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**A REMARK ON ESTIMATES OF
BILINEAR FORMS OF GRADIENTS
IN HARDY SPACE**

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A REMARK ON ESTIMATES OF BILINEAR FORMS OF GRADIENTS IN HARDY SPACE

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§0. Introduction.

In a recent interesting paper[1] L.C.Evans and S.Müller established the estimate of local Hardy space norm of gradients ψ_{x_1}, ψ_{x_2} :

$$(0.1) \quad \|\phi\psi_{x_1}\psi_{x_2}\|_{h^1} + \|\phi(\psi_{x_1}^2 - \psi_{x_2}^2)\|_{h^1} \leq C(\|\psi_{x_1}\|_{L^2(B(0,R))}^2 + \|\psi_{x_2}\|_{L^2(B(0,R))}^2)$$

provided that

$$(0.2) \quad -\Delta\psi = \omega \geq 0 \text{ in } \mathbb{R}^2.$$

Here ϕ is in $C_0^\infty(\mathbb{R}^2)$ and the constants C and R depends only on ϕ ; h^1 is a local Hardy space defined in §1 and $B(x, R)$ denotes the closed ball of radius R centered at $x \in \mathbb{R}^2$. (Another proof based on harmonic analysis is given by Semmes[2].)

This estimate is useful in proving the existence of weak solutions for the initial value problem of the two-dimensional Euler equation when the vorticity of the initial value is nonnegative measure ([1] and Delort[3]). The assumption $\omega \geq 0$ in (0.2) is essential for the estimate (0.1); in fact, Evans and Müller[1] gave a counterexample for (0.1) when the condition $\omega \geq 0$ is violated. However, in their example the set where ω is nonnegative may be complicated.

In this paper we give another counterexample for (0.1) even when ω is odd in the second variable x_2 i.e. $\omega(x_1, x_2) = -\omega(x_1, -x_2)$ and $\omega(x_1, x_2) \geq 0$ for $x_2 \geq 0$. This suggests that it is difficult to extend weak solutions for the initial-boundary value problem of the Euler equation when the domain is a half space $\mathbb{R}_+^2$ even if initial value is nonnegative in $\mathbb{R}_+^2$.

To get our counterexample we construct a sequence ψ^ϵ of form $\psi^\epsilon(x) = \psi(x/\epsilon)$. A key observation is the existence of function ψ that satisfies

$$\int_{\mathbb{R}^2} \psi_{x_1}^2 dx \neq \int_{\mathbb{R}^2} \psi_{x_2}^2 dx$$

with $-\Delta\psi = \omega$, where $\omega \in C_0^\infty(\mathbb{R}^2)$ is odd in the second variable x_2 and $\omega \geq 0$ in $\mathbb{R}_+^2$, and $\psi \in H^1(\mathbb{R}^2)$; $H^1(\mathbb{R}^2)$ denotes the Sobolev space, i.e. the space of $f \in L^2(\mathbb{R}^2)$ with $f_{x_1}, f_{x_2} \in L^2(\mathbb{R}^2)$.

§1. Definition and Main Theorem.

We begin with definition of local Hardy space as in [1].

Definition 1.1. Let η be in $C_0^\infty(\mathbb{R}^n)$ with $\text{supp}\eta \subset B(0, 1)$ and $\int_{\mathbb{R}^n} \eta dx = 1$. For a function f in $L_{loc}^1(\mathbb{R}^n)$, f^{**} is defined by

$$(1.1) \quad f^{**}(x) = \sup_{0 < r < 1} \left| \int_{\mathbb{R}^n} \eta \left(\frac{x-y}{r} \right) f(y) dy \right|.$$

The local Hardy space $\mathcal{H}_{loc}^1$ is defined by

$$(1.2) \quad \mathcal{H}_{loc}^1(\mathbb{R}^n) = \{f \in L_{loc}^1(\mathbb{R}^n) \mid f^{**} \in L_{loc}^1(\mathbb{R}^n)\}.$$

We recall the normed local Hardy space h^1 defined by

$$(1.3) \quad h^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) \mid f^{**} \in L^1(\mathbb{R}^n)\}$$

with the norm

$$\|f\|_{h^1(\mathbb{R}^n)} = \|f^{**}\|_{L^1(\mathbb{R}^n)}.$$

Definition 1.2. For a function f in $C_0^\infty(\mathbb{R}^2)$, we define the operator $(-\Delta)^{-1}$ by

$$(1.4) \quad (-\Delta)^{-1} f(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(y) \log |x-y| dy.$$

Theorem 1.3. Let T and S be the spaces of form

$$\begin{aligned} T &= \{\omega \in C_0^\infty(\mathbb{R}^2) \mid \omega(x_1, x_2) \geq 0 \text{ for } x_2 \geq 0, \omega(x_1, x_2) = -\omega(x_1, -x_2)\}, \\ S &= \{(-\Delta)^{-1} \omega \mid \omega \in T\}. \end{aligned}$$

Then there exists a sequence $\{\psi^\epsilon\}_{0 < \epsilon < 1}$ such that

$$\sup_{0 < \epsilon < 1} \|\psi^\epsilon\|_{H^1(\mathbb{R}^2)} < \infty$$

and

$$(1.5) \quad \lim_{\epsilon \downarrow 0} \|\phi \{(\psi_{x_1}^\epsilon)^2 - (\psi_{x_2}^\epsilon)^2\}\|_{h^1(\mathbb{R}^2)} = \infty$$

where $\phi \in C_0^\infty(\mathbb{R}^2)$, $0 \leq \phi \leq 1$, $\phi|_{B(0, 1/8)} \equiv 1$ and $\text{supp}\phi \subset B(0, 1/2)$.

§2. Proof of Theorem

At first, we show a fundamental estimate in normed local Hardy space; this is an extension of a result of Evans and Müller[1].

Lemma 2.1. Assume that f is in $L^1(\mathbb{R}^n)$, and $\int_{\mathbb{R}^n} f(x)dx \neq 0$. Let $f^\epsilon = \frac{1}{\epsilon^n} f\left(\frac{x}{\epsilon}\right)$. Then $\|f^\epsilon\|_{L^1} = \|f\|_{L^1} < \infty$, and

$$(2.1) \quad \lim_{\epsilon \downarrow 0} \|\phi f^\epsilon\|_{h^1(\mathbb{R}^n)} = \infty$$

for a function ϕ in $C_0^\infty(\mathbb{R}^n)$ with $0 \leq \phi \leq 1$, $\phi|_{B(0,1/8)} \equiv 1$, and $\text{supp } \phi \subset B(0,1/4)$.

Proof. L^1 -estimate is easily obtained by scaling variables. To show the estimate (2.1), assume that the function η in (1.1) satisfies $0 \leq \eta \leq 1$ and $\eta|_{B(0,1/2)} \equiv 1$. Now we estimate the function $(\phi f^\epsilon)^{**}$:

$$(2.2) \quad \begin{aligned} (\phi f^\epsilon)^{**}(x) &= \sup_{0 < r < 1} \left| \frac{1}{r^n} \int_{\mathbb{R}^n} \eta\left(\frac{x-y}{r}\right) \phi(y) f^\epsilon(y) dy \right| \\ &= \sup_{0 < r < 1} \left| \frac{1}{r^n} \int_{\mathbb{R}^n} \eta\left(\frac{x-\epsilon z}{r}\right) \phi(\epsilon z) f(z) dz \right|. \end{aligned}$$

Now take a parameter $R > 0$, and let $r = 4|x|$. Then

$$(2.3) \quad \begin{aligned} (\phi f^\epsilon)^{**}(x) &\geq \frac{1}{4^n |x|^n} \left| \int_{B(0,R)} \eta\left(\frac{x-\epsilon z}{4|x|}\right) \phi(\epsilon z) f(z) dz \right| \\ &\quad - \frac{1}{4^n |x|^n} \left| \int_{\mathbb{R}^n \setminus B(0,R)} \eta\left(\frac{x-\epsilon z}{4|x|}\right) \phi(\epsilon z) f(z) dz \right| \\ &= I_1^\epsilon - I_2^\epsilon \text{ for } \epsilon R < |x| < 1/4. \end{aligned}$$

We show that $\lim_{\epsilon \downarrow 0} \|I_2^\epsilon\|_{L^1(B(0,1/4))} = 0$ and $\lim_{\epsilon \downarrow 0} \|I_1^\epsilon\|_{L^1(B(0,1/4))} = \infty$ to complete the proof. Firstly, we estimate the term I_2^ϵ :

$$\begin{aligned} I_2^\epsilon(x) &\leq \frac{1}{(4\epsilon R)^n} \int_{\mathbb{R}^n \setminus B(0,R)} |f(z)| dz = \frac{F(R)}{(4\epsilon R)^n} \\ &\text{with } F(R) = \int_{\mathbb{R}^n \setminus B(0,R)} |f(z)| dz. \end{aligned}$$

Since $F(R)$ is continuous, nonincreasing, and $\lim_{R \rightarrow \infty} F(R) = 0$, for sufficiently small ϵ , there exists $R = R(\epsilon)$ such that

$$(2.4) \quad \frac{\{F(R)\}^{1/2}}{R^n} = \epsilon^n.$$

By (2.4), we get

$$(2.5) \quad I_2^\epsilon(x) \leq \frac{\{F(R)\}^{1/2}}{4^n} \longrightarrow 0 \text{ as } \epsilon \downarrow 0$$

and get $\lim_{\epsilon \downarrow 0} \|I_2^\epsilon\|_{L^1(B(0,1/4))} = 0$.

Secondly, we estimate the term I_1^ϵ . As $\left| \frac{x-\epsilon z}{4|x|} \right| \leq \frac{|x|}{4|x|} + \frac{\epsilon R}{4|x|} \leq \frac{1}{2}$,

$$(2.6) \quad \begin{aligned} I_1^\epsilon &= \frac{1}{4^n |x|^n} \left| \int_{B(0,R)} \phi(\epsilon z) f(z) dz \right| \\ &= \frac{1}{4^n |x|^n} \left| \int_{B(0,R)} f(z) dz \right| \text{ for } \epsilon R \leq |x| \leq 1/4. \end{aligned}$$

for $\epsilon R < 1/8$.

Now let $\epsilon \downarrow 0$. For $\epsilon R = \{F(R)\}^{1/2n} \rightarrow 0$ by (2.4),

$$(2.7) \quad \lim_{\epsilon \downarrow 0} I_1^\epsilon = \frac{1}{4^n |x|^n} \left| \int_{\mathbb{R}^n} f(z) dz \right| = \frac{\|f\|_{L^1(\mathbb{R}^n)}}{4^n |x|^n} \text{ for } 0 \leq |x| \leq 1/4.$$

Since $\frac{1}{|x|^n}$ is not in $L^1(\mathbb{R}^n)$, we get a conclusion. $\square$

Next lemma is important to show Lemma 2.3 which is the key to show Theorem 1.3:

Lemma 2.2. *For any function ψ in S , there exists a constant C depending only on ψ such that*

$$(2.8) \quad |\psi(x)| \leq \frac{C}{1+|x|}$$

$$(2.9) \quad |\psi_{x_j}(x)| \leq \frac{C}{1+|x|^2}, \quad j = 1, 2$$

for $x \in \mathbb{R}^2$.

Proof. If ψ is in S , then there exists a function ω in T such that

$$(2.10) \quad \psi(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \omega(y) \log |x-y| dy.$$

Notice that ω is in $C_0^\infty(\mathbb{R}^2)$, so there is a constant $R = R(\omega)$ such that $\text{supp } \omega \subset B(0, R)$. Since ω is odd in x_2 , we get

$$(2.11) \quad \begin{aligned} \psi(x) &= \frac{1}{2\pi} \int_{B(0,R)} \omega(y) \log |x-y| dy \\ &= \frac{1}{2\pi} \int_{B_+(0,R)} \omega(y) \log \frac{|x-y|}{|x-\bar{y}|} dy, \end{aligned}$$

$$(2.12) \quad \begin{aligned} \psi_{x_1}(x) &= \frac{1}{2\pi} \int_{B(0,R)} \omega(y) \frac{x_1 - y_1}{|x-y|^2} dy \\ &= \frac{2x_2}{\pi} \int_{B_+(0,R)} \omega(y) \frac{y_2(x_1 - y_1)}{|x-y|^2 |x-\bar{y}|^2} dy, \end{aligned}$$

$$(2.13) \quad \begin{aligned} \psi_{x_2}(x) &= \frac{1}{2\pi} \int_{B(0,R)} \omega(y) \frac{x_2 - y_2}{|x-y|^2} dy \\ &= \frac{1}{\pi} \int_{B_+(0,R)} \omega(y) \frac{y_2 \{ (x_2 - y_2)(x_2 + y_2) - (x_1 - y_1)^2 \}}{|x-y|^2 |x-\bar{y}|^2} dy, \end{aligned}$$

where $B_+(0, R)$ denotes $B(0, R) \cap \mathbb{R}_+^2$ and $\bar{y} = (y_1, -y_2)$. Now we show that ψ and ψ_{x_j} is bounded in $B(0, R)$ and that there exists a constant C such that

$$(2.14) \quad \begin{aligned} |\psi(x)| &\leq C|x|^{-1}, \\ |\psi_{x_j}(x)| &\leq C|x|^{-2} \text{ for } |x| \geq 2R. \end{aligned}$$

The boundedness of ψ and ψ_j on $B(0, 2R)$ is obtained by the fact that $\log|x|$ and $|x|^{-1}$ are in $L_{loc}^1(\mathbb{R}^2)$:

$$(2.15) \quad \begin{aligned} |\psi(x)| &\leq \frac{\sup|\omega|}{2\pi} \int_{B(0,R)} |\log|x-y|| dy \\ &= C_\omega \int_{B(x,R)} |\log|y|| dy \\ &\leq C_\omega \int_{B(0,3R)} |\log|y|| dy \\ &= C_{\omega,R} < \infty, \end{aligned}$$

$$(2.16) \quad \begin{aligned} |\psi_{x_j}(x)| &\leq \frac{\sup|\omega|}{2\pi} \int_{B(0,R)} \frac{1}{|x-y|} dy \\ &\leq C_\omega \int_{B(0,3R)} |y|^{-1} dy \\ &= C_{\omega,R} < \infty \end{aligned}$$

for $|x| \leq 2R$. (Notice that $|y| \leq |x-y| + |x| \leq 3R$.)

Now we show (2.14) to complete the proof. We may assume $x_2 \geq 0$ to estimate ψ , because ψ is odd in x_2 . By this assumption, we get the following inequality:

$$(2.17) \quad 1 \leq \frac{|x-\bar{y}|}{|x-y|} \leq \frac{|x-y| + |y-\bar{y}|}{|x-y|} \leq 1 + \frac{2R}{|x|-R} \leq 1 + \frac{4R}{|x|}$$

for $|x| \geq 2R$ and $|y| \leq R$. The inequality (2.17) leads the estimate of ψ :

$$(2.18) \quad \begin{aligned} |\psi(x)| &\leq \frac{1}{2\pi} \log \left(1 + \frac{4R}{|x|} \right) \int_{B_+(0,R)} \omega(y) dy \\ &\leq \frac{1}{2\pi} \frac{4R}{|x|} \log \left(1 + \frac{4R}{|x|} \right)^{\frac{|x|}{4R}} \|\omega\|_{L^1(B_+(0,R))} \\ &\leq \frac{2R}{\pi} \|\omega\|_{L^1(\mathbb{R}^2)} |x|^{-1}. \end{aligned}$$

Notice that the inequality

$$\frac{|x|}{2} \leq |x-y|, \quad \frac{|x|}{2} \leq |x-\bar{y}|$$

holds for $|x| \geq 2R, |y| \leq R$. This inequality leads the estimate of ψ_{x_j} :

$$(2.19) \quad \begin{aligned} |\psi_{x_1}(x)| &\leq \frac{2|x|}{\pi} \int_{B_+(0,R)} |\omega(y)| \frac{|y|}{|x-y|^2|x-\bar{y}|} dy \\ &\leq \frac{16R\|\omega\|_{L^1}}{\pi|x|^2}, \end{aligned}$$

$$(2.20) \quad \begin{aligned} |\psi_{x_2}(x)| &\leq \frac{1}{\pi} \int_{B_+(0,R)} |\omega(y)| \frac{2|y|}{|x-y||x-\bar{y}|} dy \\ &\leq \frac{8R\|\omega\|_{L^1}}{\pi|x|^2}. \end{aligned}$$

Combining the estimate (2.15), (2.16), (2.18), (2.19), and (2.20) leads the conclusion (2.8) and (2.9). $\square$

Now we are ready to show the key lemma.

Lemma 2.3. *There exists a function ψ in S such that*

$$(2.21) \quad \int_{\mathbb{R}^2} \{\psi_{x_1}(x)\}^2 dx \neq \int_{\mathbb{R}^2} \{\psi_{x_2}(x)\}^2 dx.$$

Proof. Assume that the conclusion is not true, i.e.

$$(2.22) \quad \int_{\mathbb{R}^2} \{\psi_{x_1}(x)\}^2 dx = \int_{\mathbb{R}^2} \{\psi_{x_2}(x)\}^2 dx$$

for any ψ in S . Let ψ is in S , and let $\psi^h(x) = \psi(x_1 - h, x_2)$. Then the function ψ^h and $\psi + \psi^h$ are in S . In fact, there exists a function ω in T such that $\psi = (-\Delta)^{-1}\omega$, and we can write $\psi^h = (-\Delta)^{-1}\omega^h$ and $\psi + \psi^h = (-\Delta)^{-1}(\omega + \omega^h)$, where $\omega^h(x) = \omega(x_1 - h, x_2)$. It is obvious that ω^h and $\omega + \omega^h$ are in T .

By the assumption (2.22), we get

$$(2.23) \quad \begin{aligned} \int_{\mathbb{R}^2} \{\psi_{x_1}(x)\}^2 dx &= \int_{\mathbb{R}^2} \{\psi_{x_2}(x)\}^2 dx, \\ \int_{\mathbb{R}^2} \{(\psi^h)_{x_1}(x)\}^2 dx &= \int_{\mathbb{R}^2} \{(\psi^h)_{x_2}(x)\}^2 dx, \\ \int_{\mathbb{R}^2} \{(\psi + \psi^h)_{x_1}(x)\}^2 dx &= \int_{\mathbb{R}^2} \{(\psi + \psi^h)_{x_2}(x)\}^2 dx. \end{aligned}$$

Combining the equalities (2.23), we get

$$(2.24) \quad \int_{\mathbb{R}^2} \psi_{x_1}(x)(\psi^h)_{x_1}(x)dx = \int_{\mathbb{R}^2} \psi_{x_2}(x)(\psi^h)_{x_2}(x)dx.$$

Now we integrate the equality (2.24) by h :

$$(2.25) \quad \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \psi_{x_1}(x)(\psi^h)_{x_1}(x)dx = \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \psi_{x_2}(x)(\psi^h)_{x_2}(x)dx.$$

Notice that we can change the order of integration by the estimate (2.9). Firstly, we compute the left term of (2.25):

$$\begin{aligned}
 (2.26) \quad \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \psi_{x_1}(x) (\psi^h)_{x_1}(x) dx dh &= \int_{\mathbb{R}^2} \psi_{x_1}(x) \int_{-\infty}^{\infty} \psi_{x_1}(x_1 - h, x_2) dh dx \\
 &= \int_{\mathbb{R}^2} \psi_{x_1}(x) \left[\psi(h, x_2) \right]_{-\infty}^{\infty} dx \\
 &= 0.
 \end{aligned}$$

The equality is obvious by the estimate (2.8). Secondly, we compute the right term of (2.25):

$$\begin{aligned}
 (2.27) \quad \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \psi_{x_2}(x) (\psi^h)_{x_2}(x) dx dh &= \int_{\mathbb{R}^2} \psi_{x_2}(x) \int_{-\infty}^{\infty} \psi_{x_2}(x_1 - h, x_2) dh dx \\
 &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \psi_{x_2}(x_1, x_2) dx_1 \right\}^2 dx_2 \\
 &> 0.
 \end{aligned}$$

The positivity of integrals is shown by computing the form of ψ_{x_2} . Notice that ψ_{x_2} is continuous, so it is sufficient to check that $\psi_{x_2}(x_1, 0)$ is not zero. By (2.13), we can write ψ_{x_2} as

$$(2.28) \quad \psi_{x_2}(x) = \frac{1}{\pi} \int_{B_+(0, R)} \omega(y) \frac{y_2 \{ (x_2 - y_2)(x_2 + y_2) - (x_1 - y_1)^2 \}}{|x - y|^2 |x - \bar{y}|^2} dy$$

where ω is a function in T . Putting $x_2 = 0$ in (2.28) leads

$$(2.29) \quad \psi_{x_2}(x_1, 0) = -\frac{1}{\pi} \int_{B_+(0, R)} \omega(y) \frac{y_2}{(x_1 - y_1)^2 + y_2^2} dy.$$

Since ω and the integral kernel are positive in $B_+(0, R)$, $\psi_{x_2}(x_1, 0)$ is always nonzero. The results of computation (2.26) and (2.27) leads a contradiction, and we get a conclusion of the lemma. $\square$

Proof of Theorem 1.3. Let ψ the function in S that satisfies (2.20). Let $\psi^\epsilon(x) = \psi(x/\epsilon)$. Then

$$\{(\psi^\epsilon)_{x_1}(x)\}^2 - \{(\psi^\epsilon)_{x_2}(x)\}^2 = \frac{1}{\epsilon^2} \{(\psi_{x_1})^2 - (\psi_{x_2})^2\} \left(\frac{x}{\epsilon} \right)$$

and Lemma 2.1 is applicable. (Notice that $\psi_{x_1}^2 - \psi_{x_2}^2$ is in $L^1(\mathbb{R}^2)$: the estimate (2.9) leads this fact.) $\square$

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