Weighted Norm Inequalities
For Some Singular Integral Operators

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Weighted Norm Inequalities

For

Some Singular Integral Operators

by

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Dedicated to Professor Satoru Igari on his sixtieth birthday

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Abstract. For bounded Lebesgue measurable functions α, β on the unit circle, $S_{\alpha,\beta} = \alpha P_+ + \beta P_-$ is called a singular integral operator, where P_+ is an analytic projection and P_- is a co-analytic projection. We study one-weighted norm inequalities of $S_{\alpha,\beta}$ on $L^2(W)$. We introduce a class HS_r of weights with $r = |\alpha - \beta|/\|\alpha - \beta\|_\infty$ in order to characterize those weights. For example, we show that $S_{\alpha,\beta}$ is bounded with respect to a weight W if and only if W belongs to HS_r or $|\alpha - \beta|W \equiv 0$. If r is a nonzero constant, then HS_r is just a well known class of weights due to Helson and Szegő. Moreover we study the Koosis type problem of two weights of $S_{\alpha,\beta}$ and get very simple necessary and sufficient conditions for such weights.

§1. Introduction

Let m denote the normalized Lebesgue measure on the unit circle T. Let A be the disc algebra, that is, A is the algebra of all continuous functions on T whose negative Fourier coefficients are zero. For $0 < p < \infty$, the Hardy space H^p is the closure of A in $L^p = L^p(m)$, and H^∞ is the weak*-closure of A in $L^\infty = L^\infty(m)$. A function Q in H^∞ is an inner function if $|Q| = 1$ *a.e.*. A function h is an outer function if there exists a real function t in L^1 and a real constant c such that $h = e^{t+i\tilde{t}+ic}$, where $\tilde{t}$ denotes the harmonic conjugate function of t with $\tilde{t}(0) = 0$. Let A_0 be the subspace of all functions in A whose mean value is zero, and let $\bar{A}_0$ be the subspace of all complex conjugates of functions in A_0. Let S be the singular integral operator defined by

$$Sf(\zeta) = \frac{1}{\pi i}\int_T \frac{f(z)}{z-\zeta}dz$$

the integral being a Cauchy principal value. If f is in L^1, then $Sf(\zeta)$ exists for almost everywhere ζ on T, and $Sf(\zeta)$ is a measurable function. We shall define the analytic projection P_+ and co-analytic projection P_- by

$$P_+ = (I+S)/2, P_- = (I-S)/2,$$

where I denotes the identity operator. Then

$$(P_+ - P_-)f(\zeta) = Sf(\zeta) = i\tilde{f}(\zeta) + \int_T f dm.$$

For functions α and β in L^∞,

$$S_{\alpha,\beta} = \alpha P_+ + \beta P_- = \frac{\alpha-\beta}{2}S + \frac{\alpha+\beta}{2}I$$

is called a singular integral operator (cf.[3]). For a nonnegative function W in L^1, put

$$L^2(W) = L^2(Wdm) \text{ and } \|f\|_W = \{\int_T |f|^2 W dm\}^{1/2}.$$

In this paper, we consider the Helson-Szegő type problem of one weight of $S_{\alpha,\beta}$. We treat the one-weighted norm inequalities as the following :

$$\|S_{\alpha,\beta}f\|_W \le C\|f\|_W$$

and

$$\|S_{\alpha,\beta}f\|_W \ge \varepsilon\|f\|_W$$

where C and ε are positive constants. The first one-weighted inequality was studied by H.Helson and G.Szegö [4] when $\alpha = 1$ and $\beta = 0$ or $\alpha = 1$ and $\beta = -1$. Then they introduced a class HS of weights which satisfy such an inequality. In §3 of this paper, we show that the first one-weighted inequality is true for some positive constant C if and only if the weight W belongs to a class HS_r of weights with $r = |\alpha - \beta| / \|\alpha - \beta\|_\infty$ or if $|\alpha - \beta| W \equiv 0$. HS_r contains HS and if r is a positive constant, then $HS_r = HS$ [see §2]. In §2, we study the relation between HS_r and the $(A_{2,r})$ condition which is called the weighted (A_2) condition. The (A_2) condition was introduced by M.Rosenblum in his study of the weighted continuity for the Poisson transform, and then used by B.Muckenhoupt (before [5]) for the maximal function. R.A.Hunt, B.Muckenhoupt and R.L.Wheeden [5] showed that HS consists of weights which satisfy the (A_2) condition. T.Nakazi and T.Yamamoto [8], and T.Yamamoto [12] studied the first one-weighted inequality for $C = 1$. They gave necessary and sufficient conditions that are more complicated than HS_r. A class of weights was introduced in [12]. Our class HS_r is larger and more simple than that. The second one-weighted inequality was studied by R.Rochberg [10] when the weight W is in HS. T.Yamamoto [11] and T.Nakazi [7] studied it in some special cases. In §4 of this paper, we give a simple necessary and sufficient condition for the second one-weighted inequality for some positive constant ε, when $ess. \inf |\alpha - \beta| > 0$.

In this paper, we consider the Koosis type problem of two weights, that is, given a nonnegative function W in L^1 find necessary and sufficient conditions on W such that there exists a nonnegative and nonzero function U satisfying one of the following two-weighted norm inequalities :

$$\|S_{\alpha,\beta} f\|_U \leq \|f\|_W$$

and

$$\|S_{\alpha,\beta} f\|_W \geq \|f\|_U.$$

The first two-weighted inequality was studied by P.Koosis [6] when $\alpha = 1$ and $\beta = 0$ or $\alpha = 1$ and $\beta = -1$. He showed that there exists a nonzero function U if and only if W^{-1} belongs to L^1. In §3, we can show that this type theorem is true for functions α and β in L^∞. The second two-weighted inequality has never been studied before. In §4, we give a very simple necessary and sufficient condition for that there exists a nonzero function U for the second one. There is a different problem, that is, given nonnegative function W and U in L^1 find necessary and sufficient conditions on W and U such that one of the two-weighted norm inequalities above holds. This problem was solved in the particular case of $\alpha = 1$ and $\beta = -1$ by Cotlar and Sadosky using their lifting theorem. Since this problem becomes very complicated in the general case of $\alpha, \beta \in L^\infty$, we do not consider this problem. In this paper, the Cotlar-Sadosky lifting theorem [1] is essential and is used several times.

§2. Weighted Helson-Szegő or (A_2) condition

In this section, we define a class HS_r of weights which is called a weighted Helson-Szegő class, and a class $A_{2,r}$ of weights which satisfy the weighted (A_2) condition. We will show $A_{2,r} \supseteq HS_r$. The weighted Helson-Szegő class will be used later. Let r be a function in L^∞ with $0 \le r \le 1$.

$$W \in HS_r \text{ if and only if } W = e^{u+\tilde{v}}$$

where u and v are real functions in L^1 with $|v| \le \pi/2$, and there exists a finite constant $\gamma > 0$ such that

$$r^2 e^u \le \gamma \cos v \quad \text{and} \quad e^{-u} \le \gamma \cos v.$$

$W \in A_{2,r}$ if and only if there exists a finite constant $\gamma > 0$ such that $W^{-1} \in L^1$ and

$$\widehat{r^2 W}(a) \widehat{W^{-1}}(a) \le \gamma \qquad (a \in D).$$

D denotes the open unit disc and $\hat{V}$ is the harmonic extension to D of a function V in L^1. $W \in HS$ if and only if $W = e^{u+\tilde{v}}$ where u and v are real functions in L^∞ with $\|v\|_\infty < \pi/2 (see[4])$. $W \in HS_1$ if and only if $W = e^{u+\tilde{v}}$ where u and v are real functions in L^1 with $|v| \le \frac{\pi}{2}$, and there exists a finite constant $\gamma > 0$ such that

$$2 \le e^{-u} + e^u \le \gamma \cos v.$$

Hence, if $r \equiv 1$ then $HS_r = HS$ and $A_{2,r} = A_2$. The set A_2 is the well known class of weights which satisfy the (A_2) condition and $HS = A_2 (see$ [5, Theorem 2]).

Theorem 1. Let r and r' be functions in L^∞ with $0 \le r \le 1$ and $0 \le r' \le 1$.

(1) $A_{2,r} \supseteq HS_r \supseteq HS$. In particular, if $W \in HS_r$ then both $r^2 W$ and W^{-1} belong to L^1.

(2) If r/r' is bounded, then $HS_{r'} \subseteq HS_r$ and hence if both r/r' and r'/r are bounded, then $HS_{r'} = HS_r$.

(3) If r^{-1} is bounded, then $A_{2,r} = A_2 = HS = HS_r$.

(4) If $r \equiv 0$, then $A_{2,r} \cap L^1 = HS_r \cap L^1$.

(5) If $r^{-2} \notin \bigcup_{p>1} L^p$, then $HS_r \ne HS$ and hence $A_{2,r} \ne A_2$.

(6) $W \in HS_r$ if and only if there exists a nonzero function $V \in L^1$ and a real constant c such that

$$\frac{1}{\gamma} r^2 W \le V \le \gamma W \quad \text{and} \quad (\tilde{V} + c)^2 \le \gamma V W$$

for some positive constant γ.

Proof. (1) If $W = e^{u+\tilde{v}}$, $u \in L^\infty$ and $\|v\|_\infty < \pi/2$, then $\cos v \ge \delta > 0$ for some constant $\delta, e^u \in L^\infty$ and $e^{-u} \in L^\infty$. Hence there exists a finite constant $\gamma \ge 1$ such that

$$r^2 e^u \le \gamma \cos v \quad \text{and} \quad e^{-u} \le \gamma \cos v.$$

Therefore $HS_r \supseteq HS$. If $W \in HS_r$, then $W = e^{u+\tilde{v}}, r^2 e^u \le \gamma \cos v, e^{-u} \le \gamma \cos v$ and $\gamma \ge 1$. Hence

$$r^2 W \le \gamma e^{\tilde{v}} \cos v \quad \text{and} \quad W^{-1} \le \gamma e^{-\tilde{v}} \cos v.$$

Put $f = e^{\tilde{v}-iv}$, then both f and f^{-1} belongs to H^p for some $p < 1$ because $\|v\|_\infty \le \pi/2$. Hence for any $a \in D$

$$\widehat{r^2 W}(a) \le \gamma Ref(a) \le \gamma |f(a)|$$

and

$$\widehat{W^{-1}}(a) \le \gamma Ref^{-1}(a) \le \gamma |f^{-1}(a)|.$$

This implies that $W \in A_{2,r}$ and $A_{2,r} \supseteq HS_r$. (2) If $r \le Cr'$ for some positive constant C and $W \in HS_{r'}$ and $W = e^{u+\tilde{v}}$ with $\gamma' \ge 1$, then $r^2 e^u \le (Cr')^2 e^u \le C^2 \gamma' \cos v$ and $e^{-u} \le \gamma' \cos v$. This implies that $W \in HS_r$ and $HS_{r'} \subseteq HS_r$. (3) follows trivially from (2). (4) When $r \equiv 0$, $W \in A_{2,r}$ if and only if $W^{-1} \in L^1$, and $W \in HS_r$ if and only if $W = e^{u+\tilde{v}}$ and $e^{-u} \le \gamma \cos v$. If $V = W^{-1} \in L^1$, then $V + i\tilde{V} = e^{c-\tilde{v}+iv}$ and $\|v\|_\infty \le \pi/2$ for some real constant c. Hence $V = e^{c-\tilde{v}} \cos v$. If W is in L^1, then $u = -c - \log(\cos v)$ belongs to $L^1, W = e^{u+\tilde{v}}$ and $e^{-u} = e^c \cos v$. (5) Suppose $r^{-2} \notin \bigcup_{p>1} L^p$. When $\log r \in L^1$, if $W = r^{-2}$, then we can write $W = e^{u+\tilde{v}}$ where $v = 0$, $u = -2\log r \in L^1$, $r^2 e^u = \cos v$ and $e^{-u} \le \cos v$. Hence $W \in HS_r$. If $HS_r = HS$, then $r^{-2} \in HS$ and hence $r^{-2} \in \bigcup_{p>1} L^p$. This contradicts the hypothesis on r. When $\log r \notin L^1$, there exists r' such that $r \le r' \le 1$, $\log r' \in L^1$ but $(r')^{-2} \notin \bigcup_{p>1} L^p$. By (1) and (2), $HS_r \supseteq HS_{r'} \supseteq HS$. If $HS_r = HS$, then $HS_{r'} = HS$ and hence $(r')^{-2} \in HS$. This contradicts that $(r')^{-2} \notin \bigcup_{p>1} L^p$. Thus $HS_r \ne HS$.

(6) If $W \in HS_r$, then by the proof of (1) there exists a function f in H^p with $Ref \in L^1$ such that

$$r^2 W \le \gamma Ref \quad \text{and} \quad W^{-1} \le \gamma Ref^{-1}.$$

If we put $V = Ref$, then $Ref^{-1} = V/\{V^2 + (\tilde{V} + c)^2\}$ for some real constant c. Hence $r^2 W \le \gamma V$ and

$$W^{-1} \le \gamma \frac{V}{V^2 + (\tilde{V} + c)^2} \le \gamma V^{-1}$$

and $(\tilde{V} + c)^2 \le V^2 + (\tilde{V} + c)^2 \le \gamma V W$. Conversely if $\frac{1}{\gamma} r^2 W \le V \le \gamma W$ and $(\tilde{V} + c)^2 \le \gamma V W$, then $r^2 W \le \gamma V$ and $V^2 + (\tilde{V} + c)^2 \le 2\gamma V W$. Hence $W^{-1} \le 2\gamma V/\{V^2 + (\tilde{V} + c)^2\}$. If $f = V + i(\tilde{V} + c)$, then $f = e^{\tilde{v}+s-iv}$ for some real constant s where $|v| \le \pi/2$, and hence

$$r^2W \le \gamma e^{\tilde{v}+s}\cos v \ \text{ and } \ W^{-1} \le 2\gamma e^{-(\tilde{v}+s)}\cos v.$$

Put $e^u = We^{-\tilde{v}}$, then $W \in HS_r$.

(6) of Theorem 1 with $r \equiv 1$ implies a result of V.F.Gaposkin (cf.[4]). That is, $W \in HS$ if and only if there exists a nonzero function $V \in L^1$ and a real constant c such that

$$\frac{1}{\gamma}W \le V \le \gamma W \text{ and } (\tilde{V}+c)^2 \le \gamma V^2$$

for some positive constant $\gamma \ge 1$. Hence $W \in HS$ if and only if there is a bounded function u_1 and a real constant c such that

$$|\tilde{V}+c| < \gamma V,$$

where $V = e^{-u_1}W$. We could not show that $c = 0$. We can ask two natural questions. Is the converse of (3) of Theorem 1 true ? Is (4) of Theorem 1 true in general ? The structure of HS_r is not simple. We define two more simple classes B_r^0 and B_r^1 of weights. Put

$B_r^0 = \{W > 0\ ;\ W = W_0W_1, \log W \in L^1, r^2W_0 \in L^\infty, W_0^{-1} \in L^\infty, W_1 = e^{\tilde{v}}$ and $|v| \le \pi/2 - \varepsilon$ for some $\varepsilon > 0\}$ and

$B_r^1 = \{W > 0\ ;\ W = W_0W_1, \log W \in L^1, r^2W_0 \in L^\infty, W_0^{-1} \in L^\infty, W_1 = e^{\tilde{v}}$ and $|v| \le \pi/2\}$. Then $B_r^1 \supseteq HS_r \supseteq B_r^0 \supseteq HS$. There exists v such that $|v| \le \frac{\pi}{2}$ and $e^{-\tilde{v}} \notin L^1$. Suppose $W = W_0W_1 = 1 \times e^{\tilde{v}}$, then $W \in B_r^1$ and $W^{-1} \notin L^1$. This and (1) of Theorem 1 implies that $W \notin HS_r$. Hence $B_r^1 \underset{\neq}{\supset} HS_r$ for arbitrary r. If $W = r^{-2}$ with $r^{-2} \notin \bigcup_{p>1} L^p$, then $W \in B_r^0$ and $W \notin HS$. Hence $B_r^0 \underset{\neq}{\supset} HS$ for some r. If $r^{-1} \in L^\infty$, then $HS_r = B_r^0 = HS$. It is very nice if $HS_r = B_r^0$ for arbitrary r. Unfortunately it is easy to see that it is not true. In fact, if $W \in B_r^0$ then by definition W^{-1} belongs to L^p for some $p > 1$. Then the following characterization of W with $W^{-1} \in L^1$ shows that $HS_r \ne B_r^0$ for some $r \ge 0$. Both W and W^{-1} belong to L^1 and $W > 0$ if and only if there exists a nonzero r such that $W \in HS_r \cap L^1$ if and only if $W \in HS_0 \cap L^1$. Suppose $W > 0$ and $W, W^{-1} \in L^1$. Let $k = W^{-1} + i(W^{-1})^\sim$, then k is an outer function, since $Rek > 0$. Since $|k^{-1}| \le W, k^{-1}$ is in H^1. Let $V = Re(k^{-1})$, then there exists a real constant c such that $\tilde{V} + c = Im(k^{-1})$. Hence,

$$\frac{W^{-2}}{W^{-2} + (W^{-1})^{\sim 2}}W = V \le W$$

and

$$\begin{aligned}(\tilde{V}+c)^2 &= \{Im(k^{-1})\}^2 = \frac{(W^{-1})^{\sim 2}}{\{W^{-2} + (W^{-1})^{\sim 2}\}^2} \\ &\le \frac{1}{W^{-2} + (W^{-1})^{\sim 2}} = VW.\end{aligned}$$

By (6) of Theorem 1, $W \in HS_r \cap L^1$, where

$$r = \sqrt{\frac{W^{-2}}{W^{-2} + (W^{-1})^{\sim 2}}} > 0 \qquad a.e.$$

Conversely, if $W \in HS_r \cap L^1$, then by (1) of Theorem 1, W, $W^{-1} \in L^1$. Then we can choose r such that $r > 0$ *a.e.*. Now it is easy to see that $W \in L^p$ and $W^{-1} \in L^p$ for some $p > 1$ and $W > 0$ if and only if there exists r such that $r > 0$ *a.e.*, $W \in B_r^0 \cap L^p$ for some $p > 1$, since if $W \in B_r^1$ and $0 < \varepsilon < 1$ then $W^{1-\varepsilon} \in B_r^0$.

The following two lemmas will be used in §3. The first one was used by the second author in [11] and its proof is clear.

Lemma 1. Let u, v and r be real functions. Then $|1 - e^{-u-iv}|^2 \le 1 - r^2$ if and only if $r^2 e^u + e^{-u} \le 2\cos v$.

Lemma 2. Let u, v and r be real functions, and $0 \le r \le 1$.

$$r^2 e^u \le \gamma \cos v \quad \text{and} \quad e^{-u} \le \gamma \cos v$$

for some constant $\gamma > 0$ if and only if there exists a constant $\varepsilon > 0$ such that

$$|1 - \varepsilon e^{-u-iv}|^2 \le 1 - (\varepsilon r)^2.$$

Proof. If $r^2 e^u \le \gamma \cos v$ and $e^{-u} \le \gamma \cos v$ for some $\gamma > 0$, then

$$\frac{1}{\gamma^2} r^2 e^{u + \log\gamma} \le \cos v \quad \text{and} \quad e^{-(u+\log\gamma)} \le \cos v.$$

Put $\varepsilon = 1/\gamma$, then

$$(\varepsilon r)^2 e^{u+\log\gamma} + e^{-(u+\log\gamma)} \le 2\cos v.$$

By Lemma 1, $|1 - \varepsilon e^{-u-iv}|^2 \le 1 - (\varepsilon r)^2$. Conversely, if $|1 - \varepsilon e^{-u-iv}|^2 \le 1 - (\varepsilon r)^2$, then by Lemma 1

$$(\varepsilon r)^2 e^s + e^{-s} \le 2\cos v \text{ and } s = u - \log\varepsilon.$$

Hence $(\varepsilon r)^2 e^s \le 2\cos v$ and $e^{-s} \le 2\cos v$. Therefore if $\gamma = 2/\varepsilon$, then $r^2 e^u \le \gamma \cos v$ and $e^{-u} \le \gamma \cos v$.

§3. Bounded singular integral operators

In this section, the first one-weighted norm inequality of Helson-Szegő type and the first two-weighted norm inequality of Koosis type are studied.

Theorem 2. Suppose α, β are in L^∞ and W is a nonnegative function in L^1. Then

$$\|S_{\alpha,\beta} f\|_W \leq C\|f\|_W \qquad (f \in A + \bar{A}_0)$$

for some positive constant C if and only if $\alpha - \beta \not\equiv 0$ and W belongs to HS_r with $r = |\alpha - \beta|/\|\alpha - \beta\|_\infty$, or $|\alpha - \beta|W \equiv 0$.

Proof. $\|S_{\alpha,\beta}\|_W < \infty$ if and only if $\|S_{\alpha-\beta,0}\|_W < \infty$ because $S_{\alpha,\beta} f = (\alpha - \beta)P_+ f + \beta f \quad (f \in A + \bar{A}_0)$. We shall prove that $\|S_{\alpha-\beta,0}\|_W < \infty$ if and only if $\alpha - \beta \not\equiv 0$ and $W \in HS_r$ with $r = |\alpha - \beta|/\|\alpha - \beta\|_\infty$ or $|\alpha - \beta|W \equiv 0$. Suppose $\|S_{\alpha-\beta,0}\|_W < \infty$ and $C \geq \|S_{\alpha-\beta,0}\|_W$. Then, for $f_1 \in A$ and $f_2 \in \bar{A}_0$

$$\|(\alpha - \beta) f_1\|_W^2 \leq C^2 \|f_1 + f_2\|_W^2.$$

Let $W_1 = (C^2 - |\alpha - \beta|^2)W, W_2 = W_3 = C^2 W$, then

$$\int_T \{|f_1|^2 W_1 + |f_2|^2 W_2 + 2Re(f_1 \bar{f}_2 W_3)\} dm \geq 0.$$

By the Cotlar-Sadosky lifting theorem [1], there exists a k in H^1 such that $|W_3 - k|^2 \leq W_1 W_2$ and hence $|C^2 W - k|^2 \leq C^2(C^2 - |\alpha - \beta|^2)W^2$. Suppose $|\alpha - \beta|W \not\equiv 0$. Then $0 < |k| \leq 2C^2 W$ and hence $\log W \in L^1$. Then

$$\left|1 - \frac{k}{C^2 W}\right|^2 \leq 1 - \frac{|\alpha - \beta|^2}{C^2}.$$

Put $r' = |\alpha - \beta|/C$ and $e^{-u-iv} = k/C^2 W$ where u and v are real functions, $u \in L^1$, $|v| \leq \pi/2$, then $0 \leq r' \leq 1$ and

$$|1 - e^{-u-iv}|^2 \leq 1 - (r')^2.$$

By Lemma 2, $e^{u+\tilde{v}}$ belongs to $HS_{r'}$ and hence by (1) of Theorem 1, $e^{-u-\tilde{v}} \in L^1$. Therefore $C^2 W e^{-u-\tilde{v}} = k e^{-\tilde{v}+iv}$ is a positive function in $H^{1/2}$ and hence by the Neuwirth-Newman theorem [9], it is a positive constant C'. Thus $W = C' C^{-2} e^{u+\tilde{v}}$ and W belongs to $HS_{r'}$. Put $r = |\alpha - \beta|/\|\alpha - \beta\|_\infty$, then by (2) of Theorem 1, $W \in HS_r$. For the converse, if $|\alpha - \beta|W \equiv 0$, then $\|S_{\alpha-\beta,0}\|_W = 0$ and hence we may assume $W \in HS_r$. By definition of HS_r and Lemma 2, $W = e^{u+\tilde{v}}$ and

$$|1 - \varepsilon e^{-u-iv}|^2 \leq 1 - (\varepsilon r)^2$$

for some positive constant ε. If $k = \varepsilon W e^{-u-iv}$, then $k = \varepsilon e^{\tilde{v}-iv}$ is an outer function and

$$|W-k|^2 \leq \{1-(\varepsilon r)^2\}W^2.$$

This implies $k \in H^1$. Let $W_1 = \{1-(\varepsilon r)^2\}W, W_2 = W_3 = W$, then

$$|W_3 - k|^2 \leq W_1 W_2.$$

By the Cotlar-Sadosky lifting theorem [1], for $f_1 \in A$ and $f_2 \in \bar{A}_0$

$$\int_T \{|f_1|^2 W_1 + |f_2|^2 W_2 + 2Re(f_1 \bar{f}_2 W_3)\} dm \geq 0.$$

Hence,

$$\|\varepsilon r f_1\|_W^2 \leq \|f_1 + f_2\|_W^2.$$

Thus $\|S_{\alpha-\beta,0} f\|_W^2 \leq \|\alpha - \beta\|_\infty^2 \varepsilon^{-2} \|f\|_W^2$.

If $ess.\inf|\alpha - \beta| > 0$ and $W \not\equiv 0$ in Theorem 2, then by (3) of Theorem 1, $\|S_{\alpha,\beta}\|_W < \infty$ if and only if $W \in HS$ (*see*[11]). Hence Theorem 2 shows the Helson-Szegő theorem [4]. The following theorem for $\alpha = 1$ and $\beta = 0$ is the Koosis theorem [6].

Theorem 3. Suppose α, β are in L^∞ with $\alpha - \beta \not\equiv 0$ and W is a nonnegative function in L^1. There exists a nonnegative function U with $|\alpha - \beta| U \not\equiv 0$ such that

$$\|S_{\alpha,\beta} f\|_U \leq \|f\|_W \qquad (f \in A + \bar{A}_0)$$

if and only if W^{-1} belongs to L^1.

Proof. Suppose $|\alpha-\beta|U \not\equiv 0$ and $\|S_{\alpha,\beta} f\|_U \leq \|f\|_W$. Since $\max\{|\alpha|^2, |\beta|^2\}U \leq W$, we have $\{|\alpha - \beta|U > 0\} \subseteq \{W > 0\} \subseteq T$. Put $d = (U/W + \varepsilon)^{1/2}$ for some $\varepsilon > 0$, then $|d\alpha - d\beta|W \not\equiv 0$ and $\|S_{d\alpha,d\beta} f\|_W \leq \|f\|_W$. By Theorem 2, $W \in HS_r$ for $r = d|\alpha - \beta|/\|d(\alpha - \beta)\|_\infty$. By (1) of Theorem 1, W^{-1} belongs to L^1. Conversely, if $W^{-1} \in L^1$, by the remark after Theorem 1 there exists r' with $r' > 0$ *a.e.* such that $W \in HS_{r'}$. There exists $d \in L^\infty$ such that $d > 0$ *a.e.* and $d|\alpha-\beta| \leq r'$. By (2) of Theorem 1, $W \in HS_r$ with $r = d|\alpha - \beta|/\|d(\alpha - \beta)\|_\infty$. By Theorem 2, $\|S_{d\alpha,d\beta} f\|_W \leq C\|f\|_W$. Put $U = C^{-2} d^2 W$, then $\|S_{\alpha,\beta} f\|_U \leq \|f\|_W$ and $|\alpha - \beta|U \not\equiv 0$, since $U > 0$ *a.e.* and $\alpha - \beta \not\equiv 0$.

§4. Bounded below singular integral operators

In this section, the second one-weighted norm inequality and the second two-weighted norm inequality are studied. We could not give a necessary and sufficient condition for the one-weighted inequality, in general. The second author [11] gave a necessary and sufficient condition for that $S_{\alpha,\pm\beta}$ are bounded below with respect to W. When $W \in HS$, if $S_{\alpha,\beta}$ is bounded below $w.r.t.W$, then $S_{\alpha,-\beta}$ is also bounded below $w.r.t.W$. Hence his result covers the Rochberg theorem for $p = 2$. The first author [7] generalized the above result for arbitrary $p, 1 < p < \infty$. In fact, Rochberg [10] gave the necessary and sufficient condition for the invertibility of the Toeplitz operators on weighted H^p spaces for $p, 1 < p < \infty$. In Theorem 4, $HS_r \supseteq B_r^0$ (*see* §2) but unfortunately $HS_r \neq B_r^0$, in general. If r^{-1} is bounded, then $HS_r = B_r^0$ (*see* §2).

Lemma 3. Let r be a function in L^∞ with $0 \le r \le 1$ and $ess.\inf|\alpha\beta| > 0$. The following (1) and (2) are equivalent.

(1) For any enough small positive constant ε, there exist an inner function Q and a real function t in L^1 such that

$$\frac{\alpha\bar{\beta}-\varepsilon^2}{|\alpha\bar{\beta}-\varepsilon^2|} = Qe^{i\tilde{t}} \text{ and } We^{-t} \in B_r^0.$$

(2) There exist an inner function Q and a real function ℓ in L^1 such that

$$\frac{\alpha\bar{\beta}}{|\alpha\bar{\beta}|} = Qe^{i\tilde{\ell}} \text{ and } We^{-\ell} \in B_r^0.$$

Proof. Put $e^{is} = (\alpha\bar{\beta})|\alpha\bar{\beta}-\varepsilon^2|/|\alpha\bar{\beta}|(\alpha\bar{\beta}-\varepsilon^2)$ and $\ell = t - \tilde{s}$, then

$$\frac{\alpha\bar{\beta}}{|\alpha\bar{\beta}|} = Qe^{i\tilde{t}}e^{is} = e^{ic}Qe^{i\tilde{\ell}}$$

for some real constant c and

$$\frac{\alpha\bar{\beta}-\varepsilon^2}{|\alpha\bar{\beta}-\varepsilon^2|} = e^{ic}Qe^{i\tilde{\ell}}e^{-is} = Qe^{i\tilde{t}}.$$

Since $We^{-\ell} = (We^{-t})e^{\tilde{s}}$ and $\|s\|_\infty \to 0$ as $\varepsilon \to 0$, for enough small $\varepsilon > 0$ it is easy to show that $We^{-\ell} \in B_r^0$ if and only if $We^{-t} \in B_r^0$ by definition of B_r^0 in §2. These imply the equivalence of (1) and (2).

Theorem 4. Suppose α, β are in L^∞ and W is a nonnegative function in L^1 satistying $|\alpha-\beta|W \not\equiv 0$.

(1) If there exists a positive constant δ such that

$$\|S_{\alpha,\beta}f\|_W \geq \delta\|f\|_W \qquad (f \in A + \bar{A}_0),$$

then for each positive constant $\varepsilon \not< \delta$ there exist an inner function Q and a real function t in L^1 such that

$$\frac{\alpha\bar{\beta} - \varepsilon^2}{|\alpha\bar{\beta} - \varepsilon^2|} = Qe^{i\tilde{t}} \text{ and } We^{-t} \in HS_r$$

where $r = |\alpha - \beta|/\|\alpha - \beta\|_\infty$, and $\alpha\bar{\beta} - \varepsilon^2$, α and β are invertible in L^∞.

(2) If for each enough small positive constant ε there exist an inner function Q and a real function t in L^1 such that

$$\frac{\alpha\bar{\beta} - \varepsilon^2}{|\alpha\bar{\beta} - \varepsilon^2|} = Qe^{i\tilde{t}} \text{ and } We^{-t} \in B_r^0$$

where $r = |\alpha - \beta|/\|\alpha - \beta\|_\infty$, and $\alpha\bar{\beta} - \varepsilon^2$, α and β are invertible in L^∞, then

$$\|S_{\alpha,\beta}f\|_W \geq \delta\|f\|_W \qquad (f \in A + \bar{A}_0)$$

for some positive constant δ.

Proof. (1) Suppose $\|S_{\alpha,\beta}f\|_W \geq \delta\|f\|_W$ $(f \in A + \bar{A}_0)$. Let $W_1' = (|\alpha|^2 - \delta^2)W, W_2' = (|\beta|^2 - \delta^2)W, W_3' = (\alpha\bar{\beta} - \delta^2)W$. Then, for $f_1 \in A$ and $f_2 \in \bar{A}_0$,

$$\int_T \{|f_1|^2 W_1' + |f_2|^2 W_2' + 2Re(f_1\bar{f}_2 W_3')\}dm \geq 0.$$

By the Cotlar-Sadosky lifting theorem [1], $W_1' \geq 0, W_2' \geq 0$, and there exists a k' in H^1 such that $|W_3' - k'|^2 \leq W_1'W_2'$. Hence, $(|\alpha|^2 - \delta^2)W \geq 0, (|\beta|^2 - \delta^2)W \geq 0$, and $|(\alpha\bar{\beta} - \delta^2)W - k'|^2 \leq (|\alpha|^2 - \delta^2)(|\beta|^2 - \delta^2)W^2$. If $k' \equiv 0$, then $\delta^2|\alpha - \beta|^2W^2 \leq 0$. This contradiction implies $k' \equiv 0$. Since $(|\alpha|^2 - \delta^2)(|\beta|^2 - \delta^2) \leq |\alpha\bar{\beta} - \delta^2|^2$, we have $0 < |k'| \leq 2|\alpha\bar{\beta} - \delta^2|W$ and hence $W > 0$. Hence $|\alpha| \geq \delta$ and $|\beta| \geq \delta$. If $0 < \varepsilon < \delta$, then

$$\|S_{\alpha,\beta}f\|_W \geq \varepsilon\|f\|_W \quad (f \in A + \bar{A}_0).$$

Let $W_1 = (|\alpha|^2 - \varepsilon^2)W$, $W_2 = (|\beta|^2 - \varepsilon^2)$, $W_3 = (\alpha\bar{\beta} - \varepsilon^2)W$. Then, for $f_1 \in A$ and $f_2 \in \bar{A}_0$,

$$\int_T \{|f_1|^2 W_1 + |f_2|^2 W_2 + 2Re(f_1\bar{f}_2 W_3)\}dm \geq 0.$$

By the Cotlar-Sadosky lifting theorem [1], there exists a k in H^1 such that $|W_3 - k|^2 \leq W_1W_2$. Hence,

$$|(\alpha\bar{\beta} - \varepsilon^2)W - k|^2 \leq (|\alpha|^2 - \varepsilon^2)(|\beta|^2 - \varepsilon^2)W^2.$$

Since $|\alpha| \geq \delta$, $|\beta| \geq \delta$ and $0 < \varepsilon < \delta$, we have $|\alpha\bar{\beta} - \varepsilon^2| \geq |\alpha\beta| - \varepsilon^2 \geq \delta^2 - \varepsilon^2 > 0$. Hence $\alpha\bar{\beta} - \varepsilon^2, \alpha$ and β are invertible in L^∞. If $k \equiv 0$, then $\varepsilon^2|\alpha - \beta|^2W^2 \leq 0$. This contradiction implies $k \not\equiv 0$. Since $(|\alpha|^2 - \varepsilon^2)(|\beta|^2 - \varepsilon^2) \leq |\alpha\bar{\beta} - \varepsilon^2|^2$, we have $0 < |k| \leq 2|\alpha\bar{\beta} - \varepsilon^2|W$ and hence $\log(|\alpha\bar{\beta} - \varepsilon^2|W) \in L^1$. Then

$$\left|1-\frac{k}{(\alpha\bar{\beta}-\varepsilon^2)W}\right|^2 \le \frac{(|\alpha|^2-\varepsilon^2)(|\beta|^2-\varepsilon^2)}{|\alpha\bar{\beta}-\varepsilon^2|^2}.$$

If $e^{-u-iv}=k/(\alpha\bar{\beta}-\varepsilon^2)W$ and $q=\varepsilon|\alpha-\beta|/|\alpha\bar{\beta}-\varepsilon^2|$ where u and v are real functions, then $|1-e^{-u-iv}|^2 \le 1-q^2$, $(\alpha\bar{\beta}-\varepsilon^2)W = ke^{u+iv}$ and $|\alpha\bar{\beta}-\varepsilon^2|W=|k|e^u$. Hence we can take u,v in L^1 and $|v|\le \frac{\pi}{2}$. By the inner-outer factorization of k, $k=Qexp\{\log|k| + i(\log|k|)^\sim\}$ where Q is an inner function. Put $t=-\tilde{v}+\log|k|$, then

$$\frac{\alpha\bar{\beta}-\varepsilon^2}{|\alpha\bar{\beta}-\varepsilon^2|} = \frac{k}{|k|}e^{iv} = Q'e^{i\tilde{t}}$$

where $Q'=Qe^{ic}$ for some real constant c and by Lemma 2

$$|\alpha\bar{\beta}-\varepsilon^2|We^{-t} = e^{u+\tilde{v}} \in HS_q.$$

Let $u_0=\log|\alpha\bar{\beta}-\varepsilon^2|$, then $u_0\in L^\infty$ and $We^{-t}=e^{(u-u_0)+\tilde{v}}$. Since $q^2e^u\le\gamma\cos v$ and $e^{-u}\le\gamma\cos v$, we have $q^2e^{u-u_0}\le(\gamma e^{\|u_0\|_\infty})\cos v$ and $e^{-(u-u_0)}\le(\gamma e^{\|u_0\|_\infty})\cos v$. Hence $We^{-t}\in HS_q$. If $r=|\alpha-\beta|/\|\alpha-\beta\|_\infty$, then both r/q and q/r are bounded. Hence by (2) of Theorem 1, We^{-t} belongs to HS_r.

(2) If for each enough small positive constant ε there exist an inner function Q and a real function t in L^1 such that

$$\frac{\alpha\bar{\beta}-\varepsilon^2}{|\alpha\bar{\beta}-\varepsilon^2|} = Qe^{i\tilde{t}} \text{ and } We^{-t}\in B_r^0,$$

then by Lemma 3

$$\frac{\alpha\bar{\beta}}{|\alpha\bar{\beta}|} = Qe^{i(c+\tilde{\ell})} \text{ and } We^{-\ell}\in B_r^0$$

and $\ell=t-\tilde{s}$ where $e^{is}=(\alpha\bar{\beta})|\alpha\bar{\beta}-\varepsilon^2|/|\alpha\bar{\beta}|(\alpha\bar{\beta}-\varepsilon^2)$ and c is a real constant. Note $\|s\|_\infty\to 0$ as $\varepsilon\to 0$ since $ess.\inf|\alpha\beta|>0$. By definition of B_r^0, $We^{-\ell}=e^{u_0+\tilde{v}_0}$ where $u_0\in L^1$, $\|v_0\|_\infty<\frac{\pi}{2}$,

$$r^2e^{u_0}\le\gamma_0\cos v_0 \text{ and } e^{-u_0}\le\gamma_0\cos v_0.$$

Since $t-\ell=\tilde{s}$ and $\|s\|_\infty\to 0$ as $\varepsilon\to 0$,

$$We^{-t}=We^{-\ell}e^{-\tilde{s}}=e^{u_0+(v_0-s)^\sim}$$

and $\|v_0-s\|_\infty<\pi/2$. Put $u_1=u_0$ and $v_1=v_0-s$, then for small $\varepsilon>0$

$$\frac{\alpha\bar{\beta}-\varepsilon^2}{|\alpha\bar{\beta}-\varepsilon^2|} = Qe^{i\tilde{t}} \text{ and } We^{-t}=e^{u_1+\tilde{v}_1}\in B_r^0$$

where

$$r^2e^{u_1} \leq \gamma_\varepsilon \cos v_1 \text{ and } e^{-u_1} \leq \gamma_\varepsilon \cos v_1.$$

Since $v_1 = v_0 - s$ and $\|s\|_\infty \to 0$ as $\varepsilon \to 0, \gamma_\varepsilon$ is bounded. Put $u = u_1 + \log \gamma_\varepsilon, v = v_1$ and $\rho = r/\gamma_\varepsilon$, then $\gamma_\varepsilon We^{-t} = e^{u+\tilde{v}}, \rho^2 e^u \leq \cos v$ and $e^{-u} \leq \cos v$, and hence

$$\rho^2 e^u + e^{-u} \leq 2\cos v.$$

Put $W' = \gamma_\varepsilon W/|\alpha\bar{\beta} - \varepsilon^2|$, then $|\alpha\bar{\beta} - \varepsilon^2|W'e^{-t} = \gamma_\varepsilon We^{-t}$. Put $k = (\alpha\bar{\beta} - \varepsilon^2)W'e^{-u-iv}$, then $k = Qe^{t+i\tilde{t}}e^{\tilde{v}-iv}$ is in H^p for some $p > 0$ and by Lemma 1,

$$\left|1 - \frac{k}{(\alpha\bar{\beta} - \varepsilon^2)W'}\right|^2 = |1 - e^{-u-iv}|^2 \leq 1 - \rho^2.$$

Hence $|k| \leq CW'$ for some positive constant C and so $k \in H^1$. If we put $q = \varepsilon|\alpha - \beta|/|\varepsilon^2 - \alpha\bar{\beta}|$, then

$$\frac{q}{\rho} = \gamma_\varepsilon \frac{\|\alpha - \beta\|_\infty}{|\alpha - \beta|}\frac{\varepsilon|\alpha - \beta|}{|\alpha\bar{\beta} - \varepsilon^2|} = \frac{\varepsilon\gamma_\varepsilon\|\alpha - \beta\|_\infty}{|\alpha\bar{\beta} - \varepsilon^2|}$$

and hence there exists $\varepsilon > 0$ such that $q \leq \rho$ because γ_ε and $\|(\alpha\bar{\beta} - \varepsilon^2)^{-1}\|_\infty$ are bounded. Therefore

$$|(\alpha\bar{\beta} - \varepsilon^2)W' - k|^2 \leq |\alpha\bar{\beta} - \varepsilon^2|^2(1 - q^2)W'^2 = (|\alpha|^2 - \varepsilon^2)(|\beta|^2 - \varepsilon^2)W'^2$$

Since α and β are invertible in L^∞ and ε is enough small, we have $|\alpha| \geq \varepsilon$ and $|\beta| \geq \varepsilon$, Let $W_1 = (|\alpha|^2 - \varepsilon^2)W'$, $W_2 = (|\beta|^2 - \varepsilon^2)W'$ and $W_3 = (\alpha\bar{\beta} - \varepsilon^2)W'$. Then, $|W_3 - k|^2 \leq W_1W_2$. By the Cotlar-Sadosky lifting theorem, for $f_1 \in A$ and $f_2 \in \bar{A}_0$,

$$\int_T \{|f_1|^2 W_1 + |f_2|^2 + 2Re(f_1\bar{f}_2W_3)\}dm \geq 0.$$

Hence $\|S_{\alpha,\beta}f\|_{W'} \geq \varepsilon\|f\|_{W'}(f \in A + \bar{A}_0)$ and so $\|S_{\alpha,\beta}f\|_W \geq \delta\|f\|_W$ for some positive constant δ because $W/W', W'/W \in L^\infty$.

Corollary. Suppose α, β are in L^∞ with $ess.\inf|\alpha - \beta| > 0$ and W is a nonnegative function in L^1. Then the following (1) $\sim$ (3) are equivalent.

(1) $\|S_{\alpha,\beta}f\|_W \geq \delta\|f\|_W \quad (f \in A + \bar{A}_0)$ for some positive constant δ.

(2) There exist a positive constant $\varepsilon \nless \delta$, an inner function Q and a real function ℓ in L^1 such that

$$\frac{\alpha\bar{\beta} - \varepsilon^2}{|\alpha\bar{\beta} - \varepsilon^2|} = Qe^{i\tilde{\ell}} \text{ and } We^{-\ell} \in HS$$

where $\alpha\bar{\beta} - \varepsilon^2$, α and β are invertible in L^∞.

(3) There exist an inner function Q and a real function t in L^1 such that

$$\frac{\alpha\bar{\beta}}{|\alpha\bar{\beta}|} = Qe^{i\tilde{t}} \quad \text{and} \quad We^{-t} \in HS$$

and both α and β are invertible in L^∞.

Proof. If $ess. \inf |\alpha - \beta| > 0$ and $r = |\alpha - \beta| / \|\alpha - \beta\|_\infty$, then by (3) of Theorem 1 and the remark in §2, $HS_r = B_r^0 = HS$. Hence the equivalence of (1) and (2) follows from Theorem 4. The equivalence of (2) and (3) follows from Lemma 3.

Theorem 5. Suppose α, β are in L^∞ with $\alpha - \beta \not\equiv 0$ and W is a nonnegative function in L^1. There exists a nonnegative function U with $|\alpha - \beta| U \not\equiv 0$ such that

$$\|S_{\alpha,\beta} f\|_W \geq \|f\|_U \qquad (f \in A + \bar{A}_0)$$

if and only if there exist a real constant c and a real function t in L^1 such that

$$\frac{\alpha\bar{\beta}}{|\alpha\bar{\beta}|} = e^{i(c+\tilde{t})} \quad \text{and} \quad \frac{e^t}{|\alpha\bar{\beta}|W} \in L^1$$

Proof. Suppose $\|S_{\alpha,\beta} f\|_W \geq \|f\|_U \quad (f \in A + \bar{A}_0)$. Let $W_1 = |\alpha|^2 W - U$, $W_2 = |\beta|^2 W - U$, $W_3 = \alpha\bar{\beta} W - U$. Then, for $f_1 \in A$ and $f_2 \in \bar{A}_0$,

$$\int_T \{|f_1|^2 W_1 + |f_2|^2 W_2 + 2Re(f_1 \bar{f}_2 W_3)\} dm \geq 0.$$

By the Cotlar-Sadosky lifting theorem [1], $W_1 \geq 0, W_2 \geq 0$, and there exists a k in H^1 such that $|W_3 - k|^2 \leq W_1 W_2$. Hence, $|\alpha|^2 W - U \geq 0$, $|\beta|^2 W - U \geq 0$, and

$$|(\alpha\bar{\beta} W - U) - k|^2 \leq (|\alpha|^2 W - U)(|\beta|^2 W - U).$$

By a calculation,

$$(|\alpha|^2 W - U)^{1/2} (|\beta|^2 W - U)^{1/2} \leq |\alpha\bar{\beta}| W - U$$

and hence

$$|\alpha\bar{\beta} W - k| \leq (|\alpha|^2 W - U)^{1/2} (|\beta|^2 W - U)^{1/2} + U \leq |\alpha\bar{\beta}| W.$$

If $k \equiv 0$, then $|\alpha\bar{\beta} W - U|^2 \leq (|\alpha|^2 W - U)(|\beta|^2 W - U)$ and hence $|\alpha - \beta|^2 WU \equiv 0$. Hence $|\alpha - \beta| U = 0$ on $\{W > 0\}$. Since $U \leq \min\{|\alpha|^2, |\beta|^2\} W$, we have $U = 0$ on $\{W = 0\}$, and hence $|\alpha - \beta| U \equiv 0$. This contradiction implies that k is a nonzero function. Hence $\log |\alpha\bar{\beta} W| \in L^1$ because $|k| \leq 2|\alpha\bar{\beta} W|$. There exists an outer function g in H^1 such that $|\alpha\bar{\beta} W| = |g|$. Hence $|\alpha\bar{\beta} W - k| \leq |g|$. By a lemma of Koosis $(cf.[2, p161])$ there exists an outer function f in H^1 such that

$$\frac{\alpha\bar{\beta}}{g}W = \frac{f}{|f|}.$$

gf is an outer function in H^1 and hence $gf = e^{t+i(\tilde{t}+c)}$ where $t = \log|gf|$ and c is a real constant. Thus

$$\frac{\alpha\bar{\beta}}{|\alpha\bar{\beta}|} = \frac{gf}{|gf|} = e^{i(\tilde{t}+c)}$$

and

$$\frac{e^t}{|\alpha\bar{\beta}|W} = \frac{e^t}{|g|} = |f| \in L^1.$$

We will show the converse. Put $G = e^t/|\alpha\bar{\beta}|W$, then $G \in L^1$. Put $k = e^{t+i(c+\tilde{t})}/(G+i\tilde{G})$, then $|k| \leq e^t/G = |\alpha\bar{\beta}|W$ and hence $k \in H^1$. Since $\alpha\bar{\beta}/|\alpha\bar{\beta}| = e^{i(c+\tilde{t})}$,

$$\begin{aligned}
|\alpha\bar{\beta}W - k| &= \left|\alpha\bar{\beta}W - \frac{e^{t+i(c+\tilde{t})}}{G+i\tilde{G}}\right| = \left||\alpha\bar{\beta}|W - \frac{e^t}{G+i\tilde{G}}\right| \\
&= |\alpha\bar{\beta}|W\left|1 - \frac{G}{G+i\tilde{G}}\right| = |\alpha\bar{\beta}|W\left|\frac{\tilde{G}}{G+i\tilde{G}}\right|.
\end{aligned}$$

Let $U = G^2W/(G^2+\tilde{G}^2)$, then $U > 0$ and

$$|\alpha\bar{\beta}W - k|^2 = |\alpha\bar{\beta}|^2W(W-U)$$

We may assume $|\alpha| \leq 1$ and $|\beta| \leq 1$. For all $f = f_1 + f_2 \in A + \bar{A}_0$,

$$\begin{aligned}
&\int |S_{\alpha,\beta}f|^2Wdm - \int|P_+f|^2|\alpha\bar{\beta}|^2Udm \\
&= \int|f_1|^2(|\alpha|^2W - |\alpha|^2|\beta|^2U)dm + \int|f_2|^2|\beta|^2Wdm + 2Re\int f_1\bar{f}_2(\alpha\bar{\beta}W - k)dm \\
&\geq \int|f_1|^2|\alpha|^2(W-U)dm + \int|f_2|^2|\beta|^2Wdm - 2\int|f_1\bar{f}_2||\alpha\bar{\beta}|W^{1/2}(W-U)^{1/2}dm \\
&= \int\{|f_1||\alpha|(W-U)^{1/2} - |f_2||\beta|W^{1/2}\}^2dm \geq 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int |S_{\alpha\beta}f|^2Wdm - \int|P_-f|^2|\alpha\bar{\beta}|^2Udm \\
&= \int|f_1|^2|\alpha|^2Wdm + \int|f_2|^2(|\beta|^2W - |\alpha|^2|\beta|^2U)dm + 2Re\int f_1\bar{f}_2(\alpha\bar{\beta}W - k)dm \\
&\geq \int|f_1|^2|\alpha|^2Wdm + \int|f_2|^2|\beta|^2(W-U)dm - 2\int|f_1\bar{f}_2||\alpha\bar{\beta}|W^{1/2}(W-U)^{1/2}dm \\
&\geq 0
\end{aligned}$$

Suppose $U' = |\alpha\bar{\beta}|^2U/4$, then $U' > 0$. Since $\alpha - \beta \not\equiv 0, |\alpha - \beta|U' \not\equiv 0$ and

$$\|S_{\alpha,\beta}f\|_W \geq \|P_+f\|_{U'} + \|P_-f\|_{U'} \geq \|f\|_{U'}.$$

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