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Hilbert Schemes and Simple Singularities A_n and D_n

YUKARI ITO AND IKU NAKAMURA

ABSTRACT. For any finite subgroup G of $SL(2,\mathbb{C})$ of order n, we consider a certain subscheme $\mathrm{Hilb}^G(\mathbb{A}^2)$ of $\mathrm{Hilb}^n(\mathbb{A}^2)$ consisting of G-invariant 0-dimensional subschemes of length n. We prove without using the classification of finite subgroups in $SL(2,\mathbb{C})$ that $\mathrm{Hilb}^G(\mathbb{A}^2)$ is a minimal resolution of a simple singularity $\mathbb{A}^2/G$ in the canonical manner. Any point of the exceptional set is a G-invariant 0-dimensional subscheme of $\mathbb{A}^2$ with support the origin, to which we associate the ideal sheaf I defining the subscheme. A minimal G-submodule of I generating the $\mathcal{O}_{\mathbb{A}^2}$-module I is a one dimensional trivial G-module plus one or two nontrivial mutually distinct irreducible G-modules. This gives a bijective correpondence between the set of all the irreducible components of the exceptional set and the set of all the equivalence classes of irreducible G-modules, which turns out to be the (classical) McKay correspondence.

Introduction. It is well known that to finite subgroups of $SL(2,\mathbb{C})$, there correspond certain hypersurface singularities as the quotients of a complex affine plane $\mathbb{A}^2$ by the finite groups. These singularities are called simple singularities, Kleinian singularities, rational double points or ADE. In the present article we study the simple singularities and their resolutions (of singularities) from a purely algebro-geometric point of view.

Let $\mathrm{S}^n(\mathbb{A}^2)$ ($\simeq \mathrm{Chow}^n(\mathbb{A}^2)$) be the n-th symmetric product of $\mathbb{A}^2$, that is by definition, the quotient of the product of n-copies of $\mathbb{A}^2$ by the natural action of the symmetry group $\mathcal{S}_n$ of n letters. Let $\mathrm{Hilb}^n(\mathbb{A}^2)$ be the Hilbert scheme of $\mathbb{A}^2$ parametrizing all the 0-dimensional subschemes of length n. By [Beauville][Fogarty] $\mathrm{Hilb}^n(\mathbb{A}^2)$ is a crepant resolution of $\mathrm{S}^n(\mathbb{A}^2)$ with a holomorphic symplectic structure.

Let G be an arbitrary finite subgroup of $SL(2,\mathbb{C})$. The group G operates on $\mathbb{A}^2$ so that it operates upon both $\mathrm{Hilb}^n(\mathbb{A}^2)$ and $\mathrm{S}^n(\mathbb{A}^2)$ canonically. Now we consider the particular case where n is equal to the order of G. Then it is easy to see that the G-fixed point set $\mathrm{S}^n(\mathbb{A}^2)^G$ in $\mathrm{S}^n(\mathbb{A}^2)$ is isomorphic to the quotient space $\mathbb{A}^2/G$. The G-fixed point set $\mathrm{Hilb}^n(\mathbb{A}^2)^G$ in $\mathrm{Hilb}^n(\mathbb{A}^2)$ is always nonsingular, but *a priori* can be disconnected. There is however a unique irreducible component of $\mathrm{Hilb}^n(\mathbb{A}^2)^G$ dominating $\mathrm{S}^n(\mathbb{A}^2)^G$, which we denote by $\mathrm{Hilb}^G(\mathbb{A}^2)$. (See Corollary 5.5 and the remark to it.) Since $\mathrm{Hilb}^G(\mathbb{A}^2)$ inherits a holomorphic symplectic structure from $\mathrm{Hilb}^n(\mathbb{A}^2)$, $\mathrm{Hilb}^G(\mathbb{A}^2)$ is a crepant (or minimal) resolution of an affine normal surface $\mathbb{A}^2/G$ with an isolated singularity (Theorem 1.3).

The purpose of the present article is to study the structure of $\mathrm{Hilb}^G(\mathbb{A}^2)$ in detail in the cases A_n and D_n using the symmetric tensor or homogeneous polynomial representations of G. The cases E_6,E_7 and E_8 will be studied in "the second half" [N1] of the article.

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Let $\mathfrak{m}$ (resp. $\mathfrak{m}_S$) be the maximal ideal of the origin of $\mathbb{A}^2$ (resp. $S := \mathbb{A}^2/G$) and let $\mathfrak{n} = \mathfrak{m}_S\mathcal{O}_{\mathbb{A}^2}$. Any point $\mathfrak{p}$ of the exceptional set E of $\mathrm{Hilb}^G(\mathbb{A}^2)$ is a G-invariant 0-dimensional subscheme Z of $\mathbb{A}^2$ with support the origin, to which we associate a G-invariant ideal subsheaf I of $\mathfrak{m}$ defining Z. As is easily shown, I contains $\mathfrak{n}$ (Corollary 2.3). Let $V(I) := I/(\mathfrak{m}I + \mathfrak{n})$. The finite G-module $V(I)$ is isomorphic to a minimal G-submodule of $I/\mathfrak{n}$ generating the $\mathcal{O}_{\mathbb{A}^2}$-module $I/\mathfrak{n}$.

If $\mathfrak{p}$ is a smooth point of the exceptional set E, $V(I)$ is a nontrivial irreducible G-module. Meanwhile if $\mathfrak{p}$ is a singular point of E, then $V(I)$ is a sum of two mutually distinct nontrivial irreducible G-modules. For any equivalence class of a nontrivial irreducible G-module ρ we define a subset $E(\rho)$ of E consisting of all $I \in \mathrm{Hilb}^G(\mathbb{A}^2)$ such that $V(I)$ contains ρ as a G-submodule. We will see that $E(\rho)$ is naturally identified with the set of all nontrivial proper G-submodules of $\rho^{\oplus 2}$, which is isomorphic to a smooth rational curve by Schur's lemma (Theorems 3.5 and 3.6). The map $\rho \mapsto E(\rho)$ gives a bijective correpondence (Theorem 3.1) between the set $\mathrm{Irr}(G)$ of all the equivalence classes of irreducible G-modules and the set $\mathrm{Irr}(E)$ of all the irreducible components of E, which turns out to be the classical McKay correpondence [McKay]. We will also give an explanation as to why the tensoring process by the natural representation enters the McKay correpondence. (See the paragraph 5.8 for instance.) An outline of the story is given in the paragraph 5.8. In addition to the McKay correspondence, the most remarkable is that there are two kinds of dualities (Theorems 3.2, 3.3, 3.5) in the G-module decomposition of the algebra $\mathfrak{m}/\mathfrak{n}$ called the coalgebra of the group G. It is the second duality that explain the naturality of tensoring process by the natural representation in the McKay correspondence. See also [IN2].

We remark that our results of the present article are true as well in any characteristic if the groud field k is algebraically closed and if the order of G and the characteristic of k are coprime.

The paper is organized as follows. In section one we prove that $\mathrm{Hilb}^G(\mathbb{A}^2)$ is a crepant (or minimal) resolution of $\mathbb{A}^2/G$. In section 2 we give some preparatory lemmas from representations of finite groups. In section 3 we formulate our main theorem and relevant theorems we prove in the present article and [N1]. We will give a complete description of the ideals corresponding to the points of the exceptional set E. In sections 4 and 5 we study $\mathrm{Hilb}^G(\mathbb{A}^2)$ and prove the main theorem in the cases A_n and D_n. The cases E_6,E_7 and E_8 will be discussed in [N1].

1. The Crepant (Minimal) Resolution

Lemma 1.1. *Let G be a finite group in $GL(2,\mathbb{C})$, $\mathrm{Hilb}^n(\mathbb{A}^2)^G$ the subset of $\mathrm{Hilb}^n(\mathbb{A}^2)$ consisting of all the points fixed by any element of G. Then $\mathrm{Hilb}^n(\mathbb{A}^2)^G$ is nonsingular.*

Proof. By [Fogarty] $\mathrm{Hilb}^n(\mathbb{A}^2)$ is nonsingular. Let $\mathfrak{p}$ be a point in $\mathrm{Hilb}^n(\mathbb{A}^2)^G$. Then the action of G on $\mathrm{Hilb}^n(\mathbb{A}^2)$ at $\mathfrak{p}$ is linearlized.

This is shown as follows. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{W,p}$. Then G operates on $\mathfrak{m}/\mathfrak{m}^2$ in the natural manner. Let B be a finite subset of $\mathfrak{m}$ whose natural images in $\mathfrak{m}/\mathfrak{m}^2$ generate $\mathfrak{m}/\mathfrak{m}^2$. Since G is finite, the set $B^* := \{g(a); g \in G, a \in B\}$ is also finite. Let U be a linear subspace of $\mathfrak{m}$ spannned by B^*, which is obviously finite dimensional. By the definition the linear space U is G-invariant and the natural homomorphism $\pi : U \to \mathfrak{m}/\mathfrak{m}^2$ is G-equivariant. Since any representation of the finite group G is completely reducible, there exists a G-equivariant linear subspace V of U such that $U = V \oplus \operatorname{Ker} \pi$. It follows that $V \simeq \mathfrak{m}/\mathfrak{m}^2$. This shows that the action of G is linearized by a basis of V.

Therefore the fixed locus $\mathrm{Hilb}^n(\mathbb{A}^2)^G$ at $\mathfrak{p}$ is a linear subspace of W with regards to the basis of V. Hence it is nonsingular. q.e.d.

Lemma 1.2. *Let G be a finite subgroup in $SL(2,\mathbb{C})$, n the order of G and $\mathrm{S}^n(\mathbb{A}^2)^G$ the subset of $\mathrm{S}^n(\mathbb{A}^2)$ consisting of all the points of $\mathrm{S}^n(\mathbb{A}^2)$ fixed by any element of G. Then $\mathrm{S}^n(\mathbb{A}^2)^G \simeq \mathbb{A}^2/G$.*

Proof. Let $\mathfrak{q}$ $(\neq 0) \in \mathbb{A}^2$. The point $\mathfrak{q}$ is fixed by no element of G except the identity. Therefore the set $G \cdot \mathfrak{q} := \{g(\mathfrak{q}); g \in G\}$ determines a point in $\mathrm{S}^n(\mathbb{A}^2)^G$. Conversely any point of $\mathrm{S}^n(\mathbb{A}^2)^G$ is an unordered set Σ of n points in $\mathbb{A}^2$. If Σ contains a point $\mathfrak{q}$ different from the origin, the above argument shows that Σ contains the set $G \cdot \mathfrak{q}$. Since $|\Sigma| = n = |G|$, we have $\Sigma = G \cdot \mathfrak{q}$.

We see that $G\cdot\mathfrak{q} = G\cdot\mathfrak{q}'$ for a pair of points $\mathfrak{q}$ $(\neq 0)$ and $\mathfrak{q}'$ $(\neq 0)$ if and only if $\mathfrak{q}' \in G\cdot\mathfrak{q}$. Therefore we have the isomorphism $\mathrm{S}^n(\mathbb{A}^2 \setminus \{0\})^G \simeq (\mathbb{A}^2 \setminus \{0\})/G$, which extends to a finite morphism of $\mathrm{S}^n(\mathbb{A}^2)^G$ onto $\mathbb{A}^2/G$ naturally. It follows from Zariski main theorem that $\mathrm{S}^n(\mathbb{A}^2)^G \simeq \mathbb{A}^2/G$ because $\mathbb{A}^2/G$ is normal. q.e.d.

Theorem 1.3. *Let G be a finite subgroup in $SL(2,\mathbb{C})$, n the order of G. Then there is a unique irreducible component $\mathrm{Hilb}^G(\mathbb{A}^2)$ of $\mathrm{Hilb}^n(\mathbb{A}^2)^G$ dominating $\mathbb{A}^2/G$, which is a crepant (or equivalently a minimal) resolution of $\mathbb{A}^2/G$.*

Proof. We have a natural morphism π of $\mathrm{Hilb}^G(\mathbb{A}^2)$ onto $\mathrm{S}^n(\mathbb{A}^2)^G$ defined by $\pi(Z) = \mathrm{supp}(Z)$ (with due multiplicities) for a 0-dimensional subscheme Z of $\mathbb{A}^2$.

Any point of $\mathrm{S}^n(\mathbb{A}^2)^G \setminus \{0\}$ is a G-orbit of a point $\mathfrak{p}\,(\neq 0) \in \mathbb{A}^2$. It determines a G-invariant reduced 0-dimensional subscheme. It follows that $\mathrm{Hilb}^G(\mathbb{A}^2)$ is birationally equivalent to $\mathrm{S}^n(\mathbb{A}^2)^G$, so that it is a resolution of $\mathrm{S}^n(\mathbb{A}^2)^G$.

By [Fujiki, Proposition 2.6], $\mathrm{Hilb}^G(\mathbb{A}^2)$ inherits canonically a holomorphic symplectic structure from $\mathrm{Hilb}(\mathbb{A}^2)$. Since $\dim \mathrm{Hilb}^G(\mathbb{A}^2) = \dim \mathbb{A}^2/G = 2$, this implies that the dualising sheaf of $\mathrm{Hilb}^G(\mathbb{A}^2)$ is trivial. This completes the proof. □

Lemma 1.4. *Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{\mathbb{A}^2,0}$ and $Z \in \mathrm{Hilb}^n(\mathbb{A}^2)$ with $\mathrm{supp}(Z)$ the origin. Let $I := I_Z$ be the ideal defining Z. Then $\mathfrak{m}^n \subset I$.*

Proof. We consider a descending chain of ideals $\mathfrak{m} \supset I + \mathfrak{m}^2 \supset \cdots \supset I + \mathfrak{m}^{n+1} \supset I$. Since $h^0(\mathfrak{m}/I) = h^0(\mathcal{O}_{\mathbb{A}^2}/I) - 1 = n - 1$, there exists the least positive integer $k \leq n$ such that $I + \mathfrak{m}^k = I + \mathfrak{m}^{k+1}$. Let $M := I + \mathfrak{m}^k/I$. Then we see that $\mathfrak{m}M = I + \mathfrak{m}^{k+1}/I = M$, whence $M = 0$ by Nakayama's lemma. It follows that $\mathfrak{m}^k \subset I$, a fortiori, $\mathfrak{m}^n \subset I$. □

Remark. In what follows we identify a subscheme Z and the ideal I_Z, so that we consider $I_Z \in \mathrm{Hilb}^n(\mathbb{A}^2)$ if no confusion is possible.

2. G-modules.

Lemma 2.1. *Let G be a finite group of $GL(n,\mathbb{C})$. Let S be a connected scheme, $\mathcal{I}$ an ideal of $\mathcal{O}_{\mathbb{A}^n\times S}$ with $\mathcal{O}_{\mathbb{A}^n\times S}/\mathcal{I}$ flat over S and $\mathcal{I}_s := \mathcal{I}\otimes\mathcal{O}_{\mathbb{A}^n\times\{s\}}$. Suppose that we are given a regular action of G on $\mathcal{O}_{\mathbb{A}^n\times S}$. If* $\dim\operatorname{supp}(\mathcal{O}_{\mathbb{A}^n\times\{s\}}/\mathcal{I}_s)=0$ *for any $s\in S$, then $\mathcal{O}_{\mathbb{A}^n\times\{s\}}/\mathcal{I}_s$ is independent of s as G-modules.*

Proof. Since $\dim\operatorname{supp}(\mathcal{O}_{\mathbb{A}^n\times\{s\}}/\mathcal{I}_s)=0$, $h^1(\mathcal{O}_{\mathbb{A}^n\times\{s\}}/\mathcal{I}_s)=0$. Hence $h^0(\mathcal{O}_{\mathbb{A}^n\times\{s\}}/\mathcal{I}_s)$ is constant on S. Therefore the sheaf $\mathcal{O}_{\mathbb{A}^n\times S}/\mathcal{I}$ is a locally free $\mathcal{O}_S$-module of finite rank. By the assumption, G operates upon the vector space $\mathcal{O}_{\mathbb{A}^n\times\{s\}}/\mathcal{I}_s$, depending on s regularly. Let $\Delta(g,x):=\det(x\cdot\mathrm{id}-T(g))$ be the characteristic polynomial of the action $T(g)$ of $g\in G$ on $\mathcal{O}_{\mathbb{A}^n\times S}/\mathcal{I}$. The coefficients of $\Delta(g,x)$ are fixed (or rather universal) symmetric polynomials of eigenvalues of $T(g)$. Since all the eigenvalues of $T(g)$ are roots of unity, it follows that the coefficients of $\Delta(g,x)$ are constant on S. In particular the character $\operatorname{Tr}T(g)$ is independent of $s\in S$. Since any representation of a finite group is uniquely determined by its character, the representation of G on $\mathcal{O}_{\mathbb{A}^n\times\{s\}}/\mathcal{I}_s$ is independent of $s\in S$. □

Corollary 2.2. *Let G be a finite subgroup of $SL(2,\mathbb{C})$, I an ideal of $\mathcal{O}_{\mathbb{A}^2}$ with $I\in\mathrm{Hilb}^G(\mathbb{A}^2)$. Then as G-modules $\mathcal{O}_{\mathbb{A}^2}/I\simeq\mathbb{C}[G]\simeq\sum_\rho(\deg\rho)\rho$, the group algebra.*

Corollary 2.3. *Let I be an ideal of $\mathcal{O}_{\mathbb{A}^2}$ with $I\in\mathrm{Hilb}^G(\mathbb{A}^2)$. Any G-invariant function vanishing at the origin is contained in I.*

Proof. $\mathcal{O}_{\mathbb{A}^2}/I\simeq\mathbb{C}[G]$ by Corollary 2.2. Any G-submodule of $\mathcal{O}_{\mathbb{A}^2}/I$ is viewed as a G-submodule by the complete reducibility of G-modules. This implies that $\mathcal{O}_{\mathbb{A}^2}/I$ has a unique trivial G-submodule spanned by constant fucntions of $\mathbb{A}^2$. It follows that any G-invariant function vanishing at the origin is contained in I. □

3. Main Theorem.

Any finite subgroup of $SL(2,\mathbb{C})$ is, up to conjugacy, one of the following subgroups by [Klein]

type	G		order	h
A_n	$\mathbb{Z}_{n+1}$	cyclic	$n+1$	$n+1$
D_n	$\mathbb{D}_{n-2}$	binary dihedral	$4(n-2)$	$2n-2$
E_6	$\mathbb{T}$	binary tetrahedral	24	12
E_7	$\mathbb{O}$	binary octahedral	48	18
E_8	$\mathbb{I}$	binary icosahedral	120	30

where h is the Coxeter number of the corresponding root system.

Let G be a finite subgroup of $SL(2,\mathbb{C})$, $\mathrm{Irr}(G)$ the set of all nontrivial irreducible G-modules, and $\mathrm{Irr}_0(G)$ the union of $\mathrm{Irr}(G)$ and the trivial one dimensional G-module.

Let $V(\rho)\in\mathrm{Irr}(G)$ be a G-module, ρ the corresponding homomorphism of G into $GL(V(\rho))$. If no confusion is possible we consider $V(\rho)\in\mathrm{Irr}(G)$ or $\rho\in\mathrm{Irr}(G)$ interchangeably.

Let $X = X_G := \mathrm{Hilb}^G(\mathbb{A}^2)$, $S = S_G := \mathbb{A}^2/G$, $\mathfrak{m}$ (resp. $\mathfrak{m}_S$) the maximal ideal of X (resp. S) at the origin and $\mathfrak{n} := \mathfrak{m}_S\mathcal{O}_{\mathbb{A}^2}$. Let $\pi : X \to S$ be the natural morphism and E the exceptional set of π. Let $\mathrm{Irr}(E)$ be the set of irreducible components of E. Any $I \in X$ contained in E is a G-invariant ideal of $\mathcal{O}_{\mathbb{A}^2}$ which contains $\mathfrak{n}$ by Lemma 2.2.

First we define

Definition. $V(I) := I/(\mathfrak{m}I + \mathfrak{n})$.

Definition. For any ρ,ρ', and $\rho'' \in \mathrm{Irr}(G)$ we define

$$E(\rho) := \left\{ I \in \mathrm{Hilb}^G(\mathbb{A}^2); V(I) \text{ contains a } G\text{-module } V(\rho) \right\}$$
$$P(\rho,\rho') := \left\{ I \in \mathrm{Hilb}^G(\mathbb{A}^2); V(I) \text{ contains a } G\text{-module } V(\rho)\oplus V(\rho') \right\}$$
$$Q(\rho,\rho',\rho'') := \left\{ I \in \mathrm{Hilb}^G(\mathbb{A}^2); V(I) \text{ contains a } G\text{-module } V(\rho)\oplus V(\rho')\oplus V(\rho'') \right\}.$$

Remark. If $\rho \neq \rho'$, then $P(\rho,\rho') = E(\rho)\cap E(\rho')$. Note that $\rho = \rho'$ is possible.

Definition. Two irreducible G-modules ρ and ρ' are (McKay-) adjacent if $\rho\otimes\rho_{\mathbf{nat}} \supset \rho'$ or vice versa.

Definition. The McKay graph $\Gamma(\mathrm{Irr}(G))$ of $\mathrm{Irr}(G)$ is defined to be a graph whose vertices are $\mathrm{Irr}(G)$. Two vertices ρ and ρ' of $\Gamma(\mathrm{Irr}(G))$ are connected by a single edge if and only if ρ and ρ' are adjacent.

Then our main theorem is stated as follows.

Theorem 3.1. *Let G be a finite subgroup of $SL(2,\mathbb{C})$. Then*

(1) the map $\rho \mapsto E(\rho)$ is a bijective correspondence between $\mathrm{Irr}(G)$ *and* $\mathrm{Irr}(E)$,

(2) $E(\rho)$ is a smooth rational curve for any $\rho \in \mathrm{Irr}(G)$ with $E(\rho)^2 = -2$,

(3) $P(\rho,\rho') \neq \emptyset$ if and only if ρ and ρ' are adjacent. In this case $P(\rho,\rho')$ is a (reduced) single point, where $E(\rho)$ and $E(\rho')$ intersect transversally.

(4) $P(\rho,\rho) = Q(\rho,\rho',\rho'') = \emptyset$ for any $\rho,\rho'\,\rho'' \in \mathrm{Irr}(G)$.

By Theorem 3.1 (3) we see that if $\rho\neq\rho'$, then $E(\rho)E(\rho') = 1$ or 0 if ρ and ρ' are adjacent or otherwise. Therefore $\Gamma(\mathrm{Irr}(G))$ is the same as $\Gamma(\mathrm{Irr}(E))$, the dual graph $\Gamma(\mathrm{Irr}(E))$ of E, in other words, the Dynkin diagram of S_G. Let h $(=: 2\bar{h})$ be the Coxeter number of $\Gamma(\mathrm{Irr}(E))$.

We define nonnegative integers $d(\rho)$ for any $\rho\in\mathrm{Irr}(G)$ as follows.

If G is cyclic, then we define $e(\chi^k) = k$, $d(\chi^k) = |\frac{n+1}{2} - k|$ where χ is a generator of the character group of G. Though the definition of $e(\rho)$ and $d(\rho)$ depends on the choice of the generator χ, the definition of the pair $\left(\bar{h} - d(\rho), \bar{h} + d(\rho)\right) = (e(\rho), n+1-e(\rho))$ is independent of the choice. If G is not cyclic, then $\Gamma(\mathrm{Irr}(G))$ is star-shape with a unique center. For any $\rho\in\mathrm{Irr}(G)$, we define $d(\rho)$ to be the distance from the vertex ρ to the center, where $d(\rho) = 0$ for the center ρ. It is evident that $d(\rho) = d(\rho')\pm 1$ if ρ and $\rho'\in\mathrm{Irr}(G)$ are adjecent.

For any positive integer m let $S_m := S_m(\rho_{\mathbf{nat}})$ be the symmetric m-tensors by $\rho_{\mathbf{nat}}$, that is, the space of homogeneous polynomials of degree m. We call a G-submodule W of $\mathfrak{m}/\mathfrak{n}$

homogeneous of degree m if it is generated by homogeneous polynomials of the same degree m. $\mathfrak{m}/\mathfrak{n}$ is decomposed into irreducible homogeneous G-modules because $\mathfrak{n}$ is generated by homogeneous polynomials. If W is a direct sum of homogeneous G-submodules, then we denote the homogeneous part of W of degree m by $S_m(W)$. For any G-module W in some $S_m(\mathfrak{m}/\mathfrak{n})$, we denote by $S_j \cdot W$ a G-submodule of $S_{m+j}(\mathfrak{m}/\mathfrak{n})$ generated by the products of $S_j(\mathfrak{m}/\mathfrak{n})$ and W. We denote by $W[\rho]$ the ρ-factor of W, that is, the sum of all the copies of ρ in W. We denote by $[W : \rho]$ the multiplicity of $\rho \in \mathrm{Irr}(G)$ in a G-module W. We define

Definition.

$$S_{\mathrm{McKay}}(\mathfrak{m}/\mathfrak{n}) = \sum_{\rho\in\mathrm{Irr}(G)} S_{\bar{h}\pm d(\rho)}(\mathfrak{m}/\mathfrak{n})[\rho]$$

Theorem 3.2. *Let G be any finite subgroup of $SL(2,\mathbb{C})$ and $2\bar{h}$ the Coxeter number of G. Then as G-modules, we have*

(1) $\mathfrak{m}/\mathfrak{n} = \sum_{\rho\in\mathrm{Irr}(G)} 2(\deg \rho)\rho$,

(2) $S_{\mathrm{McKay}}(\mathfrak{m}/\mathfrak{n}) \simeq \sum_{\rho\in\mathrm{Irr}(G)} 2\rho$,

(3) $S_{\bar{h}-k}(\mathfrak{m}/\mathfrak{n}) \simeq S_{\bar{h}+k}(\mathfrak{m}/\mathfrak{n})$ *for any* k, $S_k(\mathfrak{m}/\mathfrak{n}) = 0$ *for* $k \geq 2\bar{h}$,

See [IN2] for Theorem 3.2. The duality (3) in the above theorem is sharpened as follows. It is this form of the duality that we need for the explanation of the McKay observation.

Theorem 3.3. *(Duality) Assume that G is cyclic. Then for any $\rho \in \mathrm{Irr}(G)$ there exists a unique pair $V^+_{e(\rho)}(\rho)$ and $V^-_{n+1-e(\rho)}(\rho)$ of irreducible homogeneous G-submodules of $S_{e(\rho)}(\mathfrak{m}/\mathfrak{n})[\rho]$ and $S_{n+1-e(\rho)}(\mathfrak{m}/\mathfrak{n})[\rho]$ respectively with the following properties.*

(1) $V^+_{e(\rho)}(\rho) \simeq V^-_{n+1-e(\rho)}(\rho) \simeq \rho$.

(2) If ρ and ρ' are adjacent with $e(\rho) = e(\rho') + 1$, then $V^+_{e(\rho)}(\rho) = \{S_1 \cdot V^+_{e(\rho')}(\rho')\}[\rho]$ and $V^-_{n+1-e(\rho')}(\rho') = \{S_1 \cdot V^-_{n+1-e(\rho)}(\rho)\}[\rho']$,

Theorem 3.4. *(Duality) Assume that G is not cyclic. Let $2\bar{h}$ be the Coxeter number of S. Let $V_{\bar{h}\pm d(\rho)}(\rho) := S_{\bar{h}\pm d(\rho)}(\mathfrak{m}/\mathfrak{n})[\rho]$ for any $\rho \in \mathrm{Irr}(G)$. Then the following are true.*

(1) $V_{\bar{h}-d(\rho)}(\rho) \simeq V_{\bar{h}+d(\rho)}(\rho) \simeq \rho^{\oplus 2}$ *or* ρ *if* $d(\rho) = 0$ *resp.* $d(\rho) \geq 1$.

(2) If ρ and ρ' are adjacent with $d(\rho') = d(\rho)+1 \geq 2$, $V_{\bar{h}-d(\rho)}(\rho) = \{S_1 \cdot V_{\bar{h}-d(\rho')}(\rho')\}[\rho]$ and $V_{\bar{h}+d(\rho')}(\rho') = \{S_1 \cdot V_{\bar{h}+d(\rho)}(\rho)\}[\rho']$,

(3) If $d(\rho) = 0$, then there exists $\rho_i \in \mathrm{Irr}(G)$ $(i = 1, 2, 3)$ adjacent to ρ such that $\{S_1 \cdot V_{\bar{h}-1}(\rho_i)\}[\rho] \simeq \rho$ for any i, $V_{\bar{h}}(\rho) = \{S_1 \cdot V_{\bar{h}-1}(\rho_i)\}[\rho] + \{S_1 \cdot V_{\bar{h}-1}(\rho_j)\}[\rho]$ $(i \neq j)$ and $V_{\bar{h}+1}(\rho_i) = \{S_1 \cdot V_{\bar{h}}(\rho)\}[\rho_i]$.

The exceptional sets of $\mathrm{Hilb}^G(\mathbb{A}^2)$ are described in the following theorems.

Theorem 3.5. *Assume that G is cyclic. Then*

(1) $I \in E(\rho) \setminus \cup_{\rho'} P(\rho, \rho')$ if and only if $V(I)$ is a nonzero irreducible G-submodule $(\simeq \rho)$ of $V^+_{e(\rho)}(\rho) \oplus V^-_{n+1-e(\rho)}(\rho)$ different from $V^+_{e(\rho)}(\rho)$ and $V^-_{n+1-e(\rho)}(\rho)$

(2) $I \in P(\rho, \rho') \neq \emptyset$ with $e(\rho) = e(\rho')+1$ if and only if $V(I) = V^+_{e(\rho)}(\rho) \oplus V^-_{n+1-e(\rho')}(\rho')$.

Theorem 3.6. *Assume that G is not cyclic.*

(1) Assume $d(\rho) \geq 1$.

(1.1) $I \in E(\rho) \setminus \cup_{\rho'} P(\rho, \rho')$ if and only if $V(I)$ is a nonzero irreducible G-submodule ($\simeq \rho$) of $V_{\bar{h}-d(\rho)}(\rho) \oplus V_{\bar{h}+d(\rho)}(\rho)$ different from $V_{\bar{h}-d(\rho)}(\rho)$ and $V_{\bar{h}+d(\rho)}(\rho)$.

(1.2) Let ρ and ρ' be an adjacent pair with $d(\rho') = d(\rho) + 1$. Then $I \in P(\rho, \rho')$ if and only if $V(I) = V_{\bar{h}-d(\rho)}(\rho) \oplus V_{\bar{h}+d(\rho')}(\rho')$ $(=: W(\rho, \rho'))$.

(2) Assume $d(\rho) = 0$.

(2.1) $I \in E(\rho) \setminus \cup_{\rho'} P(\rho, \rho')$ if and only if $V(I)$ is a nonzero irreducible G-module ($\simeq \rho$) contained in $V_{\bar{h}-d(\rho)}(\rho) \oplus V_{\bar{h}+d(\rho)}(\rho)$ different from $\{S_1 \cdot V_{\bar{h}-1}(\rho')\}[\rho]$ for any $\rho' \in \mathrm{Irr}(G)$ adjacent to ρ.

(2.2) $I \in P(\rho, \rho') \neq \emptyset$ if and only if $V(I) = \{S_1 \cdot V_{\bar{h}-1}(\rho')\}[\rho] \oplus V_{\bar{h}+1}(\rho')$ $(=: W(\rho, \rho'))$.

Remark. Theorem 3.3-3.6 are proved in the present article for a binary dihedral group, and in the second part [N1] for the remaining cases binary tetra, octa and icosahedral groups.

Remark. One can recover I from $V(I)$ by defining $I = V(I)\mathcal{O}_{\mathbb{A}^2} + \mathfrak{n}$.

Remark. By Theorems 3.5 and 3.6, the curve $E(\rho)$ is identified with $\mathbb{P}(\rho \oplus \rho) \simeq \mathbb{P}^1$, the projective space of nontrivial proper G-submodules ρ in $\rho \oplus \rho$.

Remark. The relations in Theorem 3.4 (1.2) or (2.2) as well as the following observation explain why tensoring by the natural representation comes in the McKay correspondence. See also [IN2]. We observe

$$\begin{aligned}
W(\rho, \rho') &= V_{\bar{h}-d(\rho)}(\rho) \oplus V_{\bar{h}+d(\rho')}(\rho') \qquad (d(\rho) \geq 1, d(\rho') = d(\rho) + 1)\\
&= \{S_1 \cdot V_{\bar{h}-d(\rho')}(\rho')\}[\rho] \oplus V_{\bar{h}+d(\rho')}(\rho')\\
&= V_{\bar{h}-d(\rho)}(\rho) \oplus \{S_1 \cdot V_{\bar{h}+d(\rho)}(\rho)\}[\rho']\\
W(\rho, \rho') &= \{S_1 \cdot V_{\bar{h}-1}(\rho')\}[\rho] \oplus V_{\bar{h}+1}(\rho') \qquad (d(\rho) = 0, d(\rho') = 1)\\
&= \{S_1 \cdot V_{\bar{h}-1}(\rho')\}[\rho] \oplus \{S_1 \cdot V_{\bar{h}}(\rho)\}[\rho']
\end{aligned}$$

4. $\mathbf{A}_n$.

4.1. Character table Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{\mathbb{A}^2}$ at the origin. Let (x, y) be a system of coordinates of $\mathbb{A}^2$, G a cyclic group of order $n+1$ and σ a generator of G. Let ε be a primitive $(n+1)$-th root of unity. We define the action of the generator σ upon $\mathbb{C}^2$ by $(x, y) \mapsto (x, y) \cdot g = (\varepsilon x, \varepsilon^{-1} y)$. The simple singularity of type A_n is the quotient of $\mathbb{A}^2$ by the cyclic group G.

We recall the character table of G and relevant invariants.

The Coxeter number of A_n is equal to $n+1$.

ρ	σ	d	$h-d$	
ρ_0	1	$-$	$-$	
ρ_i	ϵ^i	i	$n+1-i$	$(1 \leq i \leq n)$

Let $\mathfrak{m}_S$ be the maximal ideal of S_G at the origin and $\mathfrak{n} := \mathfrak{m}_S\mathcal{O}_{\mathbb{A}^2}$.

Lemma 4.2. $\mathrm{Hilb}^G(\mathbb{A}^2)$ *is the union of the following G-invariant ideals of colength* $n+1$;

$$I(\Sigma) := \prod_{\mathfrak{p}\in\Sigma} \mathfrak{m}_{\mathfrak{p}} = (x^{n+1} - a^{n+1}, xy - ab, y^{n+1} - b^{n+1})$$
$$I_i(p_i : q_i) := (p_i x^i - q_i y^{n+1-i}, xy, x^{i+1}, y^{n+2-i})$$

where Σ *is a G-invariant subset of* $\mathbb{A}^2$ *disjoint from the origin with* $\mathfrak{p} := (a,b) \in \Sigma$, $\mathfrak{p} \neq (0,0)$, $1 \le i \le n$ *and* $[p_i, q_i] \in \mathbb{P}^1$.

Proof. Let $I \in \mathrm{Hilb}^G(\mathbb{A}^2)$ with $I \subset \mathfrak{m}$. Then by Corollary 2.2 $\mathcal{O}_{\mathbb{A}^2}/I \simeq \mathbb{C}[G] \simeq \oplus_{i=0}^n \rho_i$ as G-modules. $\mathfrak{n} \subset I$ by Corollary 2.3. Let $N := \mathfrak{m}/\mathfrak{n}$ and $M := I/\mathfrak{n}$. N is generated by x^i and y^j $(1 \le i,j \le n)$. Let $N[\rho_i]$ be the ρ_i-part of N. Then $N[\rho_i] \simeq \rho_i^{\oplus 2}$ and it is spanned by x^i and y^{n+1-i}. By Theorem 3.3 $M[\rho_i] \simeq \{\mathbb{C}[G] \ominus \rho_0\}[\rho_i] \simeq \rho_i$ $(\forall i \neq 0)$.

It follows that there exists i such that $u := p_i x^i - q_i y^{n+1-i} \in M$ for some $[p_i, q_i] \in \mathbb{P}^1$. If $p_i q_i \neq 0$, then $M = (u) + \mathfrak{n}/\mathfrak{n}$ and $I = (u, xy, x^{i+1}, y^{n+2-i})$.

Assume that M contains none of $p_i x^i - q_i y^{n+1-i}$ with $p_i q_i \neq 0$. It follows from $M[\rho] \simeq \rho$ $(\rho \neq \rho_0)$ that M is spanned by x^i and y^{n+2-i}. Hence $I = (x^i, y^{n+2-i}, xy)$. □

Remark. As we see in the proof of Corollary 4.4, $\mathrm{Hilb}^G(\mathbb{A}^2)$ is the disjoint union of the subsets in Lemma 4.2 except that $I_i(0:1) = I_{i+1}(1:0)$.

Theorem 4.3. *Let* a *and* b *be the parameters of* $\mathbb{A}^2$ *on which the group* G *acts by* $g(a,b) = (\varepsilon a, \varepsilon^{-1} b)$. *Let* $S := \mathbb{A}^2/G$, $\tilde{S}$ *the toric minimal resolution of* S *and* U_i *the affine charts of* $\tilde{S}$ *defined by*

$$\mathbb{A}^2/G \simeq \mathrm{Spec}\ \mathbb{C}[a^{n+1}, ab, b^{n+1}]$$
$$U_i := \mathrm{Spec}\ \mathbb{C}[s_i, t_i] \quad (1 \le i \le n+1)$$

where we denote $s_i := a^i/b^{n+1-i}$, *and* $t_i := b^{n+2-i}/a^{i-1}$ *under the usual notaion of torus embeddings. Then the isomorphism of* $\tilde{S}$ *with* $\mathrm{Hilb}^G(\mathbb{A}^2)$ *is given by (the morphism defined by the universal property of* $\mathrm{Hilb}^n(\mathbb{A}^2)$ *from) two-dimensional flat families of zero-dimensional subschemes defined by the following G-invariant ideals of* $\mathcal{O}_{\mathbb{A}^2}$ $(1 \le i \le n+1)$;

$$\mathcal{I}_i(s_i, t_i) := (x^i - s_i y^{n+1-i}, xy - s_i t_i, y^{n+2-i} - t_i x^{i-1}).$$

Proof. First we note $\mathcal{I}_i(s_i, 0) = I_i(1 : s_i)$. If $i \ge 2$, then $\mathcal{I}_i(0, t_i) = I_{i-1}(t_i : 1)$.

If $ab = s_i t_i \neq 0$, we see $\mathcal{I}_i(s_i, t_i) = (x^{n+1} - a^{n+1}, xy - ab, y^{n+1} - b^{n+1})$. In fact, let $p = (a,b) \neq (0,0) \in \mathbb{A}^2$ and $\Sigma := \{p \cdot g; g \in G\}$. It is clear that $\mathcal{I}_i(s_i, t_i) \subset \mathfrak{m}_p$ so that $\mathcal{I}_i(s_i, t_i) \subset I_\Sigma$ by the G-invariance of $\mathcal{I}_i(s_i, t_i)$. Since the colengths of $\mathcal{I}_i(s_i, t_i)$ and I_Σ in $\mathcal{O}_{\mathbb{A}^2}$ are equal to $n+1$, $\mathcal{I}_i(s_i, t_i) = I_\Sigma = (x^{n+1} - a^{n+1}, xy - ab, y^{n+1} - b^{n+1})$.

By the universality of $\mathrm{Hilb}^n(\mathbb{A}^2)$ and by Lemma 4.2, we have a finite birational morphism of $\tilde{S}$ onto $\mathrm{Hilb}^G(\mathbb{A}^2)$. It follows from Lemma 2.1 and Zariski's main theorem that $\tilde{S} \simeq \mathrm{Hilb}^G(\mathbb{A}^2)$. This completes the proof. □

Corollary 4.4. *The exceptional set of the morphism $\pi : \mathrm{Hilb}^G(\mathbb{A}^2) \to \mathbb{A}^2/G$ consists of a chain of n rational curves intersecting trasnversally.*

Proof. We define a subset E_i of Grass $:= \mathrm{Grass}(\mathfrak{m}/J + \mathfrak{m}^{n+1}, n) \simeq \mathrm{Grass}(2n, n)$ by $E_i = \{\bar{I}_i(p_i : q_i); [p_i, q_i] \in \mathbb{P}^1\} \simeq \mathbb{P}^1$ where $\bar{I}_i(p_i : q_i) := I_i(p_i : q_i)/\mathfrak{n}$.

We show that E_i and E_{i+1} intersect transversally in Grass at a point $\mathfrak{p}_i := \bar{I}_i(0:1) = \bar{I}_{i+1}(1:0)$. In fact, we see

$$\bar{I}_i(p_i : q_i) = \mathbb{C}(p_i x^i - q_i y^{n+1-i}) + \mathbb{C}x^{i+1} + W_i$$
$$\bar{I}_{i+1}(p_{i+1} : q_{i+1}) = \mathbb{C}(p_{i+1} x^{i+1} - q_{i+1} y^{n-i}) + \mathbb{C}y^{n+1-i} + W_i$$

where $W_i := \bigoplus_{j=n+2-i}^{n} \mathbb{C}x^j \oplus \bigoplus_{j=i+2}^{n} \mathbb{C}y^j$. In a down-to-earth term, with the coefficients of y^i, y^{i+1},x^{n-i} and x^{n+1-i}, the subspaces $\bar{I}_i(p_i : q_i)/W_i$ and $\bar{I}_{i+1}(p_{i+1} : q_{i+1})/W_i$ of $\mathfrak{m}/W_i$ are expressed as 2×4 matrices

$$\bar{I}_i(p_i : q_i)/W_i = \begin{pmatrix} p_i & 0 & 0 & q_i \\ 0 & 1 & 0 & 0 \end{pmatrix}$$
$$\bar{I}_{i+1}(p_{i+1} : q_{i+1})/W_i = \begin{pmatrix} 0 & p_{i+1} & q_{i+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It is easy to see using this expression that E_i and E_{i+1} intersect transversally at a point $\mathfrak{p}_i$. It remains to prove that the algebraic structure of E_i and the intersection are the same as those induced from the strucuture of $\mathrm{Hilb}^G(\mathbb{A}^2)$. We define a G-invariant ideal $\mathcal{I}$ of $\mathcal{O}_{\mathbb{A}^2 \times E}$ by $\mathcal{I} = (\mathcal{V} + \mathfrak{n})\, O_{\mathbb{A}^2 \times E}$ where $\mathcal{V}$ is the universal subbundle of $\mathcal{O}_{\mathrm{Grass}}^{\oplus 2n}$ of rank n over $\mathrm{Grass}(2n, n)$.

The coherent sheaf $O_{\mathbb{A}^2 \times E}/\mathcal{I}$ is $O_{\mathbb{A}^2 \times E}$-flat (in fact, locally free) because $\mathrm{supp}(O_{\mathbb{A}^2}/\mathcal{I}_t)$ is 0-dimensional and $\dim(\mathcal{O}_{\mathbb{A}^2}/\mathcal{I}_t))$ is constant. Thus we have a flat family of zero-dimensional subschemes in $\mathrm{Hilb}^G(\mathbb{A}^2)$ defined by G-invariant ideals $\mathcal{I}$ parametrized by the chain $E := E_1 + \cdots + E_n$ of rational curves. By the universality of $\mathrm{Hilb}^n(\mathbb{A}^2)$ we have an injective morphism of E into $\mathrm{Hilb}^G(\mathbb{A}^2)$. It follows that the algebraic structure of E_i and the intersection are the same as those induced from $\mathrm{Hilb}^n(\mathbb{A}^2)$. Therefore by Lemma 4.2 the exceptional set of π is E. □

4.5. Proof of Theorems 3.1, 3.2, 3.3 and 3.5 — A_n case

First we prove uniqueness of $V_j^{\pm}(\rho)$. Since $S_1 = \rho_1 \oplus \rho_n$, we have unique choices $V_1^+(\rho_1) = S_1[\rho_1]$ and $V_1^-(\rho_n) = S_1[\rho_n]$. Then we have

$$V_i^+(\rho_i) = \{S_1 \cdot V_{i-1}^+(\rho_{i-1})\}[\rho_i]$$
$$V_{n+1-i}^-(\rho_{n+1-i}) = \{S_1 \cdot V_{n-i}^-(\rho_{n-i})[\rho_{n+1-i}].$$

In fact, since $S_1 \cdot \rho_i = \rho_{i+1} \oplus \rho_{i-1}$, we have $V_i^+(\rho_i) \simeq \rho_i$ and $V_{n+1-i}^-(\rho_{n+1-i}) \simeq \rho_{n+1-i}$ by the induction hypothesis. This proves uniqueness.

We easily see that $V_i^+(\rho_i)$ and $V_{n+1-i}^-(\rho_i)$ are spanned respectively by x^i and y^{n+1-i} if $1 \le i \le n$. The assertions in Theorems 3.3 and 3.5 are clear from the above. If

$I \in E(\rho_i) \setminus \cup P(\rho_i, \rho_{i\pm 1})$, then $V(I) \simeq \rho_i$ and $I = I_i(1:s)$ for some $s \neq 0$, which proves (3) of Thoerem 3.2. If $I \in P(\rho_i, \rho_{i-1})$, then by Theorem 4.3 we have $I = I_i(1:0)$ and $V(I) = V_i^+(\rho_i) \oplus V_{n+2-i}^-(\rho_{i-1})$. This completes the proof of Theorems 3.2, 3.3 and 3.5.

Theorem 3.1 for G cyclic is now clear. It suffices to note $E(\rho_i) = E_i$.

5. $\mathbf{D}_5$ and $\mathbf{D}_n$.

5.1. Binary dihedral group. Let G be a subgroup of $SL(2,\mathbb{C})$ of order $4n-8$ generated by two elements σ and τ;

$$\sigma = \begin{pmatrix} \varepsilon, & 0 \\ 0, & \varepsilon^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}$$

where ε is a primitive $\ell := (2n-4)$-th root of unity. Then we have

$$\sigma^{2n-4} = 1,\ \tau^4 = 1,\ \sigma^{n-2} = \tau^2,\ \tau\sigma\tau^{-1} = \sigma^{-1}.$$

The group G is called a binary dihedral group $\mathbb{D}_{n-2}$. G operates upon $\mathbb{A}^2$ from the right by $(x,y) \mapsto (x,y)g$ $(g \in G)$. The ring of all G-invarinat polynomials is generated by $x^\ell + y^\ell$, $xy(x^\ell - y^\ell)$ and x^2y^2.

By Theorem 2.4, the surface $X_G := \mathrm{Hilb}^G(\mathbb{A}^2)$ is a minimal resolution of a singular affine surface $S_G := \mathbb{A}^2/G$ with a simple singularity of type D_n. The purpose of this section is to give two different descriptions of X_G. The first one is rather abstract, while the second is concrete as in Theorem 4.3. As a consequence of the first description of X_G, the fixed point set $\mathrm{Hilb}(\mathbb{A}^2)^G$ is connected (Corollary 5.5).

First we give an abstract description of a minimal resolution of $S_G := \mathbb{A}^2/G$ by using twice the technique in the case where the group is cyclic.

Let H be the subgroup of G generated by σ, $X_H := \mathrm{Hilb}^H(\mathbb{A}^2)$ and $S_H := \mathbb{A}^2/H$. Note that $|H| = \ell = 2n-4$. By Theorem 2.4, X_H is a minimal resolution of S_H. In view of Theorem 4.3 we have a set of affine charts U_k of X_H $(1 \le k \le 2n-4)$ defined by

$$U_k := \{\mathcal{I}_k(s_k, t_k)\,;\,(s_k,t_k) \in \mathbb{A}^2\}$$

where $s_k = a^{\ell+1-k}/b^{k-1}$, $t_k = b^k/a^{\ell-k}$. In what follows we view U_k as a toric chart of the minimal resolution of S_H or as a subset of $\mathrm{Hilb}^H(\mathbb{A}^2)$ interchangably.

Lemma 5.2.

(5.2.1) $\tau^*\mathcal{I}_k(s,t) = \mathcal{I}_{2n-3-k}((-1)^k t, (-1)^{k-1}s)$

(5.2.2) $\tau^*\mathcal{I}_{n-2}(s,t) = \mathcal{I}_{n-2}((-1)^{n-1}st^2, (-1)^n t^{-1})$ *if* $t \neq 0$

(5.2.3) $\tau^* I_k(p:q) = I_{2n-4-k}((-1)^k q : p)$.

Proof. It is easy to see that

$$\begin{aligned}\tau^* I_k(p,q) &= \tau^*(px^k - qy^{\ell-k}, xy, x^{k+1}, y^{\ell+1-k}) \\ &= (-(-1)^k q x^{\ell-k} + py^k, xy, x^{\ell+1-k}, y^{k+1}) \\ &= I_{2n-4-k}((-1)^k q, p)\end{aligned}$$

where $[p:q] \in E_k \simeq \mathbb{P}^1$, which proves (5.2.3). The rest is proved similarly. $\square$

Corollary 5.3. *We define an involution* $\mathrm{Hilb}(\tau)$ *of* X_H *by* $\mathrm{Hilb}(\tau)(I) := \tau^*(I)$. *Let* $E_k := \{I_k(p_k : q_k); [p_k, q_k] \in \mathbb{P}^1\}$ *and* $U'_{n-2} := U_{n-2} \cap U_{n-1}$.

(5.3.1) $\mathrm{Hilb}(\tau)(U'_{n-2}) = U'_{n-2}$ *and* $\mathrm{Hilb}(\tau)|_{U'_{n-2}} : (s,t) \mapsto ((-1)^{n-1}st^2, (-1)^n t^{-1})$.

(5.3.2) $\mathrm{Hilb}(\tau)$ *has exactly two fixed points* $\mathfrak{p}_\pm := I_{n-2}(1 : \pm i^{n/2})$ *on* X_H.

(5.3.3) $\mathrm{Hilb}(\tau)(E_k) = E_{2n-4-k} \quad (1 \le k \le 2n-5)$.

Proof. (5.3.1) is clear from (5.2.1), where U'_{n-2} is viewed as a subset of U_{n-2}. The union $E_H := E_1 + \cdots + E_{2n-5}$ is the exceptional set of $\pi_H : X_H \to S_H$. Since the action of $\mathrm{Hilb}(\tau)$ on $X_H \setminus E_H$ is fixed point free, $\mathrm{Hilb}(\tau)$ has exactly two fixed points on X_H by (5.2.2). (5.3.3) follows from (5.2.1). □

Lemma 5.4. *Let N be the group generated by* $\mathrm{Hilb}(\tau)$, *and* $\mathrm{Hilb}^N(X_H)$ *the unique irreducible component of the fixed points locus* $\mathrm{Hilb}^2(X_H)^N$ *of* $\mathrm{Hilb}^2(X_H)$ *by N which dominates* X_H/N $(\simeq \mathrm{S}^2(X_H)^N)$. *Then* $\mathrm{Hilb}^N(X_H) \simeq X_G := \mathrm{Hilb}^G(\mathbb{A}^2)$.

Proof. $\mathrm{Hilb}(\tau)$ operates upon X_H locally near the isolated fixed points $\mathfrak{p}_k$ by $\mathrm{Hilb}(\tau) : (u_k, v_k) \to (-u_k, -v_k)$ with suitable local coordinates. Therefore $\mathrm{Hilb}^N(X_H)$ is by Theorem 2.4 a minimal resolution of X_H/N with trivial dualising sheaf. It follows that $\mathrm{Hilb}^N(X_H)$ is a minimal resolution of $\mathbb{A}^2/G$. □

Corollary 5.5. *The set of all the fixed points* $\mathrm{Hilb}^{|G|}(\mathbb{A}^2)^G$ *is connected, in particular,* $\mathrm{Hilb}^{|G|}(\mathbb{A}^2)^G = \mathrm{Hilb}^G(\mathbb{A}^2)$.

Proof. Since H is cyclic, we see that $\mathrm{Hilb}^{|H|}(\mathbb{A}^2)^H = \mathrm{Hilb}^H(\mathbb{A}^2) = X_H$ by the proof of Theorem 4.3. Since N is cyclic, we have $\mathrm{Hilb}(X_H)^N = \mathrm{Hilb}^N(X_H)$. □

Remark. Since E_6 (resp. E_7) is a cyclic extension of D_4 (resp. E_6), we see by the same argument that $\mathrm{Hilb}^{|G|}(\mathbb{A}^2)^G$ is connected at least up to E_7.

Remark. Unfortunately the isomorphism in Lemma 5.4 is not functorial. So Lemma 5.4 appears to be useless for the study of X_G. However once we find an $I \in X_G$, say $I = (x^\ell, xy, y^\ell) \in X_G$ where $|G| = 2\ell$, we can determine X_G by deforming I in the direction of exceptional set because the exceptional set of X_G is connected. Though we give *a priori* the ideals in $X_G = \mathrm{Hilb}^G(\mathbb{A}^2)$ in the sequel, all of these were found by deformation theory.

5.6. Character table. The Coxeter number $2\bar{h}$ of D_n is equal to $2n-2$.

Table 5.1

ρ	1	σ	τ	d	$(\bar{h} \pm d)$
ρ'_0	1	1	1	$(n-3)$	$-$
ρ'_1	1	1	-1	$n-3$	$(2, \ell)$
ρ_2	2	$\epsilon + \epsilon^{-1}$	0	$n-4$	$(3, \ell-1)$
ρ_k	2	$\epsilon^{k-1} + \epsilon^{-(k-1)}$	0	$n-2-k$	$(k+1, \ell+1-k)$
ρ_{n-2}	2	$\epsilon^{n-3} + \epsilon^{-(n-3)}$	0	0	$(n-1, n-1)$
ρ'_{n-1}	1	-1	i^n	1	$(n-2, n)$
ρ'_n	1	-1	$-i^n$	1	$(n-2, n)$

5.7. Symmetric tensors mod $\mathfrak{n}$. Recall $\ell := 2n-4$. The space S_m of symmetric m-tensors by $\rho_{\mathrm{nat}} := \rho_2$, that is the space of homogeneous polynomials of degree m and the images $\bar{S}_m$ of S_m in $\mathfrak{m}/\mathfrak{n}$ are decomposed into irreducible G-modules as follows.

Let $\rho_1 := \rho'_0 + \rho'_1$, $\rho_{n-1} := \rho'_{n-1} + \rho'_n$ and $\rho_k := \rho_j$ if $k \equiv j \bmod 2n-4$. Then we have

$$S_m = \begin{cases} \rho'_0 + \rho_3 + \rho_5 + \cdots + \rho_{m-1} + \rho_{m+1} & m \equiv 0 \bmod 4 \\ \rho'_1 + \rho_3 + \rho_5 + \cdots + \rho_{m-1} + \rho_{m+1} & m \equiv 2 \bmod 4 \\ \rho_2 + \rho_4 + \rho_6 + \cdots + \rho_{m-1} + \rho_{m+1} & m \equiv 1, 3 \bmod 4 \end{cases}$$

Table 5.2

m	$\bar{S}_m$	m	$\bar{S}_m$
0	0	$\ell+2$	0
1	ρ_2	$\ell+1$	ρ_2
2	$\rho'_1+\rho_3$	ℓ	$\rho'_1+\rho_3$
3	$\rho_2+\rho_4$	$\ell-1$	$\rho_2+\rho_4$
k	$\rho_{k-1}+\rho_{k+1}$	$\ell-k+2$	$\rho_{k-1}+\rho_{k+1}$
$n-2$	$\rho_{n-3}+\rho'_{n-1}+\rho'_n$	n	$\rho_{n-3}+\rho'_{n-1}+\rho'_n$
$n-1$	$2\rho_{n-2}$		

By the above table we see that $\mathfrak{m}/\mathfrak{n} \simeq (\mathbb{C}[G] \ominus \rho_0)^{\oplus 2}$. This isomorphism is realised by giving G-submodules $2\rho'_i$ ($i = 1, n-1, n$) and $4\rho_i$ ($2 \le i \le n-2$) explicitly as follows. We define a G-submodule of $\mathfrak{m}/\mathfrak{n}$ by $\bar{V}_i(\rho_j) := S_i(\mathfrak{m}/\mathfrak{n})[\rho_j]$, and define $V_i(\rho_j)$ to be a G-submodule of S_i such that $V_i(\rho_j) \simeq \bar{V}_i(\rho_j)$, $V_i(\rho_j) \equiv \bar{V}_i(\rho_j) \bmod \mathfrak{n}$. We use $V_i(\rho_j)$ and $\bar{V}_i(\rho_j)$ without distinction when harmless. We easily see that $V_i(\rho_j) \simeq \rho_j$ except for $(i,j) = (n-1, n-2)$ if it is not zero, while $V_{n-1}(\rho_{n-2}) \simeq \rho_{n-2}^{\oplus 2}$. We list up nonzero G-submodules of $\mathfrak{m}/\mathfrak{n}$.

Table 5.3

$$\begin{array}{ll}
V_2(\rho'_1) = \{xy\}, & V_\ell(\rho'_1) = \{x^\ell - y^\ell\} \\
V_{k-1}(\rho_k) = \{x^{k-1}, y^{k-1}\}, & V_{k+1}(\rho_k) = \{x^k y, xy^k\}, \\
V_{\ell-k+1}(\rho_k) = \{x^{\ell-k+1}, y^{\ell-k+1}\}, & V_{\ell-k+3}(\rho_k) = \{x^{\ell-k+2}y, xy^{\ell-k+2}\}. \\
V_{n-3}(\rho_{n-2}) = \{x^{n-3}, y^{n-3}\}, & V_{n+1}(\rho_{n-2}) = \{x^n y, xy^n\}, \\
\multicolumn{2}{l}{V_{n-1}(\rho_{n-2}) = V'_{n-1}(\rho_{n-2}) \oplus V''_{n-1}(\rho_{n-2}) = \{x^{n-1}, y^{n-1}, x^{n-2}y, xy^{n-2}\},} \\
V'_{n-1}(\rho_{n-2}) = \{x^{n-1}, y^{n-1}\}, & V''_{n-1}(\rho_{n-2}) = \{x^{n-2}y, xy^{n-2}\}. \\
V_{n-2}(\rho'_{n-1}) = \{x^{n-2} - i^n y^{n-2}\}, & V_n(\rho'_{n-1}) = \{xy(x^{n-2} + i^n y^{n-2})\} \\
V_{n-2}(\rho'_n) = \{x^{n-2} + i^n y^{n-2}\}, & V_n(\rho'_n) = \{xy(x^{n-2} - i^n y^{n-2})\}
\end{array}$$

where $2 \le k \le n-3$. It is easy to see that $\mathfrak{n}$ is generated by $x^\ell + y^\ell$, $(x^\ell - y^\ell)xy$ and x^2y^2. We also note that $x^{\ell+2}, y^{\ell+2} \in \mathfrak{n}$ and that $\mathfrak{m}/\mathfrak{n}$ is spanned by x, y, xy, x^i, y^i, $x^i y$ and xy^i ($2 \le i \le \ell$) with a unique relation $x^\ell + y^\ell \equiv 0 \bmod \mathfrak{n}$. Hence we easily see that $\mathfrak{m}/\mathfrak{n}$ is the sum of the above $V_i(\rho_j)$. It follows that $\mathfrak{m}/\mathfrak{n} \simeq \sum_{\rho \in \mathrm{Irr}(G)} 2\deg(\rho)\rho \simeq (\mathbb{C}[G] \ominus \rho_0)^{\oplus 2}$.

5.8. A sketch for $\mathbf{D}_5$. Before entering the general case, we will sketch the case of D_5 without rigorous proofs to facilitate the reader's understanding. The Coxeter number $2\bar{h}$ of

D_5 is equal to 8. The irreducible representations of D_5 and the irreducible decompositions of S_m and $\bar{S}_m$ are tabulated as follows. The other symbols stand for the same as before. We note $\bar{S}_m = 0$ for $m \geq 8$.

Table 5.4

ρ	deg	d	$4 \pm d$	$V_{4-d}(\rho)$	$V_{4+d}(\rho)$
ρ_0	1	2	–	–	–
ρ_1'	1	2	$(2,6)$	xy	$x^6 - y^6$
ρ_2	2	1	$(3,5)$	x^2y, xy^2	x^5, y^5
ρ_3	2	0	$(4,4)$	x^4, y^4, x^3y, xy^3	
ρ_4'	1	1	$(3,5)$	$x^3 - iy^3$	$xy(x^3 + iy^3)$
ρ_5'	1	1	$(3,5)$	$x^3 + iy^3$	$xy(x^3 - iy^3)$

Table 5.5

m	$\bar{S}_m$
1	ρ_2
2	$\rho_1' + \rho_3$
3	$\rho_2 + \rho_4' + \rho_5'$
4	$2\rho_3$
5	$\rho_2 + \rho_4' + \rho_5'$
6	$\rho_1' + \rho_3$
7	ρ_2

It is clear that $\mathfrak{m}/\mathfrak{n} \simeq (\mathbb{C}[G] \ominus \rho_0)^{\oplus 2} = 2\rho_1' + 4\rho_2 + 2\rho_3 + 2\rho_4' + 2\rho_5'$.

First we consider the case where $\mathcal{I}(W) \in E(\rho_1') \setminus P(\rho_1', \rho_2)$.

Let $\mathcal{I}(W) := W\mathcal{O}_{\mathbb{A}^2} + \mathfrak{n}$ for any nonzero G-module $W \in \mathbb{P}(V_2(\rho_1') + V_6(\rho_1')) = \mathbb{P}(\{xy, x^6 - y^6\})$ such that $W \neq V_6(\rho_1')$, *i.e.* $W \neq \{x^6 - y^6\}$. Then we see that

$$\begin{aligned}\mathcal{I}(W)/\mathfrak{n} &= W + \sum_{k=1}^{5} S_1 W + \mathfrak{n}/\mathfrak{n} = W + \sum_{k=1}^{5} S_k V_2(\rho_1') + \mathfrak{n}/\mathfrak{n}\\ &\simeq W + \rho_2 + \rho_3 + (\rho_4' + \rho_5') + \rho_3 + \rho_2 \simeq \sum_{\rho \in \mathrm{Irr}(G)} \deg(\rho)\rho\end{aligned}$$

Therefore $\mathcal{I}(W) \in \mathrm{Hilb}^G(\mathbb{A}^2)$. It is clear that $V(\mathcal{I}(W)) := \mathcal{I}(W)/\mathfrak{m}\mathcal{I}(W) + \mathfrak{n} \simeq W \simeq \rho_1'$. It follows that $\mathcal{I}(W) \in E(\rho_1') \setminus P(\rho_1', \rho_2)$. Hence we have

$$\begin{aligned}\lim_{W \to V_6(\rho_1')} \mathcal{I}(W) &= V_6(\rho_1') + \sum_{k \geq 1} S_k V_2(\rho_1')\\ &= \mathcal{I}(V_6(\rho_1') \oplus S_1 V_2(\rho_1'))\\ &= \mathcal{I}(V_6(\rho_1') \oplus V_3(\rho_2)) \in P(\rho_1', \rho_2)\end{aligned}$$

where $S_1 \otimes V_2(\rho_1') \simeq S_1 V_2(\rho_1') \simeq V_3(\rho_2) \simeq \rho_2$. The factor $S_1 \otimes V_2(\rho_1') \simeq \rho_2$ among generators of $P(\rho_1', \rho_2)$ explains the connection between the tensoring process by $S_1 \simeq \rho_2$ and the intersection of two rational curves $E(\rho_1')$ and $E(\rho_2)$ in the McKay observation.

Next we consider $W \in \mathbb{P}(V_3(\rho_2) \oplus V_5(\rho_2))$ with $W \neq V_3(\rho_2), V_5(\rho_2)$. Then we have

$$\begin{aligned}\mathcal{I}(W)/\mathfrak{n} :&= W + \sum_{k \geq 1} S_k W + \mathfrak{n}/\mathfrak{n}\\ &= W + \sum_{k \geq 1}^{2} S_k V_3(\rho_2) + \bar{S}_6 + \bar{S}_7 + \mathfrak{n}/\mathfrak{n}\\ &\simeq W + \rho_3 + (\rho_4' + \rho_5') + (\rho_1' + \rho_3) + \rho_2 \simeq \sum_{\rho \in \mathrm{Irr}(G)} \deg(\rho)\rho.\end{aligned}$$

Since $\bar{S}_6 = V_6(\rho_1') + S_3V_3(\rho_2) \neq S_3V_3(\rho_2)$, we have

$$\begin{aligned}\lim_{W\to V_3(\rho_2)} \mathcal{I}(W) &= V_6(\rho_1') + V_3(\rho_2) + \sum_{k\geq 1} S_kV_3(\rho_2)\\ &= \mathcal{I}(V_6(\rho_1') \oplus V_3(\rho_2)) \in P(\rho_1', \rho_2)\end{aligned}$$

where we note that $V_6(\rho_1')$ is the unique ρ_1-factor of $S_1V_5(\rho_2) \simeq S_1 \otimes V_5(\rho_2)$. Thus and similarly we see

$$\begin{aligned}\lim_{\substack{W\to V_6(\rho_1')\\ W\simeq\rho_1'}} \mathcal{I}(W) &= \lim_{\substack{W\to V_3(\rho_2)\\ W\simeq\rho_2}} \mathcal{I}(W) \in P(\rho_1', \rho_2),\\ \lim_{W\to V_5(\rho_2)} \mathcal{I}(W) &= V_5(\rho_2) + \sum_{k\geq 1} S_kV_3(\rho_2)\\ &= \mathcal{I}(V_5(\rho_2) \oplus S_1V_3(\rho_2)) \in P(\rho_2, \rho_3)\end{aligned}$$

where we note $S_1V_3(\rho_2) \simeq \{S_1 \otimes V_3(\rho_2)\}[\rho_3] \simeq \rho_3$.

Now we consider the case $\mathcal{I}(W) \in E(\rho_3) \setminus P(\rho_3, \rho_2) \cup P(\rho_3, \rho_4') \cup P(\rho_3, \rho_5')$. Let $W_2 := S_1V_3(\rho_2) \simeq \rho_3$ and $W_k := S_1V_3(\rho_k') \simeq \rho_3$ $(k = 4, 5)$. We note that $W_2 = \{S_1 \otimes V_3(\rho_2)\}[\rho_3]$ and $W_k = \{S_1 \otimes V_3(\rho_k')\}[\rho_3]$ for $k = 4, 5$. Let $W \in \mathbb{P}(V_4(\rho_3))$ such that $W \neq W_k$. In this case we see $\mathcal{I}(W)/\mathfrak{n} = W + \bar{S}_5 + \bar{S}_6 + \bar{S}_7 \simeq \sum_{\rho\in\mathrm{Irr}(G)} \deg(\rho)\rho$. Moreover

$$\begin{aligned}\lim_{\substack{W\to V_5(\rho_2)\\ W\simeq\rho_2}} \mathcal{I}(W) &= \mathcal{I}(V_5(\rho_2) \oplus W_2) = \lim_{\substack{W\to W_2\\ W\simeq\rho_3}} \mathcal{I}(W) \in P(\rho_2, \rho_3)\\ \lim_{\substack{W\to V_5(\rho_k')\\ W\simeq\rho_k'}} \mathcal{I}(W) &= \mathcal{I}(V_5(\rho_2) \oplus W_k) = \lim_{\substack{W\to W_k\\ W\simeq\rho_3}} \mathcal{I}(W) \in P(\rho_3, \rho_k').\end{aligned}$$

The above argument explains the connection between the tensoring process by $\rho_2 = \rho_{\mathrm{nat}}$ and the intersection of two rational curves. The argument also shows that $E(\rho)$ is naturally identified with $\mathbb{P}(V_{4-d(\rho)}(\rho) + V_{4+d(\rho)}(\rho))$, the set of all nontrivial proper G-submodules of $V_{4-d(\rho)}(\rho) + V_{4+d(\rho)}(\rho) \simeq \rho^{\oplus 2}$, which is isomorphic to $\mathbb{P}^1$ by Schur's lemma.

Now we consider the general case. Theorem 3.6 is restated in the following manner.

Theorem 5.9. *Let E be the exceptional set of the morphism $\pi : X_G \to S_G$,* Sing(E) *the singular points of E. Let $E(\rho)$ be an irreducible component of E ($\rho \in$* Irr(G)*) and*

$E^0(\rho) := E(\rho) \setminus \mathrm{Sing}(E)$. Then $E^0(\rho)$ and $\mathrm{Sing}(E)$ are given by the following.

$$\begin{aligned}
E^0(\rho_1') &= \{\mathcal{I}(W);\ W \subset V_2(\rho_1') \oplus V_\ell(\rho_1'), W \neq 0, V_\ell(\rho_1')\}\\
E^0(\rho_k) &= \left\{\mathcal{I}(W);\ \begin{matrix} W \subset V_{k+1}(\rho_k) \oplus V_{\ell-k+1}(\rho_k) \\ W \neq 0, V_{k+1}(\rho_k), V_{\ell-k+1}(\rho_k)\end{matrix}\right\}\ (2 \leq k \leq n-3)\\
E^0(\rho_{n-2}) &= \left\{\mathcal{I}(W);\ \begin{matrix} W \subset V_{n-1}(\rho_{n-2}), W \neq 0, V''_{n-1}(\rho_{n-2}) \\ W \neq S_1 \cdot V_{n-2}(\rho_j')\ (j = n-1, n)\end{matrix}\right\}\\
E^0(\rho_j) &= \{\mathcal{I}(W);\ W \subset V_{n-2}(\rho_j') \oplus V_n(\rho_j'),\ W \neq 0, V_n(\rho_j')\}\\
\mathrm{Sing}(E) &= \left\{\begin{matrix} P(\rho_1', \rho_2),\ P(\rho_k, \rho_{k+1})\ (2 \leq k \leq n-3) \\ P(\rho_{n-2}, \rho_{n-1}'),\ P(\rho_{n-2}, \rho_n') \end{matrix}\right\}\\
P(\rho_1', \rho_2) &= \mathcal{I}(V_\ell(\rho_1') \oplus V_3(\rho_2)) = E(\rho_1') \cap E(\rho_2)\\
P(\rho_k, \rho_{k+1}) &= \mathcal{I}(V_{\ell-k+1}(\rho_k) \oplus V_{k+2}(\rho_{k+1}))\ (2 \leq k \leq n-4)\\
P(\rho_{n-3}, \rho_{n-2}) &= \mathcal{I}(V_n(\rho_{n-3}) \oplus V''_{n-1}(\rho_{n-2}))\\
P(\rho_{n-2}, \rho_j') &= \mathcal{I}(S_1 V_{n-2}(\rho_j') \oplus V_n(\rho_j'))
\end{aligned}$$

where $j = n-1, n$.

5.10. Proof of Theorem 5.9-Start. Let $C(\rho_i')$ (resp. $C(\rho_k)$) be the projective space of all nonzero G-submodules of $V_2(\rho_1') \oplus V_\ell(\rho_1')$ $(i = 1)$ or $V_{n-2}(\rho_i') \oplus V_n(\rho_i')$ $(i = n-1, n)$ (resp. $V_{k+1}(\rho_k) \oplus V_{\ell-k+1}(\rho_k)$ $(2 \leq k \leq n-2)$). It is clear that $C(\rho_i')$ and $C(\rho_k)$ are rational curves. As we will see in the sequel, they are embedded naturally into $\mathrm{Grass}(\mathfrak{m}/\mathfrak{n}, 2\ell-2)$.

THE CASE WHERE $\mathcal{I}(W) \in E(\rho_1') \setminus P(\rho_1', \rho_2)$.

Let $\mathcal{I}(W) := W\mathcal{O}_{\mathbb{A}^2} + \mathfrak{n}$ for any nonzero G-module $W \in C(\rho_1')$ with $W \neq V_\ell(\rho_1')$. First assume $W = V_2(\rho_1')$. Then it is easy to see that $\mathcal{I}(W)/\mathfrak{n}$ contains $V_{k+1}(\rho_k)$, $V_{\ell-k+3}(\rho_k)$, $V''_{n-1}(\rho_{n-2})$ and $V_{n+1}(\rho_{n-2})$ for any $2 \leq k \leq n-3$. Similarly $\mathcal{I}(W)/\mathfrak{n}$ contains $V_n(\rho_{n-1}')$ and $V_n(\rho_n')$ as well as $W = V_2(\rho_1')$. It follows

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{k=1}^{\ell-1} S_k \bar{V}_2(\rho_1') = W + \sum_{k=1}^{\ell-2} S_k \bar{V}_2(\rho_1') + \bar{S}_{\ell+1}$$

In particular, $\mathcal{I}(W)/\mathfrak{n} \simeq \sum_{\rho \in \mathrm{Irr}(G)} \deg(\rho)\rho$. Hence $\mathcal{I}(W) \in \mathrm{Hilb}^G(\mathbb{A}^2)$. By the above structure of $\mathcal{I}(W)/\mathfrak{n}$, we see that $V(\mathcal{I}(W)) := \mathcal{I}(W)/\{\mathfrak{m}\mathcal{I}(W) + \mathfrak{n}\} \simeq W \simeq \rho_1'$. It follows that $\mathcal{I}(W) \in E(\rho_1')$.

Next we assume $W \neq V_2(\rho_1'), V_\ell(\rho_1')$. Then we first see that $x^3y \in \mathcal{I}(W)$ because $x^3y - (x^3y - 2tx^{\ell+2}) = 2tx^{\ell+2} \in \mathfrak{n}$. It follows that $\mathcal{I}(W)/\mathfrak{n}$ contains $V_{\ell+1}(\rho_2)$, $V_{k+1}(\rho_k)$, $V_{\ell-k+3}(\rho_k)$, $V''_{n-1}(\rho_{n-2})$, $V_{n+1}(\rho_{n-2})$, $V_n(\rho_{n-1}')$ and $V_n(\rho_n')$ where $3 \leq k \leq n-3$. Since $S_1 \cdot W + V_{\ell+1}(\rho_2) = V_3(\rho_2) + V_{\ell+1}(\rho_2) \simeq \rho_2^{\oplus 2}$, $\mathcal{I}(W)/\mathfrak{n}$ contains $2\rho_2$ too. It follows that

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{m \geq 0}^{\ell-2} S_m \bar{V}_2(\rho_1') = W + \sum_{m=0}^{\ell-3} S_m \bar{V}_2(\rho_1') + \bar{S}_{\ell+1}$$

Hence we have $\mathcal{I}(W)/\mathfrak{n} \simeq \sum_{\rho\in\mathrm{Irr}(G)} \deg(\rho)\rho$. Therefore $\mathcal{I}(W) \in \mathrm{Hilb}^G(\mathbb{A}^2)$. By the above structure of $\mathcal{I}(W)/\mathfrak{n}$, $V(\mathcal{I}(W)) \simeq W \simeq \rho_1'$. It follows that $\mathcal{I}(W) \in E(\rho_1') \setminus P(\rho_1', \rho_2)$.

THE CASE WHERE $\mathcal{I}(W) \in P(\rho_1', \rho_2)$.

Let $W = W(\rho_1', \rho_2) := V_\ell(\rho_1') \oplus V_3(\rho_2)$. $\mathcal{I}(W)/\mathfrak{n}$ contains x^2y and xy^2 hence it contains $V_{i+1}(\rho_i)$, $V_{\ell-i+3}(\rho_i)$, $(3 \le i \le n-3)$, $V_{\ell+1}(\rho_2)$, $V_{n+1}(\rho_{n-2})$, $V_n(\rho'_{n-1})$ and $V_n(\rho'_n)$. Similarly $\mathcal{I}(W)/\mathfrak{n}$ contains $V''_{n-1}(\rho_{n-2})$. We note that $\{\mathcal{I}(W)/\mathfrak{n}\}[\rho_1'] = W = V_\ell(\rho_1') = \{S_1 \cdot V_{\ell-1}(\rho_2)\}[\rho_1']$ and $\{\mathcal{I}(W)/\mathfrak{n}\}[\rho_2] = V_3(\rho_2) \oplus V_{\ell+1}(\rho_2) = S_1 \cdot V_2(\rho_1') \oplus V_{\ell+1}(\rho_2)$. It follows that

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{m=0}^{\ell-2} S_m \bar{V}_3(\rho_2) = W + \sum_{m=0}^{\ell-3} S_m \bar{V}_3(\rho_2) + \bar{S}_{\ell+1}$$

Hence we have $\mathcal{I}(W)/\mathfrak{n} \simeq \sum_{\rho\in\mathrm{Irr}(G)} \deg(\rho)\rho$. Therefore $\mathcal{I}(W) \in \mathrm{Hilb}^G(\mathbb{A}^2)$. We also see

$$\begin{aligned} V(\mathcal{I}(W)) &= V_\ell(\rho_1') \oplus \{S_1 \cdot V_2(\rho_1')\}[\rho_2] \\ &= \{S_1 \cdot V_{\ell-1}(\rho_2)\}[\rho_1'] \oplus V_3(\rho_2) \simeq \rho_1' \oplus \rho_2 \end{aligned}$$

Therefore $\mathcal{I}(W) \in P(\rho_1', \rho_2)$.

THE CASE WHERE $\mathcal{I}(W) \in E(\rho_k) \setminus P(\rho_{k\pm1}, \rho_k)$ $(2 \le k \le n-3)$.

We consider now $W \in C(\rho_k) = \mathbb{P}(\rho_k \subset V_{k+1}(\rho_k) \oplus V_{\ell-k+1}(\rho_k))$ with $W \neq V_{k+1}(\rho_k)$, $V_{\ell-k+1}(\rho_k)$. Let $\mathcal{I}(W) = W\mathcal{O}_{\mathbb{A}^2} + \mathfrak{n}$. Hence we may assume that $x^{k+1}y - ty^{\ell-k+1} \in W$ for a nonzero constant t. Since $x^{k+3}y = x^2(x^{k+1}y - ty^{\ell-k+1}) + tx^2y^{\ell-k+1}$, and $x^2y^2 \in \mathfrak{n}$, $\mathcal{I}(W)$ contains $x^{k+3}y$. Similarly since $ty^{\ell-k+2} = -y(x^{k+1}y - ty^{\ell-k+1}) + x^{k+1}y^2$, we have $y^{\ell-k+2} \in \mathcal{I}(W)$. Hence we see that $\mathcal{I}(W)/\mathfrak{n}$ contains $V_{\ell-i+1}(\rho_i)$ $(2 \le i \le k-1)$, $V_{i+1}(\rho_i)$ $(k+2 \le i \le n-3)$, $V_{\ell-i+3}(\rho_i)$ $(2 \le i \le n-3)$, $V'_{n-1}(\rho_{n-2})$, $V''_{n-1}(\rho_{n-2})$, $V_\ell(\rho_1')$, $V_n(\rho'_{n-1})$ and $V_n(\rho'_n)$. Since $xy^{\ell-k+1} \in V_{\ell-k+2}(\rho_{k+1})$, we have $V_{\ell-k+3}(\rho_k) \subset \mathcal{I}(W)/\mathfrak{n}$ and $x^{k+2}y = x(x^{k+1}y - ty^{\ell-k+1}) + txy^{\ell-k+1} \in \mathcal{I}(W)/\mathfrak{n}$. Hence $V_{k+2}(\rho_{k+1}) \subset \mathcal{I}(W)/\mathfrak{n}$ if $k \le n-4$. It follows that

$$\begin{aligned} \mathcal{I}(W)/\mathfrak{n} &= W + \sum_{m=1}^{\ell-k} S_m \bar{V}_{k+1}(\rho_k) + \sum_{m=0}^{k-1} S_m \bar{V}_{\ell-k+2}(\rho_{k-1}) \\ &= W + \sum_{m=1}^{\ell-2k} S_m \bar{V}_{k+1}(\rho_k) + \sum_{m=\ell-k+2}^{\ell+1} \bar{S}_m \end{aligned}$$

It follows from $W \simeq \rho_k$ that $\mathcal{I}(W)/\mathfrak{n} \simeq \sum_{\rho\in\mathrm{Irr}(G)} \deg(\rho)\rho$. Therefore $\mathcal{I}(W) \in \mathrm{Hilb}^G(\mathbb{A}^2)$. It is easy to see that $V(\mathcal{I}(W)) \simeq W \simeq \rho_k$ so that $\mathcal{I}(W) \in E(\rho_k)$.

THE CASE WHERE $\mathcal{I}(W) \in P(\rho_k, \rho_{k+1})$.

Let $W = W(\rho_k, \rho_{k+1}) := V_{\ell-k+1}(\rho_k) \oplus V_{k+2}(\rho_{k+1})$ $(2 \le k \le n-4)$. For $k = n-3$ we let $W = W(\rho_{n-3}, \rho_{n-2}) := V_n(\rho_{n-3}) \oplus V''_{n-1}(\rho_{n-2})$. $\mathcal{I}(W)/\mathfrak{n}$ contains $V_{\ell-i+1}(\rho_i)$ $(2 \le i \le k)$,

$V_{i+1}(\rho_i)$ $(k+1 \le i \le n-3)$, $V_{\ell-i+3}(\rho_i)$ $(2 \le i \le n-2)$, $V''_{n-1}(\rho_{n-2})$ and $V_n(\rho'_i)$ $(i = n-1, n)$. Similarly $V_\ell(\rho'_1) \subset \mathcal{I}(W)/\mathfrak{n}$. Hence $\mathcal{I}(W) \in P(\rho_k, \rho_{k+1}) \subset \mathrm{Hilb}^G(\mathbb{A}^2)$. We also see that

$$V(\mathcal{I}(W)) = \begin{cases} V_{\ell-k+1}(\rho_k) \oplus \{S_1 \cdot V_{k+1}(\rho_k)\}[\rho_{k+1}] & (2 \le k \le n-4) \\ V_n(\rho_{n-3}) \oplus \{S_1 \cdot V_{n-2}(\rho_{n-3})\}[\rho_{n-2}] & (k = n-3) \end{cases}$$
$$= \begin{cases} \{S_1 \cdot V_{\ell-k}(\rho_{k+1})\}[\rho_k] \oplus V_{k+1}(\rho_k) \simeq \rho_k \oplus \rho_{k+1} \\ \{S_1 \cdot V'_{n-1}(\rho_{n-2})\}[\rho_{n-3}] \oplus V''_{n-1}(\rho_{n-2}) \simeq \rho_{n-3} \oplus \rho_{n-2}. \end{cases}$$

THE CASE WHERE $\mathcal{I}(W) \in E(\rho_{n-2}) \setminus P(\rho_{n-2}, \rho_{n-3}) \cup P(\rho_{n-2}, \rho'_{n-1}) \cup P(\rho_{n-2}, \rho'_n)$.
Let $W \in C(\rho_{n-2}) = \mathbb{P}(V_{n-1}(\rho_{n-2}))$ and define $\mathcal{I}(W) := W\mathcal{O}_{\mathbb{A}^2} + \mathfrak{n}$. Let $W_0 = S_1 \cdot V_{n-2}(\rho'_{n-1})$, $W_\infty = S_1 \cdot V_{n-2}(\rho'_n)$ and $W_1 = V''_{n-1}(\rho_{n-2})$. Let $H = x^{n-2} - i^{n/2}y^{n-2}$ and $G = x^{n-2} + i^{n/2}y^{n-2}$. Then we have $W = \{xH - txG, yH + tyG\}$ for some t. Assume $t \ne 0, 1, \infty$. Equivalently, $W \ne W_\lambda$ $(\lambda = 0, 1, \infty)$. Then $x^n \in \mathcal{I}(W)/\mathfrak{n}$, so that $V_\ell(\rho'_1)$, $V_{\ell-i+1}(\rho_i)$ $(2 \le i \le n-3)$ and $V_{\ell-i+3}(\rho_i)$ $(2 \le i \le n-2)$ are contained in $\mathcal{I}(W)/\mathfrak{n}$. We also see that $xyH \in V_n(\rho'_{n-1}) \subset \mathcal{I}(W)/\mathfrak{n}$ and $xyG \in V_n(\rho'_n) \subset \mathcal{I}(W)/\mathfrak{n}$. It follows that

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{m=n}^{\ell+1} \bar{S}_m$$

Since $W \simeq \rho_{n-2}$, we have $\mathcal{I}(W)/\mathfrak{n} \simeq \sum_{\rho \in \mathrm{Irr}(G)} \deg(\rho)\rho$ with $V(\mathcal{I}(W)) \simeq W$. It follows that $\mathcal{I}(W) \in \mathrm{Hilb}^G(\mathbb{A}^2)$.

THE CASE WHERE $\mathcal{I}(W) \in E(\rho'_{n-1}) \setminus P(\rho_{n-2}, \rho'_{n-1})$.
Let $W \in C(\rho'_{n-1}) := \mathbb{P}(V_{n-2}(\rho'_{n-1}) \oplus V_n(\rho'_{n-1}))$. Assume $W \ne V_n(\rho'_{n-1})$. Then $\mathcal{I}(W)/\mathfrak{n}$ contains $x^n y$ and hence x^n. It follows that $\mathcal{I}(W)/\mathfrak{n}$ contains $V_{\ell-i+1}(\rho_i)$, $V_{\ell-i+3}(\rho_i)$ $(2 \le i \le n-3)$ and $V_{n+1}(\rho_{n-2})$. We also see that $\mathcal{I}(W)/\mathfrak{n}$ contains $x^{n-1} - i^{n/2}xy^{n-2}$ so that $\{\mathcal{I}(W)/\mathfrak{n}\} \cap V_{n-1}(\rho_{n-2}) \simeq \rho_{n-2}$. Similarly we easily see $V_\ell(\rho'_1), V_n(\rho'_n) \subset \mathcal{I}(W)/\mathfrak{n}$. It follows that

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{m=1}^{2} S_m \bar{V}_{n-2}(\rho'_{n-1}) + \sum_{m=n+1}^{\ell+1} \bar{S}_m$$

Since $W \simeq \rho'_{n-1}$, $\mathcal{I}(W)/\mathfrak{n} \simeq \sum_{\rho \in \mathrm{Irr}(G)} \deg(\rho)\rho$. Therefore $\mathcal{I}(W) \in E(\rho'_{n-1}) \subset \mathrm{Hilb}^G(\mathbb{A}^2)$ with $V(\mathcal{I}(W)) \simeq W$.

THE CASE WHERE $\mathcal{I}(W) \in P(\rho_{n-2}, \rho'_{n-1})$.
We consider $W = W(\rho_{n-2}, \rho'_{n-1}) := S_1 \cdot V_{n-2}(\rho'_{n-1})[\rho_{n-2}] \oplus V_n(\rho'_{n-1}) = W_0 \oplus V_n(\rho'_{n-1})$. Then $\mathcal{I}(W)/\mathfrak{n}$ contains x^n so that $\mathcal{I}(W)/\mathfrak{n}$ contains $V_\ell(\rho'_1)$, $V_{\ell-i+1}(\rho_i)$, $V_{\ell-i+3}(\rho_i)$ $(2 \le i \le n-3)$, $V_{n+1}(\rho_{n-2})$ and $V_n(\rho'_n)$. Since $W \subset \mathcal{I}(W)/\mathfrak{n}$, we see $\mathcal{I}(W)/\mathfrak{n} \simeq \sum_{\rho \in \mathrm{Irr}(G)} \deg(\rho)\rho$. Hence $\mathcal{I}(W) \in P(\rho_{n-2}, \rho'_{n-1}) \subset \mathrm{Hilb}^G(\mathbb{A}^2)$ with $V(\mathcal{I}(W)) \simeq W$.

THE CASE WHERE $\mathcal{I}(W) \in E(\rho'_n) \setminus P(\rho_{n-2}, \rho'_n)$ or $\mathcal{I}(W) \in P(\rho_{n-2}, \rho'_n)$ is similar to the above. We omit the details. □

5.11. Limits of $\mathcal{I}(W)$.

We consider limits of $\mathcal{I}(W) \in E(\rho)$ when it approaches $P(\rho, \rho')$ for ρ' adjacent to ρ.

We first consider $W \in C(\rho_1')$ with $W \neq V_\ell(\rho_1')$. Then by **5.10** we see that $\mathcal{I}(W) = W + V_3(\rho_2) + \sum_{m\geq 1} S_m V_3(\rho_2)$. Hence we have

$$\begin{aligned}\lim_{\substack{W\to V_\ell(\rho_1')\\ W\in C(\rho_1')}} \mathcal{I}(W) &= V_\ell(\rho_1') + V_3(\rho_2) + \sum_{m\geq 1} S_m V_3(\rho_2)\\ &= \mathcal{I}(V_\ell(\rho_1') \oplus V_3(\rho_2))\\ &= \mathcal{I}(W(\rho_1', \rho_2))\end{aligned}$$

Next we consider $W \in C(\rho_2)$ with $W \neq V_3(\rho_2), V_{\ell-1}(\rho_2)$. Then by **5.10** we have $\mathcal{I}(W) = W + V_\ell(\rho_1') + \sum_{m\geq 0} S_m V_4(\rho_3)$. Since $V_4(\rho_3) \subset S_1 V_3(\rho_2)$, we have

$$\begin{aligned}\lim_{\substack{W\to V_3(\rho_2)\\ W\in C(\rho_2)}} \mathcal{I}(W) &= V_\ell(\rho_1') + V_3(\rho_2) + \sum_{m\geq 1} S_m V_3(\rho_2)\\ &= \mathcal{I}(W(\rho_1', \rho_2)) = \lim_{\substack{W\to V_\ell(\rho_1')\\ W\in C(\rho_1')}} \mathcal{I}(W)\end{aligned}$$

Next consider $W \in C(\rho_k) = \mathbb{P}(V_{\ell-k+1}(\rho_k) \oplus V_{k+1}(\rho_k))$ with $W \neq V_{k+1}(\rho_k), V_{\ell-k+1}(\rho_k)$. By **5.10** we have

$$\begin{aligned}\mathcal{I}(W) &= W + \sum_{m\geq 0} S_m V_{k+2}(\rho_{k+1}) + \sum_{m\geq 0} S_m V_{\ell-k+2}(\rho_{k-1})\\ \lim_{W\to V_{k+1}(\rho_k)} \mathcal{I}(W) &= \sum_{m\geq 0} S_m V_{\ell-k+2}(\rho_{k-1}) + \sum_{m\geq 0} S_m V_{k+1}(\rho_k)\\ &= \mathcal{I}(V_{\ell-k+2}(\rho_{k-1}) \oplus V_{k+1}(\rho_k))\\ &= \mathcal{I}(W(\rho_{k-1}, \rho_k)) \quad (3 \leq k \leq n-3)\\ \lim_{W\to V_{\ell-k+1}(\rho_k)} \mathcal{I}(W) &= \sum_{m\geq 0} S_m V_{\ell-k+1}(\rho_k) + \sum_{m\geq 0} S_m V_{k+2}(\rho_{k+1})\\ &= \mathcal{I}(V_{\ell-k+1}(\rho_k) \oplus V_{k+2}(\rho_{k+1}))\\ &= \mathcal{I}(W(\rho_k, \rho_{k+1})) \quad (2 \leq k \leq n-4)\end{aligned}$$

Thus for $2 \leq k \leq n-4$ we see that

$$\lim_{W\to V_{\ell-k+1}(\rho_k)} \mathcal{I}(W) = \mathcal{I}(W(\rho_k, \rho_{k+1})) = \lim_{W\to V_{k+2}(\rho_{k+1})} \mathcal{I}(W).$$

Similarly for $W \in C(\rho_{n-2})$ with $W \neq W_\lambda$ $(\lambda = 0, 1, \infty)$ we have

$$\mathcal{I}(W) = W + \sum_{m\geq 0} S_m V_n(\rho_{n-3}) + \sum_{\substack{m\geq 0\\ j=n-1,n}} S_m V_n(\rho_j') = W + \sum_{m\geq n} S_m.$$

Hence we see

$$\begin{aligned}
\lim_{W\to W_1} \mathcal{I}(W) &= \sum_{m\geq 0} S_m V_n(\rho_{n-3}) + \sum_{m\geq 0} S_m W_1 \\
&= \mathcal{I}(W_1 \oplus V_n(\rho_{n-3})) \\
\lim_{W\to W_0} \mathcal{I}(W) &= W_0 + \sum_{m\geq 0} S_m V_n(\rho_{n-3}) + \sum_{m\geq 0} S_m V_n(\rho'_{n-1}) \\
&= \sum_{m\geq 0} S_m W_0 + \sum_{m\geq 0} S_m V_n(\rho'_{n-1}) \\
&= \mathcal{I}(W_0 \oplus V_n(\rho'_{n-1}))
\end{aligned}$$

because $V_n(\rho_{n-3}) \subset S_1 W_0 + \mathfrak{n}$. Consequently

$$\begin{aligned}
\lim_{W\to V_n(\rho_{n-3})} \mathcal{I}(W) &= V_n(\rho_{n-3}) + \sum_{m\geq 0} S_m V_{n+1}(\rho_{n-4}) + \sum_{m\geq 0} S_m V''_{n-1}(\rho_{n-2}) \\
&= \sum_{m\geq 0} S_m V_n(\rho_{n-3}) + \sum_{m\geq 0} S_m W_1 \\
&= \mathcal{I}(V_n(\rho_{n-3}) \oplus W_1) = \lim_{W\to W_1} \mathcal{I}(W) \\
\lim_{W\to V_n(\rho'_{n-1})} \mathcal{I}(W) &= V_n(\rho'_{n-1}) + \sum_{m\geq 1} S_m V_{n-2}(\rho'_{n-1}) \\
&= \mathcal{I}(V_n(\rho'_{n-1}) \oplus S_1 V_{n-2}(\rho'_{n-1})) \\
&= \mathcal{I}(V_n(\rho'_{n-1}) \oplus W_0) = \lim_{W\to W_0} \mathcal{I}(W)
\end{aligned}$$

The limit when W approaches W_∞ is similar.

Thus we have proved the following

Lemma 5.12. *The limit of $\mathcal{I}(W)$ is $\mathcal{I}(W(\rho,\rho'))$ when $\mathcal{I}(W) \in E(\rho)$ approaches $P(\rho,\rho')$ for ρ' adjacent to ρ.*

See Theorem 3.6 for the definition of $\mathcal{I}(W(\rho,\rho'))$. See also Theorem 5.9.

In order to complete the proofs of Theorem 3.1 for $G = \mathbb{D}_{n-2}$ and Theorem 5.9 we need also prove

Lemma 5.13. *$E(\rho)$ and $E(\rho')$ intersects at $P(\rho,\rho')$ transversally if ρ and ρ' are adjacent.*

Proof. By the proof of Theorem 1.3 $X_G = \mathrm{Hilb}^G(\mathbb{A}^2)$ is smooth and the tangent space $T_{[I]}(X_G)$ at $[I]$ is given by the G-invariant subspace $\mathrm{Hom}_{O_{\mathbb{A}^2}}(I, O_{\mathbb{A}^2}/I)^G$ of $T_{[I]}(\mathrm{Hilb}^n(\mathbb{A}^2)) \simeq \mathrm{Hom}_{O_{\mathbb{A}^2}}(I, O_{\mathbb{A}^2}/I)$ where $n = |G|$. Assume that ρ and ρ' are adjacent with $d(\rho') = d(\rho)+1$. Let $W(\rho,\rho') = V_{\bar{h}-d(\rho)}(\rho) \oplus V_{\bar{h}-d(\rho')}(\rho')$. Then $\mathcal{I}(W(\rho,\rho')) \in P(\rho,\rho')$. We will prove the following formula

$$\begin{aligned}
T_{[I]}(X_G) \simeq \mathrm{Hom}_{O_{\mathbb{A}^2}}(I, O_{\mathbb{A}^2}/I)^G &\simeq \mathrm{Hom}_G(V_{\bar{h}-d(\rho)}(\rho), V_{\bar{h}+d(\rho)}(\rho)) \\
&\quad \oplus \mathrm{Hom}_G(V_{\bar{h}+d(\rho')}(\rho'), V_{\bar{h}-d(\rho')}(\rho'))
\end{aligned}$$

where $I = \mathcal{I}(W(\rho, \rho'))$. First assume $\rho = \rho_2$ and $\rho' = \rho_1'$. Then

$$\mathrm{Hom}_{O_{\mathbb{A}^2}}(I, O_{\mathbb{A}^2}/I)^G \subset \mathrm{Hom}_G(V_\ell(\rho_1'), V_2(\rho_1')) \oplus \mathrm{Hom}_G(V_3(\rho_2), V_1(\rho_2) \oplus V_{\ell-1}(\rho_2))$$

Let ϕ be any element of $\mathrm{Hom}_{O_{\mathbb{A}^2}}(I, O_{\mathbb{A}^2}/I)^G$. Since any G-isomorphism ϕ_0 of $V_3(\rho_2)$ onto $V_1(\rho_2)$ is given by $\phi_0(x^2y) = x$, $\phi_0(xy^3) = -y$. Therefore we may assume $\phi = c\phi_0 \mod V_{\ell-1}(\rho_2)$ for some constant c. Since ϕ defines an $O_{\mathbb{A}^2}$- homomorphism, we have $y\phi(x^2y) = x\phi(xy^2)$, so that $2cxy = 0$ in $O_{\mathbb{A}^2}/I$. It follows that $c = 0$, and $\phi(V_3(\rho_2)) \subset V_{\ell-1}(\rho_2)$. Thus the proof of the formula for $I = \mathcal{I}(W(\rho_1', \rho_2))$ is complete.

Now we consider the general case. By **5.10** we see that $\{\mathfrak{m}/I\}[\rho]$ contains $V_{\bar{h}+d(\rho)}(\rho)$ as a nontrivial factor, while $\{\mathfrak{m}/I\}[\rho']$ contains $V_{\bar{h}-d(\rho')}(\rho')$ similarly. Moreover by the proof in **5.10** we see that either of the linear subspaces $\mathrm{Hom}_G(V_{\bar{h}-d(\rho)}(\rho), V_{\bar{h}+d(\rho)}(\rho))$ and $\mathrm{Hom}_G(V_{\bar{h}+d(\rho')}(\rho'), V_{\bar{h}-d(\rho')}(\rho'))$ yield nontrivial deformations of the ideal I inside the exceptional set E. Since $\dim T_{[I]}(X_G) = 2$ by Theorem 1.3, these linear subspaces span $T_{[I]}(X_G)$. Hence we have

$$\begin{aligned} T_{[I]}(X_G) &\simeq \mathrm{Hom}_G(V_{\bar{h}-d(\rho)}(\rho), V_{\bar{h}+d(\rho)}(\rho)) \\ &\qquad \oplus \mathrm{Hom}_G(V_{\bar{h}+d(\rho')}(\rho'), V_{\bar{h}-d(\rho')}(\rho')) \\ T_{[I]}(E(\rho)) &\simeq \mathrm{Hom}_G(V_{\bar{h}-d(\rho)}(\rho), V_{\bar{h}+d(\rho)}(\rho)) \\ T_{[I]}(E(\rho')) &\simeq \mathrm{Hom}_G(V_{\bar{h}+d(\rho')}(\rho'), V_{\bar{h}-d(\rho')}(\rho')), \end{aligned}$$

This completes the proof of Lemma 5.13 for $\rho, \rho' \neq \rho_{n-2}$. The cases $\rho = \rho_{n-2}$ are proved similarly. □

Lemma 5.14. *Let $E^*(\rho)$ be the closure in E of the set $\{\mathcal{I}(W); W \in C(\rho), W \neq V_{\bar{h}\pm d(\rho)}\}$. Then $E^*(\rho)$ is a smooth rational curve.*

Remark. We will see $E(\rho) = E^*(\rho)$ in 5.15.

Proof. By Lemma 5.13 $E^*(\rho)$ is smooth at $\mathcal{I}(W(\rho, \rho'))$ for ρ' adjacent to ρ. It remains to prove the assertion elesewhere on $E^*(\rho)$. Let $C^0(\rho) := \{W \in C(\rho); W \neq V_{\bar{h}\pm d(\rho)}\}$ and $I := \mathcal{I}(W)$ for $W \in C^0(\rho)$. Since we have a flat family of ideals $\mathcal{I}(W)$, $W \in C^0(\rho)$, we have a natural morphism $\iota : C^0(\rho) \to \mathrm{Hilb}^G(\mathbb{A}^2)$, so that we have a natural homomorphism $(d\iota)_* : T_{[W]}(C(\rho)) \to T_{[I]}(\mathrm{Hilb}^G(\mathbb{A}^2))$. Equivalently there is a homomorpism

$$(d\iota)_* : \mathrm{Hom}(W, V_{\bar{h}-d(\rho)}(\rho) + V_{\bar{h}+d(\rho)}(\rho)/W) \to \mathrm{Hom}_{O_{\mathbb{A}^2}}(I, O_{\mathbb{A}^2}/I)^G$$

Let $\phi \in T_{[W]}(C(\rho))$. Then $(d\iota)_*(\phi)(I) \subset \mathfrak{m}/I$ because $C(\rho) \subset E$. Recall $\{\mathfrak{m}/I\}[\rho_0] = 0$ by Corollary 2.2. Hence $(d\iota)_*(\phi)(\mathfrak{n}) = 0$. Since $I/\mathfrak{n}$ is generated by W by **5.10**, $(d\iota)_*(\phi)$ is induced from ϕ by extending it to $\oplus S_k W$ as an $O_{\mathbb{A}^2}$-homomorphism. Note that we have $V_{\bar{h}-d(\rho)}(\rho) + V_{\bar{h}+d(\rho)}(\rho)/W \subset \mathfrak{m}/I$. It follows that $(d\iota)_*$ is injective and that $C^0(\rho)$ is immersed at $\mathcal{I}(W)$. The same argument applies as well when $W = V_{\bar{h}+d(\rho)}$ if there is no adjacent ρ' with $d(\rho') > d(\rho)$. Hence $E^*(\rho)$ is a smooth rational curve. □

5.15. Proof of Theorem 3.1 for $G = \mathbb{D}_{n-2}$ and Theorem 5.9 - Completion. Let E be the exceptional set of π, and E^* the union of all $E^*(\rho)$ ($\rho \in \mathrm{Irr}(G)$). Since $E^*(\rho) \subset E(\rho)$

by **5.10**, E^* is a subset of E. Since π is a birational morphism, E is connected and it is set-theoretically the total fiber $\pi^{-1}(0)$ over the singular point $0 \in S_G$. Hence in particular $P(\rho, \rho') \subset E$ for anyρ, ρ'. By Lemma 5.12, E^* is connected because the (McKay) graph $\Gamma(\mathrm{Irr}(G))$ of $\mathrm{Irr}(G)$ is connected. By Lemma 5.14 E^* is smooth except at $\mathcal{I}(W(\rho, \rho'))$. By Lemma 5.13 E^* has two smooth irreducible components $E^*(\rho)$ and $E^*(\rho')$ meeting transversally at $\mathcal{I}(W(\rho, \rho'))$. It follows that E^* is a connected component of E. Hence $E^* = E$. It follows that $E(\rho) = E^*(\rho)$ ($\forall \rho \in \mathrm{Irr}(G)$), $P(\rho, \rho') = \{\mathcal{I}(W(\rho, \rho'))\}$ for ρ, ρ' adjacent, and $P(\rho, \rho') = \phi$ otherwise. Similarly $Q(\rho, \rho', \rho'') = \phi$. Thus Theorem 5.9 and Theorem 3.1 is proved.

5.16. Conclusion. The proof of Theorems 3.1 and 5.9 prove also Theorems 3.2, 3.4 and 3.6 automatically. Since any subscheme in $\mathrm{Hilb}^G(\mathbb{A}^2)$ with support outside the exceptional set E is a G-orbit of distinct $|G|$ points in $\mathbb{A}^2 \setminus \{0\}$, the defining ideal I of it is given by using G-invariant functions as follows

$$I = (F(x, y) - F(a, b), G(x, y) - G(a, b), H(x, y) - H(a, b)),$$

where $F(x, y) = x^\ell + y^\ell$, $G(x, y) = xy(x^\ell + y^\ell)$, $H(x, y) = x^2y^2$ and $(a, b) \neq (0, 0)$. Thus we obtain a complete description of the ideals in $\mathrm{Hilb}^G(\mathbb{A}^2)$.

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