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Citation	Hokkaido University technical report series in mathematics, 75, 1
Issue Date	2003-01-01
DOI	https://doi.org/10.14943/633
Doc URL	https://hdl.handle.net/2115/691
Type	departmental bulletin paper
File Information	quadricabstract.pdf



NUMERICAL CHARACTERISATIONS OF HYPERQUADRICS

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We characterise the smooth n -dimensional hyperquadrics as Fano manifolds of length n .

Given a Fano manifold X [resp. a pair (X, x_0) of a Fano manifold X and a closed point x_0 on it], we define the (global) *length* $l(X)$ of X [resp. the *local length* $l(X, x_0)$ of (X, x_0)] to be the positive integer

$$\min_{C \subset X} \{(C, -K_X)\},$$

where C runs through the set of the rational curves contained in X [resp. the set of the rational curves such that $x_0 \in C \subset X$].

The local length $l(X, x_0)$ is a lower semicontinuous function in x_0 and the global length $l(X)$ is by definition equal to $\inf_{x_0 \in X} l(X, x_0)$. For a given closed point $x_0 \in X$, it is known that $l(X, x_0) \leq \dim X + 1$, the equality holding if and only if X is projective space [2].

In terms of the notions above, our main result is the following

Theorem 1. *Let X be a smooth Fano variety of dimension $n \geq 3$ defined over an algebraically closed field k of characteristic zero. Then the following three conditions are equivalent:*

- (1) X is isomorphic to a smooth hyperquadric $Q_n \subset \mathbb{P}^{n+1}$.
- (2) The global length $l(X)$ is n .
- (3) $\rho(X) = 1$ and $l(X, x_0) = n$ for a sufficiently general point $x_0 \in X$, where $\rho(X)$ stands for the Picard number.

This simple numerical result involves the preceding characterisations due to Brieskorn [1], Kobayashi-Ochiai [6], and Cho-Sato [3][4] as immediate corollaries. Namely

Theorem 2. *For a smooth X Fano n -fold ($n \geq 3$) over \mathbb{C} , the three conditions in (0.1) are also equivalent to the following four:*

- (4) There is a homotopy equivalence between X and Q_n such that the induced cohomology isomorphism $H^2(Q_n, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})$ identifies the anticanonical classes.
- (5) The anticanonical class $c_1(X)$ is divisible by n in $\text{Pic}(X)$.
- (6) The tangent bundle Θ_X is not ample, but $\wedge^2 \Theta_X$ is ample.
- (7) There is a surjective morphism $Q_m \rightarrow X$, $m \geq n$, and $X \not\cong \mathbb{P}^n$.

The outline of the proof of Theorem 1 is as follows.

Assume that a smooth Fano n -fold X satisfies the condition (3). Because smooth Fano 3-folds with Picard number one are completely classified by Iskovskih [5], we may assume that $n \geq 4$ (this assumption is of course of purely technical nature). Pick up two general points $x_+, x_- \in X$. We consider an (arbitrary) irreducible component $W\langle x_+, x_- \rangle$ of the closed subset

$$\left\{ C \subset X \mid C \text{ is a connected union of rational curves, } C \supset \{x_+, x_-\}, (C, -K_X) = 2n \right\}$$

of the Chow scheme $\text{Chow}(X)$.

Under our hypothesis, it is easy to show that $\dim W\langle x_+, x_- \rangle = n - 1$. Each closed point $w \in W\langle x_+, x_- \rangle$ represents either an irreducible rational curve $C \subset X$ or a connected union of two irreducible rational curves $L_+ \cup L_- \subset X$ with $L_\pm \ni x_\pm$, $L_\pm \not\ni x_\mp$.

Let $V\langle x_+, x_- \rangle \subset W\langle x_+, x_- \rangle \times X$ be the associated incidence variety with natural surjective projection $\text{pr}_X: V\langle x_+, x_- \rangle \rightarrow X$. Let $\bar{V}\langle x_+, x_- \rangle$ denote the normalisation of $V\langle x_+, x_- \rangle$ and $\bar{\text{pr}}_X: \bar{V}\langle x_+, x_- \rangle \rightarrow X$ the induced projection. The inverse images of x_\pm via this projection determine distinguished sections $\sigma_\pm \subset \bar{V}\langle x_+, x_- \rangle$ over the normalisation $\bar{W}\langle x_+, x_- \rangle$ of $W\langle x_+, x_- \rangle$.

Given a smooth curve T and a morphism $f: T \rightarrow \bar{W}\langle x_+, x_- \rangle$, the fibre product $T \times_{\bar{W}\langle x_+, x_- \rangle} \bar{V}\langle x_+, x_- \rangle$ is a very special conic bundle over T with two specified sections σ_+, σ_- (*strongly symmetric conic bundle*).

Definition. Let $\pi: \mathcal{C} \rightarrow T$ be a normal conic bundle over a smooth projective curve T and B a nef and big Cartier divisor on \mathcal{C} . The fibre space $\pi: \mathcal{C} \rightarrow T$ (or the total space \mathcal{C} , by abuse of terminology) is said to be an *B-symmetric* conic bundle if $B\mathcal{C}_{t+} = B\mathcal{C}_{t-}$ whenever a closed fibre \mathcal{C}_t is a union of two components $\mathcal{C}_{t+}, \mathcal{C}_{t-}$.

Assume that $\pi: \mathcal{C} \rightarrow T$ has two distinct sections σ_+, σ_- . The triple $(\mathcal{C}; \sigma_+, \sigma_-)$ is said to be *strongly B-symmetric* if the following four conditions are satisfied:

- (a) \mathcal{C} is *B-symmetric*;
- (b) σ_+ and σ_- are mutually disjoint divisors contained in the non-critical locus $\mathcal{C}^\circ = \mathcal{C} \setminus \text{Cr}(\mathcal{C})$;
- (c) $B\sigma_+ = B\sigma_-$;
- (d) For any reducible fibre $\mathcal{C}_t = \mathcal{C}_{t+} + \mathcal{C}_{t-}$, we have

$$\begin{aligned} \sigma_+\mathcal{C}_{t+} &= \sigma_-\mathcal{C}_{t-} = 1, \\ \sigma_+\mathcal{C}_{t-} &= \sigma_-\mathcal{C}_{t+} = 0, \end{aligned}$$

(possibly after suitable reindexing of the irreducible components $\mathcal{C}_{t\pm}$). \square

Given a strongly symmetric conic bundle, we can deduce the following proposition from elementary, but a little complicated, calculation of intersection numbers (the detail will be published elsewhere):

Proposition. *Let $\pi: \mathcal{C} \rightarrow T$ be a normal conic bundle over a smooth projective curve with a nef big divisor B and two sections σ_+, σ_- . Assume that $(\mathcal{C}; \sigma_+, \sigma_-)$ is strongly *B-symmetric* and let s denote the number of the singular fibres $\mathcal{C}_{t_i} = \mathcal{C}_{t_i+} + \mathcal{C}_{t_i-}$ and put $\delta_i = \mathcal{C}_{t_i+} - \mathcal{C}_{t_i-}$. Let $\mu: \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ be the minimal resolution. Then we have:*

- (1) *If its strict transform $\tilde{\sigma} \subset \tilde{\mathcal{C}}$ has negative self intersection, then a section $\sigma \subset \mathcal{C}$ coincides with one of the two specified sections σ_\pm . In particular,*

σ is one of the σ_{\pm} once a section $\sigma \in \mathcal{C}$ satisfies $\sigma^2 < 0$. If $\sigma \neq \sigma_{\pm}$ and its strict transform $\tilde{\sigma}$ satisfies $\tilde{\sigma}^2 = 0$, then σ is disjoint with σ_{\pm} and with $\text{Sing}(\mathcal{C})$.

- (2) If there are two sections $\sigma_1, \sigma_2 \neq \sigma_{\pm} \in \mathcal{C}$ such that the strict transforms $\tilde{\sigma}_1, \tilde{\sigma}_2$ are mutually disjoint in $\tilde{\mathcal{C}}$, then $\sigma_1 \cup \sigma_2$ is away from $\sigma_+ \cup \sigma_-$.

With the aid of this proposition, we show (compare [2], Proof of Theorem 4.2, Step 3) that the projection $\overline{\text{pr}}_X: \overline{V}\langle x_+, x_- \rangle \rightarrow X$ has the following remarkable property.

Lemma. *Let C be a general fibre of the conic bundle $\overline{\text{pr}}_W: \overline{V}\langle x_+, x_- \rangle \rightarrow \overline{W}\langle x_+, x_- \rangle$. Then $\overline{\text{pr}}_X^{-1}(\overline{\text{pr}}_X(C))$ is a union of C , σ_+ , σ_- and possibly an extra closed subset which is away from σ_{\pm} .*

In view of this, we can apply a similar argument as in [*ibid*, Steps 4–8] to prove that $\overline{\text{pr}}_X$ lifts to an isomorphism between $\overline{V}\langle x_+, x_- \rangle$ and the two-point blowup $\text{Bl}_{\{x_+, x_-\}}X$ of X . This induces isomorphisms

$$\overline{W}\langle x_+, x_- \rangle \simeq \sigma_{\pm} \simeq E_{\pm} \simeq \mathbb{P}^{n-1},$$

where $E_{\pm} \subset \text{Bl}_{\{x_+, x_{\pm}\}}X$ is the exceptional divisor over $x_{\pm} \in X$. The pullback $\tilde{H}_0 = \text{pr}_{\overline{W}}^*L$ of the hyperplane divisor $L \subset \overline{W}\langle x_+, x_- \rangle \simeq \mathbb{P}^{n-1}$ is a semiample divisor on $\overline{V}\langle x_+, x_- \rangle \simeq \text{Bl}_{\{x_+, x_-\}}X$. Then we easily see that \tilde{H}_0 contracts to an ample divisor H_0 on X and that the complete linear system $|H_0|$ defines an isomorphism from X to a hyperquadric in \mathbb{P}^{n+1} .

The parameter space $W\langle x_+, x_- \rangle$ eventually turns out to be the dual projective space of the complete linear system $|\mu^*H_0 - E_+ - E_-| \simeq \mathbb{P}^{n-1}$ on $\text{Bl}_{\{x_+, x_-\}}X$, which is viewed as the sublinear system $|H_0(-x_+ - x_-)| \subset |H_0| \simeq \mathbb{P}^{n+1}$ on X . To be more explicit, for each $n - 1$ -dimensional linear subspace Λ of

$$H^0(X, \mathcal{I}_{x_+} \mathcal{I}_{x_-}(H_0)) \subset H^0(Q_n, \mathcal{O}(1)),$$

we associate $[C] \in W\langle x_+, x_- \rangle$, where C is the plane conic cut out of Q_n by the $n - 1$ hyperplanes $\in \Lambda$ through x_+, x_- .

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