L^2(\mathbb{R}^d)$ and a Boson Fock space $\mathcal{F}(\mathcal{W})$ over $\mathcal{W} = \oplus_{r=1}^{d-1} L^2(\mathbb{R}^d)$. Introducing scalings with respect to parameters c (the speed of light), m (the mass of the particle) and e (the charge of the particle), the authors have shown the existence of the strong resolvent limits of the scaled self-adjoint operators $H_\rho^{REN}(\kappa) + V \otimes I$ with an infinite self-energy of the nonrelativistic particle subtracted with a scalar potential V, (we call the limit "the scaling limit of $H_\rho + V \otimes I$ "): In [6] we have proved the following:

Let V and ρ satisfy some conditions and Δ be the Laplacian in $L^2(\mathbb{R}^d)$. Then $H_\rho^{REN}(\kappa) + V \otimes I$ is self-adjoint and bounded from below uniformly in sufficiently large $\kappa > 0$ with

$$s - \lim_{\kappa \to \infty} (H_\rho^{REN}(\kappa) + V \otimes I - z)^{-1} = \mathcal{S} \left\{ \left(-\frac{1}{2m_\infty} \Delta + V_{eff} - z \right)^{-1} \otimes P_0 \right\} \mathcal{S}^{-1},$$

where $z \in \mathbb{C} \setminus \mathbb{R}$, m_∞ is a positive constant, $\mathcal{S}$ a unitary operator on $L^2(\mathbb{R}^d) \otimes \mathcal{F}(\mathcal{W})$, P_0 a *projection on $\mathcal{F}(\mathcal{W})$ and V_{eff} a multiplication operator defined by*

$$V_{eff}(x) = (2\pi\alpha)^{-\frac{d}{2}} \int dy e^{-|x-y|^2/2\alpha} V(y),$$

where α is a positive constant. The multiplication operator V_{eff} is called "the effective potential".

One of the strongest methods to analyze the scaling limits in [5,6] was to find Bogoliubov transformations $\mathcal{U}$, which implements a unitary equivalence between the Pauli-Fierz Hamiltonians H_ρ and decoupled Hamiltonians of the form

$$\widetilde{H} = -\frac{1}{2\widetilde{m}}\Delta \otimes I + I \otimes H_b + constant,$$

where $\widetilde{m}$ is a positive constant and H_b is the free Hamiltonian of the quantized radiation field in $\mathcal{F}(\mathcal{W})$; the authors of [5,6] show equations of the following type:

$$\left(H_\rho^{REN} + V \otimes I - z\right)^{-1} = \mathcal{U}\left(\widetilde{H} + \mathcal{U}^{-1}(V \otimes I)\mathcal{U} - z\right)^{-1}\mathcal{U}^{-1}. \tag{1. 1}$$

In this paper, the Pauli-Fierz Hamiltonian $H_{\vec{\rho}}$ with N-nonrelativistic charged particles in the Coulomb gauge with the dipole approximation are defined as operators acting in the Hilbert space $\underbrace{L^2(\mathbb{R}^d) \otimes ... \otimes L^2(\mathbb{R}^d)}_{N} \otimes \mathcal{F}(\mathcal{W}) \cong L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}(\mathcal{W})$ by

$$\begin{aligned} H_{\vec{\rho}} &= \frac{1}{2m}\sum_{j=1}^{N}\sum_{\mu=1}^{d}\left(-i\hbar D_\mu^j \otimes I - eI \otimes A_\mu(\rho_j)\right)^2 + I \otimes H_b \\ &= -\frac{\hbar^2}{2m}\Delta \otimes I + I \otimes H_b + \frac{1}{2m}\sum_{j=1}^{N}\sum_{\mu=1}^{d}\left(2e\hbar i D_\mu^j \otimes A_\mu(\rho_j) + e^2 I \otimes A_\mu^2(\rho_j)\right), \end{aligned}$$

where D_μ^j is the differential operator with respect to the j-th variable in the μ-th direction, Δ the Laplacian in $L^2(\mathbb{R}^{dN})$, $\hbar$ the Planck constant divided 2π and $A_\mu(\rho_j)$ the quantized radiation field in the μ-th direction with an ultraviolet cut-off function ρ_j in the Coulomb gauge. Problems arising in the many particles system are as follows:

(i) Do there any Bogoliubov transformations such as (1.1) exist?

(ii) What kind of scalar potentials V and sets of ultraviolet cut-off functions $(\rho_1, ..., \rho_N)$ do a scaling limit of the Hamiltonian $H_{\vec{\rho}} + V \otimes I$ exist for ? Furthermore, what kind of infinite self-energy should be subtracted from the original Hamiltonian $H_{\vec{\rho}} + V \otimes I$?

(iii) If the scaling limit exists, what form does the effective potential have?

With this motivation, we continue here to analyze a scaling limit of the Pauli-Fierz model with N-nonrelativistic charged particles.

We introduce the same scaling as [6] as follows;

$$c(\kappa) = c\kappa, e(\kappa) = e\kappa^{-\frac{1}{2}}, m(\kappa) = m\kappa^{-2}. \tag{1. 2}$$

Introducing a pseudo differential operator $E^{REN}(D, \kappa)$ in $L^2(\mathbb{R}^{dN})$ with a symbol $E^{REN}(p, \kappa)$ such that $E^{REN}(p, \kappa) \to \infty$ as $\kappa \to \infty$, we define a Hamiltonian $H_{\vec{\rho}}^{REN}(\kappa)$ by

$$H_{\vec{\rho}}^{REN}(\kappa) = -E^{REN}(D, \kappa) \otimes I + \kappa I \otimes H_b + \frac{1}{2m} \sum_{j=1}^{N} \sum_{\mu=1}^{d} \left(\kappa 2e\hbar i D_\mu^j \otimes A_\mu(\rho_j) + e^2 I \otimes A_\mu^2(\rho_j) \right).$$

For sufficiently large $\kappa > 0$ and a scalar potential V with some conditions, we shall show that $H_{\vec{\rho}}^{REN}(\kappa) + V \otimes I$ is essentially self-adjoint on $D(-\Delta \otimes I) \cap D(I \otimes H_b)$ and bounded from below uniformly in sufficiently large $\kappa > 0$, and the existence of Bogoliubov transformations $\mathcal{U}(\kappa)$, which gives a unitary equivalence of $H_{\vec{\rho}}^{REN}(\kappa) + V \otimes I$ and a self-adjoint operator $\widetilde{H_{\vec{\rho}}}(\kappa) + C_\kappa(V)$ as follows;

$$(H_{\vec{\rho}}^{REN}(\kappa) + V \otimes I - z)^{-1} = \mathcal{U}(\kappa)(\widetilde{H_{\vec{\rho}}}(\kappa) + C_\kappa(V) - z)^{-1}\mathcal{U}^{-1}(\kappa),$$

where $\widetilde{H_{\vec{\rho}}}(\kappa) = \widetilde{E}(D, \kappa) \otimes I + \kappa I \otimes H_b$, $\widetilde{E}(D, \kappa)$ is a pseudo differential operator in $L^2(\mathbb{R}^{dN})$ and $C_\kappa(V) = \mathcal{U}^{-1}(\kappa)(V \otimes I)\mathcal{U}(\kappa)$ (Theorem 3.5). Then we see that $\mathcal{U}(\kappa) \to \mathcal{U}(\infty)$ as $\kappa \to \infty$ strongly (Theorem 3.4) and hence we get

$$s - \lim_{\kappa \to \infty} (H_{\vec{\rho}}^{REN}(\kappa) + V \otimes I - z)^{-1} = \mathcal{U}(\infty) \left\{ (E^\infty(D) + V_{eff} - z)^{-1} \otimes P_0 \right\} \mathcal{U}^{-1}(\infty),$$

where $E^\infty(D)$ is a pseudo differential operator in $L^2(\mathbb{R}^{dN})$ and V_{eff} a multiplication operator. (Theorems 3.6, 3.7). In the case of one-particle system the effective potential V_{eff} is a Gaussian transformation of a given scalar potential V. However, we shall see that in the N-particle system, V_{eff} is not necessary to be a Gaussian transformation. Actually it is determined by a matrix $\tilde{\Delta}^\infty = (\tilde{\Delta}^\infty_{ij})_{1\le i,j\le N}$ defined by

$$\tilde{\Delta}^\infty_{ij} = \frac{1}{2}\frac{d-1}{d}\left(\frac{\hbar}{mc}\right)\frac{e^2}{\hbar c}\int_{\mathbb{R}^d} dk \frac{\hat{\rho}_i(k)\hat{\rho}_j(k)}{\omega(k)^3}, \tag{1. 3}$$

where $\omega(k) = |k|, k \in \mathbb{R}^d$. In the case where $\tilde{\Delta}^\infty$ is non-degenerate, the effective potential V_{eff} is Gaussian transformations of V.

The outline of this paper is as follows. In section 2, we define the Pauli-Fierz Hamiltonian with N-nonrelativistic charged particles in the Coulomb gauge with the dipole approximation and show its self-adjointness. Moreover we construct an exact solution to the Heisenberg equation from the point of view of the operator theory (Corollary 2.9). In section 3, when the scaling parameter $\kappa > 0$ is sufficiently large, we show that a Bogoliubov transformation can be constructed, and define a renormalized self-adjoint operator $H_{\vec{\rho}}^{REN}(\kappa)$ which is the original Hamiltonian $H_{\vec{\rho}}(\kappa)$ with an infinite self-energy of the nonrelativistic charged particles subtracted. We shall show the existence of the scaling limit of $H_{\vec{\rho}}+V\otimes I$ and give an explicit form of the effective potential. We give a typical example of a scalar potential and a set of ultraviolet cut-off functions. In section 4, we give a physical interpretation of the matrix $\tilde{\Delta}^\infty$.

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2 THE PAULI-FIERZ MODEL AND EXACT SOLUTION

To begin with, let us introduce some preliminary notations. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. We denote the inner product and the associated norm by $< *, \cdot >_{\mathcal{H}}$ and $||\cdot||_{\mathcal{H}}$ respectively.

The inner product is linear in $\cdot$ and antilinear in $*$. The domain of an operator A in $\mathcal{H}$ is denoted by $D(A)$. A notation The Fourier transformation of a function f is denoted by $\hat{f}$ (resp.$\check{f}$) and $\bar{f}$ the complex conjugate of f. In this paper, summations over repeated Greek letters are understood. Let

$$\mathcal{W} \equiv \underbrace{L^2(\mathbb{R}^d) \oplus ... \oplus L^2(\mathbb{R}^d)}_{d-1}.$$

We define the Boson Fock space over $\mathcal{W}$ by

$$\mathcal{F}(\mathcal{W}) \equiv \bigoplus_{n=0}^{\infty} \otimes_s^n \mathcal{W} \equiv \bigoplus \mathcal{F}_n(\mathcal{W}),$$

where $\otimes_s^0 \mathcal{W} \equiv \mathbb{C}$ and $\otimes_s^n \mathcal{W}$ $(n \geq 1)$ denotes the n-fold symmetric tensor product. Put

$$\mathcal{F}^N(\mathcal{W}) \equiv \bigoplus_{n=0}^{N} \mathcal{F}_n(\mathcal{W}) \bigoplus_{n \geq N+1} \{0\}.$$

Moreover we define the finite particle subspace of $\mathcal{F}(\mathcal{W})$ by

$$\mathcal{F}^{\infty}(\mathcal{W}) \equiv \bigcup_{N=0}^{\infty} \mathcal{F}^N(\mathcal{W}).$$

The annihilation operator $a(f)$ and the creation operator $a^{\dagger}(f)$ $(f \in \mathcal{W})$ act on the finite particle subspace and leave it invariant with the canonical commutation relations (CCR): for $f, g \in \mathcal{W}$

$$\begin{aligned} [a(f), a^{\dagger}(g)] &= \left\langle \bar{f}, g \right\rangle_{\mathcal{W}}, \\ [a^{\sharp}(f), a^{\sharp}(g)] &= 0, \end{aligned}$$

where $[A, B] = AB - BA$, $a^{\sharp}$ denotes either a or $a^{\dagger}$. Furthermore,

$$\left\langle a^{\dagger}(f)\Phi, \Psi \right\rangle_{\mathcal{F}(\mathcal{W})} = \left\langle \Phi, a(\bar{f})\Psi \right\rangle_{\mathcal{F}(\mathcal{W})}, \quad \Phi, \Psi \in \mathcal{F}^{\infty}(\mathcal{W}).$$

We define polarization vectors $e^r (r = 1, ..., d-1)$ as measurable functions $e^r : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that

$$e^r(k) e^s(k) = \delta_{rs}, \quad e^r(k) k = 0, \quad a.e. k \in \mathbb{R}^d.$$

In this paper, we fix polarization vectors e^r. The μ-th direction time-zero smeared radiation field in the Coulomb gauge with the dipole approximation is defined as operators acting in $\mathcal{F}(\mathcal{W})$ by

$$A_\mu(f) = \frac{1}{\sqrt{2}}\left\{a^\dagger\left(\oplus_{r=1}^{d-1}\frac{\sqrt{\hbar}e^r_\mu \hat{f}}{\sqrt{c\omega}}\right) + a\left(\oplus_{r=1}^{d-1}\frac{\sqrt{\hbar}e^r_\mu \tilde{\hat{f}}}{\sqrt{c\omega}}\right)\right\}, \tag{2. 1}$$

and the conjugate momentum

$$\Pi_\mu(f) = \frac{i}{\sqrt{2}}\left\{a^\dagger\left(\oplus_{r=1}^{d-1}\sqrt{\hbar}\sqrt{c\omega}e^r_\mu \hat{f}\right) - a\left(\oplus_{r=1}^{d-1}\sqrt{\hbar}\sqrt{c\omega}e^r_\mu \tilde{\hat{f}}\right)\right\}, \tag{2. 2}$$

where $\tilde{g}(k) = g(-k)$. Note that in the case where f is real-valued, $A_\mu(f)$ and $\Pi_\mu(f)$ are symmetric operators. Let $\Omega = (1, 0, 0, ...) \in \mathcal{F}(\mathcal{W})$. It is well known that

$$\mathcal{L}\left\{a^\dagger(f_1)...a^\dagger(f_n)\Omega, \Omega | f_j \in \mathcal{W}, j = 1, ..., n, n \geq 1\right\}$$

is dense in $\mathcal{F}(\mathcal{W})$. For a nonnegative self-adjoint operator $h : \mathcal{W} \to \mathcal{W}$, an operator $\Gamma(e^{-th})$ is defined by

$$\begin{aligned}\Gamma(e^{-th})a^\dagger(f_1)...a^\dagger(f_n)\Omega &= a^\dagger(e^{-th}f_1)...a^\dagger(e^{-th}f_n)\Omega,\\ \Gamma(e^{-th})\Omega &= \Omega.\end{aligned}$$

The operator $\Gamma(e^{-th})$ defines a unique strongly continuous one-parameter semigroup on $\mathcal{F}(\mathcal{W})$. Hence, by Stone's theorem, there exists a nonnegative self-adjoint operator $d\Gamma(h)$ in $\mathcal{F}(\mathcal{W})$ such that

$$\Gamma(e^{-th}) = e^{-td\Gamma(h)}.$$

The operator $d\Gamma(h)$ is called "the second quantization of h". Put $\tilde{\omega} = \underbrace{\omega \oplus ... \oplus \omega}_{d-1}$. The free Hamiltonian H_b in $\mathcal{F}(\mathcal{W})$ is defined by

$$H_b \equiv \hbar c d\Gamma(\tilde{\omega}).$$

Let M_d be a Hilbert space defined by

$$M_d = \left\{ f \left| \int |f(k)|^2 \omega(k)^d dk < \infty \right. \right\},$$

with the inner product

$$< f, g >_n = \int_{\mathbb{R}^d} \bar{f}(k) g(k) \omega(k)^n dk.$$

We have the following commutation relations on $\mathcal{F}^\infty(\mathcal{W})$,

$$\begin{aligned}
&[A_\mu(f), A_\nu(g)] = 0, && \hat{f}, \hat{g} \in M_{-1}, \\
&[\Pi_\mu(f), \Pi_\nu(g)] = 0, && \hat{f}, \hat{g} \in M_1, \\
&[A_\mu(f), \Pi_\nu(g)] = i\hbar \left\langle d_{\mu\nu} \hat{\bar{f}}, \hat{g} \right\rangle_{L^2(\mathbb{R}^d)}, && \hat{f}, \hat{g} \in M_{-1} \cap M_0 \cap M_1,
\end{aligned}$$

and on $D(H_b^{\frac{3}{2}})$,

$$\begin{aligned}
&[H_b, A_\mu(f)] = -i\hbar \Pi_\mu(f), && \hat{f} \in M_{-1} \cap M_1, \\
&[H_b, \Pi_\mu(f)] = i\hbar c^2 A_\mu(-\Delta f), && \hat{f} \in M_3 \cap M_1,
\end{aligned}$$

where Δ is the Laplacian in the L^2-sense and $d_{\mu\nu}(k) = \sum_{r=1}^d e_\mu^r(k) e_\nu^r(k)$. The Pauli-Fierz Hamiltonian with N-nonrelativistic charged particles interacting with the quantized radiation field in the Coulomb gauge with the dipole approximation is defined by

$$H_{\vec{\rho}} \equiv H_{\rho_1, \ldots, \rho_N} \equiv \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left(-i\hbar D_\mu^j \otimes I - eI \otimes A_\mu(\rho_j) \right)^2 + I \otimes H_b,$$

acting in

$$\underbrace{L^2(\mathbb{R}^d) \otimes \ldots \otimes L^2(\mathbb{R}^d)}_{N} \bigotimes \mathcal{F}(\mathcal{W}) \cong L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}(\mathcal{W}) \cong \int_{\mathbb{R}^{dN}}^{\oplus} \mathcal{F}(\mathcal{W}) dx,$$

where D_μ^j is the L^2-derivative with respect to the j-th variable in the μ-th direction, $\rho_j' s$ serve as ultraviolet cut-off functions. We introduce a scaling with respect to the parameters c, e, m as (1.2). Throughout this paper, for objects $A = A(c, e, m)$ containing the parameters c, e, m, we denote the scaled object by $A(\kappa) \equiv A(c(\kappa), e(\kappa), m(\kappa))$. We define a class of sets of functions as follows:

Definition 2.1 $\vec{\rho} = (\rho_1, ..., \rho_N)$ *is in* P *if and only if*

(1) $\hat{\rho}_j, j = 1, ..., N$ *are rotation invariant,* $\hat{\rho}_j(k) = \hat{\rho}_j(|k|)$*, and real-valued,*

(2) $\hat{\rho}_j/\omega, \hat{\rho}_j/\sqrt{\omega}, \hat{\rho}_j, \sqrt{\omega}\hat{\rho}_j \in L^2(\mathbb{R}^d)$.

Moreover $\vec{\rho}$ *is in* $\widetilde{P}$ *if and only if in addition to* **(1)** *and* **(2)** *above*

(3) *For all* $j = 1, ..., N$*,* $\hat{\rho}_j/\omega\sqrt{\omega} \in L^2(\mathbb{R}^d)$ *and there exist* $0 < \alpha < 1$ *and* $1 \leq \epsilon$ *such that* $\hat{\rho}_i(\sqrt{s})\hat{\rho}_j(\sqrt{s})(\sqrt{s})^{d-2} \in Lip(\alpha) \cap L^\epsilon([0, \infty))$*, where* $Lip(\alpha)$ *is the set of the Lipschitz continuous functions on* $[0, \infty)$ *with order* α*,*

(4) $\sup_k |\hat{\rho}_j(k)\omega^{\frac{d}{2}-\frac{3}{2}}(k)| < \infty, \sup_k |\hat{\rho}_j(k)\omega^{\frac{d}{2}-\frac{1}{2}}(k)| < \infty, j = 1, ..., N.$

Observe that Definition 2.1 (**1**) implies that ρ_j's are real-valued functions. Hence $A_\mu(\rho_j)$'s are symmetric operators. Put

$$H_0 = -\frac{1}{2m}\hbar^2\Delta \otimes I + I \otimes H_b,$$

where Δ is the Laplacian in $L^2(\mathbb{R}^{dN})$. It is well known that H_0 is a nonnegative self-adjoint operator on $D(H_0) = D\left(-\frac{1}{2m}\hbar^2\Delta \otimes I\right) \cap D(I \otimes H_b)$.

Theorem 2.2 ([3,4]) *For* $\vec{\rho} \in P$ *and* $\kappa > 0$*, the operator* $H_{\vec{\rho}}(\kappa)$ *is self-adjoint on* $D(H_0)$ *and essentially self-adjoint on any core of* H_0 *and nonnegative.*

Let $\mathbf{F} = F \otimes I$, where F denotes the Fourier transform in $L^2(\mathbb{R}^{dN})$. It is clear that operators $\mathbf{F}H_{\vec{\rho}}\mathbf{F}^{-1}$ can be decomposable as follows:

$$\mathbf{F}H_{\vec{\rho}}(\kappa)\mathbf{F}^{-1} = \int_{\mathbb{R}^{dN}}^{\oplus} H_{\vec{\rho}}(p, \kappa)dp,$$

where

$$H_{\vec{\rho}}(p, \kappa) = \frac{1}{2m}\sum_{j=1}^{N}\sum_{\mu=1}^{d}\left(\kappa\hbar p_\mu^j - eA_\mu(\rho_j)\right)^2 + \kappa H_b.$$

Theorem 2.3 ([3,4]) *For $\vec{\rho} \in P$ and $\kappa > 0$, the operator $H_{\vec{\rho}}(p, \kappa)$ is self-adjoint on $D(H_0)$ and essentially self-adjoint on any core of H_b and nonnegative.*

Following [3,4,6], we shall construct a Heisenberg field concretely. The Heisenberg field $A_\mu(f, t, \kappa)$ with the scaling parameter κ is defined by a solution to the Heisenberg equation:

$$\begin{aligned} \frac{d}{dt} A_\mu(f, t, \kappa) &= \frac{i}{\hbar}[H_{\vec{\rho}}(p, \kappa), A_\mu(f, t, \kappa)], \\ A_\mu(f, 0, \kappa) &= A_\mu(f, \kappa). \end{aligned}$$

In order to construct the Heisenberg field in a rigorous way, we shall prepare some technical lemmas. We define an $N \times N$ matrix-valued function $\mathbb{D}(z) = (D_{ij}(z))_{1 \le i,j \le N}$ by

$$D_{ij}(z) = m\delta_{ij} - \frac{e^2}{c^2}\frac{d-1}{d}\int_{\mathbb{R}^d} \frac{\hat{\rho}_i(k)\hat{\rho}_j(k)}{z - |k|^2} dk, \quad z \in \mathbb{C} \setminus [0, \infty).$$

Lemma 2.4 *Let (,) denote the Euclidean inner product. Suppose $\vec{\rho} \in \tilde{P}$. Then the followings hold:*

(1) *The functions $D_{ij}(z, \kappa), 1 \le i, j \le N, \kappa > 0$ are analytic in $\mathbb{C} \setminus [0, \infty)$.*

(2) *For $s \in [0, \infty)$ and $\kappa > 0$, the pointwise limit $D_{\pm ij}(s, \kappa) \equiv \lim_{h \to 0} D_{ij}(s \pm ih, \kappa)$ exists and has the following form*

$$\begin{aligned} D_{\pm ij}(s, \kappa) &= \frac{m}{\kappa^2}\delta_{ij} - \frac{1}{\kappa^3}\left(\frac{e^2}{c^2}\frac{\mathbf{V}_d}{2}\frac{d-1}{d}\right) H_{ij}(s) \pm \frac{2\pi i}{\kappa^3}\left(\frac{e^2}{c^2}\frac{\mathbf{V}_d}{2}\frac{d-1}{d}\right) K_{ij}(s), \\ K_{ij}(s) &= \hat{\rho}_i(\sqrt{s})\hat{\rho}_j(\sqrt{s}) s^{\frac{d}{2}-1}, \\ H_{ij}(s) &= \lim_{\epsilon \to 0+} \int_{|s-x|>\epsilon} \frac{K_{ij}(x)}{s-x} dx, \end{aligned}$$

where $\mathbf{V}_d = 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$ ($\Gamma(z)$ is the gamma function). The convergence is uniform in $s \in [0, \infty)$; for any $\delta > 0$, there exists $h_0 > 0$ independent of s, κ, such that for $0 <^\forall h \le h_0$,

$$|D_{ij}(s \pm ih, \kappa) - D_{\pm ij}(s, \kappa)| \le \frac{\delta}{\kappa^3}.$$

Moreover $H_{ij}(s)$ is Lipschitz continuous in $s \in [0,\infty)$ with the same order as that of K_{ij} and contained in $L^{\epsilon}(\mathbb{R}^d)$ with some $\epsilon \geq 1$.

(3) *Let $\kappa > 0$ be sufficiently large. Put $\mathbb{D}_{\pm}(s,\kappa) = (D_{\pm ij}(s,\kappa))_{1\leq i,j\leq N}$. Then there exists a positive constant $d_1(\kappa)$ such that for $(w_1,...,w_N) = \vec{w} \in \mathbb{C}^N$,*

$$\inf_{s\in[0,\infty)} |(\mathbb{D}_{\pm}(s,\kappa)\vec{w},\vec{w})| > d_1(\kappa)|\vec{w}|^2.$$

(4) *Let $\kappa > 0$ be sufficiently large. Then there exists a positive constant $d_2(\kappa)$ such that for $\vec{w} \in \mathbb{C}^N$,*

$$\inf_{z\in\mathbb{C}\setminus[0,\infty)} |(\mathbb{D}(z,\kappa)\vec{w},\vec{w})| > d_2(\kappa)|\vec{w}|^2.$$

Proof: The statements (1) and (2) are fundamental facts([9]). We shall prove (3). From (2) it follows that

$$(\mathbb{D}_{\pm}(s,\kappa)\vec{w},\vec{w}) = \frac{m}{\kappa^2}\left\{|\vec{w}|^2 - \frac{1}{\kappa}\frac{\lambda}{m}(H(s)\vec{w},\vec{w})\right\} \pm 2\pi i\frac{\lambda}{\kappa^3}(K(s)\vec{w},\vec{w}),$$

where $\lambda = \frac{e^2}{c^2}\frac{\mathbf{V}_d}{2}\frac{d-1}{d}$, $H(s) = (H_{ij}(s))_{1\leq i,j\leq d}$, $K(s) = (K_{ij}(s))_{1\leq i,j\leq d}$. Since H_{ij} is a Lipschitz continuous function and contained in $L^{\epsilon}([0,\infty))$, it is bounded. Hence we have

$$|(H(s)\vec{w},\vec{w})| \leq N \times \sup_{s\in[0,\infty),1\leq i,j\leq d} |H_{ij}(s)|\cdot|\vec{w}|^2 \equiv \alpha|\vec{w}|^2.$$

Thus we can see that for sufficiently large $\kappa > 0$

$$|(\mathbb{D}_{\pm}(s,\kappa)\vec{w},\vec{w})| \geq \frac{m}{\kappa^2}\left(1 - \frac{1}{\kappa}\frac{\lambda}{m}\alpha\right)|\vec{w}|^2.$$

Hence we get (3). We shall prove (4). From (2) it follows that for any $\eta > 0$, there exists $\epsilon_0 > 0$ independent of $s \in [0,\infty)$ and $\kappa > 0$ such that for $0 <^{\forall} \epsilon \leq \epsilon_0$,

$$|(\mathbb{D}_{\pm}(s,\kappa)\vec{w},\vec{w})| - \frac{\eta}{\kappa^3}|\vec{w}|^2 \leq |(\mathbb{D}(s \pm i\epsilon,\kappa)\vec{w},\vec{w})|.$$

Hence we have

$$|(\mathbb{D}(s\pm i\epsilon,\kappa)\vec{w},\vec{w})| \geq \left\{\frac{m}{\kappa^2}\left(1-\frac{1}{\kappa}\frac{\lambda}{m}\alpha\right)-\frac{\eta}{\kappa^3}\right\}|\vec{w}|^2. \qquad (2.\ 3)$$

On the other hand, put $\Pi_{\epsilon_0}=\mathbb{C}\setminus\{x+iy|x\geq 0,|y|\leq\epsilon_0\}$. Then we see that for $x+iy\in\Pi_{\epsilon_0}$

$$\begin{aligned}(\mathbb{D}(x+iy,\kappa)\vec{w},\vec{w}) &= \frac{m}{\kappa^2}\left(|\vec{w}|^2-\frac{1}{\kappa}\frac{\lambda}{m}\int_0^\infty\frac{(x-s)|\sum_{i=1}^N w_i\hat{\rho}_i(\sqrt{s})|^2 s^{\frac{d}{2}-1}}{(x-s)^2+y^2}ds\right)\\ &\quad +i\frac{\lambda}{\kappa^3}\int_0^\infty\frac{y|\sum_{i=1}^N w_i\hat{\rho}(\sqrt{s})|^2 s^{\frac{d}{2}-1}}{(x-s)^2+y^2}ds.\end{aligned}$$

Noting that $|ab|/a^2+b^2\leq 1/2$, we have

$$\begin{aligned}\left|\int_0^\infty\frac{(x-s)|\sum_{i=1}^N w_i\hat{\rho}_i(\sqrt{s})|^2 s^{\frac{d}{2}-1}}{(x-s)^2+y^2}ds\right| &\leq \frac{1}{2|y|}\int_0^\infty|\sum_{i=1}^N w_i\hat{\rho}_i(\sqrt{s})|^2 s^{\frac{d}{2}-1}ds,\\ &\leq \frac{1}{2\epsilon_0}\int_0^\infty\sum_{i=1}^N|\hat{\rho}_i(\sqrt{s})|^2 s^{\frac{d}{2}-1}ds|\vec{w}|^2\\ &\equiv \frac{\beta}{\epsilon_0}|\vec{w}|^2.\end{aligned}$$

Since ϵ_0 is independent of $\kappa>0$, we see that for sufficiently large $\kappa>0$,

$$|(\mathbb{D}(x+iy,\kappa)\vec{w},\vec{w})|\geq\frac{m}{\kappa^2}\left(1-\frac{1}{\kappa}\frac{\lambda}{m}\frac{\beta}{\epsilon_0}\right)|\vec{w}|^2,\quad x+iy\in\Pi_{\epsilon_0}. \qquad (2.\ 4)$$

Combining (2.3) and (2.4), we get (4). □

From Lemma 2.4 (3) and (4) it follows that for sufficiently large $\kappa>0$, there exist the inverse matrices to $\mathbb{D}(z,\kappa)$ and $\mathbb{D}_\pm(s,\kappa)$, which satisfy

$$\sup_{s\in[0,\infty)}\left|\left(\mathbb{D}_\pm^{-1}(s,\kappa)\vec{w}_1,\vec{w}_2\right)\right|<\frac{1}{d_1(\kappa)}|\vec{w}_1||\vec{w}_2|, \qquad (2.\ 5)$$

$$\sup_{z\in\mathbb{C}\setminus[0,\infty)}\left|\left(\mathbb{D}^{-1}(z,\kappa)\vec{w}_1,\vec{w}_2\right)\right|<\frac{1}{d_2(\kappa)}|\vec{w}_1||\vec{w}_2|. \qquad (2.\ 6)$$

We set for $\vec{\rho}\in\widetilde{P}$ and sufficiently large $\kappa>0$

$$Q(k,\kappa)\equiv\mathbb{D}_+^{-1}(k^2,\kappa)\begin{pmatrix}\hat{\rho}_1(k)\\ \vdots\\ \hat{\rho}_N(k)\end{pmatrix}\equiv(Q_1(k,\kappa),...,Q_N(k,\kappa)).$$

For later use in Appendix, we note that for all $s \in [0, \infty)$,

$$D_{+ij}(s,\kappa) - D_{-ij}(s,\kappa) = 2\pi i \frac{1}{\kappa^3}\left(\frac{e^2}{c^2}\mathbf{V}_d\frac{d-1}{d}\right)\hat{\rho}_i(\sqrt{s})\hat{\rho}_j(\sqrt{s})s^{\frac{d}{2}-1}. \tag{2. 7}$$

Put $\mathbb{D}^{-1}(z) = (D_{ij}^{-1}(z))_{1\leq ij\leq N}$, $\mathbb{D}_{\pm}^{-1}(s) = (D_{\pm ij}^{-1}(s))_{1\leq ij\leq N}$. Then (2.7) implies that

$$\frac{2\pi i}{\kappa^3}\left(\frac{e^2}{c^2}\mathbf{V}_d\frac{d-1}{d}\right)\sum_{k,l=1}^{N} D_{-ik}^{-1}(s,\kappa)D_{+jl}^{-1}(s,\kappa)\hat{\rho}_k(\sqrt{s})\hat{\rho}_l(\sqrt{s})s^{\frac{d}{2}-1} = D_{-ij}(s,\kappa) - D_{+ji}(s,\kappa). \tag{2. 8}$$

Remark 2.5

(1) *In [3,4,6], the authors define functions $D_{\pm}(s)$ corresponding to $\mathbb{D}_{\pm}(s)$ defined in this paper. The function $1/D_{\pm}(s,\kappa)$ can be well defined for some ρ and any $\kappa > 0$. However, in our case, we do not know whether $\mathbb{D}_{\pm}(s,\kappa)$ has the inverse or not for all $\kappa > 0$. But since, in this paper, we focus on an asymptotic behavior as $\kappa \to \infty$, it is sufficient to consider the case where κ is sufficiently large.*

(2) *For the proof of Lemma 2.4, we do not need Definition 2.1 (4).*

We define operators G_h $(h > 0)$ by

$$(G_h f)(k) = \int_{\mathbb{R}^d} \frac{f(k')}{(k^2 - k'^2 + ih)(kk')^{\frac{d}{2}-1}}dk'.$$

It is well known and not so hard to see that G_h are bounded linear operators on $L^2(\mathbb{R}^d)$ and the strong limits $\lim_{h\to 0} G_h \equiv G$ exists ([4]). Furthermore G is skew symmetric ($G^* = -G$). For sufficiently large $\kappa > 0$, we can define the following operators:

$$T_{\mu\nu}(\kappa)f \equiv \delta_{\mu\nu}f + \frac{1}{\kappa^3}\frac{e^2}{c^2}\sum_{j=1}^{N} Q_j(\kappa)\omega^{\frac{d}{2}-1}G\omega^{\frac{d}{2}-1}d_{\mu\nu}\hat{\rho}_j f, \quad 1 \leq \mu,\nu \leq d.$$

Lemma 2.6 *Suppose that $\vec{\rho} \in \tilde{P}$ and $\kappa > 0$ is sufficiently large. Then the following holds.*

(1) *$T_{\mu\nu}(\kappa)$ and $T_{\mu\nu}^*(\kappa)$ are bounded operators on M_α, $\alpha = -1,0,1$ and $(T_{\mu\nu}(\kappa)f)\tilde{} = T_{\mu\nu}(\kappa)\tilde{f}$.*

(2) *Put* $D_{ij}^{-1}(0,\kappa) \equiv D_{\pm ij}^{-1}(0,\kappa)$ *and let* $f \in M_{-1}$. *Then*

$$\left\langle d_{\nu\alpha}\frac{Q_i(\kappa)}{\sqrt{\omega^3}}, \frac{1}{\sqrt{\omega}}T_{\mu\nu}(\kappa)f\right\rangle_{L^2(\mathbb{R}^d)} = \left\langle d_{\mu\alpha}\sum_{j=1}^N D_{ij}^{-1}(0,\kappa)\frac{\hat{\rho}_j}{\sqrt{\omega^3}}, \frac{f}{\sqrt{\omega}}\right\rangle_{L^2(\mathbb{R}^d)}, i=1,...,N.$$

(3) $[\ \omega^2, T_{\mu\nu}^*(\kappa)\ ] = -\sum_{i=1}^N \frac{1}{\kappa^3}\frac{e^2}{c^2}\langle Q_i(\kappa), \cdot\rangle_{L^2(\mathbb{R}^d)}\, d_{\mu\nu}\hat{\rho}_i.$

(4) $T_{\mu\nu}(\kappa)\hat{\rho}_j = \delta_{\mu\nu}\frac{m}{\kappa^2}Q_j(\kappa).$

(5) $T_{\mu\nu}^*(\kappa)d_{\nu\alpha}T_{\alpha\beta}(\kappa) = d_{\mu\beta}.$

(6) $e_\mu^r T_{\mu\nu}(\kappa)d_{\nu\alpha}T_{\alpha\beta}(\kappa)e_\beta^s = \delta_{rs}.$

Proof: See Appendix. □

In the rest of this section, we fix sufficiently large $\kappa > 0$ and omit κ in notations for simplicity. Define $\widehat{A}_\mu(f) = A_\mu(\hat{f})$ and $\widehat{\Pi}_\mu(f) = \Pi_\mu(\hat{f})$. We put

$$\begin{aligned} B^{(r)}(f,p) &= \frac{1}{\sqrt{2}}\left\{\widehat{A}_\mu\left(\frac{1}{\sqrt{\hbar}}T_{\mu\nu}^* e_\nu^r\sqrt{c\omega}f\right) + i\widehat{\Pi}_\mu\left(\frac{1}{\sqrt{\hbar}}T_{\mu\nu}^* e_\nu^r\frac{f}{\sqrt{c\omega}}\right)\right. \\ &\quad \left. + \sum_{j=1}^N \hbar p_\nu^j\left\langle\frac{e}{\sqrt{\hbar}}\frac{Q_j e_\nu^r}{(c\omega)^{\frac{3}{2}}}, f\right\rangle_{L^2(\mathbb{R}^d)}\right\}, \quad f \in M_0, \\ B^{\dagger(r)}(f,p) &= \frac{1}{\sqrt{2}}\left\{\widehat{A}_\mu\left(\frac{1}{\sqrt{\hbar}}\overline{T}_{\mu\nu}^* \tilde{e}_\nu^r\sqrt{c\omega}\tilde{f}\right) - i\widehat{\Pi}_\mu\left(\frac{1}{\sqrt{\hbar}}\overline{T}_{\mu\nu}^* \tilde{e}_\nu^r\frac{\tilde{f}}{\sqrt{c\omega}}\right)\right. \\ &\quad \left. + \sum_{j=1}^N \hbar p_\nu^j\left\langle\frac{e}{\sqrt{\hbar}}\frac{\overline{Q}_j e_\nu^r}{(c\omega)^{\frac{3}{2}}}, f\right\rangle_{L^2(\mathbb{R}^d)}\right\}, f \in M_0, p=(p^1,...,p^N) \in \mathbb{R}^{dN}. \end{aligned}$$

By the definition of $A_\mu(f)$ and $\Pi_\mu(f)$, for the vector of the form $\mathbf{f} = f_1 \oplus ... \oplus f_{d-1} \in \mathcal{W}$, we see that for $p = (p^1,...,p^N) \in \mathbb{R}^{dN}$

$$\begin{aligned} B(\mathbf{f},p) &\equiv \sum_{r=1}^{d-1} B^{(r)}(f_r,p) = a^\dagger(\mathbf{W}_-\mathbf{f}) + a(\mathbf{W}_+\mathbf{f}) + \sum_{j=1}^N\left\langle \mathbf{L}_j p^j, \mathbf{f}\right\rangle_{L^2(\mathbb{R}^d)}, \\ B^\dagger(\mathbf{f},p) &\equiv \sum_{r=1}^{d-1} B^{\dagger(r)}(f_r,p) = a^\dagger(\overline{\mathbf{W}}_+\mathbf{f}) + a(\overline{\mathbf{W}}_-\mathbf{f}) + \sum_{j=1}^N\left\langle \overline{\mathbf{L}}_j p^j, \mathbf{f}\right\rangle_{L^2(\mathbb{R}^d)}, \end{aligned} \qquad (2.\ 9)$$

where

$$\begin{aligned}
\mathbf{W}_{\pm} &= (W_{\pm}^{(r,s)})_{1\le r,s\le d-1}, \\
\mathbf{L}_j &= (L_{\mu j}^r)_{1\le \mu\le d,\ 1\le r\le d-1} = \begin{pmatrix} L_{1j}^1 & \cdots & L_{dj}^1 \\ \vdots & & \vdots \\ L_{1j}^{d-1} & \cdots & L_{dj}^{d-1} \end{pmatrix}, j = 1, ..., N,
\end{aligned}$$

where

$$\begin{aligned}
W_{+}^{(r,s)} f &= \frac{1}{2}\left(\frac{1}{\sqrt{\omega}} e_\mu^r T_{\mu\nu}^* e_\nu^s \sqrt{\omega} + \sqrt{\omega} e_\mu^r T_{\mu\nu}^* e_\nu^s \frac{1}{\sqrt{\omega}}\right) f, \\
W_{-}^{(r,s)} f &= \frac{1}{2}\left(\frac{1}{\sqrt{\omega}} e_\mu^r T_{\mu\nu}^* \tilde{e}_\nu^s \sqrt{\omega} - \sqrt{\omega} e_\mu^r T_{\mu\nu}^* \tilde{e}_\nu^s \frac{1}{\sqrt{\omega}}\right) \tilde{f}, \\
L_{\mu j}^r &= \frac{e\sqrt{\hbar} e_\mu^r Q_j}{\sqrt{2c^3\omega^3}}.
\end{aligned}$$

We see that, by Lemma 2.6 (1), $\mathbf{W}_{\pm}$ is a bounded operator on $\mathcal{W}$. By virtue of Lemma 2.6 (5) and (6), one can easily see that $\mathbf{W}_{\pm}$ satisfy the following algebraic relations:

$$\begin{aligned}
\mathbf{W}_+^* \mathbf{W}_+ - \mathbf{W}_-^* \mathbf{W}_- &= I, \\
\overline{\mathbf{W}}_+^* \mathbf{W}_- - \overline{\mathbf{W}}_-^* \mathbf{W}_+ &= 0, \\
\mathbf{W}_+ \mathbf{W}_+^* - \overline{\mathbf{W}}_- \overline{\mathbf{W}}_-^* &= I, \\
\mathbf{W}_- \mathbf{W}_+^* - \overline{\mathbf{W}}_+ \overline{\mathbf{W}}_-^* &= 0. \qquad (2.\ 10)
\end{aligned}$$

We put $\mathcal{W}_\alpha = \underbrace{M_\alpha \oplus ... \oplus M_\alpha}_{d-1}$, $\alpha \in \mathbb{R}$. These relations (2.10) imply that on $\mathcal{F}^\infty(\mathcal{W})$ for $\mathbf{f}, \mathbf{g} \in \mathcal{W}_0$,

$$\begin{aligned}
[B(\mathbf{f}, p), B^\dagger(\mathbf{g}, p)] &= \langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{W}}, \\
[B^\sharp(\mathbf{f}, p), B^\sharp(\mathbf{g}, p)] &= 0,
\end{aligned}$$

and for $\Phi, \Psi \in \mathcal{F}^\infty(\mathcal{W})$,

$$\left\langle B^\dagger(\mathbf{f}, p)\Phi, \Psi \right\rangle_{\mathcal{F}(\mathcal{W})} = \left\langle \Phi, B(\bar{\mathbf{f}}, p)\Psi \right\rangle_{\mathcal{F}(\mathcal{W})}.$$

Lemma 2.7 *For $\mathbf{f} \in \mathcal{W}_0 \cap \mathcal{W}_2$, $p \in \mathbb{R}^{dN}$ and $\vec{\rho} \in \widetilde{P}$, we have*

$$[\ H_{\vec{\rho}}(p), B^\sharp(\mathbf{f}, p)\ ] = \pm B^\sharp(\hbar c\tilde{\omega}\mathbf{f}, p), \ \ on \ \ \mathcal{F}^\infty(\mathcal{W}) \cap D(H_b^{\frac{3}{2}}), \tag{2. 11}$$

where + (resp.-) corresponds to $B^\dagger$ (resp.B).

Proof: Suppose that $\mathbf{f} \in \mathcal{W}_{-2} \cap \mathcal{W}_0 \cap \mathcal{W}_2$. Then by Lemma 2.6 (3) and (4), one can directly see that (2.11) holds. Next by a limiting argument, one can get (2.11) for $\mathbf{f} \in \mathcal{W}_0 \cap \mathcal{W}_2$. □

Define

$$\begin{aligned} A(\mathbf{f}, p) &\equiv \frac{1}{\sqrt{2}}\left(B^\dagger(\mathbf{f}, p) + B(\bar{\mathbf{f}}, p)\right), \\ \Pi(\mathbf{f}, p) &\equiv \frac{i}{\sqrt{2}}\left(B^\dagger(\mathbf{f}, p) - B(\bar{\mathbf{f}}, p)\right), \quad \mathbf{f} \in \mathcal{W}_0. \end{aligned}$$

We can easily see that the operators $A(\mathbf{f}, p)|_{\mathcal{F}^\infty(\mathcal{W})}$ and $\Pi(\mathbf{f}, p)|_{\mathcal{F}^\infty(\mathcal{W})}$ are essentially self-adjoint by the Nelson analytic vector theorem [10,Theorem X.39]. We denote the self-adjoint extensions by the same symbols.

Theorem 2.8 *Suppose $\vec{\rho} \in \widetilde{P}$. Then for $\mathbf{f} \in \mathcal{W}_0$*

$$\exp\left(i\frac{t}{\hbar}H_{\vec{\rho}}(p)\right) A(\mathbf{f}, p) \exp\left(-i\frac{t}{\hbar}H_{\vec{\rho}}(p)\right) = A(e^{ic\tilde{\omega}t}\mathbf{f}, p), \tag{2. 12}$$

$$\exp\left(i\frac{t}{\hbar}H_{\vec{\rho}}(p)\right) \Pi(\mathbf{f}, p) \exp\left(-i\frac{t}{\hbar}H_{\vec{\rho}}(p)\right) = \Pi(e^{ic\tilde{\omega}t}\mathbf{f}, p). \tag{2. 13}$$

Proof: We only show an outline of the proof. For simplicity, put $A(e^{ic\tilde{\omega}t}\mathbf{f}, p) = A(\mathbf{f}, p, t)$. Let $C^\infty(H_b) = \cap_{n=1}^\infty D(H_b^n)$. We can easily see that, by Lemma 2.7, $\left\langle e^{iA(\mathbf{f},p,t)}\Psi, \Phi\right\rangle, \Psi, \Phi \in C^\infty(H_b) \cap \mathcal{F}^\infty(\mathcal{W}), \mathbf{f} \in \mathcal{W}_{-2} \cap \mathcal{W}_0 \cap \mathcal{W}_2$, is differentiable in t with

$$\frac{d}{dt}\left\langle e^{iA(\mathbf{f},p,t)}\Psi, \Phi\right\rangle_{\mathcal{F}(\mathcal{W})} = \left\langle \frac{i}{\hbar}e^{iA(\mathbf{f},p,t)}\Psi, H_{\vec{\rho}}(p)\Phi\right\rangle_{\mathcal{F}(\mathcal{W})} - \left\langle \frac{i}{\hbar}H_{\vec{\rho}}(p)\Psi, e^{-iA(\mathbf{f},p,t)}\Phi\right\rangle_{\mathcal{F}(\mathcal{W})} \tag{2. 14}$$

From (2.14) it follows that

$$\frac{d}{dt}\left\langle e^{-i\frac{t}{\hbar}H_{\vec{\rho}}(p)}e^{iA(\mathbf{f},p,t)}e^{i\frac{t}{\hbar}H_{\vec{\rho}}(p)}\Psi, \Phi\right\rangle_{\mathcal{F}(\mathcal{W})} = 0, \quad \Psi, \Phi \in D(H_b), \mathbf{f} \in \mathcal{W}_0.$$

Hence

$$e^{isA(\mathbf{f},p,0)} = e^{i\frac{t}{\hbar}H_{\vec{\rho}}(p)}e^{isA(\mathbf{f},p,t)}e^{-i\frac{t}{\hbar}H_{\vec{\rho}}(p)}, \quad on\ C^\infty(H_b)\cap\mathcal{F}^\infty(\mathcal{W}). \tag{2. 15}$$

By a limiting argument, one can see that (2.15) holds for $\Phi,\Psi\in D(H_b),\mathbf{f}\in\mathcal{W}_0$. Since the both sides of (2.15) are one parameter unitary groups in $s\in\mathbb{R}$, Stone's theorem yields (2.12). (2.13) is quite similar to (2.12). Thus we get the desired result. □

For $t\in\mathbb{R}$, we define operators in $\mathcal{F}(\mathcal{W})$ by

$$\begin{aligned} A_\mu(f,t|p) &= \frac{1}{\sqrt{2}}\sum_{r=1}^{d-1}\left\{B^{\dagger(r)}\left(\frac{\sqrt{\hbar}}{\sqrt{c\omega}}e^{ic\omega t}e_\nu^r\overline{T}_{\mu\nu}\hat{f},p\right)+B^{(r)}\left(\frac{\sqrt{\hbar}}{\sqrt{c\omega}}e^{-ic\omega t}e_\nu^rT_{\mu\nu}\tilde{\hat{f}},p\right)\right\} \\ &\quad -e\sum_{i,j=1}^N\hbar p_\nu^i\left\langle d_{\mu\nu}D_{ij}^{-1}(0)\frac{\hat{\rho}_j}{\sqrt{(c\omega)^3}},\frac{\hat{f}}{\sqrt{c\omega}}\right\rangle_{L^2(\mathbb{R}^d)}, \quad \hat{f}\in M_{-1},\mu=1,...,d. \end{aligned}$$

Form Lemma 2.5 (2) (5) and (6) it follows that

$$A_\mu(f,0|p) = A_\mu(f). \tag{2. 16}$$

Corollary 2.9 *Suppose $\vec{\rho}\in\widetilde{P}$. Then the operator $A_\mu(f,t|p)$ is the Heisenberg field with*

$$\exp\left(i\frac{t}{\hbar}H_{\vec{\rho}}(p)\right)A_\mu(f)\exp\left(-i\frac{t}{\hbar}H_{\vec{\rho}}(p)\right) = A_\mu(f,t|p). \tag{2. 17}$$

Proof: It is enough to show (2.17) for a real-valued function f. For a real-valued function f such that $\hat{f}\in M_{-1}$, we can see that

$$\begin{aligned} A_\mu(f) &= \frac{1}{\sqrt{2}}\sum_{r=1}^{d-1}\left\{B^{\dagger(r)}\left(\frac{\sqrt{\hbar}}{\sqrt{c\omega}}e_\nu^r\overline{T}_{\mu\nu}\hat{f},p\right)+B^{(r)}\left(\overline{\frac{\sqrt{\hbar}}{\sqrt{c\omega}}e_\nu^r\overline{T}_{\mu\nu}\hat{f},p}\right)\right\} \\ &\quad -e\sum_{i,j=1}^N\hbar p_\nu^i\left\langle d_{\mu\nu}D_{ij}^{-1}(0)\frac{\hat{\rho}_j}{\sqrt{(c\omega)^3}},\frac{\hat{f}}{\sqrt{c\omega}}\right\rangle. \end{aligned}$$

Hence (2.17) follows from Theorem 2.8. □

Corollary 2.10 *Suppose $\vec{\rho}\in\widetilde{P}$. Then for $\Phi\in D(H_b)$,*

$$\exp\left(i\frac{t}{\hbar}H_{\vec{\rho}}(p)\right)B^\sharp(\mathbf{f},p)\exp\left(-i\frac{t}{\hbar}H_{\vec{\rho}}(p)\right)\Phi = B^\sharp(e^{ic\tilde{\omega}t}\mathbf{f},p)\Phi$$

Proof: Note that $D(B^\sharp(\mathbf{f},p)) \subset D(H_b) = D(H_{\vec\rho}(p))$. Thus from (2.12) and (2.13) it follows that on $D(H_b)$

$$\begin{aligned} e^{i\frac{t}{\hbar}H_{\vec\rho}(p)}A(\mathbf{f},p)e^{-i\frac{t}{\hbar}H_{\vec\rho}(p)} &= \frac{1}{\sqrt2}\left\{e^{i\frac{t}{\hbar}H_{\vec\rho}(p)}B^\dagger(\mathbf{f},p)e^{-i\frac{t}{\hbar}H_{\vec\rho}(p)} + e^{i\frac{t}{\hbar}H_{\vec\rho}(p)}B(\overline{\mathbf{f}},p)e^{-i\frac{t}{\hbar}H_{\vec\rho}(p)}\right\}, \\ &= \frac{1}{\sqrt2}\left\{B^\dagger(e^{ic\widetilde\omega t}\mathbf{f}) + B(e^{-ic\widetilde\omega t}\overline{\mathbf{f}})\right\}, \\ e^{i\frac{t}{\hbar}H_{\vec\rho}(p)}\Pi(\mathbf{f},p)e^{-i\frac{t}{\hbar}H_{\vec\rho}(p)} &= \frac{i}{\sqrt2}\left\{e^{i\frac{t}{\hbar}H_{\vec\rho}(p)}B^\dagger(\mathbf{f},p)e^{(-i\frac{t}{\hbar}H_{\vec\rho}(p)} - e^{i\frac{t}{\hbar}H_{\vec\rho}(p)}B(\overline{\mathbf{f}},p)e^{-i\frac{t}{\hbar}H_{\vec\rho}(p)}\right\} \\ &= \frac{i}{\sqrt2}\left\{B^\dagger(e^{ic\widetilde\omega t}\mathbf{f}) - B(e^{-ic\widetilde\omega t}\overline{\mathbf{f}})\right\}. \end{aligned}$$

Thus the corollary follows. □

3 BOGOLIUBOV TRANSFORMATIONS AND SCALING LIMITS

In this section, we construct a unitary operator which implements a unitary equivalence of the Pauli-Fierz Hamiltonian and a decoupled Hamiltonian. Moreover we investigate a scaling limit of the Pauli-Fierz Hamiltonian. Unless otherwise stated in this section, we suppose that $\kappa > 0$ is sufficiently large. Since the bounded operators $W_-^{r,s}(\kappa)$ have integral kernels

$$W_-^{(r,s)}(k,k',\kappa) = \frac{1}{\kappa^3}\frac{e^2}{c^2}\frac{e_\mu^r(k)e_\mu^s(k')\sum_{j=1}^N \hat\rho_j(k)\overline{Q}_j(k',\kappa)}{2(|k|+|k'|)(|k||k'|)^{\frac12}},$$

such that $W_-^{(r,s)}(\kappa) \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$, the operator $\mathbf{W}_-(\kappa)$ is a Hilbert Schmidt operator on $\mathcal{W}$. Then from (2.9) and (2.10) it follows that there exist two unitary operators $U(\kappa)$ (p independent) and $S(p,\kappa)$ such that ([6,Section III])

$$U^{-1}(\kappa)S(p,\kappa)^{-1}B^\sharp(\mathbf{f},p,\kappa)S(p,\kappa)U(\kappa) = a^\sharp(\mathbf{f}), \quad \mathbf{f} \in \mathcal{W}. \tag{3. 1}$$

Concretely $S(p,\kappa)$ is given by

$$S(p,\kappa) = \exp\left(\sum_{i,j=1}^N \frac{e\hbar}{\kappa^2}p_\mu^i\left\{a\left(\oplus_{r=1}^{d-1}\frac{e_\mu^r D_{ij}^{-1}(0,\kappa)\hat\rho_j}{\sqrt{2\hbar c^3\omega^3}}\right) - a^\dagger\left(\oplus_{r=1}^{d-1}\frac{e_\mu^r D_{ij}^{-1}(0,\kappa)\hat\rho_j}{\sqrt{2\hbar c^3\omega^3}}\right)\right\}\right).$$

Theorem 3.1 *Suppose $\vec{\rho} \in \widetilde{P}$. Then putting $S(p,\kappa)U(\kappa) = \mathcal{U}(p,\kappa)$, we see that $\mathcal{U}(p,\kappa)$ maps $D(H_b)$ onto itself with*

$$\mathcal{U}(p,\kappa)H_{\vec{\rho}}(p,\kappa)\mathcal{U}^{-1}(p,\kappa) = \kappa H_b + E(p,\kappa), \tag{3. 2}$$

where

$$\begin{aligned}
E(p,\kappa) &= \frac{\hbar^2}{2m}\sum_{i=1}^{N}\sum_{\mu=1}^{d}\left(\kappa p_\mu^i + \kappa\widetilde{p}_\mu^i(\kappa)\right)^2 + \square(\kappa), \\
\widetilde{p}_\mu^i(\kappa) &= \sum_{j=1}^{N} p_\nu^j \Delta_{\nu\mu}^{ji}(\kappa), \\
\Delta_{\nu\mu}^{ji}(\kappa) &= \frac{1}{\kappa^3}\frac{e^2}{2c^2}\sum_{k=1}^{N}\sum_{r,s=1}^{d-1}\left\langle \frac{e_\nu^r D_{jk}^{-1}(0,\kappa)\hat{\rho}_k}{\sqrt{\omega^3}}, \left(I + \mathbf{W}_-(\kappa)\mathbf{W}_+^{-1}(\kappa)\right)^{(r,s)}\frac{e_\mu^s\hat{\rho}_i}{\sqrt{\omega}}\right\rangle_{L^2(\mathbb{R}^d)}, \\
\square(\kappa) &= \frac{e^2\hbar}{4mc}\sum_{i=1}^{N}\sum_{r,s=1}^{d-1}\left\langle \frac{e_\mu^r\hat{\rho}_i}{\sqrt{\omega}}, \left(I - \mathbf{W}_-(\kappa)\mathbf{W}_+^{-1}(\kappa)\right)^{(r,s)}\frac{e_\mu^s\hat{\rho}_i}{\sqrt{\omega}}\right\rangle_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Proof: For simplicity, we omit the symbol κ. Put $\mathcal{U}(p)\Omega \equiv \Omega(p)$. From [6,Proposition 2.4, Lemma 5.9] it follows that $\Omega(p) \in D(H_b)$. Then $\Omega(p) \in D(B(\mathbf{f},p))$. By virtue of Corollary 2.10 and (3.1), we can see that for all $\mathbf{f} \in \mathcal{W}$

$$B(\mathbf{f},p)\exp\left(i\frac{t}{\hbar}H_{\vec{\rho}}(p)\right)\Omega(p) = 0. \tag{3. 3}$$

The equation (3.3) implies that there exists a positive constant $E(p)$ such that

$$\exp\left(i\frac{t}{\hbar}H_{\vec{\rho}}(p)\right)\Omega(p) = \exp\left(i\frac{t}{\hbar}E(p)\right)\Omega(p). \tag{3. 4}$$

Hence from Corollary 2.10, (3.1), (3.4) and the denseness of

$$\mathcal{L}\left\{ B^\dagger(\mathbf{f}_1)....B^\dagger(\mathbf{f}_n)\Omega(p), \Omega(p) \middle| \mathbf{f}_j \in \mathcal{W}, j = 1,...,n, n \geq 1\right\},$$

one can get (3.2)(we refer to [6,Lemma 5.12]). Noting that ([6,Lemma 2.2])

$$a\left(\oplus_{r=1}^{d-1}\frac{\sqrt{\hbar}e_\mu^r\hat{\rho}_i}{\sqrt{2c\omega}}\right)\Omega(p) = \left\{-\widetilde{p}_\mu^i - a^\dagger\left(\oplus_{r=1}^{d-1}\sum_{s=1}^{d-1}\left(\mathbf{W}_-\mathbf{W}_+^{-1}\right)^{(r,s)}\frac{\sqrt{\hbar}e_\mu^s\hat{\rho}_i}{\sqrt{2c\omega}}\right)\right\}\Omega(p),$$

one can easily get $E(p)$ by

$$E(p)=\frac{<H_{\vec{\rho}}(p)\Omega(p),\Omega>_{\mathcal{F}(\mathcal{W})}}{<\Omega(p),\Omega>_{\mathcal{F}(\mathcal{W})}}.$$

This completes the proof. $\square$

The positive constant $E(p,\kappa)$ can be rewritten as

$$E(p,\kappa)=\frac{\kappa^2\hbar^2}{2m}p^2+E^{REN}(p,\kappa)+\widetilde{E}(p,\kappa),$$

where

$$\begin{aligned}\widetilde{E}(p,\kappa) &= \frac{\kappa^2\hbar^2}{2m}\sum_{i,j=1}^{N}\sum_{\mu,\nu=1}^{d}p_\mu^i b_{\mu\nu}^{ij}(\kappa)p_\nu^j, \qquad (3.\ 5)\\ b_{\mu\nu}^{ij}(\kappa) &= \sum_{k=1}^{N}\sum_{\alpha=1}^{d}\left(\frac{\Delta_{\nu\alpha}^{jk}(\kappa)+\overline{\Delta_{\nu\alpha}^{jk}(\kappa)}}{2}\right)\left(\frac{\Delta_{\mu\alpha}^{ik}(\kappa)+\overline{\Delta_{\mu\alpha}^{ik}(\kappa)}}{2}\right),\\ E^{REN}(p,\kappa) &= E(p,\kappa)-\frac{\kappa^2\hbar^2}{2m}p^2-\widetilde{E}(p,\kappa).\end{aligned}$$

Let $M(K)$ be the set of $K\times K$ complex matrices. Note that since $(b_{\mu\nu}^{ij}(\kappa))_{1\le i,j\le N,1\le\mu,\nu\le d}\in M(N)\otimes M(d)\cong M(dN)$ is nonnegative and symmetric, we have $\widetilde{E}(p,\kappa)\ge 0$ for $p\in\mathbb{R}^{dN}$. We define $H_{\vec{\rho}}{}^{REN}(p,\kappa)$ and $\widetilde{H_{\vec{\rho}}}(p,\kappa)$ by

$$\begin{aligned}H_{\vec{\rho}}{}^{REN}(p,\kappa) &= -E^{REN}(p,\kappa)+\kappa H_b+\frac{1}{2m}\sum_{j=1}^{N}\sum_{\mu=1}^{d}\left(-2\kappa e\hbar p_\mu^j A_\mu(\rho_j)+e^2A_\mu(\rho_j)^2\right),\\ \widetilde{H_{\vec{\rho}}}(p,\kappa) &= \widetilde{E}(p,\kappa)+\kappa H_b.\end{aligned}$$

Then one can see that

$$\begin{aligned}H_{\vec{\rho}}{}^{REN}(\kappa) &\equiv \mathbf{F}^{-1}\left(\int_{\mathbb{R}^{dN}}^{\oplus}H_{\vec{\rho}}^{REN}(p,\kappa)dp\right)\mathbf{F}\\ &= -E^{REN}(D,\kappa)\otimes I+\kappa I\otimes H_b\\ &\quad+\frac{1}{2m}\sum_{j=1}^{N}\sum_{\mu=1}^{d}\left(-2\kappa e\hbar iD_\mu^j\otimes A_\mu(\rho_j)+e^2I\otimes A_\mu(\rho_j)^2\right),\\ \widetilde{H_{\vec{\rho}}}(\kappa) &\equiv \mathbf{F}^{-1}\left(\int_{\mathbb{R}^{dN}}^{\oplus}\widetilde{H}(p)dp\right)\mathbf{F}\\ &= \widetilde{E}(D,\kappa)\otimes I+\kappa I\otimes H_b,\end{aligned}$$

where $E^{REN}(D,\kappa)$ and $\widetilde{E}(D,\kappa)$ are pseudo differential operators on $L^2(\mathbb{R}^{dN})$ with symbols $E^{REN}(p,\kappa)$ and $\widetilde{E}(p,\kappa)$ respectively.

Theorem 3.2 *Suppose $\vec{\rho} \in \widetilde{P}$. Then $H_{\vec{\rho}}^{REN}(\kappa)$ and $\widetilde{H_{\vec{\rho}}}(\kappa)$ are essentially self-adjoint on any core of H_0 and bounded from below.*

Proof: By the definition of $E^{REN}(D,\kappa)$ and $\widetilde{E}(D,\kappa)$, $H_{\vec{\rho}}^{REN}(\kappa)$ and $\widetilde{H_{\vec{\rho}}}(\kappa)$ are symmetric. For $f \in D(-\Delta)$, there exist $d_1(\kappa)$ and $d_2(\kappa)$ such that

$$||\widetilde{E}(D,\kappa)f||_{L^2(\mathbb{R}^{dN})} \leq d_1(\kappa)||-\Delta f||_{L^2(\mathbb{R}^{dN})},$$
$$||E^{REN}(D,\kappa)f||_{L^2(\mathbb{R}^{dN})} \leq d_2(\kappa)||-\Delta f||_{L^2(\mathbb{R}^{dN})}.$$

Hence, similar to the proof of Theorem 2.2, the Nelson commutator theorem yields desired results. □

Remark 3.3 *Write*

$$E(p,\kappa) = \frac{\hbar^2\kappa^2}{2m}p^2 + \sum_{\mu=1}^{d}\sum_{i=1}^{N}\frac{\hbar^2\kappa^2}{m}p_\mu^i\widetilde{p}_\mu^i(\kappa) + \sum_{\mu=1}^{d}\sum_{i=1}^{N}\frac{\hbar^2\kappa^2}{2m}\widetilde{p}_\mu^i(\kappa)^2 + \Box(\kappa). \qquad (3.\ 6)$$

Then the first and second terms on the right hand side of (3.6) diverge as $\kappa \to \infty$ for $p \neq 0$, but the rest terms not. Actually we see that

$$\begin{aligned}\lim_{\kappa\to\infty}\frac{\hbar^2\kappa^2}{2m}\sum_{\mu=1}^{d}\sum_{i=1}^{N}\widetilde{p}_\mu^i(\kappa)^2 &= \frac{1}{2m}\left(\frac{e^2}{2mc^2}\right)\left(\frac{d-1}{d}\right)^2\sum_{\alpha=1}^{d}\sum_{k=1}^{N}\left(\sum_{j=1}^{N}\hbar p_\alpha^j\left\langle\frac{\hat{\rho}_j}{\sqrt{\omega^3}},\frac{\hat{\rho}_k}{\sqrt{\omega}}\right\rangle_{L^2(\mathbb{R}^d)}\right)^2,\\ &\equiv E^\infty(p).\end{aligned}$$

Then, by (3.2), concerning an asymptotic behavior of $H_{\vec{\rho}}(\kappa)$ as $\kappa \to \infty$, we should subtract the first and second terms in the right hand side of (3.6) from the original Hamiltonian $H_{\vec{\rho}}(\kappa)$. However one can not say that $\widetilde{p}_\mu^i(\kappa)^2$ is real and nonnegative for any $p \in \mathbb{R}^{dN}$. To guarantee the nonnegative self-adjointness of the Hamiltonian $H_{\vec{\rho}}^{REN}(p,\kappa)$ with the divergence terms subtracted, we should define $\widetilde{E}(p,\kappa)$ such as (3.5). In this sense, we may say that the operator $H_{\vec{\rho}}^{REN}(\kappa)$ has an interpretation of the Hamiltonian $H_{\vec{\rho}}(\kappa)$ with the infinite self-energy of the nonrelativistic particles subtracted.

We define

$$\mathcal{U}(\kappa) \;=\; \mathbf{F}^{-1}\left(\int_{\mathbb{R}^{dN}}^{\oplus} \mathcal{U}(\kappa, p)dp\right)\mathbf{F}.$$

Then we have the following theorem.

Theorem 3.4 *Suppose that $\vec{\rho} \in \widetilde{P}$. Then*

$$\begin{aligned} s-\lim_{\kappa\to\infty} \mathcal{U}(\kappa) &= \exp\left(\sum_{j=1}^{N} \frac{e\hbar}{m} D_{\mu}^{j} \otimes \left\{ a\left(\oplus_{r=1}^{d-1} \frac{e_{\mu}^{r}\hat{\rho}_{j}}{\sqrt{2\hbar c^{3}\omega^{3}}}\right) - a^{\dagger}\left(\oplus_{r=1}^{d-1} \frac{e_{\mu}^{r}\hat{\rho}_{j}}{\sqrt{2\hbar c^{3}\omega^{3}}}\right)\right\}\right), \\ &\equiv \mathcal{U}(\infty). \end{aligned} \tag{3. 7}$$

Proof: From [6,Theorem 3.11] it follows (3.7). □

We take scalar potentials V to be real-valued measurable functions on $\mathbb{R}^{dN}$ and put

$$C_{\kappa}(V) = \mathcal{U}^{-1}(\kappa)(V \otimes I)\mathcal{U}(\kappa), \quad C(V) = \mathcal{U}^{-1}(\infty)(V \otimes I)\mathcal{U}(\infty). \tag{3. 8}$$

We introduce conditions $(\mathbf{V-1})$ and $(\mathbf{V-2})$ as follows.

(V-1) For sufficiently large $\kappa > 0$, $D(\widetilde{E}(D,\kappa)) \subset D(V)$ and for $\lambda > 0$, $V(\widetilde{E}(D,\kappa)+\lambda)^{-1}$ is bounded with

$$\lim_{\lambda\to\infty} ||V(\widetilde{E}(D,\kappa)+\lambda)^{-1}|| = 0, \tag{3. 9}$$

where the convergence is uniform in sufficiently large $\kappa > 0$.

(V-2) For $\lambda > 0$, $V(\widetilde{E}(D,\kappa)+\lambda)^{-1}$ is strongly continuous in κ and

$$s-\lim_{\kappa\to\infty} V(\widetilde{E}(D,\kappa)+\lambda)^{-1} = V(E^{\infty}(D)+\lambda)^{-1}.$$

The condition (3.9) yields that, by the Kato-Rellich theorem and commutativity of $\mathcal{U}(\kappa)$ and $(\widetilde{E}(D,\kappa)+\lambda)^{-1}$, operators $\widetilde{E}(D,\kappa)\otimes I + C_{\kappa}(V)$ are essentially self-adjoint on any core of $D(\widetilde{E}(D,\kappa)\otimes I)$ and uniformly bounded from below in sufficiently large $\kappa > 0$. Moreover since $I \otimes H_b$ is nonnegative and commute with $\widetilde{E}(D,\kappa)\otimes I$, one can see that

$$\widetilde{H}_{\vec{\rho}}(V,\kappa) \equiv \widetilde{E}(D,\kappa)\otimes I + C_{\kappa}(V) + \kappa I \otimes H_b$$

is essentially self-adjoint on any core of $D(\tilde{E}(D,\kappa)\otimes I+\kappa I\otimes H_b)$ and uniformly bounded from below in sufficiently large $\kappa>0$. In particular, $D(H_0)$ is a core of $\widetilde{H_{\vec{p}}}(V,\kappa)$. Put

$$H_{\vec{p}}^{REN}(V,\kappa)\equiv H_{\vec{p}}^{REN}(\kappa)+V\otimes I.$$

Theorem 3.5 *Let $\vec{p}\in\tilde{P}$. Suppose that V satisfies $(\mathbf{V-1})$ and $(\mathbf{V-2})$. Then, for sufficiently large $\kappa>0$, the operator $H_{\vec{p}}^{REN}(V,\kappa)$ is essentially self-adjoint on $D(H_0)$ and bounded from below uniformly in sufficiently large $\kappa>0$. Moreover the unitary operator $\mathcal{U}(\kappa)$ maps $D(H_0)$ onto itself and for $z\in\mathbb{C}\setminus\mathbb{R}$ or $z<0$ with $|z|$ sufficiently large,*

$$\left(H_{\vec{p}}^{REN}(V,\kappa)-z\right)^{-1}=\mathcal{U}(\kappa)\left(\widetilde{H_{\vec{p}}}(V,\kappa)-z\right)^{-1}\mathcal{U}^{-1}(\kappa). \tag{3. 10}$$

Proof: Since $\mathcal{U}(\kappa)$ maps $D(I\otimes H_b)$ onto itself (see Theorem 3.1) and $-\Delta\otimes I$ commutes with $\mathcal{U}(\kappa)$ on $D(-\Delta\otimes I)$, $\mathcal{U}(\kappa)$ maps $D(H_0)$ onto itself. Put

$$S_0^\infty(\mathbb{R}^{dN})=\left\{f\in L^2(\mathbb{R}^{dN})|\hat{f}\in C_0^\infty(\mathbb{R}^{dN})\right\}.$$

At first , by Theorem 3.1, we see that for $\Phi\in S_0^\infty(\mathbb{R}^{dN})\hat{\otimes}D(H_b)$,

$$H_{\vec{p}}^{REN}(V,\kappa)\Phi=\mathcal{U}(\kappa)\widetilde{H_{\vec{p}}}(V,\kappa)\mathcal{U}^{-1}(\kappa)\Phi. \tag{3. 11}$$

By a limiting argument we can extend (3.11) to $\Phi\in D(H_0)$. Since $D(H_0)$ is a core of $\widetilde{H_{\vec{p}}}(V,\kappa)$ and $\mathcal{U}(\kappa)$ maps $D(H_0)$ onto itself, the right hand side of (3.11) is essentially self-adjoint on $D(H_0)$. So is the left hand side of (3.11). (3.10) can be easily shown. □

We want to consider a scaling limit of $H_{\vec{p}}^{REN}(V,\kappa)$ as $\kappa\to\infty$. In [5], a general theory of the strong resolvent limit of self-adjoint operators including abstract versions like as the self-adjoint operator $\widetilde{H_{\vec{p}}}(V,\kappa)$ has been established. We shall apply the theory in [5] with a little modification. Let V satisfy $(\mathbf{V-1})$. Then since $D(C(V))\supset D(-\Delta)\hat{\otimes}D(H_b)$, one can define, for $\Phi\in\mathcal{F}(\mathcal{W})$ and $\Psi\in D(H_b)$, a symmetric operator $E_{\Phi,\Psi}(C(V))$ with $D(E_{\Phi,\Psi}(C(V))=D(-\Delta)$ by

$$\langle f,E_{\Phi,\Psi}(C(V))g\rangle_{L^2(\mathbb{R}^{dN})} = \langle f\otimes\Phi,C(V)(g\otimes\Psi)\rangle_{\mathcal{F}},\quad f\in L^2(\mathbb{R}^{dN}),g\in D(-\Delta).$$

In particular, we call $E_{\Omega,\Omega}(C(V)) \equiv E_\Omega(C(V))$ "the partial expectation of $C(V)$ with respect to Ω"([5, Section II]).

Theorem 3.6 *Let $\vec{p} \in \widetilde{P}$. Suppose that V satisfies the conditions* $(\mathbf{V-1})$ *and* $(\mathbf{V-2})$. *Then for $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ sufficiently large,*

$$s - \lim_{\kappa\to\infty}(H_{\vec{p}}{}^{REN}(V,\kappa) - z)^{-1} = \mathcal{U}(\infty)\left\{(E^\infty(D) + E_\Omega(C(V)) - z)^{-1} \otimes P_0\right\}\mathcal{U}^{-1}(\infty), \tag{3. 12}$$

where P_0 is the projection from $\mathcal{F}(\mathcal{W})$ to the one dimensional subspace $\{\alpha\Omega|\alpha \in \mathbb{C}\}$.

Proof: By $(\mathbf{V-1})$ and $(\mathbf{V-2})$, we see that

(V-1)' For sufficiently large $\kappa > 0$, $D(\widetilde{E}(D,\kappa)) \subset D(C_\kappa(V))$ and for $\lambda > 0$, $C_\kappa(V)(\widetilde{E}(D,\kappa) + \lambda)^{-1}$ is bounded with

$$\lim_{\lambda\to\infty} ||C_k(V)(\widetilde{E}(D,\kappa) + \lambda)^{-1}|| = 0,$$

where the convergence is uniform in sufficiently large $\kappa > 0$.

(V-2)' For $\lambda > 0$, $C_\kappa(V)(\widetilde{E}(D,\kappa) + \lambda)^{-1}$ is strongly continuous in κ and

$$s - \lim_{\kappa\to\infty} C_k(V)(\widetilde{E}(D,\kappa) + \lambda)^{-1} = C(V)(E^\infty(D) + \lambda)^{-1}.$$

From [5,Section II], $(\mathbf{V-1})'$ and $(\mathbf{V-2})'$ imply that

$$s - \lim_{\kappa\to\infty}\left(\widetilde{H_{\vec{p}}}(V,\kappa) - z\right)^{-1} = (E^\infty(D) + E_\Omega(C(V)) - z)^{-1} \otimes P_0.$$

Thus by Theorems 3.4 and 3.5, we get (3.12). □

We want to see $E_\Omega(C(V))$ more explicitly. For $\vec{p} \in \widetilde{P}$, let $\widetilde{\Delta}^\infty = (\widetilde{\Delta}^\infty_{ij})_{1\le i,j\le d}$, where $\widetilde{\Delta}^\infty_{ij}$ is defined in (1.3). Let $\mathbf{I}_{d\times d}$ denote $d \times d$-identity matrix. Since $\Delta^\infty \equiv \widetilde{\Delta}^\infty \otimes \mathbf{I}_{d\times d} \in$

$M(N) \otimes N(d) \cong M(dN)$ is a nonnegative symmetric matrix, there exist unitary matrices $\mathbf{T} \in M(dN)$ so that

$$\mathbf{T}\Delta^{\infty}\mathbf{T}^{-1} = \begin{pmatrix} \lambda_1 \mathbf{I}_{d\times d} & & & \\ & \lambda_2 \mathbf{I}_{d\times d} & & \\ & & \ddots & \\ & & & \lambda_N \mathbf{I}_{d\times d} \end{pmatrix}, \tag{3. 13}$$

where $\lambda_1 \geq \lambda_2 ... \geq \lambda_N \geq 0$.

Theorem 3.7 *Suppose $\lambda_1 \geq \lambda_2 ... \geq \lambda_K > 0$, $\lambda_{K+1} = ... = \lambda_N = 0$ and fix a unitary operator $\mathbf{T}$ in (3.13). Let $x = (x_1, ..., x_N)$, $x_j \in \mathbb{R}^d$, $j = 1, ..., N$ and V satisfy*

$$\int_{\mathbb{R}^{dK}} dy_1 ... dy_K |V| \circ \mathbf{T}^{-1} (y_1, ..., y_K, (\mathbf{T}x)_{K+1}, ..., (\mathbf{T}x)_N) \exp\left(-\frac{\sum_{j=1}^{K} |(\mathbf{T}x)_j - y_j|^2}{2\lambda_1 ... \lambda_K} \right) < \infty. \tag{3. 14}$$

Moreover we suppose that the left hand side of (3.14) is locally bounded. Then the partial expectation $E_\Omega(C(V))$ is given by a multiplication operator V_{eff};

$$\begin{aligned} V_{eff}(x) &= (2\pi\lambda_1 ... \lambda_K)^{-\frac{d}{2}} \int_{\mathbb{R}^{dK}} dy_1 ... dy_K V \circ \mathbf{T}^{-1} (y_1, ..., y_K, (\mathbf{T}x)_{K+1}, ..., (\mathbf{T}x)_N) \\ & \quad \times \exp\left(-\frac{\sum_{j=1}^{K} |(\mathbf{T}x)_j - y_j|^2}{2\lambda_1 ... \lambda_K} \right). \end{aligned}$$

In particular, in the case where $\tilde{\Delta}^{\infty}$ is non-degenerate, V_{eff} is given by

$$V_{eff}(x) = (2\pi \det \tilde{\Delta}^{\infty})^{-\frac{d}{2}} \int_{\mathbb{R}^{dN}} V(y) \exp\left(-\frac{|x-y|^2}{2 \det \tilde{\Delta}^{\infty}} \right) dy.$$

Proof: Suppose $V \in \mathcal{S}(\mathbb{R}^{dN})$, which is the set of the rapidly decreasing infinitely continuously differentiable functions on $\mathbb{R}^{dN}$. Then the direct calculation shows that for $f, g \in L^2(\mathbb{R}^{dN})$

$$\langle f, E_\Omega(C(V))g \rangle_{L^2(\mathbb{R}^{dN})} = \langle f, V_{eff} g \rangle_{L^2(\mathbb{R}^{dN})}. \tag{3. 15}$$

We next consider the case where V is bounded. In this case we can approximate V by a sequence $\{V_n\}_{n=1}^{\infty}$, $V_n \in \mathcal{S}(\mathbb{R}^{dN})$, such that

$$||V - V_n||_\infty \to 0 \ (n \to \infty),$$

where $||\cdot||_\infty$ denotes the sup norm. Then we have

$$E_\Omega(C(V_n)) \to E_\Omega(C(V)) \ (n \to \infty),$$

strongly. Moreover $(V_n)_{eff}(x) \to V_{eff}(x)$ for all $x \in \mathbb{R}^{dN}$. Thus for $f, g \in L^2(\mathbb{R}^{dN})$, (3.15) follows for such V. Finally, let V satisfy (3.14). Define

$$V_n = \begin{cases} V(x) & |V(x)| \le n, \\ n & |V(x)| > n. \end{cases}$$

Hence for $f \in L^2(\mathbb{R}^{dN})$ and $g \in D(-\Delta)$, we have

$$\langle f, E_\Omega(C(V_n))g\rangle_{L^2(\mathbb{R}^{dN})} \to \langle f, E_\Omega(C(V))g\rangle_{L^2(\mathbb{R}^{dN})} \ (n \to \infty).$$

On the other hand, since the left hand side of (3.14) is locally bounded, we can see that for $f \in C_0^\infty(\mathbb{R}^{dN})$ and $g \in D(-\Delta)$,

$$\langle f, (V_n)_{eff} g\rangle_{L^2(\mathbb{R}^{dN})} \to \langle f, V_{eff} g\rangle_{L^2(\mathbb{R}^{dN})} \ (n \to \infty),$$

which completes the proof. □

Remark 3.8 *In Theorem 3.7, in the case where $\widetilde{\Delta}^\infty$ is non-degenerate, since the left hand side of (3.14) is continuous in $x \in \mathbb{R}^{dN}$, it is necessarily locally bounded.*

We call V_{eff} "the effective potential with respect to V". We give some examples of scalar potentials V and ultraviolet cut off functions ρ.

Example 3.9 ([non-degenerate case]) *Let*

$$\widetilde{\Delta}_{ij}^\infty = \delta_{ij}\frac{1}{2}\frac{d-1}{d}\left(\frac{\hbar}{mc}\right)^2 \frac{e^2}{\hbar c}\int_{\mathbb{R}^d} dk \frac{\hat{\rho}_i(k)^2}{\omega(k)^3}.$$

Then there exist positive constants δ_1 and δ_2 such that for sufficiently large $\kappa > 0$

$$\delta_1|p|^2 \le \widetilde{E}(p,\kappa) \le \delta_2|p|^2. \tag{3. 16}$$

Let $d = 3$ and V be the Coulomb potential;

$$V(x_1, ..., x_N) \quad = \quad -\sum_{j=1}^{N} \frac{\alpha_j}{|x_j|} + \sum_{i \neq j} \frac{\beta_{ij}}{|x_i - x_j|}, \quad \alpha_j \geq 0, \beta_{ij} \geq 0.$$

Then V is the Kato class potential ([10],Theorem X.16). Namely for any $\epsilon > 0$, there exists $b \geq 0$ such that $D(V) \supset D(-\Delta)$ and

$$||V\Phi||_{L^2(\mathbb{R}^{3N})} \leq \epsilon|| - \Delta\Phi||_{L^2(\mathbb{R}^{3N})} + b||\Phi||_{L^2(\mathbb{R}^{3N})}. \tag{3. 17}$$

Together with (3.16) and (3.17), one can see that V satisfies $(\mathbf{V-1})$, $(\mathbf{V-2})$ and for any $t > 0$

$$\int_{\mathbb{R}^{3d}} |V|(y) e^{-t|x-y|^2} dy < \infty.$$

Then the scaling limit of the Pauli-Fierz Hamiltonian with the Coulomb potential exists and has the effective potential given by

$$\begin{aligned} V_{eff}(x) &= (2\pi\gamma)^{-\frac{3}{2}} \int_{\mathbb{R}^{3N}} V(y) e^{-\frac{|x-y|^2}{2\gamma}} dy, \\ \gamma &= \left\{ \frac{1}{3} \left(\frac{\hbar}{mc} \right)^2 \frac{e^2}{\hbar c} \right\}^N \Pi_{j=1}^N \left(\int_{\mathbb{R}^3} dk \frac{\hat{\rho}_j^2(k)}{\omega(k)^3} \right). \end{aligned}$$

Moreover

$$E^\infty(D) = -\frac{1}{2m} \left(\frac{e^2}{2mc^2} \right) \left(\frac{d-1}{d} \right)^2 \otimes_{j=1}^N \left|\left| \frac{\hat{\rho}_j}{\omega} \right|\right|^4 \hbar^2 \Delta_{j,},$$

where Δ_j, $j = 1, ..., N$, is the Laplacian in $L^2(\mathbb{R}^d)$.

Example 3.10 ([non-degenerate case]) *Let $\widetilde{\Delta}^\infty$ be non-degenerate and V be the Phillips perturbation with respect to $-\Delta$([12]). Then (3.16) holds with some δ_1 and δ_2. Hence V satisfies $(\mathbf{V-1})$, $(\mathbf{V-2})$ and for any $t > 0$*

$$\int_{\mathbb{R}^{dN}} |V|(y) e^{-t|x-y|^2} dy < \infty.$$

Hence the scaling limit of the Pauli-Fierz Hamiltonian with Phillips perturbation exists and has the effective potential in Theorem 3.7.

Example 3.11 ([degenerate case]) *Let V be a real-valued bounded function. Then V satisfies the conditions $(V-1)$ and $(V-2)$. Hence the scaling limit of the Pauli-Fierz Hamiltonian with the scalar potential V exists for all $\vec{\rho} \in \tilde{P}$.*

Example 3.12 ([degenerate case]) *Let $\rho_i = \rho$, $i = 1, ..., N$ and V satisfy* $(\mathbf{V}-\mathbf{1})$, $(\mathbf{V}-\mathbf{2})$ *and the assumption stated in Theorem 3.7. Then* rank $\tilde{\Delta}^\infty = 1$ *and the non-zero eigenvalue C is given by*

$$C = \frac{N}{2}\frac{d-1}{d}\left(\frac{\hbar}{mc}\right)^2 \frac{e^2}{\hbar c}\int_{\mathbb{R}^d} dk \frac{\hat{\rho}(k)^2}{\omega(k)^3}.$$

Thus the scaling limit of the Pauli-Fierz Hamiltonian with the ultraviolet cut-off function ρ exists and has the following effective potential:

$$V_{eff}(x) = (2\pi C)^{-\frac{d}{2}} \int_{\mathbb{R}^d} dy_1 V \circ \mathbf{T}^{-1}(y_1, (\mathbf{T}x)_2, ..., (\mathbf{T}x)_N) \exp\left(-\frac{|(\mathbf{T}x)_1 - y_1|^2}{2C}\right).$$

Moreover

$$E^\infty(D) = -\frac{1}{2m}\left(\frac{N^2 e^2}{2mc^2}\right)\left(\frac{d-1}{d}\right)^2 \left\|\frac{\hat{\rho}}{\omega}\right\|^4 \hbar^2 \Delta,$$

where Δ is the Laplacian in $L^2(\mathbb{R}^{dN})$.

4 CONCLUDING REMARK

As is seen in Theorem 3.7, the effective potential V_{eff} is characterized by the matrix-valued functional $\tilde{\Delta}^\infty = \tilde{\Delta}^\infty(\vec{\rho})$, which has the following mathematical meaning; putting

$$\mathcal{U}(\infty)(x_i \otimes I)\mathcal{U}^{-1}(\infty) - x_i \otimes I \equiv \Delta x_i, \quad i = 1, ..., N,$$

we see that the partial expectation of $\Delta x_i \Delta x_j$ with respect to Ω is as follows;

$$E_\Omega[(\Delta x_i \Delta x_j)] = \tilde{\Delta}^\infty_{ij}(\vec{\rho})I.$$

In one-nonrelativistic particle case, A.Arai [5] shows that the partial expectation $E_{\Omega}[(\Delta x)^2]$ with respect to Ω may be interpreted as the mean square fluctuation in position of one-nonrelativistic particle ([2]). In this sense, $\widetilde{\Delta}_{ij}^{\infty}(\vec{\rho})$ may also be interpreted as correlation of fluctuations in position of the i-th and the j-th nonrelativistic particles under the action of quantized radiation fields.

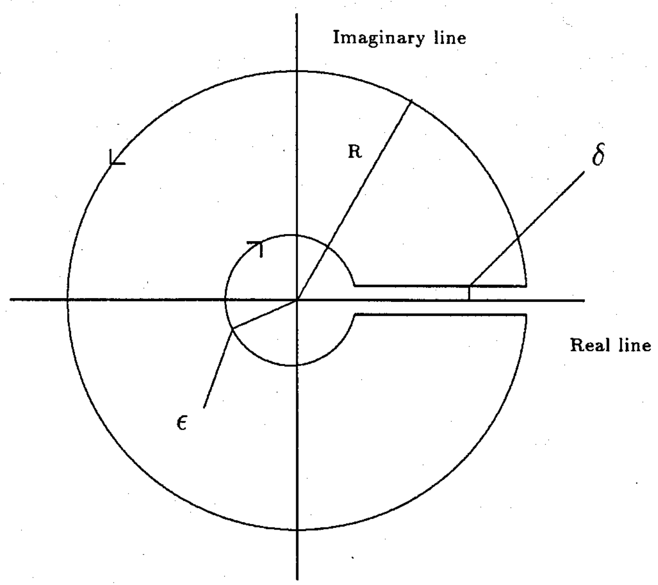

Figure 1: Cut Plane $\mathbb{C}_{R,\delta,\epsilon}$

5 APPENDIX

In this Appendix we prove Lemma 2.6. For simplicity, in this proof, we omit κ in notations and put

$$\lambda = \frac{e^2}{c^2}, \quad \widehat{G} = \omega^{1-\frac{d}{2}} G \omega^{1-\frac{d}{2}}, \quad \widehat{G}_t = \omega^{1-\frac{d}{2}} G_t \omega^{1-\frac{d}{2}}, t > 0.$$

(1) : This follows from the definition of $T_{\mu\nu}$ and $T^*_{\mu\nu}$, and Definition 2.1 (4).

(2) :For $f \in M_{-1}$,

$$\begin{aligned} \left\langle \omega^{-\frac{3}{2}} Q_i, \omega^{-\frac{1}{2}} d_{\nu\alpha} T_{\mu\nu} f \right\rangle_{L^2(\mathbb{R}^d)} &= \left\langle d_{\mu\alpha} \omega^{-\frac{3}{2}} Q_i, \omega^{-\frac{1}{2}} f \right\rangle_{L^2(\mathbb{R}^d)} \\ &\quad + \sum_{j=1}^{N} \left\langle d_{\nu\alpha} \omega^{-\frac{3}{2}} Q_i, \lambda \omega^{-\frac{1}{2}} Q_j \widehat{G} d_{\mu\nu} \hat{\rho}_j f \right\rangle_{L^2(\mathbb{R}^d)} \\ &= I + II \end{aligned}$$

Using (2.8), one can see that

$$II = \lim_{t \to 0} \lambda \sum_{j=1}^{N} \int \frac{\overline{Q}_i(k) Q_j(k) d_{\mu\nu}(k') d_{\nu\alpha}(k) \hat{\rho}_j(k') f(k')}{(k^2 - k'^2 + it) k^2} dk dk'$$

$$= \lim_{t\to 0}\lambda\sum_{j=1}^{N}\frac{d-1}{d}\mathbf{V}_d\int\frac{\sum_{k,l=1}^{N}D_{-ik}^{-1}(s)D_{+jl}^{-1}(s)\hat{\rho}_k(\sqrt{s})\hat{\rho}_l(\sqrt{s})s^{\frac{d}{2}-1}F_j(k')}{(s-k'^2+it)s}dsdk'$$

$$= \lim_{t\to 0}\frac{1}{2\pi i}\sum_{j=1}^{N}\int\frac{1}{(s-k'^2+it)s}\left(D_{-ij}^{-1}(s)-D_{+ji}^{-1}(s)\right)F_j(k')dsdk',$$

where $F_j(k')=d_{\mu\alpha}(k')\hat{\rho}_j(k')f(k')$. Using the contour integral on the cut plane $\mathbb{C}_{R,\delta,\epsilon}$ (Figure 1), by (2.5) and (2.6), we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_0^\infty\frac{1}{(s-k'^2+it)s}\left(D_{-ij}^{-1}(s)-D_{+ji}^{-1}(s)\right)ds\\
&=\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_\epsilon^\infty\frac{1}{(s-k'^2+it)s}\left(D_{-ij}^{-1}(s)-D_{+ji}^{-1}(s)\right)ds\\
&=\lim_{\epsilon\to 0}\lim_{R\to\infty,\delta\to 0}-\frac{1}{2\pi i}\int_{\mathbb{C}_{R,\delta,\epsilon}}\frac{D_{ij}^{-1}(z)}{(z-k'^2+it)z}dz-\frac{D_{ij}^{-1}(0)}{-k'^2+it}\\
&=-\frac{D_{ij}^{-1}(k'^2-it)}{k'^2-it}-\frac{D_{ij}^{-1}(0)}{-k'^2+it}.
\end{aligned}$$

Then

$$\begin{aligned}
II &= \lim_{t\to 0}\sum_{j=1}^{N}\int\frac{F_j(k')D_{ij}^{-1}(0)}{k'^2-it}-\frac{D_{ij}^{-1}(k'^2-it)F_j(k')}{k'^2-it}dk'\\
&= -\left\langle d_{\mu\alpha}\omega^{-\frac{3}{2}}Q_i,\omega^{-\frac{1}{2}}f\right\rangle_{L^2(\mathbb{R}^d)}+\left\langle d_{\mu\alpha}\omega^{-\frac{3}{2}}\sum_{j=1}^{N}D_{ij}^{-1}(0)\hat{\rho}_j,\omega^{-\frac{1}{2}}f\right\rangle_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Hence we get (2).

(3),(4) : They are direct calculations.

(5) : For $f,g\in M_0$,

$$\begin{aligned}
<T_{\mu\nu}^{*}d_{\nu\alpha}T_{\alpha\beta}f,g>_{L^2(\mathbb{R}^d)} &= <d_{\mu\beta}f,g>_{L^2(\mathbb{R}^d)}+\lambda\sum_{j=1}^{N}<d_{\nu\beta}f,Q_i\widehat{G}d_{\mu\nu}\hat{\rho}_i g>_{L^2(\mathbb{R}^d)}\\
&\quad+\lambda\sum_{j=1}^{N}<d_{\mu\alpha}Q_j\widehat{G}d_{\alpha\beta}\hat{\rho}_j f,g>_{L^2(\mathbb{R}^d)}\\
&\quad+\lambda^2\frac{d-1}{d}\sum_{i,j=1}^{N}<Q_i\widehat{G}d_{\alpha\beta}\hat{\rho}_i f,Q_j\widehat{G}d_{\mu\alpha}\hat{\rho}_j g>_{L^2(\mathbb{R}^d)}\\
&= I+II+III+IV.
\end{aligned}$$

Then,

$$
\begin{aligned}
IV &= \lim_{t\to 0}\lambda^2\sum_{i,j,k,l=1}^{N}\mathbf{V}_d\frac{d-1}{d}\int\frac{D^{-1}_{-ik}(s)D^{-1}_{+jl}(s)\hat\rho_k(\sqrt{s})\hat\rho_l(\sqrt{s})s^{\frac{d}{2}-1}F_{ij}(k',k'')}{(s-k'^2-it)(s-k''^2+it)}dsdk'dk'',\\
&\equiv \lim_{t\to 0} IV_t,
\end{aligned}
$$

where $F_{ij}(k',k'')=d_{\mu\alpha}(k'')\hat\rho_j(k'')d_{\alpha\beta}(k')\hat\rho_i(k')\bar f(k')g(k'')$. By using the cut plane integral method as in (2), we have

$$
\begin{aligned}
IV_t &= \sum_{i,j=1}^{N}\int\frac{-\lambda D^{-1}_{ij}(k'+it)F_{ij}(k',k'')}{k'^2-k''^2+2it}+\frac{-\lambda D^{-1}_{ij}(k''-it)F_{ij}(k',k'')}{k''^2-k'^2-2it}dk'dk''\\
&= -\lambda\sum_{i,j=1}^{N}\left\langle d_{\alpha\beta}f, D^{-1}_{ji}(\cdot+it)\hat\rho_i\widehat{G}_{2t}d_{\mu\alpha}\hat\rho_j g\right\rangle_{L^2(\mathbb{R}^d)}\\
&\quad -\lambda\sum_{i,j=1}^{N}\left\langle d_{\mu\alpha}D^{-1}_{ij}(\cdot+it)\hat\rho_j\widehat{G}_{2t}d_{\alpha\beta}\hat\rho_i f, g\right\rangle_{L^2(\mathbb{R}^d)}.
\end{aligned}
$$

By a limiting argument as $t\to 0$, we get

$$\lim_{t\to 0} IV_t = -II-III.$$

(6) : For $f,g\in M_0$,

$$
\begin{aligned}
\left\langle e^r_\mu T_{\mu\nu}d_{\nu\alpha}T_{\alpha\beta}e^s_\beta f, g\right\rangle_{L^2(\mathbb{R}^d)} &= <\delta_{rs}f,g>_{L^2(\mathbb{R}^d)} -\lambda<e^r_\beta\rho_j\widehat{G}\overline{Q}_j e^s_\beta f,g>_{L^2(\mathbb{R}^d)}\\
&\quad -\lambda<f, e^s_\mu\rho_j G\overline{Q}_j e^r_\mu g>_{L^2(\mathbb{R}^d)}\\
&\quad +\lambda^2<d_{\mu\beta}\rho_j\widehat{G}\overline{Q}_j e^s_\beta f, \rho_j\widehat{G}\overline{Q}_j e^r_\mu g>_{L^2(\mathbb{R}^d)}\\
&= I-II-III+IV.
\end{aligned}
$$

We see that

$$
\begin{aligned}
IV &= \lambda^2\lim_{\epsilon\to 0}\sum_{i,j=1}^{N}\mathbf{V}_d\frac{d-1}{d}\int\frac{\hat\rho_j(\sqrt{s})\hat\rho_i(\sqrt{s})s^{\frac{d}{2}-1}H_{ij}(k',k'')}{(s-k'^2-i\epsilon)(s-k''^2+i\epsilon)}dsdk'dk''\\
&= \lambda\lim_{\epsilon\to 0}\sum_{i,j=1}^{N}\int\left(\frac{D_{ij}(k'^2+i\epsilon)}{k'^2-k''^2+2i\epsilon}+\frac{D_{ij}(k''^2-i\epsilon)}{k''^2-k'^2-2i\epsilon}\right)H_{ij}(k',k'')dk'dk''\\
&= \lambda\lim_{\epsilon\to 0}\sum_{i,j=1}^{N}<f, e^s_\mu Q_j D_{ij}(\cdot+i\epsilon)\widehat{G}_{2\epsilon}\overline{Q}_i e^r_\mu g>_{L^2(\mathbb{R}^d)}\\
&\quad +\lambda\sum_{i,j=1}^{N}<e^r_\mu Q_i\overline{D}_{ij}(\cdot-i\epsilon)\widehat{G}_{2\epsilon}\overline{Q}_j e^s_\mu f, g>_{L^2(\mathbb{R}^d)},
\end{aligned}
$$

where $H_{ij}(k',k'') = Q_j(k')\overline{Q}_i(k'')e^s_\mu(k')e^r_\mu(k'')\bar{f}(k')g(k'')$. Note that $\sum_{j=1}^N Q_j D_{+ij} = \hat{\rho}_i$. Then

$$
\begin{aligned}
&= \lambda \sum_{i=1}^N < f, e^s_\mu \hat{\rho}_i \widehat{G}\overline{Q}_i e^r_\mu g >_{L^2(\mathbb{R}^d)} + \lambda \sum_{j=1}^N < e^r_\mu \hat{\rho}_j \widehat{G}\overline{Q}_j e^s_\mu f, g >_{L^2(\mathbb{R}^d)} \\
&= II + III.
\end{aligned}
$$

Hence we get the desired results. □

6 REFERENCES

[1]H.A.Bethe, The electromagnetic shift of energy levels, Phys.Rev.72,(1947)339-342.

[2] T.A.Welton, Some observable effects of the quantum mechanical fluctuations of the electromagnetic field, Phys.Rev.74(1948)1157-1167.

[3] A.Arai, A note on scattering theory in non-relativistic quantum electrodynamics, J.Phys.A.Math.Gen.16,(1983)49-70.

[4] A.Arai, Rigorous theory of spectra and radiation for a model in a quantum electrodynamics, J.Math.Phys.24(1983)1896-1910.

[5] A.Arai, An asymptotic analysis and its applications to the nonrelativistic limit of the Pauli-Fierz and a spin-boson model, J.Math.Phys.31(1990)2653-2663.

[6] F.Hiroshima, Scaling limit of a model in quantum electrodynamics, J.Math.Phys.34(1993)4478-4578.

[7] F.Hiroshima, Diamagnetic inequalities for a systems of nonrelativistic particles with a quantized radiation field. to appear Rev.Math.Phys.Vol.8 No.2.

[8] F.Hiroshima, Functional integral representation of a model in quantum electrodynamics, submitted to J.Funct.Anal..

[9] E.C.Titchmarsh, Theory of Fourier Integrals, 2nd.ed.(Oxford Univ.Press, Oxford,1948).

[10] M.Reed, B.Simon, Method of Modern Mathematical Physics II Fourier Analysis and

Self-Adjoint operator, Academic Press(1975).

[11] E.A. Berezin, "Second quantization", Academic Press(1966).

[12] E.B.Davies, Properties of the Green's functions of some Schrödinger operators, J.London Math.Soc.7,483-493(1976).